Photon dispersion in a strong magnetic field with positronium formation: Theory

Article *in* Astrophysics and Space Science · December 1986

DOI: 10.1007/BF00644586

CITATIONS
40

READS
15

2 authors:

Anatoly E. Shabad
Russian Academy of Sciences
90 PUBLICATIONS **1,150** CITATIONS

SEE PROFILE

V. V. Usov
Weizmann Institute of Science
172 PUBLICATIONS **4,075** CITATIONS

SEE PROFILE

PHOTON DISPERSION IN A STRONG MAGNETIC FIELD WITH POSITRONIUM FORMATION: THEORY

A. E. SHABAD

P. N. Lebedev Physical Institute, Moscow, U.S.S.R.

and

V. V. USOV

Space Research Institute, Moscow, U.S.S.R.

(Received 10 July, 1986)

Abstract. Positrinium atom is considered in a strong magnetic field $B \gg \alpha^2 B_{cr} = m^2 ce^3/\hbar^3 = 2.35 \times 10^9\ G$ in a vector-potential gauged as $A_x = -By$. The energy spectrum is obtained including its dependence on the centre-of-mass wave vector across the magnetic field. The pole-like contributions into the photon polarization operator coming from the positronium states are calculated and dispersion curves of joint photon–positronium states are obtained as trajectories of poles of the photon Green function in momentum space.

When propagating in a strong magnetic field ($B \gtrsim 0.1 B_{cr} \simeq 4 \times 10^{12}$ G) with curved lines of force, a photon is canalized along the magnetic field by adiabatically transforming into a bound electron–positron pair, which is a stronger effect than the analogous photon capture by transforming into an unbound pair at the edge of the continuum discussed previously by the authors. The effect of bound pair formation by γ-quanta in a strong magnetic field may be important near pulsars, γ-burst sources, powerful X-ray sources in close binary systems and other astronomical objects, recognized as magnetic neutron stars.

1. Introduction

In the recent years a great deal of effort was devoted to study of polarizational properties of the vacuum with a strong constant magnetic field. Although some important properties, such as the tensor decomposition of the polarization operator and photon Green function, as well as the complete description of photon eigenmodes have been obtained using the first principles alone, i.e., in an approximation-independent way (Batalin and Shabad, 1971; Shabad, 1975), other facts associated to the photon propagation and absorption are left to the tender mercies of the one-loop Furry perturbative approximation. This was calculated for the polarization operator beyond the unperturbed photon mass shell by many authors (Batalin and Shabad, 1971; Baier *et al.*, 1975; Tsai, 1974; Tsai and Erber, 1974; Melrose and Stoneham, 1976, 1977; Melrose, 1981, 1982, 1983). It was shown (Shabad, 1972, 1975) that the polarization operator is infinite at each threshold of creation by a photon of an electron–positron pair on Landau levels. This cyclotronic resonance was a reason for an essential change in the shape of the photon dispersion curves. These were obtained (Shabad, 1972, 1975) by finding the trajectories of the photon Green function poles in momentum space, i.e., by solving the dispersion equations for the photon eigenmodes near each singularity of the polarization operator. Before doing so, the polarization operator, initially calculated in the proper time representation, was given a spectral form, i.e., represented as a sum

Astrophysics and Space Science **128** (1986) 377–409.

over pairs of Landau numbers of individual singular contributions coming from each semi-discrete electron–positron state.

In the present paper we go far beyond the one-loop accuracy for the polarization operator in a strong magnetic field by taking into account the mutual Coulomb attraction between the virtual electron and positron inside the loop, which exchange an infinite number of virtual photon to form a mutually bound state, a positronium atom in a magnetic field. In Section 4 of the present paper we extend the expressions of (Shabad, 1975) by presenting the polarization operator as a sum of contributions from intermediate electron–positron bound states (the contribution of the continuous spectrum is not included), labelled by two Landau numbers n, n' (one for electron and one for positron) and an extra integer n_c which is the label of a Coulomb-bound level. A likewise expression for the imaginary (absorptive) part of the polarization operator is also given. The individual contributions of positronia provide poles in the polarization operator and δ-functions in its imaginary part. Trajectories of these poles in the energy-momentum space are the spectral curves of positronum in a strong magnetic field, which are found in Section 3 for every n, n', n_c using the Bethe–Salpeter equation which makes the job of summing the ladder of photon exchanges cementing the pair into the bound state. Previously, Leinson and Orayevsky (1985) applied this equation to the same problem of finding the positronium spectrum in a strong magnetic field. The spectrum they found relates to different quantum numbers which are not convenient for the present purpose of considering the contribution into the polarization operator as being not connected with the photon quantum numbers by the momentum conservation law. This point will be discussed in Section 2.

It is crucial that the spectral curves of the positronium intersect the unperturbed photon mass shall $K_0^2 - \mathbf{K} = 0$ (here K_μ is the photon 4-momentum, $\mu = 0, 1, 2, 3$). This is the manifestation of the fact that one-photon positronium creation and annihilation are in a magnetic field not only virtual, but also real processes. We devote Section 2 to discussion of this not completely kinematic question, as well as to the choice of the gauge and appropriate quantum numbers. As a result of this intersection the final dispersion curves obtained in Section 5 by solving the photon dispersion equations in the vicinities of the positronium poles of the polarization operator show essential deviations from the shape $K_0^2 = \mathbf{K}^2$ more strong than what they were before the bound pair formation has been taken into account (Shabad, 1972, 1975). The new dispersion curves describe mixed, or joint, photon–positronium states, like polariton states well known in the solid body theory (see monographs (Nikitin and Umansky, 1979; Davydov, 1976; Agranovich and Ginzburg, 1979), for example). The appearance of these states enhances the effect of photon capture by the strong magnetic field of a pulsar described previously (Shabad and Usov, 1982, 1984). Moving along the curved magnetic lines of force of a pulsar the photon shifts along its dispersion curve, converting adiabatically into a positronium atom. This qualitative result, without the derivation of the polarization operator representation and the positronium spectrum, was communicated by us in a letter (Usov and Shabad, 1985) and in a somewhat more detailed form in (Shabad and Usov, 1985, 1986). The same conclusion was drawn in a short

publication by Herold *et al.* (1985) who refer to numerical computations of the lowest-lying positronium spectral curve and of the photon dispersion curve adjoint to it. In Section 6 we present the general astrophysical context for the application of the present results to the theory of pulsar.

2. One-Photon Positronium Creation and Annihilation in a Magnetic Field

For kinematic reasons a free electron–positron pair can neither annihilate into one photon, nor be produced by one photon. The same refers to a Coulomb-bound pair, the positronium atom. In a strong magnetic field, however, neither one-photon annihilation nor production of a free e^+e^--pair is kinematically forbidden. As concerns the kinematic possibility for these processes to proceed in a strong magnetic field for a bound pair, positronium, the seemingly contradictory literature data (Carr and Sutherland, 1978; Wunner and Herold, 1979; Wunner *et al.*, 1981) have led us to the following judgement: one-photon annihilation and creation of a positronium in a strong magnetic field is possible (Carr and Sutherland, 1978; Leinson and Orayevsky, 1984, 1985), but neither from, nor to the state of the minimum energy (Wunner and Herold, 1979; Wunner *et al.*, 1981). Carr and Sutherland (1978) have calculated in a strong magnetic field B the probability of one-photon annihilation of a mutually-bound e^+e^--pair with zero-Landau quantum numbers and zero projection of angular momenta of the electron M_e and positron M_p, onto the magnetic field. They have not come across any kinematical prohibition of the reaction. On the other hand, appealing to the translation invariance in a constant and homogeneous magnetic field of a system of two particles – electron and positron (one should add: taken in coinciding space-time points) – Wunner and Herold (1979), Wunner *et al.* (1981) have advocated the kinetmatic impossibility of one-photon annihilation of a pair from the ground state, under the assumption (although not self-evident, but confirmed by our data) that the Coulomb interaction between the two particles removes the known energy degeneracy in the magnetic field so that the energy minimum would correspond to zero value of a quantum number K which is analogous to the momentum of the centre of mass of the pair. This assumption is clearly in contradiction with the supposition due to Carr and Sutherland (1978) that the ground state is the one for which $M_e = M_0 = 0$. Consequently, one should attribute the probability of one-photon annihilation, calculated by Carr and Sutherland (1978), as well as in the later paper (Leinson and Orayevsky, 1984, 1985) (where better wave functions are used) to one of the excited states of the pair. Such a state is in no way distinguished among other states, which belong to the same complete set of states and can annihilate too.

Unlike other authors, we are using the gauge in which the vector-potential of the external field is chosen in the form $A_x = -By, A_0 = A_y = A_z = 0$ ($\mathbf{B}_x = \mathbf{B}_y = 0, \mathbf{B} = \mathbf{B}_z$). Although no results concerning the one-photon annihilation and pair creation depend on the gauge*, each gauge suggests in a natural way this or that complete set of states

* The state of a pair may depend on the gauge until its wave function is taken at coinciding space-time arguments, which is just required for considering its transformations into and from a photon.

is the value of the wave function of the relative motion along the magnetic field in coinciding points $|\varphi(0)|^2$. The same coefficient determines therefore the matrix element of one-photon annihilation of a positronium or its creation by a photon, and also the and corresponding quantum numbers. In the gauge we are using here, the translation invariance can be taken into account explicitly and makes an analogue to momentum conservation law. It turns out that in the corresponding complete set of states of the pair there always exist some quite define states, the ground state being not among them, which, exclusively, can one-photon annihilate. To establish these states, i.e., to determine the values of corresponding quantum numbers is the dynamical problem whose solution should answer what seems a purely kinematical question: from what states the annihilation is possible, if at all?

Let us explain this in more precise terms. In a sufficiently strong magnetic field $B \gg (e^2/c\hbar)^2(m^2c^3/e\hbar) = \alpha^2 B_{cr}$ where the electron orbit radius $L = (\hbar c/eB)^{1/2}$ is much less than the Bohr radius $a_B = \hbar^2/me^2$, the dependence of the electron and positron wave functions on the degrees of freedom transversal to the magnetic field may be taken in the same form as when they do not form a positronium. The Coulomb interaction between them modifies the wave functions of the relative motion along the magnetic field (the motion of the centre of mass in this direction remains, naturally, free). Such an approximation is referred to as adiabatic in the literature. In this approximation the conservation law which follows from the translational invariance takes the form

$$\hbar K_\perp = p_x^p + p_x^e = -(y_0^p - y_0^e)\frac{eB}{c}, \tag{2.1}$$

where $\hbar K_\perp$ is projection of the photon momentum on the xy-plane ortogonal to $\mathbf{B}$, the x-direction being chosen along $\mathbf{K}_\perp$: $K_x = K_\perp$, $K_y = 0$ (this does not imply any loss of generality due to the gauge invariance of quantities to be calculated and conclusions to be drawn in the paper). The continuous quantum numbers $p_x^{p,e}$ describe the motion of the electron and positron phases along the x-axis. The quantum numbers $p_x^{p,e}$ are connected with the y-coordinates of the electron and positron orbit centres $y_0^{e,p}$ by the relation $y_0^{e,p} = \mp p^{e,p}c/eB$. The transverse motion is described here by the product of transverse parts of electron and positron Dirac wave functions $\psi_\perp$ which have the structure

$$\psi_\perp = \frac{1}{\sqrt{L_x}} \exp\left(\frac{ip_x x}{\hbar}\right) \lambda_\alpha(\xi, n, p_z), \tag{2.2}$$

where L_x is the normalization length along the x-axis, $\xi = \sqrt{eB/c\hbar}\,(y - y_0)$, n is the Landau quantum number $n = 0, 1, 2, \ldots$ and λ_α is a spinor consisting of Hermite functions with the numbers n and $n - I$. The wave functions of a pair as they are obtained in the adiabatic approximation are written completely in Section 3 (see Equation (3.23)).

A photon with the transverse momentum component $\hbar K_\perp$ produces only those (is produced only by those) states – out of the total set of states of the transverse motion described by pairs of the wave functions (2.2) – for which the projections of the electron

and positron orbit centres onto the xy-plane in the direction y orthogonal to the photon motion are separated by the distance $c\hbar K_\perp/eB$ from one another. The photon, which obeys the dispersion equation

$$K_0^2 - K_\parallel^2 = K_\perp^2 \tag{2.3}$$

(where $c\hbar K_0 = \hbar\omega$ is the energy, and $K_\parallel$ is the wave vector of the photon along **B**) can produce a bound or a free pair if

$$K_0^2 = \left(\frac{P_z}{\hbar}\right)^2 + \frac{1}{c^2\hbar^2}\,\varepsilon_{nn'}^2(n_c, P_x^2\hbar^{-2})\,, \tag{2.4}$$

where $P_z = p_z^p + p_z^e$ is momentum of longitudinal motion of the centre of mass of the pair, $P_x = p_x^p + p_x^e$, and $\varepsilon_{nn'}(n_c, P_x^2\hbar^{-2})$ is the energy* of the pair, whose centre of mass does not move along the z-axis, the electron and the positron occupy Landau levels with the numbers n, n', the number of their Coulomb-bound state is denoted by n_c. Using the momentum conservation law for the longitudinal motion $P_z = \hbar K_\parallel$ and the relations (2.1), (2.3), (2.4), we obtain the kinematic condition for the pair production by a photon in the form of an equation for $K_\perp$, given by

$$K_\perp^2 = \frac{1}{c^2\hbar^2}\,\varepsilon_{nn'}^2(n_c, K_\perp^2)\,. \tag{2.5}$$

In order to establish whether this equation admit of any solutions, one should know the positronium dispersion law, i.e., the energy dependence on the variable P_x, the transverse wave vector of the centre of mass, or on the transverse distance $(y_0^p - y_0^e)$ between the centres of orbits. This is a quantitative dynamic problem to be solved in this paper.

