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STRONG-FIELD CYCLOTRON SCATTERING. I. SCATTERING AMPLITUDES AND NATURAL LINE WIDTH

CARLO GRAZIANI[1,2]
Departments of Physics and Astronomy and Astrophysics, University of Chicago
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ABSTRACT

The introduction of resonance line width into the QED cyclotron scattering amplitudes is considered. It is shown that the width arises from loop corrections to the electron propagator, which also bring about shifts in the Landau energy levels. A formalism is developed that allows the dressed electron propagator to be derived. It is shown that the states of Herold, Ruder, & Wunner, and of Sokolov & Ternov, which diagonalize the component of the magnetic moment operator parallel to the external magnetic field, are appropriate for calculation of the scattering amplitudes, whereas the states of Johnson & Lippmann are not. In addition, it is shown that the Breit-Wigner broadening approximation $E \to E - i\Gamma/2$ is consistent with the perturbation-theoretic order of the calculation, if the former basis states are chosen, but not the latter. It is further shown that the decay rates obey the expected "time dilation" scaling relation $\Gamma_{\rm Lab} = \gamma^{-1}\Gamma_{\rm Rest}$ if the states of Herold et al. or of Sokolov & Ternov are chosen, but that they do not if the states of Johnson & Lippmann are used.

Subject headings: line: formation — pulsars: general — radiation mechanisms: cyclotron and synchrotron

1. INTRODUCTION

The physics of cyclotron scattering has been of great astrophysical interest since the observation of cyclotron lines in the spectra of accretion-powered X-ray pulsars. Such lines were first seen in spectral data from the pulsar Her X-1 (Trümper et al. 1978), and from 4U 0115+63 (Wheaton et al. 1979), and subsequently in several other pulsars by the *Ginga* Large Area Counter (Makishima 1991). Cyclotron scattering is also important in gamma-ray bursts. Early reports of lines in bursts detected by *Konus* (Mazets et al. 1981) have been confirmed by observations made by the *Ginga* gamma-ray burst detector (Murakami et al. 1988).

The magnetic field strengths deduced from cyclotron line centroid energies lie in the range $0.5–5 \times 10^{12}$G for accretion-powered pulsars, and $2–4 \times 10^{12}$G for gamma-ray bursts (Lamb 1991). There is evidence from spin behavior of many accretion-powered pulsars for field strengths of $1–5 \times 10^{12}$G (Lamb 1991), or even as large as $2–5 \times 10^{13}$G (Parmar et al. 1989; Lamb 1988). Such field strengths are an appreciable fraction of the critical field strength $B_c \equiv m^2c^3/e\hbar = 4.414 \times 10^{13}$ G which heralds the necessity of a fully relativistic treatment of the scattering process. Indeed, there is no known reason that surface field strengths comparable to or greater than B_c may not occur in neutron stars. Consequently, the quantum electrodynamic (QED) treatment of cyclotron scattering can be important in the interpretation of lines in the spectra of accretion-powered pulsars and gamma-ray bursts.

There has been some investigation of strong-field cyclotron scattering. Herold (1979) gave leading-order perturbation theory expressions for scattering amplitudes and cross sections in the case where both the initial and final electron states are the Landau ground state. Daugherty & Harding (1986) generalized this result to include excited final states. These leading-order calculations make no provision for the natural line width of the resonances, and as a consequence the scattering amplitudes diverge at the resonances. This is a serious difficulty in using these scattering amplitudes in the analysis of cyclotron resonant scattering phenomena: the QED scattering cross sections are known to good accuracy everywhere except where they are most interesting, where they are not known at all.

Harding & Daugherty (1991) attempted to remedy this defect by introducing the Breit-Wigner line shape "by hand," using the broadening prescription

$$E \to E - i\Gamma/2 \,, \tag{1.1}$$

where Γ is the decay rate of the intermediate excited state in the amplitudes of Daugherty & Harding (1986). Bussard, Alexander, & Mészáros (1986) had earlier used equation (1.1) to introduce the natural line width into the electron propagator. This complexification of the energy is motivated by the analogous result in the theory of a discrete state coupled to a continuum in nonrelativistic quantum mechanics (Weisskopf & Wigner 1930; or see Gasiorowicz 1974; pp. 473–480).

While this prescription is physically well-motivated, it is objectionable on philosophical grounds. The complexification of the energy described by equation (1.1) should be derived from QED, rather than imposed ad hoc. As I show in this work, this may be done by considering loop corrections to the electron propagator. That this makes sense may be seen qualitatively as follows: Weisskopf & Wigner (1930) found that the complex correction to the energy of the discrete state includes not only an imaginary width, but also a real self-energy term. In free QED, the self-energy of the electron is computed by considering loop corrections to the electron propagator. The matrix elements of the mass operator are purely real in that case, so the electron mass suffers only a real correction. It might be expected, then, that in the case where an external magnetic field is present, the real correction to the

[1] Mailing address: Department of Astronomy and Astrophysics, University of Chicago, 5640 S. Ellis Avenue, Chicago, IL 60637.
[2] Presented as a thesis to the Department of Physics, The University of Chicago, in partial fulfillment of the requirements for the Ph.D. degree.

electron energy induced by loop corrections to the propagator will be accompanied by an imaginary correction which will comprise the lifetime of the state. This was pointed out by Constantinescu (1972).

A second, practical objection to the broadening prescription of Harding & Daugherty (1991) is that it is incomplete. The solutions of the Dirac equation in a constant external magnetic field have a twofold degeneracy associated with the spin of the electron. The choice of basis states which span this degenerate subspace is thus to some extent arbitrary (Melrose & Parle 1983). This arbitrariness is irrelevant in the case of the zero line width scattering amplitudes of Daugherty & Harding (1986) because there the electron is initially in the ground state (which has no spin degeneracy), while a summation over spins is applied to the intermediate and final states. When finite line width is introduced, however, different states have different decay rates, and thus different resonant energy denominators. In this case, the result of the summation over intermediate spin states is no longer insensitive to the choice of basis states, and it matters a great deal what basis is used.

The basis states of Johnson & Lippmann (1949) have gained some currency in the literature and were used in particular by Constantinescu (1972) and by Harding & Daugherty (1991). Unfortunately, this basis is not a likely candidate to resolve the ambiguity caused by spin degeneracy, as the states of Johnson & Lippmann (1949) are not eigenstates of any physically meaningful spin operator (Melrose & Parle 1983). Sokolov & Ternov (1968) derived a basis of states by diagonalizing the component of the magnetic moment operator parallel to the external magnetic field, which they showed commuted with both the Dirac Hamiltonian and the radiative correction to the Dirac equation postulated by Pauli (1941). Herold, Ruder, & Wunner (1982) obtained a basis of states differing from those derived by Sokolov & Ternov (1968) only in the choice of gauge and of coordinates—Herold et al. (1982) used Cartesian coordinates, while Sokolov & Ternov (1968) used cylindrical coordinates. For a lucid discussion of the solutions of the Landau-Dirac equation, see Melrose & Parle (1983).

In this paper I show that in computing electron-photon scattering amplitudes in a constant external magnetic field, it is *essential* to choose a basis of states which diagonalize the component of the magnetic moment operator parallel to the external magnetic field. Thus, the states of Sokolov & Ternov (1968) or of Herold et al. (1982) are appropriate, while the Johnson & Lippmann (1949) states are not. I also show that when the correct basis states are used, the Breit-Wigner broadening approximation (eq. [1.1]) is consistent with the perturbation-theoretic order of the calculation.

In § 2 the radiative corrections to the Landau-Dirac equation (that is, the Dirac equation in a constant, external magnetic field) are discussed, as well as the inverse relation between the modified Landau-Dirac equation and the dressed propagator. In § 3 a technique for inverting wave operators is outlined and applied as an example to the Landau-Dirac operator. In § 4 the inversion technique is applied to the modified Landau-Dirac operator, and an expression for the dressed propagator is derived. It is shown that the states of Herold et al. (1982), and of Sokolov & Ternov (1968), are appropriate for calculation of the scattering amplitudes, whereas the states of Johnson & Lippmann (1949) are not. In § 5 the scattering amplitudes are considered, and it is shown that the Breit-Wigner broadening approximation $E \to E - i\Gamma/2$ is consistent with the perturbation-theoretic order of the calculation. It is further shown that the decay rates obey the expected "time dilation" scaling relation $\Gamma_{\rm Lab} = \gamma^{-1}\Gamma_{\rm Rest}$ if the states of Herold et al. (1982) or of Sokolov & Ternov (1968) are chosen, but do not if the states of Johnson & Lippmann (1949) are used.

