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CLASSICAL CALCULATION OF THOMSON CROSS-SECTIONS IN THE PRESENCE OF A STRONG MAGNETIC FIELD

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Abstract—We compute the differential and total Thomson scattering cross-sections in a fully ionized plasma with a strong magnetic field, and discuss its polarization dependence. Approximate expressions for the cross-sections in the quasi-longitudinal and quasi-transverse propagation approximation are given.

1. INTRODUCTION

The existence of very strong ($B \sim 10^{12}$ Gauss) magnetic fields in cosmic objects has been considered since the discovery of neutron stars 10 years ago, and more recently, direct experimental evidence in the form of a cyclotron line in the X-ray range has been reported by Trümper *et al.* (1977) for the regularly pulsating X-ray source Her X-1, believed to be a rotating, magnetized neutron star. In the treatment of the radiative transfer in such objects, the most important opacity source, outside of resonances such as at the cyclotron frequency, is the scattering of photons on electrons moving in the magnetic field. In laboratory conditions, such as in magnetically contained plasmas, in fusion research or solid state physics, a knowledge of scattering cross sections in a strongly magnetoactive plasma is also of great interest.

In this paper we concentrate on the coherent Thomson scattering cross section (i.e. without change of frequency). As is known, for the non-relativistic energies we shall consider here, the problem is similar to scattering by linear oscillators, and a classical treatment leads to the same answer as the quantum treatment. Starting from the work of Canuto, Lodenquai and Ruderman (1971) and Gnedin and Sunyaev (1974), we extend it to calculate explicit expressions for the differential and polarization change cross-sections. The correct series expansions for quasi-longitudinal and quasi-transverse propagation are then derived, in powers of the parameter $b = \dfrac{\omega_H \sin^2 \theta}{\omega\, 2 \cos \theta}$.

2. DIFFERENTIAL CROSS-SECTION

The propagation of electromagnetic waves in a strongly magnetized plasma is characterized by a dielectric tensor depending generally on the angle between the wave propagation vector and the magnetic field H, and on the cyclotron and plasma frequencies through the parameters

$$u^{1/2} = (\omega_H/\omega) = eH/mc\omega, \qquad v = (\omega_p/\omega)^2 = 4\pi N_e e^2/m\omega^2. \tag{1}$$

For frequencies much higher than the proton cyclotron frequency, as is assumed throughout this paper, the refractive index of the normal wave of polarization α

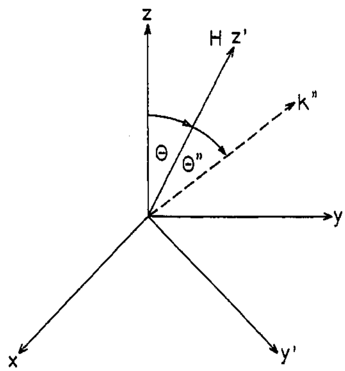

FIG. 1.—The geometrical configuration: The magnetic field H (in direction of the z'-axis) makes an angle θ with the z-axis. The wave vector $\mathbf{k}''$ of the scattered wave is at an angle θ'' with respect to H. The azimuthal angle ϕ is not shown.

propagating along the z-axis and at angle θ to the magnetic field (see Fig. 1) is given by GINZBURG (1964) as

$$n_\alpha^2(\theta)=1-\frac{2v(1-v)}{2(1-v)-u\sin^2\theta+(-)^\alpha[u^2\sin^4\theta+4u(1-v)^2\cos^2\theta]^{1/2}}. \qquad (2)$$

Such a wave is in general purely transverse only in the limit where $v\to 0$. The two normal modes are $\alpha=1$ (extraordinary wave), corresponding to the electric vector rotating clock-wise when looking along the field if $\theta\neq\pi/2$, or perpendicular to H if $\theta=\pi/2$, and $\alpha=2$ (ordinary wave), rotating counter clock-wise along H if $\theta\neq\pi/2$, and parallel to H if $\theta=\pi/2$. The ratio of the electric field components is given by

$$\frac{E_{y,\alpha}}{E_{x,\alpha}}=i\kappa_\alpha(\theta)=-i\frac{2u^{1/2}(i-v)\cos\theta}{u\sin^2\theta-(-)^\alpha[u^2\sin^4\theta+4u(1-v)^2\cos^2\theta]^{1/2}}, \qquad (3)$$

$$\frac{E_{z,\alpha}}{E_{x,\alpha}}=i\lambda_\alpha(\theta)=-i\frac{u^{1/2}v\sin\theta}{u-(1-v)-uv\cos^2\theta}+i\frac{uv\cos\theta\sin\theta\kappa_\alpha(\theta)}{u-(1-v)-uv\cos^2\theta}. \qquad (4)$$