The value $\hbar K_\perp = P_x = 0$ is obviously not a solution of Equation (2.5), since $\varepsilon_{nn'}(n_c, 0)$ cannot be equal to zero even for $n = n' = n_c = 0$. The opposite would mean that the mass defect of the Coulomb attraction has completely absorbed the mass of the pair, and this is of course impossible (see, however, the discussion in Section 3 of the fall-down-onto-the centre effect and the associated vacuum instability in the exponentially strong magnetic field). On this basis it was concluded in (Wunner and Herold, 1979; Wunner *et al.*, 1981), where only the point $P_x = 0$ (for $n = n' = n_c = 0$) as corresponding to the energy minimum was considered, that both one-photon annihilation and creation of a positronium of minimum energy are kinematically forbidden. The result of the present paper (Section 3), shows that the energy $\varepsilon_{nn'}(n_c, P_x^2\hbar^{-2})$ depends on the transverse centre-of-mass wave vector $P_x\hbar^{-1}$ in such a way that equation (2.5) has a solution for every n, n', n_c and thus the process under consideration is allowed for $P_x \neq 0$. It will be shown at the same time that the energy of the pair is a weakly increasing function of P_x, so that the state for which the reaction is possible, does not correspond to the energy minimum point of any of the dispersion curves labelled by the discrete quantum numbers n, n', n_c.

* The energy $\varepsilon_{nn'}$ might depend also on another quantum number, $p_x \equiv (p_x^e - p_x^p)/2$ but, as shown in Section 3, the degeneracy with respect to this quantum number is retained in the adiabatic approximation.

For the energy of a pair which does not move along the magnetic field one can write

$$\varepsilon_{nn'}(n_c, P_x^2\hbar^{-2}) = mc^2[\sqrt{1 + (2nB/B_{cr})} + \sqrt{1 + (2n'B/B_{cr})}] - \\ - \Delta\varepsilon_{nn'}(n_c, P_x^2\hbar^{-2})\,. \tag{2.6}$$

The first term here, the energy of an unbound pair mutually at rest along z, gives the boundary of the continuum at any given n, n', while the second term $\Delta\varepsilon_{nn'}$ is the binding energy. The boundary of the continuum does not depend on the transverse wave vector $P_x\hbar^{-1}$. This is a result of the degeneracy of the Landau levels for an electron and a positron treated separately. The Coulomb interaction between them lifts the degeneracy – i.e., the binding energy $\Delta\varepsilon_{nn'}$ – and also the whole energy (2.6) depends on the wave vector $P_x\hbar^{-1}$. It is shown below that in the domain $B \gg \alpha^2 B_{cr}$ the energy $\varepsilon_{nn'}(n_c, P_x^2\hbar^{-2})$

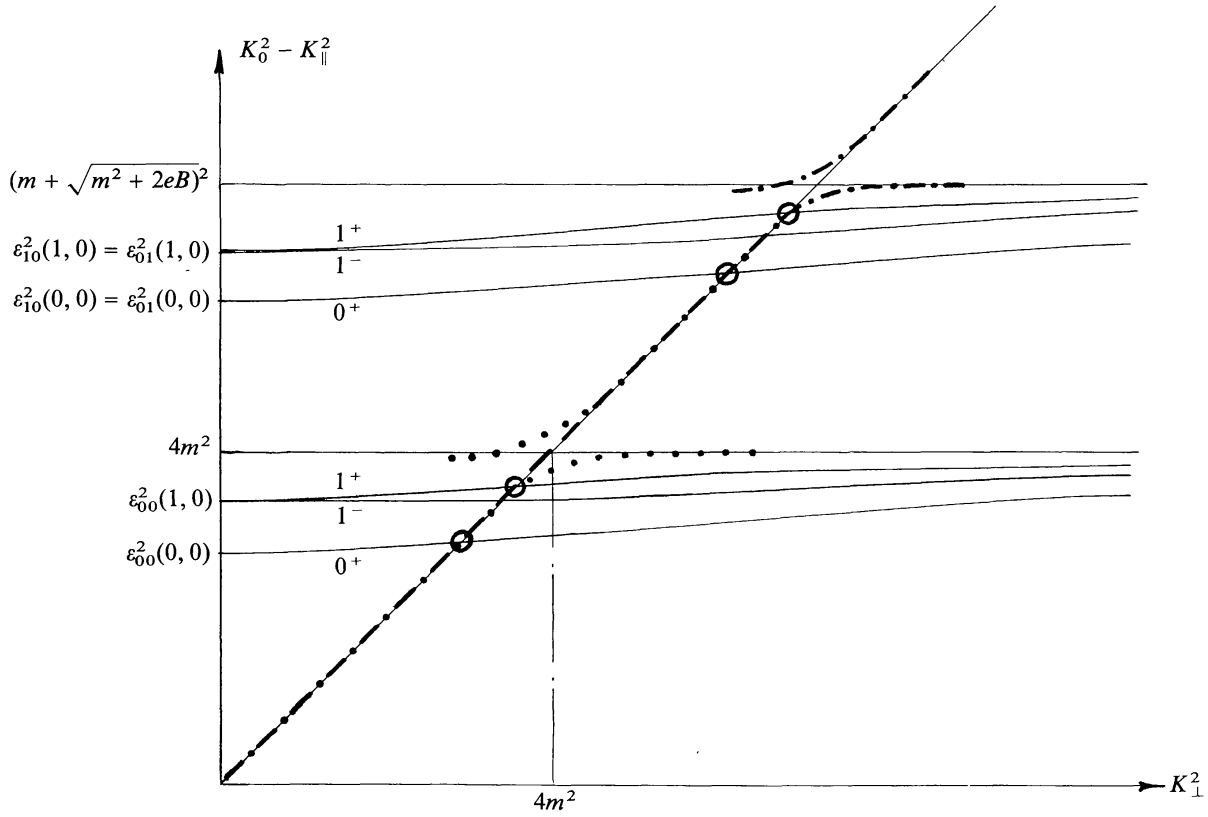

Fig. 1. Photon and positronium spectral curves treated as independent. The vertical axis bears the photon (positronium) energy squared in the reference frame comoving with the centre of mass along the magnetic field. The horizontal axis bears the photon (positronium) momentum squared across the magnetic field. The sloped straight line is the light cone, Equation (2.3). The horizontal straight lines are the boundaries of continua for the first two series of bound states with the Landau numbers $n = n' = 0$ and $n = 0, n' = 1$ or $n = 1, n' = 0$. The positronium spectral curves, Equation (2.6) are light curved lines. Three Coulomb-bound levels ($n_c^\pm = 0^+, 1^\pm$) are shown for two series of states: the one for electron and positron with antiparallel spin projections onto the magnetic field ($n = n' = 0$) and the other for the both cases of parallel and antiparallel spin projections ($n = 0, n' = 1$ or $n = 1, n' = 0$). The plus or minus sign denotes the parity of state under the z-axis reflection. Higher series ($n + n' > 1$) are not shown. Dispersion curves for mode-2 photons are dotted and for mode-3 photons are dashed. Mutual transformations of the photon and positronium are kinematically allowed in the quasi-intersection points, shown as circles.

tends asymptotically to the constant first term in (2.6) – the boundary of the continuum – as P_x grows infinitely, i.e., the binding energy $\Delta\varepsilon_{nn'}$ tends to zero. This is illustrated in Figure 1, where the photon dispersion law is given not in the form (2.3), but taking into account the effect produced by the resonance behaviour of the vacuum polarization in a magnetic field near the threshold of pair production by a photon (Shabad, 1972, 1975; Shabad and Usov, 1982, 1984). This has been done only for completeness because in this context this does not affect qualitatively our conclusion. It is seen from Figure 1 that Equation (2.5), the same as the more exact equation which replaces (2.5) once the above vacuum polarization effect is taken into account, does have a solution which is the intersection* point of the photon and positronum dispersion curves. Note that the qualitative conclusion we have drawn that the one-photon reactions are allowed remains valid also beyond the scope of the adiabatic approximation – i.e., for weaker fields, since, as indicated in Section 3 – the labeling of states by the three discrete (n, n', n_c) and three continuous (P_z, P_x, p_x) quantum numbers continues to hold; only their interpretation in terms of the parameters (quantum numbers) of mutually unperturbed electron and positron in a magnetic field is no longer valid.

3. Bethe–Salpeter Equation for Electron–Positron State in a Strong Magnetic Field

Leinson and Orayevsky (1984) were the first to use the Bethe–Salpeter equation for solving the problem of the positronium spectrum in a strong magnetic field. In this Section we follow these authors, but work in another gauge and with another set of wave functions of the transverse motion which is more adequate to the problem, as has already been explained above.

Consider the four-fermion Green's function in an external magnetic field

$$K_{\alpha\beta\gamma\delta}(x_1, x_2, x_3, x_4) = \langle 0 |\, T({}^e\psi_\alpha(x_1)\,{}^p\psi_\beta(x_2)\,{}^e\bar{\psi}_\gamma(x_3)\,{}^p\bar{\psi}_\delta(x_4)\,|\, 0 \rangle\,, \quad (3.1)$$

where $x_i = (r_i, t_i)$ are space-time points, α, β, γ, δ are spinor indices, ${}^e\psi_\alpha$, ${}^p\psi_\alpha$ are, respectively, electron and positron Heisenberg field operators related by the charge conjugation ${}^p\psi = C\,{}^e\bar{\psi}$. The contribution into (3.1) of a bound electron–positron state $|n, n', n_c, P_z, P_x, p_x\rangle$ with fixed discrete quantum numbers n, n', n_c has the form

$$\frac{1}{4\pi i E}\int \mathrm{d}p_x\,\mathrm{d}P_x\,\mathrm{d}P_z\,\mathrm{d}E'\;\frac{\chi^{E'}_{\alpha\beta}(x_1, x_2)\bar{\chi}^{E'}_{\gamma\delta}(x_3, x_4)}{E' - E(P_x, P_z, p_x)}\,, \quad (3.2)$$

which implies that, for the spectrum of the bound state,

$$E(P_x, P_z, p_x) = \sqrt{\varepsilon^2_{nn'}(n_c, P_x, p_x) + P_z^2 c^2}\,. \quad (3.3)$$

At this stage we have made the only assumption that out of the three (apart from P_x) quantum numbers responsible for the transverse motion two quantum numbers n, n' are

* We mean the intersection which would take place unless the interaction between the photon and the positronium curves is taken into account. This interaction leads to repulsion and reconnection of the dispersion curves. This question is investigated in Section 5.

discrete, and one, p_x, is continuous, and that the relative motion along the magnetic field is quantized by the discrete quantum number n_c. This assumption is compatible with the strong field limit, but allows to formulate consistent equations for wave functions also beyond this limit, what makes it *a posteriori* justified. In (3.2)

$$\chi^{E'}_{\alpha\beta}(x_1, x_2) = \langle 0 | \, T({}^e\psi_\alpha(x_1)\, {}^p\psi_\beta(x_2)) | \, E', n, n', n_c, P_z, P_x, p_x \rangle \tag{3.4}$$

is the wave function of the bound pair off its mass shell $E' = E$. As not to make equations cumbersome, we do not show explicitly the dependence of the function $\chi^{E'}(x_1, x_2)$ on the quantum numbers $n, n', n_c, P_z, P_x, p_x$. In deriving (3.2) we have taken into account the time translation invariance which is due to the constancy of the external field and its 4-potential. If we use the ladder approximation and disregard the retardation effects (nonrelativity of the internal motion) we shall find that the wave function $\chi^{E'}(x_1, x_2)$ of the e^+e^--pair satisfies in the neighbourhood of the point $E' = E$ the following Bethe–Salpeter equation in an external field (Karplus and Klein, 1952)

$$\chi^E_{\alpha\beta}(x_1, x_2) = -i\alpha c \int G^e_{\alpha\alpha'}(x_1, x_1') G^p_{\beta\beta'}(x_2, x_2') \gamma^0_{\alpha'\rho} \gamma^0_{\beta'\sigma} \frac{\delta(t_1' - t_2')}{|\mathbf{r}_1' - \mathbf{r}_2'|} \times$$

$$\times \chi^E_{\rho\sigma}(x_1', x_2') \, \mathrm{d}^4x_1' \, \mathrm{d}^4x_2' \,, \tag{3.5}$$

where the electron and positron Green's functions related as $G^p_{\alpha\beta}(x, x') = -C_{\alpha\alpha'} C^{-1}_{\beta\beta'} G^e_{\beta'\alpha'}(x', x)$ satisfy the equations (the radiative mass corrections being neglected)

$$c\left(-i\hat{\partial}_\mu \mp \frac{e}{\hbar c} \hat{A}_\mu + \frac{mc}{\hbar} \right) G^{e,p}(x, x') = \delta^{(4)}(x - x') \,. \tag{3.6}$$

Time translation invariance allows us to write

$$\chi^E_{\alpha\beta}(x_1, x_2) = \exp\left[-\frac{iE}{2\hbar}(t_1 + t_2) \right] \eta_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, t_1 - t_2) \,,$$

$$G^{e,p}(x, x') = G^{e,p}(\mathbf{r}, \mathbf{r}', t - t') \,. \tag{3.7}$$

Passing in (3.5) to Fourier-transform according to the rule

$$G(\mathbf{r}, \mathbf{r}', t - t') = \frac{1}{2\pi\hbar} \int \tilde{G}(\mathbf{r}, \mathbf{r}', \varepsilon) \exp\left[-\frac{i\varepsilon}{\hbar} (t - t') \right] \mathrm{d}\varepsilon \,,$$

carrying out the needed integrations and assuming $t_1 = t_2$, we obtain a closed equation for $\eta_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, 0)$:

$$\eta_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, 0) = \frac{i\alpha c}{2\pi\hbar} \int \tilde{G}^e_{\alpha\alpha'}(\mathbf{r}_1, \mathbf{r}_1', \varepsilon) C^{-1}_{\beta\lambda} C_{\beta'\lambda'} \tilde{G}^e_{\lambda\lambda'}(\mathbf{r}_2', \mathbf{r}_2, \varepsilon - E) \times$$

$$\times \, \mathrm{d}\varepsilon \gamma^0_{\alpha'\rho} \gamma^0_{\beta'\sigma} \frac{1}{|\mathbf{r}_1' - \mathbf{r}_2'|} \eta_{\rho\sigma}(\mathbf{r}_1', \mathbf{r}_2', 0) \, \mathrm{d}\mathbf{r}_1', \mathrm{d}\mathbf{r}_2' \,. \tag{3.8}$$

