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Quantum Electrodynamics in Strong Magnetic Fields. III* Electron–Photon Interactions

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Abstract

A version of QED is developed which allows one to treat electron–photon interactions in the magnetized vacuum exactly and which allows one to calculate the responses of a relativistic quantum electron gas and include these responses in QED. Gyromagnetic emission and related crossed processes, and Compton scattering and related processes are discussed in some detail. Existing results are corrected or generalized for nonrelativistic (quantum) gyroemission, one-photon pair creation, Compton scattering by electrons in the ground state and two-photon excitation to the first Landau level from the ground state. We also comment on maser action in one-photon pair annihilation.

1. Introduction

A full synthesis of quantum electrodynamics (QED) and the classical theory of plasmas requires that the responses of the medium (plasma + vacuum) be included in the photon properties and interactions in QED and that QED be used to calculate the responses of the medium. It was shown by Melrose (1974) how this synthesis could be achieved in the unmagnetized case. In the present paper we extend the synthesized theory to include the magnetized case. Such a synthesized theory is desirable even when the effects of a material medium are negligible. The magnetized vacuum is birefringent with a full hierarchy of nonlinear response tensors, and for many purposes it is convenient to treat the magnetized vacuum as though it were a material medium.

Our starting point in this paper is the S-matrix expansion with two terms in the interaction Hamiltonian (Section 2). One term is the usual Dirac interaction Hamiltonian modified by the re-interpretation of the wavefunctions as exact solutions of Dirac's equation in the presence of a static magnetic field. The other term is the nonlinear interaction Hamiltonian introduced in Part II (Melrose 1983, present issue p. 775). It is shown in Section 2 how one may develop a momentum space representation. It is important to formulate the theory in momentum space because it is only in momentum space that the responses of the medium may be included in a simple way. However, it is not possible to develop a momentum space representation directly, as familiar in QED, because the wavefunctions do not have momentum space representations in the usual sense. This is connected with the lack

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of translational invariance across the field or alternatively in the lack of conservation of perpendicular momentum. The way this difficulty is overcome is by using momentum space representations only of pairs of wavefunctions, specifically in terms of the vertex function evaluated in Part I (Melrose and Parle 1983, present issue p. 755). A result established in Part I implies that gauge and coordinate dependent quantities need appear only in connection with external electrons and positrons and these may be included in density of states factors.

The terms in the S-matrix expansion have diagrammatic interpretations in the usual sense, with m-photon vertices included as outlined in Part II. One point which has not been emphasized previously concerns the role of closed electron loop diagrams. Any closed electron loop is to be replaced by the appropriate m-photon vertex, and the m-photon vertex function is to be calculated for a vacuum from the amplitude for the closed loop diagram. This point is discussed in Section 3 where it is shown that the response of an electron gas may be included by replacing the electron propagator by a statistically averaged propagator, as first done in this context by Tsytovich (1961). In Section 4 our synthesized theory is completed by formulating rules for writing down the S-matrix amplitude corresponding to specific diagrams and for evaluating probabilities from the amplitudes.

In Section 5 gyromagnetic emission and crossed processes related to it are discussed. In particular it is shown how emission by positrons and electrons is related in the nonrelativistic quantum limit, and how the exact result may be used to reproduce a known result for one-photon pair creation in the ultra-relativistic limit. In Section 6 Compton scattering and related processes are discussed. In particular, two-photon absorption in which the electron is initially in its ground state and jumps to its first excited level is evaluated explicitly; this process is of interest in connection with hard X-ray lines from neutron stars (Melrose and Parle 1981).

2. S-matrix Expansion

The S-matrix expansion, when the interaction Hamiltonian consists of the single-particle term

$$\hat{\mathscr{H}}_{\mathrm{I}}(x) = -e\colon \hat{\bar{\psi}}(x)\, A_\mu(x)\gamma^\mu\, \hat{\psi}(x)\colon \tag{1}$$

and the nonlinear term $\hat{\mathscr{H}}_{\mathrm{NL}}(x)$ introduced in Part II, reduces to

$$\begin{aligned} S = \sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{(-\mathrm{i})^n}{r!(n-r)!}\int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_r\, \mathrm{d}^4x_{r+1} \ldots \mathrm{d}^4x_n \\ \times\, T\{\hat{\mathscr{H}}_{\mathrm{I}}(x_1)\ldots\hat{\mathscr{H}}_{\mathrm{I}}(x_r)\,\hat{\mathscr{H}}_{\mathrm{NL}}(x_{r+1})\ldots\hat{\mathscr{H}}_{\mathrm{NL}}(x_n)\}\,. \end{aligned} \tag{2}$$

It is shown in Part II that a product of terms involving $\hat{\mathscr{H}}_{\mathrm{NL}}(x)$ may be reexpanded:

$$\begin{aligned} \sum_{n=1}^{\infty}(-\mathrm{i})^n\int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n\, T\{\hat{\mathscr{H}}_{\mathrm{NL}}(x_1)\ldots\hat{\mathscr{H}}_{\mathrm{NL}}(x_n)\} = \sum_{m=3}^{\infty}(-\mathrm{i})\int \mathrm{d}^4x\, \hat{\mathscr{H}}_{\mathrm{NL}}^{(m-2)}(x) \\ + \text{disconnected terms} + \text{multiply contracted terms}\,. \end{aligned} \tag{3}$$

Then in equation (2) a term $\hat{\mathscr{H}}_{\mathrm{NL}}^{(n)}(x)$, which involves an $(n+2)$-photon vertex, is of the same order as a product of n terms $\hat{\mathscr{H}}_{\mathrm{I}}$.

The expansion (2) when rearranged using (3) leads to connected terms of three kinds. Those which involve no term $\hat{\mathscr{H}}_{\mathrm{NL}}$ are directly analogous to the usual terms in QED. Now, however, many processes which are kinematically forbidden in vacuo are allowed. The simplest corresponds to an electron line connected to a photon line: this describes gyromagnetic emission on absorption. A second kind of term includes those which involve only $\hat{\mathscr{H}}_{\mathrm{NL}}$. These describe photon–photon interactions and are discussed in Part II. The third type of term describes 'nonlinear' modifications to the usual photon–electron and electron–electron interactions. The simplest example is a modification to Compton scattering. The scattering particle gives rise to vacuum fluctuations, described say by $A_{\mathrm{f}}^{\mu}(k'')$. The incoming ($k$) and outgoing photons ($k'$) interact with this fluctuating field through the nonlinear $\alpha^{\mu\nu\rho}(k,k',k'')$. The amplitude for this process is to be added to those for the familiar diagrams contributing to Compton scattering (Stoneham 1980*a*).

(*a*) *Momentum Space*

As pointed out in the Introduction we cannot proceed directly to momentum space because only pairs and not individual wavefunctions have momentum space representations in the usual sense. Before discussing this we write down the momentum space representation of the 4-potential, the photon propagator and $\hat{\mathscr{H}}_{\mathrm{NL}}^{(n)}$. These are given in Part II [equations (II.35), (II.41) and (II.42)]:

$$\hat{A}^{\mu}(k) = \sum_{M} a_M(\boldsymbol{k})\{e_M^{\mu}(\boldsymbol{k})\,\hat{c}_M(\boldsymbol{k})(2\pi)^4\delta^4(k-k_M) + e_M^{*\mu}(\boldsymbol{k})\,\hat{c}_M^{\dagger}(\boldsymbol{k})(2\pi)^4\delta^4(k+k_M)\}\,. \tag{4}$$

$$\underbrace{\hat{A}^{\mu}(k)\,\hat{A}^{\nu}(k')} = -\mathrm{i}(2\pi)^4\delta^4(k+k')\,D^{\mu\nu}(k)\,, \tag{5}$$

$$-\mathrm{i}\int \mathrm{d}^4x\;\hat{\mathscr{H}}_{\mathrm{NL}}^{(n)}(x) = -\frac{\mathrm{i}}{n+2}\int\frac{\mathrm{d}^4k_0}{(2\pi)^4}\cdots\frac{\mathrm{d}^4k_{n+1}}{(2\pi)^4} \times\tilde{\alpha}^{\nu_0\ldots\nu_{n+1}}(k_0,\ldots,k_{n+1})\,\hat{A}_{\nu_0}(k_0)\ldots\hat{A}_{\nu_{n+1}}(k_{n+1})\,. \tag{6}$$

(*b*) *Electron Line with One Vertex*

Electron terms appear either as external lines or closed loops, with the external lines being an electron line from the initial to the final state, a positron line conventionally from the final to the initial state or a pair in the initial or final state. An external line with one vertex arises from the term $:\hat{\bar{\psi}}(x)\,\gamma^{\mu}\,\hat{\psi}(x):$. The definition (I.46) in Part I of the vertex function with (I.49) or (I.52) allows us to write

$$[\gamma_{q'q}^{\varepsilon'\varepsilon}(\boldsymbol{k})]^{\mu} \equiv \frac{1}{V}\int \mathrm{d}\boldsymbol{x}\,\exp(-\mathrm{i}\,\boldsymbol{k}\,.\,\boldsymbol{x})\,\bar{\psi}_{q'}^{\varepsilon'}(\boldsymbol{x})\,\gamma^{\mu}\,\psi_q^{\varepsilon}(\boldsymbol{x}) = d_{q'q}^{\varepsilon'\varepsilon}(\boldsymbol{k})[\Gamma_{q'q}^{\varepsilon'\varepsilon}(\boldsymbol{k})]^{\mu}\,, \tag{7}$$

where

$$d_{q'q}^{\varepsilon'\varepsilon}(\boldsymbol{k}) = \{1/V(eB)^{\frac{1}{2}}\}\exp\{\mathrm{i}\,k_x(\varepsilon p_y+\varepsilon' p_y')/2eB\} \times 2\pi\delta(\varepsilon p_y-\varepsilon' p_y'-k_y)\,2\pi\delta(\varepsilon p_z-\varepsilon' p_z'-k_z) \tag{8a}$$

$$= (2\pi/VeB)\{-\mathrm{i}\exp(-\mathrm{i}\,\psi)\}^{s-s'}J_{s'-s}^{s}(k_\perp^2/2eB)\,2\pi\delta(\varepsilon p_z-\varepsilon' p_z'-k_z) \tag{8b}$$

is different for cartesian (8a) and cylindrical (8b) coordinates. For conciseness of notation we write the second quantized wavefunction (cf. equations I.67) in the form

$$\hat{\psi}(x) = \sum_{q,\varepsilon} \hat{a}_q^{\varepsilon} \psi_q^{\varepsilon}(\boldsymbol{x}) \exp(-\mathrm{i}\,\varepsilon \mathscr{E}_q t), \tag{9a}$$

$$\hat{\bar{\psi}}(x) = \sum_{q,\varepsilon} \hat{\bar{a}}_q^{\varepsilon} \bar{\psi}_q^{\varepsilon}(\boldsymbol{x}) \exp(\mathrm{i}\,\varepsilon \mathscr{E}_q t), \tag{9b}$$

so that we have

$$\hat{a}_q^{+} = \hat{a}_q, \quad \hat{a}_q^{-} = \hat{b}_q^{\dagger}, \quad \hat{\bar{a}}_q^{+} = \hat{a}_q^{\dagger}, \quad \hat{\bar{a}}_q^{-} = \hat{b}_q, \tag{10}$$

where $\hat{a}_q$ and $\hat{b}_q$ are the electron and positron annihilation operators and $\hat{a}_q^{\dagger}$ and $\hat{b}_q^{\dagger}$ are the corresponding creation operators. Then from the definition of the normal product we have

$$:\hat{\bar{\psi}}(x)\gamma^{\mu}\hat{\psi}(x): = \sum_{\varepsilon'\varepsilon,q'q} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \exp(-\mathrm{i}\,kx)[\hat{G}_{q'q}^{\varepsilon'\varepsilon}(k)]^{\mu}, \tag{11}$$

with

$$[\hat{G}_{q'q}^{\varepsilon'\varepsilon}(k)]^{\mu} = :\hat{\bar{a}}_{q'}^{\varepsilon'} \hat{a}_q^{\varepsilon}: D_{q'q}^{\varepsilon'\varepsilon}(k)[\Gamma_{q'q}^{\varepsilon'\varepsilon}(\boldsymbol{k})]^{\mu}, \tag{12}$$

where

$$D_{q'q}^{\varepsilon'\varepsilon}(k) = V\, d_{q'q}^{\varepsilon'\varepsilon}(\boldsymbol{k})\, 2\pi\delta(\varepsilon\mathscr{E}_q - \varepsilon'\mathscr{E}_{q'} - \omega) \tag{13}$$

includes conservation of energy and parallel momentum. The normal product in (12) has the following form:

$$:\hat{\bar{a}}_{q'}^{+}\hat{a}_q^{+}: = \hat{a}_{q'}^{\dagger}\hat{a}_q \qquad \text{electron line}, \tag{14a}$$

$$:\hat{\bar{a}}_{q'}^{-}\hat{a}_q^{-}: = -\hat{b}_q^{\dagger}\hat{b}_{q'} \qquad \text{positron line}, \tag{14b}$$

$$:\hat{\bar{a}}_{q'}^{+}\hat{a}_q^{-}: = \hat{a}_{q'}^{\dagger}\hat{b}_q^{\dagger} \qquad \text{final pair}, \tag{14c}$$

$$:\hat{\bar{a}}_{q'}^{-}\hat{a}_q^{+}: = \hat{b}_{q'}\hat{a}_q \qquad \text{initial pair}. \tag{14d}$$

In the usual diagrammatic interpretation a solid line represents an electron or a positron with an arrow along the line from the initial state on the right towards the final state on the left for an electron, and in the opposite direction for a positron. The vertices corresponding to equations (14) are illustrated in Fig. 1. Note that for a positron line there is an additional minus sign in (14b) and the interpretation of q' and q as the initial and final states is opposite to that for an electron line.

