Aust. J. Phys., 1983, 36, 775–98

Quantum Electrodynamics in Strong Magnetic Fields. II* Photon Fields and Interactions

D. B. Melrose

School of Physics, University of Sydney,
Sydney, N.S.W. 2006.

Abstract

A theory is developed which synthesizes the classical theory of wave–wave interactions in a medium and the QED theory of photon–photon interactions in a vacuum with or without static fields. First, a covariant version of the theory of wave dispersion in an arbitrary medium is outlined. Then the photon propagator is calculated both by solving the wave equation directly and from the vacuum expectation value, and the latter is generalized by introducing a statistically averaged propagator. The hierarchy of nonlinear responses of the medium is introduced and used to define a nonlinear interaction Hamiltonian. The expansion of the S matrix with this interaction Hamiltonian is re-ordered to correspond to a hierarchy of processes ordered according to the number of external photons. The theory is used to treat three-wave and four-wave interactions and both classical and QED applications are discussed. A covariant form of the Onsager relations and calculations of the hierarchy of response tensors using covariant classical kinetic theory and the Heisenberg–Euler Lagrangian are presented in appendices.

1. Introduction

It is well known that in the presence of a magnetostatic field $\boldsymbol{B}$ the vacuum is birefringent (see e.g. Toll 1952; Minguzzi 1957, 1958; Klein and Nigam 1964; Erber 1966). For $\sin\theta \neq 0$, where θ is the angle between the wavevector $\boldsymbol{k}$ and $\boldsymbol{B}$, both wave modes have $|\boldsymbol{k}| > \omega$; one is linearly polarized along $\boldsymbol{k} \times \boldsymbol{B}$ and the other is linearly polarized along $\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{B})$. In any version of QED in which the effect of a magnetostatic field is taken into account exactly, 'photons' must correspond to wave quanta in these two wave modes. Thus there are two kinds of photons and neither has the dispersion relation $k^2 = \omega^2 - |\boldsymbol{k}|^2 = 0$ in general. It follows that the photon propagator, whose poles correspond to the dispersion relations, cannot be proportional to $1/k^2$ and must be proportional to $1/(k^2-k_+^2)(k^2-k_-^2)$, where $k^2 = k_\pm^2$ denote the dispersion relations for the modes labelled + and − here. Both the wave properties and the photon propagator are found here by solving a wave equation which includes the response of the magnetized vacuum. This response may be described in terms of a polarization 4-tensor $\alpha^{\mu\nu}(k)$, which was calculated explicitly for the magnetized vacuum by Melrose and Stoneham (1976, 1977). The techniques required to derive the wave properties are essentially those used in the theory of wave dispersion in plasmas. These techniques may be applied to the magnetized vacuum alone or indeed to any system whose linear response may

* Part I, *Aust. J. Phys.*, 1983, 36, 755.

0004-9506/83/060775$02.00

be described in terms of a polarization 4-tensor $\alpha^{\mu\nu}(k)$. In effect a magnetized vacuum is like a material medium in that the properties of radiation in the system are intrinsically different from those of radiation in vacuo. Moreover, just as in a medium, one may define a hierarchy of nonlinear responses of the magnetized vacuum, and these responses allow the possibility of nonlinear interactions such as photon splitting and photon scattering.

In this paper a QED formulation is developed for the theory of photon–photon interactions in an arbitrary medium. The main motivation is connected with the development of a version of QED which takes the effects of a magnetostatic field into account exactly (Melrose and Parle 1983*a*, 1983*b*; present issue pp. 755 and 799). However, as formulated here, the theory depends only on the response tensors and may be applied to any system for which they are known. For a theory of the magnetized vacuum with an electron gas present, to be fully internally consistent one must use QED to calculate the response tensors, for example as done by Melrose (1974) and in Part III (Melrose and Parle 1983*b*). However, for many practical purposes one is well justified in using a simpler theory to calculate the response tensors. Indeed the theory developed here has two obvious applications, one as an alternative procedure for treating nonlinear photon–photon interactions in QED, and another as a covariant version of the theory of wave–wave interactions in dispersive media such as classical plasmas.

In Section 2 a covariant theory for wave dispersion (Melrose 1973, 1981, 1982) is summarized. The radiation field is then second quantized simply by associating an annihilation operator $\hat{c}_M(\boldsymbol{k})$ and a creation operator $\hat{c}_M^\dagger(\boldsymbol{k})$ with, respectively, the positive and negative frequency parts of the wave 4-potential for waves in the mode M. This quantization is effectively that of a scalar field, with the 4-vector nature of the field determined by the polarization 4-vector $e_M^\mu(\boldsymbol{k})$ for waves in the mode M. The photon propagator is calculated in Section 3 in two ways. One method involves solving the wave equation directly. The other method is based on the vacuum expectation value $\langle 0 \,|\, \hat{A}^\mu(x)\,\hat{A}^\nu(x') \,|\, 0\rangle$ and depends explicitly on the quantization of the photon field. The fact that the two methods produce equivalent results justifies the quantization procedure. In Section 3 a statistically averaged propagator is also defined and evaluated where the average is over the distributions of waves in various modes. A subsystem of waves in a particular mode M may be defined by writing down its Lagrangian, for example in terms of Whitham's (1965) averaged Lagrangian (Dewar 1977). It has been shown earlier (Melrose 1981) that the resulting classical wave action satisfies all the properties required for it to be reinterpreted as the photon occupation number, and this fact allows one to carry out the quantization procedure trivially. The statistical average here involves the photon occupation number.

In Section 4 the nonlinear response tensors are defined and used to construct a nonlinear Hamiltonian density $\hat{\mathscr{H}}_{\rm NL}(x)$ which describes the effects of wave–wave interactions. Proceeding as in QED, the S matrix is evaluated in Section 5 for interactions described by $\hat{\mathscr{H}}_{\rm NL}(x)$ alone. Interactions which involve the usual interaction Hamiltonian in QED and $\hat{\mathscr{H}}_{\rm NL}(x)$ together are discussed in Part III. Specific formulae describing three-wave interactions ('photon splitting' in QED) and four-wave interactions ('photon scattering' in QED) are derived in Section 6. Some aspects of the use of the theory are discussed and illustrated in Section 7. A number of more detailed points are treated in appendices. The covariant theory for wave dispersion is developed in more detail in Appendix 1 where it is applied to the important 'weak-anisotropy' limit where the refractive index is close to unity. A covariant form of the Onsager relations is presented in Appendix 2. The response tensors are calculated in Appendix 3 for classical unmagnetized and magnetized plasmas using a covariant kinetic theory, and they are calculated in Appendix 4 for static fields in vacuo using the Heisenberg–Euler Lagrangian.

The notation used here is essentially the same as that used in earlier papers (Melrose 1973, 1981, 1982). The main exceptions are the use of natural units here ($\hbar = c = 1$) and of electromagnetic units from the SI system. The combination of natural and SI units leads to $\varepsilon_0 = 1/\mu_0$, with $\varepsilon_0 = 1/4\pi$ in rationalized gaussian units and $\varepsilon_0 = 1$ in unrationalized (Heaviside–Lorentz) units. The charge of the electron ($-e$ with $e > 0$) appears in the fine structure constant $\alpha = e^2/4\pi\varepsilon_0$, the critical magnetic field $B_c = m^2/e = 4{\cdot}4\times 10^9$ T ($=4{\cdot}4\times 10^{13}$ G), the electron gyrofrequency $\Omega_0 = eB/m$ and the plasma frequency $\omega_{p0} = (e^2 n_0/\varepsilon_0 m)^{\frac{1}{2}}$.

2. Wave Properties

The covariant theory for wave dispersion used here was summarized by Melrose (1981). Briefly, given the linear response 4-tensor $\alpha^{\mu\nu}(k)$ one retains only the hermitian (H) part of it in writing

$$\Lambda^{\mu\nu}(k) \equiv k^2 g^{\mu\nu} - k^\mu k^\nu + \mu_0\, \alpha^{\mu\nu(\mathrm{H})}(k)\,. \tag{1}$$

One then constructs the tensor $\lambda^{\mu\nu}(k)$ whose elements are the cofactors of $\Lambda^{\nu\mu}(k)$, and the tensor $\lambda^{\mu\nu\alpha\beta}(k)$ of cofactors of the 2×2 minors of $\Lambda^{\mu\nu}(k)$. Due to the gauge invariance and charge continuity conditions one has $k_\nu \Lambda^{\mu\nu}(k) = 0 = k_\mu \Lambda^{\mu\nu}(k)$, and hence $\lambda^{\mu\nu}(k) = k^\mu k^\nu \lambda(k)$ where $\lambda(k)$ is an invariant. The dispersion equation may be written in the invariant form

$$\lambda(k) = 0\,. \tag{2}$$

Any particular solution of (2), $\omega = \omega_M(\boldsymbol{k})$ say, is the dispersion relation for a wave mode M. The solutions appear in pairs, one corresponding to a positive frequency and the other to a negative frequency. One is free to choose $\omega_M(-\boldsymbol{k}) = -\omega_M(\boldsymbol{k})$.

The polarization 4-vector $e^\mu_M(\boldsymbol{k})$ for mode M may be constructed using the identity

$$\lambda^{\mu\nu\alpha\beta}(k_M) \propto \{k^\mu_M\, e^\nu_M(\boldsymbol{k}) - k^\nu_M\, e^\mu_M(\boldsymbol{k})\}\{k^\alpha_M\, e^{*\beta}_M(\boldsymbol{k}) - k^\beta_M\, e^{*\alpha}_M(\boldsymbol{k})\}\,, \tag{3}$$

where k^μ_M denotes k^μ evaluated at $\omega = \omega_M(\boldsymbol{k})$. We are free to choose $e^\mu_M(-\boldsymbol{k}) = e^{*\mu}_M(\boldsymbol{k})$. The normalization is specified in the temporal gauge [$e^0_M(\boldsymbol{k}) = 0$]:

$$e^\mu_M(\boldsymbol{k})\, e^*_{M\mu}(\boldsymbol{k}) = -1\,. \tag{4}$$

The form of $e^\mu_M(\boldsymbol{k})$ in any other gauge is found by adding a constant times k^μ_M to $e^\mu_M(\boldsymbol{k})$ in the temporal gauge with the constant determined by the gauge condition in the desired gauge. The other wave quantity of interest is related to the constant of proportionality in (3) and may be interpreted as the ratio of electric to total energy in waves in mode M:

$$R_M(\boldsymbol{k}) = \left(-\frac{\lambda^{0\tau}{}_{0\tau}(k)}{\omega\, \partial\lambda(k)/\partial\omega}\right)_{\omega=\omega_M(\boldsymbol{k})}. \tag{5}$$

One has $R_M(-\boldsymbol{k}) = R_M(\boldsymbol{k})$.

(a) Explicit Expressions for $\lambda(k)$ and $\lambda^{\mu\nu\alpha\beta}(k)$

Some explicit expressions for $\lambda(k)$ and $\lambda^{\mu\nu\alpha\beta}(k)$ were quoted in Appendix 1 of Melrose (1981). These explicit forms involved $\Lambda^{\mu\nu}(k)$. In practice it is more convenient to have explicit forms involving traces of products of $\alpha^{\mu\nu(\mathrm{H})}(k)$. Relevant results are written down in Appendix 1 below.

An important limiting case is that in which the wave properties are not greatly different from those of transverse waves in vacuo. If $\alpha^{\mu\nu(\mathrm{H})}(k)$ is small in some meaningful sense then one can expand in powers of the components of $\alpha^{\mu\nu(\mathrm{H})}(k)$. The degeneracy between the two transverse states of polarization in vacuo is broken when terms up to second order in $\alpha^{\mu\nu(\mathrm{H})}(k)$ are retained. We refer to the approximation in which only terms up to this order are retained as the *weak anisotropy limit*. The wave properties in this limit are determined explicitly in Appendix 1.