Equation (3.8) will be reduced to the Schrödinger equation if, while integrating over ε in the kernel, we keep only the positive energy pole contribution in the expansion of the electron and positron Green functions into series of the Dirac operator eigenfunctions

$$G^{e,p}_{\alpha\alpha'}(x, x') \simeq \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\varepsilon \sum_q \frac{{}^{e,p}\psi^+_\alpha(q, \mathbf{r}) {}^{e,p}\bar{\psi}^+_{\alpha'}(q, \mathbf{r}')}{\varepsilon - \varepsilon_q} \exp\left[-\frac{i\varepsilon}{\hbar}(t - t')\right], \tag{3.9}$$

where q is the complete set of electron quantum numbers in a magnetic field $q = (n, p_x, p_z, s)$, and the energy is

$$\varepsilon_q \equiv \varepsilon_n = \sqrt{m^2c^4 + p_z^2c^2 + 2neBc\hbar}\,.$$

The sum Σ_q implies summation over discrete and integration over continuous components of q. Using (3.9) in (3.8) and integrating over ε by closing the contour in the upper half-plane, we have ($\alpha = e^2/\hbar c$)

$$\eta_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, 0) = -e^2 \sum_{q_1, q_2} \frac{{}^e\psi^+_\alpha(q_1, \mathbf{r}_1) {}^p\psi^+_\beta(q_2, \mathbf{r}_2)}{E - \varepsilon_{q_1} - \varepsilon_{q_2}} \int d\mathbf{r}'_1\, d\mathbf{r}'_2 \times \tag{3.10}$$

$$\times \frac{1}{|\mathbf{r}'_1 - \mathbf{r}'_2|} {}^e\bar{\psi}^+_{\alpha'}(q_1, \mathbf{r}'_1)\gamma^0_{\alpha'\rho} {}^p\bar{\psi}^+_{\beta'}(q_2, \mathbf{r}'_2)\gamma^0_{\beta'\sigma}\eta_{\rho\sigma}(\mathbf{r}'_1, \mathbf{r}'_2, 0)\,.$$

The translation invariance along the z- and x-axes allows us to write in addition to (3.7), also

$$\eta_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, 0) = \exp\left[\frac{i}{\hbar}(P_zZ + P_xX)\right] \tilde{\eta}_{\alpha\beta}(x_1 - x_2, y_1, y_2, z_1 - z_2)\,, \tag{3.11}$$

where $Z = \frac{1}{2}(z_1 + z_2)$, $X = \frac{1}{2}(x_1 + x_2)$.

The representation (3.11) is formally the reason for appearance of the δ-functions expressing the momentum conservation laws for x- and z-components in the equations to follow.

Making the dependence on x and z in the functions ${}^{e,p}\psi$, ${}^{e,p}\bar{\psi}$ in (3.10) explicit (see (2.2)), and using (3.11) (recall that $L = (c\hbar/eB)^{1/2}$) we get

$$\exp\left[\frac{i}{\hbar}(P_zZ + P_xX)\right] \tilde{\eta}_{\alpha\beta}(x_1 - x_2, y_1, y_2, z_1 - z_2) =$$

$$= \frac{-e^2}{(2\pi\hbar)^4} \sum_{n, n'} \int \frac{\exp[(i/\hbar)(p^e_z(z_1 - z'_1) + p^p_z(z_2 - z'_2))]}{E - \varepsilon_n(p^e_z) - \varepsilon_{n'}(p^p_z)} \times$$

$$\times \exp\left[\frac{i}{\hbar}(p^e_x(x_1 - x'_1) + p^p_x(x_2 - x'_2) {}^e\lambda^+_\alpha\left(\frac{y_1}{L} - \frac{p^e_xL}{\hbar}, n\right)\right] \times$$

$$\times {}^{e}\lambda_{\rho}^{*+}\left(\frac{y_1'}{L}-\frac{p_x^e L}{\hbar},n\right){}^{p}\lambda_{\beta}^{+}\left(\frac{y_2}{L}+\frac{p_x^p L}{\hbar},n'\right){}^{p}\lambda_{\sigma}^{*+}\left(\frac{y_2'}{L}+\frac{p_x^p L}{\hbar},n'\right)\times$$

$$\times \mathrm{d}p_x^e\,\mathrm{d}p_x^p\,\mathrm{d}p_z^e\,\mathrm{d}p_z^p\exp\left[\frac{i}{\hbar}(P_z Z'+P_x X')\right]\tilde{\eta}_{\rho\sigma}(x_1'-x_2',y_1',y_2',z_1'-z_2')\times$$

$$\times\frac{\mathrm{d}X'\,\mathrm{d}(x_1'-x_2')\,\mathrm{d}y_1'\,\mathrm{d}y_2'\,\mathrm{d}Z'\,\mathrm{d}(z_1'-z_2')}{\sqrt{(x_1'-x_2')^2+(y_1'-y_2')^2+(z_1'-z_2')^2}}\,. \tag{3.12}$$

Integrating over $X'=\frac{1}{2}(x_1'+x_2')$ and $Z'=\frac{1}{2}(x_1'+z_2')$ we obtain

$$\tilde{\eta}_{\alpha\beta}(x_1-x_2,y_1,y_2,z_1-z_2)=$$

$$=\frac{-e^2}{(2\pi\hbar)^2}\int\exp\left[\frac{i}{\hbar}(p_x(x_1-x_2)+p_z(z_1-z_2))\right]\times$$

$$\times\delta(p_x^e+p_x^p-P_x)\delta(p_z^e+p_z^p-P_z)\sum_{nn'}\frac{\exp[-(i/\hbar)(p_x w+p_z u)]}{E-\varepsilon_n(p_z^e)-\varepsilon_{n'}(p_z^p)}\times$$

$$\times{}^{e}\lambda_{\alpha}^{+}\left(\frac{y_1}{L}-\frac{p_x^e L}{\hbar},n\right){}^{e}\lambda_{\rho}^{*+}\left(\frac{y_1'}{L}-\frac{p_x^e L}{\hbar},n\right){}^{p}\lambda_{\beta}^{+}\left(\frac{y_2}{L}+\frac{p_x^p L}{\hbar},n'\right)\times$$

$$\times{}^{p}\lambda_{\sigma}^{*+}\left(\frac{y_2'}{L}+\frac{p_x^p L}{\hbar},n'\right)\frac{\tilde{\eta}_{\rho\sigma}(w,y_1',y_2',u)}{\sqrt{w^2+(y_1'-y_2')^2+u^2}}\times$$

$$\times\mathrm{d}p_x^e\,\mathrm{d}p_x^p\,\mathrm{d}p_z^e\,\mathrm{d}p_z^p\,\mathrm{d}w\,\mathrm{d}y_1'\,\mathrm{d}y_2'\,\mathrm{d}u\,, \tag{3.13}$$

where $p_z=(p_z^e-p_z^p)/2$, $p_x=(p_x^e-p_x^p)/2$,

$$\mathrm{d}p_{x,z}^e\,\mathrm{d}p_{x,z}^p=\mathrm{d}P_{x,z}\,\mathrm{d}p_{x,z}\,. \tag{3.14}$$

We shall seek for a solution to Equation (3.13) in the form of expansion in the complete set of functions ${}^{e,p}\lambda_{\alpha}^{+}$:

$$\tilde{\eta}_{\alpha\beta}(x_1-x_2,y_1,y_2,z_1-z_2)=$$

$$=\sum_{nn'}\int\tilde{\tilde{\eta}}(p_x,n,n',z_1-z_2){}^{e}\lambda_{\alpha}^{+}\left(\frac{y_1}{L}-\frac{p_x^e L}{\hbar},n\right)\times$$

$$\times{}^{p}\lambda_{\beta}^{+}\left(\frac{y_2}{L}+\frac{p_x^p L}{\hbar},n'\right)\exp\left[\frac{i(x_1-x_2)p_x}{\hbar}\right]\delta(p_x^e+p_x^p-P_x)\,\mathrm{d}p_x^e\,\mathrm{d}p_x^p\,. \tag{3.15}$$

Then the coefficient of this expansion satisfies the equation

$$\tilde{\tilde{\eta}}(p_x, n, n', z_1 - z_2)\,\delta(p_x^e + p_x^p - P_x) = \frac{-e^2}{(2\pi\hbar)^2}\int \mathrm{d}p_z^e\,\mathrm{d}p_z^p \times$$

$$\times\,\delta(p_z^e + p_z^p - P_z)\,\frac{\exp[(i/\hbar)p_z(z_1 - z_2)]}{E - \varepsilon_n(p_z^e) - \varepsilon_{n'}(p_z^p)}\,\delta(p_x^e + p_x^p - P_x) \times$$

$$\times \int \sum_{\bar{n},\bar{n}'} \mathrm{d}\bar{p}_x^e\,\mathrm{d}\bar{p}_x^p\,\tilde{\tilde{\eta}}(\bar{p}_x, \bar{n}, \bar{n}', u)\,\delta(\bar{p}_x^e + \bar{p}_x^p - P_x)\,{}^e\lambda_\rho^{+*}\left(\frac{y_1}{L} - \frac{p_x^e c}{LeB}, n\right) \times$$

$$\times\,{}^p\lambda_\sigma^{+*}\left(\frac{y_2}{L} + \frac{p_x^p c}{LeB}, n'\right){}^e\lambda_\rho^{+}\left(\frac{y_1}{L} - \frac{\bar{p}_x^e c}{LeB}, \bar{n}\right){}^p\lambda_\sigma^{+}\left(\frac{y_2}{L} + \frac{\bar{p}_x^p c}{LeB}, \bar{n}'\right) \times$$

$$\times\,\mathrm{d}u\,\mathrm{d}w\,\frac{\exp\left[-\frac{i}{\hbar}\left(p_z u + p_x w - \frac{(\bar{p}_x^e - \bar{p}_x^p)}{2}w\right)\right]}{\sqrt{w^2 + (y_1 - y_2)^2 + u^2}}\,\mathrm{d}y_1\,\mathrm{d}y_2\,. \qquad (3.16)$$

This is equivalent to the Schrödinger equation in the variable $(z_1 - z_2)$ in the limit of small $p_z^{e,p}$ (in this limit the spinor functions λ_α do not depend on p_z, and $\varepsilon_n(p_z) = m_n c^2 + p_z^2/2m_n$, where $m_n = (m^2 + 2eBn\hbar/c^3)^{1/2}$)

$$\left\{E - (m_n + m_{n'})c^2 - \frac{1}{8}P_z^2\,\frac{(m_{n'} + m_n)}{m_{n'}m_n} + \frac{i\hbar}{2}P_z\,\frac{(m_{n'} - m_n)}{m_{n'}m_n}\,\frac{\mathrm{d}}{\mathrm{d}(z_1 - z_2)} + \right.$$

$$\left. + \frac{\hbar^2}{2}\,\frac{m_{n'} + m_n}{m_{n'}m_n}\,\frac{\mathrm{d}^2}{\mathrm{d}(z_1 - z_2)^2}\right\}\tilde{\tilde{\eta}}(p_x, n, n', z_1 - z_2) =$$

$$= -\frac{e^2}{2\pi\hbar}\sum_{\bar{n},\bar{n}'}\int \mathrm{d}\bar{p}_x\,\tilde{\tilde{\eta}}(\bar{p}_x, \bar{n}, \bar{n}', z_1 - z_2) \times$$

$$\times \int \mathrm{d}w\,\mathrm{d}y_1\,\mathrm{d}y_2\,\frac{\exp[-(i/\hbar)w(p_x - \bar{p}_x)]}{\sqrt{w^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}} \times$$

$$\times\,{}^e\lambda_\rho^{+*}\left(y_1\left(\frac{eB}{c\hbar}\right)^{1/2} - \frac{P_x + 2p_x}{2(eB\hbar/c)^{1/2}}, n\right) \times$$

$$\times\,{}^e\lambda_\rho^{+}\left(y_1\left(\frac{eB}{c\hbar}\right)^{1/2} - \frac{P_x + 2\bar{p}_x}{2(eB/\hbar c)^{1/2}}, \bar{n}\right) \times$$

$$\times\,{}^p\lambda_\sigma^{+*}\left(y_2\left(\frac{eB}{c\hbar}\right)^{1/2} + \frac{P_x - 2p_x}{2(eB\hbar/c)^{1/2}}, n'\right) \times$$

$$\times\,{}^p\lambda_\sigma^{+}\left(y_2\left(\frac{eB}{c\hbar}\right)^{1/2} + \frac{P_x - 2\bar{p}_x}{2(eB\hbar/c)^{1/2}}, \bar{n}'\right). \qquad (3.17)$$

Introducing the dimensionless integration variables $q_x' = (\bar{p}_x - p_x)L/\hbar$, $y_1' = L^{-1}(y_1 - L^2\hbar^{-1}p_x^e)$, $y_2' = L^{-1}(y_2 + L^2\hbar^{-1}p_x^p)$, $w' = wL^{-1}$ and the variable $q_x = p_x L\hbar^{-1}$, we come to the conclusion that Equation (3.17) has the structure (here $F(\mathrm{d}/\mathrm{d}(z_1 - z_2))$ is the same differential operator as in the left-hand side of (3.17))

$$F\left(\frac{\mathrm{d}}{\mathrm{d}(z_1 - z_2)}\right)\tilde{\tilde{\eta}}(q_x, n, n', z_1 - z_2) =$$

$$= \sum_{\bar{n}, \bar{n}'} \int \mathrm{d}q_x' \, \tilde{\tilde{\eta}}(q_x' + q_x, \bar{n}, \bar{n}', z_1 - z_2) V(z_1 - z_2, q_x', n, n', \bar{n}, \bar{n}') \quad (3.18)$$

in which the integral kernel does not depend on q_x and is equal to (all the integration variables are dimensionless)

$$V(z_1 - z_2, q_x', n, n', \bar{n}, \bar{n}') = -\frac{e^2L^2}{2\pi} \int \mathrm{d}w \, \mathrm{d}y_1 \, \mathrm{d}y_2 \times$$

$$\times \frac{\exp(iq_x'w)}{\sqrt{L^2w^2 + [(y_1 - y_2)L + L^2P_x\hbar^{-1}]^2 + (z_1 - z_2)^2}} \times$$

$$\times {}^{e}\overset{*}{\lambda}{}_{\rho}^{+}(y_1, n)\,{}^{e}\lambda_{\rho}^{+}(y_1 - q_x', \bar{n})\,{}^{p}\overset{*}{\lambda}{}_{\sigma}^{+}(y_2, n')\,{}^{p}\lambda_{\sigma}^{+}(y_2 - q_x', \bar{n}'). \quad (3.19)$$

Equation (3.18) is an infinite set of integro-differential equations for determining the eigenfunctions $\tilde{\tilde{\eta}}$ and eigenvalues E. Motion along the relative $z_1 - z_2$ coordinate is described by the differential 'part' of the equation, motion along the y-coordinates of the two particles, y_1 and y_2, is described by a set of equations with respect to the variables n, n' and, finally, for the motion along the coordinate $q_x = (p_x^e - p_x^p)L/2\hbar$ which, for a free pair, has the meaning of the sum of the y-coordinates of the electron and positron orbit centres, the equation is integral. The latter is a kind of secular equation, for it determines the weight with which the initially energetically degenerate states of mutually free particles must enter into the wave function (3.15). The integral part of Equation (3.18) is obviously satisfied by q_x-independent $\tilde{\tilde{\eta}}$-functions, i.e., if all the positions of the pair as a whole with respect to the origin of the y-axis are taken as equiprobable. There is, however, no reason to restrict oneself to this solution. Fourier transformation of Equation (3.18) with respect to the variable q_x resolves its integral part and transforms it into differential in $z_1 - z_2$ equations with potentials depending on v, the variable Fourier-conjugate to q_x, as a parameter. Now the energy eigenvalue will also depend on v – the degeneracy is eliminated (but, as will be seen below, not in the adiabatic approximation). This is the situation with the exact Equation (3.18).