In a future paper, the scattering amplitudes, decay rates, and cross sections will be computed, and the cross sections will be compared with the cross sections obtained by Daugherty & Harding (1991) using the states of Johnson & Lippmann (1949). In a second, future paper, thermally averaged profiles will be exhibited.

2. THE DRESSED ELECTRON PROPAGATOR

Throughout this work, $\hbar$ and c are set equal to 1. The Dirac equation in a constant external magnetic field is $D\Phi_A^{(\xi)} = 0$, where D is the Landau-Dirac operator

$$D \equiv i\not{\partial} - e\not{A} - m \,. \tag{2.1}$$

In the expression $\Phi_A^{(\xi)}$, the subscript A refers to the set of quantum numbers that characterize the state, while the superscript $\xi = \pm 1$ specifies whether the state is a positive or negative energy solution of the Dirac equation. The choices of gauge, states, and state normalizations are given in Appendix A. The bare electron propagator is

$$G_F(x, y) = \sum_A \sum_{\xi = \pm 1} \int \frac{dp^0}{2\pi} e^{-ip^0(x^0 - y^0)} \frac{\phi_A^{(\xi)}(\boldsymbol{x})\overline{\phi_A^{(\xi)}}(\boldsymbol{y})}{p^0 - \xi(E_A - i0)} \tag{2.2}$$

(Bjorken & Drell 1964, eq. [6.48], where the θ functions have been replaced by their Fourier representation). It is related to D by $G_F = (D + i0)^{-1}$. In this expression, G_F is viewed as an operator, whose action on a spinor Ψ is given by $(G_F \cdot \Psi)(x) \equiv \int d^4y G_F(x, y)\Psi(y)$.

The mass operator $\Sigma(x, y)$ is defined as the sum of the amplitudes corresponding to all one-particle irreducible Feynman diagrams (i.e., those diagrams that cannot be severed into two unconnected diagrams by the removal of a single line) with only one incoming and one outgoing external electron lines. The first few such diagrams are depicted in Figure 1. To leading order in e^2,

$$\Sigma(x, y) = ie^2 D_F^{\mu\nu}(x - y)\gamma_\mu G_F(x, y)\gamma_\nu \,, \tag{2.3}$$

where $D_F^{\mu\nu}(x - y)$ is the photon propagator. As is well known, this expression for Σ suffers from ultraviolet divergences. Technically, the integrals appearing in the propagators in equation (2.3) should be regularized by some kind of cutoff, which should be removed in a suitably delicate manner at the end of the calculation. Alternatively, following the methods of Epstein & Glaser (1973), the distributions which are used to construct Σ could be combined in such a way that no divergences appear in the first place (see Scharf 1989). Fortunately, renormalization is not central to the purpose at hand. I am chiefly concerned with the imaginary part of the matrix elements of Σ, which is responsible for the appearance of natural line width, whereas the divergences occur in the real part of

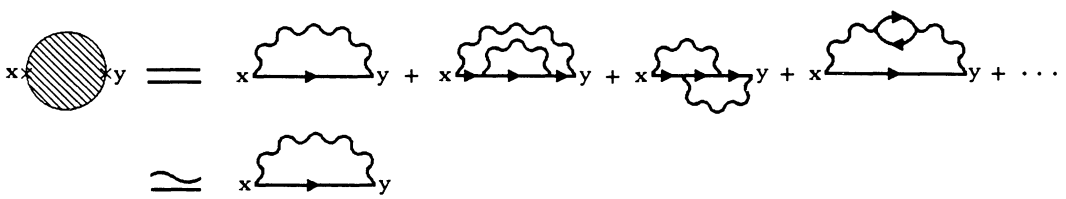

FIG. 1.—Feynman diagrams representing contributions to the mass operator $\Sigma(x, y)$. Σ is represented by the hatched circle. To leading order in α, Σ is given by eq. (2.3). This is represented by the second line in the "graphical equation" of the figure.

the matrix elements. I will therefore adopt equation (2.3) as the definition of the (leading-order) mass operator, with due caution with respect to issues of renormalization.

The dressed electron propagator $G_{\rm Tot}(x, y)$ is represented by the Feynman diagrams shown in Figure 2. It satisfies the operator equation

$$G_{\rm Tot} = G_F + G_F \cdot \Sigma \cdot G_{\rm Tot} , \tag{2.4}$$

which may be solved straightforwardly for $G_{\rm Tot}$:

$$\begin{aligned} G_{\rm Tot} &= (1 - G_F \cdot \Sigma)^{-1} \cdot G_F \\ &= (D + i0 - \Sigma)^{-1} . \end{aligned} \tag{2.5}$$

That is, the dressed propagator is the inverse of the modified Landau-Dirac operator

$$D_M \equiv D + i0 - \Sigma . \tag{2.6}$$

3. OPERATOR INVERSION TECHNIQUE

The inversion of operators such as D_M is not as simple as it is in the case when the external field is absent. In the latter case, the translation invariance of the system implies that operators such as $G_{\rm Tot}(x, y)$ actually depend on x and y only through their difference $x - y$. As a result, the problem may be transformed to a four (spinor)-dimensional matrix inversion by a Fourier transformation. When an external field is present, however, translational invariance is lost. If the field is constant, the Landau-Dirac operator D is invariant under a combination of translation and gauge transformation, but this is not sufficient to recover the utility of the Fourier transformation technique.

The inversion technique adopted here rests upon the following simple observation: if W is an operator in a Hilbert space and is self-adjoint with respect to the inner product on the Hilbert space, then it has a spectral decomposition

$$W = \sum_\lambda \lambda P_\lambda \tag{3.1}$$

where the λ are the eigenvalues of W, and P_λ is the projection operator onto the eigensubspace of λ. If none of the λ are zero, then we have

$$W^{-1} = \sum_\lambda \frac{1}{\lambda} P_\lambda . \tag{3.2}$$

When applying these simple ideas to the inversion of wave operators, it should be borne in mind that the operators of interest live in spaces of functions of the four-dimensional spacetime, rather than in spaces of functions of three-dimensional space such as the usual spaces of quantum states. As a consequence, the integrals that appear in inner products and norms are four-dimensional (as opposed to the more familiar three-dimensional integrals that appear in ordinary state normalization statements, such as the ones in Appendix A).

Before tackling the inversion of D_M, it will be helpful to consider a simpler example: the inversion of the unmodified Landau-Dirac operator, D. In order to carry out the program outlined above, it is necessary to specify a Hilbert space, equipped with a positive-definite inner product. This is easily done: the Hilbert space is $[L^2(\boldsymbol{R}^4)]^4$, the space of square-integrable spinorial functions

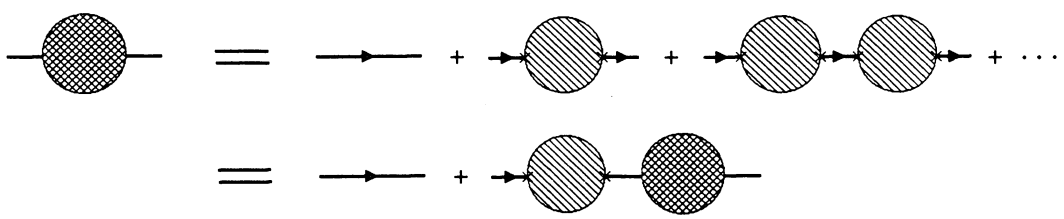

FIG. 2.—Feynman diagrams representing contributions to the dressed electron propagator. The propagator is represented by the cross-hatched circle connected to two electron lines. The first line of the "graphical equation" of the figure represents the operator equation $G_{\rm Tot} = G_F + G_F \cdot \Sigma \cdot G_F + G_F \cdot \Sigma \cdot G_F \cdot \Sigma \cdot G_F + \cdots$. The second line represents the equivalent eq. (2.4).

