# Photon splitting above the pair creation threshold in a strong magnetic field

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#### **Abstract**

The process of photon splitting  $\gamma \to \gamma \gamma$  in a strong magnetic field is investigated both below and above the pair creation threshold. Contrary to the statement by Baier et al., the "allowed" channel  $\gamma_{\parallel} \to \gamma_{\perp} \gamma_{\perp}$  is shown not to be a comprehensive description of splitting in the strong field because the "forbidden" channel  $\gamma_{\parallel} \to \gamma_{\parallel} \gamma_{\perp}$  is also essential. The partial amplitudes and the splitting probabilities are calculated taking account of the photon dispersion and large radiative corrections near the resonance.

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Nowadays it is generally recognized that an external electromagnetic field plays the role of an active medium. It can induce new interactions and can also change the dispersion properties of particles, thus allowing processes which are forbidden in vacuum. The photon splitting process  $\gamma \to \gamma \gamma$  could be a prominent example of the magnetic field effect.

The theoretical study of this process has a rather long history [1, 2, 3]. Recent progress in astrophysics has drawn attention again to the photon splitting induced by a magnetic field. This process may explain the peculiarity of the spectra of the observed  $\gamma$ -bursts [4, 5]. The origin of these  $\gamma$ -bursts is not clear yet. Some models exist where astrophysical cataclysms like a supernova explosion or a coalescence of neutron stars could be the sources of such  $\gamma$ -bursts. It is generally supposed that these objects have strong magnetic fields  $\sim 10^{16} - 10^{17} G$  [6] much greater than the Schwinger value,  $B_e = m^2/e \simeq 4.41 \cdot 10^{13} G$  (hereafter m is the electron mass, e > 0 is the elementary charge).

In our opinion, continued investigation of the process  $\gamma \to \gamma \gamma$  is important not only because of its possible applications, but to also gain an improved understanding of radiative processes in intense external fields. The study of the photon splitting  $\gamma \to \gamma \gamma$  in a strong magnetic field [7, 8, 9] has so far considered only the collinear limit of the process, when the only allowed transition with respect to photon polarizations is,  $\gamma_{\parallel} \rightarrow \gamma_{\perp} \gamma_{\perp}$  (in Adler's notation [2]). However, photon dispersion in a strong magnetic field,  $B \gg B_e$ , leads to significant deviations from the collinearity of the kinematics of this process. This is due to the fact that the eigenvalues of the photon polarization operator (the photon effective mass squared) become large near the so-called cyclotron resonances [10]. The lowest of them is only essential in a strong field and corresponds to the pair creation threshold  $\omega_0 = 2m$  (in the frame where the photon momentum is perpendicular to the field direction). A photon of the  $\perp$  mode acquires in this region a significant effective mass,  $k^2 = \omega^2 - \mathbf{k}^2 < 0$ ,  $|k^2| \sim \omega^2$ , and this defines the kinematics of the photon splitting, which is far from collinearity.

On the other hand, a large value of the polarization operator near the resonance requires taking account of large radiative corrections which reduce to a renormalization of the photon wave-function

$$\varepsilon_{\alpha}^{(\lambda)} \to \varepsilon_{\alpha}^{(\lambda)} \sqrt{Z_{\lambda}}, \quad Z_{\lambda}^{-1} = 1 - \frac{\partial \mathcal{P}^{\lambda}}{\partial \omega^{2}}, \quad \lambda = \parallel, \perp.$$
 (1)

Here  $\varepsilon_{\alpha}^{(\parallel)}$ ,  $\varepsilon_{\alpha}^{(\perp)}$  are the polarization four-vectors of the photon modes and  $\mathcal{P}^{\lambda}$  are the eigenvalues of the photon polarization operator, corresponding to

these modes [10].