For the subsequent analysis of Equation (3.18) we realize the adiabatic approximation announced in the Introduction, i.e., we assume that the transverse part of the wave function is a superposition of the products of the wave functions of independent electron and positron with a fixed pair of values of the Landau quantum numbers n, n' (in other words, in the sum (3.15) we take only one term, with n, n' equal to the respective numbers which must label the eigenfunctions $\tilde{\tilde{\eta}}_{\alpha\beta}(x_1 - x_2, y_1, y_2, z_1 - z_2)$. Formally,

this approximation becomes possible because for strong fields, as the Bohr radius is much larger than the orbit radius, the terms L^2w^2 and $(y_1 - y_2)L$ can be neglected in the kernel (3.19) in two cases: (1) if $(z_1 - z_2)$ is of the order of or larger than the Bohr radius, (2) if the third independent parameter with dimensionality of length $L^2P_x\hbar^{-1}$ is of the order of the Bohr radius or larger. Then due to completeness and orthonormality of the exploited set of solutions of the Dirac equation the integration over $dy_1\, dy_2\, dw$ gives the factor $\delta_{n\bar{n}}\delta_{n'\bar{n}'}\,\delta(q_x')$. Thus, in the zero the order in the ratio (L/a_B), we have one Schrödinger equation

$$F\left(\frac{d}{d(z_1 - z_2)}\right)\tilde{\tilde{\eta}}(q_x, z_1 - z_2, n, n') = \tilde{\tilde{\eta}}(q_x, z_1 - z_2, n, n')V_0(z_1 - z_2) \quad (3.20)$$

with the potential

$$V_0(z_1 - z_2) = -\frac{e^2}{\sqrt{(z_1 - z_2)^2 + L^4P_x^2\hbar^{-2}}}\,; \quad (3.21)$$

which is valid for all $z_1 - z_2$ (including $z_1 - z_2 = 0$) only for sufficiently large $\hbar^{-1}L^2P_x \gtrsim a_B$ and for all $\hbar^{-1}L^2P_x$ (including $P_x = 0$) for distances between the particles $(z_1 - z_2)$ not smaller than the Bohr radius.

The term in the operator F in (3.17), linear in the derivative, is eliminated by the substitution of the unknown function

$$\tilde{\tilde{\eta}} = \exp\left[\left(\frac{iP_z}{2\hbar}\right)(m_n - m_{n'})\,(z_1 - z_2)/(m_n + m_{n'})\right]\varphi\,,$$

where $m_n = m\sqrt{1 + (2nB/B_{cr})}$, $m_{n'} = m\sqrt{1 + (2n'B/B_{cr})}$, and the Schrödinger equation (3.20) takes the form

$$\left[(E - c^2m_n - c^2m_{n'}) + \frac{\hbar^2}{2}\left(\frac{m_nm_{n'}}{m_n + m_{n'}}\right)^{-1}\frac{d^2}{d(z_1 - z_2)^2} - \right.$$

$$\left. - \frac{P_z^2}{2(m_n + m_{n'})} - V_0\right]\varphi(z_1 - z_2) = 0\,. \quad (3.22)$$

Since the differential operator (including the potential) does not depend on q_x, the same refers to a solution of Equation (3.22). Using (3.7), (3.11), (3.15), we see that the complete equal-time wave function (3.4) of the pair is a superposition of the functions

$$\varkappa_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2, t) = \exp\left(-\frac{iEt}{\hbar}\right)\exp\left[\frac{iP_zz_{\text{c.m.}}}{\hbar} + iP_x(x_1 + x_2)/2\hbar\right]\times$$

$$\times \exp\left[\frac{ip_x(x_1 - x_2)}{\hbar}\right]{}^e\lambda_\alpha\left(\frac{y_1}{L} - p_x\frac{L}{\hbar} - \frac{P_xL}{2\hbar}, n\right)\times$$

$$\times {}^p\lambda_\beta\left(\frac{y_2}{L} - p_x\frac{L}{\hbar} + \frac{P_xL}{2\hbar}, n'\right)\varphi(z_1 - z_2)\,, \quad (3.23)$$

weighted with arbitrary coefficients depending on $p_x = \hbar q_x/L$. We have separated here the motion along the magnetic field of the centre of mass of the electron at the Landau level n and of the positron at the level n' (or vice versa) with the z-coordinate $Z_{\text{c.m.}} = (z_1 m_n + z_2 m_{n'})/(m_n + m_{n'})$. Separation of the centre of mass motion along x would have no practical importance because the potential is p_x-independent. We have, therefore, chosen the necessary phase in (3.11), (3.23) in the simplest form, whereas the translational invariance allows us to write it in the form $iP_x X/\hbar$, where X is any linear combination of x_1 and x_2 with the condition that $X = x_1 = x_2$ at $x_1 = x_2$. The use of a more general X would lead only to a redefinition of the quantum number p_x in (3.23).

We have seen that the dependence of the pair wave function on the transverse degree of freedom (described by the variable p_x equal to the sum of the y-coordinates of the centres of the electron and positron orbits) remains arbitrary, the same as for a non-interacting pair. Since in this case the potential (3.21) does not depend on p_x the energy eigenvalue E of the pair does not depend on $p_x = q_x\hbar/L$ either. Thus, we have established that in the adiabatic (i.e., zero-order with respect to L/a_B) approximation the degeneracy in p_x is not eliminated, and the secular equation does not lead to any limitations on superposition of states with different quantum numbers p_x. On the contrary, the degeneracy with respect to the difference of the orbit centre positions along y is lifted: the potential (3.21) and, accordingly, the energy and the wave function depend on the quantum number P_x, but only for its very high values. In this situation it is reasonable to consider the wave function directly in the form (3.23) with a definite value of the quantum number p_x. Whereas for an unbound pair the quantity $\varphi(z_1 - z_2) = L_z^{-1/2} \exp[i(z_1 - z_2)(p_z^e - p_z^p)]$ is a function normalized to unity in a box of length L_z along the z-coordinate (we can take $L_z = a_B$, since $a_B \gg L$ in our case), for a bound pair we have, instead, the normalization condition

$$\int_{-\infty}^{+\infty} |\varphi(z_1 - z_2)|^2 \, \mathrm{d}(z_1 - z_2) = 1 \,. \tag{3.24}$$

Note that, in order to retain the next non-vanishing contribution of the order of L^2/a_B^2, one would have to solve, instead of (3.22), a set of finite number of Equations (3.18) in which n differs from $\bar{n}$, and n' from $\bar{n}'$ by one and two units, and one and two derivatives of $\tilde{\tilde{\eta}}$ with respect to p_x are involved, apart from $\tilde{\tilde{\eta}}$ itself. Thus, the use of the adiabatic Ansatz, i.e., of Equation (3.22), in which the expression (3.19) with n', $n = \bar{n}'$, $\bar{n}$ is taken for a potential, falls beyond the accepted accuracy and cannot, therefore, be used for making the zeroth approximation (3.21) more precise. Such an expression may be still useful for guidance. Its use for extrapolating the potential (3.21) into the domain of small $(z_1 - z_2)^2 + L^4 P_x^2 \hbar^{-2}$ where the latter is not otherwise defined reduces, within the accuracy adopted, by adding a term $\sim L^2$ to $(z_1 - z_2)^2 + L^4 P_x^2 \hbar^{-2}$ to:

$$V_0'(z_1 - z_2) = -\frac{e^2}{\sqrt{(z_1 - z_2)^2 + L^2 + L^4 P_x^2 \hbar^{-2}}} \,. \tag{3.25}$$

On the contrary, the extension of the potential by Equation (3.21) would lead to a non-physical result showing the fall-down onto the centre for $P_x = 0$ as is seen from (3.27). In other words, for $P_x = 0$ the Schrödinger equation cannot be expanded with respect to L/a_B uniformly in the entire region of $(z_1 - z_2)$ because at the point $z_1 = z_2$ the solution of the zeroth approximation is singular, whereupon the eigenvalue of the ground state ($n_c = 0$) energy is also singular in the zeroth approximation. This singularity is cut off at the orbit length L. Nondiagonal corrections, the same as the excited state ($n_c > 0$) energies are non-singular in the zeroth approximation and, therefore, the corrections to the adiabatic approximation for them need not be considered here.

Let us approximate the potential by the expression

$$V(z_1 - z_2) = -\frac{e^2}{|z_1 - z_2| + \sqrt{L^2 + L^4 P_x^2 \hbar^{-2}}} \,. \tag{3.26}$$

This expression makes it possible to solve the Schrödinger equation (Loudon, 1959). In the frame where the centre of mass of the electron and positron which have the Landau quantum numbers n and n' does not move along the z-axis ($P_z = 0$) one obtains the following approximate value for the ground state energy ($n_c = 0$):

$$\varepsilon_{nn'}(0, P_x^2) = E|_{P_z = 0} =$$

$$= (m_n + m_{n'})c^2 - \frac{\alpha^2 M_{nn'} c^2}{2}\left[2 \ln \frac{a^B_{nn'}}{2\sqrt{L^2 + L^4 P_x^2 \hbar^{-2}}}\right]^2, \tag{3.27}$$

where $M_{nn'}$ is the reduced mass

$$M_{nn'} = \frac{m_n m_{n'}}{m_n + m_{n'}}$$

and

$$a_B^{nn'} = \frac{\hbar^2}{e^2 M_{nn'}} \tag{3.28}$$

is the Bohr radius of the mutually bound electron and positron which are at the Landau levels with numbers n, n'. For excited states of the same series $n, n'(n_c = 1, 2, 3, \ldots)$ which are even under the reflection $z_1 \leftrightarrow z_2$ we have

$$\varepsilon_{nn'}(n_c, P_x^2) = E|_{P_z = 0} = \tag{3.29}$$

$$= (m_n + m_{n'})c^2 - \frac{\alpha^2 M_{nn'} c^2}{2}\left[n_c + \left(\ln \frac{n_c a_B^{nn'}}{2\sqrt{L^2 + L^4 P_x^2 \hbar^{-2}}}\right)^{-1}\right]^{-2},$$

and for the odd ones we have

$$\varepsilon_{nn'}(n_c, P_x^2) = E|_{P_z = 0} =$$

$$= (m_n + m_{n'})c^2 - \frac{\alpha^2 M_{nn'} c^2}{2}\left[n_c + \frac{2\sqrt{L^2 + L^4 P_x^2 \hbar^{-2}}}{a_B^{nn'}}\right]^{-2}. \tag{3.30}$$

We should note that, in accordance with Loudon (1959), there is the parity degeneracy in the point $P_x = 0$ in the limit $(L/a_B) \to 0$. If either L/a_B or P_x are finite, the energy (3.30) of the odd state is lower than (3.29) for the same n. Equations (2.27), (3.29) describe the spectrum as long as the logarithms involved essentially exceed the unity. As P_x^2, increases, the mass defects of all the levels labelled by n_c decrease according to (3.27), (3.29), (3.30). Formally they vanish as soon as $L^2 P_x \hbar^{-1}$ reaches the value of the order of $a_B^{nn'}$ in (2.27), $n_c a_B^{nn'}$ in (3.29) and ∞ in (2.30). For such values of P_x these formulae are already not literally valid. However, from the form of the potentials (3.21), (3.25), (3.26) this result is seen to be valid qualitatively: the potentials decrease with the growth of P_x and, therefore, the levels must approach the boundary of the continuum.

For $P_x = 0$ the energy eigenvalue (3.27) coincides for $n = n' = 0$ with the result of Leinson and Orayevski (1984, 1985). Our results for the spectra are distinguished by their dependence on P_x.