The momentum space representation then reduces to

$$-\mathrm{i}\int \mathrm{d}^4x\, \hat{\mathscr{H}}_{\mathrm{I}}(x) = \sum_{q\varepsilon,q'\varepsilon'} \mathrm{i}\,e \int \frac{\mathrm{d}^4k}{(2\pi)^4} [\hat{G}_{q'q}^{\varepsilon'\varepsilon}(-k)]^{\mu} \hat{A}_{\mu}(k). \tag{15}$$

(c) *Electron–Positron Line with Several Vertices*

Now consider the S matrix element with two factors $\hat{\mathscr{H}}_{\mathrm{I}}$, and one contraction between electron operators. The contraction corresponds to a propagator $\mathrm{i}\,G(x_2, x_1)$

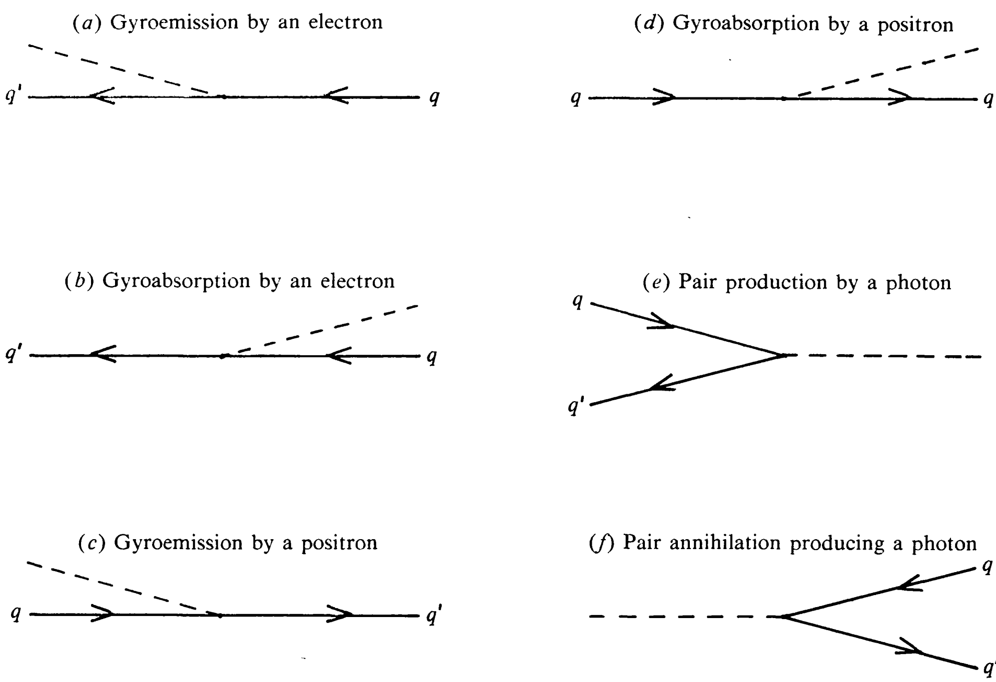

Fig. 1. Six lowest order processes in a magnetic field. The labelling of external states is the same as in equations (14). All these processes are related by crossing symmetries: for example, transition rates for (*b*) and (*c*) differ only by densities of the external states.

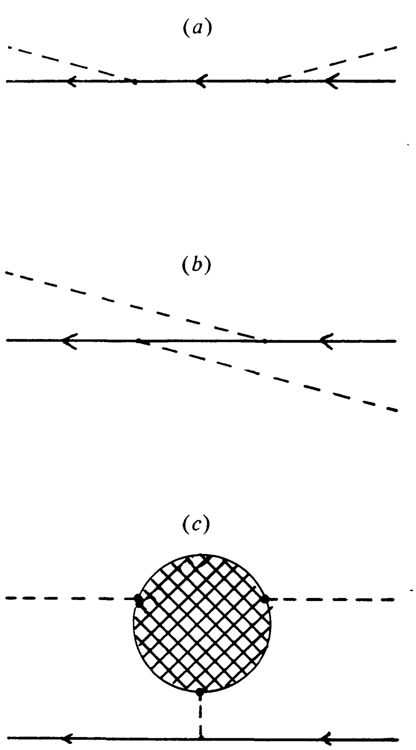

Fig. 2. The three diagrams that contribute to lowest order Compton scattering; these reflect the three terms in the transition probability given by equation (70). Diagrams (*a*) and (*b*) are the usual terms; (*c*) involves the quadratic response of the medium. As discussed in Section 4, the 3-photon vertex is of order 1, so that diagram (*c*) contributes to the same order as the first two diagrams.

between the two points in coordinate space. (Here we order points from right to left; diagrammatically x_1 and x_2 are connected by a solid line with the arrow directed from x_1 to x_2.) From (I.70) and (I.71) we have

$$\mathrm{i}\,G(x_2,x_1) = \mathrm{i}\sum_{q''\varepsilon''}\psi^{\varepsilon''}_{q''}(x_2)\,\bar{\psi}^{\varepsilon''}_{q''}(x_1)\int\frac{\mathrm{d}E}{2\pi}\,\frac{\exp\{-\mathrm{i}\,E(t_2-t_1)\}}{E-\varepsilon''(\mathscr{E}_{q''}-\mathrm{i}0)}. \tag{16}$$

The S matrix element then reduces to

$$(-\tfrac{1}{2}\mathrm{i})^2\int\mathrm{d}^4x_2\,\mathrm{d}^4x_1\,\hat{\mathscr{H}}_{\mathrm{I}}(x_2)\,\hat{\mathscr{H}}_{\mathrm{I}}(x_1)$$

$$= \sum_{q\varepsilon,q'\varepsilon'}(\tfrac{1}{2}\mathrm{i}\,e)^2\int\frac{\mathrm{d}^4k_2}{(2\pi)^4}\,\frac{\mathrm{d}^4k_1}{(2\pi)^4}[\hat{G}^{\varepsilon'\varepsilon}_{q'q}(-k_2,-k_1)]^{\nu\mu}\hat{A}_\nu(k_2)\,\hat{A}_\mu(k_1), \tag{17}$$

with

$$[\hat{G}^{\varepsilon'\varepsilon}_{q'q}(k_2,k_1)]^{\nu\mu} = :\hat{a}^{\varepsilon'}_{q'}\,\hat{a}^{\varepsilon}_{q}:D^{\varepsilon'\varepsilon}_{q'q}(k_2,k_1)\sum_{Q''}\frac{\mathrm{i}[\Gamma^{\varepsilon'\varepsilon''}_{q'q''}(\boldsymbol{k}_2)]^\nu[\Gamma^{\varepsilon''\varepsilon}_{q''q}(\boldsymbol{k}_1)]^\mu}{\varepsilon\mathscr{E}_q-\omega_1-\varepsilon''(\mathscr{E}_{q''}-\mathrm{i}0)}, \tag{18}$$

where Q'' denotes n'', σ'' and ε'' collectively. In deriving (18) we use properties established in Section 5*b* of Part I: specifically the sum over the intermediate state is performed over p''_z and either p''_y or s'' as indicated there. The sum over p''_z implies conservation of parallel momentum at the vertex with

$$\varepsilon''p''_z = \varepsilon p_z - k_{1z} = \varepsilon' p'_z + k_{2z}. \tag{19}$$

The quantity $D^{\varepsilon'\varepsilon}_{q'q}(k_2,k_1)$ is defined for the case $n=2$ by (24) below. The diagrammatic representations of the terms in (18) are illustrated in Fig. 2.

The result (17) with (18) generalizes to n factors of $\hat{\mathscr{H}}_{\mathrm{I}}$, with $n-1$ contractions. We have

$$\frac{(-\mathrm{i})^n}{n!}\int\mathrm{d}^4x_n\ldots\mathrm{d}^4x_1\,\hat{\mathscr{H}}_{\mathrm{I}}(x_n)\ldots\hat{\mathscr{H}}_{\mathrm{I}}(x_1) = \sum_{q\varepsilon,q'\varepsilon'}\frac{(\mathrm{i}\,e)^n}{n!}$$

$$\times\int\frac{\mathrm{d}^4k_n}{(2\pi)^4}\cdots\frac{\mathrm{d}^4k_1}{(2\pi)^4}[\hat{G}^{\varepsilon'\varepsilon}_{q'q}(-k_n,\ldots,-k_1)]^{\nu_n\ldots\nu_1}\hat{A}_{\nu_n}(k_n)\ldots\hat{A}_{\nu_1}(k_1), \tag{20}$$

with

$$[\hat{G}^{\varepsilon'\varepsilon}_{q'q}(k_n,\ldots,k_1)]^{\nu_n\ldots\nu_1} = :\hat{a}^{\varepsilon'}_{q'}\,\hat{a}^{\varepsilon}_{q}:D^{\varepsilon'\varepsilon}_{q'q}(k_n,\ldots,k_1)\sum_{Q_1\ldots Q_{n-1}}$$

$$\times\frac{\mathrm{i}^{n-1}[\Gamma^{\varepsilon'\varepsilon_{n-1}}_{q'q_{n-1}}(\boldsymbol{k}_n)]^{\nu_n}\ldots[\Gamma^{\varepsilon_r\varepsilon_{r-1}}_{q_rq_{r-1}}(\boldsymbol{k}_r)]^{\nu_r}\ldots[\Gamma^{\varepsilon_1\varepsilon}_{q_1q}(\boldsymbol{k}_1)]^{\nu_1}}{\{E_{n-1}-\varepsilon_{n-1}(\mathscr{E}_{q_{n-1}}-\mathrm{i}0)\}\ldots\{E_{r-1}-\varepsilon_{r-1}(\mathscr{E}_{q_{r-1}}-\mathrm{i}0)\}\ldots\{E_1-\varepsilon_1(\mathscr{E}_{q_1}-\mathrm{i}0)\}}, \tag{21}$$

with E_i and p_{zi} with $i=1$ to $n-1$ determined by

$$E_1 = \varepsilon\mathscr{E}_q-\omega_1, \qquad E_{r-1}-E_r = \omega_r, \qquad E_{n-1} = \varepsilon'\mathscr{E}_{q'}+\omega_n, \tag{22}$$

$$\varepsilon_r p_{zr} = \varepsilon p_z-(\boldsymbol{k}_1+\ldots+\boldsymbol{k}_r)_z = \varepsilon' p'_z+(\boldsymbol{k}_{r+1}+\ldots+\boldsymbol{k}_n)_z. \tag{23}$$

The quantity $D^{\varepsilon'\varepsilon}_{q'q}(k_n,\ldots,k_1)$ includes the gauge dependent factors $d^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k}_r)$ (cf. equations 8) for each vertex and the energy δ functions. The product of these is summed or integrated over the energies, parallel momenta and quantum numbers p_y or s for the intermediate states. The result is the gauge dependent factor

$$D^{\varepsilon'\varepsilon}_{q'q}(k_n, \ldots, k_1) = 2\pi\delta\Big(\varepsilon\mathscr{E}_q - \varepsilon'\mathscr{E}_{q'} - \sum_{i=1}^{n} \omega_i\Big)\frac{2\pi}{L_z}\delta\Big(\varepsilon p_z - \varepsilon' p'_z - \sum_{i=1}^{n} k_{iz}\Big)$$

$$\times \exp\Big(-\frac{\mathrm{i}}{2eB}\sum_{j<i}(\boldsymbol{k}_i \times \boldsymbol{k}_j)_z\Big)$$

$$\times \begin{cases} \dfrac{L_y L_z}{V(eB)^{\frac{1}{2}}}\dfrac{2\pi}{L_y}\delta\Big(\varepsilon p_y - \varepsilon' p'_y - \sum_{i=1}^{n} k_{iy}\Big)\exp\Big(\dfrac{\mathrm{i}}{2eB}\sum_{i=1}^{n} k_{ix}(\varepsilon p_y + \varepsilon' p'_y)\Big) & \text{(24a)} \\ (2\pi L_z/VeB)\{-\mathrm{i}\exp(-\mathrm{i}\,\Psi)\}^{s-s'} \mathrm{J}^{s}_{s'-s}(X), & \text{(24b)} \end{cases}$$

with

$$X \equiv K_\perp^2/2eB, \qquad \sum_{i=1}^{n}(k_{ix} + \mathrm{i}\,k_{iy}) \equiv K_\perp \exp(\mathrm{i}\,\Psi). \tag{25}$$

3. Response Tensors for the Vacuum and for Magnetized Plasmas

The response tensors for the vacuum are closely related to the amplitude for closed electron loop diagrams. After renormalization the amplitude for the two-sided (bubble), three-sided ('triangle') and four-sided ('box') loop diagrams give the linear, quadratic and cubic response tensors respectively. When an electron gas is present, its responses may be included in the amplitude for the loop diagrams in a way suggested by Tsytovich (1961). As explained in the Introduction, one relates the response to the forward scattering amplitude and one calculates this amplitude by introducing a statistically averaged propagator.