Both the effect of noncollinearity and radiative corrections have not, so far, been taken into account. For example, in the paper by Baier et al. [7] the amplitude and the probability of the photon splitting were obtained for the photon energies up to the threshold  $\omega_0 = 2m$ . Strictly speaking, that result is incorrect because the noncollinearity of the photon momenta and the photon wave-function renormalization were not considered. In fact, their amplitude describes the photon splitting correctly only in the limit  $\omega \ll m$ . The same criticism is appropriate for the paper [9].

It should be stressed also, that the channel  $\gamma_{\parallel} \to \gamma_{\perp} \gamma_{\perp}$  is not adequate to describe photon splitting because of the noncollinearity of the photon momenta. In particular, the transition  $\gamma_{\parallel} \to \gamma_{\parallel} \gamma_{\perp}$  forbidden in the collinear limit, gives an essential contribution into the splitting probability, contrary to the statement by Baier et al. [7].

In this paper, photon splitting in a strong magnetic field is investigated both below and above the pair-creation threshold, with an emphasis on the noncollinearity of the kinematics, and taking account of large radiative corrections. The splitting probability of a photon with high energy is of particular importance for calculations of the spectra of strongly magnetized cosmic objects.

The process of photon splitting in an external field is depicted by two Feynman diagrams, see Fig.1. The electron propagator in a magnetic field could be presented in the form

$$S(x,y) = e^{i\Phi(x,y)} \hat{S}(x-y), \tag{2}$$

$$\Phi(x,y) = -e \int_{x}^{y} d\xi_{\mu} \left[ A_{\mu}(\xi) + \frac{1}{2} F_{\mu\nu}(\xi - y)_{\mu} \right], \qquad (3)$$

where  $A_{\mu}$  is a 4-potential of the uniform magnetic field. The translational invariant part  $\hat{S}(x-y)$  of the propagator has several representations. It is convenient for our purposes to take it in the form of a partial Fourier integral expansion

$$\hat{S}(X) = -\frac{i}{4\pi} \int_{0}^{\infty} \frac{d\tau}{th\tau} \int \frac{d^{2}p}{(2\pi)^{2}} \left\{ [(p\gamma)_{\parallel} + m] \Pi_{-}(1 + th\tau) + \right. \\ + \left. [(p\gamma)_{\parallel} + m] \Pi_{+}(1 - th\tau) - (X\gamma)_{\perp} \frac{ieB}{2th\tau} (1 - th^{2}\tau) \right\} \times$$
(4)

$$\times \exp\left(-\frac{eBX_{\perp}^{2}}{4\,th\tau} - \frac{\tau(m^{2} - p_{\parallel}^{2})}{eB} - i(pX)_{\parallel}\right),$$

$$d^{2}p = dp_{0}dp_{3}, \quad \Pi_{\pm} = \frac{1}{2}(1 \pm i\gamma_{1}\gamma_{2}), \quad \Pi_{\pm}^{2} = \Pi_{\pm}, \quad [\Pi_{\pm}, (a\gamma)_{\parallel}] = 0,$$

where  $\gamma_{\alpha}$  are the Dirac matrices in the standard representation, the fourvectors with the indices  $\bot$  and  $\parallel$  belong to the (1, 2) plane and the Minkowski (0, 3) plane correspondingly, when the field **B** is directed along the third axis. Then for arbitrary 4-vectors  $a_{\mu}$ ,  $b_{\mu}$  one has

$$\begin{array}{lll} a_{\perp} & = & (0,a_1,a_2,0), & a_{\parallel} = (a_0,0,0,a_3), \\ (ab)_{\perp} & = & (a\Lambda b) = a_1b_1 + a_2b_2, & (ab)_{\parallel} = (a\widetilde{\Lambda}b) = a_0b_0 - a_3b_3, \end{array}$$

where the matrices are introduced  $\Lambda_{\alpha\beta} = (\varphi\varphi)_{\alpha\beta}$ ,  $\tilde{\Lambda}_{\alpha\beta} = (\tilde{\varphi}\tilde{\varphi})_{\alpha\beta}$ , connected by the relation  $\tilde{\Lambda}_{\alpha\beta} - \Lambda_{\alpha\beta} = g_{\alpha\beta} = diag(1, -1, -1, -1)$ ,  $\varphi_{\alpha\beta} = F_{\alpha\beta}/B$ ,  $\tilde{\varphi}_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}\varphi_{\mu\nu}$  are the dimensionless tensor of the external magnetic field and the dual tensor,  $(a\Lambda b) = a_{\alpha}\Lambda_{\alpha\beta}b_{\beta}$ .