Let us now summarize our consideration. In the lowest adiabatic approximation the eigenfunctions η and eigenvalues E of the Bethe–Salpeter equation are labelled as follows. Each pair of Landau numbers n, n' sets up a series of bound states, whose binding energies $E - (m_n + m_{n'})c^2 = -\Delta\varepsilon_{nn'}$ are eigenvalues of the Schrödinger equation (3.22) for $P_z = 0$. They are numbered by a discrete quantum number n_c for $\Delta\varepsilon_{nn'} > 0$ and by a continuous one in the opposite case (that is, the values $(m_n + m_{n'})c^2$ are the boundaries of the continua responsible for the infinite relative motion along the magnetic field of the electron and positron in the n, n'-states). The binding energies $\Delta\varepsilon_{nn'}$ also depend on the distance between the y-coordinates of the centres of the electron and positron orbits $|y_0^p - y_0^e| = L^2 P_x \hbar^{-1}$ and tend to zero for every level if it exceeds the Bohr radius (see Figure 1). The energy remains degenerate with respect to the motion along the degree of freedom associated to the sum of coordinates of the orbit centres. The quantity $P_x \hbar^{-1}$ has also the meaning of the total wave vector of the pair in the x-direction. For a pair which is not mutually bound the energy E does not depend on P_x (degeneration) and, hence, the group velocity $\partial E/\partial P_x$ across the magnetic field is zero – i.e., no real transverse motion is associated with the wave vector P_x. For a mutually bound pair the results (3.27)–(3.29), (3.30) show that due to the removal of degeneracy the transverse group velocity is nonzero although it remains very small as compared to the velocity of light. This explains why the above non-relativistic approximations have not, in fact, concerned the motion across the magnetic field, but only along it. In our case the internal motion along z is nonrelativistic, and the energy dependence on the momentum of the free centre of mass motion along z can be easily relativized by Equation (3.3), as shown in Figure 1.

The mass defect in the ground state (2.7) can reach high values for small P_x due to the singularity in the potential V_0 (3.21) of the adiabatic approximation for $P_x = 0$ (fall-down onto the centre in the one-dimensional Schrödinger equation (3.20)). As has already been mentioned, beyond the framework of adiabatic approximation the singularity of the potential is cut off at the values of $(z_1 - z_2)$ of the order of the orbit radius L (see (3.25), (3.26)). The mass defect in (3.27), therefore, contains the characteristic large quantity $\ln(a_B/L)$. Similar is the origin of some other large quantities which will appear

below for the value of the wave function in coinciding points, the residue of the plarization operator in the positronium pole. Note that the value of the magnetic field

$$B \sim \alpha^2 \exp\left(\frac{2^{3/2}}{\alpha}\right) B_{cr}$$

formally provides the vanishing of the ground-state energy of the positronium – i.e., a complete compensation of the rest energy by the mass defect. This implies the vacuum instability in a magnetic field under the positronium creation.

4. Contribution of Bound States Into the Photon Polarization Operator in a Strong Magnetic Field*

Now we are going to find the contribution of a bound positronium state with fixed discrete quantum numbers n, n', n_c and even under the sign reversal of $(z_1 - z_2)$, into the polarization operator of a photon $\Pi_{\mu\nu}(K)\,\delta(K - K')$, where K is the photon 4-momentum, and into its imaginary part, responsible for mutual transformations of the positronium and a photon. The calculations will be made beyong the non-perturbed photon mass shell $K_0^2 - \mathbf{K}^2 = 0$ in order that the results could be used further both for establishing the deviations of the photon dispersion law from this form and for modifying the dispersion law of a bound pair (3.27), (3.29) through the annihilation interaction.

The contribution of the bound state into the polarization operator is obtained if in the four-fermion Green's function (3.1), (3.2) we take the initial and the final pairs of Fermi-fields in coinciding points $x_1 = x_2 = x$, $x_3 = x_4 = x'$, multiply by two Dirac matrices γ_μ and γ_ν indexed as α, β and γ, δ, respectively, by the exponentials $\exp[i(Kx - K'x')]$, sum up over the spinor indices and integrate over $\mathrm{d}^4x\,\mathrm{d}_x^4{}'$. The use of the wave function (3.23) leads to the δ-functions expressing the conservation laws – the same as when the contribution of unbound pairs into the polarization operator is calculated

$$\delta(K_\parallel - P_z)\,\delta(K_0 - E)\,\delta(K_\perp - P_x)\,\delta^{(4)}(K - K')$$

(the reference frame is so chosen that $K_y = 0$ and thus $K_x = K_\perp$). Our calculation must differ from the above-mentioned case, by the facts that first $L_z^{-1/2} \exp[i(p_z^e - p_z^p)(z_1 - z_2)/2]$ should be replaced in the point $z_1 = z_2$ by $\varphi(0)$* and, second, that instead of the integration over the momentum z-component of the relative motion $(p_z^e - p_z^p)$ (with the sum $p_z^e + p_z^p = P_z = K_\parallel$ fixed) which is a continuous quantum number characterizing the intermediate two-particle state with fixed n, n' in the expansion of the function (3.1) over the complete set of physical states, the polarization operator should involve the sum over n_c, where we are interested mostly

* In this and the next sections we put $c = \hbar = 1$.

* For this reason the states with the energy (3.30), which are odd under the sign reversal of $(z_1 - z_2)$ do not contribute into the polarization operator, since $\varphi(0) = 0$ for them.

in contributions of individual terms. With these remarks in mind we can use the ready results for obtaining the contributions of interest, Batalin and Shabad (1971) calculated the photon polarization operator approximated as a loop constituted by the Green functions in the external field of electron and positron with their interactions between themselves disregarded (calculations of that paper involved a constant electric field E in arbitrary combination with the magnetic field B, so that the both invariants of the external field $\mathbf{E}^2 - \mathbf{B}^2$ and $\mathbf{E} \cdot \mathbf{B}$ were nonzero). The individual contributions from the states with given values of the Landau quantum number of electron and positron into the polarization operator eigenvalues which correspond to the propagation of each of the three possible polarization modes were determined by Shabad (1975). These contributions are characterized by two factors

$$I_{1,2}^{nn'} = \int_{-1}^{1} \frac{\mathrm{d}\eta \psi_{1,2}(\eta)}{(n+n') + \eta(n-n') - [(1-\eta^2)/4eB](K_0^2 - K_\parallel^2) + (m^2/eB) - iO}, \quad (4.1)$$

where $\psi_1 = 1$, $\psi_2 = 1 - \eta^2$, while the contributions into the cut discontinuities (imaginary parts doubled) are characterized by

$$\Delta I_{1,2}^{nn'} = \int_{-1}^{1} 2\pi i \delta\left(n + n' + \eta(n-n') - \frac{(1-\eta^2)}{4eB}(K_0^2 - K_\parallel^2) + \frac{m^2}{eB}\right) \psi_{1,2}(\eta)\,\mathrm{d}\eta. \quad (4.2)$$

The variable η is responsible for the degree of freedom of the relative motion of electron and positron along the magnetic field. The relation between η and p_z is of the form

$$\eta = \frac{2p_z}{\sqrt{K_0^2 - K_\parallel^2}} - \frac{2(n-n')eB}{K_0^2 - K_\parallel^2}. \quad (4.3)$$

Substituting (4.3) into (4.2) we have

$$\Delta I_{1,2}^{nn'} = \int_{-\infty}^{+\infty} 2\pi ieB\,\delta\left(p_z^2 + \frac{K_\parallel^2 - K_0^2}{4} + eB(n+n') + m^2 - \frac{(n-n')^2 e^2 B^2}{K_0^2 - K_\parallel^2}\right) \frac{2\psi_{1,2}\,\mathrm{d}p_z}{\sqrt{K_0^2 - K_\parallel^2}} =$$

$$= \int_{-\infty}^{+\infty} \frac{8\pi ieB(K_0^2 - K_\parallel^2)\psi_{1,2}}{(K_0^2 - K_\parallel^2)^2 - 4e^2B^2(n-n')^2} \times$$

$$\times\, \delta\left(-\sqrt{K_0^2 - K_\parallel^2} + \sqrt{m^2 + p_z^2 + 2enB} + \sqrt{m^2 + p_z^2 + 2en'B}\right) \mathrm{d}p_z =$$

$$= \frac{16\pi ieB\sqrt{K_0^2 - K_\parallel^2}\,\psi_{1,2}}{\left[K_0^2 - K_\parallel^2 - \dfrac{4e^2B^2(n-n')^2}{K_0^2 - K_\parallel^2}\right]} \times$$

$$\times \int_{-\infty}^{+\infty} dp_z \delta((\sqrt{m^2 + p_z^2 + 2enB} + \sqrt{m^2 + p_z^2 + 2en'B})^2 + K_\parallel^2 - K_0^2), \tag{4.4}$$

$$\psi_1 = 1, \qquad \psi_2 = \frac{4}{K_0^2 - K_\parallel^2}\left[eB(n + n') + m^2 - \frac{2(n - n')^2}{K_0^2 - K_\parallel^2} e^2B^2\right]. \tag{4.5}$$

(We have taken into account that the odd powers of p_z in ψ_2 do not contribute after integrated over dp_z). The integral of the δ-function in (4.5) originates from the sum over the intermediate states in the expansion of the polarization operator in a complete set of these states. The sum of the square roots in the argument of the δ-function is the total energy of the electron and positron which do not interact with one another and have the momenta p_z and $-p_z$ in the centre of mass frame with respect to the z-axis. In the case of a bound pair this sum should be replaced by the energy of the discrete even quantum state $\varepsilon_{nn'}(n_c, P_x^2)$ (3.27), (3.29), in which the momentum squared P_x^2 is replaced by $K_\perp^2$ due to the corresponding conservation law δ-function. The momentum differential dp_z must be replaced here by $2\pi/L_z$, where L_z is the normalization dimension of the system along the z-axis, while the quantity $1/L_z$, which is the square of the wave function $(1/L_z)^{1/2} \exp[i(z_1 - z_2)p_z]$ (normalized in a box of the same size) of the relative motion along the magnetic field, must in its turn be replaced by $|\varphi_{nn'n_c}(0)|^2$, where $\varphi_{nn'n_c}(z_1 - z_2)$ is the solution of the Schrödinger equation (3.22) normalized to unity according to (3.24). With these prescriptions and using equations of Shabad (1975), we obtain the expressions for the sum of the contributions from the positronium bound states (without the continuum) into the imaginary parts $\Delta_{1,2,3}$ of the three eigenvalues of the polarization operator, which are responsible for absorption of electromagnetic waves of the three different polarization modes as they were described in Shabad (1975). For the mode 1 we have* ($K^2 \equiv K_\perp^2 + K_\parallel^2 - K_0^2$)

$$\Delta_1 = \frac{e^2K^2}{8\pi^2} \exp\left(-\frac{K_\perp^2}{2eB}\right) \sum_{n_c=0}^{\infty} \left\{\left[\sum_{n,n'=0}^{\infty} \frac{n!}{n'!} \Phi^2\left(-n', n - n' + 1, \frac{K_\perp^2}{2eB}\right) + \right.\right.$$

$$\left. + \sum_{n,n'=1}^{\infty} \frac{(n-1)!}{(n'-1)!} \Phi^2\left(-n' + 1, n - n' + 1, \frac{K_\perp^2}{2eB}\right)\right] \times$$

$$\times \frac{1}{2eB}\left[\frac{(K_\parallel^2 - K_0^2)}{4} \Delta I_2^{nn'n_c} + m^2 \Delta I_1^{nn'n_c}\right] +$$

$$\left. + 2 \sum_{n,n'=1}^{\infty} \frac{n!}{(n'-1)!} \Phi\left(-n' + 1, n - n' + 1, \frac{K_\perp^2}{2eB}\right) \times\right.$$

* Note that in Shabad (1972) and in the equations hereafter the charge is normalized so that $(e^2/4\pi) = \alpha = \frac{1}{137}$, but at the same time, one should assume that $eB = B/\sqrt{137}$, since only in this case the term eA_μ in the Dirac equation (3.6) does correspond to the interaction with the electromagnetic field whose energy is $(E^2 + B^2)/8\pi$ and not $(E^2 + B^2)/2$.

$$\times \Phi\left(-n', n-n'+1, \frac{K_\perp^2}{2eB}\right) \Delta I_1^{nn'n_c}\Big\} \times$$

$$\times \left(\frac{K_\perp^2}{2eB}\right)^{n-n'-1} \frac{1}{\Gamma^2(n-n'+1)} . \tag{4.6}$$

The term $n = n' = 0$ is not to be included here in the first sum. In (4.6) and in equations to follow the sign Σ sums up all the factors to the right of it. The absorption parts $\Delta_{2,3}$ of modes 2, 3 are expressed with the help of the relations

$$\Delta_2 = \bar{\Delta}_2 - K_\perp^2 \frac{\bar{\Delta}_2 - \Delta_1}{K^2} , \qquad \Delta_3 = \Delta_3' + \frac{K_\parallel^2 - K_0^2}{K^2} \Delta_1 , \tag{4.7}$$

where Δ_1 is given by (4.6) and

$$\bar{\Delta}_2 = -\frac{e^2K^2}{16\pi^2} \exp\left(-\frac{K_\perp^2}{2eB}\right) \times$$

$$\times \sum_{n_c=0}^{\infty} \left[\sum_{n,n'=0}^{\infty} \frac{n!}{n'!} \Phi^2\left(-n', n-n'+1, \frac{K_\perp^2}{2eB}\right) + \right.$$

$$\left. + \sum_{n,n'=1}^{\infty} \frac{(n-1)!}{(n'-1)!} \Phi^2\left(-n'+1, n-n'+1, \frac{K_\perp^2}{2eB}\right)\right] \times$$

$$\times \left(\frac{K_\perp}{2eB}\right)^{n-n'} \frac{\Delta I_2^{nn'n_c}}{\Gamma^2(n-n'+1)} , \tag{4.8}$$

$$\Delta_3' = \frac{e^2K_\perp^2}{2\pi^2} \exp\left(-\frac{K_\perp^2}{2eB}\right) \times$$

$$\times \sum_{n_c=0}^{\infty} \sum_{n,n'=1}^{\infty} \frac{n!}{(n'-1)!} \frac{\Phi(-n', n-n', K_\perp^2/2eB)}{\Gamma(n-n')} \times$$

$$\times \frac{\Phi(-n'+1, n-n'+2, K_\perp^2/2eB)}{\Gamma(n-n'+2)} \left(\frac{K_\perp^2}{2eB}\right)^{n-n'-1} \Delta I_1^{nn'n_c} . \tag{4.9}$$

In (4.6), (4.8), (4.9), Φ stands for degenerate hypergeometric functions related to the generalized Laguerre polynomials L_n^α by

$$L_{n'}^{n-n'}(y) = \frac{n!}{n'!} \frac{\Phi(-n', n-n'+1, y)}{\Gamma(n-n'+1)} . \tag{4.10}$$