(a) *Statistically Averaged Electron Propagator*

Let $G(x', x)$ denote the electron propagator in vacuo. Its evaluation as a vacuum expectation value follows from

$$G(x', x) = \mathrm{i}\,\mathrm{Tr}[\rho_0\, T\{\hat{\psi}(x')\,\hat{\bar{\psi}}(x)\}], \tag{26}$$

where $\rho_0 \equiv |0\rangle\langle 0|$ is a density matrix corresponding to the vacuum and 'Tr' denotes the trace. An arbitrary electron gas may be described in terms of a density matrix ρ. In practice the density matrix is diagonal in the states of interest, i.e. we have $\rho_{qq'} = 0$ for $q' \neq q$, and although this need not be the case in principle we assume it to be the case here. The statistically averaged propagator is defined by replacing ρ_0 in (26) by ρ:

$$\bar{G}(x', x) = -\mathrm{i}\,\mathrm{Tr}[\rho\, T\{\hat{\psi}(x')\,\hat{\bar{\psi}}(x)\}]. \tag{27}$$

After inserting equations (9) with (10) in (27) the traces are evaluated using

$$\mathrm{Tr}[\rho\hat{a}^\dagger_q\,\hat{a}_{q'}] = n^+_q\,\delta_{qq'}, \qquad \mathrm{Tr}[\rho\hat{a}_q\,\hat{a}^\dagger_{q'}] = (1-n^+_q)\delta_{qq'}, \tag{28a, b}$$

$$\mathrm{Tr}[\rho\hat{b}^\dagger_q\,\hat{b}_{q'}] = n^-_q\,\delta_{qq'}, \qquad \mathrm{Tr}[\rho\hat{b}_q\,\hat{b}^\dagger_{q'}] = (1-n^-_q)\delta_{qq'}, \tag{28c, d}$$

where n_q^ε denotes the occupation numbers of electrons ($\varepsilon = 1$) and positrons ($\varepsilon = -1$). The propagator in the form (16) is then modified to

$$\mathrm{i}\,G(x_2,x_1) = \mathrm{i}\sum_{q''\varepsilon''} \psi_{q''}^{\varepsilon''}(x_2)\,\bar{\psi}_{q''}^{\varepsilon''}(x_1)\int\frac{\mathrm{d}E}{2\pi}\exp\{-\mathrm{i}\,E(t_2-t_1)\}$$

$$\times\left(\frac{1-n_{q''}^{\varepsilon''}}{E-\varepsilon''(\mathscr{E}_{q''}-\mathrm{i}0)}+\frac{n_{q''}^{\varepsilon''}}{E-\varepsilon''(\mathscr{E}_{q''}+\mathrm{i}0)}\right). \tag{29}$$

The principal value part of the integral is independent of the electron gas, and the semi-residue part includes a factor $1-2n_{q''}^{\varepsilon''}$ with the unit term arising from the vacuum.

(b) Closed Loop Diagrams

Firstly let us calculate the amplitude for loop diagrams ignoring the electron gas. An n-sided loop corresponds to an S matrix element with n factors of $\hat{\mathscr{H}}_\mathrm{I}$, and n contractions. We write

$$\frac{(-\mathrm{i})^n}{n!}\int \mathrm{d}^4x_n \dots \mathrm{d}^4x_1 :\hat{\bar{\psi}}(x_n)\,\hat{A}_\mu(x_n)\,\gamma^\mu\,\hat{\psi}(x_n)\,\hat{\bar{\psi}}(x_{n-1})$$

$$\dots\hat{\psi}(x_2)\,\hat{\bar{\psi}}(x_1)\,\hat{A}_\mu(x_1)\,\gamma^\mu\,\hat{\psi}(x_1):$$

$$=\frac{(\mathrm{i}\,e)^n}{n!}\int\frac{\mathrm{d}^4k_n}{(2\pi)^4}\dots\frac{\mathrm{d}^4k_1}{(2\pi)^4}\,L^{\nu_n\dots\nu_1}(-k_n,\dots,-k_1)\,\hat{A}_{\nu_n}(k_n)\dots\hat{A}_{\nu_1}(k_1). \tag{30}$$

The calculation of $L^{\nu_n\dots\nu_1}$ is similar to the calculation of $[\hat{G}]^{\nu_n\dots\nu_1}$ in (21), but with one important complication: the sum of the 4-momenta is zero, i.e. $\sum_{i=1}^n k_i = 0$, and one integral over an arbitrary loop 4-momentum remains. Using (7) one has a factor

$$d_{q_1q_n}^{\varepsilon_1\varepsilon_n}(k_n)\,d_{q_nq_{n-1}}^{\varepsilon_n\varepsilon_{n-1}}(k_{n-1})\dots d_{q_2q_1}^{\varepsilon_2\varepsilon_1}(k_1)\,,$$

which is to be summed or integrated over the intermediate states. There is no difficulty with the E and p_z integrations, and the remaining integral over p_y or sum over s leads to the following result: the integral over the undetermined loop momentum is equivalent to operation with

$$\frac{eB}{2\pi}\int\frac{\mathrm{d}E}{2\pi}\int\frac{\mathrm{d}p_z}{2\pi}\exp\left(-\frac{\mathrm{i}}{2eB}\sum_{j<i}(\boldsymbol{k}_i\times\boldsymbol{k}_j)_z\right)(2\pi)^4\delta^4\left(\sum_{i=1}^n k_i\right),$$

where E and p_z are the undetermined energy and parallel momentum around the loop. Hence we find

$$L^{\nu_n\dots\nu_1}(k_n,\dots,k_1) = \frac{eB}{2\pi}\int\frac{\mathrm{d}E}{2\pi}\int\frac{\mathrm{d}p_z}{2\pi}\sum_{Q_1\dots Q_n}(-\mathrm{i})^n\exp\left(-\frac{\mathrm{i}}{2eB}\sum_{j<i}(\boldsymbol{k}_i\times\boldsymbol{k}_j)_z\right)$$

$$\times\frac{[\Gamma_{q_1q_n}^{\varepsilon_1\varepsilon_n}(k_n)]^{\nu_n}\dots[\Gamma_{q_{r+1}q_r}^{\varepsilon_{r+1}\varepsilon_r}(k_r)]^{\nu_r}\dots[\Gamma_{q_2q_1}^{\varepsilon_2\varepsilon_1}(k_1)]^{\nu_1}}{\{E_n-\varepsilon_n(\mathscr{E}_{q_n}-\mathrm{i}0)\}\dots\{E_r-\varepsilon_r(\mathscr{E}_{q_r}-\mathrm{i}0)\}\dots\{E_1-\varepsilon_1(\mathscr{E}_{q_1}-\mathrm{i}0)\}}, \tag{31}$$

where Q_r denotes $\sigma_r, n_r, \varepsilon_r$ collectively. We are free to choose E and p_z arbitrarily and a possible choice is $E = E_1$ and $p_z = p_{1z}$. Conservation of energy and momentum are implicit in (31) with

$$E_r = E_{r-1} - \omega_{r-1}, \qquad \varepsilon_{r+1}\, p_{(r+1)z} = \varepsilon_r\, p_{rz} - k_{rz}. \tag{32a, b}$$

The contributions to the energy integral are of three kinds, those involving only principal values and no semi-residues, those involving $n-1$ principal values and one semi-residue, and those involving more than one semi-residue. The contribution of the first kind gives zero, and contributions of the third kind are nonphysical due to our use of the Feynman propagator which is acausal. The physical terms arise from the n terms in which the semi-residue at each of the denominators is taken sequentially. The causal condition then needs to be imposed. This may be done in the usual way simply by adding i0 to each frequency in the denominator in the resulting expression.

The inclusion of the electron gas at this stage is straightforward. Consider the term arising from the semi-residue in the rth denominator. It is proportional to $-\mathrm{i}\pi\varepsilon_r\,\delta(E_r - \varepsilon_r\,\mathscr{E}_{qr})$, and as explained following equation (29), on performing the statistical average over an electron gas the net effect is to multiply this term by $1-2n_{qr}^{\varepsilon_r}$. The unit term then gives the vacuum contribution and the term $-2n_{qr}^{\varepsilon_r}$ gives the contribution of the electron gas.

(c) *Response Tensors*

The response tensors may now be identified by noting that the terms in the nth order nonlinear response (6) must arise from the $(n+1)$-sided loop diagrams as in (30). There are $n+1$ terms of the form (30) arising from permutations of the subscripts 0 to n. Thus we identify

$$-\{\mathrm{i}/(n+1)\}\alpha^{\nu_0\nu_1\dots\nu_n}(k_0, k_1, \dots, k_n)$$
$$= \frac{(\mathrm{i}\,e)^{n+1}}{(n+1)!}\sum_P P^{1\dots n}_{1'\dots n'}\, L^{\nu_{n'}\dots\nu_{1'}\nu_{0'}}(-k_{n'}, \dots, -k_{1'}, -k_{0'}). \tag{33}$$

The linear response tensor follows from (31) and (30). In this case we have

$$\alpha^{\mu\nu}(k) = -\frac{e^3B}{2\pi}\sum_{Q,Q'}\int\frac{\mathrm{d}p_z}{2\pi}\,\frac{\{\frac{1}{2}(\varepsilon'-\varepsilon)+\varepsilon n_q^{\varepsilon}-\varepsilon' n_{q'}^{\varepsilon'}\}}{\omega-\varepsilon\mathscr{E}_q+\varepsilon'\mathscr{E}_{q'}+\mathrm{i}0}$$
$$\times\,[\Gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^{\mu}[\Gamma^{\varepsilon\varepsilon'}_{qq'}(\boldsymbol{k})]^{*\nu}, \tag{34}$$

with $\varepsilon' p_z' = \varepsilon p_z - k_z$ now implicit. In the final factor in (31) we have appealed to the symmetry property (I.60). The term $\frac{1}{2}(\varepsilon'-\varepsilon)$ leads to the vacuum polarization tensor after renormalization (Melrose and Stoneham 1977).

The quadratic response tensor is given by

$$\alpha^{\mu\nu\rho}(k_1, k_2, k_3) = \alpha_1^{\mu\nu\rho}(k_1, k_2, k_3) + \alpha_1^{\mu\rho\nu}(k_1, k_3, k_2), \tag{35}$$

with

$$\alpha_1^{\mu\nu\rho}(k_1,k_2,k_3) = -\frac{e^4B}{4\pi}\int\frac{\mathrm{d}p_z}{2\pi}\exp\Big(-\frac{\mathrm{i}}{2eB}\sum_{i>j}(\boldsymbol{k}_i\times\boldsymbol{k}_j)_z\Big)\sum_{Q_1Q_2Q_3}$$

$$\times\bigg(\frac{\frac{1}{2}\varepsilon_1(1-n_{q1}^{\varepsilon_1})}{(\omega_1+\varepsilon_1\mathscr{E}_{q1}-\varepsilon_3\mathscr{E}_{q3})(\omega_2+\varepsilon_2\mathscr{E}_{q2}-\varepsilon_1\mathscr{E}_{q1})}$$

$$+\frac{\frac{1}{2}\varepsilon_2(1-2n_{q2}^{\varepsilon_2})}{(\omega_2+\varepsilon_2\mathscr{E}_{q2}-\varepsilon_1\mathscr{E}_{q1})(\omega_3+\varepsilon_3\mathscr{E}_{q3}-\varepsilon_2\mathscr{E}_{q2})}$$

$$+\frac{\frac{1}{2}\varepsilon_3(1-2n_{q3}^{\varepsilon_3})}{(\omega_3+\varepsilon_3\mathscr{E}_{q3}-\varepsilon_2\mathscr{E}_{q2})(\omega_1+\varepsilon_1\mathscr{E}_{q1}-\varepsilon_3\mathscr{E}_{q3})}\bigg)$$

$$\times[\Gamma_{q3q1}^{\varepsilon_3\varepsilon_1}(-\boldsymbol{k}_1)]^\mu[\Gamma_{q1q2}^{\varepsilon_1\varepsilon_2}(-\boldsymbol{k}_2)]^\nu[\Gamma_{q2q3}^{\varepsilon_2\varepsilon_3}(-\boldsymbol{k}_3)]^\rho\,. \tag{36}$$

The cubic response tensor is given by

$$\begin{aligned}\alpha^{\mu\nu\rho\tau}(k_1,k_2,k_3,k_4) = {} & \alpha_1^{\mu\nu\rho\tau}(k_1,k_2,k_3,k_4)\\ & +\alpha_1^{\mu\nu\tau\rho}(k_1,k_2,k_4,k_3)+\alpha_1^{\mu\tau\nu\rho}(k_1,k_4,k_2,k_3)\\ & +\alpha_1^{\mu\tau\rho\nu}(k_1,k_4,k_3,k_2)+\alpha_1^{\mu\rho\tau\nu}(k_1,k_3,k_4,k_2)\\ & +\alpha_1^{\mu\rho\nu\tau}(k_1,k_3,k_2,k_4)\,,\end{aligned} \tag{37}$$

with

$$\alpha_1^{\mu\nu\rho\tau}(k_1,k_2,k_3,k_4) = \frac{e^5B}{12\pi}\int\frac{\mathrm{d}p_z}{2\pi}\sum_{Q_1Q_2Q_3Q_4}\exp\Big(-\frac{\mathrm{i}}{2eB}\sum_{i>j}(\boldsymbol{k}_i\times\boldsymbol{k}_j)_z\Big)$$

$$\times\bigg(\sum_{i=1}^{4}\{\tfrac{1}{2}\varepsilon_i(1-2n_{qi}^{\varepsilon_i})/(\omega_i+\varepsilon_i\mathscr{E}_{qi}-\varepsilon_{i-1}\mathscr{E}_{qi-1})$$

$$\times(\omega_{i+1}+\varepsilon_{i+1}\mathscr{E}_{qi+1}-\varepsilon_i\mathscr{E}_{qi})(\omega_{i+1}+\omega_{i+2}+\varepsilon_{i+2}\mathscr{E}_{qi+2}-\varepsilon_i\mathscr{E}_{qi})\}\bigg)$$

$$\times[\Gamma_{q4q1}^{\varepsilon_4\varepsilon_1}(-\boldsymbol{k}_1)]^\mu[\Gamma_{q1q2}^{\varepsilon_1\varepsilon_2}(-\boldsymbol{k}_2)]^\nu[\Gamma_{q2q3}^{\varepsilon_2\varepsilon_3}(-\boldsymbol{k}_3)]^\rho[\Gamma_{q3q4}^{\varepsilon_3\varepsilon_4}(-\boldsymbol{k}_4)]^\tau\,. \tag{38}$$

Expressions equivalent to (35)–(38) have been written down previously by Stoneham (1978).