of four-dimensional spacetime. The positive-definite inner product is

$$(\Psi_1, \Psi_2) = \int d^4x \Psi_1(x)^\dagger \Psi_2(x) . \tag{3.3}$$

Unfortunately, D is not self-adjoint with respect to this inner product. D would be self-adjoint if $\Psi^\dagger$ were replaced by $\overline{\Psi}$ in equation (3.3), but this would not yield a positive-definite inner product. The problem is resolved by modifying the Dirac operator. Consider the operator

$$\begin{aligned} D' &\equiv \gamma^0 D \\ &= i\partial_0 - \boldsymbol{\pi}\cdot\boldsymbol{\alpha} - m\beta \\ &= i\partial_0 - H , \end{aligned} \tag{3.4}$$

where $\alpha_i = \gamma^0\gamma^i$, $\beta = \gamma^0$, $\boldsymbol{\pi} = -i\nabla - e\boldsymbol{A}$, and H is the Landau-Dirac Hamiltonian. D' is self-adjoint with respect to the positive-definite inner product of equation (3.3). In order to find $(D + i0)^{-1}$ observe that

$$\begin{aligned} (D + i0)^{-1} &= (\gamma^0 D' + i0)^{-1} \\ &= (D' + \gamma^0 i0)^{-1}\gamma^0 . \end{aligned} \tag{3.5}$$

It is clear from equations (3.5) and (3.4) that in order to compute $(D + i0)^{-1}$, it is necessary to find D'^{-1}, right-multiply it by γ^0, and let $m \to m - i0$ in the final result to recover the Feynman boundary conditions. Now, in the spirit of equations (3.1) and (3.2), consider the spectral structure of D'. The eigenvector equation

$$D'\Psi^{(\xi)}_{A\lambda} = \lambda\Psi^{(\xi)}_{A\lambda} , \tag{3.6}$$

may be solved by separating the x^0 dependence in $\Psi^{(\xi)}_{A\lambda}$:

$$\Psi^{(\xi)}_{A\lambda}(x) = e^{-ip^0x^0}\phi^{(\xi)}_A(\boldsymbol{x}) , \tag{3.7}$$

where as explained in Appendix A, $A \equiv (N_A, \sigma_A, p_A, a_A)$, and $\phi^{(\xi)}_A(\boldsymbol{x})$ is an eigenfunction of H with eigenvalue ξE_A. Inserting equation (3.7) into equation (3.6), the result is

$$p^0 = \lambda + \xi E_A . \tag{3.8}$$

ξ and E_A are defined in Appendix A. As a consequence of the normalization of the $\phi^{(\xi)}_A$ (stated in Appendix A), the $\Psi^{(\xi)}_{A\lambda}$ have the following normalization

$$(\Psi^{(\xi)}_{A\lambda}, \Psi^{(\xi')}_{A'\lambda'}) = (2\pi)\delta(\lambda - \lambda')\delta_{\xi\xi'}\delta_{AA'} . \tag{3.9}$$

The meaning of the symbol $\delta_{AA'}$ is explained in Appendix A. Acceptable plane wave states are obtained for all λ in the range $-\infty < \lambda < \infty$. The spectral representation of D' is thus

$$D'(x, y) = \sum_A \sum_{\xi=\pm1} \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} \lambda e^{-i(\lambda+\xi E_A)(x^0-y^0)} \phi^{(\xi)}_A(\boldsymbol{x})\phi^{(\xi)}_A(\boldsymbol{y})^\dagger , \tag{3.10}$$

and D'^{-1} is

$$\begin{aligned} D'^{-1}(x, y) &= \sum_A \sum_{\xi=\pm1} \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi}\frac{1}{\lambda} e^{-i(\lambda+\xi E_A)(x^0-y^0)} \phi^{(\xi)}_A(\boldsymbol{x})\phi^{(\xi)}_A(\boldsymbol{y})^\dagger \\ &= \sum_A \sum_{\xi=\pm1} \int_{-\infty}^{+\infty} \frac{dp^0}{2\pi} e^{-ip^0(x^0-y^0)} \frac{\phi^{(\xi)}_A(\boldsymbol{x})\phi^{(\xi)}_A(\boldsymbol{y})^\dagger}{p^0 - \xi E_A} . \end{aligned} \tag{3.11}$$

After right-multiplication by γ^0 and replacement of m by $m - i0$, the expression that results for $G_F(x, y)$ is the one given in equation (2.2). It is instructive (and amusing) to repeat the above reasoning with the free Dirac operator $D_{\text{Free}} \equiv i\not{\partial} - m$ and with the Klein-Gordon operator $K \equiv \Box + m^2$: the resulting inverses are the well-known free electron propagator and the Feynman propagator, respectively.

4. INVERSION OF THE MODIFIED LANDAU-DIRAC OPERATOR

The inversion begins by premultiplying the modified Landau-Dirac operator by γ^0 as was done in § 3. Define

$$D'_M \equiv \gamma^0 D - \gamma^0\Sigma \equiv D' - \Sigma' . \tag{4.1}$$

D'_M must now be diagonalized. For this purpose it is convenient to use the Parle representation (Parle 1987). The essential features of the Parle representation are summarized in Appendix A. The Parle representation of D' is

$$\tilde{D}'(N, p, a, p^0; N', p', a', p^{0'}) = (p^0 - p_N\alpha_1 - p\alpha_3 - m\beta)\delta_{NN'}(2\pi)\delta(p^0 - p^{0'})\left(\frac{2\pi}{L}\right)\delta(p - p')\left(\frac{2\pi}{Lm\omega_B}\right)\delta(a - a') \tag{4.2}$$

(compare eqs. [3.4], [A13], and [A15]). Here, $p_N \equiv (2Nm\omega_B)^{1/2}$. The diagonal character of $\tilde{D}'$ is nothing more than a reflection of the fact that the Parle representation is constructed from states that diagonalize D'.

Parle (1987) established that the Parle representation of Σ' has the form

$$\tilde{\Sigma}'(N, p, a, p^0; N', p', a', p^{0'}) = \hat{\Sigma}'(N, p, p^0)\delta_{NN'}(2\pi)\delta(p^0 - p^{0'})\left(\frac{2\pi}{L}\right)\delta(p - p')\left(\frac{2\pi}{Lm\omega_B}\right)\delta(a - a') . \tag{4.3}$$

The δ-function structure of $\tilde{\Sigma}'$ reflects the invariance of Σ' under x^0, x^3, and x^2 translations (the last a consequence of the choice of gauge). The diagonality with respect to N is less easy to prove. Parle (1987) did so by explicit computation of the azimuthal integration that appears when calculating $\tilde{\Sigma}'$.

Another way of understanding the diagonality with respect to N is as a consequence of the symmetries of the problem under rotations and translations in the $x^1 - x^2$ plane. As I pointed out in § 3, the Landau-Dirac operator is not invariant under simple coordinate rotations and translations in the $x^1 - x^2$ plane, but rather under a combination of such transformations and gauge transformations. The generators of these "physical" rotations and translation are easily found by the usual methods (see, for example, Bjorken & Drell 1964, § 2.2). The generators of physical translations in the x^i direction ($i = 1, 2$) are

$$\xi_1 \equiv -i\frac{\partial}{\partial x^1} + m\omega_B x^2 , \tag{4.4}$$

$$\xi_2 \equiv -i\frac{\partial}{\partial x^2} . \tag{4.5}$$

ξ_1 and ξ_2 are proportional to the operators corresponding to the x^2 and x^1 coordinates of the orbit center, respectively. The generator of physical rotations in the $x^1 - x^2$ plane is

$$J \equiv \frac{1}{2}\sigma_{12} - m\omega_B[(x^2)^2 - (x^1)^2] + x^1\left(-i\frac{\partial}{\partial x^2}\right) - x^2\left(-i\frac{\partial}{\partial x^1}\right), \tag{4.6}$$

where $\sigma_{12} \equiv \mathbf{1}_2 \otimes \sigma_3$, $\mathbf{1}_2$ denotes the two-dimensional unit matrix, σ_3 is the third Pauli matrix, and $\otimes$ denotes the tensor product.

