The "Distributed Ghost" with independence - a study on computational ability of skewed asynchronous cellular automata

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This paper explores the computational ability of "distributed ghost" cellular automata (CA) [14] after introducing independence in the updating scheme. Traditionally, the CA system dictates all cells to update together following the concept of the global clock. To introduce independence in the system, CA researchers have introduced the notion of fully asynchronous updating scheme with atomicity property where, again, the CA system dictates two neighbouring cells not to update together. In this study, we explore the skewed asynchronous system after breaking the atomicity property. Specifically, we study the computational ability of the proposed skewed asynchronous system in the context of the density classification problem. In this direction, the first theoretical study includes identification of two attractor (all 0 and all 1) skewed asynchronous system (here, 13 candidate ECA rules) after considering the problem statement of density classification. Next, following the basins of attraction dynamics of the two attractor systems, we consider ECA rules 170, 178 and 184 as candidates for this distributed consensus problem. Finally, we demonstrate the computational ability (i.e., efficiency) of the proposed cellular system with independence during density classification.

Keywords: Cellular Automata (CA), Skewed asynchronous CA, Density classification problem, Basins of attraction, Convergence.

1 Introduction

During late 1940s, John von Neumann [22] introduced (one of the) oldest model of natural computing - cellular automata (CA). The beauty of CA is simplicity where simple local interaction depicts huge complex global behaviour [11]. Because of this parallel and distributed computation ability, this "distributed ghost" cellular automata [14] still captures the attention of research community [1]. However, in this model of decentralized computing, the cells are not independent, i.e., this model follows a global clock which forces the cells to get updated synchronously. During the last two decades [6, 15], CA researchers argues that this assumption of global clock is not very natural. In this direction, to introduce independence in cell update, the CA research community [6] investigated the notion of asynchronous system, where the most studied updating scheme is fully asynchronous cellular automata [8, 21].

In fully asynchronous system, at each time step, we select a random cell for update following *uniform* distribution. According to Fatès [6], this update scheme is the most natural one after considering the continuous nature of real time. Moreover, fully asynchronism follows (a special case) *atomicity property* where no two neighbouring cells can not be updated together at the same time step. Note that, this system with atomicity property is also capable to model distributed and concurrent systems [16, 2]. However, in our recent work [19], we question this assumption of atomicity property after considering cellular automata as a model of societal phenomenon. According to our argument [19], "in society, some neighbouring populations follow the same societal norms which violate the independence introduced by atomicity property". On the other hand, in terms of perturbation or noise, fully asynchronous system can not also be able to capture the notion of *burst* noise. In this direction, in our early work [19], we introduced the *skewed* asynchronous system after breaking the notion of atomicity property. In skewed asynchronous CA, at each time step, we select a random cell, and update the selected cell and it's neighbouring cell (say, right neighbouring cell). That is, two neighbouring cells can be updated together at the same time step. According to the study of [19], presence (or, absence) of atomicity property depicts huge impact on the convergence (non-convergence) property of *elementary* cellular automata (ECA) system. In details, some convergent elementary CA system[1] shows non-convergent dynamics in the absence of atomicity property. Moreover, some non-convergent recurrent elementary CA system[2] with internal return property (i.e., reversible system) depicts convergent behaviour in the absence of atomicity property. In a different direction, Fatès [7] explored parity problem [3] with 1-dimensional 2-state 4-neighbourhood

[1]Elementary CA rules 26, 58, 90, and 122.

[2]Elementary CA rules 38, 54, 134, and 150.

[3]In the computational task of solving parity problem, if the parity of the initial configuration (i.e., number of 1's in the initial configuration) is even (resp. odd), the cellular system converges to all 0 (resp. all 1) configuration.

cellular automata [4] where the system can able to compute the parity problem after following skewed updating scheme. Note that, the same is not true neither for traditional synchronous system nor for fully asynchronous system [20, 7].

With this background, the current work explore the computational ability of the proposed skewed system. Specifically, we explore the computational ability of skewed system in solving density classification problem (i.e., distributed consensus problem) [10, 3, 13, 4]. In the computational task of density classification problem, if the initial configuration is associated with more number of 1's in comparison with number of 0's, then the cellular system converges to all 1; otherwise, the system converges to all 0. The work of [12] reported the impossibility result of density classification after considering (traditional) synchronous elementary CA system. However, it is possible to solve the problem with 'good' accuracy by using combination of two elementary CA rules[5] following temporally non-uniform [9] and stochastic (probabilistic) [5] rule selection mechanism. Here, Section 2 revisits the notion of skewed asynchronous system [19] in the context of density classification problem [4]. Recall that, in our early [19], we identified the convergence of elementary CA rules under skewed asynchronous updating scheme following the experimental qualitative and quantitative approach. According to the problem statement of density classification task, two attractor (all 0 and all 1) elementary skewed asynchronous systems are the capable candidate for this problem. In this direction, Section 3 theoretically identifies the eligible candidate two-attractor systems. However, a two attractor system does not 'always' guarantee good accuracy in terms of solving density classification. Therefore, we explore the *basins of attraction* dynamics for these two-attractor systems to identify 'more' suitable skewed systems (see Section 3). Here, we use the notion of *clouds* [17] to classify the basins of attraction dynamics of the system into *deterministic*, *eccentric*, and *partially eccentric* cloud systems. Finally, Section 4 reports the density classification capability of skewed asynchronous system.