(b) Onsager Relations

It is well known that the Onsager relations imply the identity (see e.g. Melrose 1980, p. 36)

$$\varepsilon_{ij}(\omega,\boldsymbol{k})\,|_{B} = \varepsilon_{ji}(\omega,\,-\boldsymbol{k})\,|_{-B}$$

for the equivalent dielectric tensor. This symmetry property ensures that, apart from an arbitrary phase factor, the polarization 3-vector is of the form

$$\boldsymbol{e}_M(\boldsymbol{k}) = \{K_M(\boldsymbol{k})\,\boldsymbol{\kappa} + T_M(\boldsymbol{k})\,\boldsymbol{t} + \mathrm{i}\,\boldsymbol{a}\}/\{K_M^2(\boldsymbol{k}) + T_M^2(\boldsymbol{k}) + 1\}^{\frac{1}{2}}, \tag{6}$$

with

$$\boldsymbol{\kappa} \equiv \boldsymbol{k}/|\boldsymbol{k}|, \qquad \boldsymbol{a} \equiv -\boldsymbol{\kappa}\times\boldsymbol{B}/|\boldsymbol{\kappa}\times\boldsymbol{B}|, \qquad \boldsymbol{t} \equiv \boldsymbol{a}\times\boldsymbol{\kappa}. \tag{7}$$

The polarization 4-vector in the temporal gauge is $e_M^\mu(\boldsymbol{k}) = \big(0, \boldsymbol{e}_M(\boldsymbol{k})\big)$ with $\boldsymbol{e}_M(\boldsymbol{k})$ satisfying (6) with real $K_M(\boldsymbol{k})$ and $T_M(\boldsymbol{k})$.

It is aesthetically displeasing to need to resort to non-covariant arguments to deduce the property (6). One may avoid this by identifying the covariant form of the Onsager relations, i.e. by identifying the time-reversal-invariance property of $\alpha^{\mu\nu}(k)$. This is done in Appendix 2. The generalization of (6) with (7) to an arbitrary frame and gauge is then found in terms of basis 4-vectors introduced by Shabad (1975), as discussed in Appendix 2.

(c) Second Quantization

Melrose (1981) wrote the Fourier transform of the 4-potential for waves in the mode M in the form, apart from an arbitrary phase factor,

$$A_M^\mu(k_M) = a_M(\boldsymbol{k})\,e_M^\mu(\boldsymbol{k})\,2\pi\delta\big(\omega-\omega_M(\boldsymbol{k})\big), \tag{8}$$

and showed that the electric energy in waves in the range $\mathrm{d}\boldsymbol{k}/(2\pi)^3$ is given by

$$W_M^{[E]}(\boldsymbol{k}) = \varepsilon_0\{\omega_M(\boldsymbol{k})\,a_M(\boldsymbol{k})\}^2. \tag{9}$$

On second quantizing we wish to normalize to one photon in the range (density of photon states) $V\mathrm{d}\boldsymbol{k}/(2\pi)^3$. This then corresponds to a total energy $W_M(\boldsymbol{k}) = W_M^{[E]}(\boldsymbol{k})/R_M(\boldsymbol{k})$ equal to $\omega_M(\boldsymbol{k})/V$. The desired normalization then corresponds to

$$a_M(\boldsymbol{k}) = \{R_M(\boldsymbol{k})/V\varepsilon_0\,\omega_M(\boldsymbol{k})\}^{\frac{1}{2}}. \tag{10}$$

The second quantization procedure is now straightforward. We introduce annihilation operators $\hat{c}_M(\boldsymbol{k})$ and creation operators $\hat{c}_M^\dagger(\boldsymbol{k})$ and require that they satisfy the commutation relations

$$[\hat{c}_M(\boldsymbol{k}), \hat{c}_{M'}^\dagger(\boldsymbol{k}')] = \delta_{MM'}\{(2\pi)^3/V\}\delta^3(\boldsymbol{k}-\boldsymbol{k}'), \tag{11a}$$

$$[\hat{c}_M(\boldsymbol{k}), \hat{c}_{M'}(\boldsymbol{k}')] = 0 = [\hat{c}_M^\dagger(\boldsymbol{k}), \hat{c}_{M'}^\dagger(\boldsymbol{k}')]. \tag{11b}$$

The 4-potential $A^\mu(x)$ for the radiation field is then rewritten as the operator

$$\hat{A}^\mu(x) = V\int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} \sum_M a_M(\boldsymbol{k})[e_M^\mu(\boldsymbol{k})\,\hat{c}_M(\boldsymbol{k})\exp\{-\mathrm{i}\,\omega_M(\boldsymbol{k})t + \mathrm{i}\,\boldsymbol{k}.\boldsymbol{x}\}$$

$$+ e_M^{*\mu}(\boldsymbol{k})\,\hat{c}_M^\dagger(\boldsymbol{k})\exp\{\mathrm{i}\,\omega_M(\boldsymbol{k})t - \mathrm{i}\,\boldsymbol{k}.\boldsymbol{x}\}], \tag{12}$$

where the sum is over all wave modes. Note that the negative frequency solutions are implicit in (8) due to $\omega_M(-\boldsymbol{k}) = -\omega_M(\boldsymbol{k})$, and these are explicit in (12) where only the positive solution $\omega_M(\boldsymbol{k}) > 0$ is to be retained in each term.

3. Photon Propagator

The inhomogeneous wave equation in the present context is

$$\Lambda^{\mu\nu}(k)\,A_\nu(k) = -\mu_0\,J^\mu_{\mathrm{ext}}(k), \tag{13}$$

where $J^\mu_{\mathrm{ext}}(k)$ is an arbitrary source term. The photon propagator (in momentum space) is defined as any quantity $D^{\mu\nu}(k)$ which satisfies

$$\Lambda^\mu{}_\nu(k)\,D^{\nu\rho}(k) = -\mu_0(g^{\mu\rho} - k^\mu k^\rho/k^2). \tag{14}$$

The final terms $k^\mu k^\rho/k^2$ on the right-hand side of (14) have no net effect due to charge continuity requiring $k_\mu J^\mu_{\mathrm{ext}}(k) = 0$ for any source term.

From the identity (A7) of Melrose (1981) and the form $\lambda^{\mu\nu}(k) = k^\mu k^\nu \lambda(k)$ for the matrix of cofactors, we have

$$\Lambda^\mu{}_\nu(k)\,\lambda^{\nu\rho\alpha\beta}(k) = \lambda(k)(g^{\mu\alpha}k^\rho k^\beta - g^{\mu\beta}k^\rho k^\alpha). \tag{15}$$

By inspection an acceptable form for the propagator is

$$D^{\mu\nu}(k) = \mu_0 \frac{k_\alpha k_\beta}{k^4}\frac{\lambda^{\mu\alpha\nu\beta}(k)}{\lambda(k)}. \tag{16}$$

As expected the poles of $D^{\mu\nu}(k)$ at $\lambda(k) = 0$ define the natural wave modes.

On using (16) and (14) to solve (13) one finds that the solution $A^\mu(k)$ is in the Lorentz gauge. If we desire $A^\mu(k)$ to be in an arbitrary gauge, with gauge condition

$$G_\alpha^*\,A^\alpha(k) = 0 \qquad (G\text{ gauge}), \tag{17}$$

say, then in place of (16) we may choose

$$D^{\mu\nu}(k) = \mu_0 \frac{G_\alpha^*\,G_\beta}{|Gk|^2}\frac{\lambda^{\mu\alpha\nu\beta}(k)}{\lambda(k)}. \tag{18}$$

More generally $D^{\mu\nu}(k)$ is defined only to within a transformation of the kind

$$D^{\mu\nu}(k) = D^{\mu\nu}(k) + k^{\mu}\zeta^{\nu} + \xi^{\mu}k^{\nu} + \chi k^{\mu}k^{\nu}, \tag{19}$$

where ζ^{ν}, ξ^{μ} and χ may depend on the components of k^{μ}.

In constructing the propagator in coordinate space, which we shall not do here, one may close the ω contour and use contour integration. The contribution from the poles is found using the Landau prescription. Near $\omega = \omega_M(\boldsymbol{k})$ one has

$$\lambda(k) \approx \{\partial\lambda(k)/\partial\omega\}_{\omega=\omega_M(\boldsymbol{k})}\{\omega - \omega_M(\boldsymbol{k}) + \mathrm{i}0\}. \tag{20}$$

We refer to the part of $D^{\mu\nu}(k)$ arising from the semiresidue at each pole as the resonant (res) part. One finds

$$D^{\mu\nu}_{(\mathrm{res})}(k) = \sum_M \frac{\mathrm{i}\,\pi\mu_0\, R_M(\boldsymbol{k})}{\omega_M(\boldsymbol{k})} e^{\mu}_M(\boldsymbol{k})\, e^{*\nu}_M(\boldsymbol{k})\,\delta\big(\omega - \omega_M(\boldsymbol{k})\big), \tag{21}$$

where the sum is over all the modes. At this stage both solutions $\omega = \omega_M(\boldsymbol{k}) = -\omega_M(-\boldsymbol{k})$ of (2) are implicit. The negative $\boldsymbol{k}$ solution is assumed to be related to the positive $\boldsymbol{k}$ solution by

$$\omega_M(-\boldsymbol{k}) = -\omega_M(\boldsymbol{k}), \qquad e^{\mu}_M(-\boldsymbol{k}) = e^{*\mu}_M(\boldsymbol{k}), \qquad R_M(-\boldsymbol{k}) = R_M(\boldsymbol{k}). \tag{22}$$

The two solutions contribute equally when evaluating the propagator in coordinate space.

We have invoked the causal condition in evaluating (21), and hence its Fourier inversion describes the propagator $D^{\mu\nu}(x-x')$ only for $t-t' > 0$. We have

$$\begin{aligned}\theta(t-t')\,D^{\mu\nu}(x-x') &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\exp\{-\mathrm{i}\,k(x-x')\}\,D^{\mu\nu}_{(\mathrm{res})}(k)\\ &= \sum_M \mathrm{i}\,\mu_0 \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\frac{R_M(\boldsymbol{k})}{\omega_M(\boldsymbol{k})} e^{\mu}_M(\boldsymbol{k})\, e^{*\nu}_M(\boldsymbol{k})\\ &\qquad \times \exp\{-\mathrm{i}\,\omega_M(\boldsymbol{k})(t-t') + \mathrm{i}\,\boldsymbol{k}.(\boldsymbol{x}-\boldsymbol{x}')\},\end{aligned} \tag{23a}$$

where a factor of 2 arises from the two solutions implicit in (21) with $\omega_M(\boldsymbol{k}) > 0$ now in (23a). The propagator for $t' > t$ is obtained by replacing $+\mathrm{i}0$ by $-\mathrm{i}0$ in (20), and hence by complex conjugating (21). Thus, we find

$$\begin{aligned}\theta(t'-t)\,D^{\mu\nu}(x-x') &= \sum_M \mathrm{i}\,\mu_0 \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\frac{R_M(\boldsymbol{k})}{\omega_M(\boldsymbol{k})} e^{*\mu}_M(\boldsymbol{k})\, e^{\nu}_M(\boldsymbol{k})\\ &\qquad \times \exp\{\mathrm{i}\,\omega_M(\boldsymbol{k})(t-t') - \mathrm{i}\,\boldsymbol{k}.(\boldsymbol{x}-\boldsymbol{x}')\},\end{aligned} \tag{23b}$$

where we again rewrite the result in terms of positive frequencies $\omega_M(\boldsymbol{k}) > 0$ for comparison with the vacuum expectation value.

(a) *Vacuum Expectation Value*

The definition of $D^{\mu\nu}$ as a vacuum expectation value is

$$D^{\mu\nu}(x-x') = \mathrm{i}\langle 0\,|\,T\{\hat{A}^{\mu}(x)\,\hat{A}^{\nu}(x')\}\,|\,0\rangle. \tag{24}$$

On using (10)–(12), and also the definition $\hat{c}_M(\boldsymbol{k})\,|\,0\rangle = 0$ of the vacuum, one obtains

$$D^{\mu\nu}(x-x') = \sum_M \mathrm{i} \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} \frac{R_M(\boldsymbol{k})}{\varepsilon_0\, \omega_M(\boldsymbol{k})} [\theta(t-t') e_M^{\mu}(\boldsymbol{k}) e_M^{*\nu}(\boldsymbol{k})$$
$$\times \exp\{-\mathrm{i}\,\omega_M(\boldsymbol{k})(t-t') + \mathrm{i}\,\boldsymbol{k}.(\boldsymbol{x}-\boldsymbol{x}')\} + \theta(t'-t)\, e_M^{*\mu}(\boldsymbol{k})\, e_M^{\nu}(\boldsymbol{k})$$
$$\times \exp\{\mathrm{i}\,\omega_M(\boldsymbol{k})(t-t') - \mathrm{i}\,\boldsymbol{k}.(\boldsymbol{x}-\boldsymbol{x}')\}], \tag{25}$$

which reproduces equations (23) as required.