In spite of the translational and gauge noninvariance of the phase  $\Phi(x, y)$  in the propagator (2), the total phase of three propagators in the loop of Fig.1 is translational and gauge invariant

$$\Phi(x,y) + \Phi(y,z) + \Phi(z,x) = -\frac{e}{2}(z-x)_{\mu}F_{\mu\nu}(x-y)_{\nu}.$$

The amplitude of the photon splitting  $\gamma \to \gamma \gamma$  takes the form

$$\mathcal{M} = e^{3} \int d^{4}X \, d^{4}Y \, Sp\{\hat{\varepsilon}(k)\hat{S}(Y)\hat{\varepsilon}(k'')\hat{S}(-X-Y)\hat{\varepsilon}(k')\hat{S}(X)\} \times e^{-ie(XFY)/2} \, e^{i(k'X-k''Y)} + (\varepsilon(k'), k' \leftrightarrow \varepsilon(k''), k''), \tag{5}$$

where  $k_{\alpha} = (\omega, \mathbf{k})$  is the 4-vector of the momentum of initial photon with the polarization vector  $\varepsilon_{\alpha}$ , k' and k'' are the 4-momenta of final photons, X = z - x, Y = x - y.

Substitution of the propagator (4) into the amplitude (5) leads to a very cumbersome expression in a general case. Relatively simple results were obtained only in the two limits of a weak field [2] and of the strong field with collinear kinematics [7].

It is advantageous to use the asymptotic expression of the electron propagator for an analysis of the amplitude (5) in the strong field without the

collinear approximation. This asymptotic could be easily derived from Eq. (4) by evaluation of the integral over  $\tau$  in the limit  $eB/|m^2 - p_{\parallel}^2| \gg 1$ . In this case the propagator takes the simple form

$$\hat{S}(X) \simeq S_a(X) = \frac{ieB}{2\pi} \exp(-\frac{eBX_{\perp}^2}{4}) \int \frac{d^2p}{(2\pi)^2} \frac{(p\gamma)_{\parallel} + m}{p_{\parallel}^2 - m^2} \Pi_{-} e^{-i(pX)_{\parallel}}, \quad (6)$$

which was obtained for the first time in Ref. [11]. Substituting this in Eq. (5) and integrating, one would expect to obtain the amplitude which depends linearly on the field strength, namely, as  $B^3/B^2$ , where  $B^2$  in the denominator arises from the integration over  $d^2X_{\perp}d^2Y_{\perp}$ . However, two parts of the amplitude (5) cancel each other exactly. Thus, the asymptotic form of the electron propagator (6) only shows that the linear-on-field part of the amplitude is zero and provides no way of extracting the next term of expansion over the field strength.

As the analysis shows, that could be done by the insertion of two asymptotic (6) and one exact propagator (4) into the amplitude (5), with all interchanges. It is worthwhile now to turn from the general amplitude (5) to the partial amplitudes corresponding to definite photon modes,  $\parallel$  and  $\perp$ , which are just the stationary photon states with definite dispersion relations in a magnetic field. There are 6 independent amplitudes and only two of them are of physical interest. We have obtained the following expressions, to the terms of order 1/B