2 Skewed asynchronous system, Dynamics, and Convergence

Here, we consider one-dimensional two-state three-neighbourhood (self, left, and right neighbours) cellular automata under periodic boundary condition, i.e., elementary cellular automata (ECA). In this system, each cell can be represented by set of indices $\mathcal{L} = \mathbb{Z}/n\mathbb{Z}$, where n is the number of cells or CA size. Each cell is associated with a state at each time step where the set of states is $\mathcal{Q} = \{0, 1\}$. Here,

[4] Following the local transition function: f(a, b, c, d) = $(1-b, 1-c)$ if $a \neq b$ and $c \neq d$; otherwise, f(a, b, c, d) = (b, c).

[5] Elementary CA rules 184 and 232.

a *configuration* represents the collection of all states at a given time. Therefore, $\varepsilon_n = \{0,1\}^{\mathcal{L}}$ depicts the set of configurations. Here, the local transition function can be defined as $f : \{0,1\}^3 \to \{0,1\}$. That is, a cell updates its state following the self state and neighbours (left and right) state. We represent this local transition function following a look-up table (see Table 1). In the look-up table, each argument (say, (x,y,z)) of local transition function is denoted by Rule Min Term (RMT) (say, r), where $r = 4 \times x + 2 \times y + z$. Moreover, the decimal equivalent of the eight outputs of the local transition function is called *rule*, i.e., $f(1,1,1) \cdot 2^7 + f(1,1,0) \cdot 2^6 + \cdots + f(0,0,0) \cdot 2^0$. Here, if the state of a cell remains same, then we call the corresponding RMT as *passive*, i.e., $f(x,y,z) = y$; otherwise, we call the corresponding RMT as *active*, i.e., $f(x,y,z) \neq y$. We have $2^8 = 256$ ECAs, out of which 88 are minimal representative ECAs and the remaining are their equivalent. Table 1 depicts the example look-up table for elementary CA 170 and 184. In Table 1, RMT 6 $(110 \to 0)$ is active and RMT 3 $(011 \to 1)$ is passive for both these ECAs.

(x,y,z) (RMT)	111 (7)	110 (6)	101 (5)	100 (4)	011 (3)	010 (2)	001 (1)	000 (0)	Rule
$f(x,y,z)$	1	0	1	0	1	0	1	0	170
	1	0	1	1	1	0	0	0	184

Table 1: Look-up table for elementary CA rules 170 and 184.

Note that, a configuration can also be written as a sequence of RMTs. However, it is not an arbitrary sequence of RMTs. In our early work [21], we have also introduced the notion of *primary RMT sets*, say T, where $\{0\}$, $\{7\}$, $\{2,5\}$, $\{4,1,2\}$, $\{5,3,6\}$, $\{1,3,6,4\}$ are the primary RMT sets after considering elementary CA system. Now, one or more combination of primary RMT sets is responsible for any CA configuration where a configuration can not be formed with a subset of a primary RMT set T. That is, if a RMT $r \in T$ is appeared m times $(m > 0)$ in a RMT sequence or configuration, then the other members of that primary RMT set T should appear m times in the corresponding configuration (if applicable). Here, we call the corresponding configuration of each primary set as *homogeneous configuration*, i.e., **0**, **1**, **01**, **001**, **011** and **0011** respectively [21]. Here, we denote configuration all 0 or 0^n (resp. all 1 or 1^n) as **0** (resp. **1**). Similarly, $(01)^{n/2}$ is denoted as **01**, and so on.

Next, we introduce the proposed skewed asynchronous system, where at each time step, we randomly and uniformly select one cell (say i) and update cell i and its right neighbouring cell $i+1$ following the local transition function. Let us consider, $(U_t)_{t \in \mathbb{N}} \in \mathcal{L}^{\mathbb{N}}$ represents the random sequence of selected cells for update. Here, evolution of the system starting from an initial configuration x can be reported by the stochastic process $(x^t)_{t\in\mathbb{N}}$, and defined recursively by: $x^0 = x$ and $x^{t+1} = F(x^t, U_t)$ with $x_i^{t+1} = \begin{cases} f(x_{i-1}^t, x_i^t, x_{i+1}^t) & \text{if } i = U_t \text{ or } i = U_t + 1 \\ x_i^t & \text{otherwise.} \end{cases}$

Note that, for fully asynchronous system, following the similar notation, we write $x_i^{t+1} = \begin{cases} f(x_{i-1}^t, x_i^t, x_{i+1}^t) & \text{if } i = U_t \\ x_i^t & \text{otherwise.} \end{cases}$

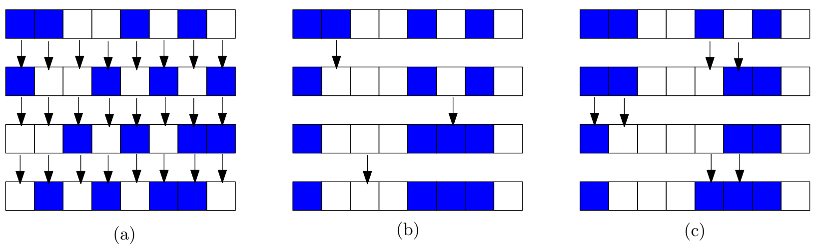


Figure 1: Evolution of elementary CA 170 following (a) traditional synchronous; (b) fully asynchronous; (c) skewed asynchronous updating scheme.