(*b*) *Statistically Averaged Propagator*

If photons are present we may define a statistically averaged propagator by replacing the vacuum state $|\,0\rangle$ in (24) by the actual state $|\,\rangle$. Using the density matrix $\rho \equiv |\,\rangle\langle\,|$ we replace (24) by

$$D^{\mu\nu}(x-x') = \mathrm{i}\,\mathrm{Tr}\{\rho\, \hat{A}^{\mu}(x)\, \hat{A}^{\nu}(x')\}, \tag{26}$$

where 'Tr' denotes the trace. The evaluation of the trace reduces to

$$\mathrm{Tr}\{\rho\, \hat{c}_M^{\dagger}(\boldsymbol{k})\, \hat{c}_{M'}(\boldsymbol{k}')\} = N_M(\boldsymbol{k})(2\pi)^3 \delta^3(\boldsymbol{k}-\boldsymbol{k}')\delta_{MM'}, \tag{27a}$$

$$\mathrm{Tr}\{\rho\, \hat{c}_M(\boldsymbol{k})\, \hat{c}_{M'}^{\dagger}(\boldsymbol{k}')\} = \{1+N_M(\boldsymbol{k})\}(2\pi)^3 \delta^3(\boldsymbol{k}-\boldsymbol{k}')\delta_{MM'}. \tag{27b}$$

Using these properties one finds in place of (21)

$$\bar{D}^{\mu\nu}_{(\mathrm{res})}(k) = \sum_M \frac{\mathrm{i}\,\pi\mu_0\, R_M(\boldsymbol{k})}{\omega_M(\boldsymbol{k})} e_M^{\mu}(\boldsymbol{k})\, e_M^{*\nu}(\boldsymbol{k})\, \delta(\omega-\omega_M(\boldsymbol{k}))\{1+2N_M(\boldsymbol{k})\}. \tag{28}$$

4. Nonlinear Interactions

The expansion of the induced current in powers of the amplitude of the electromagnetic field defines not only the linear response tensor $\alpha^{\mu\nu}(k)$ but also a hierarchy of nonlinear response tensors:

$$J^{\mu}(k) = \alpha^{\mu\nu}(k)\, A_{\nu}(k) + \sum_{n=2}^{\infty} \int \mathrm{d}\lambda^{(n)}$$
$$\times\, \alpha^{\mu\nu_1\ldots\nu_n}(-k, k_1, \ldots, k_n)\, A_{\nu_1}(k_1) \ldots A_{\nu_n}(k_n), \tag{29}$$

$$\mathrm{d}\lambda^{(n)} \equiv \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \cdots \frac{\mathrm{d}^4 k_n}{(2\pi)^4} (2\pi)^4 \delta^4(k-k_1 \ldots -k_n). \tag{30}$$

(Note the convention that k is given a minus in the argument of $\alpha^{\mu\nu_1\ldots\nu_n}$.) The quadratic ($n=2$), cubic ($n=3$) etc. nonlinear responses imply the possibility of three-photon, four-photon etc. interactions respectively.

Without loss of generality we may assume that the nth order nonlinear response tensor is symmetric under permutations of the labels 1 to n. It then follows that, provided one neglects intrinsically nonlinear dissipative processes, the tensor is symmetric under permutations of all $n+1$ indices and arguments (Melrose 1972). That is, one has

$$\alpha^{\nu_0\nu_1\ldots\nu_n}(k_0, k_1, \ldots, k_n) = P^{01\ldots n}_{0'1'\ldots n'}\, \alpha^{\nu_{0'}\nu_{1'}\ldots\nu_{n'}}(k_{0'}, k_{1'}, \ldots, k_{n'}), \tag{31}$$

where $P^{01\ldots n}_{0'1'\ldots n'}$ denotes an arbitrary permutation of $0'1'\ldots n'$ amongst $01\ldots n$, and where we have

$$\sum_{i=0}^{n} k_i = \sum_{i'=0'}^{n'} k_{i'} = 0. \tag{32}$$

We may impose the symmetry property (31) explicitly by writing

$$\alpha^{\nu_0\nu_1\ldots\nu_n}(k_0, k_1, \ldots, k_n) = \frac{1}{(n+1)!}\sum_P P^{01\ldots n}_{0'1'\ldots n'} \times \alpha^{\nu_{0'}\nu_{1'}\ldots\nu_{n'}}(k_{0'}, k_{1'}, \ldots, k_{n'}), \tag{33}$$

where the sum is over all $(n+1)!$ permutations.

To second quantize (29) we simply replace $A^\mu(k)$ by the operator

$$\hat{A}^\mu(k) = \int \mathrm{d}^4x \exp(\mathrm{i}\,kx)\,\hat{A}^\mu(x), \tag{34}$$

with $\hat{A}^\mu(x)$ given by (12). Explicit evaluation gives

$$\hat{A}^\mu(k) = \sum_M a_M(\boldsymbol{k})\{e^\mu_M(\boldsymbol{k})\,\hat{c}_M(\boldsymbol{k})(2\pi)^4\delta^4(k-k_M) + e^{*\mu}_M(\boldsymbol{k})\,\hat{c}^\dagger_M(\boldsymbol{k})(2\pi)^4\delta^4(k+k_M)\}, \tag{35}$$

where $\omega_M(\boldsymbol{k})$ is now assumed positive.

Nonlinear Hamiltonian Density

The Hamiltonian density corresponding to the nonlinear interaction is

$$\mathscr{H}_{\mathrm{NL}}(x) = J^\mu_{\mathrm{NL}}(x)\,A_\mu(x), \tag{36}$$

with $J^\mu_{\mathrm{NL}}(x)$ given by the inverse Fourier transform of the sum in (29). In the following we shall be concerned only with

$$\int \mathrm{d}^4x\, \mathscr{H}_{\mathrm{NL}}(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, J^\mu_{\mathrm{NL}}(k)\,A_\mu(-k). \tag{37}$$

On inserting the sum from (29) into (37) we need to take account of the different meanings ascribed to $A^\mu(k)$. In equation (29), $A_{\nu_1}(k_1)$ etc. describe the components of a test field, and $A_\mu(-k)$ in (37) describes the nonlinear interaction with another test field. If we combine the two separate test fields into a single test field, then the self-interaction for the nth term in (29) is just $1/n$ of the result obtained by direct substitution, i.e. direct substitution corresponds to counting the self-interaction n times. Hence, we make the identification

$$\int \mathrm{d}^4x\, \hat{\mathscr{H}}_{\mathrm{NL}}(x) = \sum_{n=2}^{\infty}\frac{1}{n+1}\int\frac{\mathrm{d}^4k_0}{(2\pi)^4}\cdots\frac{\mathrm{d}^4k_n}{(2\pi)^4} \times (2\pi)^4\delta^4(k_0+\ldots+k_n)\alpha^{\nu_0\ldots\nu_n}(k_0,\ldots,k_n)\,\hat{A}_{\nu_0}(k_0)\ldots\hat{A}_{\nu_n}(k_n), \tag{38}$$

for the total nonlinear self-interaction, where we now second quantize denoting the operation by a circumflex.

5. *S*-matrix Expansion

Consider systems of photons interacting due to the nonlinear responses described by (29) and included in $\hat{\mathscr{H}}_{\mathrm{NL}}$. A dynamical theory for such nonlinear wave–wave interactions may be developed by using the conventional S-matrix expansion of QED with the interaction Hamiltonian density taken to be $\hat{\mathscr{H}}_{\mathrm{NL}}$. Formally, the expansion gives

$$\hat{S} = \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int \mathrm{d}^4x_1 \dots \mathrm{d}^4x_n\, T\{\hat{\mathscr{H}}_{\mathrm{NL}}(x_1) \dots \hat{\mathscr{H}}_{\mathrm{NL}}(x_n)\}. \tag{39}$$

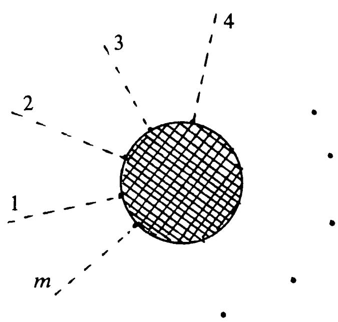

Fig. 1. An m-photon vertex is represented by a hatched circle with m external photon (dotted) lines.

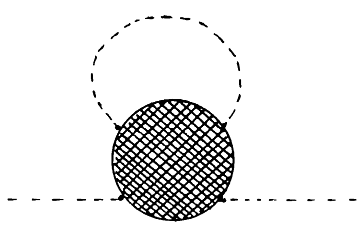

Fig. 2. A 4-photon vertex with two vertices joined by a photon propagator may represent (i) a radiative correction to the vacuum polarization or (ii) a nonlinear correction to the linear response tensor (provided the statistically averaged photon propagator is used).

A diagrammatic interpretation of terms in (39) involves associating an m-photon vertex (Fig. 1) with a response involving $\alpha^{\nu_0 \dots \nu_{m-1}}$ (Melrose 1974). We argue below that each such vertex is equivalent to a term of order $m-2$ in the usual expansion in QED. We need to rearrange the expansion (39) so that all terms of a given order are grouped together. Firstly, we use Wick's theorem to rearrange the integrand in normal order. The usual separation into connected and disconnected parts applies (see e.g. Bjorken and Drell 1965, p. 188) and we need consider only the sum of connected terms. The connected part involves terms with no contractions, one contraction, two contractions etc., with each contraction being associated with the photon propagator through (24). The resulting diagrams may be put into three classes. One class consists of an m-photon vertex with no propagators; these diagrams correspond to no contractions in the S-matrix expansion. The second class consists of m_1 photon, m_2 photon etc. vertices, each of which has one vertex connected to a photon propagator. The third class consists of diagrams in which one or more m-photon vertices has two or more vertices connected to a photon propagator. The simplest example of a diagram of the third class is the 4-photon vertex with two vertices connected by a photon propagator (Fig. 2). This corresponds either to a radiative or a nonlinear correction to the photon propagator.

(*a*) *Order of an S-matrix Element*

Let us define the order of an S matrix element in terms of the number of external lines. (All lines here are photon lines.) The photon propagator has two external lines and it is clearly of order zero in that it involves no photon interactions. The term involving the 3-photon vertex and no propagators is then of order one, and the term involving the m-photon vertex and no propagators is of order $m-2$. Diagrams of the second class involving m_1 photon, m_2 photon etc. vertices connected by propagators correspond to elements of order $(m_1-2)+(m_2-2)+\dots$. Then to second order we have one diagram of the first class involving the 4-photon vertex, and three diagrams of the second class involving two 3-photon vertices and a propagator (Fig. 3).