In case the Γ-functions in the above equations involve non-positive arguments, it is

convenient to pass over to positive arguments using the relations

$$\frac{\Phi(-n', n-n'+1, y)}{\Gamma(n-n'+1)} = (-y)^{n'-n}\,\frac{n'!}{n!}\,\frac{\Phi(-n, n'-n+1, y)}{\Gamma(n'-n+1)}\,. \tag{4.11}$$

The quantities $\Delta I_{1,2}^{nn'n_c}$ are equal to

$$\Delta I_{1,2}^{nn'n_c} = \frac{32\pi^2 ieB\,|\varphi_{nn'n_c}(0)|^2\,\psi_{1,2}}{\sqrt{K_0^2 - K_\parallel^2} - \dfrac{4e^2B^2(n-n')^2}{(K_0^2 - K_\parallel^2)^{3/2}}}\,\delta(\varepsilon_{nn'}^2(n_c, K_\perp^2) - K_0^2 + K_\parallel^2)\,, \tag{4.12}$$

where $\psi_{1,2}$ are expressed according to (4.5). The contribution of positronium states into the polarization operator can be reconstructed by its discontinuities using the optical theorem. To this end we introduce the quantities

$$\begin{aligned} I_{1,2}^{nn'n_c} &= \frac{16\pi\,|\varphi_{nn'n_c}(0)|^2 eB}{\sqrt{K_0^2 - K_\parallel^2} - \dfrac{4e^2B^2(n-n')^2}{(K_0^2 - K_\parallel^2)^{3/2}}}\,\frac{\psi_{1,2}}{\varepsilon_{nn'}^2(n_c, K_\perp^2) + K_\parallel^2 - K_0^2 - iO} \\ &\simeq \frac{16\pi\,|\varphi_{nn'n_c}(0)|^2 eB}{\varepsilon_{nn'}(n_c, K_\perp^2) - \dfrac{4e^2B^2(n-n')^2}{\varepsilon_{nn'}^3(n_c, K_\perp^2)}} \\ &\qquad\qquad \frac{\psi_{1,2}}{\varepsilon_{nn'}^2(n_c, K_\perp^2) + K_\parallel^2 - K_0^2 - iO}\,, \end{aligned} \tag{4.13}$$

where $\psi_{1,2}$ is either (4.5) or, alternatively,

$$\psi_1 = 1\,, \qquad \psi_2 = \frac{4}{\varepsilon_{nn'}^2(n_c, K_\perp^2)}\left[eB(n+n') + m^2 - \frac{2(n-n')^2e^2B^2}{\varepsilon_{nn'}^2(n_c, K_\perp^2)}\right]. \tag{4.14}$$

Then in the neighbourhood of the poles corresponding to the discrete levels of the positronium the behaviour of the eigenvalues $\varkappa_{1,2,3}$ of the polarization operator responsible for the three polarization modes (1, 2, 3) is obtained if for $\varkappa_{1,2,3}$ one takes the same expressions as (4.6)–(4.9) but with the substitution $\Delta I_{1,2}^{nn'n_c} \to I_{1,2}^{nn'n_c}$ and without the summation over n, n', n_c.

Note that the expressions obtained for the polarization operator are by themselves exact relativistic consequences of the adiabatic approximation. The wave function of the transverse motion, which is the basis of the calculations, is the exact relativistic function of an unbound pair in a magnetic field. The nonrelativity is introduced only when the solution and the eigenvalue of the Schrödinger equation (3.23) (with $P_z = 0$) are taken, respectively, for $\varphi_{nn'n_c}(0)$ and $\varepsilon_{nn'}(n_c, P_x^2)$.

As is seen from the above equations, the contribution of the bound positronium state into the polarization operator and into its imaginary part enters with a coefficient which

residue of the positronium pole in the polarization operator. This contribution also determines the influence of this pole on the spacing between the joint dispersion curves of the photon and the positronium. As discussed in the previous section, the wave function at $z_1 - z_2$ can be estimated by the potential extrapolation (3.25), (3.26). Then, according to Loudon (1959) for the state $n_c = 0$ (the lowest in each series) we have

$$\varphi_{nn'0}(z_1 - z_2) = \frac{\exp(-|z_1 - z_2|/a_B^{nn'}\delta)}{\sqrt{\delta a_B^{nn'}}}, \tag{4.15}$$

which is valid as long as the quantity

$$\delta = \frac{1}{2}\left(\ln \frac{a_B^{nn'}}{2\sqrt{L^2 + L^4 P_x^2}}\right)^{-1}$$

is small. Here $a_B^{nn'}$ is the Bohr radius (3.28). As $\delta \to n_c = 0$, $|\varphi_{nn'0}|^2 \to \delta(z_1 - z_2)$ which corresponds to the above-mentioned falling down onto the centre. It is of importance that for $z_1 = z_2$ the wave function (4.15) has the order $(\delta a_B^{nn'})^{-1/2}$. Due to the smallness of δ this value is much larger than the inverse root of the normalization length along the z-axis, $L_z^{-1/2} \sim (a_B^{nn'})^{-1/2}$. This must enhance the matrix element of creation of a bound pair as compared with a free pair by a photon. The wave functions of the excited states $n_c = 1, 2, 3, \ldots$ in each series are, on the contrary, non-singular at $\delta \to n_c$. Moreover, in this limit, which corresponds to a purely Coulomb one-dimensional potential $-e^2/|z_1 - z_2|$, i.e., $L/a_B^{nn'} = 0$, $P_x = 0$ in (3.25), the wave functions of the states even under the reflection $z_1 \leftrightarrow z_2$ vanish simultaneously with L/a_B (cf. Loudon, 1959) so that the coefficients in (4.12), (4.13) disappear for $n_c \geq 1$ like

$$|\varphi_{nn'n_c}(0)|^2 \simeq \frac{2}{n_c^3 a_B^{nn'}} \frac{L^2 + L^4 K_\perp^2}{(a_B^{nn'})^2}. \tag{4.16}$$

It is, therefore, clear that the pole contribution into the polarization operator coming from the excited, $n_c \geq 1$, states is essentially suppressed due to the small residue as compared with the contribution of the ground (even) state $n_c = 0$ and even with the contribution of an unbound pair. As concerns odd states, their contribution is identical zero, $\varphi_{nn'n_c}^{\text{odd}}(0) = 0$. Therefore, they can neither one-photon annihilate, nor be one-photon created. They do not provide poles to the photon Green function, and their dispersion curves can well intersect the dispersion curve of the photon without repulsing from it, contrary to the dispersion curves of even states, especially the one with $n_c = 0$. The latter case is discussed in the next section.

Let us write separately the contributions into the polarization operator and into its imaginary part from the ground state of an electron and a positron at zero Landau levels. For $n = n' = 0$ the positronium can one-photon annihilate by transforming only to a photon of mode 2, a linearly polarized wave in which the electric field vector $\mathbf{e}^{(2)}$ lies in the plane spanned by the external magnetic field B direction and the photon wavevector $\mathbf{K}$. With our choice of coordinates this is the xz-plane. Mode 2 is purely

transverse, $\mathbf{e}^{(2)} \cdot \mathbf{K} = 0$, only if the annihilated positronium has not been moving along the z-axis (that is, if $K_\parallel = P_z = 0$). The contribution of the ground state $n = n' = n_c = 0$ into the cut discontinuities of the eigenvalues of the polarization operator becomes

$$\Delta\varkappa_1^{(000)} = \Delta\varkappa_3^{(000)} = 0\,,$$

$$\Delta\varkappa_2^{(000)} = i8\pi\alpha eB\varepsilon_{00}(0, K_\perp^2)\,|\varphi_{000}(0)|^2 \times$$

$$\times \exp\left(-\frac{K_\perp^2}{2eB}\right)\delta(\varepsilon_{00}^2(0, K_\perp^2) - K_0^2 + K_\parallel^2)\,, \tag{4.17}$$

and the eigenvalue $\varkappa_2$ near the pole corresponding to the ground state has the form

$$\varkappa_2 = 4\alpha eB\varepsilon_{00}(0, K_\perp^2)\,|\varphi_{000}(0)|^2\,\frac{\exp(-K_\perp^2/2eB)}{\varepsilon_{00}^2(0, K_\perp^2) - K_0^2 + K_\parallel^2 - iO}\,; \tag{4.18}$$

where, according to (3.27), the ground-state energy, which increases with an increase of $P_x^2 = K_\perp^2$, is given by

$$\varepsilon_{00}(0, K_\perp^2) = 2m - \alpha^2 m\left(\ln\frac{L}{a_B}\sqrt{1 + K_\perp^2 L^2}\right)^2, \tag{4.19}$$

and according to (4.15), the wave function squared at coincident points is

$$|\varphi_{000}(0)|^2 = \frac{1}{a_B}\left|\ln\frac{a_B}{L\sqrt{1 + K_\perp^2 L^2}}\right|. \tag{4.20}$$

We recall that $a_B = (\alpha m)^{-1}$, $L = (eB)^{-1/2}$, $a_B \gg L$. The most tightly ($n_c = 0$) bound state of the pair of particles, electron and positron, one of which is at the Landau level $n = 0$ and the other at $n' = 1$ (and vice versa), contributes poles already into all the three eigenvalues $\varkappa_{1,2,3}$.

In the rest of this section we list some formal results concerning the annihilation and creation probabilities treated within the perturbation theory with respect to positronium interaction with the electromagnetic field. This is done only for the sake of completeness, since in the next section we proceed without appealing to these expressions. To obtain the total probability for a positronium with the quantum numbers n, n', n_c to be created per unit time by one photon of the ith mode, one should take

$$W^i = \left.\frac{\Delta_i^{(nn'n_c)}}{4K_0}\right|_{K^2=0}. \tag{4.21}$$

$W^1 = 0$ due to the factor K^2 in (4.6). Comparing (4.21) with the expression for the same probability written through the modulus of the matrix element of $e^+e^-\gamma$ squared (the pair is assumed here to be bound), summed up over (discrete in the normalization volume) quantum numbers of the pair p_x, P_x, P_z after the conservation laws

$\delta(P_x - K_\perp)\,\delta(P_z - K_z)$ and the formula

$$\sum_{p_x} = (L_x/2\pi) \int_{-L_y eB/2}^{L_y eB/2} \mathrm{d}p_x = \frac{L_x L_y eB}{2\pi}$$

are taken into account

$$W^i = 2\pi(2eB)\alpha M_i^2 \delta(K_0^2 - \varepsilon_{nn'}^2(n_c, K_\perp^2) - K_\parallel^2)|_{K^2=0}\,, \tag{4.22}$$

we find that to pick up the values of the matrix elements squared, one should, omitting summation over discrete quantum numbers, substitute equations (4.7)–(4.9) for Δ_i into the relation

$$\alpha M_i^2 \delta(K_0^2 - K_\parallel^2 - \varepsilon_{nn'}) = \frac{\Delta_i^{(nn'n_c)}(K_\perp^2, \varepsilon_{nn'})}{16\pi eBK_0}\,. \tag{4.23}$$

The probability of a bound pair annihilation into a photon of the ith mode is obtained by integrating over K_0

$$W^i_{nn'n_c} = \frac{2\pi}{L_x} \frac{\alpha M_i^2}{2\sqrt{\varepsilon_{nn'}^2(n_c, P_x^2) + P_z^2}} =$$

$$= \frac{\alpha}{L_x} \frac{1}{\varepsilon_{nn'}^2(n_c, P_x^2) + P_z^2} \Theta_i^{(nn'n_c)}(P_x^2, \varepsilon_{nn'})\,. \tag{4.24}$$

The newly-introduced quantities Θ_i are expressed by the same Equations (4.6)–(4.9), as Δ_i, but in the corresponding quantities $\Delta I_{1,2}$ (4.12) the factor $16eB\,\delta(\varepsilon_{nn'}^2 - K_0^2 + K_\parallel^2)$ is omitted and $K_\parallel^2$, $K_\perp^2$ are everywhere replaced by P_z^2, P_x^2, respectively, and $\varepsilon_{nn'}^2$ is substituted for $K_0^2 - K_\parallel^2$. The value of $P_x^2 = K_\perp^2$ is understood as that obtained by solving Equation (2.5). Equation (4.24) involves the normalization length L_x. This is due to the fact that the motion along the x-axis remains non-quantized and for each of the two particles, electron and positron, is described by the wave function $L^{-1/2}\exp(ipx)$. In the 'positronium' state, whose annihilation we are considering, the linear density $n = N/L_x$, where $N = (N_e N_p)^{1/2}$ is the geometrical average of the electron and positron numbers, can be given arbitrarily, the same as the volume density in annihilation of free pairs without a magnetic field or the surface density in annihilation of mutually unbound pairs on Landau levels in a magnetic field (Klepikov, 1954). Hence, the positronium lifetime under annihilation is given by the equation $\tau = W^{-1}\sqrt{N_e N_p} = W^{-1} L_x n$ from where the length L_x has been cancelled.

5. The Poles of the Green's Function for the Photon

The 'probabilities' of positronium creation by a photon and one-photon positronium annihilation calculated in the previous section refer to the lowest-order perturbation theory with respect to the interaction between a photon and a positronium. The

corresponding expressions for the probabilities do not however correspond to their direct purpose since if the spectral curves of photon and positronium quasi-intersect*, the positronium state gives, as we have seen, an infinite pole contribution into the polarization operator, which means a very strong interaction between the photon and positronium dispersion curves. (This interaction is very small in a more traditional case (Tarasov, 1957) where there is no external field and, thus, no quasi-intersection.) In accord with the known general theorem (see, e.g., Nikitin and Umansky, 1979) dispersion curves of quasi-intersecting states should repulse from one another and reconnect, i.e., mixed (or joint) states are formed which are stationary to the extent the original positronium and photon were so. In this case the 'probabilities' of the previous section characterize only the fractions of the initial unperturbed photon and positronium states in the exact mixed photon–positronium state.