For such a plasma, CANUTO *et al.* (1971) have derived (for the sake of completeness, in Appendix 1 we sketch the derivation of this formula) the differential Thomson cross-section for an incoming normal wave α to be scattered into solid angle $\mathrm{d}\,\Omega''$ around the outgoing wave vector making an angle θ'' with respect to H (see Fig. 1), to be

$$\frac{\mathrm{d}\sigma_\alpha}{\mathrm{d}\Omega''}=r_0^2(1-u)^{-2}[n_\alpha\cos\delta_\alpha(1+\kappa_\alpha^2+\lambda_\alpha^2)]^{-1}$$
$$\times\sum_{\beta=1}^{2}n_\beta''\cos\delta_\beta''(1+\kappa_\beta''^2+\lambda_\beta''^2)^{-1}G_{\alpha\beta}(\theta,\theta'',\phi'') \qquad (5)$$

where $r_0=e^2/mc^2$ and δ_α is the angle between the wave vector k and the

Poynting vector S_α, defined through

$$\tan \delta_\alpha = -[2n_\alpha(\theta)]^{-1}\, \partial n_\alpha^2(\theta)/\partial\theta. \tag{6}$$

The summation in (5) is over the outgoing normal polarization states, and

$$\begin{aligned} G_{\alpha\beta}(\theta, \theta'', \phi'') = {} & A^2[\cos^2\phi'' + \kappa_\beta''^2 \cos^2\theta'' \sin^2\phi'' + \lambda_\beta''^2 \sin^2\theta'' \sin^2\phi'' \\ & +2\kappa_\beta''\lambda_\beta'' \sin\theta'' \cos\theta'' \sin^2\phi''] + B^2[\sin^2\phi'' + \kappa_\beta''^2 \cos^2\theta'' \cos^2\phi'' \\ & +\lambda_\beta''^2 \sin^2\theta'' \cos^2\phi'' + 2\kappa_\beta''\lambda_\beta'' \sin\theta'' \cos\theta'' \cos^2\phi''] \\ & +C^2(1-u)^2[\kappa_\beta''^2 \sin^2\theta'' + \lambda_\beta''^2 \cos^2\theta'' - 2\kappa_\beta''\lambda_\beta'' \sin\theta'' \cos\theta''] \\ & -2AB[\kappa_\beta'' \cos\theta'' + \lambda_\beta'' \sin\theta''] + 2AC(u-1) \\ & \times[\kappa_\beta'' \sin\theta'' \cos\phi'' - \lambda_\beta'' \cos\theta'' \cos\phi''] + 2BC(u-1) \\ & \times[\kappa_\beta''\lambda_\beta''(\cos^2\theta'' - \sin^2\theta'') \cos\phi'' \\ & -(\kappa_\alpha''^2 - \lambda_\alpha''^2) \sin\theta'' \cos\theta'' \cos\phi''] \end{aligned} \tag{7}$$

where

$$A = 1 - u^{1/2}(\kappa_\alpha \cos\theta - \lambda_\alpha \sin\theta) \tag{7a}$$

$$B = u^{1/2} - (\kappa_\alpha \cos\theta - \lambda_\alpha \sin\theta) \tag{7b}$$

$$C = \kappa_\alpha \sin\theta + \lambda_\alpha \cos\theta. \tag{7c}$$

Although this expression is rather lengthy, in the limit where $v \to 0$, the λ_α, λ_α'' terms vanish and $n_\alpha \cos\delta_\alpha \to 1$, leading to somewhat simpler expressions. The $v \to 0$ limit is adequate in most applications of X-ray propagation in astrophysical plasmas.