For example, Figure 1 depicts evolution of cellular system following elementary CA rule 170 (in Table 1, see the look-up table). Here, in Figure 1(a), the system follows a global clock, that is, synchronous updating scheme. In other words, the cellular system dictates all cells to update together. In Figure 1(b), we select a random cell for update at each time step following fully asynchronous updating scheme. For evidence, we select the second cell during the first time step, sixth cell during the second time step, and so on. Note that, here, the cellular system dictates two neighbouring cells not to update together. Figure 1(c) depicts the skewed asynchronous updating scheme where we select fifth cell during first time step, and update both the fifth and sixth (right neighbouring cell) cell together.

In this context, in our early work [19], we have explored the space-time dynamics of the proposed skewed asynchronous system in comparison with fully asynchronous and classical (synchronous) CA system. Figure 2 displays evidence space-time diagrams for ECA rules 4, 54, 58 and 142. Here, many elementary CA rules have shown *strong* resistance against this skewed perturbation (after considering both fully asynchronous and synchronous system). For evidence, in Figure 2, ECA 4 shows similar *fixed point* dynamics for all these updating schemes. Similarly, ECA 142 also shows (a kind of *week*) resistance against this perturbation, see the *periodic* dynamics of ECA 142 in Figure 2. However, the same is not applicable for ECA rules 54 and 58. Observe that, ECA 54 depicts non-convergent (specifically, recurrent or reversible [18]) dynamics under fully asynchronous updating scheme and non-convergent *complex* dynamics under classical synchronous updating scheme. However, in Figure 2, ECA 54 shows convergent (i.e., converges to all 0) dynamics following the proposed skewed asynchronous updating scheme. Note that, the same is also applicable for ECA rules 38, 134, and 150. In a opposite direction, ECA 58 respectively depicts convergent and non-convergent dynamics under fully and skewed updating scheme, see Figure 2. Here again, the same is also applicable for ECA rules 26, 90, and 122. To sum up, the cellular system may show brutal change in dynamics with absence (or presence) of this atomicity property.

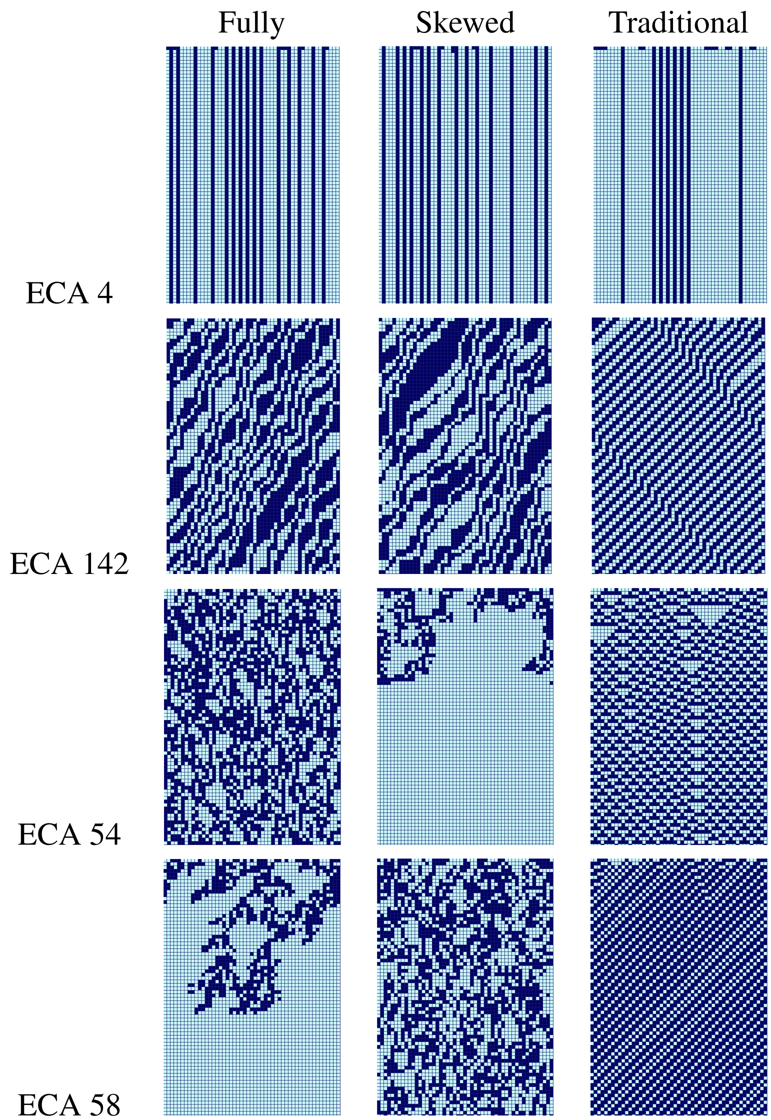


Figure 2: Dynamics of elementary CA system for changing updating schemes (i.e., fully asynchronous, skewed asynchronous, and traditional CA system).

Recall that, in this study, we want to explore density classification capability of skewed asynchronous system. Here, a cellular system should evolve towards point attractor all 1 when the initial configuration is associated with more number of 1's in comparison with number of 0's. On the other hand, if the initial configuration is associated with more number of 0's in comparison with number of 1's, then the cellular system should converge to point attractor all 0. Note that, here, we call a configuration $x \in \varepsilon_n$ as *point attractor* following $\forall u \in \mathcal{L}$, $F(x, u) = x$. In other words, x is associated with only passive RMTs. Here, this property of point attractor is independent of the updating environment. Next, following the problem statement of density classification problem, the task is to theoretically explore two attractor (all 0 and all 1) skewed asynchronous system using the notion of *primary RMT sets* and *homogeneous configurations*.