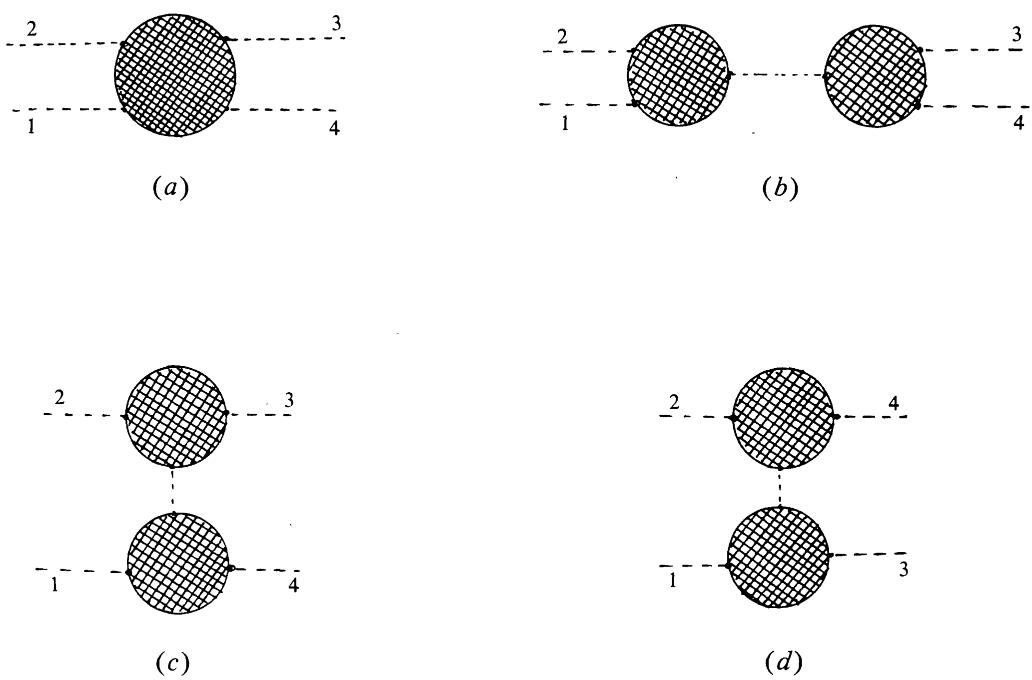

Fig. 3. Effective cubic response tensor is composed of the true cubic response tensor (*a*), and three pairs of quadratic response tensors (*b*), (*c*) and (*d*), each connected by a propagator.

(*b*) *Equivalent Response Tensors*

We may combine the terms with no contractions, those with one contraction between two m-photon vertices, two contractions between three m-photon vertices, and so on (but with no closed loops) and rearrange the expansion in the form

$$\sum_{n=1}^{\infty}\frac{(-\mathrm{i})^n}{n!}\int \mathrm{d}^4x_1 \dots \mathrm{d}^4x_n\, T\{\hat{\mathscr{H}}_{\mathrm{NL}}(x_1)\dots\hat{\mathscr{H}}_{\mathrm{NL}}(x_n)\}$$

$$=\sum_{m=3}^{\infty}(-\mathrm{i})\int \mathrm{d}^4x\, \hat{\mathscr{H}}_{\mathrm{NL}}^{(m-2)}(x) + \text{multiply contracted terms}. \qquad (40)$$

The equivalent m-photon vertex function involves a sum of terms including the m-photon vertex function and singly connected m_1 photon, m_2 photon etc. vertex functions, such that one has $(m_1-2)+(m_2-2)+\ldots = m-2$. In evaluating the contractions in momentum space, one uses

$$\underbrace{\hat{A}^{\mu}(k)\,\hat{A}^{\nu}}(k') = -\mathrm{i}\,D^{\mu\nu}(k)(2\pi)^4\delta^4(k+k'). \tag{41}$$

Then by writing

$$-\mathrm{i}\int \mathrm{d}^4x\, \hat{\mathscr{H}}_{\mathrm{NL}}^{(n-1)}(x) = -\frac{\mathrm{i}}{n+1}\int \frac{\mathrm{d}^4k_0}{(2\pi)^4}\cdots\frac{\mathrm{d}^4k_n}{(2\pi)^4}(2\pi)^4\delta^4(k_0+\ldots+k_n)$$
$$\times\,\tilde{\alpha}^{\nu_0\ldots\nu_n}(k_0,\ldots,k_n)\,\hat{A}_{\nu_0}(k_0)\ldots\hat{A}_{\nu_n}(k_n), \tag{42}$$

one defines a set of equivalent response tensors $\tilde{\alpha}$.

For $n = 2$ we have

$$\tilde{\alpha}^{\nu_0\nu_1\nu_2}(k_0,k_1,k_2) = \alpha^{\nu_0\nu_1\nu_2}(k_0,k_1,k_2). \tag{43a}$$

For $n = 3$ there is a contribution from pairs of 3-photon vertices (Figs 3*b*–*d*):

$$\tilde{\alpha}^{\nu_0\nu_1\nu_2\nu_3}(k_0,k_1,k_2,k_3) = \alpha^{\nu_0\nu_1\nu_2\nu_3}(k_0,k_1,k_2,k_3)$$
$$-2\{\alpha^{\nu_0\nu_1\eta}(k_0,k_1,k_2+k_3)\,D_{\eta\theta}(k_2+k_3)\,\alpha^{\theta\nu_2\nu_3}(k_2+k_3,k_2,k_3)+\text{perm.}\}, \tag{43b}$$

where '+ perm.' implies the addition of further terms obtained from that written by the interchanges $(\nu_1,k_1)\leftrightarrow(\nu_2,k_2)$ and $(\nu_1,k_1)\leftrightarrow(\nu_3,k_3)$.

(*c*) *Multiple Contractions*

The definition of the equivalent response tensors covers cases where two or more m-photon vertices are connected by photon propagators without any closed (photon) loops. In practice one is interested in cases where only the nonresonant part of these propagators contributes; the resonant part of a propagator splits the diagram into diagrams for two independent processes. More generally closed photon loops may occur. The simplest example is a propagator connecting two of the vertices in the 4-photon vertex, and analogous diagrams involving pairs of 3-photon vertices, as illustrated in Fig. 3. These terms correspond to nonlinear corrections to the linear response tensor. The best known physical consequences of such nonlinear corrections are self-focusing of light and collapse of Langmuir turbulence.

It follows from (29), with the response tensors replaced by equivalent response tensors, that for one contraction over two of the A factors in the cubic response we have a nonlinear correction to the linear response tensor:

$$\alpha_{\mathrm{NL}}^{\mu\nu}(k) = -3\mathrm{i}\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\,\tilde{\alpha}^{\mu\nu\theta\eta}(-k,k,k',-k')\,\bar{D}_{\theta\eta}(k'), \tag{44}$$

where we introduce the statistically averaged photon propagator (28). In the expression (28) for $\bar{D}^{\mu\nu}$ the unit term corresponds to a radiative correction in a QED calculation of the linear response tensor for the vacuum, and the term proportional to $N_M(\boldsymbol{k})$ corresponds to the nonlinear correction which allows the possibility of self-focusing of light and collapse of Langmuir turbulence.

(d) S Matrix in Momentum Space

In writing down the S matrix in momentum space we ignore the radiative and/or nonlinear corrections and retain only the connected diagrams. Then we have, finally,

$$\hat{S}^{(0)} = 1\,, \tag{45a}$$

$$\hat{S}^{(1)} = -\frac{\mathrm{i}}{3}\int \frac{\mathrm{d}^4k_0}{(2\pi)^4}\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\,\tilde{\alpha}^{\nu_0\nu_1\nu_2}(k_0,k_1,k_2)\,\hat{A}_{\nu_0}(k_0)\,\hat{A}_{\nu_1}(k_1)\,\hat{A}_{\nu_2}(k_2)\,, \tag{45b}$$

$$\hat{S}^{(n)} = -\frac{\mathrm{i}}{n+1}\int \frac{\mathrm{d}^4k_0}{(2\pi)^4}\cdots\frac{\mathrm{d}^4k_n}{(2\pi)^4}\,\tilde{\alpha}^{\nu_0\ldots\nu_n}(k_0,\ldots,k_n)\,\hat{A}_{\nu_0}(k_0)\ldots\hat{A}_{\nu_n}(k_n)\,. \tag{45c}$$

6. Photon Splitting and Photon–Photon Scattering

We may use the theory developed above to derive formulae describing photon–photon interactions. To first order one has photon splitting or photon coalescence, i.e. one photon in and two photons out or vice versa. In next order one has photon–photon scattering, i.e. two photons in and two photons out, and also one photon splitting into three and the inverse coalescence of three photons into one.

(a) Photon Splitting

Consider a photon in the mode M splitting into two photons in the modes M' and M'', with wavevectors $\boldsymbol{k}$, $\boldsymbol{k}'$ and $\boldsymbol{k}''$ respectively. Our initial state is $\hat{c}^{\dagger}_M(\boldsymbol{k})\,|\,0\rangle$ and our final state is $\langle 0\,|\,\hat{c}_{M'}(\boldsymbol{k}')\,\hat{c}_{M''}(\boldsymbol{k}'')$. On inserting (12) in (45b) and taking the matrix elements between these states one finds

$$S_{\mathrm{fi}} = -2\mathrm{i}(2\pi)^4\delta^4(k_M-k'_{M'}-k''_{M''})\,a_M(\boldsymbol{k})\,a_{M'}(\boldsymbol{k}')\,a_{M''}(\boldsymbol{k}'')\,\alpha^{MM'M''}(\boldsymbol{k},-\boldsymbol{k}',-\boldsymbol{k}'')\,, \tag{46}$$

with

$$\alpha^{MM'M''}(\boldsymbol{k},-\boldsymbol{k}',-\boldsymbol{k}'') = e^{\mu}_M(\boldsymbol{k})\,e^{*\nu}_{M'}(\boldsymbol{k}')\,e^{*\rho}_{M''}(\boldsymbol{k}'')\,\alpha_{\mu\nu\rho}(k_M,-k'_{M'},-k''_{M''})\,. \tag{47}$$

The factor 2 in (46) arises from the factor $\frac{1}{3}$ in (45b) and a factor 6 from the six terms of the form $A_M A_{M'} A_{M''}$ in $(A_M+A_{M'}+A_{M''})^3$.

The rate per unit time at which the splitting proceeds is given by

$$\mathrm{d}w_{\mathrm{fi}} = \lim_{T\to\infty}\frac{|\,S_{\mathrm{fi}}\,|^2}{T}\frac{V\,\mathrm{d}\boldsymbol{k}'}{(2\pi)^3}\frac{V\,\mathrm{d}\boldsymbol{k}''}{(2\pi)^3}\,, \tag{48}$$

where density of states factors for the final electrons are included. On using (10) in (46), one finds

$$\mathrm{d}w_{\mathrm{fi}} = u^{MM'M''}(\boldsymbol{k},-\boldsymbol{k}',-\boldsymbol{k}'')\frac{\mathrm{d}\boldsymbol{k}'}{(2\pi)^3}\frac{\mathrm{d}\boldsymbol{k}''}{(2\pi)^3}\,, \tag{49}$$

with

$$u^{MM'M''}(\boldsymbol{k},-\boldsymbol{k}',-\boldsymbol{k}'') = 4\frac{(2\pi)^4}{\varepsilon_0^3}\frac{R_M(\boldsymbol{k})\,R_{M'}(\boldsymbol{k}')\,R_{M''}(\boldsymbol{k}'')}{|\,\omega_M(\boldsymbol{k})\,\omega_{M'}(\boldsymbol{k}')\,\omega_{M''}(\boldsymbol{k}'')\,|}$$

$$\times|\,\alpha^{MM'M''}(\boldsymbol{k},-\boldsymbol{k}',-\boldsymbol{k}'')\,|^2\delta\big(\omega_M(\boldsymbol{k})-\omega_{M'}(\boldsymbol{k}')-\omega_{M''}(\boldsymbol{k}'')\big)\,\delta^3(\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{k}'')\,. \tag{50}$$

In (49) and (50) we adopt the convention that positive $\boldsymbol{k}$ denotes an outgoing photon and negative $\boldsymbol{k}$ denotes an ingoing photon. With this convention (50) includes all the crossed processes with the relation between the positive and negative $\boldsymbol{k}$ defined by (22).

(b) The Case $M' = M''$

Suppose the two final photons are in the same mode M'. Then the factor 2 is absent in (46). However, there is a factor of 2 which arises from $\langle 0 | \hat{c}_{M'} \hat{c}_{M'} \hat{c}^{\dagger}_{M'} \hat{c}^{\dagger}_{M'} | 0 \rangle = 2$. The net effect is that the factor 4 in (50) is replaced by 2. Alternatively one may adopt the convention that in this case one is to use (50) as it stands and to integrate over only half the final available phase space.