(*d*) *Radiative and Nonlinear Corrections*

So far we have been concerned only with closed electron loop diagrams in which all the external photon vertices have different k values. Radiative and nonlinear corrections involve cases where the sum of two k values is zero. An example of such a term is given in (II.44). Such corrections can be important in leading to new physical effects, such as self-focusing of light and collapse of Langmuir turbulence, and they should be included in a completely general expansion. However, we ignore them here.

4. Rules for Feynman Diagrams

We may now formulate rules for drawing Feynman diagrams, writing down the scattering amplitude corresponding to each diagram and evaluating rate coefficients from the scattering amplitudes.

The structure of the theory is not changed in any essential way by the inclusion of a magnetic field and of responses of an ambient medium. Consequently any standard set of rules for drawing Feynman diagrams may be applied to the present theory. The important changes relate to the interpretation of the diagrams rather than to their construction. There is however one change in the construction associated with the introduction of m-photon vertices. As discussed in the previous section, in QED an m-photon vertex is associated with an m-sided closed electron loop diagram. As in Part II an m-photon vertex is drawn as a hatched circle with m photon lines attached to it (see Fig. 1 on p. 783). A rule governing their introduction is:

> Any closed m-sided electron loop diagram is replaced by an m-photon vertex. For the purpose of counting the order of a process, an m-photon vertex counts as order $m-2$.

We adopt the rules for drawing diagrams in the form summarized by Berestetskii *et al.* (1971, §78). The simplest diagram involves an electron or positron line with one vertex connecting it to a photon line. As illustrated in Fig. 1 on p. 803, this describes six different processes, all of which are related by crossing symmetry. These are gyromagnetic emission and absorption by electrons or positrons, and one-photon pair creation or annihilation. In our theory splitting of one photon into two and coalescence of two photons into one are of the same order as these processes; the processes which involve only photons are discussed in Part II. Next in order one has Compton scattering and various crossed processes, and also photon–photon scattering (Part II). The diagrams contributing to Compton scattering are shown in Fig. 2 (on p. 803). Note that the 3-photon vertex contributes through 'nonlinear' scattering.

(*a*) *Scattering Amplitude in Momentum Space*

It is not possible to write down a momentum space representation for processes in a magnetic field in the same way as one can in the absence of an external field. The reason for this is that the electron propagator is a function of the two space–time points separately and not simply a function of the difference between them. Consequently the electron propagator may not be Fourier transformed with a single wave vector k. A momentum space representation may be obtained by taking the product of the two electron wavefunctions at a single vertex. In this case the Fourier transform leads to the vertex function (7). With the momentum space elements associated with vertices rather than propagators, the external particle states are automatically included and sums over intermediate states need to be performed. With the detailed properties of the electron states being included in the vertex functions, the electron propagator is replaced by a scalar function like the energy denominators in nonrelativistic perturbation theory. We shall refer to them as 'energy denominators'.

Our convention for the directions of 4-momenta are implicit in the definitions of the vertex functions. According to (15), when k is directed towards an electron–photon vertex the argument of the vertex function has a minus sign. Consequently, when the argument k of the vertex function is positive the 4-momentum is directed away from the vertex. In other words positive k is associated with emission of a photon. For the m-photon vertices the signs of the arguments in the relation (33) imply that all 4-momenta are directed towards the vertex. In this case a positive argument is associated with absorption of a photon.

Consider the convention that an electron–positron line has its arrow directed from the state q to the state q'. For an electron line ($\varepsilon' = \varepsilon = 1$), q corresponds to the initial state and q' to the final state, and for a positron line, q' corresponds to the initial state and q to the final state. The other possibilities correspond to pair creation and pair annihilation with both q and q' in the final and initial states respectively. The initial state is always on the right and the final state on the left. We label a sequence of states along an electron–positron line $q, q_1, ..., q_n, q'$ in the same direction as the arrow.

(*b*) *Rules*

(1) With each electron–photon vertex is associated an index μ and a 4-momentum k directed away from the vertex. The vertex contributes to S_{fi} a factor

$$\mathrm{i}\, e[\Gamma^{\varepsilon_2\varepsilon_1}_{q_2 q_1}(\boldsymbol{k})]^{\mu},$$

where (ε_2, q_2) and (ε_1, q_1) label the states in the opposite direction to the arrow along the electron–positron line.

(2) With each pair (ε_2, q_2) and (ε_1, q_1) of electron–positron states, ordered in the opposite direction to the arrow along the electron–positron line and at the end of the line, is associated in S_{fi} a factor $D^{\varepsilon'\varepsilon}_{q'q}(k_n, ..., k_1)$ given by (24).

(3) Energy and parallel momentum are conserved at each vertex and in the rth line they are given by

$$E_r = \varepsilon\mathscr{E}_q - (\omega_1 + \ldots + \omega_r) = \varepsilon'\mathscr{E}_{q'} + (\omega_{r+1} + \ldots + \omega_n)\,, \tag{39a}$$

$$\varepsilon_r\, p_{rz} = \varepsilon p_z - (\boldsymbol{k}_1 + \ldots + \boldsymbol{k}_r)_z = \varepsilon' p'_z + (\boldsymbol{k}_{r+1} + \ldots + \boldsymbol{k}_n)_z\,. \tag{39b}$$

The rth internal electron–positron line contributes to S_{fi} an energy denominator

$$\mathrm{i}/\{E_r - \varepsilon_r(\mathscr{E}_{q_r} - \mathrm{i}\,0)\}\,.$$

(4) An m-photon vertex with all 4-momenta $k_1, ..., k_m$ directed towards the vertex contributes to S_{fi} a factor

$$-\mathrm{i}(m-1)!\,\tilde{\alpha}^{\nu_1 \ldots \nu_m}(k_1, ..., k_m)(2\pi)^4\delta^4(k_1 + \ldots + k_m)\,.$$

[The factor $(m-1)!$ arises from (6), with $0 \ldots n+1$ replaced by $1 \ldots m$, due to a factor m in the denominator and a factor $m!$ in the numerator from the term in $(A^{(1)} + \ldots + A^{(m)})^m$ involving $m!\,A^{(1)} \ldots A^{(m)}$, i.e. assuming m different photon fields.]

(5) An internal photon line with 4-momentum k directed from vertex μ to vertex ν contributes to S_{fi} a factor

$$-\mathrm{i}\, D_{\mu\nu}(k)\,,$$

associated with vertex functions $[\Gamma^{\varepsilon'\varepsilon''}_{q'q''}(\boldsymbol{k})]^{\mu}$ and $[\Gamma^{\varepsilon''\varepsilon}_{q''\varepsilon}(-\boldsymbol{k})]^{\nu}$.

(6) An external photon line corresponding to mode M contributes to S_{fi} a factor

$$a_M(\boldsymbol{k})\, e_{M\mu}(\boldsymbol{k}) \qquad \text{or} \qquad a_M(\boldsymbol{k})\, e^*_{M\mu}(\boldsymbol{k})$$

for absorption or emission respectively, with

$$a_M(\boldsymbol{k}) = \{R_M(k)/V\varepsilon_0 \,|\, \omega_M(\boldsymbol{k})\,|\}^{\frac{1}{2}}. \tag{40}$$

(c) *Averaged Transition Rate*

In the absence of a magnetic field, 4-momentum is conserved and it is conventional to write $S_{\mathrm{fi}} = \delta_{\mathrm{fi}} + \mathrm{i}(2\pi)^4\delta^4(P_{\mathrm{f}} - P_{\mathrm{i}})T_{\mathrm{fi}}$, where P_{f} and P_{i} denote the final and initial 4-momenta respectively. In the presence of a magnetic field only energy and parallel momentum are necessarily conserved and one cannot define a T_{fi} in this way. Nevertheless, it is convenient to seek a reduced matrix element which plays a role similar to that of T_{fi}. Specifically, we would like M_{fi} to be independent of the choice of gauge. In general one cannot describe a given process without introducing a gauge (or some equivalent concept) because the process may depend on the initial position of the particles in the xy plane. In cartesian and cylindrical coordinates, this position is included in the initial data through p_y and s respectively. In practice one is rarely interested in the dependence on initial position and it is convenient to average over it.

The transition rate $w_{\mathrm{i}\to\mathrm{f}}$ for a transition from state i to state f is given by

$$w_{\mathrm{i}\to\mathrm{f}} = T^{-1}|\,S_{\mathrm{fi}}\,|^2(\Pi\, D_{\mathrm{i}})(\Pi\, D_{\mathrm{f}}). \tag{41}$$

The densities of the initial and final states are given as follows: For an electron we have (cf. equations I.33)

$$D_{\mathrm{i}} = V(eB)^{\frac{1}{2}}/L_y L_z\,, \qquad D_{\mathrm{i}} = VeB/2\pi L_z\,, \tag{42a, b}$$

and (cf. equations I.34)

$$D_{\mathrm{f}} = D_{\mathrm{i}}\, L_y(\mathrm{d}p_y/2\pi)L_z(\mathrm{d}p_z/2\pi)\,, \qquad D_{\mathrm{f}} = D_{\mathrm{i}}\, L_z(\mathrm{d}p_z/2\pi)\,, \tag{43a, b}$$

in cartesian and cylindrical coordinates respectively. For photons we have $D_{\mathrm{i}} = 1$ and

$$D_{\mathrm{f}} = V\,\mathrm{d}\boldsymbol{k}/(2\pi)^3\,. \tag{44}$$

Note that each electron–positron line (other than closed loops which have already been discussed) is described by a factor $D^{\varepsilon'\varepsilon}_{q'q}$ in S_{fi}, and that the factors D_{i} given by (42) appear in the denominators in the expressions (24) for $D^{\varepsilon'\varepsilon}_{q'q}$. It follows that in the transition rate the factors D_{i} all cancel. Equivalently we could renormalize our wavefunctions such that D_{i} is unity in (42) and the analogous factors $L_y L_z/V(eB)^{\frac{1}{2}}$ and $2\pi L_z/VeB$ do not appear in (24).

Included in the initial data is the value of p_y or of s for each initial electron or positron. These quantum numbers provide information on the location of the particle in the xy plane, and in practice we are not interested in this information. Consequently we average over this position. We sum or integrate over the quantum numbers which determine the position of the final particle. Using (I.63) and (I.65) in the case of cartesian and cylindrical coordinates, this average corresponds to operation with

$$\hat{O}_{\mathrm{av}} = L_x^{-1}\int_{-\frac{1}{2}L_x}^{\frac{1}{2}L_x}\mathrm{d}x = \frac{2\pi L_z}{VeB}L_y\int\frac{\mathrm{d}p_y}{2\pi}, \tag{45a}$$

$$= \frac{2\pi}{\pi R^2}\int_0^R \mathrm{d}r\, r = \frac{2\pi L_z}{VeB}\sum_s, \tag{45b}$$

with $V = L_x L_y L_z$ and $V = \pi R^2 L_z$ respectively. This averaging process is relevant only if the value of p_y or of s is determined. For example, if there is only one particle in the initial state then $w_{\mathrm{i}\to\mathrm{f}}$ does not depend on its position and it is not necessary to average. It is convenient to average over all initial positions however, and to adjust the result by making the replacements

$$L_y\int\frac{\mathrm{d}p_y}{2\pi}\to\frac{VeB}{2\pi L_z}, \qquad \sum_s\to\frac{VeB}{2\pi L_z}, \tag{46a, b}$$

in any cases where the function being integrated or summed is independent of p_y or s.

