1. $\psi(x) = Nx \exp\left(-\frac{1}{2}\alpha x^2\right)$, $\alpha > 0$

a) This is first excited state for harmonic oscillator,

so $E_2 = \frac{3}{2}\hbar\omega$; $\alpha = \frac{m\omega}{\hbar}$

so that $N = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{2}{\sqrt{2^1 \cdot 1!}}$ $\left(\frac{1}{\sqrt{2^n n!}} \text{ for } n^{th} \text{ state}\right)$

$\langle \hat{x} \rangle = 0$ by symmetry of $\psi(x)$

$\langle \hat{x}^2 \rangle = \frac{1}{2}\frac{\hbar}{m\omega}(2n+1)$ (general case)

so for our $\psi(x)$; $\langle x^2 \rangle = \frac{3}{2}\frac{\hbar}{m\omega}$

b) $\langle \hat{p} \rangle = 0$ since $\psi(x)$ is odd so $\psi'(x)$ is even so the function integrated for $\langle \hat{p} \rangle$ will be odd $\Rightarrow \langle \hat{p} \rangle = 0$

$\langle \hat{p}^2 \rangle = \frac{\hbar m\omega}{2}(2n+1)$ (general case)

$\Rightarrow \langle \hat{p}^2 \rangle = \frac{3\hbar m\omega}{2}$ for our $\psi(x)$

Comments : i) for harmonic oscillator, expectations are best computed in energy eigen-basis and not x or p-space

ii) note that $\langle \hat{x}^2 \rangle = \frac{1}{2}\frac{\hbar}{m\omega}(2n+1)$ — (1)

while $\langle \hat{p}^2 \rangle = \frac{1}{2}\hbar m\omega(2n+1)$ is obtained

by : $\boxed{m \to \frac{1}{m\omega},\ \omega \to \omega,\ x \to p}$, this is general recipe

to go from x- space to p-space

c) Only $\psi(x) = \delta(x-x')$ are position eigenstate.
Only complex exponentials are momentum eigenstate

d) $V(x) = 0$

$$\Rightarrow \quad H = \frac{p^2}{2m}$$

$$\Rightarrow \quad \langle H \rangle = \frac{\langle p^2 \rangle}{2m} = \frac{\hbar m\omega}{2 \cdot 2m}(2n+1) = \frac{\hbar\omega}{2}\left(n+\frac{1}{2}\right)$$

e) i) Here, $\hat{H}\psi = E\psi$

$$\Rightarrow \quad -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + V\psi = E\psi$$

$$\Rightarrow \quad -\frac{\hbar^2}{2m}\left[-3\alpha \cancel{x e^{-\alpha x^2/2}} + \alpha^2 x^{\cancel{3}\,2} \cancel{e^{-\alpha x^2/2}}\right] + V(x)\cancel{x e^{-\alpha x^2/2}} = E\cancel{x e^{-\alpha x^2/2}}$$

(At $x=0$, eqn is zero trivially)

$$\Rightarrow \quad \frac{3\hbar^2\alpha}{2m} - \frac{\hbar^2\alpha^2 x^2}{2m} + V(x) = E$$

given $V(0) = 0$

$$\Rightarrow \quad E = \frac{3\hbar^2\alpha}{2m}, \quad V(x) = \frac{\hbar^2\alpha^2 x^2}{2m}$$

if we put $\alpha = m\omega/\hbar$ as claimed in a),

$$E = \frac{3\hbar\omega}{2}, \quad V(x) = \frac{1}{2}m\omega^2 x^2$$

ii) No, ground state cannot have nodes