3. PARTIAL AND TOTAL CROSS-SECTION

The differential cross-section for an incoming mode α to be scattered into polarization state β is given by (5) without the summation. In the $v \to 0$ limit, to which we limit ourselves from here on, n_α and $\cos\delta_\alpha$ tend to unity, and from (3) one can write

$$\kappa_\alpha = b + (-)^\alpha(1+b^2)^{1/2} \tag{8}$$

$$[1+\kappa_\alpha^2]^{-1} = 2^{-1}[1-(-)^\alpha b(1+b^2)^{-1/2}] \tag{9}$$

where

$$b = 2^{-1}u^{1/2}x^2y^{-1} \tag{10}$$

and we introduce the definitions

$$x = \sin\theta, \qquad y = \cos\theta. \tag{11}$$

The following frequently appearing expressions are useful:

$$[1+\kappa_\alpha^2]^{-1}[1+\kappa_\alpha^2 y^2] = \tfrac{1}{2}\{1+y^2-(-)^\alpha(y^2-1)^2[y^4-2y^2(1-2/u)+1]^{-1/2}\} \tag{12a}$$

$$[1+\kappa_\alpha^2]^{-1}[\kappa_\alpha^2 x^2] = \tfrac{1}{2}\{1-y^2+(-)^\alpha(y^2-1)^2[y^4-2y^2(1-2/u)+1]^{-1/2}\} \tag{12b}$$

$$[1+\kappa_\alpha^2]^{-1}[\kappa_\alpha y] = (-)^\alpha u^{-1/2}y^2[y^4-2y^2(1-2/u)+1]^{-1/2}. \tag{12c}$$

In this limit we can then integrate (5) over ϕ'' to obtain

$$\frac{d\sigma_{\alpha\beta}}{d(\cos\theta'')}=\sigma_T(3/8)(1-u)^{-2}[1+\kappa_\alpha^2]^{-1}[1+\kappa_\beta''^2]^{-1}\cdot\{[(1-u^{1/2}\kappa_\alpha\cos\theta)^2$$
$$+(u^{1/2}-\kappa_\alpha\cos\theta)^2][1+\kappa_\beta''^2\cos^2\theta'']$$
$$+2(1-u)^2\kappa_\alpha^2\sin^2\theta[\kappa_\beta''^2\sin^2\theta'']$$
$$+4(u^{1/2}\kappa_\alpha\cos\theta-1)(u^{1/2}-\kappa_\alpha\cos\theta)[\kappa_\beta''\cos\theta'']\}. \tag{13}$$

In Figs 2 and 3 the behaviour of this differential polarization change cross-section (13) is plotted for various values of θ. Integrating also over θ'', with the aid of (12a,b,c) we obtain the total polarization change cross-section

$$\frac{\sigma_{\alpha\beta}}{\sigma_T}=[1+\kappa_\alpha^2]^{-1}\left[\left\{\frac{(1+u)}{(u-1)^2}(1+\kappa_\alpha^2y^2)-\frac{u^{1/2}}{(u-1)^2}4\kappa_\alpha y\right\}d_\beta\right.$$
$$\left.+\kappa_\alpha^2x^2f_\beta+\left\{\frac{(1+u)}{(u-1)^2}\kappa_\alpha y-\frac{u^{1/2}}{(u-1)^2}(1+\kappa_\alpha^2y^2)\right\}g_\beta\right] \tag{14}$$

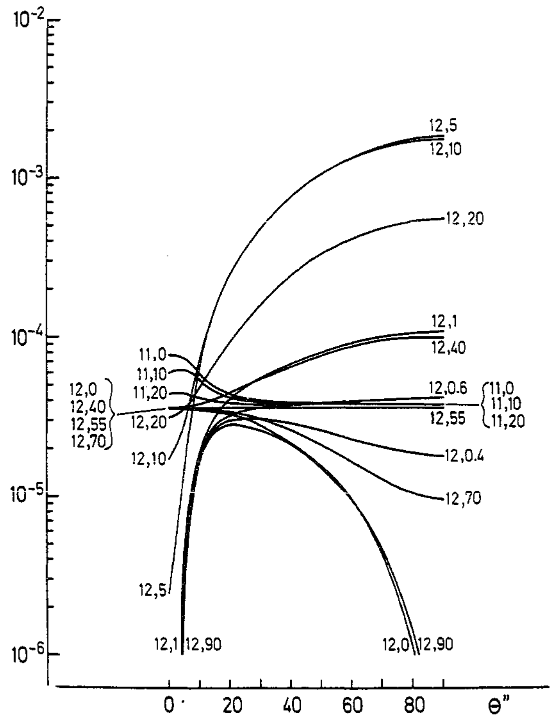

FIG. 2.—Differential cross-section $\frac{d\sigma_{\alpha\beta}}{d(\cos\theta'')}$ for incoming extraordinary mode (polarization 1) and various incoming angle. The curves are labelled $(\alpha\beta,\theta)$ where $\alpha=1$: incoming extraordinary polarization mode, β = outgoing polarization mode, θ = incoming angle in degrees, θ'' = outgoing angle in degrees, and the cross-section is given in units of σ_T. In this example we set $u=10^4$.