For obtaining the spectral curves of the mixed photon–positronium state in the neighbourhood of quasi-intersections it is sufficient to use the contribution of the individual pole term (4.13) into $\varkappa_{1,2,3}$ in the right-hand side of the photon dispersion equation, which takes into account the vacuum polarization in the magnetic field (Batalin and Shabad, 1971; Shabad, 1972, 1975), due to creation of positronium: i.e.

$$K_{\parallel}^2 - K_0^2 + K_{\perp}^2 = \varkappa_i , \qquad i = 1, 2, 3 . \tag{5.1}$$

Solution of the dispersion equation (5.1) for each i supplies a pole to one of the three terms in the photon Green function (Shabad, 1972, 1975)

$$D_{\mu\nu}(K) = \sum_{i=1}^{3} \frac{a_\mu^{(i)}(K) a_\nu^{(i)}(K)}{K^2 - \varkappa_i} , \quad \mu, \nu = 0, 1, 2, 3 , \tag{5.2}$$

which corresponds to the ith mode. The quantities $a_\mu^{(i)}(K)$ here are three orthonormalized eigenvectors of the polarization operator which describe the vector-potential of the corresponding mode found in (Batalin and Shabad, 1971) in an approximation-independent way (see Shabad, 1975; Shabad and Usov, 1984; for detailed description of the eigenmodes).

For each set of discrete quantum numbers n, n', n_c, Equation (5.1) is quadratic with respect to the variable $K_0^2 - K_{\perp}^2$ and can be easily solved. For instance, using (4.18) as the right-hand side of (5.1), we obtain two dispersion curves

$$K_0^2 - K_{\parallel}^2 = \tfrac{1}{2}\{\varepsilon_{00}^2(0, K_{\perp}^2) + K_{\perp}^2 \pm \sqrt{(\varepsilon_{00}^2(0, K_{\perp}^2) - K_{\perp}^2)^2 + 4A(K_{\perp}^2)}\} , \tag{5.3}$$

where

$$A(K_{\perp}^2) = 4\alpha eB\varepsilon_{00}(0, K_{\perp}^2) \, |\varphi_{000}(0)|^2 \exp\left(-\frac{K_{\perp}^2}{2eB}\right), \tag{5.4}$$

separated from one another by a gap with its width in the narrowest part (i.e. approximately at $K_{\perp}^2 = \varepsilon_{00}^2(0, K_{\perp}^2) \sim 4m^2$) of about

$$\Delta(K_0^2 - K_{\parallel}^2) \simeq 2\sqrt{A(4m^2)} . \tag{5.5}$$

* Quasi-intersection is an intersection of levels with identical quantum numbers before the interaction between them is taken into account.

As $K_\perp^2$ increases, one of the branches (the upper sign in (5.3)) asymptotically approaches with an exponential speed the light cone $K_0^2 - K_\parallel^2 = K_\perp^2$ (we disregard for a moment the influence of the neighbouring positronium states) and becomes a purely photon state, whereas at $K_\perp^2 = 0$ it comes to a point close to the positronium energy at $K_\perp^2 = 0$, that is, becomes close to a purely positronium state. On the contrary, the other branch (the lower sign in (5.3)) approaches the positronium spectral curve at an exponential speed as $K_\perp^2$ increases, and becomes the spectrum of the positronium, whereas it approaches the light cone, as $K_\perp^2$ decreases – i.e., becomes the spectrum of the photon. (For the photon one must also have $K_0^2 - K_\parallel^2 = 0$ at $K_\perp^2 = 0$ from the gauge invariance. The deviation of the lower branch of Equation (5.3) from this point is an artefact which occurs because the pole approximation used for the polarization operator in the dispersion equation (5.1) is not precise at this point, far away from the pole. In Figure 2 this defect is removed graphically.)

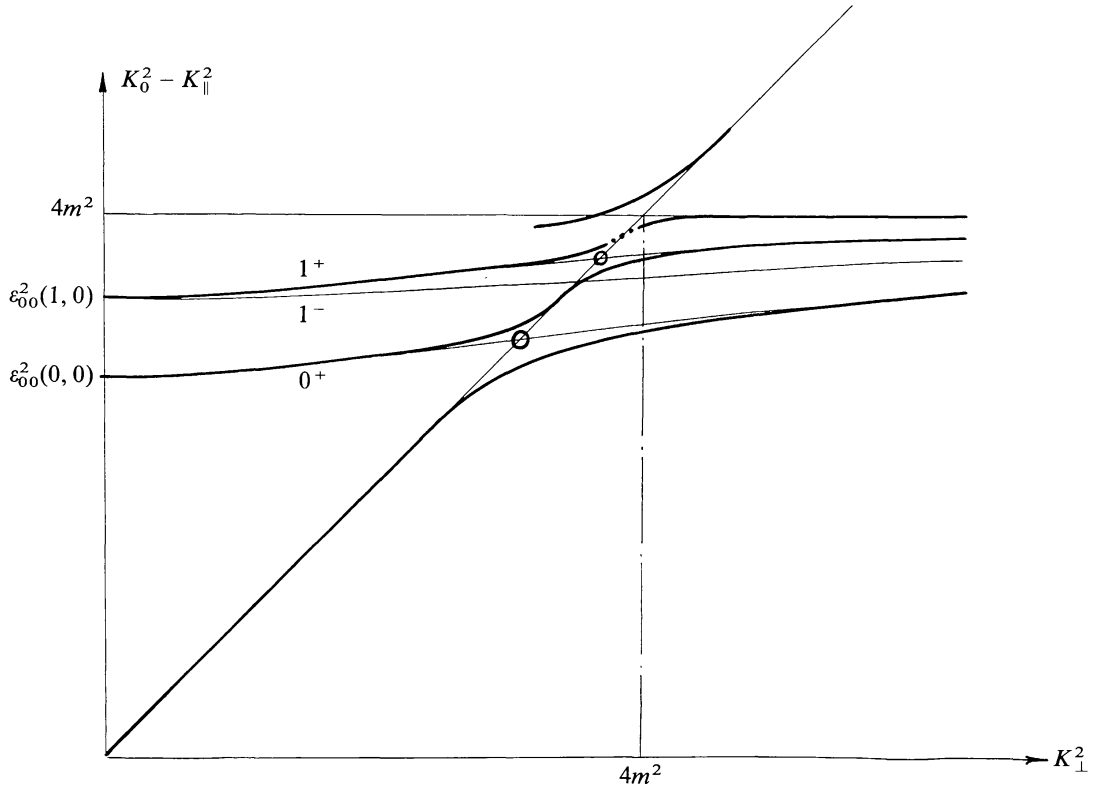

Fig. 2. Dispersion curves for the mode-2 photon mixed with two positronium states ($n_c = 0^+, 1^+$) of the lowest series ($n = n' = 0$) and with the continuum boundary of the same series of states. The dots indicate the smoothened dispersion curve in the region of quasi-continuum.

Other quasi-intersections are pushed apart in a similar way. The general picture is presented schematically in Figure 2 for quasi-interactions between the mode-2 photon and two positronium bound states with $n_c = 0, 1$ for which the first pair creation threshold $K_0^2 - K_\parallel^2 = 4m^2$ is the border of continuum. As already mentioned, the

positronium states, odd under the z-coordinate reversal, truly (not quasi-) intersect the photon curves. (They are not poles of the photon propagator.) This is seen in Figure 2 for the odd excited positronium spectral curve 1^-. When n_c grows, he bound states (3.29) condence towards the border of continuum. As the resolution power of any observation process is necessarily restricted, they make, for sufficiently large $n_c \geq n_0$, the so-called quasi-continuum. By approximately replacing the summation over n_c in the polarization operator by the integration over $x = 1/n_c$, $\mathrm{d}x = -(1/n_c^2)$ one can show that the quasi-continuum is represented by an additional fragment of a cut (in the analyticity domain of the polarization operator), located within the range (we are considering only $n = n' = 0$ for simplicity) $4m^2(1 - \alpha^2/8n_0^2)^2 < K_0^2 - K_\parallel^2 < 4m^2$, which prolongs the usual cut for whose appearance the true continuum in the region above the free pair creation threshold $K_0^2 - K_\parallel^2 > 4m^2$ is responsible. It is important that neither the lower edge (branching point) of the quasi-continuum cut at $4m^2(1 - \alpha^2/8n_0^2)^2$, nor the upper edge at $4m^2$ introduce singular behaviour into polarization operator. (The finiteness of the polarization operator at the upper edge is a consequence of the fast, $1/n_c^3$, decrease of the residues in the bound state poles (4.16) as $n_c \to \infty$.) Hence, the quasi-continuum does not affect any strongly the photon dispersion curves. Consequently, the above procedure for its introduction proposes in fact a way for smoothening out the infinite oscillations of the dispersion curve caused by the densely located bound states. The smoothened dispersion curve in the region of quasi-continuum is shown by the dots in Figure 2, taking $n_0 = 2$ for example. It is also shown in Figure 2 that the top dispersion curve of the series $n = n' = 0$ approaches asymptotically (also with an exponential speed) the boundary of the continuum (the threshold line) as $K_\perp^2$ increases, i.e., the photon becomes a pair which is not mutually bound, but at the same time has no relative motion along the z-axis. This phenomenon is a consequence of the inverse square root singular behaviour of the polarization operator near the threshold of the continuum. It was established in (Shabad, 1972), disregarding the Coulomb interaction between the electron and positron, then studied in more detail in Shabad (1975) or Shabad and Usov (1982, 1984); also in application to semi-conductor and electron–positron plasma optics in Korovina and Shabad (1975) or Pérez Rojas and Shabad (1982), respectively. In Shabad (1975) or Shabad and Usov (1982, 1984) one can find an analytical expression for the corresponding dispersion curve near the threshold, as well as for the real and imaginary parts of the dispersion curve for the photon above the threshold (the corresponding pole of the photon Green function lies on non-physical sheet of the complex plane of the variable $K_0^2 - K_\parallel^2$). The latter branch when moving away from the threshold, asymptotically approaches the light cone (Figure 2). The imaginary part of this curve is responsible for creation by a photon of an unbound electron–positron pair at Landau levels $n = n' = 0$. For fuller details see Shabad (1975) or Shabad and Usov (1982, 1984).

The Coulomb interaction between the electron and positron in the continuum (where it does not bind them into positronium) cannot significantly change these results, derived basically from the inverse square root singularity, since the latter is the phase volume effect, reflecting the character of the unbound motion of the pair in a magnetic field. Note

that as far as strong magnetic fields $B \gg \alpha^2 B_{cr}$ are concerned the influence of the first, $n_c = 1$ (and higher $n_c > 1$) excited states on the dispersion curves, although shown in Figure 2, should not be taken seriously because of the vanishing contribution (4.16). The contribution of $n_c = 0$ is, however, important.

The positronium states of the lowest series ($n = n' = 0$), including the boundary of its continuum, do not contribute into $\varkappa_{1,3}$. For this reason the spectral curves of photons of modes 1 and 3 truly intersect the positronium spectra of the lowest series (the latter are not generally poles of propagators for these modes).

For the rest of the series (where at least one of the integers n, n' is greater than zero) the picture of quasi-intersection between the positronium and photons of modes 2 and 3 is qualitatively the same as in Figure 2. An essential difference is, however, observed for mode 1. As is seen from (4.6), the discontinuity Δ_1 and the eigenvalue of the polarization operator $\varkappa_1$ has a factor K^2. Accordingly, the dispersion equation (5.1) for mode 1 always has the trivial solution $K^2 = 0$, which, does not, however, correspond to any electromagnetic wave due to the polarization properties inherent in this mode (see Shabad, 1975) four-dimensionally longitudinal wave of the vector-potential with $K^2 = 0$). If $K^2 \neq 0$, K^2 can be cancelled out of the dispersion equation for mode 1. Near each of the pole contributions into $\varkappa_1$ the solution of the dispersion equation thus obtained has the structure

$$1 = -\frac{C_{nn'n_c} \exp(-K_\perp^2/eB)}{\varepsilon_{nn'}^2(n_c, K_\perp^2) - K_0^2 + K_\parallel^2 + iO}, \tag{5.6}$$

where the positive values of $C_{nn'n_c}$ are found, as described above, from Equation (4.6) with (4.12) replaced by (4.13). From this it is seen that the dispersion curves (propagator poles) of mode 1 pass close along the initial dispersion curves of the even positronium states of every series except for the lowest. A massless excitation branch, which passes through the point $K_0 = K_\parallel = K_\perp = 0$, is thus absent from mode 1. On the whole we see that only two (2 and 3) of the three photon modes have massless branches. This is in accordance with the principle of invariability of the number of degrees of freedom: a photon has only two degrees of freedom. As concerns the massive branches, i.e., those for which $K_0 \neq 0$ for $K_\parallel = K_\perp = 0$, they occur in all the three modes. The values of the rest masses for them are close to the energies of positronium states which are at rest as a whole along the magnetic field and have zero phase velocity across the magnetic field, $P_x = 0$. It is worth noticing that no elementary excitations had been observed in the vacuum in mode 1 (Shabad, 1975) until the contribution of bound states was taken into account. We see, that this fact has proved to be approximation-dependent, and not a consequence of first principles.

The polarization properties of eigenmodes 1, 2, 3 are described in Shabad (1975) or Shabad and Usov (1984). Here we only mention for guidance that for transversal propagation, $\mathbf{K} \perp \mathbf{B}$, $K_\parallel = 0$ modes 2, 3 are transversely polarized, the electric field $\mathbf{e}_2$ of mode 2 is parallel with $\mathbf{B}$, $\mathbf{e}_2 \perp \mathbf{K}$, while the electric field of mode 3 is orthogonal to $\mathbf{B}$ and $\mathbf{K}$. For $K_\parallel = 0$, mode 1 is a purely electric longitudinal wave.

