1-Pset-2
MIT OCW
8.05-2013

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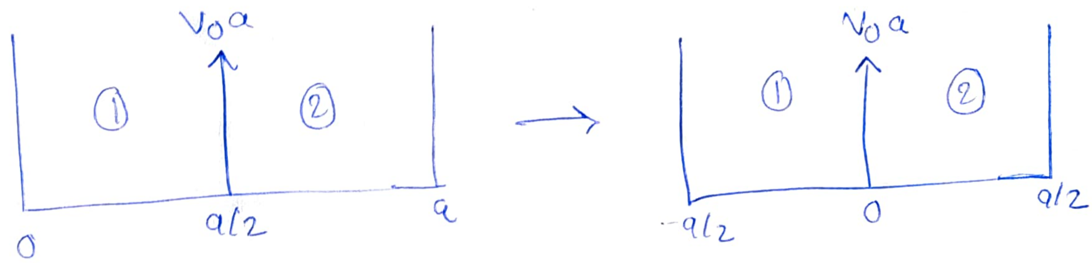
1.



in ①: $\psi(x) = A\cos[k(x+a/2)] + B\sin[k(x+a/2)]$ $\quad k = \sqrt{\frac{2mE}{\hbar^2}}$

in ②: $\psi(x) = C\cos[k(x+a/2)] + D\sin[k(x+a/2)]$

$\psi(-a/2) = 0 \Rightarrow A = 0$

$\psi(a/2) = 0 \Rightarrow C = 0$

$\psi(0^-) = \psi(0^+) \Rightarrow B = D$

$$\frac{d\psi}{dx}\Big|_{0^-}^{0^+} = \frac{2m}{\hbar^2} V_0 a\, \psi(0)$$

$$\Rightarrow -k\cancel{B}\cos(ka/2) - k\cancel{B}\cos(ka/2) = \frac{2mV_0 a}{\hbar^2}\cancel{B}\sin(ka/2)$$

$$\Rightarrow -k\cot(ka/2) = \frac{mV_0 a}{\hbar^2} = \frac{\gamma}{a} \quad ; \gamma \gg 1$$

$$\Rightarrow -\frac{\tan(ka/2)}{k} = \frac{a}{\gamma}$$

lowest state will occur when $(ka/2)$ is just less than π

$$\Rightarrow -\frac{\tan(ka/2)}{k}\Bigg|_{\frac{ka}{2}=\pi} + \frac{d}{d(ka/2)}\left[-\frac{\tan(ka/2)}{k}\right]\Bigg|_{\frac{ka}{2}=\pi} \cdot \delta = \frac{a}{\gamma}$$

where $\delta = \frac{ka}{2} - \pi$ or deviation

now $$-\frac{\tan(ka/2)}{k}\Bigg|_{\frac{ka}{2}=\pi} = 0$$

$$\frac{d}{d(ka/2)}\left[-\frac{\tan(ka/2)}{k}\right] = \frac{a}{2}\frac{d}{du}\left[-\frac{\tan u}{u}\right]$$

$$= \frac{a}{2} \; -\frac{\sec^2 u}{u} + \frac{\tan u}{u^2}\Bigg|_{\frac{ka}{2}=\pi} = -\frac{a}{2\pi}$$

$$\Rightarrow -\frac{a\delta}{2\pi} = \frac{1}{\gamma} \Rightarrow \delta = -\frac{2\pi\cdot\cancel{a}}{ra}$$

or $$\frac{ka}{2} = \pi + \delta = \pi \mp \frac{2\pi}{ra}$$

$$\Rightarrow k = \frac{2\pi}{a} \mp \frac{4\pi}{ra^2}$$

$$\Rightarrow E = \frac{\hbar^2}{2m}\left[\frac{4\pi^2}{a^2} + \frac{16\pi^2}{r^2a^2} \mp \frac{16\pi^2}{ra^2}\right]$$

$$\Rightarrow E \approx \frac{4\hbar^2}{2ma^2} - \frac{8\hbar^2\pi^2}{2mra^2}$$, energy of first excited state is $\frac{4\hbar^2}{2ma^2}$, as $V_0 \to \infty$, $E \to \frac{4\hbar^2}{2ma^2}$