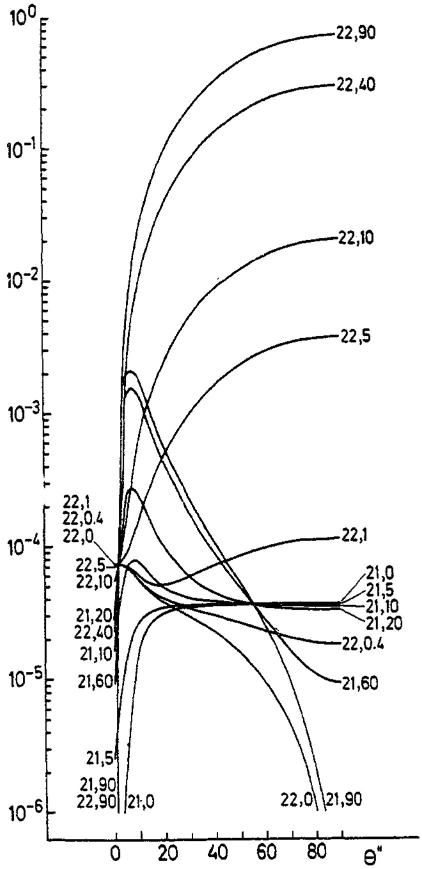

FIG. 3.—Same as Fig. 2, except incoming polarization mode is the ordinary wave $\alpha = 2$, and $u = 10^4$.

where $\sigma_T = (8\pi r_0^2/3)$, and the other constants are

$$\begin{aligned} d_\beta &= \tfrac{1}{2} - (-)^\beta \tfrac{3}{16} I \\ f_\beta &= \tfrac{1}{2} + (-)^\beta \tfrac{3}{8} I \\ g_\beta &= (-)^\beta u^{-1/2} \tfrac{3}{2} J \end{aligned} \tag{15}$$

with I and J being integrals over $dy'' = d(\cos\theta'')$

$$\begin{aligned} I &= \int_{-1}^{1} (y''^2 - 1)^2 [y''^4 - 2(1 - 2/u) y''^2 + 1]^{-1/2} \, dy'' \\ J &= \int_{-1}^{1} y''^2 [y''^4 - 2(1 - 2/u) y''^2 + 1]^{-1/2} \, dy''. \end{aligned} \tag{16}$$

The values of these integrals, computed for different values of $u = (\omega_H/\omega)^2$ are given in Table 1. If we sum (14) over outgoing polarizations β, the terms in $(-)^\beta$ in (15) cancel, so that $g = 0$, $d = f = 1$ and we obtain the integrated total

Table 1.

u	I	J
10^{-4}	0.052417	0.004999
10^{-3}	0.129400	0.015797
10^{-2}	0.295158	0.049693
10^{-1}	0.588520	0.152611
10^{1}	1.2061	1.000998
10^{2}	1.304960	1.863650
10^{3}	1.32849	2.900037
10^{4}	1.33263	4.011065
10^{5}	1.333240	5.159039
10^{6}	1.333321	6.305361
10^{7}	1.333332	6.743244

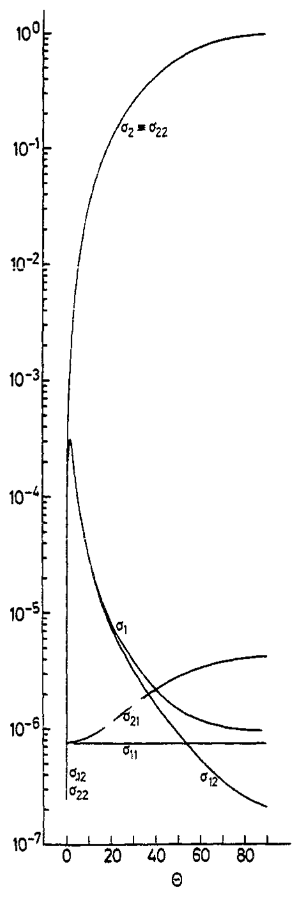

Fig. 4.—Total cross-section in units of σ_T for $u = 10^6$, with θ in degrees.