3 Two attractor skewed systems and cloud dynamics

Following the problem statement of density classification problem, here, we first target to identify skewed asynchronous system with following theoretical properties:

(a) skewed asynchronous system with the only point attractors **0** (all 0) and **1** (all 1); and

(b) the skewed asynchronous system need to be convergent.

In this direction, we report the following results.

Theorem 1 *Following* 13 *minimal representative ECA rules, out of* 88 *minimal ECA rules, are associated with the only point attractors* ***0*** *and* ***1****:*
`128, 130, 136, 138, 146, 152, 154, 160, 162, 168, 170, 178, 184`

Proof : Here, the discussion is trivial following the notion of primary RMT sets and homogeneous configurations. Following primary RMT sets (combination of one or more) are responsible for the formation of any CA configuration - $\{0\}$, $\{7\}$, $\{2{,}5\}$, $\{4{,}1{,}2\}$, $\{5{,}3{,}6\}$, $\{1{,}3{,}6{,}4\}$. We call a primary RMT set passive if all member RMTs of the primary RMT set are passive. Note that, for a point attractor, all the RMTs are passive. Therefore, passive primary RMT sets are responsible for formation of point attractors. Therefore, ECA R is associated with point attractors **0** and **1** when primary RMT sets $\{0\}$ and $\{7\}$ are passive, respectively. Moreover, **0** and **1** are the *only* point attractors of the system when the rest primary RMT sets (i.e., $\{2{,}5\}$, $\{4{,}1{,}2\}$, $\{5{,}3{,}6\}$, $\{1{,}3{,}6{,}4\}$) are not passive. This is only true for the above 13 minimal representative ECA (out of 88 minimal rules). $\square$

However, the presence of point attractors **0** and **1** does not guarantee that the system is convergent. It is also possible that a point attractor is only *self-reachable*. For evidence, **0** (all 0) configuration is a point attractor for ECA 60, however, this point attractor is only self-reachable under fully asynchronous updating scheme, see our early work of [18]. Therefore, to claim a convergent system, we need to show that the point attractor(s) are reachable from any configuration. However, under the proposed skewed asynchronous system, the system follows different path for different update pattern. Therefore, we need to show that the point attractor(s) are reachable from any configuration after considering all possible update pattern. In this context, note the following.

Lemma 2 *[19] Convergent skewed asynchronous CA are absorbing Markov chain.*

Now, following the properties of absorbing Markov chain, we need to only show that the point attractor(s) are reachable from any configuration after considering any update pattern. In our early work [19], we have identified the conditions of convergence after considering individual point attractors **0** (all 0) and **1** (all 1), see the following.

Theorem 3 *[19] Skewed asynchronous CA R with passive RMT 0 (resp. RMT 7) converges to point attractor **0** (resp. **1**) after satisfying one of the following conditions:*

(a) RMT 2 (resp. RMT 5) of R is active and at least one RMT from the pair of RMTs $\{1, 4\}$ (resp. $\{3, 6\}$) of R is passive.

(b) RMTs 3 and 6 (resp. 1 and 4) of R are active.

According to the Theorem 3, following are the conditions for convergence after considering point attractor **0** (all 0): *Condition A:* active RMT 2 and (at least) one passive RMT form the pair $\{1, 4\}$; and *Condition B:* active RMTs 3 and 6. Here, after satisfying (at least) one of the above conditions, the system converges to all 0. In a similar direction, following are the conditions for convergence after considering point attractor **1** (all 1): *Condition C:* active RMT 5 and (at least) one passive RMT form the pair $\{3, 6\}$; and *Condition D:* active RMTs 1 and 4. Here again, after satisfying (at least) one of the above conditions, the system converges to all 1. Note that, condition A and condition D contradicts with each other. Therefore, both can not be true together. Moreover, the same is also true for condition B and condition C. To sum up, here, a two attractor skewed system with point attractors **0** and **1** converges after satisfying one of the above four conditions.

Proposition 4 *All the* 13 *minimal representative two attractor skewed CA system of Theorem 1 are convergent.*

Proof : To establish the convergence property of the 13 minimal representative ECA rules of Theorem 1, we need to satisfy at least one of the above four conditions for each ECA rule. Note that, if the system is convergent after considering either point attractor **0** or point attractor **1**, then the system is convergent. In other words, it is not required to fulfil the convergence criteria for both of the point attractors (i.e., **0** and **1**). Following Table depicts the situations for these 13 ECA rules after considering the conditions A, B, C and D. In this Table, if ECA R follows condition X (where, $X \in \{A, B, C, D\}$), then we mark with "✓"; otherwise, we note with "×". Here, the Table also reports the active and passive RMTs for each ECA rule, see the following.