It is straightforward to establish that the ξ_i and J commute with D and with G_F. It is somewhat more laborious (but still straightforward) to prove that they also commute with Σ, and therefore with Σ'. Now, there is a useful identity relating the ξ_i and J to the operator

$$\begin{aligned} H_\perp^2 &\equiv (\boldsymbol{\pi}_\perp \cdot \boldsymbol{\alpha})^2 \\ &= \pi_1^2 + \pi_2^2 + m\omega_B\sigma_{12} , \end{aligned} \tag{4.7}$$

whose eigenvalues are $2Nm\omega_B$. The identity is

$$H_\perp^2 = 2m\omega_B J + \xi_1^2 + \xi_2^2 . \tag{4.8}$$

Since ξ_1, ξ_2, and J all commute with Σ', it follows that

$$[H_\perp, \Sigma'] = 0 , \tag{4.9}$$

so that Σ' is indeed diagonal in N, as asserted.

Parle (1987) showed that the matrix $\hat{\Sigma}'(N, p, p^0)$ in equation (4.3) has the general form

$$\hat{\Sigma}'(N, p, p^0) = c_1 m\beta + c_2 p_N\alpha_1 + c_3(p\alpha_3 - p^0) + c_4 m\beta\sigma_{12} + c_5\sigma_{12}(p\alpha_3 - p^0) , \tag{4.10}$$

where the c_i are functions of N, p, and p^0. This expression differs from that of Parle (1987) in the term proportional to $p_N\alpha_1$—Parle obtained $p_N\alpha_2$ instead. The difference is due to the different choice of phase of the electron wavefunctions, as explained in Appendix A.

The c_i are formally infinite if the expression for Σ given in equation (2.3) is used. In fact, their calculation involves regularizing the divergences in equation (2.3), subtracting real counterterms of the form $(am + bD)\delta^4(x - y)$ from the regularized expression for $\Sigma(x, y)$, and choosing the real constants a and b in such a way that certain physical normalization conditions will be satisfied after the regularization is removed. Note that as a consequence of the form of the counterterms, the commutation properties of Σ' with the ξ_i, J, and $H_\perp$ are unaltered by the renormalization. The resulting c_i are finite, dimensionless numbers, having the order of magnitude α, the fine-structure constant.

It is now clear from equations (4.1), (4.2), (4.3), and (4.10) that the diagonalization of D'_M is equivalent to the diagonalization of the matrix

$$\begin{aligned} \hat{D}'_M(N, p, p^0) &\equiv (1 + c_3 + c_5\sigma_{12})(p^0 - p\alpha_3) - (1 + c_2)p_N\alpha_1 - (1 + c_1)m\beta - c_4 m\beta\sigma_{12} \\ &= (1 + c_3)[(1 + b_5\sigma_{12})(p^0 - p\alpha_3) - (1 + b_2)p_N\alpha_1 - (1 + b_1)m\beta - b_4 m\beta\sigma_{12}] , \end{aligned} \tag{4.11}$$

where the b_i are all of the same order of magnitude as the c_i. Following Parle (1987), $\hat{D}'_M$ is transformed using the matrix

$$Y \equiv \text{Diag}\,[(1 + b_5)^{-1/2}, (1 - b_5)^{-1/2}, (1 + b_5)^{-1/2}, (1 - b_5)^{-1/2}] . \tag{4.12}$$

Thus,

$$\hat{D}'_M(N, p, p^0) = (1 + c_3)Y^{-1}ZY^{-1}$$
$$Z = p^0 - p\alpha_3 - p'_N\alpha_1 - m'\beta - \Delta\beta\sigma_{12}\,, \tag{4.13}$$

where

$$p'_N = p_N[1 + \mathcal{O}(\alpha)]; \quad m' = m[1 + \mathcal{O}(\alpha)]; \quad \Delta = m\mathcal{O}(\alpha)\,. \tag{4.14}$$

Z has four nondegenerate eigenvalues, and commutes with the operator

$$\rho \equiv m'\sigma_{12} - p'_N\sigma_2 \otimes \sigma_2\,, \tag{4.15}$$

which is just $\hat{\mu}_3$, the Parle representation of μ_3, the x^3-component of the magnetic moment operator (eq. [A18]), with the substitutions $m \to m'$, $p_N \to p'_N$. Thus, in the limit $\alpha \to 0$, the eigenvectors of Z tend to eigenvectors of $\hat{\mu}_3$. In general, the eigenvectors of Z differ from those of $\hat{\mu}_3$ only by terms of order α, as a consequence of equation (4.14). Since the eigenvectors of $\hat{D}'_M$ are obtained from those of Z by means of the linear transformation Y, and since Y differs from the unit matrix by terms of order α, it follows that the eigenvectors of $\hat{D}'_M$ also differ from those of $\hat{\mu}_3$ only by terms of order α. Therefore, to leading order in α, *the states that diagonalize D'_M are the eigenstates of μ_3*.

The states of Herold et al. (1982) satisfy this criterion, as do those of Sokolov & Ternov (1968) (which differ from those of Herold et al. [1982] only in the choice of gauge and of coordinates). The states of Johnson & Lippmann (1949) do not, and are therefore inappropriate for the purpose of calculating scattering amplitudes. The states that I adopt in this work are those of Herold et al. (1982). They are summarized in Appendix A.

Now that the eigenstates of D'_M are in hand, it is still necessary to determine the eigenvalues, to leading order in α. This can be done by considering the expectation value of $D'_M = D' - \Sigma'$ in each of the states

$$(\Psi^{(\xi)}_{Ap^0}, (D' - \Sigma')\Psi^{(\eta)}_{Bq^0}) = \lambda_{Ap^0\xi}(2\pi)\delta(p^0 - q^0)\delta_{\xi\eta}\,\delta_{AB}\,, \tag{4.16}$$

where λ is the eigenvalue, and $\Psi^{(\xi)}_{Ap^0}(x) = e^{-ip^0x^0}\phi^{(\xi)}_A(\boldsymbol{x})$. Consequently,

$$\lambda_{Ap^0\xi} = \sum_{B,\eta}\int\frac{dq^0}{2\pi}\,(\Psi^{(\xi)}_{Ap^0}, (D' - \Sigma')\Psi^{(\eta)}_{Bq^0})\,. \tag{4.17}$$

Now, the leading-order contribution to the real part of $\lambda_{Ap^0\xi}$ is due to the D' term. It is simply $p^0 - \xi E_A$, and is $\mathcal{O}(\alpha^0)$. The leading-order contribution to the imaginary part of $\lambda_{Ap^0\xi}$ is due to the Σ' term (since the diagonal matix elements of D' are real), and is $\mathcal{O}(\alpha)$. Thus,

$$\lambda_{Ap^0\xi} = p^0 - \xi E_A + i\xi\Lambda(p^0, \xi, A)/2\,,$$
$$\frac{\xi\Lambda(p^0, \xi, A)}{2} \equiv -\mathrm{Im}\left\{\sum_{B,\eta}\int\frac{dq^0}{2\pi}\,(\Psi^{(\xi)}_{Ap^0}, \Sigma'\Psi^{(\eta)}_{Bq^0})\right\}\,, \tag{4.18}$$

to leading order in α.