$$\mathcal{M}_{\parallel \to \parallel \perp} = i4\pi \left(\frac{\alpha}{\pi}\right)^{3/2} \frac{(k'\varphi k'')(k'\tilde{\varphi}k'')}{[(k')_{\parallel}^{2}(k'')_{\perp}^{2}k_{\perp}^{2}]^{1/2}} H\left(\frac{4m^{2}}{(k')_{\parallel}^{2}}\right), \tag{7}$$

$$\mathcal{M}_{\parallel \to \perp \perp} = i4\pi \left(\frac{\alpha}{\pi}\right)^{3/2} \frac{(k'\tilde{\Lambda}k'')}{[(k')_{\parallel}^{2}(k'')_{\parallel}^{2}k_{\perp}^{2}]^{1/2}} \left\{ (k\tilde{\Lambda}k'') H\left(\frac{4m^{2}}{(k')_{\parallel}^{2}}\right) + (k\tilde{\Lambda}k') H\left(\frac{4m^{2}}{(k'')_{\parallel}^{2}}\right) \right\}, \tag{8}$$

where

$$H(z) = \frac{z}{\sqrt{z-1}} \arctan \frac{1}{\sqrt{z-1}} - 1, \ z > 1,$$

$$H(z) = -\frac{1}{2} \left( \frac{z}{\sqrt{1-z}} \ln \frac{1+\sqrt{1-z}}{1-\sqrt{1-z}} + 2 + i\pi \frac{z}{\sqrt{1-z}} \right), \ z < 1.$$
(9)

As for remaining amplitudes, we note that  $\mathcal{M}_{\parallel \to \parallel \parallel}$  is equal to zero in this approximation. On the other hand, the photon of the  $\perp$  mode due to its dispersion can split into two photons only in the kinematic region  $k_{\parallel}^2 > 4m^2$  where the tree-channel  $\gamma_{\perp} \to e^+ e^-$  [12] strongly dominates.

Thus we will analyse further the photon splitting of the  $\parallel$  mode in the region  $k_{\parallel}^2 < (m + \sqrt{m^2 + 2eB})^2$  where the tree-channel  $\gamma_{\parallel} \to e^+ e^-$  does not exist. In the formal limit of the collinearity of the photon momenta, the amplitude  $\mathcal{M}_{\parallel \to \parallel \perp}$  goes to zero while the amplitude  $\mathcal{M}_{\parallel \to \perp \perp}$  coincides with the amplitude obtained in Ref. [7].

Although the process involves three particles, its amplitude is not a constant, because it contains the external field tensor in addition to the photon 4-momenta. The general expression for the splitting probability can be written in the form

$$W_{\lambda \to \lambda' \lambda''} = \frac{g}{32\pi^2 \omega} \int |\mathcal{M}_{\lambda \to \lambda' \lambda''}|^2 Z_{\lambda} Z_{\lambda'} Z_{\lambda''} \times \delta(\omega_{\lambda}(\mathbf{k}) - \omega_{\lambda'}(\mathbf{k}') - \omega_{\lambda''}(\mathbf{k} - \mathbf{k}')) \frac{d^3 k'}{\omega_{\lambda'} \omega_{\lambda''}}, \tag{10}$$

where the factor  $g = 1 - \frac{1}{2} \delta_{\lambda'\lambda''}$  is inserted to account for possible identity of the final photons. The factors  $Z_{\lambda}$  account for the large radiative corrections which reduce to the wave-function renormalization of a real photon with definite dispersion  $\omega = \omega_{\lambda}(\mathbf{k})$ . The integration over phase space of two final photons in Eq. (10) has to be performed using the photon energy dependence on the momenta,  $\omega = \omega_{\lambda}(\mathbf{k})$ , which can be found from the dispersion equations

$$\omega_{\lambda}^{2}(\mathbf{k}) - \mathbf{k}^{2} - \mathcal{P}^{\lambda} = 0,$$

where  $\mathcal{P}^{\lambda}$  are the eigenvalues of the photon polarization operator [10]. In the limit of a strong field and in the kinematic region  $k_{\parallel}^2 = \omega^2 - k_3^2 \ll eB$  one obtains

$$\mathcal{P}^{\parallel} \simeq -\frac{\alpha}{3\pi} k_{\parallel}^2, \tag{11}$$

$$\mathcal{P}^{\perp} \simeq -\frac{2\alpha}{\pi} eB H\left(\frac{4m^2}{k_{\parallel}^2}\right),$$
 (12)

where H(z) is the function defined in Eq. (9).