ECA	Active RMTs	Passive RMTs	Condition A	Condition B	Condition C	Condition D
128	2,3,6	0,1,4,5,7	✓	✓	×	×
130	1,2,3,6	0,4,5,7	✓	✓	×	×
136	2,6	0,1,3,4,5,7	✓	×	×	×
138	1,2,6	0,3,4,5,7	✓	×	×	×
146	1,2,3,4,6	0,5,7	×	✓	×	✓
152	2,4,6	0,1,3,5,7	✓	×	×	×
154	1,2,4,6	0,3,5,7	×	×	×	✓
160	2,3,5,6	0,1,4,7	✓	✓	×	×
162	1,2,3,5,6	0,4,7	✓	✓	×	×
168	2,5,6	0,1,3,4,7	✓	×	✓	×
170	1,2,5,6	0,3,4,7	✓	×	✓	×
178	1,2,3,4,5,6	0,7	×	✓	×	✓
184	2,4,5,6	0,1,3,7	✓	×	✓	×

According to the above Table, all these 13 ECA rules are marked with at least one "✓". Therefore, all these 13 ECA rules are convergent. □

To sum up, all these 13 ECA rules (of Theorem 1) are convergent where the only point attractors are all 0 and all 1. Next, we explore the *basins of attraction* dynamics of these ECA rules following the problem statement of density classification.

Proposition 5 *The point attractor **1** is not reachable from any configuration (excluding itself) for following ECA rules:* `128`, `130`, `136`, `138`, `152`, `160`, `162`

Proof : Under the skewed asynchronous updating scheme, the immediate possible previous configurations (i.e., predecessor) of all 1 configurations are $(\cdots 1110111\cdots)$ and $(\cdots 11100111\cdots)$. Here, the corresponding RMT sequences respectively are $(\cdots 7765377\cdots)$ and $(\cdots 77641377\cdots)$. In this context, recall that, skewed asynchronous scheme only allows to update two neighbouring cells in each time step. Here, to reach all 1 configuration starting from configuration $(\cdots 1110111 \cdots)$, we need either active RMT 5 and passive RMT 6 or active RMT 5 and passive RMT 3. However, that is not true for the above ECA rules. On the other hand, to reach all 1 configuration starting from configuration $(\cdots 11100111 \cdots)$, we need active RMTs 1 and 4. Again, that is not true for the above ECA rules. Therefore, all 1 configuration is not the successor of any configuration (excluding itself) for the above ECA rules. Lastly, it is possible to reach **1** form itself because **1** is a point attractor of the system (i.e., all RMTs are passive). □

To sum up, according to Proposition 4, ECA rules 128, 130, 136, 138, 152, 160, 162 are convergent with point attractors **0** and **1**. However, according to Proposition 5, point attractor **1** is not reachable from any configuration (excluding itself) for the above ECA rules. Therefore, here, all configurations (excluding **1**) reaches to point attractor all 0 in the basins of attraction dynamics. However, it violates the requirement of density classification task. Therefore, we exclude ECA rules 128, 130, 136, 138, 152, 160 and 162 as the candidate of density classification task.

Proposition 6 *The point attractor **0** is not reachable from any configuration (excluding itself) for ECA rule* `154`.

Proof : Under the skewed asynchronous updating scheme, the immediate possible previous configurations (i.e., predecessor) of all 0 configurations are $(\cdots 0001000 \cdots)$ and $(\cdots 00011000 \cdots)$. The corresponding RMT sequences respectively are $(\cdots 0012400 \cdots)$ and $(\cdots 00136400 \cdots)$. Here, to reach all 0 configuration starting from configuration $(\cdots 0001000 \cdots)$, we need either active RMT 2 and passive RMT 1 or active RMT 2 and passive RMT 4. However, that is not true for ECA 154. On the other hand, to reach all 0 configuration starting from configuration $(\cdots 00011000 \cdots)$, we need active RMTs 3 and 6. Again, that is not true for ECA 154. Therefore, all 0 configuration is not the successor of any configuration (excluding itself) for the ECA 154. In the context of self reachability, **0** is a point attractor. □

Therefore, here, all configurations (excluding **0**) reaches to point attractor all 1 in the basins of attraction dynamics for ECA 154. Again, it violates the requirement of density classification task. Therefore, we also exclude ECA 154 as the candidate of density classification task. In this context, in our early work [17], we have introduced the *cloud* property of asynchronous cellular system which contradicts with the deterministic predecessor-successor relationship of (traditional) synchronous system. In asynchronous system, the system may reach to different point attractor in different run starting from a single initial configuration. Following this, we denote a configuration as *cloud configuration* when it is possible to reach more than one point attractors starting from the corresponding configuration; otherwise, we call the configuration as *deterministic configuration*. Next, for a skewed asynchronous system, if every configuration is a deterministic configuration, then we call the skewed asynchronous system as a *deterministic cloud skewed system*. Further, we denote a skewed system as an *eccentric cloud skewed system* when every configuration (except the point attractors) are cloud configuration. Lastly, if some configurations of the system are deterministic and some configurations are cloud, then we name the system as a *partially eccentric cloud system*, for the formal definition see [17]. Clearly, here, all configurations of ECA rules 128, 130, 136, 138, 152, 160, 162 are deterministic configurations (travelling towards only point attractor all 0). The same is also applicable for ECA 154 (travelling towards only point attractor all 1). Therefore, these ECA rules show deterministic cloud dynamics under skewed asynchronous updates. Note that the same is not true for fully asynchronous systems, for example, ECA 162 depicts eccentric cloud behaviour under fully asynchronous update. Now, according to the above discussion, we remark the following.