The spectral representation of G_{Tot} is thus

$$G_{\mathrm{Tot}}(x, y) = \sum_A\sum_{\xi=\pm1}\int\frac{dp^0}{2\pi}\,\frac{\Psi^{(\xi)}_{Ap^0}(x)\bar{\Psi}^{(\xi)}_{Ap^0}(y)}{\lambda_{Ap^0\xi}}$$
$$= \sum_A\sum_{\xi=\pm1}\int\frac{dp^0}{2\pi}\,\frac{\phi^{(\xi)}_A(\boldsymbol{x})e^{-ip^0(x^0-y^0)}\overline{\phi^{(\xi)}_A}(\boldsymbol{y})}{p^0 - \xi[E_A - i\Lambda(p^0, \xi, A)/2]}\,. \tag{4.19}$$

The explicit form of Λ may be found using the expression in equation (2.3) for Σ, since the purely real counterterms that are subtracted from Σ to renormalize it have no effect on the imaginary part of its diagonal matrix elements. The result is

$$\mathrm{A}(p^0, \xi, A) = -e^2\xi g_{\mu\nu}\int\frac{d^3\boldsymbol{k}}{2\omega_k(2\pi)^3}\sum_B\sum_{\eta=\pm1}2\pi\eta\delta[p^0 - \eta(E_B + \omega_k)][T^{(\xi\eta)}_{AB}(\boldsymbol{k})^\mu][T^{(\xi\eta)}_{AB}(\boldsymbol{k})^\nu]^*\,, \tag{4.20}$$

where the notation

$$T^{(\xi\eta)}_{AB}(\boldsymbol{k})^\mu \equiv \int d^3\boldsymbol{x}\,\overline{\phi^{(\xi)}_A}(\boldsymbol{x})\gamma^\mu\phi^{(\eta)}_B(\boldsymbol{x})e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{4.21}$$

has been introduced.

The imaginary part of the pole in the propagator, which is responsible for broadening the resonance, is manifest in equation (4.19). It is not difficult to show that the resulting pole structure in the complex p^0 plane preserves the Feynman boundary conditions, so that the $i0$ prescription of the bare propagator may now be omitted.

5. SCATTERING AMPLITUDES AND THE BREIT-WIGNER LINE SHAPE

The amplitude for Compton scattering is given by the sum of the amplitudes represented by the Feynman diagrams in Figures 3*a* and 3*b*. The amplitude corresponding to Figure 3*a* is

$$S^{(1)}_{fi} = -e^2\int d^4x\,d^4y\,b^\mu_{k_f\epsilon_f}(x)^*\overline{\Phi^{(+)}_{A_f}}(x)\gamma_\mu\,iG_{\mathrm{Tot}}(x, y)\gamma_\nu\,\Phi^{(+)}_{A_i}(y)b^\nu_{k_i\epsilon_i}(y)\,. \tag{5.1}$$

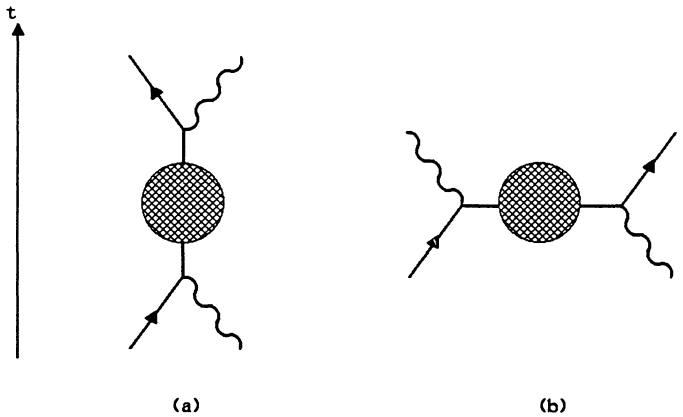

FIG. 3.—Feynman diagrams corresponding to electron-photon scattering. (*a*) "Direct" term, responsible for cyclotron resonance, and corresponding to the scattering amplitude $S^{(1)}$, given by eq. (5.1). (*b*) "Crossed" term, corresponding to the scattering amplitude $S^{(2)}$.

The photon states $b^{\mu}_{k\epsilon}$ and the electron states $\Phi^{(\xi)}_{A}$ are described in Appendix A. The amplitude $S^{(2)}$ corresponding to Figure 3*b* may be obtained from the expression for $S^{(1)}$ by the crossing symmetry substitutions $\boldsymbol{k}_i \leftrightarrow -\boldsymbol{k}_f$, $\omega_i \leftrightarrow -\omega_f$, $\epsilon_i \leftrightarrow \epsilon_f{}^*$.

With some computation, one finds

$$S^{(1)}_{fi} = -ie^2 L^{-3}(4\omega_i\omega_f)^{-1/2}(2\pi)\delta(E_f + \omega_f - E_i - \omega_i)\sum_A \sum_{\xi=\pm 1} \frac{[T^{(\xi+)}_{AA_f}(\boldsymbol{k}_f)^{\mu}(\epsilon_f)_{\mu}]^*[T^{(\xi+)}_{AA_i}(\boldsymbol{k}_i)^{\nu}(\epsilon_i)_{\nu}]}{E_i + \omega_i - \xi[E_A - i\Lambda(E_i + \omega_i, \xi, A)/2]} . \tag{5.2}$$

In order to establish the correspondence with the broadening prescription $E \to E - i\Gamma/2$, note that the term $i\Lambda/2$ in the energy denominator will be important only near resonance, that is, when $E_i + \omega_i - E_C \simeq 0$ for some state C. Suppose that this relation is satisfied exactly, that is

$$E_i + \omega_i - E_C = 0 . \tag{5.3}$$

The resonant term is $\xi = +1$, $A = C$; $\Lambda(E_C, +1, C)$ is then given by

$$\begin{aligned}\Lambda(E_C, +1, C) &= -e^2 g_{\mu\nu}\int \frac{d^3\boldsymbol{k}}{2\omega_k(2\pi)^3}\sum_B 2\pi\delta(E_C - E_B - \omega_k)[T^{(++)}_{CB}(\boldsymbol{k})^{\mu}][T^{(++)}_{CB}(\boldsymbol{k})^{\nu}]^* \\ &= -e^2\int \frac{d^3\boldsymbol{k}}{2\omega_k(2\pi)^3}\sum_B 2\pi\delta(E_C - E_B - \omega_k)\{[|\, T^{(++)}_{CB}(\boldsymbol{k})_{\mu}(\epsilon_0)^{\mu}\,|^2 - |\, T^{(++)}_{CB}(\boldsymbol{k})_{\mu}(\epsilon_3)^{\mu}\,|^2] - \sum_{j=1,2} |\, T^{(++)}_{CB}(\boldsymbol{k})_{\mu}(\epsilon_j)^{\mu}\,|^2\} .\end{aligned} \tag{5.4}$$

The standard tetrad of polarization vectors has been introduced: ϵ_0 has components $(1, \boldsymbol{0})$, ϵ_3 has components $(0, \boldsymbol{k}/\omega)$, and for $j = 1, 2$, ϵ_j has components $(0, \boldsymbol{e}_j)$, where $\boldsymbol{e}_j \cdot \boldsymbol{k} = 0$ and $\boldsymbol{e}_j \cdot \boldsymbol{e}_l = \delta_{jl}$. It is readily shown that the term in square brackets in equation (5.4) vanishes

$$\begin{aligned} T^{(++)}_{CB}(\boldsymbol{k})_{\mu}(\epsilon_3)^{\mu} &= (\epsilon_3)_{\mu}\int d^3\boldsymbol{x}\overline{\phi^{(+)}_C}(\boldsymbol{x})\gamma^{\mu}\phi^{(+)}_B(\boldsymbol{x})e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \\ &= -\omega_k^{-1}\int d^3\boldsymbol{x}\overline{\phi^{(+)}_C}(\boldsymbol{x})\boldsymbol{k}\cdot\boldsymbol{\gamma}\phi^{(+)}_B(\boldsymbol{x})e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \\ &= -\omega_k^{-1}\int d^3\boldsymbol{x}\phi^{(+)}_C(\boldsymbol{x})^{\dagger}\boldsymbol{\alpha}\phi^{(+)}_B(\boldsymbol{x})\cdot(-i\nabla)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \\ &= -\omega_k^{-1}\int d^3\boldsymbol{x}i\nabla\cdot[\phi^{(+)}_C(\boldsymbol{x})^{\dagger}\boldsymbol{\alpha}\phi^{(+)}_B(\boldsymbol{x})]e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \\ &= -\omega_k^{-1}\int d^3\boldsymbol{x}(E_C - E_B)\phi^{(+)}_C(\boldsymbol{x})^{\dagger}\phi^{(+)}_B(\boldsymbol{x})e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \\ &= -\int d^3\boldsymbol{x}\overline{\phi^{(+)}_C}(\boldsymbol{x})\gamma^0\phi^{(+)}_B(\boldsymbol{x})e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \\ &= -(\epsilon_0)_{\mu}\int d^3\boldsymbol{x}\overline{\phi^{(+)}_C}(\boldsymbol{x})\gamma^{\mu}\phi^{(+)}_B(\boldsymbol{x})e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \\ &= -T^{(++)}_{CB}(\boldsymbol{k})_{\mu}(\epsilon_0)^{\mu} . \end{aligned} \tag{5.5}$$

The resonance condition, enforced by the δ-function in equation (5.4), was used to obtain the sixth line.