Consider the case where there is only one electron–positron line. The foregoing discussion together with the form (24) of $D^{\varepsilon'\varepsilon}_{q'q}$ leads to the following result for the averaged (denoted by a bar) transition rate:

$$\bar{w}_{\mathrm{i}\to\mathrm{f}} = (2\pi/VeB)2\pi\delta(E_{\mathrm{f}}-E_{\mathrm{i}})\,2\pi\delta(P_{z\mathrm{f}}-P_{z\mathrm{i}})|\,M_{\mathrm{fi}}\,|^2(\Pi\,\bar{D}_{\mathrm{f}}), \tag{47}$$

where $E_{\mathrm{f}}, P_{z\mathrm{f}}$ and $E_{\mathrm{i}}, P_{z\mathrm{i}}$ denote the final and initial values of energy and parallel momentum respectively, and with M_{fi} constructed as with the rules above, but with rule (2) modified as follows:

(2′) For the purpose of constructing M_{fi}, $D^{\varepsilon'\varepsilon}_{q'q}(k_n, ..., k_1)$ is to be replaced by

$$V\exp\Big(-\frac{\mathrm{i}}{2eB}\sum_{j<i}(\boldsymbol{k}_i\times\boldsymbol{k}_j)_z\Big).$$

The modified density of final states in (47) is

$$\bar{D}_{\mathrm{f}} = (VeB/2\pi)\mathrm{d}p_z/2\pi. \tag{48}$$

The result (47) is readily derived by considering the three cases of an initial pair, a final pair and a line joining initial and final states separately, and by treating each in the two gauges. It is not difficult to generalize (47) to cases where there is more than one electron–positron line but we do not do so here.

The transition rate (47) is closely related to the probability per unit time for emission of a photon. This quantity has been used extensively in semiclassical theory. For present purposes it may be defined by writing

$$w = 2\pi\delta(E_{\mathrm{f}}-E_{\mathrm{i}})|\,M_{\mathrm{fi}}\,|^2, \tag{49}$$

setting $V = 1$ and assuming implicitly that parallel momentum is conserved. In the classical limit the probability (49) reproduces well-known semiclassical results (see e.g. Tsytovich 1970, 1972; Melrose 1980), as shown explicitly below.

5. Gyromagnetic Emission and One-photon Pair Creation

The simplest processes in the present theory are those which correspond to an electron–positron line with a single vertex. There are six such processes: gyromagnetic emission and absorption by an electron or a positron and one-photon pair creation or annihilation. These processes are all related by crossing symmetries and may be described in terms of a single probability. The probabilities which have been written down previously do not show these crossing symmetries explicitly. Sokolov and Ternov (1968, p. 76) indicated an expression for the probability of gyromagnetic emission by an electron in vacuo and then concentrated on the case of synchrotron radiation. Melrose (1974) wrote down a probability in terms of the Γ function derived using Johnson and Lippmann wavefunctions (see Part I), and when taking the nonrelativistic limit (Melrose and Zheleznyakov 1981) the expected symmetry between electron and positron processes is not apparent.

Here we write down the exact expression for the relevant probability, discuss the crossing symmetry and then discuss the nonrelativistic and ultra-relativistic limits. The usual expressions for one-photon pair creation may be derived in the ultra-relativistic limit. The possibility of maser emission of γ rays is then discussed briefly.

(a) *The Probability*

The probability for processes corresponding to a single photon in the mode M interacting with an electron or a positron is

$$w^M_{q'q}(\boldsymbol{k}) = \frac{e^2 R_M(\boldsymbol{k})}{\varepsilon_0 \,|\, \omega_M(\boldsymbol{k}) \,|} \,|\, e^*_{M\mu}(\boldsymbol{k}) [\Gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k})]^\mu \,|^2 2\pi\delta\big(\varepsilon\mathscr{E}_q - \varepsilon'\mathscr{E}_{q'} - \omega_M(\boldsymbol{k})\big). \tag{50}$$

As written the probability is for emission of the photon, and it includes emission by an electron ($\varepsilon = \varepsilon' = 1$) or a positron ($\varepsilon = \varepsilon' = -1$) and one-photon pair annihilation ($\varepsilon = -\varepsilon' = 1$). The corresponding results for absorption and one-photon pair creation are obtained by reversing the sign of $\boldsymbol{k}$ and using

$$\omega_M(-\boldsymbol{k}) = -\omega_M(\boldsymbol{k}), \qquad e^\mu_M(-\boldsymbol{k}) = e^*_M(\boldsymbol{k}), \tag{51a, b}$$

$$R_M(-\boldsymbol{k}) = R_M(\boldsymbol{k}), \qquad [\Gamma^{\varepsilon'\varepsilon}_{q'q}(-\boldsymbol{k})]^\mu = [\Gamma^{\varepsilon\varepsilon'}_{qq'}(\boldsymbol{k})]^{*\mu}. \tag{51c, d}$$

The reversal of the order of the states in (51d), which follows from (I.60), reflects the usual detailed-balance relation. Specifically, if $w^M_{q'q}(\boldsymbol{k})$ denotes the probability for emission by an electron with transition from state q to q', then the same probability applies to absorption between the same two states, i.e. with a transition from q' back to q.

Conservation of parallel momentum is implicit in (50) in the form

$$\varepsilon p_z - \varepsilon' p'_z - k_z = 0. \tag{52}$$

The p_z describe the physical momenta. For emission by an electron, $\varepsilon = \varepsilon' = 1$ in (52) gives $p'_z = p_z - k_z$ as required, and for emission by a positron, $\varepsilon = \varepsilon' = -1$ in (52) gives $p_z = p'_z - k_z$, which is also the required result in view of the states q and q' being final and initial states for positrons.

An explicit expression for the vertex function in (50) is written down in (I.57). This cumbersome expression simplifies considerably in the nonrelativistic and ultra-relativistic limits.

(*b*) *Nonrelativistic Limit*

The nonrelativistic limit here corresponds to $p_z^2/m^2 \ll 1$, $p_n^2/m^2 = 2nB/B_c \ll 1$ and $x = k_\perp^2/2eB \ll 1$. Gyromagnetic emission then favours low harmonics, with $\omega \approx j\Omega_e$ and $j = 1, 2, 3, \ldots$. We assume that the dispersion is weak and write $k_z \approx \omega\cos\theta$, $x \approx j^2(B/2B_c)\sin^2\theta$. When only the leading terms in an expansion in x are retained for each j [cf. equations (A18) of Part I] the vertex function (I.57) for electrons gives

$$\Gamma^{++}_{q'q}(\boldsymbol{k}) = \delta_{\sigma'\sigma}\left\{(-\mathrm{i})^j\left(\frac{B}{2B_c}\right)^{\frac{1}{2}}\left(\frac{l!}{(l-j)!}\right)^{\frac{1}{2}}\frac{x^{\frac{1}{2}(j-1)}}{(j-1)!}(1,\mathrm{i},0)\right\}$$

$$+\delta_{\sigma'-\sigma}\left\{(-\mathrm{i})^{j-\sigma}\frac{B\cos\theta}{2B_c}\left(\frac{l!}{(l-j+\sigma)!}\right)^{\frac{1}{2}}\frac{x^{\frac{1}{2}(j-\sigma)}}{(j-\sigma)!}\sigma(j,\mathrm{i}\sigma j,-\tan\theta)\right\}, \tag{53}$$

with $n' = n-j$. For positrons q' labels the initial state and hence we would write $n = n'-j$ for emission at the jth harmonic. However, it is more useful to note that the symmetry properties (I.60) and (I.61) imply

$$\Gamma^{--}_{qq'}(\boldsymbol{k}) = (-)^{l'-l}[\Gamma^{++}_{q'q}(\boldsymbol{k})]^*. \tag{54}$$

As is obvious on physical grounds, (53) with (54) implies that the only difference between emission by a positron and by an electron is in the handedness of the emitted radiation.

As pointed out by Melrose and Zheleznyakov (1981), the terms in (53) with no spin-flip ($\sigma' = \sigma$) correspond to 2^j electric multipole emission and the terms with spin-flip correspond to magnetic multipole emission. The result (53) corrects the corresponding result quoted by Melrose and Zheleznyakov (1981) for the terms involving a spin-flip for $j > 1$.

The form (53) is relevant for electrons or positrons in low Landau orbitals, i.e. for small values of l and l'. In the opposite limit the particles may be treated classically. It is straightforward to show that the leading term in the classical limit [cf. equation (A19) of Part I] leads to $\Gamma^{++}_{q'q}(\boldsymbol{k})$ and $\Gamma^{--}_{qq'}(\boldsymbol{k})$, reducing to the classical quantity $V(j,\boldsymbol{k})$ given by equation (A38b) of Part II, apart from an arbitrary phase factor. The classical theory of gyromagnetic emission has been discussed in detail (see e.g. Melrose 1980) using the classical counterpart of (50).

(*c*) *One-photon Pair Creation*

The probability (50) may be used to treat one-photon pair creation by setting $\varepsilon = -1$ and $\varepsilon' = 1$. The absorption coefficient $\alpha^M(\boldsymbol{k})$ is given by

$$\alpha^M(\boldsymbol{k}) = \sum_{\substack{n,n'\\\sigma,\sigma'}}\frac{eB}{2\pi}\int\frac{\mathrm{d}p_z}{2\pi}\,w^M_{q'q}(\boldsymbol{k}). \tag{55}$$

In the absence of a material medium we are free to choose a frame in which $\boldsymbol{k}$ is along the 1-axis ($k_z = 0$, $\psi = 0$). The two natural modes of the birefringent vacuum then correspond to polarizations along the 2- and 3-axes; we refer to these as the perpendicular and parallel modes respectively. We have

$$\alpha^{\perp,\parallel}(\boldsymbol{k}) = \sum_{\substack{n,n'\\\sigma,\sigma'}}\frac{eB}{2\pi}\int\frac{\mathrm{d}p_z}{2\pi}\frac{e^2}{2\varepsilon_0\,\omega}\,|\{\Gamma^{+-}_{q'q}(\boldsymbol{k})\}^{2,3}|^2 2\pi\delta(\mathscr{E}_q+\mathscr{E}_{q'}-\omega), \tag{56}$$

where we assume that the effect of the magnetic field on the dispersion of the photons is weak. After summing over the spin states the resulting expression is independent of the choice of spin eigenfunctions. Using either (I.57) or the form written down by Melrose (1974) one obtains

$$\alpha^{\perp,\parallel}(\boldsymbol{k}) = \sum_{n,n'} \frac{eB}{2\pi} \int \frac{\mathrm{d}p_z}{2\pi} \frac{e^2}{2\varepsilon_0\,\omega} F_{nn'}^{\perp,\parallel}(\boldsymbol{k})\, 2\pi\delta(\mathscr{E}_n + \mathscr{E}'_{n'} - \omega)\,, \tag{57}$$

with

$$\mathscr{E}_n \equiv (m^2 + p_z^2 + 2neB)^{\frac{1}{2}}\,, \qquad \mathscr{E}'_{n'} \equiv (m^2 + p_z'^2 + 2n'eB)^{\frac{1}{2}}\,, \tag{58a, b}$$

and with

$$F_{n'n}^{\perp} = (1/2\mathscr{E}'_{n'}\mathscr{E}_n)[(\mathscr{E}'_{n'}\mathscr{E}_n + m^2 - p'_z p_z)\{(\mathrm{J}^n_{n'-n-1})^2 + (\mathrm{J}^{n-1}_{n'-n+1})^2\}$$
$$+ 4n'neB\,\mathrm{J}^n_{n'-n-1}\,\mathrm{J}^{n-1}_{n'-n+1}]\,, \tag{59a}$$

$$F_{n'n}^{\parallel} = (1/2\mathscr{E}'_{n'}\mathscr{E}_n)[(\mathscr{E}'_{n'}\mathscr{E}_n + m^2 + p'_z p_z)\{(\mathrm{J}^n_{n'-n})^2 + (\mathrm{J}^{n-1}_{n'-n})^2\}$$
$$+ 4n'neB\,\mathrm{J}^n_{n'-n}\,\mathrm{J}^{n-1}_{n'-n}]\,, \tag{59b}$$

where arguments are omitted. Conservation of parallel momentum requires $p'_z + p_z = 0$ in this case.