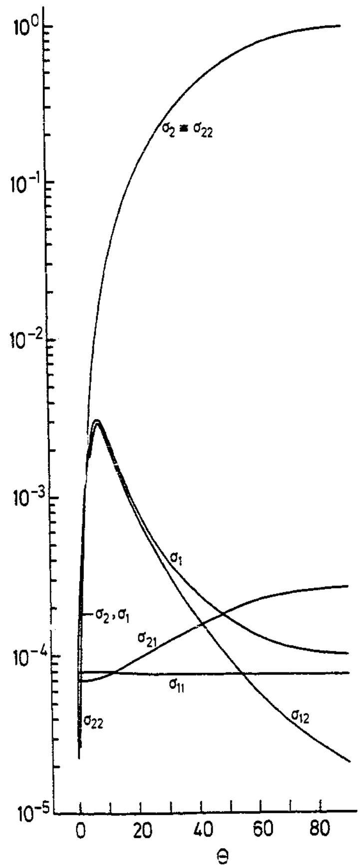

FIG. 5.—Total cross-section in units of σ_T for $u = 10^4$, with θ in degrees.

cross-section

$$\frac{\sigma_\alpha}{\sigma_T} = [1+\kappa_\alpha{}^2]^{-1}\left[\frac{(1+u)}{(u-1)^2}(1+\kappa_\alpha{}^2 y^2) - \frac{u^{1/2}}{(u-1)^2}4\kappa_\alpha y + \kappa_\alpha{}^2 x^2\right]. \tag{17}$$

Using (11)–(13), we can also write (14) and (17) as

$$\begin{aligned}\frac{\sigma_{\alpha\beta}}{\sigma_T} = {} & \tfrac{1}{2}(md_\beta - qg_\beta + f_\beta) + \tfrac{1}{2}(md_\beta - qg_\beta - f_\beta) \\ & \times\{y^2 - (-)^\alpha (y^2-1)^2[y^4 - 2(1-2/u)y^2 + 1]^{-1/2}\} \\ & + (-)^\alpha u^{-1/2}(mg_\beta - 4qd_\beta)\{y^2[y^4 - 2(1-2/u)y^2+1]^{-1/2}\}\end{aligned} \tag{18}$$

$$\begin{aligned}\frac{\sigma_\alpha}{\sigma_T} = {} & \tfrac{1}{2}(m+1) + \tfrac{1}{2}(m-1)\{y^2 - (-)^\alpha (y^2-1)^2[y^4 - 2(1-2/u)y^2+1]^{-1/2}\} \\ & + (-)^\alpha 4p\{y^2[y^4 - 2(1-2/u)y^2+1]^{-1/2}\},\end{aligned} \tag{19}$$

where

$$m = (1+u)(u-1)^{-2}, \qquad p = (u-1)^{-2}, \qquad q = u^{1/2}(u-1)^{-2}. \tag{20}$$

The behaviour of (18) and (19) is shown in Figs 4 and 5.

4. SERIES EXPANSIONS

The expressions (18) and (19) are exact in θ and u, except very near the resonance $u=1$, but are rather unwieldy, and it is desirable to have approximate expressions in the limit when θ is very close to 0 or $\pi/2$, or for u very large or small. The change in behaviour of (18) and (19), in particular the peak of the extraordinary wave for large u (cf. Figs 3 and 4) occurs in fact at $b_c \cong \sqrt{2}/2$. An expansion in powers of b or $1/b$ therefore is required, for angles small and large with respect to θ_c, where $\theta_c \equiv \theta_c(u)$ is defined setting $b=\sqrt{2}/2$.