Remark 7 *Depending on the above discussion, ECA rules* 128*,* 130*,* 136*,* 138*,* 152*,* 154*,* 160*, and* 162 *with deterministic cloud dynamics are not the appropriate candidate for density classification problem.*

Therefore, for the further discussion, we only consider two attractor skewed asynchronous ECA rules 146, 168, 170, 178, and 184 as the candidate for density classification problem. Here, we first focus on ECA 168 where the RMTs 2, 5, 6 are active and RMTs 0, 1, 3, 4, 7 are passive. Therefore, it satisfies the convergence condition A (for the point attractor **0**) and condition C (for the point attractor **1**). Here, to reach point attractor **0** starting from any configuration, we need to disappear state 1 from the configuration. Similarly, we need to disappear state 0 from the configuration to reach point attractor **1**. Now, starting from a configuration with isolated state 1 ($\cdots 00100 \cdots$), i.e., RMT sequences ($\cdots 01240 \cdots$), there are two options which disappear isolated state 1 (following either active RMT 2 and passive 1 or active RMT 2 and passive 4). However, to disappear isolated state 0, the system is associated with only one option (active RMT 5 and passive RMT 6). Therefore, as an initial comment, the system is more biased towards point attractor **0**. Moreover, if we consider two or more consecutive 1's in the configuration ($\cdots 0011 \cdots 1100 \cdots$), i.e., RMT sequences ($\cdots 0137 \cdots 7640 \cdots$), then it is possible to disappear state 1 following active 6 and passive 7/4. However, if we consider two or more consecutive 0's in the configuration ($\cdots 1100 \cdots 0011 \cdots$), i.e., RMT sequence ($\cdots 7640 \cdots 0137 \cdots$), it is not possible to disappear state 0. In fact, it is possible to disappear only state 1 following the previous approach. Therefore, these configurations (with two or more consecutive 0's) act like a deterministic configuration. And, the ECA 168 shows partially eccentric dynamics which is not suitable for density classification problems.

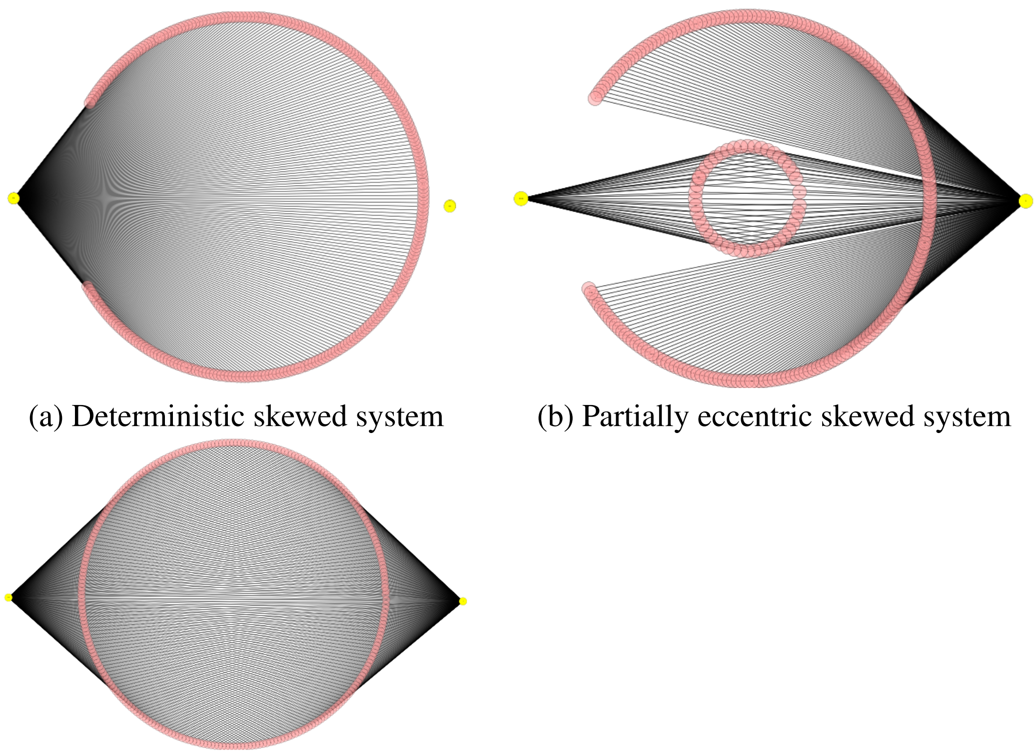
In a more theoretical direction, if a two attractor skewed ECA system satisfies condition A and C for convergence, then the equal participation of RMT sets $\{1,4\}$ and $\{3,6\}$ in the passive RMT set is a desirable condition for density classification problem. In other words, the equal participation of RMT sets $\{1,4\}$ and $\{3,6\}$ in the passive RMT set is responsible for eccentric cloud dynamics; otherwise, the system depicts partially eccentric dynamics. Here, ECA rules 170 and 184 also follow only conditions A and C. According to the above discussion, ECA 168 shows partially eccentric dynamics, and ECA 170 and 184 show eccentric cloud dynamics.

