PSET-6 P3
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Shlok Vaibhav Singh

3. a) $H = -\gamma B\widehat{S_z} = -\frac{-\gamma B\hbar\sigma_3}{2}$

since $\sigma^2 = 1$, and in PSET-2, we have shown that :

$$e^{iM\theta} = \cos\theta + iM\sin\theta, \qquad \text{if } M^2 = I \tag{1}$$

$\Rightarrow e^{-iHt/\hbar} = \cos\frac{\gamma Bt}{2} + i\sin\frac{\gamma Bt}{2}\sigma_3$

Given $|\Psi, 0\rangle = |x, +\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$|\Psi, t\rangle = \mathcal{U}(t, 0)|\Psi, 0\rangle = e^{-iHt/\hbar}$

$= \left[\cos\frac{\gamma Bt}{2} + i\sigma_3\sin\frac{\gamma Bt}{2}\right]\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$= \frac{1}{\sqrt{2}}\begin{pmatrix}\cos\frac{\gamma Bt}{2} + i\sin\frac{\gamma Bt}{2}\\ \cos\frac{\gamma Bt}{2} - i\sin\frac{\gamma Bt}{2}\end{pmatrix}$

$= \frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\frac{\gamma Bt}{2}}\\ e^{-i\frac{\gamma Bt}{2}}\end{pmatrix}$

$= \cos\frac{\pi}{4}|+\rangle + \sin\frac{\pi}{4}e^{-i\gamma Bt}$

$\Rightarrow \theta = \frac{\pi}{4}, \phi = -\gamma Bt$

Thus direction of spin rotates clockwise in x-y plane

b) Measurement along z-axis gives $\hbar/2$ with probabilty of half

$\Rightarrow \langle H\rangle = 1/2\frac{-\gamma B\hbar}{2} + 1/2\frac{\gamma B\hbar}{2} = 0$

$\langle H^2\rangle = \frac{1}{2}\frac{\gamma^2B^2\hbar}{4} + \frac{1}{2}\frac{\gamma^2B^2\hbar^2}{4} = \frac{\gamma^2B^2\hbar^2}{2}$

$\Rightarrow \Delta H = \frac{\gamma B\hbar}{2}$

while $\cos^2\phi(t) = |\langle\Psi(0)|\Psi(t)\rangle|^2$

for evolution to orthogonal state, $t = \frac{\pi}{\gamma B}$

$$\Longrightarrow \Delta H \Delta t = \frac{h}{4}$$

Inequality is saturated thus