(c) Photon–Photon Scattering

For photon–photon scattering, say of waves in the modes M_1 and M_2 into waves in the modes M_3 and M_4 with wavevectors $\boldsymbol{k}_1$ to $\boldsymbol{k}_4$ respectively, one finds in place of (46)

$$\begin{aligned} S_{\mathrm{fi}} = & -6\mathrm{i}(2\pi)^4 \delta^4(k_{M_1} + k_{M_2} - k_{M_3} - k_{M_4})\, a_{M_1}(\boldsymbol{k}_1)\, a_{M_2}(\boldsymbol{k}_2) \\ & \times a_{M_3}(\boldsymbol{k}_3)\, a_{M_4}(\boldsymbol{k}_4)\, \tilde{\alpha}^{M_1 M_2 M_3 M_4}(-\boldsymbol{k}_1, -\boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4), \end{aligned} \tag{51}$$

with

$$\begin{aligned} \tilde{\alpha}^{M_1 M_2 M_3 M_4}(-\boldsymbol{k}_1, -\boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4) \equiv & \; e^{*\mu}_{M_1}(\boldsymbol{k}_1)\, e^{*\nu}_{M_2}(\boldsymbol{k}_2)\, e^{\rho}_{M_3}(\boldsymbol{k}_3)\, e^{\sigma}_{M_4}(\boldsymbol{k}_4) \\ & \times \tilde{\alpha}_{\mu\nu\rho\sigma}(k_{M_1}, k_{M_2}, -k_{M_3}, -k_{M_4}). \end{aligned} \tag{52}$$

The rate per unit time for this process is given by

$$\mathrm{d}w_{\mathrm{fi}} = V^{-1} u^{M_1 M_2 M_3 M_4}(\boldsymbol{k}_1, \boldsymbol{k}_2, -\boldsymbol{k}_3, -\boldsymbol{k}_4) \frac{\mathrm{d}\boldsymbol{k}_1}{(2\pi)^3} \frac{\mathrm{d}\boldsymbol{k}_3}{(2\pi)^3}, \tag{53}$$

with

$$\begin{aligned} u^{M_1 M_2 M_3 M_4}(\boldsymbol{k}_1, \boldsymbol{k}_2, -\boldsymbol{k}_3, -\boldsymbol{k}_4) = & \; \{36(2\pi)^4/\varepsilon_0^4\} \\ & \times \frac{R_{M_1}(\boldsymbol{k}_1)\, R_{M_2}(\boldsymbol{k}_2)\, R_{M_3}(\boldsymbol{k}_3)\, R_{M_4}(\boldsymbol{k}_4)}{|\, \omega_{M_1}(\boldsymbol{k}_1)\, \omega_{M_2}(\boldsymbol{k}_2)\, \omega_{M_3}(\boldsymbol{k}_3)\, \omega_{M_4}(\boldsymbol{k}_4) \,|} \,|\, \tilde{\alpha}^{M_1 M_2 M_3 M_4}(\boldsymbol{k}_1, \boldsymbol{k}_2, -\boldsymbol{k}_3, -\boldsymbol{k}_4) \,|^2 \\ & \times \delta\big(\omega_{M_1}(\boldsymbol{k}) + \omega_{M_2}(\boldsymbol{k}_2) - \omega_{M_3}(\boldsymbol{k}_3) - \omega_{M_4}(\boldsymbol{k}_4)\big)\, \delta^3(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4). \end{aligned} \tag{54}$$

7. Specific Applications

The theory developed here reproduces known results in the classical theory of wave–wave interactions in plasmas and in the QED theory of photon–photon interactions in a vacuum with (or without) a static electromagnetic field. Here we discuss (a) the possible significance of a covariant and gauge invariant classical theory and (b) photon splitting in QED in the presence of static magnetic and electric fields and a cold electron gas.

(a) Classical Wave–Wave Interactions

The results derived in Section 6 differ from those derived by semiclassical methods only in the use of covariant notation. Before commenting on the covariance and gauge invariance of the theory there is one intrinsically quantum effect which might be pointed out.

In a three-wave interaction $M \leftrightarrow M' + M''$ the rate of the process is proportional to

$$N_M(\boldsymbol{k})\{1+N_{M'}(\boldsymbol{k}')\}\{1+N_{M''}(\boldsymbol{k}'')\}-\{1+N_M(\boldsymbol{k})\}N_{M'}(\boldsymbol{k}')\,N_{M''}(\boldsymbol{k}'').$$

This includes a term $N_M(\boldsymbol{k})$ and other terms involving the product of two occupation numbers. In semiclassical theory the term proportional to $N_M(\boldsymbol{k})$ is discarded because it is impossible to describe the corresponding process without using Planck's constant. In other words the process described by the term $N_M(\boldsymbol{k})$ is intrinsically quantum mechanical. In the usual treatment of photon splitting in a magnetized vacuum the occupation numbers of the final photons are assumed negligible, and this corresponds to ignoring all but the $N_M(\boldsymbol{k})$ term. Thus, although 'photon splitting' and 'three-wave interactions' may be described by the same probability (47), the conventional meanings of these terms imply quite distinct processes.

A covariant and gauge invariant theory has the obvious advantages in that one may choose the frame and the gauge for convenience. However, these advantages are relevant only if the response tensors $\alpha^{\mu\nu}$, $\alpha^{\mu\nu\rho}$ etc. are available in covariant form. (If they are not available one may construct them from the corresponding 3-tensors, but this is sufficiently tedious to outweigh any calculational advantages.) It is straightforward to calculate the tensors using covariant versions of cold plasma theory and of kinetic (Vlasov) theory, for example in the form used by Melrose (1982). The calculation using covariant kinetic theory has a major advantage over the non-covariant form in that it is relatively simple to partially integrate and reduce the resulting cumbersome expressions to relatively simple forms in which the symmetry properties of the tensors are manifest. The final results, which have not been written down before (except for $\alpha^{\mu\nu}$ in an unmagnetized plasma), are given in Appendix 3 for both the unmagnetized and magnetized cases. The cold plasma case follows trivially by writing $F(p) = n_0\,\delta^4(p-p_0)$, where n_0 and p_0^μ are constants. To obtain the cold plasma limit in the magnetized case, one is also to take the small gyroradius limit $R = 0$, which corresponds to setting $s_0 = 0$, $s_1 = 0$ etc., with $\mathrm{J}_0(0) = 1$ and $\mathrm{J}_s(0) = 0$ for $s \neq 0$ in equations (A34)–(A38). Using these results the theory is automatically in covariant form.

The advantages of a gauge invariant theory are rather limited. It is well known that waves in an isotropic plasma are either longitudinal or transverse. However, this is the case only in the rest frame of the plasma and in any other frame the waves are not longitudinal or transverse. In the rest frame it may be convenient to use the Coulomb gauge to describe longitudinal waves. These waves have $e_{\mathrm{L}}^\mu = (0,\boldsymbol{\kappa})$ in the temporal gauge, and hence

$$e_{\mathrm{L}}^\mu = -\{\omega_{\mathrm{L}}(\boldsymbol{k})/|\boldsymbol{k}|\}(1,\mathbf{0}) \tag{55}$$

in the Coulomb gauge. Thus, for example, the matrix element for a three-wave interaction between three longitudinal waves is

$$\alpha^{\mathrm{LLL}}(k_0,k_1,k_2) = -\{\omega_0\,\omega_1\,\omega_2/|\boldsymbol{k}_0|\,|\boldsymbol{k}_1|\,|\boldsymbol{k}_2|\}\alpha^{000}(k_0,k_1,k_2),$$

and so on.

(*b*) *Photon Splitting in a Strong Magnetic Field*

The conventional treatment of photon splitting in a strong magnetic field (e.g. Adler *et al.* 1970; Białynicka-Birula and Białynicki-Birula 1970; Adler 1971) involves expanding in powers of B. The lowest order Feynman diagram is the box with vertices corresponding to the three photons and to $\boldsymbol{B}$, but this gives zero in the nondispersive case, i.e. when the photons are assumed to have $k^2 = 0$. The hexagon diagram, with three vertices corresponding to $\boldsymbol{B}$, gives the lowest order nonvanishing result. In the simplest treatment the amplitudes for the box and hexagon diagrams are calculated indirectly using the Heisenberg–Euler Lagrangian, and this restricts the validity to 'low' frequencies. Exact calculations are possible (Adler 1971; Stoneham 1979), and an exact treatment of photon dispersion (Stoneham 1978) suggests that 'low' frequencies correspond to

$$\frac{\omega}{2m}\frac{(E^2+B^2)^{\frac{1}{2}}}{B_{\mathrm{c}}}\sin\theta \ll 1\,, \tag{56}$$

which is adequately satisfied for most purposes.

The method developed here allows one to treat photon splitting in a relatively simple way, and it also allows one to include plasma effects. The response tensors $\alpha^{\mu\nu}$, $\alpha^{\mu\nu\rho}$ and $\alpha^{\mu\nu\rho\tau}$ are calculated in Appendix 4 from the Heisenberg–Euler Lagrangian. In the weak anisotropy limit the tensor $t^{\mu\nu}$ introduced in Appendix 1 describes the dispersive properties: $t^{\mu\nu}$ is equal to $\mu_0\,\alpha^{\mu\nu}(k)$ evaluated at $k^2 = 0$ and with only the components orthogonal both to the time axis and to k^μ retained. Choosing axes along

$$e_1^\mu = (0,\boldsymbol{a})\,, \qquad e_2^\mu = (0,\boldsymbol{t})\,, \tag{57a, b}$$

cf. equations (7), in a frame in which $\boldsymbol{E}$ and $\boldsymbol{B}$ are parallel, one finds

$$t^{\mu\nu} = \frac{\alpha}{90\pi}\frac{\omega^2\sin^2\theta}{B_{\mathrm{c}}^2}\begin{pmatrix} 8E^2+14B^2 & -6EB \\ -6EB & 14E^2+8B^2 \end{pmatrix}, \tag{58}$$

with $\alpha \equiv e^2/4\pi\varepsilon_0$ the fine structure constant. The results derived in Appendix 1 then imply

$$k^2 = k_\pm^2 = (\alpha/90\pi)\omega^2\sin^2\theta\{(E^2+B^2)/B_{\mathrm{c}}^2\}(-11\pm3)\,, \tag{59}$$

$$e_+^\mu = (0, E\boldsymbol{a}-B\boldsymbol{t})/(E^2+B^2)^{\frac{1}{2}}\,, \qquad e_-^\mu = (0, B\boldsymbol{a}+E\boldsymbol{t})/(E^2+B^2)^{\frac{1}{2}}\,. \tag{60a, b}$$

The well-known results for a magnetized vacuum are reproduced for $E = 0$. Note that one has $e_+^\mu \propto k_\alpha F^{(\mathrm{D})\alpha\mu} \propto b_2^\mu$ and $e_-^\mu \propto k_\alpha F_0^{\alpha\mu} \propto b_1^\mu$, where b_1^μ and b_2^μ are defined by equations (A21a) and (A21b).

It might be remarked that a more conventional derivation of the wave properties (59) and (60) using non-covariant methods is considerably more complicated than the covariant method used here. Although the vacuum response for $E = 0$ and $B \neq 0$ may be described in terms of a dielectric tensor and magnetic permeability tensor, for $E \neq 0$ and $B \neq 0$ there are also magneto–electric responses and the two magneto–electric susceptibility tensors must be included. Here the magneto–electric responses appear in the terms $6EB$ in (58).