Remark 8 *Depending on the above discussion, ECA* 168 *with partially eccentric cloud dynamics is not the appropriate candidate for density classification problem. Moreover, ECA rules* 170 *and* 184 *with eccentric cloud dynamics are the appropriate candidate for the density classification problem.*

Next, we focus on ECA rules 146 and 178 which satisfy the convergence condition B (active RMTs 3 and 6; for the point attractor **0**) and condition D (active RMTs 1 and 4; for the point attractor **1**). Note that, ECA 146 (resp. 178) is associated with passive RMTs 0, 5, 7 (resp. 0, 7). Here, passive RMT 5 of ECA 146 plays an important role. Again, to reach point attractor **0** (resp. **1**), the target is to disappear state 1 (resp. 0). For both of the rules, the disappearance of state 0(s) and 1(s) (isolated/two or more consecutive) follow a similar direction. However, presence of the flip-flop pattern (010/101) in the configuration acts differently for these two ECA rules. For ECA 178, during update of a flip-flop pattern, active RMTs 2 and 5 apply; pattern 10 moves to 01. That is, the overall number of 1 in the system is conserved during that step. However, for ECA 146, active RMT 2 and passive RMT 5 apply, where state 1 disappears, i.e., pattern 10 moves to 00. That is, the system moves towards point attractor **0**. Therefore, overall, the system of ECA 146 is more biased towards point attractor all 0 which violates the desirable situation for density classification problem. However, the same is not true for ECA 178. In this context, we have also classified the cloud system into a *biased* and *unbiased* cloud system in our early work, see [17]. Following the above discussion, we remark the following.

Remark 9 *Depending on the above discussion, ECA* 146 *with biased cloud dynamics is not the appropriate candidate for density classification problem.*

Remark 10 *Finally, in this study, we consider ECA rules* 170*,* 178 *and* 184 *as the appropriate candidate for the density classification problem.*



(a) Deterministic skewed system

(b) Partially eccentric skewed system

(c) Eccentric skewed system

Figure 3: (a) Deterministic skewed ECA 130 system; (b) Partially eccentric skewed ECA 168 system; and (c) Eccentric skewed ECA 178 system.

Next, to visualize this theoretical understanding, we plot the basins of attraction dynamics of two attractor skewed systems following an experimental approach. In this experiment, we run an initial configuration, say x, 100 times to capture their destination point attractor(s). Following this, we calculate the probability to reach point attractor $\mathbf{0}$ and $\mathbf{1}$ starting from initial configuration x; denoted as $\mathbb{P}(\mathbf{0})_x$ and $\mathbb{P}(\mathbf{1})_x$. Next, we compute the same for each initial configuration $x \notin \{\mathbf{0}, \mathbf{1}\}$ (excluding the point attractors). Hereafter, we compute the overall probability of the system to reach the point attractors after considering every initial configuration following their individual probabilities. Here, $\mathbb{P}(\mathbf{0})$ and $\mathbb{P}(\mathbf{1})$ denote the overall probability of the system to reach point attractor $\mathbf{0}$ and $\mathbf{1}$. In this study, for ECA rules 128, 130, 136, 138, 152, 160, 162, $\mathbb{P}(\mathbf{0}) = 1$ and $\mathbb{P}(\mathbf{1}) = 0$. Similarly, for ECA 154, $\mathbb{P}(\mathbf{0}) = 0$ and $\mathbb{P}(\mathbf{1}) = 1$. Moreover, for partially eccentric and biased ECA rules 146 and 168, $\mathbb{P}(\mathbf{0}) \gg \mathbb{P}(\mathbf{1})$. However, for the candidate (for density classification) ECA rules 170, 178, and 184, $\mathbb{P}(\mathbf{0}) \approx \mathbb{P}(\mathbf{1})$ which validates our theoretical understanding. For visualization, next, we consider every initial configuration as the node of the system. The corresponding node for all 0 (resp. all 1) point attractor is connected with corresponding node for configuration x following edge, if $\mathbb{P}(\mathbf{0})_x > 0$ (resp. $\mathbb{P}(\mathbf{1})_x > 0$). Figure 3 depicts the basins of attraction dynamics for skewed ECA 130, 168 and 178 where the corresponding node for point attractor **0** and **1** are marked with yellow. Observe that, for ECA 130, every configuration converges to one point attractor, which depicts the deterministic cloud dynamics, see Figure 3(a). On the contrary, for ECA 178, every configuration is associated with a chance to converge both the point attractors which shows the eccentric cloud dynamics, see Figure 3(c). On the other hand, for ECA 168, some configurations are deterministic and some configurations show cloud dynamics, see Figure 3(b).

4 Results and Discussions

This section reports the efficiency of the proposed skewed asynchronous system following the density classification problem. According to the theoretical results, ECA rules 170, 178 and 184 with eccentric cloud dynamics are the appropriate candidate for density classification. Figure 4 shows the density classification ability of ECA rules 130, 152, 168, 170, 178 and 184 which reports the efficiency as a function of chances.

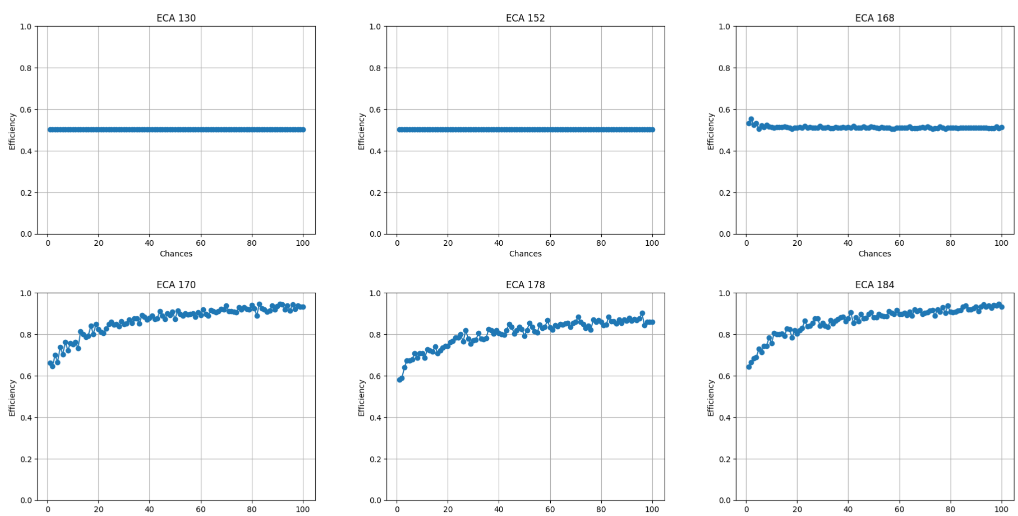


Figure 4: Density classification efficiency of skewed ECA rules 130, 152, 168, 170, 178 and 184.

