There always exists mass-zero branch

The gauge invariance implies the 4-transversality of the polarization tensor both in the vacuum and in a medium

$$\Pi_{\alpha\beta}(k)k_\beta = 0. \tag{1}$$

By differentiating it over k_γ we get

$$\Pi_{\alpha\gamma}(k) + k_\beta \frac{\partial \Pi_{\alpha\beta}(k)}{\partial k_\gamma} = 0. \tag{2}$$

Setting all components $k_\beta = 0$ we obtain

$$\Pi_{\alpha\gamma}(0) = 0, \tag{3}$$

provided that there is no singularity: $\lim_{k\to 0} \frac{\partial \Pi_{\alpha\beta}(k)}{\partial k_\gamma} < \lim_{k\to 0} \frac{1}{k}$. This option is not excluded, but it is very exotic and anyway must be subjected to verification.

Once all components of $\Pi_{\alpha\gamma}(0)$ are zero, so are all its eigenvalues, $\varkappa_i(0) = 0$. Hence for every mode the dispersion relation $k^2 = \varkappa_i(k)$ has the point $k_\alpha = 0$ as its solution, in other words there always exists a dispersion curve that passes through the origin in every mode. The abovesaid does not exclude that there may be other branches that come to the point $\mathbf{k} = 0$, but with $k_0 \neq 0$, because in this case from (2) it only follows that $\Pi_{\alpha\gamma}(k_0, 0) + k_0 \lim_{\mathbf{k}\to 0} \frac{\partial \Pi_{\alpha 0}(k)}{\partial k_\gamma} = 0$, then $\Pi_{\alpha\gamma}(k_0, 0)$ may be nonzero, for instance $\Pi_{\alpha 0}(k_0, 0) = -k_0 \frac{\partial \Pi_{\alpha 0}(k_0, 0)}{\partial k_0} \neq 0$. This is just the case with positronium: its dispersion curve comes to a finite mass value at $\mathbf{k} = 0$.