In the non-dispersive limit the three wavevectors must be collinear and hence we have $\sin\theta = \sin\theta' = \sin\theta''$. Using this fact, on considering the small contributions to $k_\pm^2$, one finds that $k^\mu = k'^\mu + k''^\mu$ can be satisfied only for $M = -$ and $M' = M'' = +$. Using the explicit expression (A49) for $\alpha^{\mu\nu\rho}$ one finds that the term of $O(e^4)$ does not contribute (due to terms like $k_0 k_1$ being proportional to k^2 which is assumed negligible and terms like $k^\mu e_\mu$ vanishing due to the waves being assumed transverse) and, as a result of the orthogonality of b_1^μ and b_2^μ, only the final explicit term in (A49) contributes for $M = -$ and $M' = M'' = +$. Substitution into (47) and (50) then gives

$$u^{-++}(\boldsymbol{k}, -\boldsymbol{k}', -\boldsymbol{k}'') = \frac{(2\pi)^4}{4\varepsilon_0^3\,\omega\omega'\omega''}\left(\frac{13e^6}{315\pi^2 m^8}(E^2+B^2)^{3/2}\omega\omega'\omega''\sin^3\theta\right)^3$$
$$\times\,\delta(\omega-\omega'-\omega'')\,\delta^3(\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{k}''), \tag{61}$$

which reproduces Adler's (1971) result (cf. his equations 21 and 22), with the minor generalization that a nonzero parallel electric field is included. Further generalizations have been discussed in detail by Stoneham (1979).

A plasma can affect photon splitting in at least three ways: it affects the dispersion of the waves, the polarization of the waves and it contributes to $\alpha^{\mu\nu\rho}$. The first two effects may be taken into account in the cold plasma approximation by adding to (58) the contribution

$$t_{\mathrm{p}}^{\mu\nu} = \omega_{\mathrm{p}0}^2\begin{pmatrix} \omega^2/(\omega^2-\Omega_0^2) & \mathrm{i}\,\omega\Omega_0\cos\theta/(\omega^2-\Omega_0^2) \\ -\mathrm{i}\,\omega\Omega_0\cos\theta/(\omega^2-\Omega_0^2) & \omega^2\cos^2\theta/(\omega^2-\Omega_0^2)+\sin^2\theta \end{pmatrix} \tag{62}$$

from the cold plasma in its rest frame with $\omega_{\mathrm{p}0}$ and Ω_0 the plasma frequency and electron gyrofrequency respectively. For $\omega^2 \gg \Omega_0^2$ the contribution to k^2 is positive and of order $\omega_{\mathrm{p}0}^2$. Inspection of (59) shows that the contribution of the electron gas and of the magnetized vacuum ($B \gg E$) are in a ratio of order $(90\pi/\alpha)(\omega_{\mathrm{p}0}^2/\Omega_0^2)(m/\omega)^2$. If the plasma dominates one finds that photon splitting is kinematically forbidden. Even when the effect of the plasma is weak it causes the natural modes to be slightly elliptically polarized rather than strictly linearly polarized, as (A6) implies. However, this has only a minor effect on the numerical coefficient in (61).

The contribution of the plasma to $\alpha^{\mu\nu\rho}$ has not been recognized in this context previously. The relevant form of $\alpha^{\mu\nu\rho}$ is for a cold magnetized plasma and is given by setting $R = 0$ and $F(p) = n_0\,\delta^4(p)$ in equation (A34) with (A35)–(A38). For $\omega \gg \Omega_0$ expanding in Ω_0/ω gives the unmagnetized case to lowest order, i.e. (A29) with $F(p) = n_0\,\delta^4(p)$, and this gives zero for three transverse waves. To next order in Ω_0/ω one obtains a nonzero result and comparing it with the vacuum contribution to $\alpha^{\mu\nu\rho}$ one finds them to be in a ratio of order $(315\pi^2/13\alpha)(\omega_{\mathrm{p}0}^2/\Omega_0^2)(m/\omega)^4$. One may conclude that there is a small range of parameters where the nonlinear contribution of the plasma is important but the dispersion of the plasma is not. This range corresponds roughly to

$$\left(\frac{\omega}{m}\right)^2 \ll \frac{90\pi}{\alpha}\frac{\omega_{\mathrm{p}0}^2}{\Omega_0^2}\left(\frac{m}{\omega}\right)^2 \ll 1\,. \tag{63}$$

It is unlikely that (63) is satisfied in cases of practical interest.

8. Conclusions

The theory developed here synthesizes the classical theory of wave–wave interactions in plasmas and photon–photon interactions in QED. The ideas and methods involved in treating nonlinear electromagnetic interactions here are also relevant in the treatment of particle–wave interactions, which are discussed in the following Part III.

Acknowledgment

I thank A. J. Parle for helpful comments on the manuscript.

References

Adler, S. L. (1971). *Ann. Phys. (New York)* **67**, 599.
Adler, S. L., Bahcall, J. N., Callan, C. C., and Rosenbluth, M. N. (1970). *Phys. Rev.* **25**, 1061.
Białynicka-Birula, Z., and Białynicki-Birula, I. (1970). *Phys. Rev.* D **2**, 2341.
Bjorken, J. D., and Drell, S. D. (1965). 'Relativistic Quantum Fields' (McGraw–Hill: New York).
Dewar, R. L. (1977). *Aust. J. Phys.* **30**, 533.
Erber, T. (1966). *Rev. Mod. Phys.* **38**, 626.
Jauch, J. M., and Rohrlich, F. (1955). 'The Theory of Photons and Electrons' (Addison–Wesley: Reading, Mass.).
Klein, J. J., and Nigam, B. P. (1964). *Phys. Rev.* B **135**, 1279.
Melrose, D. B. (1972). *Plasma Phys.* **14**, 1035.
Melrose, D. B. (1973). *Plasma Phys.* **15**, 99.
Melrose, D. B. (1974). *Plasma Phys.* **16**, 845.
Melrose, D. B. (1980). 'Plasma Astrophysics', Vol. I (Gordon and Breach: New York).
Melrose, D. B. (1981). *Aust. J. Phys.* **34**, 563.
Melrose, D. B. (1982). *Aust. J. Phys.* **35**, 67.
Melrose, D. B., and Parle, A. J. (1983*a*). *Aust. J. Phys.* **36**, 755.
Melrose, D. B., and Parle, A. J. (1983*b*). *Aust. J. Phys.* **36**, 799.
Melrose, D. B., and Stoneham, R. J. (1976). *Nuovo Cimento* A **32**, 435.
Melrose, D. B., and Stoneham, R. J. (1977). *J. Phys.* A **10**, 1211.
Minguzzi, A. (1957). *Nuovo Cimento* **6**, 501.
Minguzzi, A. (1958). *Nuovo Cimento* **9**, 145.
Schwinger, J. (1951). *Phys. Rev.* **82**, 664.
Shabad, A. E. (1975). *Ann. Phys. (New York)* **90**, 166.
Stoneham, R. J. (1978). Thesis, Australian National University.
Stoneham, R. J. (1979). *J. Phys.* A **12**, 2187.
Toll, J. S. (1952). Thesis, Princeton University.
Whitham, G. B. (1965). *J. Fluid Mech.* **22**, 273.

Appendix 1. Weak Anisotropy Limit

Let us write

$$t^{\mu\nu}(k) \equiv \mu_0\, \alpha^{(\mathrm{H})\mu\nu}(k), \tag{A1}$$

omit arguments k and define the traces

$$t^{(1)} \equiv t^{\mu}{}_{\mu}, \qquad t^{(2)} \equiv t^{\mu}{}_{\nu}t^{\nu}{}_{\mu}, \qquad t^{(3)} \equiv t^{\mu}{}_{\nu}t^{\nu}{}_{\rho}t^{\rho}{}_{\mu}. \tag{A2a, b, c}$$

One then finds, for example using equations (A13) and (A14) of Melrose (1981),

$$\lambda = (1/6k^2)[6k^6+6k^4t^{(1)}+3k^2\{(t^{(1)})^2-t^{(2)}\} +(t^{(1)})^3-3t^{(1)}t^{(2)}+2t^{(3)}], \tag{A3}$$

$$
\begin{aligned}
\lambda^{\mu\nu\alpha\beta} = {} & \tfrac{1}{2}(g^{\mu\alpha}g^{\nu\beta}-g^{\mu\beta}g^{\nu\alpha})\{(t^{(1)})^2-t^{(2)}\}+t^{\mu\alpha}t^{\nu\beta}-t^{\mu\beta}t^{\nu\alpha} \\
& +k^{\mu}k^{\alpha}\{(k^2+t^{(1)})g^{\nu\beta}-t^{\nu\beta}\}-k^{\mu}k^{\beta}\{(k^2+t^{(1)})g^{\nu\alpha}-t^{\nu\alpha}\} \\
& +k^{\nu}k^{\beta}\{(k^2+t^{(1)})g^{\mu\alpha}-t^{\mu\alpha}\}-k^{\nu}k^{\alpha}\{(k^2+t^{(1)})g^{\mu\beta}-t^{\mu\beta}\} \\
& +g^{\mu\alpha}(-t^{(1)}t^{\nu\beta}+t^{(2)\nu\beta})-g^{\mu\beta}(-t^{(1)}t^{\nu\alpha}+t^{(2)\nu\alpha}) \\
& +g^{\nu\beta}(-t^{(1)}t^{\mu\alpha}+t^{(2)\mu\alpha})-g^{\nu\alpha}(-t^{(1)}t^{\mu\beta}+t^{(2)\mu\beta})\,,
\end{aligned}
\tag{A4}
$$

with $t^{(2)\mu\nu} \equiv t^{\mu}{}_{\rho}t^{\rho\nu}$. It is usually simpler to calculate the determinant of the 3-tensor rather than the trace of the cube of the 4-tensor:

$$
\begin{aligned}
t^{(\mathrm{D})} &\equiv (1/\omega^2)\det\{t^{i}{}_{j}(k)\} \\
&= (1/6k^2)\{(t^{(1)})^3-3t^{(1)}t^{(2)}+2t^{(3)}\}\,.
\end{aligned}
\tag{A5}
$$

Case $|t^{(1)}| \ll 1$

Now we may solve $\lambda(k) = 0$ under the assumption that $t^{\mu\nu}(k)$ is proportional to a small parameter. In (A3) we neglect the terms cubic in this parameter and find

$$
k^2 = k_{\pm}^2 \equiv -\tfrac{1}{2}t_0^{(1)}\pm\tfrac{1}{2}\{2t_0^{(2)}-(t_0^{(1)})^2\}^{\frac{1}{2}}\,,
\tag{A6}
$$

where the subscript 0 indicates that the quantity is evaluated at $k^2 = 0$. The two modes are now labelled $M = \pm$. The result (A6) simplifies further if we choose a particular frame. We choose an orthonormal set of basis vectors

$$
u_0^{\mu} \equiv (1,\mathbf{0}), \quad e_{\mathrm{L}}^{\mu} \equiv (0,\boldsymbol{\kappa}), \quad e_1^{\mu}, \quad e_2^{\mu}.
\tag{A7}
$$

The longitudinal (L) part of $t^{\mu\nu}$ does not contribute in the limit $k^2 = 0$ due to

$$
t^{(1)} = (k^2/\omega^2)t^{(\mathrm{L})}+t^{1}{}_{1}+t^{2}{}_{2}\,,
\tag{A8}
$$

and similarly for $t^{(2)}$ and $t^{(3)}$. Then (A6) reduces to

$$
k^2 = k_{\pm}^2 = -\tfrac{1}{2}(t^{1}{}_{1}+t^{2}{}_{2})\pm\tfrac{1}{2}\{(t^{1}{}_{1}-t^{2}{}_{2})^2+4t^{1}{}_{2}t^{2}{}_{1}\}^{\frac{1}{2}}\,,
\tag{A9}
$$

which involves only the transverse part of $t^{\mu\nu}$. In (A9) and below we now omit subscripts to denote $k^2 = 0$.

The polarization 4-vectors may be found in this case by noting that the wave equation has been effectively reduced to the two-dimensional equation

$$
(k^2g^{\mu\nu}+t^{\mu\nu})e_{\nu} = 0
\tag{A10}
$$

in the radiation gauge ($u^{\mu}e_{\mu} = 0$, $e_{\mathrm{L}}^{\mu}e_{\mu} = 0$). One finds

$$
e_{\pm}^{\mu} = e_{\pm}^{1}\,e_{1}^{\mu}+e_{\pm}^{2}\,e_{2}^{\mu}\,,
\tag{A11}
$$

with

$$
e_{\pm}^{1} = -t^{1}{}_{2}/A\,, \qquad e_{\pm}^{2} = (t^{1}{}_{1}+k_{\pm}^2)/A\,,
\tag{A12a, b}
$$

and with

$$
A \equiv \{k_{\pm}^4+2t^{1}{}_{1}k_{\pm}^2+(t^{1}{}_{1})^2+t^{1}{}_{2}t^{2}{}_{1}\}^{\frac{1}{2}}\,.
\tag{A13}
$$

The polarization is elliptical with axial ratio T (>0 for RH and <0 for LH polarization) relative to an axis at an angle ϕ to the direction e_1^μ, with

$$2t^1{}_2/(t^1{}_1-t^2{}_2) = \{(T^2-1)\sin 2\phi - 2\mathrm{i}\,T\}/(T^2-1)\cos 2\phi . \tag{A14}$$

In (A12)–(A14) the indices 1 and 2 refer to components along e_1^μ and e_2^μ respectively.

