7 - Pset 2
MZTOCW
2013-8.05

4. $V(x) < 0 \qquad \forall x \qquad \lim_{|x|\to\infty} V(x) = 0$

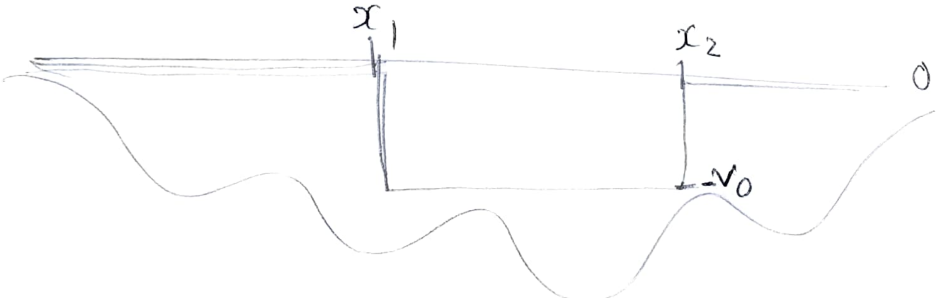
$$\psi_\alpha(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2}$$

$$E(\alpha) = \int dx\, \psi_\alpha^*(x)\, \hat{H}\, \psi_\alpha(x)$$

$$\hat{H}\psi_\alpha(x) = \frac{\hbar^2\alpha}{2m} e^{-\alpha x^2/2} - \frac{\hbar^2\alpha^2 x^2}{2m} e^{-\alpha x^2/2} + V(x)\psi_\alpha(x)$$

$$\Rightarrow E(\alpha) = \frac{\hbar^2\alpha}{4m} + \int V(x)\,\psi_\alpha(x)\,\psi_\alpha(x)\,dx$$

now, we approximate $V(x)$ by $V_1(x)$:



$$\Rightarrow E(\alpha) \le \frac{\hbar^2\alpha}{4m} - \left(\frac{\alpha}{\pi}\right)^{1/2} V_0 \int_{x_1}^{x_2} e^{-\alpha x^2}\, dx$$

now $\quad -\left(\frac{\alpha}{\pi}\right)^{1/2} V_0 \int_{x_1}^{x_2} e^{-\alpha x^2}\, dx \le -\left(\frac{\alpha}{\pi}\right)^{1/2} V_0\, e^{-\alpha x_3^2}\, |x_2 - x_1|$

where $x_3 = \max\{|x_2|, |x_1|\}$

since $\int_{x_1}^{x_2} e^{-\alpha x^2}\, dx \ge e^{-\alpha x_3^2}\, |x_2 - x_1|$

$$\Rightarrow E(\alpha) \le \frac{\hbar^2\alpha}{4m} - \left(\frac{\alpha}{\pi}\right)^{1/2} V_0\, e^{-\alpha x_3^2}\, |x_2 - x_1|$$

$$\le \sqrt{\alpha}\left(\frac{\hbar^2\sqrt{\alpha}}{4m} - \frac{V_0}{\sqrt{\pi}} e^{-\alpha x_3^2}\, |x_2 - x_1|\right)$$

if $\alpha = 0$, rhs $= 0$, but ψ will not be normalizable

but $\left(\frac{\hbar^2\sqrt{\alpha}}{4m} - \frac{V_0}{\sqrt{\pi}} e^{-\alpha x_3^2}\, |x_2 - x_1|\right)_{\alpha=0} = -\frac{V_0}{\sqrt{\pi}}\, |x_2 - x_1|$

and $E(\alpha_1) = 0$ when $\alpha_1 : \frac{\hbar^2 \sqrt{\alpha_1}}{4m} = \frac{V_0}{\sqrt{\pi}} e^{-\alpha_1 x_3^2} |x_2 - x_1|$

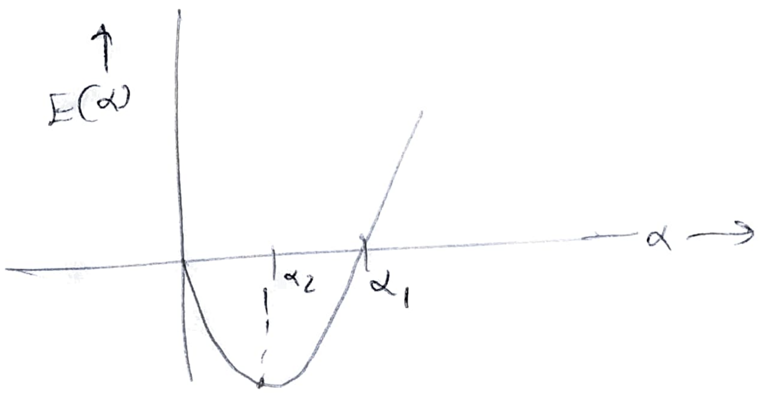
$\Rightarrow$ an α_2 can be found such that

$$0 < \alpha_2 < \alpha_1$$

and $E(\alpha_2) < 0$

since we see that $\frac{\hbar^2 \sqrt{\alpha}}{4m} - \frac{V_0}{\sqrt{\pi}} e^{-\alpha x_3^2} |x_2 - x_1| < 0$

while $\sqrt{\alpha} > 0$ $\Rightarrow$ $E(\alpha_2) < 0$



Thus we found a bound of E_{gs} by variational principle thus