In the usual treatment of this process the ultra-relativistic approximation is made (see e.g. Erber 1966). To rederive the known result we assumed that n and n' are large and continuous and replaced the sums in (57) by integrals. Next we assume $2neB$, $2n'eB \gg m^2, p_z^2$ and expand ε_n, $\varepsilon'_{n'}$ with the leading terms being $(2neB)^{\frac{1}{2}}$, $(2n'eB)^{\frac{1}{2}}$ respectively. The variables of integration are then changed to u, w and E with

$$\sinh u \equiv p_z/m\,, \qquad \sinh w \equiv \{(n')^{\frac{1}{2}} - n^{\frac{1}{2}}\}/4(n'n)^{\frac{1}{4}}\,, \tag{60a, b}$$

$$E \equiv \mathscr{E}_n + \mathscr{E}'_{n'} - \omega\,. \tag{60c}$$

The E integral is performed over the δ function giving

$$(n')^{\frac{1}{2}} + n^{\frac{1}{2}} \approx x^{\frac{1}{2}}\{1 - (m^2/\omega^2)\cosh^2 u \cosh^2 w\}\,, \tag{61}$$

where $x = \omega^2/2eB$ is the argument of the J functions. We find

$$\alpha^{\perp,\parallel}(\boldsymbol{k}) \approx \frac{e^3 B}{4\pi\varepsilon_0\,\omega} \int_0^\infty \mathrm{d}u \int_0^\infty \mathrm{d}w\, \frac{\omega^3 m \cosh u}{8e^2 B^2 \cosh^4 w} F^{\perp,\parallel}(\boldsymbol{k})\,, \tag{62}$$

with

$$F^{\perp} \approx \tfrac{1}{2}(\mathrm{J}^n_{n'-n-1} + \mathrm{J}^{n-1}_{n'-n+1})^2 + (2m^2/\omega^2)\cosh^2 w\{2\cosh^2 w \cosh^2 u - \sinh^2 u\}$$
$$\times \{(\mathrm{J}^n_{n'-n-1})^2 + (\mathrm{J}^{n-1}_{n'-n+1})^2\}\,, \tag{63a}$$

$$F^{\parallel} \approx \tfrac{1}{2}(\mathrm{J}^n_{n'-n} + \mathrm{J}^{n-1}_{n'-n})^2 + (2m^2/\omega^2)\cosh^2 w\{2\cosh^2 w \cosh^2 u - \sinh^2 u\}$$
$$\times \{(\mathrm{J}^n_{n'-n})^2 + (\mathrm{J}^{n-1}_{n'-n})^2\}\,. \tag{63b}$$

The final steps involve using equations (A5)–(A7) and (A21) of Part I to re-express the J functions in terms of $\mathrm{J}^{n}_{n'-n}$ and its derivative, and then making the Airy integral approximation to them:

$$\mathrm{J}^{n}_{n'-n}(x) \approx (1/\pi\surd 3)(2m/\omega)\cosh u \cosh w \,\mathrm{K}_{1/3}(z)\,, \tag{64a}$$

$$(\mathrm{d}/\mathrm{d}x)\mathrm{J}^{n}_{n'-n}(x) \approx (1/\pi\surd 3)(2m^2/\omega^2)\cosh^2 u \cosh w \,\mathrm{K}_{2/3}(z)\,, \tag{64b}$$

with

$$z \equiv (2/3\chi)\cosh^2 u \cosh^2 w\,, \qquad \chi \equiv \omega eB/2m^3 = B\omega/2B_{\mathrm{c}}\, m\,. \tag{65a, b}$$

This results in

$$\begin{pmatrix} F^{\perp} \\ F^{\parallel} \end{pmatrix} = \frac{32}{3\pi^2}\frac{m^4}{\omega^4}\left\{\begin{pmatrix} \cosh^2 w \\ \sinh^2 w \end{pmatrix}\cosh^4 w \cosh^4 u \,\mathrm{K}^2_{2/3}(z) \right.$$
$$\left. +\tfrac{1}{2}\{2\cosh^2 w \cosh^2 u - \sinh^2 u\}\cosh^4 w \cosh^2 u\, \mathrm{K}^2_{1/3}(z)\right\}. \tag{66}$$

The result quoted by Erber (1966) is for unpolarized photons and is reproduced by evaluating $\frac{1}{2}(\alpha^{\perp}+\alpha^{\parallel})$ using (62) and (66).

(*d*) *Maser Emission of γ Rays*

It is well known that amplification of gyromagnetic emission is possible (see e.g. the literature cited by Hewitt *et al.* 1982). Recently it has been pointed out that maser emission of γ rays is possible in principle due to pair annihilation (Ramaty *et al.* 1982). Here we examine the possibility of γ-ray maser emission due to one-photon pair annihilation.

To illustrate the approach let us first consider gyromagnetic emission. The rate at which the process proceeds in the direction $q \to q'$ involves factors $n_q^+(1-n_{q'}^+)$ from the electron states and $1+N_M(\boldsymbol{k})$ from the photon states, and the rate it proceeds in the opposite direction involves factors $(1-n_q^+)n_{q'}^+$ and $N_M(\boldsymbol{k})$. Hence the net rate is given by

$$\mathrm{d}N_M(\boldsymbol{k})/\mathrm{d}t \propto n_q^+(1-n_{q'}^+)+(n_q^+ - n_{q'}^+)N_M(\boldsymbol{k})\,. \tag{67}$$

The first term on the right-hand side of (67) describes spontaneous emission and the second term describes absorption. The absorption is negative for $n_q^+ > n_{q'}^+$.

Now in the case of one-photon pair annihilation the rates of emission and absorption of the photons are proportional respectively to $n_q^+ n_{q'}^-\{1+N_M(\boldsymbol{k})\}$ and $(1-n_q^+)(1-n_q^-)N_M(\boldsymbol{k})$, and hence the net rate is given by

$$\mathrm{d}N_M(\boldsymbol{k})/\mathrm{d}t \propto n_q^+ n_{q'}^- +(n_q^+ + n_{q'}^- - 1)N_M(\boldsymbol{k})\,. \tag{68}$$

Thus amplification of γ rays is possible only for $n_q^+ + n_{q'}^- > 1$, which corresponds to the electrons and positrons being degenerate. One readily confirms that the condition $n_q^+ + n_{q'}^- > 1$ cannot be satisfied for thermal distributions of electrons and positrons in equilibrium, but it can be satisfied for Fermi–Dirac distributions of electrons and positrons with $\mu^+ + \mu^- > 0$, where μ^+ and μ^- are the chemical potentials and where $\mu^+ + \mu^- = 0$ corresponds to thermal equilibrium.

We conclude that maser emission of γ rays due to one-photon pair annihilation is possible only in a relativistically degenerate electron–positron plasma which is nonthermal in the sense $\mu^+ + \mu^- > 0$.

6. Compton Scattering and Related Processes

In the absence of a magnetic field the allowed second order processes are Compton scattering, two-photon pair creation and annihilation and electron–electron, electron–positron and positron–positron scattering. In the presence of a magnetic field additional crossed processes are allowed. Two-photon emission and absorption are related to Compton scattering and decay of one photon into a pair plus another photon and its inverse are related to two-photon pair creation. All the processes related to Compton scattering can be described in terms of a single probability, which we write down below. We then discuss Compton scattering for electrons in low Landau orbitals and two-photon absorption when the initial electron is in its ground state and the final electron is in its first excited state.

The diagrams which contribute to Compton scattering are illustrated in Fig. 2 (see p. 803). Besides the usual diagrams (*a*) and (*b*), there is an additional one (*c*) involving the quadratic nonlinear response of the medium. In the language of plasma physics this diagram describes scattering off the shielding or self-consistent electromagnetic field of the particle; it is also called nonlinear scattering.

(*a*) *The Probability*

We choose to write the probability so that for $\varepsilon' = \varepsilon = 1$ it describes two-photon emission:

$$w_{q'q}^{M'M}(\boldsymbol{k}',\boldsymbol{k}) = \frac{e^4 R_{M'}(\boldsymbol{k})\, R_M(\boldsymbol{k})}{\varepsilon_0^2\{\omega_{M'}(\boldsymbol{k}')\,\omega_M(\boldsymbol{k})\}} \,|\, e^*_{M\mu}(\boldsymbol{k})\, e^*_{M'\nu}(\boldsymbol{k}')\, M^{\mu\nu}(\boldsymbol{k},\boldsymbol{k}')\,|^2$$
$$\times\, 2\pi\delta\big(\varepsilon\mathscr{E}_q - \varepsilon'\mathscr{E}_{q'} - \omega_M(\boldsymbol{k}) - \omega_{M'}(\boldsymbol{k}')\big)\,; \tag{69}$$

$$M^{\mu\nu}(\boldsymbol{k},\boldsymbol{k}') = \sum_{Q''} \frac{[\Gamma^{\varepsilon'\varepsilon''}_{q'q''}(\boldsymbol{k})]^\mu [\Gamma^{\varepsilon''\varepsilon}_{q''q}(\boldsymbol{k}')]^\nu}{\varepsilon\mathscr{E}_q - \omega_{M'}(\boldsymbol{k}') - \varepsilon''\mathscr{E}_{q''}} \exp\Big(-\frac{\mathrm{i}}{2eB}(\boldsymbol{k}'\times\boldsymbol{k})_z\Big)$$
$$+ \sum_{Q''} \frac{[\Gamma^{\varepsilon'\varepsilon''}_{q'q''}(\boldsymbol{k})]^\nu [\Gamma^{\varepsilon''\varepsilon}_{q''q}(\boldsymbol{k})]^\mu}{\varepsilon\mathscr{E}_q - \omega_M(\boldsymbol{k}) - \varepsilon''\mathscr{E}_{q''}} \exp\Big(\frac{\mathrm{i}}{2eB}(\boldsymbol{k}'\times\boldsymbol{k})_z\Big)$$
$$+ (2/e)[\Gamma^{\varepsilon'\varepsilon}_{q'q}(\boldsymbol{k}+\boldsymbol{k}')]^\theta D_{\theta\eta}(k_M + k'_{M'})\,\alpha^{\eta\mu\nu}(k_M + k'_{M'}, -k_M, -k'_{M'}), \tag{70}$$

where k_M denotes $(\omega_M(\boldsymbol{k}), \boldsymbol{k})$ and similarly for $k'_{M'}$. In the following calculations we neglect the final term in (70) which is due to so-called nonlinear scattering. Its effect has been discussed by Stoneham (1980*a*, 1980*b*). For Compton scattering $M \to M'$, the probability is $w_{q'q}^{M'M}(\boldsymbol{k}', -\boldsymbol{k})$, which is to be evaluated using (70) and (51).

(*b*) *Initial Electron in Its Ground State in Vacuo*

We now specialize to the case of scattering by an electron initially at rest in its ground state in vacuo. In the final state the electron may have any n' for the present:

$$\varepsilon = 1, \qquad \mathscr{E}_q = m, \qquad p_z = 0; \tag{71a}$$

$$\varepsilon' = 1, \qquad \mathscr{E}_{q'} = (m^2 + p_z'^2 + 2n'eB)^{\frac{1}{2}}, \qquad p'_z = k_z - k'_z. \tag{71b}$$

We neglect the dispersion of the waves setting

$$R_M = \tfrac{1}{2}, \qquad R_{M'} = \tfrac{1}{2}, \tag{72a}$$

$$k_x = \omega\sin\theta\cos\psi, \qquad k_y = \omega\sin\theta\sin\psi, \qquad k_z = \omega\cos\theta. \tag{72b}$$

The δ function in (69) then implies

$$\omega' = (1/\sin^2\theta')[m+\omega(1-\cos\theta\cos\theta')-\{m^2+2n'eB\sin^2\theta' \\ +2m\omega\cos\theta'(\cos\theta'-\cos\theta)+\omega^2(\cos\theta-\cos\theta')^2\}^{\frac{1}{2}}]. \tag{73}$$

We consider only the cases $n' = 0$ and $n' = 1$ explicitly.

Case $n' = 0$. To reduce the probability (70) further we need to evaluate the scalar products involving the Γ functions. We do this in the temporal gauge, denoting the polarization vectors $\boldsymbol{e}$ and $\boldsymbol{e}'$ by their cartesian components. Using (I.57) and (I.58) we have, for arbitrary p_z,

$$\Gamma^{\varepsilon''+}_{q''0}(\boldsymbol{k}) = \left(\frac{(\mathscr{E}_{q''}+\mathscr{E}^0_{q''})(\mathscr{E}^0_{q''}+m)}{4\mathscr{E}_{q''}\mathscr{E}^0_{q''}}\right)^{\frac{1}{2}}\left(\frac{\mathscr{E}_q+m}{2\mathscr{E}_q}\right)^{\frac{1}{2}}(-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\psi})^{n''} \\ \times\big(\tfrac{1}{2}(1-\sigma'')[-\rho''_{n''}\mathrm{e}^{\mathrm{i}\psi}\mathrm{J}^0_{n''-1}(x)\sqrt{2}\,\boldsymbol{e}_{\mathrm{L}}\{\tfrac{1}{2}(1+\varepsilon'')(1-\rho''_z\rho'_z)-\tfrac{1}{2}(1-\varepsilon'')(\rho''_z-\rho'_z)\} \\ +\mathrm{J}^0_{n''}(x)\boldsymbol{b}\{\tfrac{1}{2}(1+\varepsilon'')(\rho''_z-\rho'_z)-\tfrac{1}{2}(1-\varepsilon'')(1+\rho''_z\rho'_z)\}] \\ +\tfrac{1}{2}(1+\sigma'')[\mathrm{i}\,\mathrm{e}^{\mathrm{i}\psi}\mathrm{J}^0_{n''-1}(x)\sqrt{2}\,\boldsymbol{e}_{\mathrm{L}}\{\tfrac{1}{2}(1+\varepsilon'')(\rho''_z-\rho'_z)+\tfrac{1}{2}(1-\varepsilon'')(1-\rho''_z\rho'_z)\} \\ +\mathrm{i}\,\rho''_{n''}\mathrm{J}^0_{n''}(x)\boldsymbol{b}\{\tfrac{1}{2}(1+\varepsilon'')(1+\rho''_z\rho'_z)+\tfrac{1}{2}(1-\varepsilon'')(\rho''_z+\rho'_z)\}]\big), \tag{74}$$

with

$$\sqrt{2}\,\boldsymbol{e}_{\mathrm{L}} \equiv (1,-\mathrm{i},0), \qquad \boldsymbol{b} \equiv (0,0,1), \tag{75a}$$

$$x = \frac{\omega^2\sin^2\theta}{2eB}, \qquad \rho_z = \frac{p_z}{\mathscr{E}_q+\mathscr{E}^0_q}, \qquad \rho_n = \frac{(2neB)^{\frac{1}{2}}}{\mathscr{E}^0_q+m}. \tag{75b}$$