Appendix 2. Covariant Form of the Onsager Relations

Let quantities be denoted by a bar after the time reversal operation has been applied to them. We have $\bar{x}^\mu = (-t, \boldsymbol{x})$, $\bar{k}^\mu = (-\omega, \boldsymbol{k})$, $\bar{F}^{\mu\nu} = (\boldsymbol{E}, -\boldsymbol{B})$ and so on. Now the hermitian (H) and antihermitian (A) parts of $\alpha^{\mu\nu}(k)$ describe the time reversible and time irreversible parts of the response respectively. Hence, denoting the dependence on $F^{\mu\nu}$ explicitly, we have

$$\bar{\alpha}^{\mu\nu(\mathrm{H})}(\bar{k})\,|_{\bar{F}} = \alpha^{\mu\nu(\mathrm{H})}(k)\,|_F\,, \qquad \bar{\alpha}^{\mu\nu(\mathrm{A})}(\bar{k})\,|_{\bar{F}} = -\alpha^{\mu\nu(\mathrm{A})}(k)\,|_F\,, \tag{A15a, b}$$

which combine to give

$$\bar{\alpha}^{\nu\mu}(\bar{k})\,|_{\bar{F}} = \alpha^{\mu\nu}(k)\,|_F\,, \tag{A16}$$

which is a covariant version of the Onsager relations. In component form (A16) implies, for a magnetostatic field,

$$\alpha^{00}(\omega, \boldsymbol{k})\,|_{\boldsymbol{B}} = \alpha^{00}(\omega, -\boldsymbol{k})\,|_{-\boldsymbol{B}}\,, \tag{A17a}$$

$$\alpha^{0i}(\omega, \boldsymbol{k})\,|_{\boldsymbol{B}} = -\alpha^{i0}(\omega, -\boldsymbol{k})\,|_{-\boldsymbol{B}}\,, \tag{A17b}$$

$$\alpha^{ij}(\omega, \boldsymbol{k})\,|_{\boldsymbol{B}} = \alpha^{ji}(\omega, -\boldsymbol{k})\,|_{-\boldsymbol{B}}\,. \tag{A17c}$$

A set of 4-vectors b_1^μ to b_4^μ may be constructed from $F^{\mu\nu}$ and k^μ (Shabad 1975) and used as a set of basis vectors. We write

$$f^{\mu\nu} \equiv F^{\mu\nu}/B\,, \qquad B \equiv (\tfrac{1}{2}F^{\mu\nu}F_{\mu\nu})^{\frac{1}{2}}\,, \tag{A18a, b}$$

$$g_\perp^{\mu\nu} \equiv -f^\mu{}_\alpha f^{\alpha\nu}\,, \qquad g_\parallel^{\mu\nu} \equiv g^{\mu\nu} - g_\perp^{\mu\nu}\,, \tag{A19a, b}$$

$$f^{(\mathrm{D})\mu\nu} \equiv F^{(\mathrm{D})\mu\nu}/B\,, \quad F^{(\mathrm{D})\mu\nu} \equiv \tfrac{1}{2}\varepsilon^{\mu\nu}{}_{\alpha\beta}F^{\alpha\beta}\,, \tag{A20a, b}$$

where D denotes the dual. Then we introduce

$$b_1^\mu \equiv f^{\mu\alpha}k_\alpha\,, \qquad b_2^\mu \equiv f^{(\mathrm{D})\mu\alpha}k_\alpha\,, \tag{A21a, b}$$

$$b_3^\mu \equiv g_\perp^{\mu\alpha}k_\alpha - k^\mu k_\beta k_\gamma g_\perp^{\beta\gamma}/k^2\,, \quad b_4^\mu \equiv k^\mu\,. \tag{A21c, d}$$

Inspection shows that under time reversal b_2^μ, b_3^μ and b_4^μ transform similarly and that b_1^μ has an additional change of sign. Hence on writing

$$\alpha^{\mu\nu}(k) = \sum_{A,B=1}^{3} \alpha_{AB}(k)\, b_A^\mu\, b_B^\nu\,, \tag{A22}$$

where the components along k^μ and k^ν vanish due to charge continuity and gauge invariance, the Onsager relations (A16) imply

$$\alpha_{12}(k) = -\alpha_{21}(k)\,, \quad \alpha_{13}(k) = -\alpha_{31}(k)\,, \quad \alpha_{23}(k) = \alpha_{32}(k)\,. \tag{A23}$$

The tensor $\Lambda^{\mu\nu}(k)$, cf. equation (1), also satisfies (A23). It then follows that if we write a polarization 4-vector in the form

$$e^{\mu}_{M}(\boldsymbol{k}) = \sum_{A=1}^{4} E^{(A)}_{M}(\boldsymbol{k})\, b^{\mu}_{A}\,, \tag{A24}$$

then, apart from an arbitrary overall phase factor, $E^{(1)}_{M}(\boldsymbol{k})$ is imaginary and the other three components are real.

Let us note that in a frame in which the static field is a magnetostatic field along the 3-axis we have

$$f^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad f^{(\mathrm{D})\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \tag{A25a, b}$$

$$g^{\mu\nu}_{\perp} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad g^{\mu\nu}_{\parallel} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{A26a, b}$$

$$b^{\mu}_{1} = (0, k_2, -k_1, 0)\,, \qquad b^{\mu}_{2} = (-k_3, 0, 0, \omega)\,, \tag{A27a, b}$$

$$b^{\mu}_{3} = (1/k^2)\big(k_1^2+k_2^2,\, k_1(\omega^2-k_3^2),\, k_2(\omega^2-k_3^2),\, k_3(k_1^2+k_2^2)\big)\,, \tag{A27c}$$

$$b^{\mu}_{4} = (\omega, k_1, k_2, k_3)\,, \tag{A27d}$$

where k_1^2, k_2^2, k_3^2 denote here squares of the space components of the 3-vector $\boldsymbol{k}$.

Appendix 3. Classical Response Tensors

A covariant version of Vlasov theory (see e.g. Melrose 1982) leads to the following expressions for the response tensors for a distribution of electrons $F_{+}(p)$ and of positrons $F_{-}(p)$. (Recall that $q = -e$ for electrons.)

Case $\boldsymbol{B} = 0$

In the unmagnetized case electrons and positrons contribute similarly with either the same or opposite signs and we write $F(p) \equiv F_{+}(p)+F_{-}(p)$ and $F^{\mathrm{D}}(p) \equiv F_{+}(p)-F_{-}(p)$:

$$\alpha^{\mu\nu}(k) = -\frac{e^2}{m}\int \mathrm{d}^4p\, F(p)\, a^{\mu\nu}(k,k,u) \tag{A28}$$

$$\alpha^{\mu\nu\rho}(k_0,k_1,k_2) = \frac{e^3}{2m^2}\int \mathrm{d}^4p\, F^{\mathrm{D}}(p)\bigg(a^{\mu\nu}(k_0,k_1,u)\frac{k_{2\alpha}}{k_2 u}G^{\alpha\rho}(k_2,u) + a^{\mu\rho}(k_0,k_2,u)\frac{k_{1\alpha}}{k_1 u}G^{\alpha\nu}(k_1,u) + a^{\nu\rho}(k_1,k_2,u)\frac{k_{0\alpha}}{k_0 u}G^{\alpha\mu}(k_0,u)\bigg)\,, \tag{A29}$$

$$\alpha^{\mu\nu\rho\tau}(k_0,k_1,k_2,k_3) = -\frac{e^4}{6m^3}\int \mathrm{d}^4p\, F(p)$$

$$\times\left\{\frac{(k_2+k_3)^2}{(k_2u+k_3u)^2}a^{\mu\nu}(k_0,k_1,u)\,a^{\rho\tau}(k_2,k_3,u)\right.$$

$$+\frac{a^{\mu\nu}(k_0,k_1,u)}{k_2u+k_3u}\left(\frac{k_{2\alpha}(k_{2\beta}+k_{3\beta})}{k_2u}+\frac{(k_{2\alpha}+k_{3\alpha})k_{3\beta}}{k_3u}\right)G^{\alpha\rho}(k_2,u)\,G^{\beta\tau}(k_3,u)$$

$$+\frac{a^{\rho\tau}(k_2,k_3,u)}{k_0u+k_1u}\left(\frac{k_{0\alpha}(k_{0\beta}+k_{1\beta})}{k_0u}+\frac{(k_{0\alpha}-k_{1\alpha})k_{1\beta}}{k_1u}\right)G^{\alpha\mu}(k_0,u)\,G^{\beta\nu}(k_1,u)$$

$$\left.+(\nu,k_1)\leftrightarrow(\rho,k_2)+(\nu,k_1)\leftrightarrow(\tau,k_3)\right\}, \tag{A30}$$

where $(\nu,k_1)\leftrightarrow(\rho,k_2)$ denotes another three terms obtained from those written by the indicated interchanges, and $(\nu,k_1)\leftrightarrow(\tau,k_3)$ denotes a further three terms obtained similarly. In (A28)–(A30) we use the notation $u^\mu = p^\mu/m$ and

$$G^{\mu\nu}(k,u) = g^{\mu\nu}-k^\mu u^\nu/ku\,, \tag{A31}$$

$$\begin{aligned} a^{\mu\nu}(k_1,k_2,u) &= G^{\alpha\mu}(k_1,u)\,G_\alpha{}^\nu(k_2,u) \\ &= g^{\mu\nu}-\frac{k_2^\mu u^\nu}{k_1u}-\frac{k_1^\nu u^\mu}{k_2u}+\frac{(k_1k_2)u^\mu u^\nu}{(k_1u)(k_2u)}. \end{aligned} \tag{A32}$$