The final expression for the line width at resonance is thus

$$\Lambda(E_C, +1, C) = e^2 \int \frac{d^3\boldsymbol{k}}{2\omega_k(2\pi)^3} \sum_{j=1,2} \sum_B |T_{CB}^{(+,+)}(\boldsymbol{k})_\mu(\epsilon_j)^\mu|^2 2\pi\delta(E_C - E_B - \omega_k)$$
$$= \Gamma_C , \tag{5.6}$$

where Γ_C is the decay rate of the state C.

By itself, this is not enough to confirm the validity of the broadening prescription of equation (1.1). It is still necessary to examine the line shape when the resonance condition (5.3) is not exactly met. Suppose that instead of equation (5.3) the following relation held

$$E_i + \omega_i + \Delta - E_C = 0 , \tag{5.7}$$

where $\Delta \sim \Gamma_C$. Now, in the argument of the δ-function in equation (5.4), E_C must be replaced by $E_C - \Delta$. As a consequence, the result in equation (5.6) will be altered. There are two sources of alteration: the matrix elements T will be evaluated at a slightly different value of $\boldsymbol{k}$, and the contributions from the scalar and longitudinal polarizations will no longer cancel exactly. Since the characteristic frequency scale of the matrix elements T is $(m\omega_B)^{1/2}$, the fractional change in the line width due to changes in T is of order $\Delta/(m\omega_B)^{1/2} \sim \Gamma_C/(m\omega_B)^{1/2}$. It is not hard to see that the fractional change due to the nonvanishing contribution from the scalar and longitudinal polarization modes is of order $\Delta/\omega_k \sim \Gamma_C/\omega_k$. It may be shown that for all B/B_c, $\Gamma_C/(m\omega_B)^{1/2} \lesssim \alpha$ and $\Gamma_C/\omega_k \lesssim \alpha$ (Graziani, Lamb, & Wang 1993). Consequently, when equation (5.7) holds, the following estimate is valid:

$$|\Lambda(E_C - \Delta, +1, C) - \Gamma_C|/\Gamma_C \lesssim \alpha . \tag{5.8}$$

Thus the deviations from Breit-Wigner line shape near resonance are certainly small. Indeed, they are of the same order as the corrections due to the real energy shift and as the state vector corrections of § 4, both of which were ignored. Thus, the Breit-Wigner approximation yields a result near resonance that is consistent with the perturbation-theory order of the calculation. Since the radiative corrections to the energy denominator are unimportant far from resonance, the Breit-Wigner approximation to the line shape is satisfactory.

It should be emphasized, however, that this is only the case if an appropriate set of basis states is chosen for the calculation. As established in § 4, the set of basis states *must* diagonalize μ_3. The states of Johnson & Lippmann (1949) do not satisfy this condition, and their use leads to erroneous expressions for the scattering cross sections near resonance. The errors come about because the transition matrix T_{AA_i}, T_{AA_f} in equation (5.2) and the decay rates Γ_A are affected by the choice of basis states, so that the final result is sensitive to this choice.

The error introduced into the scattering cross sections by the choice of the Johnson & Lippmann (1949) states is readily estimated when $B/B_c \lesssim 1$. Denoting by Ω the triplet of quantum numbers (N, p, a), and by Θ the states of Johnson & Lippmann (1949), and assuming that $\omega_i \sim \omega_B$ in the center of x^3-momentum frame (so that $p \sim \omega_i \sim \omega_B$), it is not hard to show that

$$\Theta^{(\xi)}_{\Omega,\sigma=\pm1} = \kappa(\Phi^{(\xi)}_{\Omega,\sigma=\pm1} + \epsilon\Phi^{(\xi)}_{\Omega,\sigma=\mp1}) , \tag{5.9}$$

where κ is a normalization constant, and

$$|\epsilon| = \frac{|p|(2Nm\omega_B)^{1/2}}{(E_{Np} + E_{N0})(E_{N0} + m)} \sim \mathcal{O}[(B/B_c)^{3/2}] . \tag{5.10}$$

If the states Θ are used instead of the states Φ in computing the matrix elements T, equation (5.10) may be used to establish that the fractional eror introduced into the scattering amplitudes at the resonances is $\mathcal{O}(B/B_c)$ at the first harmonic and $\mathcal{O}[(B/B_c)^2]$ at the higher harmonics when $B/B_c \lesssim 1$. The difference between the first and higher harmonics is due to the fact that the decay rate of the $N = 1$ spin-up state is suppressed by a factor of B/B_c with respect to the decay rate of the $N = 1$ spin-down state, so that at the first harmonic the "spin-flip" transition amplitude is comparable to the "no spin-flip" amplitude (it is suppressed by a factor of B/B_c at the higher harmonics). The "spin-flip" amplitude contributes an error $\sim\mathcal{O}(B/B_c)$, while the "no spin-flip" term contributes an error $\mathcal{O}[(B/B_c)^2]$. For scattering into the $N = 0$ final state, numerical studies of the relative errors as a function of B/B_c confirm that when $B/B_c \ll 1$ the error has a slope of $\simeq 1$ at the first harmonic and $\simeq 2$ at higher harmonics (Graziani et al. 1993). These studies also establish that the error at the first harmonic reaches a maximum of $\simeq 45\%$ when $B/B_c \simeq 3$ and tapers off thereafter, while the error at the second harmonic peaks at $\simeq 25\%$ when $B/B_c \simeq 2$ (Graziani et al. 1993).

It is worth pointing out another problem that follows from an inappropriate choice of basis states, having to do with the behavior of the decay rates as a function of the initial x^3-momentum of the electron. If the states of Herold et al. (1982) are used, the expression of equation (5.6) may be manipulated to establish that the decay rates obey the expected "Lorentz time dilation" scaling:

$$\Gamma_{N\sigma}(p) = \gamma^{-1}\Gamma_{N\sigma}(0) , \tag{5.11}$$

where $\gamma = E_{Np}/E_{N0}$. However, the proof (given in Appendix B) depends crucially on the fact that the states with x^3-momentum p are related by a Lorentz boost to the rest frame states. This is a property that the states of Herold et al. (1982) and of Sokolov & Ternov (1968) have, since these states are eigenstates of the x^3-component of the magnetic moment operator (Melrose & Parle 1983), which is invariant under boosts in the x^3-direction. On the other hand, the states of Johnson & Lippmann (1949) do not have this property. Thus, the choice of Johnson & Lippmann (1949) states results in decay rates that do not obey the simple scaling relation of equation (5.11), a relation which is clearly necessary for logical consistency with classical special relativity.

6. CONCLUSIONS

Three main results emerge from this work. The first is that in order to introduce the natural line width into the resonant cyclotron scattering amplitudes, it is essential to choose the correct basis of solutions of the Landau-Dirac equation. These states must diagonalize the component of the magnetic moment operator parallel to the external magnetic field. The solutions found by Sokolov & Ternov (1968), and those found by Herold et al. (1982) satisfy this criterion. Use of the solutions of Johnson & Lippmann (1949) results in relative errors in the resonance amplitudes and cross sections which are $\mathcal{O}(B/B_c)$ at the first harmonic and $\mathcal{O}[(B/B_c)^2]$ at the higher harmonics when $B/B_c \lesssim 1$.

The second result is that the Breit-Wigner line shape that results from the broadening prescription in equation (1.1) is an approximation which is consistent with the perturbation-theory order of the calculation.

The third result is that use of the states of Sokolov & Ternov (1968), or of Herold et al. (1982), leads to decay rates that satisfy the scaling expected from time dilation, given in equation (5.11). Decay rates computed using the Johnson & Lippmann (1949) states do not obey this relation.