For $b \ll 1$, i.e. $\theta \ll \theta_c$, but no other assumption as to θ or u, we obtain from (14)–(17), up to second order in b the expansions

$$\begin{aligned}\frac{\sigma^{(1)}_{\alpha\beta}}{\sigma_T} = &\{md_\beta - qg_\beta\} + y\left\{\frac{m}{2}(-)^\alpha g_\beta - 2q(-)^\alpha d_\beta\right\}\\ &+\frac{x^2}{2}\{f_\beta - md_\beta + qg_\beta\}\\ &+\frac{x^4}{4y}(-)^\alpha\left\{\sqrt{u}f_\beta - gd_\beta - \frac{m}{4}ug_\beta + pug_\beta\right\}\end{aligned} \tag{21}$$

$$\frac{\sigma^{(1)}_{\alpha}}{\sigma_T} = m - y\{2q(-)^\alpha\} + \frac{x^2}{2}\{1-m\} + \frac{x^4}{4y}(-)^\alpha\{\sqrt{u}-q\}. \tag{22}$$

Terms proportional to b^2 appear only in the x^4 brackets, namely the last two factors and the last factor of this bracket in (21) and (22) respectively. The case θ very close to 0°, provided $\theta \ll \theta_c$, is covered by this expansion (in this case the x^4 may be dropped altogether), but it also covers cases of intermediate or large θ, if θ_c itself is large.

For $b \gg 1$, i.e. $\theta \gg \theta_c$, but no other assumption as to θ or u, up to second order in 1/b the expansions are

$$\begin{aligned}\frac{\sigma^{(2)}_{\alpha\beta}}{\sigma_T} = &\left\{\left[m + (-)^\alpha\left(4p - \frac{m}{u}\right)\right]d_\beta + \frac{(-)^\alpha}{u}f_\beta + \left[-q + \frac{(-)^\alpha}{u^{1/2}}p - m)\right]g_\beta\right\}\\ &+\frac{x^2}{2}\{(1+(-)^\alpha)(f_\beta - md_\beta + qg_\beta)\}\\ &+\frac{1}{2x^2}(-)^\alpha\left\{\left[2\frac{m}{u} - 8p\right]d_\beta - \frac{2}{u}f_\beta + \left[\frac{2}{u^{1/2}}(m-p)\right]g_\beta\right\}\end{aligned} \tag{23}$$

$$\begin{aligned}\frac{\sigma^{(2)}_{\alpha}}{\sigma_T} = &\left\{m + (-)^\alpha 4p + \frac{(-)^\alpha}{u}(1-m)\right\} + \frac{x^2}{2}\{(1+(-)^\alpha)(1-m)\}\\ &+\frac{1}{2x^2}(-)^\alpha\left\{-8p - \frac{2}{u}(1-m)\right\}.\end{aligned} \tag{24}$$

The terms in $1/b^2$ are the expressions with $1/u$ factors of the x-independent and of the $1/x^2$ brackets, in both (23) and (24). The case of θ close to $\pi/2$ is covered by this approximation, but also the intermediate or even small θ case when $\theta \gg \theta_c$ (i.e. $0 < \theta_c \ll \theta$). Notice that for $\alpha = 1$, extraordinary wave, the x^2 terms disappear but the $1/x^2$ remain.

5. SMALL AND LARGE ANGLE BEHAVIOUR

The expansions (21)–(24) are *not yet* expansions for small or large angle, because this depends on the value of u as well. Since $b=(u^{1/2}\sin^2\theta/2\cos\theta)$, for $u\gg 1$ the transition region $b=2/\sqrt{2}$ occurs at small θ, whereas for $u\ll 1$ it occurs close to $\pi/2$. Since the behaviour of b is fairly steep, one can have the situation where only one of the two approximations $b\ll 1$ or $b\gg 1$ is not enough to cover a small range of angles near 0 or $\pi/2$. In this respect, the small and large angle approximations to the total cross-section given by CANUTO, LODENQUAI and RUDERMAN (1971) are particular cases. To compare our expressions (22) and (24) with theirs, we rewrite ours as

$$\frac{\sigma_\alpha^{(1)}}{\sigma_T}=\left[\frac{1}{(1+(-)^\alpha u^{1/2})^2}\right]+\frac{x^2}{2}\left[1-\frac{1}{(1+(-)^\alpha u^{1/2})^2}\right]+\frac{x^4}{4y}\left[(-)^\alpha u^{1/2}\left(1-\frac{1}{(u-1)^2}\right)\right]. \tag{25}$$