In using (74) in (70) we need to take account of the sign ε'' in the value of p''_z, which follows from the relevant δ function in (8) for example. In the first term in (70) we have $p''_z = \varepsilon''(p_z-k'_z)$ and in the second we have $p''_z = \varepsilon''(p_z+k_z)$ (after changing the sign of $\boldsymbol{k}$ as indicated). A lengthy calculation then leads to the result

$$w_{0'0}(\boldsymbol{k}',-\boldsymbol{k}) = \frac{e^4}{8\varepsilon_0^2\omega\omega'}\,\frac{\exp\{-(\omega^2\sin^2\theta+\omega'^2\sin^2\theta')/2eB\}}{(m+\omega-\omega')(2m+\omega-\omega')} \\ \times|\sum_n a_1(n)+a_2(n)|^2 2\pi\delta(\mathscr{E}_{q'}-m-\omega+\omega'), \tag{76}$$

with

$$a_1(n) = \frac{\exp\{-(\mathrm{i}/2eB)\omega\omega'\sin\theta\sin\theta'\sin(\psi-\psi')\}}{2m\omega'-\omega'^2\sin^2\theta'+2neB}\left\{\frac{1}{n!}\left(\frac{\omega\omega'\sin\theta\sin\theta'}{2eB}\right)^n\right. \\ \times\,\mathrm{e}^{\mathrm{i}n(\psi-\psi')}\{\omega'(2m+\omega-\omega')+\omega'\cos\theta'(\omega\cos\theta-\omega'\cos\theta')\}e_z e'^*_z+$$

$$+\frac{1}{(n-1)!}\left(\frac{\omega\omega'\sin\theta\sin\theta'}{2eB}\right)^{n-1}\mathrm{e}^{\mathrm{i}(n-1)(\psi-\psi')}[\{\omega'(2m+\omega-\omega')$$

$$-\omega'\cos\theta'(\omega\cos\theta-\omega'\cos\theta')\}(e_x+\mathrm{i}\,e_y)(e'_x+\mathrm{i}\,e'_y)^*$$

$$+\omega'\sin\theta'\mathrm{e}^{-\mathrm{i}\psi'}(\omega\cos\theta-\omega'\cos\theta')(e_x+\mathrm{i}\,e_y)e'^*_z$$

$$+\omega\sin\theta\,\mathrm{e}^{\mathrm{i}\psi}(\omega\cos\theta-\omega'\cos\theta')e_z(e'_x+\mathrm{i}\,e'_y)^*]\Bigg\}, \tag{77a}$$

$$a_2(n)=\frac{\exp\{(\mathrm{i}/2eB)\omega\omega'\sin\theta\sin\theta'\sin(\psi-\psi')\}}{2m\omega+\omega^2\sin^2\theta-2neB}\Bigg\{\frac{1}{n!}\left(\frac{\omega\omega'\sin\theta\sin\theta'}{2eB}\right)^{n}$$

$$\times\,\mathrm{e}^{-\mathrm{i}n(\psi-\psi')}\{\omega(2m+\omega-\omega')+\omega\cos\theta(\omega\cos\theta-\omega'\cos\theta')\}e_z\,e'^*_z$$

$$+\frac{1}{(n-1)!}\left(\frac{\omega\omega'\sin\theta\sin\theta'}{2eB}\right)^{n-1}\mathrm{e}^{-\mathrm{i}(n-1)(\psi-\psi')}[\{\omega(2m+\omega-\omega')$$

$$-\omega\cos\theta(\omega\cos\theta-\omega'\cos\theta')\}(e_x-\mathrm{i}\,e_y)(e'_x-\mathrm{i}\,e'_y)^*$$

$$+\omega'\sin\theta'\mathrm{e}^{\mathrm{i}\psi'}(\omega\cos\theta-\omega'\cos\theta')(e_x-\mathrm{i}\,e_y)e'^*_z$$

$$+\omega\sin\theta\,\mathrm{e}^{\mathrm{i}\psi}(\omega\cos\theta-\omega'\cos\theta')e_z(e'_x-\mathrm{i}\,e'_y)^*]\Bigg\}. \tag{77b}$$

The sum over n in (76) is from $n = 0$ to ∞ for the terms proportional to $e_z\,e'^*_z$ and from $n = 1$ to ∞ for the other terms.

Apart from notation, (76) with (77) reproduces an expression derived by Herold (1979), who expressed his result in terms of a cross section.

Case $n' = 1$. An electron initially in the ground state may be left in the first excited state as a result of a scattering event. The states other than the ground are doubly degenerate and only the transition rate summed over the two degenerate states is independent of the choice of spin eigenfunctions. Here we identify the spin states as the eigenvalues of the magnetic moment operator, as discussed in Part I. In our convention the ground state has spin down ($\sigma = -1$) and we describe transitions to the states $n' = 1$, $l' = 1$, $\sigma' = -1$ and $n' = 1$, $l' = 0$, $\sigma' = 1$ as being without a spin-flip and with a spin-flip respectively.

The probability analogous to (76) is

$$w_{10}(\boldsymbol{k}',-\boldsymbol{k})=\frac{e^4}{16\varepsilon_0^2\omega\omega'}\frac{\exp\{-(\omega^2\sin^2\theta+\omega'^2\sin^2\theta')/2eB\}}{\mathscr{E}^0_{q'}(\mathscr{E}^0_{q'}+m)(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})(m+\omega-\omega')}$$

$$\times\sum_{\sigma'=\pm1}|\sum_{n}b_1^{\sigma'}(n)+b_2^{\sigma'}(n)|^2 2\pi\delta(\mathscr{E}_{q'}-m-\omega+\omega'), \tag{78}$$

with $\mathscr{E}^0_{q'} = (m^2+2eB)^{\frac{1}{2}}$ here. The functions $b_1^{\sigma'}(n)$ and $b_2^{\sigma'}(n)$ are rather lengthy and are given in Appendix 1.

Transitions to $n' > 1$ may be described analogously with the explicit expressions being similar in form to (78).

(c) *Double Absorption*

We have suggested elsewhere (Melrose and Parle 1981) that two-photon absorption leading to a transition from $n = 0$ to $n' = 1$ might be important in the generation of X-ray cyclotron lines in some X-ray pulsars. The probability for double absorption is $w_{q'q}^{M'M}(-\boldsymbol{k}', -\boldsymbol{k})$, and the detailed case of interest here follows from (78) by replacing ω', $\cos\theta'$ and ψ' by $-\omega'$, $-\cos\theta'$ and $\psi'+\pi$ respectively.

We evaluate the transition rate for double absorption,

$$\Gamma^{\mathrm{D}} = \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\frac{\mathrm{d}\boldsymbol{k}'}{(2\pi)^3}\, w_{1'0}(-\boldsymbol{k}', -\boldsymbol{k})\, N(\boldsymbol{k})\, N(\boldsymbol{k}'), \tag{79}$$

by assuming (a) that the photons come from a single distribution [this leads to an extra factor of $\frac{1}{2}$, i.e. 16 is replaced by 32 in (78)], (b) that the photons are distributed with axial symmetry and we average over azimuthal angles, (c) that x, x' and B/B_c are all $\ll 1$ and (d) that we may average over states of polarization. The average over polarization states is achieved by writing

$$\sum_n \{b_1^{q'}(n) + b_2^{q'}(n)\} = e_i e_j' G_{ij}^*,$$

and making the replacement

$$|e_i e_j' G_{ij}^*|^2 \rightarrow \tfrac{1}{4}(\delta_{ir} - \kappa_i \kappa_r)(\delta_{js} - \kappa_j' \kappa_s') G_{ij}^* G_{rs}.$$

After averaging over polarizations we obtain a relatively cumbersome averaged probability which is written down in Appendix 2.

We may evaluate the rate (79) explicitly given the form of $N(\boldsymbol{k})$. For an isotropic distribution of photons independent of ω over a narrow range $\Delta\omega$ about $\omega = \frac{1}{2}\Omega_e$, we find

$$\Gamma^{\mathrm{D}}/\Gamma = 0{\cdot}316\alpha(B/B_c)^2(\Delta\omega/\Omega_e)N^2, \tag{80}$$

where $\Gamma = \frac{4}{3}\alpha\Omega_e(B/B_c)$ is the transition rate $1 \rightarrow 0$ due to one-photon emission. The significance of this result has been discussed by Melrose and Parle (1981).

7. Concluding Remarks

In this paper and the two preceding papers in the series we have shown how quantum electrodynamics (QED) may be extended to treat the effects of an ambient magnetic field and an ambient electron gas exactly. This generalized version of QED is in effect a synthesis of QED and the kinetic theory of plasmas, with a magnetized vacuum treated as though it were a plasma-like medium. We have illustrated the uses of this theory by rederiving a variety of results otherwise treated using diverse approaches. We have extended some of the known results. Specifically we have (i) corrected existing results for nonrelativistic gyromagnetic emission to exhibit the symmetry between electron and positron emission, (ii) rederived formulae which describe one-photon pair creation presenting general (rather than strictly ultra-relativistic) formulae and including polarization in the ultra-relativistic case, (iii) discussed γ-ray maser emission due to one-photon pair annihilation, (iv) treated Compton scattering from the ground to the first excited state, and (v) treated two-photon absorption from the ground to the first excited state. However, our main purpose has been to present the development of the theory in a systematic way rather than to obtain intrinsically new detailed results.

In one sense our theory is still incomplete. QED needs to be renormalized. Although the inclusion of an ambient magnetic field (or of an electron gas) does not alter the singular terms which are removed by renormalization, it does lead to finite corrections to the otherwise singular terms. In the case of the vacuum polarization the magnetic field leads to birefringence and to nonzero contributions to the nonlinear response tensors, as discussed in Part II. The other singular terms in QED are the electron self-energy and the vertex correction. To include all the effects of an ambient magnetic field we should include the finite corrections due to $B \neq 0$ to these functions. Moreover, in evaluating them, we should also include the vacuum polarization in the photon propagator. However, these corrections are usually not important. For example, classically the electron self-energy in a plasma is different from that in vacuo [the difference for an electron at rest is found by integration of $-e\phi(x)$ over all space for $\phi(x) = -e/r$ and $\phi(x) = (-e/r)\exp(-r/\lambda_D)$ and subtracting one from the other]; in practice this is unimportant except when the electron moves from one medium to another and then the emission of transition radiation may be related to the change in the self-energy. By analogy our neglect of the contribution from $B \neq 0$ to the electron self-energy should not affect the processes discussed in this series of papers.

References

Berestetskii, V. B., Lifshitz, E. M., and Pitaevskii, L. P. (1971). 'Relativistic Quantum Theory', Part 1 (Pergamon: Oxford).

Erber, T. (1966). *Rev. Mod. Phys.* **38**, 626.

Herold, H. (1979). *Phys. Rev.* D **19**, 2868.

Hewitt, R. G., Melrose, D. B., and Rönnmark, K. G. (1982). *Aust. J. Phys.* **35**, 447.

Melrose, D. B. (1974). *Plasma Phys.* **16**, 845.

Melrose, D. B. (1980). 'Plasma Astrophysics', Vol. I (Gordon and Breach: New York).

Melrose, D. B. (1983). *Aust. J. Phys.* **36**, 775.

Melrose, D. B., and Parle, A. J. (1981). *Proc. Astron. Soc. Aust.* **4**, 214.

Melrose, D. B., and Parle, A. J. (1983). *Aust. J. Phys.* **36**, 755.

Melrose, D. B., and Stoneham, R. J. (1977). *J. Phys.* A **10**, 1211.

Melrose, D. B., and Zheleznyakov, V. V. (1981). *Astron. Astrophys.* **95**, 86.

Ramaty, R., McKinley, J. M., and Jones, F. C. (1982). *Astrophys. J.* **256**, 238.

Sokolov, A. A., and Ternov, I. M. (1968). 'Synchrotron Radiation' (Pergamon: Oxford).

Stoneham, R. J. (1978). Ph.D. Dissertation, Australian National University.

Stoneham, R. J. (1980*a*). *Optica Acta* **27**, 537.

Stoneham, R. J. (1980*b*). *Optica Acta* **27**, 545.

Tsytovich, V. N. (1961). *Sov. Phys. JETP* **13**, 1249.

Tsytovich, V. N. (1970). 'Nonlinear Effects in Plasma' (Plenum: New York).

Tsytovich, V. N. (1972). 'An Introduction to the Theory of Plasma Turbulence' (Pergamon: Oxford).