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APPENDIX A

STATES, NORMALIZATIONS, AND THE PARLE REPRESENTATION

The Dirac equation in an external constant magnetic field is given by

$$D\Phi = 0 , \tag{A1}$$

where D is the Landau-Dirac operator, given in equation (2). The choice of gauge used in this work is the Landau gauge

$$A^0 = 0 ,$$
$$\boldsymbol{A} = Bx^1 \boldsymbol{e}_2 . \tag{A2}$$

The spinor solutions of equation (A1) are denoted by $\Phi_A^{(\xi)} \cdot \xi = \pm 1$ denotes whether the solution has positive or negative energy. A denotes the set of quantum numbers that characterize the state: $A \equiv (N_A, \sigma_A, p_A, a_A)$, where N_A is the Landau level (a positive integer), $\sigma_A = \pm 1$ is the spin quantum number, p_A is the z-momentum, and a_A is the x-coordinate of the orbit center. The summation $\sum_A$ has the following meaning

$$\sum_A \equiv \sum_{N_A=0}^{\infty} \sum_{\sigma_A=\pm 1} \left(\frac{L}{2\pi}\right) \int dp_A \left(\frac{Lm\omega_B}{2\pi}\right) \int da_A . \tag{A3}$$

The energy eigenvalues E_A are given by

$$E_A \equiv E_{N_A p_A} = (m^2 + p_A^2 + 2N_A m\omega_B)^{1/2} , \tag{A4}$$

where ω_B is the cyclotron frequency, $\omega_B = |e|B/m$.

The time dependence of the states $\Phi_A^{(\xi)}(x)$ may be separated out:

$$\Phi_A^{(\xi)}(x) = e^{-i\xi E_A x^0} \phi_A^{(\xi)}(\boldsymbol{x}) . \tag{A5}$$

The spinors $\phi_A^{(\xi)}$ are given by Melrose & Parle (1983), and are most conveniently expressed in terms of the spinor representation of Parle (1987):

$$\phi_A^{(\xi)}(\boldsymbol{x}) = V_{N_A p_A a_A}(\boldsymbol{x}) u_{N_A \sigma_A p_A}^{(\xi)} . \tag{A6}$$

The following definitions have been introduced: $V_{Npa}(x^1)$ is the diagonal matrix

$$V_{Npa}(\boldsymbol{x}) \equiv \begin{pmatrix} \eta_{N-1\,pa}(\boldsymbol{x}) & 0 & 0 & 0 \\ 0 & \eta_{Npa}(\boldsymbol{x}) & 0 & 0 \\ 0 & 0 & \eta_{N-1\,pa}(\boldsymbol{x}) & 0 \\ 0 & 0 & 0 & \eta_{Npa}(\boldsymbol{x}) \end{pmatrix} , \tag{A7}$$

where the functions η are the well-known eigenfunctions of a charged scalar particle in a constant external magnetic field (in the Landau gauge),

$$\eta_{Npa}(\boldsymbol{x}) \equiv i^N L^{-1} \left(\frac{m\omega_B}{\pi^{1/2} 2^N N!}\right)^{1/2} e^{-m\omega_B (x^1 - a)^2/2} H_N[(m\omega_B)^{1/2}(x^1 - a)] e^{i(px^3 - m\omega_B a x^2)} , \tag{A8}$$

where the H_N are Hermite polynomials, and L is a quantization length that allows normalization of the plane wave in the $x^2 - x^3$ plane; and the u are column vectors

$$u^{(+)}_{N,+1,p} = \kappa \begin{bmatrix} (E_{Np} + E_{N0})(E_{N0} + m) \\ -p_N p \\ (E_{N0} + m)p \\ p_N(E_{Np} + E_{N0}) \end{bmatrix}, \tag{A9a}$$

$$u^{(+)}_{N,-1,p} = \kappa \begin{bmatrix} p_N p \\ (E_{Np} + E_{N0})(E_{N0} + m) \\ p_N(E_{Np} + E_{N0}) \\ -(E_{N0} + m)p \end{bmatrix}, \tag{A9b}$$

$$u^{(-)}_{N,+1,p} = \kappa \begin{bmatrix} (E_{N0} + m)p \\ p_N(E_{Np} + E_{N0}) \\ -(E_{Np} + E_{N0})(E_{N0} + m) \\ p_N p \end{bmatrix}, \tag{A9c}$$

$$u^{(-)}_{N,-1,p} = \kappa \begin{bmatrix} -p_N(E_{Np} + E_{N0}) \\ (E_{N0} + m)p \\ p_N p \\ (E_{Np} + E_{N0})(E_{N0} + m) \end{bmatrix}, \tag{A9d}$$

where $E_{N0} \equiv (m^2 + 2Nm\omega_B)^{1/2}$, $p_N \equiv (2Nm\omega_B)^{1/2}$, and the normalization constant κ is given by $\kappa = [4E_{Np}E_{N0}(E_{Np} + E_{N0})(E_{N0} + m)]^{-1/2}$.

The normalization of the states $\phi^{(\xi)}_A(x)$ is

$$\int_R d^3x \phi^{(\xi)}_A(x)^\dagger \phi^{(\eta)}_B(x) = \delta_{\xi\eta}\delta_{N_A N_B}\delta_{\sigma_A\sigma_B}\left(\frac{2\pi}{L}\right)\delta(p_A - p_B)\left(\frac{2\pi}{Lm\omega_B}\right)\delta(a_A - a_B)$$
$$\equiv \delta_{\xi\eta}\delta_{AB}\,, \tag{A10}$$

where R is the region $-\infty < x_1 < \infty$, $-L/2 < x_2 < L/2$, $-L/2 < x_3 < L/2$.

The matrices $V_{Npa}(x)$ obey obvious orthonormality and completeness relations

$$\int d^3x V^\dagger_{Npa}(x)V_{N'p'a'}(x) = \delta_{NN'}\left(\frac{2\pi}{L}\right)\delta(p - p')\left(\frac{2\pi}{Lm\omega_B}\right)\delta(a - a') \times \begin{cases} \mathbf{1}_4, & N \neq 0; \\ (\frac{1}{2})(\mathbf{1}_4 - \sigma_{12}), & N = 0, \end{cases} \tag{A11}$$

(where $\sigma_{12} \equiv \mathbf{1}_2 \otimes \sigma_3$, $\mathbf{1}_n$ denotes the n-dimensional unit matrix, σ_3 is the third Pauli matrix, and $\otimes$ denotes the tensor product), and

$$\sum_N \int L\frac{dp}{2\pi} L\frac{d(m\omega_B a)}{2\pi} V_{Npa}(x)V^\dagger_{Npa}(y) = \delta^3(x - y)\,. \tag{A12}$$

The Parle representation of an arbitrary Dirac operator Z is

$$\tilde{Z}(N, p, a, p^0; N', p', a', p^{0'}) = \int d^4x e^{ip^0x^0}V^\dagger_{Npa}(x)ZV_{N'p'a'}(x)e^{-ip^{0'}x^0}\,. \tag{A13}$$

Z may be reconstructed from its Parle representation as a consequence of the completeness and orthonormality of the V. The Landau-Dirac Hamiltonian has a particularly simple form in the Parle representation. It may be shown that

$$HV_{Npa}(x) = (\pi_\perp \cdot \alpha + \pi_3\alpha_3 + m\beta)V_{Npa}(x)$$
$$= V_{Npa}(x)(p_N\alpha_1 + p\alpha_3 + m\beta)\,, \tag{A14}$$

where $\pi = -i\nabla - eA$, and $\perp$ refers to components perpendicular to the 3-axis. Consequently,

$$\tilde{H}(N, p, a, p^0; N', p', a', p^{0'}) = (p_N\alpha_1 + p\alpha_3 + m\beta)\delta_{NN'}(2\pi)\delta(p^0 - p^{0'})\left(\frac{2\pi}{L}\right)\delta(p - p')\left(\frac{2\pi}{Lm\omega_B}\right)\delta(a - a')\,. \tag{A15}$$

This result differs from that of Parle in the term $p_N\alpha_1$—Parle obtained $p_N\alpha_2$ instead. The difference is due to the different choice of phase of the η functions. The factor i^N in equation (A8) is absent in the definitions of Melrose & Parle (1983). The present choice of the phase of the η functions is the one made by Johnson & Lippmann (1949) and has the virtue that the resulting spinors in the Parle representation are purely real (eqs. [A9]).