6. Effect of Photon Capture and Positronium Creation for Pulsar Physics

Consider the application of the obtained results concerning the photon-positronium dispersion curves to investigation of propagation and absorption of γ-quanta in a pulsar magnetosphere. The following formulation is adequate for the problem. There exists a strong magnetic field with curved lines of force, but with its strength almost constant. Quanta of electromagnetic radiation in the γ-range, which are generated via the curvature radiation mechanism by the charged particles (called primary) moving fast along the magnetic field are originally directed almost tangentially to the magnetic force lines. Moving onward, these γ-quanta change their momentum components along ($K_\parallel$) and across ($K_\perp$) the magnetic field, their frequence K_0 remaining constant. To the extent the geometrical optics is a good approximation, the photon shifts simultaneously upwards along the lower branch of the dispersion curve shown in Figure 2. Within this formulation this problem was investigated in our previous papers (cf. Shabad and Usov, 1982, 1984). In these papers, however, we disregarded the influence of electron–positron bound states upon the photon spectral curves, but took into account only that of the lower boundaries of their continua. This led us to the conclusion that the lower branch of the dispersion curve first went from the origin along the light cove $K_0^2 - K_\parallel^2 = K_\perp^2$ (the same as in Figure 2), but later it was bent and, as $K_\perp^2 \to \infty$, asymptotically approached the horizontal line $K_0^2 - K_\parallel^2 = 4m^2$. Simultaneously, the photon state transformed gradually and with certainty into the state of electron–positron pair at the boundary of the continuum, i.e., into one with no internal relative motion of the electron and positron along the magnetic field. This disproved the opinion, which was then conventional, that in the course of the curvature γ-quantum propagation along the magnetic field of pulsars, free electron–positron pairs may be created with any relative velocity along the magnetic field after the curve $K_0^2 - K_\parallel^2 = K_\perp^2$ 'intersects' the threshold $K_0^2 - K_\parallel^2 = 4m^2$. This creation became possible only beyond the scope of applicability of geometrical optics, that is, beyond the adiabaticity of the interaction between the electromagnetic radiation and the external field. For γ-quanta of the energy $K_0 \gg m$ the geometrical optics has been shown to be applicable for $B \gtrsim 0.1B_{cr} \simeq 4.10^{12}$ G. It has also been shown that the gradual transformation of a γ-quantum into an electron–positron pair at the continuum boundary is a capture of the γ-quantum by the magnetic field, since the group velocity of the mixed photon-pair state across the magnetic field, $\partial K_0/\partial K_\perp$, tends to zero with exponential velocity as $K_\perp^2$ increases. The trajectories of the packet centre in a magnetic field with the lines of force in the form of concentric circles have been obtained also analytically (Shabad and Usov, 1982, 1984).

The influence of positronium states enhances these effects because, first, the singularity of the polarization operator in the bound state is stronger than at the threshold of free pair creation and, second, because the coefficient $|\varphi(0)|^2$ in the residue of the ground state pole is large in accordance with the tendency to the falldown onto the centre, as mentioned in Section 4.

Relations (38), (45) from Shabad and Usov (1984) which govern the motion of the photon in the space and time, remain valid, with the function $K_\parallel^2(K_\perp^2)$ in them replaced

by $K_{\parallel}^2$ expressed in terms of $K_{\perp}^2$ from Equation (5.3). The applicability limit of the geometrical optics for the motion along the dispersion curve near the ground state of the lowest series is estimated in the same way as in Shabad and Usov (1984). According to (Shabad and Usov, 1984), the time τ during which the photon–positronium mixed state transforms from the photon-dominated state into the one where the positronium component prevails (during the same time the mixed state changes its propagation direction in the space) is given by

$$\tau \simeq \frac{K_0 r}{K_{\parallel}^2} \Delta K_{\perp} \simeq \frac{K_0 r \Delta(K_{\perp}^2)}{2K_{\parallel}^2 \varepsilon_{00}(0,0)} , \tag{6.1}$$

where r is distance of the photon from the curvature centre of field lines in the turning point. It is equal to the curvature radius in this point and may be assumed for pulsars to be about 10^7–10^8 cm. In the neighbourhood of the turning point we have $K_0^2 - K_{\parallel}^2 \simeq K_{\perp}^2 \simeq \varepsilon_{00}^2(0,0)$. In Equation (6.1) $\Delta(K_{\perp}^2) \simeq 2K_{\perp}\Delta K_{\perp}$ is the distance along the $K_{\perp}^2$ axis, measured from the point $K_{\perp}^2 = \varepsilon_{00}^2(0,0)$, such that the dispersion curve becomes indistinguishable from that of the positronium after $K_{\perp}^2 \gtrsim \varepsilon_{00}^2(0,0) + \Delta K_{\perp}^2$. It follows from (5.3) that $\Delta K_{\perp}^2$ is found from the condition

$$(\Delta K_{\perp}^2)^2 \simeq 16\alpha eB\varepsilon_{00}(0,0)\,|\varphi_{000}(0)|^2 \exp\left(-\frac{\varepsilon_{00}(0,0)}{2eB}\right)\exp\left(-\frac{\Delta K_{\perp}^2}{2eB}\right), \tag{6.2}$$

whence

$$\Delta K_{\perp}^2 \simeq 4(\alpha eB\varepsilon_{00}(0,0))^{1/2}\,|\varphi_{000}(0,0)|\exp\left(-\frac{\varepsilon_{00}(0,0)}{4eB}\right); \tag{6.3}$$

provided that (6.3) is much smaller than $2eB$.

For energetic γ-quanta, $K_0 \gg \varepsilon_{00}(0,0)$, we have for the time necessary for the turning to occur

$$\tau \simeq \frac{2r}{K_0}\left(\frac{\alpha eB}{\varepsilon_{00}(0,0)}\right)^{1/2}|\varphi_{000}(0)|\exp\left(-\frac{\varepsilon_{00}(0,0)}{4eB}\right). \tag{6.4}$$

Here, roughly, $K_{\perp}^2 \sim \varepsilon_{00}(0,0) \sim 4m^2$.

In order that the geometrical optics could be applied, the turning must occupy the time which includes many periods $\tau \gg 1/K_0$. This holds for sufficiently large fields B independently of the photon energy.

For small fields B a transition from one branch of the dispersion curve to another becomes possible, like is the case with adiabaticity violations in atomic collisions (see Nikitin and Umansky, 1979). In this case the capture of a photon by a magnetic field does not occur with certainty – probabilistic the transformation of a photon into positronium, creation of unbound pairs and the rectilinear photon propagation are possible. These phenomena can also be due to quasi-stationarity of the positronium and

the photon induced by the effects other than their interconversion. For the photon this is, first of all, its decay into two. For the ground state positronium this is two-photon annihilation, and for the excited state this is also the transition into a lower state with a photon emision. In line with Shabad and Usov (1984) one can say that the fact that the magnetic field is restricted in the space in the longitudinal direction also leads effectively to a non-stationarity of the positronium. All these effects can be empirically taken into account by introducing the Breit–Wigner broadening $i\Gamma(K)$ instead of iO into the right-hand side (4.18) of the dispersion equation (5.1), and also an imaginary correction to K_0^2 in the left-hand side of this equation. Then, by solving the dispersion equation (5.1) with real $K_\perp^2$ and complex $K_0^2 - K_\parallel^2$, one can make sure that as long as the broadenings introduced are small as compared with the gap (4.5), the picture described does not change any essentially. As the broadenings increase, the gap narrows, the dispersion curves for the real part of $K_0^2 - K_\parallel^2$ draw nearer until they ultimately cross at some point. Simultaneously, the curves for the imaginary parts of $K_0^2 - K_\parallel^2$ stop crossing and reconnect. In this case, when propagating in a curved magnetic field photons do not transform into a positronium, but move almost rectilinearly. Simultaneously, in the momentum space the photons move all the way close to the light cone.

Let us find the value of the magnetic field above which the broadening of the dispersion curves due to decay of the mixed photon-positronium state into γ-quanta can be neglected.

For kinematical reasons the state described by the lower dispersion curve of (5.3) cannot decay into two photons as belonging to polarization mode 2, according to Adler's (1971) selection rules which are not affected by the resonance situation (Usov and Shabad, 1983). The number of photons into which it can decay is larger than or equal to three. The positronium-dominated state described by the left part of the upper dispersion curve of (5.3) may undergo the 2γ-decay to the left of the crossing point of this dispersion curve with that of the mode 3 photon, which coincides, to a good accuracy, with the light cone in this place. The probability of a two-photon positronium annihilation in a strong magnetic field ($B \sim 10^{12} - 10^{13}$ G) is approximately equal to $W_{2\gamma} \simeq 8 \times 10^{12}\ (B/10^{12}\ \mathrm{G})\ \mathrm{s}^{-1}$ (Wunner and Herold, 1979). The broadening of the dispersion curves can be neglected if the energy gap between the $\pm$ branches in (5.3) (from now on the dependence on $\hbar$ and c is made explicit)

$$\Delta\varepsilon_\gamma^R = \alpha mc^2 \left(\frac{B}{B_{cr}}\right)^{1/2} \left[\ln \frac{B}{\alpha^2 B_{cr}\left(1 + \dfrac{4B_{cr}}{B}\right)}\right]^{1/2} \exp\left(-\frac{B_{cr}}{B}\right) \tag{6.5}$$

in the rest frame of the photon–positronium state $K_\parallel = 0$, essentially exceeds $\hbar W_{2\gamma}$. This limitation on B is more restrictive than the one induced by the geometrical optics applicability condition.

Using expressions for $\Delta\varepsilon_\gamma^R$ and $W_{2\gamma}$ we find the value of the magnetic field B_b, for which $\Delta\varepsilon_\gamma^R/\hbar W_{2\gamma} = 1$: $B_b \simeq 3.6 \times 10^{12}$ G. The dependence of the ratio $\Delta\varepsilon_\gamma^R/\hbar W_{2\gamma}$ on the

strength of the magnetic field B is fairly strong: it is about 10 for $B = 0.1B_{cr}$ and 10^3 for $B = 0.2B_{cr}$. For $B \gtrsim 0.1B_{cr}$ the conditions for applicability of the dispersion curve (5.3) can be regarded as fulfilled and, accordingly – with such magnetic fields – the main fraction of mode-2 γ-quanta of curvature radiation will be captured by turning into bound electron–positron pairs in the ground state ($n = n' = n_c = 0$). The energy gap between the dispersion curves for the a mode-3 photon mixed with the ground state positronium of the second series $n_c = n = 0, n' = 1$ or $n' = n_c = 0, n = 1$ differs but little from (6.5). That is why the condition of predominant creation of bound pairs by mode-3 γ-quanta is similar to the one obtained above for mode 2, viz.$B \gtrsim 0.1B_{cr}$.

Some applications of the γ-quantum capture by a strong magnetic field through adiabatic transformation into a positronium atom were discussed in Shabad and Usov (1985, 1986). The further details of astrophysical applications will be considered in a forthcoming publication.

References

Adler, S. L.: 1971, *Ann. Phys. N.Y.* **67**, 599.

Agranovich, V. M. and Ginzburg, V. L.: 1979, *Kristallooptika s uchetom prostranstvennoy dispersii i teoria eksitonov* (*Optics of Crystalls with the Spatial Dispersion, and Exciton Theory*), Nauka, Moscow.

Baier, V. N., Katkov, V. M., and Strakhovenko, V. M.: 1975, *Zh. Eksperim. Theor. Fiz.* **68**, 405.

Bakshi, P., Cover, R. A., and Kalman, G.: 1976, *Phys. Rev.* **D14**, 2532.

Bakshi, P., Cover, R. A., and Kalman, G.: 1979, *Phys. Rev.* **D20**, 3015.

Batalin, I. A. and Shabad, A. E.: 1971, *JETP* **33**, 483.

Carr, S. and Sutherland, P.: 1978, *Astrophys. Space Sci.* **58**, 83.

Davydov, A. S.: 1976, *Teoria tverdogo tela* (*Solid Body Theory*), Nauka, Moscow.

Herold, H., Ruder, H., and Wunner, G.: 1985, *Phys. Rev. Letters* **54**, 1452.

Karplus, R. and Klein, A.: 1952, *Phys. Rev.* **87**, 848.

Klepikov, N. P.: 1954, *Zh. Eksperim. Teor. Fiz.* **26**, 19.

Korovina, L. I. and Shabad, A. E.: 1975, *JETP* **40**, 512.

Leinson, L. I. and Oraevsky, V. N.: 1984, *Izmiran*, No. 14 (488) (preprint).

Leinson, L. I. and Oraevsky, V. N.: 1985, *Yad. Fiz.* **42**, 401.

Loudon, R.: 1959, *Am. J. Phys.* **27**, 648.

Melrose, D. B.: 1981, *Australian J. Phys.* **34**, 563.

Melrose, D. B.: 1982, *Australian J. Phys.* **35**, 67.

Melrose, D. B.: 1983, *Australian J. Phys.* **36**, 775.

Melrose, D. B. and Stoneham, R. J.: 1976, *Nuovo Cimento* **A32**, 435.

Melrose, D. B. and Stoneham, R. J.: 1977, *J. Phys.* **A10**, 1211.

Nikitin, E. E. and Umansky, S. Y.: 1979, *Neadiabaticheskie perekhody pri medlennykh atomnykh stolknoveniakh* (*Nonadiabatic Transitions under Slow Atomic Collisions*), Atomizdat, Moscow.

Pérez Rojas, H. and Shabad, A. E.: 1982, *Ann. Phys. N.Y.* **138**, 1.

Shabad, A. E.: 1972, *Nuovo Cimento Letters* **3**, 457.

Shabad, A. E.: 1975, *Ann. Phys. N.Y.* **90**, 166.

Shabad, A. E. and Usov, V. V.: 1982, *Nature* **295**, 215.

Shabad, A. E. and Usov, V. V.: 1984, *Astrophys. Space Sci.* **102**, 327.

Shabad, A. E. and Usov, V. V.: 1985, *Astrophys. Space Sci.* **117**, 309.

Shabad, A. E. and Usov, V. V.: 1986, *Proceedings of Course and Workshop on 'Plasma Astrophysics'*, Sukhumi, May 19–28, 1986, 399.

Tarasov, Yu. A.: 1957, *Zh. Experim. Teor. Fiz.* **33**, 706.

Tsai, W. Y.: 1974, *Phys. Rev.* **D10**, 2699.

Tsai, W. Y. and Erber, T.: 1974, *Phys. Rev.* **D10**, 492.

Urrutia, L. F.: 1978, *Phys. Rev.* **D17**, 1977.
Usov, V. V. and Shabad, A. E.: 1983, *Pis'ma Astron. Zh.* **9**, 401.
Usov, V. V. and Shabad, A. E.: 1985, *JETP Letters* **42**, 17.
Wunner, G. and Herold, H.: 1979, *Astrophys. Space Sci.* **63**, 503.
Wunner, G., Ruder, H. and Herold, H.: 1981, *J. Phys. B. At. Mol. Phys.* **14**, 765.