A calculation of the splitting probability (10) is rather complicated in the general case. In the limit  $m^2 \ll \omega^2 \sin^2 \theta \ll eB$ , where  $\theta$  is an angle

between the initial photon momentum  $\mathbf{k}$  and the magnetic field direction, we derive the following analytical expression for the probability of the channel  $\gamma_{\parallel} \to \gamma_{\parallel} \gamma_{\perp}$ :

$$W_{\parallel \to \parallel \perp} \simeq \frac{\alpha^3 \omega \sin^2 \theta}{16} (1 - x) [1 - x + 2x^2 + 2(1 - x)(1 + x)^2 \ln(1 + x) - 2x^2 \frac{2 - x^2}{1 - x} \ln \frac{1}{x}], \qquad x = \frac{2m}{\omega \sin \theta} \ll 1.$$
 (13)

Within the same approximation we obtain the spectrum of final photons in the frame where the initial photon momentum is orthogonal to the field direction:

$$\frac{dW_{\parallel \to \parallel \perp}}{d\omega'} \simeq \frac{\alpha^3}{2} \cdot \frac{\sqrt{(\omega - \omega')^2 - 4m^2}}{\omega' + \sqrt{(\omega' - \omega)^2 - 4m^2}},$$

$$\frac{\omega}{2} - \frac{2m^2}{\omega} < \omega' < \omega - 2m,$$
(14)

where  $\omega, \omega'$  are the energies of the initial and final photons of the  $\parallel$  mode.

We have made numerical calculations of the process probabilities for both channels, which are valid in the limit  $\omega^2 \sin^2 \theta \ll eB$ . Our results are represented in Figs. 2,3. The photon splitting probabilities below and near the pair-creation threshold are depicted in Fig. 2. In this region the channel  $\gamma_{\parallel} \to \gamma_{\perp} \gamma_{\perp}$  (allowed in the collinear limit) is seen to dominate the channel  $\gamma_{\parallel} \to \gamma_{\parallel} \gamma_{\perp}$  (forbidden in this limit). For comparison we show here the probability obtained without considering the noncollinearity of the kinematics and radiative corrections (the dotted line 3) which is seen to be inadequate. For example, this probability becomes infinite just above the threshold. As is seen from Fig. 3, both channels give essential contributions to the probability at high photon energies, with the "forbidden" channel dominating. It should be stressed that taking account of the photon polarization leads to the essential dependence of the splitting probabilities on the magnetic field, while the amplitudes (7), (8) do not depend on the field strength value.

In conclusion, we have investigated the photon splitting  $\gamma \to \gamma \gamma$  below and above the pair-creation threshold in the strong magnetic field,  $B \gg B_e$ . We have found the amplitudes and the probabilities of two channels,  $\gamma_{\parallel} \to \gamma_{\perp} \gamma_{\perp}$  and  $\gamma_{\parallel} \to \gamma_{\parallel} \gamma_{\perp}$ , taking account of the essential influence of the field on the process kinematics, and of large radiative corrections. The photon spectrum and the splitting probability are calculated analytically for the dominating channel  $\gamma_{\parallel} \to \gamma_{\parallel} \gamma_{\perp}$  at high energies of the initial photon. The splitting probabilities are calculated numerically for both channels. The results should be used for quantitative studies of the role of the exotic field-induced process of photon splitting in high-energy astrophysics.

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## Figure captions

to the paper: M.V. Chistyakov et al., "Photon splitting ..."