Appendix 1. Compton Scattering

The coefficients in (78) for no spin-flip ($\sigma' = -$) and for spin-flip ($\sigma' = +$) are:

$$b_1^-(n) = -\frac{\exp\{\mathrm{i}(xx')^{\frac{1}{2}}\sin(\psi-\psi')\}}{2m\omega'-\omega'^2\sin^2\theta'+2neB}x^{-\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\psi}\{(1/n!)(xx')^{\frac{1}{2}n}\mathrm{e}^{\mathrm{i}n(\psi-\psi')}$$

$$\times\big([\{\omega'(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})+\omega'\cos\theta'(\omega\cos\theta-\omega'\cos\theta')\}(m+\mathscr{E}^0_{q'})(n-x)$$

$$+2neB(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})]e_z e_z'^*$$

$$+\omega\sin\theta\,\mathrm{e}^{\mathrm{i}\psi}\{\omega'(\omega\cos\theta-\omega'\cos\theta')-\omega'\cos\theta'(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})\}(e_x-\mathrm{i}\,e_y)e_z'^*)$$

$$+\{1/(n-1)!\}(xx')^{\frac{1}{2}(n-1)}\,\mathrm{e}^{\mathrm{i}(n-1)(\psi-\psi')}$$

$$\times\,(\omega'\sin\theta'\,\mathrm{e}^{-\mathrm{i}\psi'}(m+\mathscr{E}^0_{q'})(\omega\cos\theta-\omega'\cos\theta')(n-1-x)(e_x+\mathrm{i}\,e_y)e_z'^*$$

$$+\omega\sin\theta\,\mathrm{e}^{\mathrm{i}\psi}[(\omega\cos\theta-\omega'\cos\theta')\{(m+\mathscr{E}^0_{q'})(n-x)-\omega'\}$$

$$-\omega'\cos\theta'(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})]e_z(e_x'+\mathrm{i}\,e_y')^*$$

$$-\omega^2\sin^2\theta\,\mathrm{e}^{2\mathrm{i}\psi}(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})(e_x-\mathrm{i}\,e_y)(e_x'+\mathrm{i}\,e_y')^*$$

$$+(m+\mathscr{E}^0_{q'})\{\omega'(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})+\omega'\cos\theta'(\omega'\cos\theta'-\omega\cos\theta)\}$$

$$\times\,(n-1-x)(e_x+\mathrm{i}\,e_y)(e_x'+\mathrm{i}\,e_y')^*)\}\,,\qquad\text{(A1)}$$

$$b_2^-(n)=\frac{\exp\{-\mathrm{i}(xx')^{\frac{1}{2}}\sin(\psi-\psi')\}x'^{-\frac{1}{2}}\,\mathrm{e}^{-\mathrm{i}\psi'}}{2m\omega+\omega^2\sin^2\theta-2neB}\{(1/n!)(xx')^{\frac{1}{2}n}\,\mathrm{e}^{-\mathrm{i}n(\psi-\psi')}$$

$$\times\,([\{\omega\cos\theta(\omega\cos\theta-\omega'\cos\theta')+\omega(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})\}(m+\mathscr{E}^0_{q'})(n-x')-2neB(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})]e_z\,e_z'^*$$

$$+\omega'\sin\theta'\,\mathrm{e}^{\mathrm{i}\psi'}\{\omega\cos\theta(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})-\omega(\omega\cos\theta-\omega'\cos\theta')\}e_z(e_x'+\mathrm{i}\,e_y')^*)$$

$$+\{1/(n-1)!\}(xx')^{\frac{1}{2}(n-1)}\,\mathrm{e}^{-\mathrm{i}(n-1)(\psi-\psi')}$$

$$\times\,(\omega\sin\theta\,\mathrm{e}^{-\mathrm{i}\psi}(m+\mathscr{E}^0_{q'})(\omega\cos\theta-\omega'\cos\theta')(n-1-x')e_z(e_x'-\mathrm{i}\,e_y')^*$$

$$+\omega'\sin\theta'\,\mathrm{e}^{\mathrm{i}\psi'}[\omega\cos\theta(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})+\{(m+\mathscr{E}^0_{q'})(n-x')+\omega\}$$

$$\times\,(\omega\cos\theta-\omega'\cos\theta')](e_x-\mathrm{i}\,e_y)e_z'^*$$

$$+\omega'^2\sin^2\theta'\,\mathrm{e}^{2\mathrm{i}\psi'}(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})(e_x-\mathrm{i}\,e_y)(e_x'+\mathrm{i}\,e_y')^*$$

$$+(m+\mathscr{E}^0_{q'})(n-1-x')\{\omega(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})-\omega\cos\theta(\omega\cos\theta-\omega'\cos\theta')\}$$

$$\times\,(e_x-\mathrm{i}\,e_y)(e_x'-\mathrm{i}\,e_y')^*)\}\,,\qquad\text{(A2)}$$

$$b_1^+(n)=-\frac{\exp\{-\mathrm{i}(xx')^{\frac{1}{2}}\sin(\psi-\psi')\}}{2m\omega'-\omega'^2\sin^2\theta+2neB}\,\frac{x^{-\frac{1}{2}}\,\mathrm{e}^{-\mathrm{i}\psi}}{(2eB)^{\frac{1}{2}}}\{(1/n!)(xx')^{\frac{1}{2}n}\,\mathrm{e}^{\mathrm{i}n(\psi-\psi')}$$

$$\times\,(2eB[\{n(m+\mathscr{E}^0_{q'})-\omega'(n-x)\}(\omega'\cos\theta'-\omega\cos\theta)+\omega'\cos\theta'(n-x)(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})]e_z\,e_z'^*$$

$$-\omega\sin\theta(m+\mathscr{E}^0_{q'})\{\omega'(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})+\omega'\cos\theta'(\omega'\cos\theta'-\omega\cos\theta)\}\,(e_x-\mathrm{i}\,e_y)e_z'^*)$$

$$+\{1/(n-1)!\}(xx')^{\frac{1}{2}(n-1)}\,\mathrm{e}^{\mathrm{i}(n-1)(\psi-\psi')}$$

$$\times\,(\omega'\sin\theta'\,\mathrm{e}^{-\mathrm{i}\psi'}2eB(n-1-x)(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})(e_x+\mathrm{i}\,e_y)e_z'^*$$

$$+\omega\sin\theta'\mathrm{e}^{\mathrm{i}\psi}[(m+\mathscr{E}^0_{q'})\{\omega'(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})+\omega'\cos\theta'(\omega\cos\theta-\omega'\cos\theta')\}$$

$$+2eB(n-x)(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})]e_z(e_x'+\mathrm{i}\,e_y')^*$$

$$+\omega^2\sin^2\theta\,\mathrm{e}^{2\mathrm{i}\psi}(m+\mathscr{E}^0_{q'})(\omega\cos\theta-\omega'\cos\theta')(e_x-\mathrm{i}\,e_y)(e_x'+\mathrm{i}\,e_y')^*$$

$$+2eB(n-1-x)\{\omega'(\omega\cos\theta-\omega'\cos\theta')-\omega'\cos\theta'(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})\}$$

$$\times\,(e_x+\mathrm{i}\,e_y)(e_x'+\mathrm{i}\,e_y')^*)\}\,,\qquad\text{(A3)}$$

$$
\begin{aligned}
b_2^+(n) = &-\frac{\exp\{\mathrm{i}(xx')^{\frac{1}{2}}\sin(\psi-\psi')\}}{2m\omega+\omega^2\sin^2\theta-2neB}\,\frac{x'^{-\frac{1}{2}}\,\mathrm{e}^{-\mathrm{i}\psi'}}{(2eB)^{\frac{1}{2}}}\{(1/n!)(xx')^{\frac{1}{2}n}\,\mathrm{e}^{-\mathrm{i}n(\psi-\psi')}\\
&\times\big(2eB[\{n(m+\mathscr{E}^0_{q'})+\omega(n-x')\}(\omega'\cos\theta'-\omega\cos\theta)-\omega\cos\theta(n-x')(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})]e_z e_z'^*\\
&-\omega'\sin\theta'\,\mathrm{e}^{\mathrm{i}\psi'}(m+\mathscr{E}^0_{q'})\{\omega(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})+\omega\cos\theta(\omega'\cos\theta'-\omega\cos\theta)\}\,e_z(e_x'+\mathrm{i}\,e_y')^*\big)\\
&+\{1/(n-1)!\}(xx')^{\frac{1}{2}(n-1)}\,\mathrm{e}^{-\mathrm{i}(n-1)(\psi-\psi')}\\
&\times\big(-\omega\sin\theta\,\mathrm{e}^{-\mathrm{i}\psi}2eB(n-1-x')(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})e_z(e_x'-\mathrm{i}\,e_y')^*\\
&+\omega'\sin\theta'\,\mathrm{e}^{\mathrm{i}\psi'}[\{\omega(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})+\omega\cos\theta(\omega\cos\theta-\omega'\cos\theta')\}(m+\mathscr{E}^0_{q'})\\
&\qquad\qquad-2eB(n-x')(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})](e_x-\mathrm{i}\,e_y)e_z'^*\\
&+\omega'^2\sin^2\theta'\,\mathrm{e}^{2\mathrm{i}\psi'}(m+\mathscr{E}^0_{q'})(\omega\cos\theta-\omega'\cos\theta')(e_x-\mathrm{i}\,e_y)(e_x'+\mathrm{i}\,e_y')^*\\
&+2eB(n-1-x')\{\omega(\omega'\cos\theta'-\omega\cos\theta)+\omega\cos\theta(\mathscr{E}_{q'}+\mathscr{E}^0_{q'})\}\\
&\qquad\qquad\times(e_x-\mathrm{i}\,e_y)(e_x'-\mathrm{i}\,e_y')^*\big)\}\,. \qquad (\mathrm{A4})
\end{aligned}
$$

Appendix 2. Double Absorption

We calculate the rate of double absorption by an electron in its ground state at rest, to its first excited state with spin down ($\sigma = -1$) and spin up ($\sigma = +1$). We assume that the photon frequencies are close to the cyclotron frequency:

$$w = \tfrac{1}{2}\Omega_{\mathrm{e}}(1+\delta)\,, \qquad \omega' = \tfrac{1}{2}\Omega_{\mathrm{e}}(1-\delta)\,, \qquad (\mathrm{A5a,b})$$

with $\delta \ll 1$. Then to lowest order in the fine structure constant α, B/B_{c} and δ, we have, after averaging over photon polarizations,

$$w^{\sigma}_{1'0}(\boldsymbol{k},\boldsymbol{k}')\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{k}' = \alpha^2 Z^{\sigma}\delta(E_{\mathrm{i}}-E_{\mathrm{f}})\,\mathrm{d}\omega\,\mathrm{d}\omega'\,\mathrm{d}\theta\,\mathrm{d}\theta'\,\mathrm{d}\psi\,\mathrm{d}\psi'\,N(\boldsymbol{k})N(\boldsymbol{k}')\,, \qquad (\mathrm{A6})$$

with

$$
\begin{aligned}
Z^- = \left(\frac{B}{B_{\mathrm{c}}}\right)^3\frac{(2\pi)^3}{512}&\sin\theta\sin\theta'\{\tfrac{1}{9}[472(\sin^2\theta+\sin^2\theta')-760\sin^2\theta\sin^2\theta'\\
&-20(\sin^4\theta+\sin^4\theta')+46\sin^2\theta\sin^2\theta'(\sin^2\theta+\sin^2\theta')\\
&+72\cos\theta\cos\theta'\{2(\sin^2\theta+\sin^2\theta')-\sin^2\theta\sin^2\theta'\}]\\
&+\tfrac{1}{27}\delta(\sin^2\theta-\sin^2\theta')\{-3424-16(\sin^2\theta+\sin^2\theta')\\
&\qquad+440\sin^2\theta\sin^2\theta'-864\cos\theta\cos\theta'\}\\
&+\sin\theta\sin\theta'\cos(\psi-\psi')\big(\tfrac{1}{3}[208-72(\sin^2\theta+\sin^2\theta')+26\sin^2\theta\sin^2\theta'\\
&\qquad+\cos\theta\cos\theta'\{160-8(\sin^2\theta+\sin^2\theta')\}]\\
&+\tfrac{1}{9}\delta(\sin^2\theta-\sin^2\theta')(-32-32\cos\theta\cos\theta')\big)+
\end{aligned}
$$

$$+\sin^2\theta\sin^2\theta'\cos 2(\psi-\psi')[\tfrac{1}{3}\{16-2(\sin^2\theta+\sin^2\theta')+8\cos\theta\cos\theta'\}$$

$$-\tfrac{8}{9}\delta(\sin^2\theta-\sin^2\theta')]+\tfrac{2}{9}\sin^3\theta\sin^3\theta'\cos 3(\psi-\psi')\}, \tag{A7}$$

and

$$Z^+ = \left(\frac{B}{B_c}\right)^4\frac{(2\pi)^3}{1024}\sin\theta\sin\theta'\{\tfrac{1}{9}[544(\sin^2\theta+\sin^2\theta')-904\sin^2\theta\sin^2\theta'$$

$$-20(\sin^4\theta+\sin^4\theta')+46\sin^2\theta\sin^2\theta'(\sin^2\theta+\sin^2\theta')$$

$$+72\cos\theta\cos\theta'(\sin^2\theta+\sin^2\theta')$$

$$+\tfrac{1}{27}\delta(\sin^2\theta-\sin^2\theta')\{-4048-136(\sin^2\theta+\sin^2\theta')+716\sin^2\theta\sin^2\theta'\}$$

$$+\sin\theta\sin\theta'\cos(\psi-\psi')\big(\tfrac{1}{3}[160-40(\sin^2\theta+\sin^2\theta')$$

$$+\cos\theta\cos\theta'\{208-8(\sin^2\theta+\sin^2\theta')\}]$$

$$+\tfrac{1}{9}\delta(\sin^2\theta-\sin^2\theta')(-376-80\cos\theta\cos\theta')\big)$$

$$+\sin^2\theta\sin^2\theta'\cos 2(\psi-\psi')[\tfrac{1}{3}\{16-2(\sin^2\theta+\sin^2\theta')\}$$

$$-\tfrac{20}{9}\delta(\sin^2\theta-\sin^2\theta')]\}. \tag{A8}$$

On assuming isotropic distributions of photons and carrying out the integrals over angles and over frequencies from $\delta = -\Delta\omega/\Omega_e$ to $\delta = +\Delta\omega/\Omega_e$, we obtain the result (80).

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