Case $\boldsymbol{B}\neq 0$

In the magnetized case electrons (+) and positrons (−) contribute differently:

$$\alpha^{\mu\nu}(k) = \sum_{\pm}(-e^2/m)\int \mathrm{d}^4p\,F_\pm(p)\sum_{s=-\infty}^{\infty}G^{\alpha\mu}(s,k,u)\,\tau_{\alpha\beta}^{(\pm)}(s,k,u)\,G^{*\beta\nu}(s,k,u)\,, \tag{A33}$$

$$\alpha^{\mu\nu\rho}(k_0,k_1,k_2) = \sum_{\pm}(-e^3/2m^2)\int \mathrm{d}^4p\,F_\pm(p)$$

$$\times\sum_{s_0,s_1,s_2=-\infty}^{\infty}\delta_{s_0+s_1+s_2,0}\exp\{\mp\mathrm{i}(s_0\psi_0+s_1\psi_1+s_2\psi_2)\}$$

$$\times\left(\frac{k_1^\alpha}{(k_0u)_\parallel-s_0\Omega_0}G^{\beta\mu}(s_0,k_0,u)\,\tau_{\beta\alpha}^{(\pm)}(s_0,k_0,u)\,G^{*\gamma\nu}(s_1,k_1,u)\right.$$

$$\left.\times\,\tau_{\gamma\delta}^{(\pm)}(s_2,k_2,u)\,G^{*\delta\rho}(s_2,k_2,u)+\dots\right), \tag{A34}$$

where + ... denotes five other terms obtained from that written by permuting (μ,s_0,k_0), (ν,s_1,k_1) and (ρ,s_2,k_2). In (A33) and (A34) we introduce the notations

$$G^{\mu\nu}(s,k,u)\equiv g^{\mu\nu}\mathrm{J}_s(k_\perp R)-\frac{k^\mu U^\nu(s,k)}{(ku)_\parallel-s\Omega_0}, \tag{A35}$$

$$\tau^{(\pm)\mu\nu}(s,k,u) \equiv g_{\parallel}^{\mu\nu} + \frac{(ku)_{\parallel} - s\Omega_0}{\{(ku)_{\parallel} - s\Omega_0\}^2 - \Omega_0^2}$$

$$\times [\{(ku)_{\parallel} - s\Omega_0\} g_{\perp}^{\mu\nu} \mp \mathrm{i}\,\Omega_0 f^{\mu\nu}], \tag{A36}$$

where $g_{\parallel}^{\mu\nu}$, $g_{\perp}^{\mu\nu}$ and $f^{\mu\nu}$ are defined by (A18) and (A19) and with

$$\Omega_0 \equiv eB/m, \qquad (AB)_{\parallel} \equiv g_{\parallel}^{\mu\nu} A_\mu B_\nu, \tag{A37a, b}$$

$$R \equiv \gamma v_\perp/\Omega_0 = p_\perp/eB, \qquad k^\mu = (\omega, k_\perp \cos\psi, k_\perp \sin\psi, k_\parallel), \tag{A37c, d}$$

and

$$U^\mu(s,k) = (\gamma \mathrm{J}_s(k_\perp R), \gamma \boldsymbol{V}(s,\boldsymbol{k})), \tag{A38a}$$

$$\boldsymbol{V}(s,\boldsymbol{k}) = \big(\tfrac{1}{2} v_\perp \{\exp(\pm \mathrm{i}\psi) \mathrm{J}_{s-1}(k_\perp R) + \exp(\mp \mathrm{i}\psi) \mathrm{J}_{s+1}(k_\perp R)\},$$

$$\pm \tfrac{1}{2}\mathrm{i}\, v_\perp \{\exp(\pm \mathrm{i}\psi) \mathrm{J}_{s-1}(k_\perp R) - \exp(\mp \mathrm{i}\psi) \mathrm{J}_{s+1}(k_\perp R)\}, v_\parallel \mathrm{J}_s(k_\perp R)\big). \tag{A38b}$$

Appendix 4. Response Tensors from the Heisenberg–Euler Lagrangian

The Heisenberg–Euler Lagrangian describes a vacuum with static $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. It may be used to derive the response tensors for such a vacuum at 'low' frequencies, which is usually assumed to mean $\omega \ll m$, but, at least for the linear response, requires only that (56) be satisfied.

One form of the Heisenberg–Euler Lagrangian is (Schwinger 1951)

$$\mathscr{L} = \tfrac{1}{2}\varepsilon_0(E^2 - B^2) - \frac{1}{8\pi}\int_0^\infty \frac{\mathrm{d}s}{s^3} \exp(-m^2 s)\bigg(e^2 s^2 \boldsymbol{E}.\boldsymbol{B} \frac{\mathrm{Re}\{\cosh(es\chi)\}}{\mathrm{Im}\{\cosh(es\chi)\}}$$

$$-1 + \tfrac{1}{3} e^2 s^2 (E^2 - B^2)\bigg), \tag{A39}$$

with

$$\chi^2 \equiv -E^2 + B^2 + 2\mathrm{i}\,\boldsymbol{E}.\boldsymbol{B}. \tag{A40}$$

An expansion gives

$$\mathscr{L} = \tfrac{1}{2}\varepsilon_0(E^2 - B^2) + \frac{\varepsilon_0 \alpha}{90\pi B_c^2}\{(E^2 - B^2)^2 + 7(\boldsymbol{E}.\boldsymbol{B})^2\}$$

$$+ \frac{\varepsilon_0 \alpha}{315\pi B_c^4}\{2(E^2 - B^2)^3 + 13(E^2 - B^2)(\boldsymbol{E}.\boldsymbol{B})^2\}$$

$$+ \frac{4\varepsilon_0 \alpha}{945\pi B_c^6}\{3(E^2 - B^2)^4 + 22(E^2 - B^2)^2(\boldsymbol{E}.\boldsymbol{B})^2 + 19(\boldsymbol{E}.\boldsymbol{B})^4\} + \dots, \tag{A41}$$

with $\alpha \equiv e^2/4\pi\varepsilon_0$ the fine structure constant and $B_c \equiv m^2/e$ the critical magnetic field. The result (A41) may be written in covariant form using

$$E^2 - B^2 = -\tfrac{1}{2} F^{\alpha\beta} F_{\alpha\beta}, \qquad \boldsymbol{E}.\boldsymbol{B} = -\tfrac{1}{4} F^{(\mathrm{D})\alpha\beta} F_{\alpha\beta}, \tag{A42a, b}$$

where the dual $F^{(\mathrm{D})\alpha\beta}$ is defined by (A20b).

We assume that $F^{\mu\nu}$ consists of a static part $F_0^{\mu\nu}$ and a fluctuating part

$$\delta F^{\mu\nu}(x) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \exp\{\mathrm{i}\, kx\}\{\mathrm{i}\, k^\mu A^\nu(k) - \mathrm{i}\, k^\nu A^\mu(k)\}. \tag{A43}$$

An expansion of $\mathscr{L}$ in powers of $A(k)$ gives

$$\mathscr{L} = \sum_{n=0}^{\infty} (2\mathrm{i})^n \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \cdots \frac{\mathrm{d}^4 k_n}{(2\pi)^4} \exp\{\mathrm{i}(k_1 + \ldots + k_n)x\}$$

$$\times\, k_1^{\alpha_1} \ldots k_n^{\alpha_n} A^{\beta_1}(k_1) \ldots A^{\beta_n}(k_n) (\partial^n \mathscr{L} / \partial F^{\alpha_1 \beta_1} \ldots \partial F^{\alpha_n \beta_n})_{F=F_0}. \tag{A44}$$

If $F^{\mu\nu} = (\boldsymbol{E}, \boldsymbol{B})$ denotes the construction of $F^{\mu\nu}$ from $\boldsymbol{E}$ and $\boldsymbol{B}$, then the tensors $G^{\mu\nu} = (\boldsymbol{D}, \boldsymbol{H})$ and $M^{\mu\nu} = (\boldsymbol{P}, -\boldsymbol{M})$ may be used to describe the response, with $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$ and $\boldsymbol{H} = \boldsymbol{B}/\mu_0 - \boldsymbol{M}$ as usual. We have

$$\mu_0 G^{\mu\nu} = F^{\mu\nu} + \mu_0 M^{\mu\nu} = -2\mu_0\, \partial \mathscr{L} / \partial F_{\mu\nu}. \tag{A45}$$

The 4-current is related to $M^{\mu\nu}(k)$ by

$$J^\mu(k) = -\mathrm{i}\, k_\alpha M^{\alpha\mu}(k). \tag{A46}$$

On substituting (A44) and (A45) and writing $k_0 = -k$, one identifies the response tensors defined by (26):

$$\alpha^{\nu_0 \nu_1 \ldots \nu_n}(k_0, k_1, \ldots, k_n) = (2\mathrm{i})^{n+1} k_0^{\alpha_1} k_1^{\alpha_1} \ldots k_n^{\alpha_n}$$

$$\times \{\partial^{n+1}(\mathscr{L} - \mathscr{L}^0) / \partial F^{\alpha_0}{}_{\nu_0} \partial F^{\alpha_1}{}_{\nu_1} \ldots \partial F^{\alpha_n}{}_{\nu_n}\}_{F=F_0}, \tag{A47}$$

with $\mathscr{L}^0 \equiv \frac{1}{2}\varepsilon_0(E^2 - B^2)$.

For $n = 1$ equation (A47) gives the linear response tensor

$$\alpha^{\mu\nu}(k) = 4k^\alpha k^\beta\, \partial^2(\mathscr{L} - \mathscr{L}^0) / \partial F_{\alpha\mu} \partial F_{\beta\nu}$$

$$= \frac{\varepsilon_0 \alpha}{90\pi B_c^2} \{2(k^\mu k^\nu - k^2 g^{\mu\nu}) F_0^{\alpha\beta} F_{0\alpha\beta}$$

$$+ 8k_\alpha F_0^{\alpha\mu} k_\beta F_0^{\beta\nu} + 14 k_\alpha F^{(\mathrm{D})\alpha\mu} k_\beta F^{(\mathrm{D})\beta\nu}\}, \tag{A48}$$

where only terms of order $(E^2 + B^2)/B_c^2$ are retained. As required $\alpha^{\mu\nu}$ is diagonal when the axes are oriented along the 4-vectors b_1^μ to b_4^μ introduced in equations (A21). For $n = 2$ it is important to retain the terms of next order in $(E^2 + B^2)/B_c^2$; one finds

$$\alpha^{\mu\nu\rho}(k_0, k_1, k_2) = -\frac{8\mathrm{i}\, \varepsilon_0 \alpha}{90\pi B_c^2} \{(k_0 k_1 g^{\mu\nu} - k_0^\nu k_1^\mu) k_{2\gamma} F_0^{\gamma\rho}$$

$$+ \tfrac{7}{4} k_{0\alpha} k_{1\beta} \varepsilon^{\alpha\mu\beta\nu} k_{2\gamma} F_0^{(\mathrm{D})\gamma\rho} + \text{perm.}\}$$

$$+ \frac{8\mathrm{i}\, \varepsilon_0 \alpha}{315\pi B_c^4} \{4 k_{0\alpha} F_0^{\alpha\mu} k_{1\beta} F_0^{\mu\nu} k_{2\gamma} F_0^{\gamma\rho}$$

$$+\tfrac{13}{2}k_{0\alpha}F_0^{(\mathrm{D})\alpha\mu}k_{1\beta}F_0^{(\mathrm{D})\beta\nu}k_{2\gamma}F_0^{(\mathrm{D})\gamma\rho}$$

$$+(k_0 k_1 g^{\mu\nu}-k_0^{\nu}k_1^{\mu})(3F_0^{\alpha\beta}F_{0\alpha\beta}k_{2\gamma}F_0^{\gamma\rho}$$

$$+\tfrac{13}{8}F_0^{(\mathrm{D})\alpha\beta}F_{0\alpha\beta}k_{2\gamma}F_0^{(\mathrm{D})\gamma\rho})$$

$$+\tfrac{13}{8}k_{0\alpha}k_{1\beta}\varepsilon^{\alpha\mu\beta\nu}(F_0^{(0)\alpha\beta}F_{0\alpha\beta}k_{2\gamma}F_0^{\gamma\rho}+F_0^{\alpha\beta}F_{0\alpha\beta}k_{2\gamma}F_0^{(\mathrm{D})\gamma\rho})$$

$$+\mathrm{perm.}\} \qquad \text{(A49)}$$

where '+ perm.' implies adding further terms obtained from those written by the interchanges $(1,\nu)\leftrightarrow(2,\rho)$ and $(0,\mu)\leftrightarrow(2,\rho)$, and where $\varepsilon^{\alpha\mu\beta\nu}$ is the permutation symbol. For $n = 3$ the lowest order term gives a response independent of $F_0^{\alpha\beta}$:

$$\alpha^{\mu\nu\rho\tau}(k_0,k_1,k_2,k_3) = \frac{\varepsilon_0\,\alpha}{90\pi B_c^2}\{8(k_0 k_1 g^{\mu\nu}-k_0^{\nu}k_1^{\mu})(k_2 k_3 g^{\rho\tau}-k_2^{\tau}k_3^{\rho})$$

$$+14k_{0\alpha}k_{1\beta}\varepsilon^{\alpha\mu\beta\nu}k_{2\gamma}k_{3\delta}\varepsilon^{\gamma\rho\delta\tau}+\mathrm{perm.}\}\,, \qquad \text{(A50)}$$

where '+ perm.' here means adding further terms obtained from those written by the replacements $(1,\nu)\leftrightarrow(2,\rho)$ and $(1,\nu)\leftrightarrow(3,\tau)$. Photon–photon scattering in vacuo in the 'low' frequency limit may be treated by inserting equation (A50) in (50); the resulting cross section is well known (see e.g. Jauch and Rohrlich 1955, p. 295).

Manuscript received 7 March, accepted 9 June 1983