The x^3-component of the magnetic moment operator of Sokolov & Ternov (1968) is

$$\begin{aligned}\mu_3 &\equiv m\sigma_{12} + i(\gamma^2\pi_1 - \gamma^1\pi_2) \\ &= \beta\sigma_{12}(m\beta + \boldsymbol{\pi}_\perp \cdot \boldsymbol{\alpha}) .\end{aligned} \tag{A16}$$

μ_3 also has a simple form in the Parle representation:

$$\mu_3 V_{Npa}(\boldsymbol{x}) = V_{Npa}(\boldsymbol{x})(m\sigma_{12} - p_N \sigma_2 \otimes \sigma_2) , \tag{A17}$$

so that

$$\tilde{\mu}_3(N, p, a, p^0; N', p', a', p^{0'}) = (m\sigma_{12} - p_N \sigma_2 \otimes \sigma_2)\delta_{NN'}(2\pi)\delta(p^0 - p^{0'})\left(\frac{2\pi}{L}\right)\delta(p - p')\left(\frac{2\pi}{Lm\omega_B}\right)\delta(a - a') . \tag{A18}$$

The photon states are

$$b^\mu_{k\epsilon}(x) \equiv L^{-3/2}(2\omega_k)^{-1/2}\epsilon^\mu e^{-i(\omega_k x^0 - \boldsymbol{k}\cdot\boldsymbol{x})} . \tag{A19}$$

They are normalized using the Klein-Gordon inner product:

$$\begin{aligned}(b_{k\epsilon}, b_{k'\epsilon'})_{KG} &\equiv -ig_{\mu\nu}\int_{R'} d^3\boldsymbol{x}\left[b^\mu_{k\epsilon}(x)^* \frac{\partial b^\nu_{k'\epsilon'}(x)}{\partial t} - \frac{\partial b^\mu_{k\epsilon}(x)^*}{\partial t} b^\nu_{k'\epsilon'}(x)\right] \\ &= \left(\frac{2\pi}{L}\right)^3 \delta^3(\boldsymbol{k} - \boldsymbol{k}')\delta_{\epsilon\epsilon'} .\end{aligned} \tag{A20}$$

Here, the spatial region R' is given by $-L/2 < x^1, x^2, x^3 < L/2$.

APPENDIX B

PROOF OF TIME DILATION OF DECAY RATES

Consider the decay of an electron in a state $L_A \equiv (N_A, \sigma_A)$ to a state $L_B \equiv (N_B, \sigma_B)$ when the momentum of the initial state is p. After restoring the timelike and longitudinal polarization modes to the expression for the decay rate given in equation (5.6), the decay rate may be written as follows

$$\Gamma_{L_A L_B}(p) = \frac{\alpha}{2\pi}\int d\Omega\omega\, d\omega\left(\frac{L}{2\pi}\right)dq\left(\frac{Lm\omega_B}{2\pi}\right)da_B\delta(\omega + E_{N_Bq} - E_{N_Ap})g_{\mu\nu}[T^{(+,+)}_{BA}(\boldsymbol{k})^\mu]^*[T^{(+,+)}_{BA}(\boldsymbol{k})^\nu] , \tag{B1}$$

where the T functions are defined in equation (4.20). If the expressions for the spinors given in Appendix A are substituted into equation (4.20), and the result is inserted in equation (B1), one finds

$$\Gamma_{L_A L_B}(p) = \frac{\alpha}{2\pi}\int d\Omega\omega\, d\omega\, dq\delta(\omega + E_{N_Bq} - E_{N_Ap})\delta(q + k_z - p)Q_{L_B L_A}(q, p, \boldsymbol{k}_\perp) , \tag{B2}$$

where $\boldsymbol{k}_\perp$ is the projection of $\boldsymbol{k}$ perpendicular to the z direction, and the following notation has been introduced

$$\begin{aligned}Q_{L_B L_A}(q, p, \boldsymbol{k}_\perp) &\equiv g_{\mu\nu}[R_{BA}(k^1)^\mu]^*[R_{BA}(k^1)^\nu]\,|_{a_B = a_A + k^2/m\omega_B} \\ R_{BA}(k^1)^\mu &\equiv \int dx^1 e^{ik^1 x^1}\overline{\rho^{(+)}_B}(x^1)\gamma^\mu\rho^{(+)}_A(x^1) .\end{aligned} \tag{B3}$$

The ρ are analogous to the spinors ϕ given in Appendix A, and differ from them only in that the x^2-x^3 dependence is removed by dropping the factor $L^{-1}e^{i(px^3 - m\omega_B a x^2)}$ from the η functions in equations (A8).

Now consider how equation (B2) transforms under a boost in the x^3-direction by a velocity β. The energies and momenta transform as follows

$$\begin{aligned} p' &= \gamma p + \gamma\beta E_{N_Ap}; \quad & E_{N_Ap'} &= \gamma E_{N_Ap} + \gamma\beta p \\ q' &= \gamma q + \gamma\beta E_{N_Bq}; \quad & E_{N_Bq'} &= \gamma E_{N_Bq} + \gamma\beta q \\ k'_z &= \gamma k_z + \gamma\beta\omega; \quad & \omega' &= \gamma\omega + \gamma\beta k_z .\end{aligned} \tag{B4}$$

Note that $\boldsymbol{k}_\perp$ is unaffected by the boost.

It is a simple matter to verify that the states given in Appendix A satisfy the following property: states that differ only in their x^3-momentum are related to each other by a boost, that is,

$$\rho^{(\xi)}_{Ap'}(x^1) = \kappa S\rho^{(\xi)}_{Ap}(x^1) , \tag{B5}$$

where $S \equiv e^{(\eta/2)\alpha_3}$, $\cosh\eta = \gamma$, and $\kappa \equiv \gamma^{-1/2}(1 + \beta p/E_{N_Ap})^{-1/2}$ is a normalization constant made necessary by the fact that the finite

dimensional representations of the Lorentz group are nonunitary. As a consequence of this property, it is possible to show that

$$Q_{L_B L_A}(q', p', \boldsymbol{k}_\perp) = \gamma^{-2}(1 + \beta q/E_{N_{Bq}})^{-1}(1 + \beta p/E_{N_{Ap}})^{-1} Q_{L_B L_A}(q, p, \boldsymbol{k}_\perp) \,. \tag{B6}$$

Furthermore, the product of δ-functions in equation (B2) obeys a particularly simple transformation relation,

$$\delta(\omega' + E_{N_{Bq'}} - E_{N_{Ap'}})\delta(q' + k'_z - p') = \delta(\omega + E_{N_{Bq}} - E_{N_{Ap}})\delta(q + k_z - p) \,, \tag{B7}$$

as a consequence of the fact that the Jacobian of the transformation described by equation (B4) is 1.

Finally, equation (B4) may be used to establish the following transformation relations:

$$\begin{aligned} \omega' \, d\omega' &= \gamma^2(1 + \beta \cos\theta)^2 \omega \, d\omega \\ dq' &= \gamma(1 + \beta q/E_{N_{Bq}}) dq \\ d\Omega' &= \gamma^{-2}(1 + \beta \cos\theta)^{-2} d\Omega \,. \end{aligned} \tag{B8}$$

By making use of relations (B6), (B7), and (B8), it is not hard to show that

$$\Gamma_{L_A L_B}(p') = \gamma^{-1}(1 + \beta p/E_{N_{Ap}})^{-1} \Gamma_{L_A L_B}(p) \,. \tag{B9}$$

Setting $p = 0$, so that the unboosted electron is at rest we find the expected "time dilation" result,

$$\Gamma_{L_A L_B}(p') = \gamma^{-1} \Gamma_{L_A L_B}(0) \,. \tag{B10}$$

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