In this study, we have considered every possible configuration of the system.

Moreover, for an initial configuration (say x), we run the system c (chances) times to capture the notion of many possible update patterns. Let us consider, for c_0 times (out of c chances) the configuration x converges to the point attractor $\mathbf{0}$; similarly, for c_1 times (out of the c chances) the configuration x moves towards point attractor $\mathbf{1}$. If $c_0 > c_1$, we consider all 0 as the attractor for configuration x; otherwise, all 1 is considered as the attractor for configuration x. Next, after computing the (initial) density (say d_x) of configuration x, we note the correctness of the classification. If $d_x < \frac{1}{2}$, then $c_0 > c_1$ indicates a correct classification. In other words, if $d_x \geqslant \frac{1}{2}$, then $c_0 < c_1$ reports a correct classification. Hereafter, the efficiency reports the percentage of correct classification after considering all possible initial configuration.

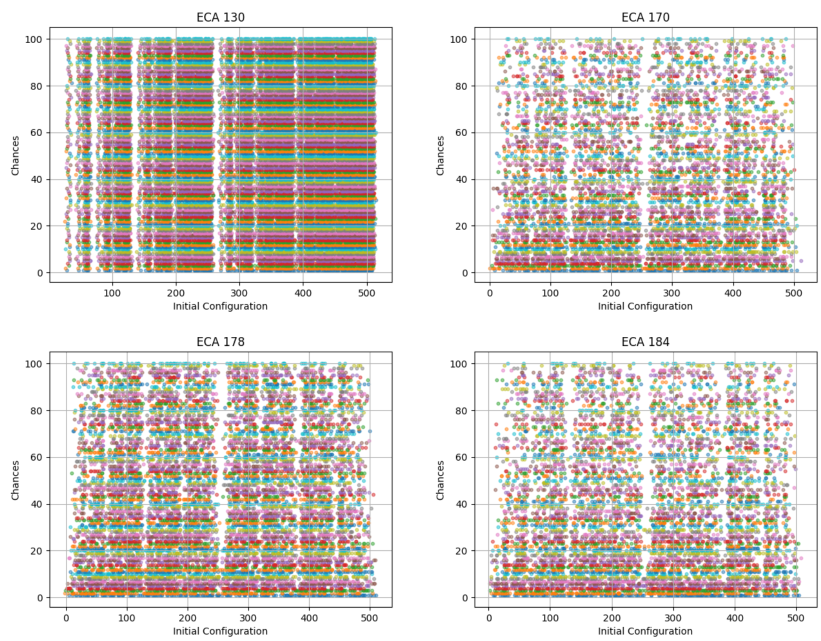


Figure 5: Incorrect results in density classification (colour dots) for increasing chances c after considering skewed ECA rules 130, 170, 178 and 184.

According to Figure 4, ECA 170 shows more than 90% efficiency after 50 chances ($c = 50$). Moreover, after 100 chances, ECA 170 depicts 97% efficiency. The similar result is also depicted by skewed ECA 184. On the other hand, ECA 178 reports 90% efficiency in density classification after 100 chances. Note that the system shows stable efficiency after 75 chances after considering ECA rules 170, 178 and 184, see Figure 4. Here, these results of Figure 4 validate the theoretical understanding. Moreover, according to theoretical results, ECA rules 130, 152

and 168 are not appropriate candidates for density classification. In Figure 4, these ECA rules show static (approx) 50% efficiency in density classification. In Figure 4, we consider $n = 13$ (here, we consider every possible initial configuration). For relatively large $n = 49$, if we consider random initial configuration, then the performance of the system remains the same. Figure 5 reports incorrect results in density classification (colour dots) for increasing chances c after considering skewed ECA rules 130, 170, 178 and 184 and $n = 8$. For ECA 130, there is no effect of increasing chances in the system. Observe that, for ECA rules 170, 178 and 184, only few initial configurations show incorrect results after getting 100 chances. Note that, here, the skewed asynchronous system with independence performs much better in comparison with traditional synchronous systems [4]. Moreover, skewed asynchronous systems (ECA rules 170 and 184) show competitive results in comparison with best performing temporally non-uniform [9] and stochastic (probabilistic) [5] cellular automata. However, the reason behind the relatively bad result of eccentric skewed ECA 178 is still open to us. Next, this research can also be extended in the following directions.

1. Here, to break the atomicity property, we have introduced the skewed asynchronous system, where two (say $k = 2$) neighbouring cells are updated together. What can be said about the $k > 2$ system (i.e., k neighbouring cells are updated together to capture the notion of burst error)?

2. Here, ECA rules 170, 178, 184 show good performance. However, the convergence time of the system is also an important factor (for example, if the system takes exponential time during convergence, then it is not desirable) which is still theoretically open to us. In this context, note that the experimental results report fast convergence for these ECA rules.

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