(In the above $x=\sin\theta\ll 1$)

$$\frac{\sigma_\alpha^{(2)}}{\sigma_T}=\left[(1+(-)^\alpha)\frac{(1+u)}{(u-1)^2}\right]+\frac{x^2}{2}\left[(1+(-)^\alpha)\left(1-\frac{1+u}{(u-1)^2}\right)\right]+\frac{1}{2x^2}\left[-(-)^\alpha\frac{2(1+u)}{(u-1)^2}\right]. \tag{26}$$

If in the latter we further assume $y=\cos\theta\ll 1$, we get

$$\frac{\sigma_\alpha^{(3)}}{\sigma_T}=\left[\frac{1}{2}\left(1+\frac{1+u}{(u-1)^2}\right)+\frac{(-)^\alpha}{2}\left(1-\frac{1+u}{(u-1)^2}\right)\right]-\frac{y^2}{2}\left[1+(-)^\alpha-(1-(-)^\alpha)\left(\frac{1+u}{(u-1)^2}\right)\right]. \tag{27}$$

One sees that, if we specialize to $u\gg 1$, our (25) and (27) agree with the small and large angle approximation of CANUTO, LODENQUAI and RUDERMAN (1971), after a miswriting is corrected in their equation (25), which should read $\sigma_1\simeq\dfrac{\sigma_T\omega^2}{(\omega-\omega_H)^2}(1+\cos^2\theta)$. An equivalent of our (26), characterized by the $1/\sin^2\theta$ term, was not given in this paper. Such an expression is required, even at small angles, if $u\gg 1$, to represent the decaying portion of σ_1.

For the partial cross-sections $\sigma_{\alpha\beta}$, no explicit expressions seem to have been computed previously. For small or intermediate angles ($b\ll 1$ and $b\gg 1$ really) we can use for $\sigma_{\alpha\beta}^{(1)}$ and $\sigma_{\alpha\beta}^{(2)}$ the expressions (21) and (23), whereas assuming $\cos\theta\equiv y\ll 1$ we get from this latter for angles close to $\pi/2$,

$$\begin{aligned}\frac{\sigma_{\alpha\beta}^{(3)}}{\sigma_T}&=\left\{\frac{m}{2}(1-(-)^\alpha)d_\beta-\frac{q}{2}(1-(-)^\alpha)g_\beta+\tfrac{1}{2}(1+(-)^\alpha)f_\beta\right\}\\&+\frac{y^2}{2}\left\{\left[m+(-)^\alpha\frac{m}{u}(u+2)-(-)^\alpha 8p\right]d_\beta-\left[1+\frac{(-)^\alpha}{u}(u+2)\right]f_\beta\right.\\&\left.+\left[-(1+(-)^\alpha)q+\frac{(-)^\alpha}{u^{1/2}}2(m-p)\right]g_\beta\right\}.\end{aligned} \tag{28}$$

These expansions (25)–(28) and (21), (23) cover the whole range of angles for σ_α and $\sigma_{\alpha\beta}$, except for the neighbourhood of the peak in the extraordinary wave, and they are valid for any value of u outside resonance ($u\gtrless 1$).

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APPENDIX

Derivation of the differential cross-section

The derivation of $d\sigma_\alpha/d\Omega''$ in the Canuto *et al.* paper, uses simultaneously two different co-ordinate conventions, one where time variations are proportional to $e^{-i\omega t}$ and H is in the x,z-plane, if not along z, as in their equations (11), (12), (B5) and dielectric tensor $\varepsilon_{\alpha\beta}$, e.g. STIX (1967), and one where the time factor is $e^{i\omega t}$ and H is in the y,z-plane, as in their equations (1) to (8) for κ_α, κ_α, λ_α, e.g. GINZBURG (1964). The derivation below is done using consistently the Ginzburg-type convention throughout.

We assume a magnetic field in the y,z-plane, at an angle θ to the z-axis, and an incoming photon propagating along z (Fig. 1). The equation of motion of the electrons (where one must take $e=-|e|$ for electrons)

$$m\dot{\mathbf{v}}=e\mathbf{E}(t)+(1/c)e\mathbf{v}\mathbf{H} \tag{A1}$$

is best solved in an x', y', z' system obtained from x, y, z by rotating around $x\equiv x'$ an angle θ, so that H is along z'. Taking

$$\mathbf{E}(t)=(E_{x'}\hat{\mathbf{e}}_{x'}+E_{y'}\hat{\mathbf{e}}_{y'}+E_{z'}\hat{\mathbf{e}}_{z'})e^{i\omega t} \tag{A2}$$