- **Fig. 1** The Feynman diagram for photon splitting in a magnetic field. The double line corresponds to the exact propagator of an electron in an external field.
- Fig. 2 The dependence of the probability of photon splitting  $\gamma \to \gamma \gamma$  on energy, below and near the pair-creation threshold: 1a, 1b for the "forbidden" channel  $\gamma_{\parallel} \to \gamma_{\parallel} \gamma_{\perp}$  and for the magnetic field strength  $B=10^2\,B_e$  and  $10^3\,B_e$ , correspondingly; 2a, 2b for the "allowed" channel  $\gamma_{\parallel} \to \gamma_{\perp} \gamma_{\perp}$  for the field strength  $B=10^2\,B_e$  and  $10^3\,B_e$ ; 3 for the channel  $\gamma_{\parallel} \to \gamma_{\perp} \gamma_{\perp}$  in the collinear limit without taking account of large radiative corrections. Here  $W_0=(\alpha/\pi)^3\,m$ .
- Fig. 3 The probability of photon splitting above the pair-creation threshold: 1a, 1b for the "forbidden" channel  $\gamma_{\parallel} \to \gamma_{\parallel} \gamma_{\perp}$  and for the magnetic field strength  $B = 10^2 \, B_e$  and  $10^3 \, B_e$ , correspondingly; 2a, 2b for the "allowed" channel  $\gamma_{\parallel} \to \gamma_{\perp} \gamma_{\perp}$  for the field strength  $B = 10^2 \, B_e$  and  $10^3 \, B_e$ .

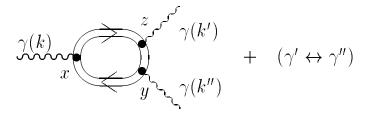


Figure 1: M.V. Chistyakov et al., "Photon splitting ..."

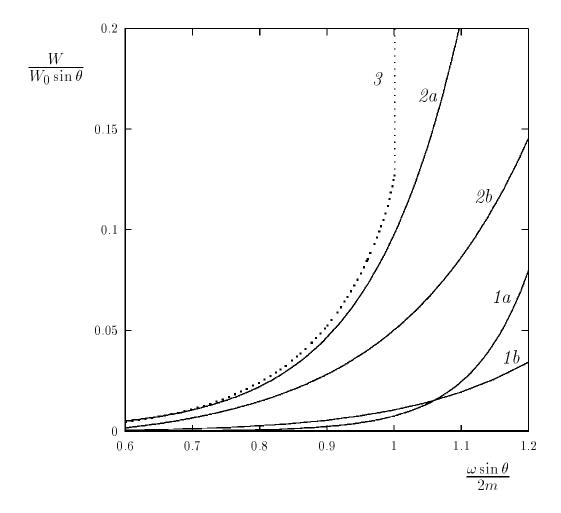


Figure 2: M.V. Chistyakov et al., "Photon splitting  $\dots$  "

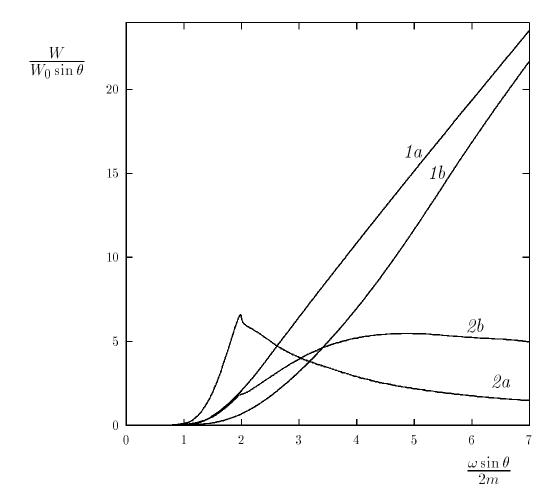


Figure 3: M.V. Chistyakov et al., "Photon splitting  $\dots$  "