$$\mathbf{v}(t)=(v_{x'}\hat{\mathbf{e}}_{x'}+v_{y'}\hat{\mathbf{e}}_{y'}+v_{z'}\hat{\mathbf{e}}_{z'})e^{i\omega t} \tag{A3}$$

we get as solution of (A1), for $\omega\neq\omega_H$

$$v_{x'}=\frac{e\omega}{im(\omega^2-\omega_H{}^2)}\left[E_{x'}+iE_{y'}\frac{\omega_H}{\omega}\right] \tag{A4}$$

$$v_{y'}=-\frac{e\omega}{m(\omega^2-\omega_H{}^2)}\left[E_{x'}\frac{\omega_H}{\omega}+iE_{y'}\right] \tag{A5}$$

$$v_{z'}=\frac{eE_{z'}}{im\omega}. \tag{A6}$$

We take, as in CANUTO *et al.* (1971) the time-averaged power (mode β, $\beta=1,2$) emitted per unit solid angle in the (θ'',ϕ'') direction by the electron accelerated in the incident wave (mode α, $\alpha=1,2$) with wave vector k in the y', z'-plane at an angle θ with the z'-axis

$$\frac{dP_{\alpha\beta}(\theta\to\theta'',\phi'')}{d\Omega''}=\frac{e^2\omega^2}{8\pi c^3}\mathrm{Re}\frac{1}{|\mathbf{E}(\theta'',\phi'')|^2}(\mathbf{E}_\beta{}^*\cdot\mathbf{v}_\alpha)(\mathbf{E}_\beta\cdot\mathbf{v}_\alpha{}^*)n_\beta(\theta'')\cos\delta_\beta \tag{A7}$$

We evaluate the expression in the x'', y'', z'' co-ordinate system, which has the z''-axis along $\mathbf{k}''$, and $x''=x'$. Now for $E_{x'',\beta}$, $E_{y'',\beta}$, $E_{z'',\beta}$ expressions similar to (3) and (4) are valid, and one has further

$$v_{x'',\alpha}=\cos\phi''v_{x',\alpha}-\sin\phi''v_{y',\alpha} \tag{A8}$$

$$v_{y'',\alpha}=\sin\phi''\cos\theta''v_{x',\alpha}+\cos\phi''\cos\theta''v_{y',\alpha}-\sin\theta''v_{z',\alpha} \tag{A9}$$

$$v_{z'',\alpha}=\sin\phi''\sin\theta''v_{x',\alpha}+\cos\phi''\sin\theta''v_{y',\alpha}+\cos\theta''v_{z',\alpha}. \tag{A10}$$

Taking into account (A4) to (A6) and

$$E_{x'}=E_x \tag{A11}$$

$$E_{y'}=\cos\theta E_y-\sin\theta E_z \tag{A12}$$

$$E_{z'}=\sin\theta E_y+\cos\theta E_z \tag{A13}$$

as well as (3) and (4), and the definition of the Poynting vector,

$$\langle s_\alpha \rangle = c(8\pi)^{-1}|E_{x,\alpha}|^2(1+\kappa_\alpha{}^2(\theta)+\lambda_\alpha{}^2(\theta))n_\alpha(\theta)\cos\delta_\alpha(\theta), \tag{A14}$$

we obtain finally using (A7)

$$\begin{aligned}\frac{\mathrm{d}\sigma_{\alpha\beta}(\theta\to\theta'',\phi'')}{\mathrm{d}\Omega''} &\equiv \langle s_\alpha\rangle^{-1}\left\langle\frac{\mathrm{d}P_{\alpha\beta}(\theta\to\theta'',\phi'')}{\mathrm{d}\Omega''}\right\rangle \\ &= r_0{}^2(u-1)^{-2}(n_\alpha\cos\delta_\alpha)^{-1}(n_\beta''\cos\delta_\beta'')(1+\kappa_\alpha{}^2+\lambda_\alpha{}^2)^{-1} \\ &\quad\times(1+\kappa_\beta''^2+\lambda_\beta''^2)^{-1}G_{\alpha\beta}(\theta,\theta'',\phi'')\end{aligned} \tag{A15}$$

where $G_{\alpha\beta}$ is given in (7). Summing over outgoing polarizations β gives then $\mathrm{d}\sigma_\alpha/\mathrm{d}\Omega''$ as in (5).