MZI
2013

6. a) $\Psi(x,t)$ obeys

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t)$$

a) $\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0$

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}\psi^*\psi = \psi^*\frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t}\psi$$

$$= \psi^*\left(\frac{i\hbar}{2m}\frac{\partial^2\psi}{\partial x^2} - \frac{i}{\hbar}V\psi\right) + \psi\left(-\frac{i\hbar}{2m}\frac{\partial^2\psi^*}{\partial x^2} + \frac{i}{\hbar}V\psi^*\right)$$

$$= \frac{i\hbar}{2m}\left[\psi^*\frac{\partial^2\psi}{\partial x^2} - \psi\frac{\partial^2\psi^*}{\partial x^2}\right] = \frac{\hbar}{m}\operatorname{Im}\left(\psi\frac{d^2\psi^*}{\partial x^2}\right)$$

$$\frac{\partial J}{\partial x} = \frac{\hbar}{m}\cdot\frac{\partial}{\partial x}\operatorname{Im}\left(\psi^*\frac{\partial\psi}{\partial x}\right)$$

$$= -\frac{i\hbar}{2m}\frac{\partial}{\partial x}\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right)$$

$$= -\frac{i\hbar}{2m}\left(\cancel{\frac{\partial\psi^*}{\partial x}\frac{\partial\psi}{\partial x}} - \cancel{\frac{\partial\psi}{\partial x}\frac{\partial\psi^*}{\partial x}} + \psi^*\frac{\partial^2\psi}{\partial x^2} - \psi\frac{\partial^2\psi^*}{\partial x^2}\right)$$

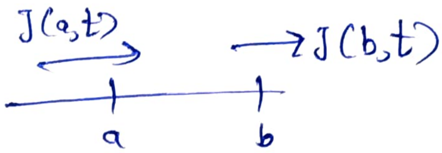
$$= \frac{\hbar}{m}\operatorname{Im}\left(\psi^*\frac{\partial^2\psi}{\partial x^2}\right)$$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0}$$ since $\operatorname{Im}\left(\psi^*\frac{\partial^2\psi}{\partial x^2}\right) = -\operatorname{Im}\left(\psi\frac{\partial^2\psi^*}{\partial x^2}\right)$

b) $P_{ab}(t) = \int_a^b dx\, \rho(x,t)$

$$\frac{dP_{ab}(t)}{dt} = \frac{d}{dt}\int_a^b dx\, \rho(x,t) = \int_a^b dx\, \frac{\partial \rho(x,t)}{\partial t} \quad \left(\frac{d}{dt} \text{ becomes } \frac{\partial}{\partial t}\right)$$

$$= -\int_a^b dx\, \frac{\partial J(x,t)}{\partial x} = J(a,t) - J(b,t)$$



Thus, we can relate rate of change of probability of locating a particle over a region in terms of currents permeating in through the boundaries

By normalization, we mean, at time t

$$\int_{-\infty}^{\infty} dx\, \psi(x,t)^* \psi(x,t)\, dx = 1$$

to find its time evol^n, we compute derivative wrt time:

$$\frac{d}{dt}\int_{-\infty}^{\infty} dx\, \psi(x,t)^* \psi(x,t)\, dx$$

$$= \int_{-\infty}^{\infty} dx\, \frac{\partial \rho(x,t)}{\partial t} = -\int_{-\infty}^{\infty} dx\, \frac{\partial J(x,t)}{\partial x}$$

$$= -\lim_{x\to\infty} J(x,t) + \lim_{x\to-\infty} J(x,t) = 0$$

(since $\psi, \psi' \to 0$ as $x \to \infty$)

$\Rightarrow$ once normalized, remains normalized

17-Pset-1
MZ7OCW
8·05-2013

c) for $\psi(x) = e^{i\alpha(x)}\phi(x)$,

$$\rho(x) = \phi^2(x)$$

$$J(x) = \frac{\hbar}{m}\,\mathrm{Im}\left(e^{-i\alpha x}\phi(x)\left[i\alpha'(x)e^{i\alpha(x)}\phi(x) + \phi'(x)e^{i\alpha(x)}\right]\right)$$

$$= \frac{\hbar}{m}\,\mathrm{Im}\left(i\alpha'(x)\phi^2(x) + \phi'\phi\right)$$

$$= \frac{\hbar}{m}\alpha'(x)\phi^2(x)$$

$$\Rightarrow \boxed{\frac{J(x)}{\rho(x)} = \frac{\hbar}{m}\alpha'(x)}$$

We have seen in 5c, that $\langle p\rangle$ gets contribution only from derivative of exponential term (follow from fact that $\langle p\rangle$ should be real)

i.e. $$\langle p\rangle = -i\hbar\int_{-\infty}^{\infty} e^{-i\alpha x}\phi(x)\left[i\alpha'(x)e^{i\alpha x}\phi(x) + \phi'(x)e^{i\alpha(x)}\right]dx$$

$$\langle p\rangle = \hbar\int_{-\infty}^{\infty}\alpha'(x)\phi^2(x)\,dx$$

$$\Rightarrow \langle v\rangle = \int_{-\infty}^{\infty}\frac{\hbar}{m}\alpha'(x)\phi^2(x)\,dx = \int_{-\infty}^{\infty} J(x)/\rho(x)\,\phi^2(x)\,dx$$

so maybe like x is the local coordinate for particle, we can see $\frac{\hbar}{m}\alpha'(x)$ as local velocity for particle and just like we compute $\langle x\rangle$, we can compute $\langle v\rangle$ (If states are normalizable)

d) $\psi(x) = Ae^{ipx/\hbar} + Be^{-ipx/\hbar}$

$$\psi^* \frac{\partial \psi}{dx} = \left(A^* e^{-ipx/\hbar} + B^* e^{ipx/\hbar}\right) \frac{ip}{\hbar}\left(Ae^{ipx/\hbar} - Be^{-ipx/\hbar}\right)$$

$$= \frac{ip}{\hbar}|A|^2 - \frac{ip}{\hbar}|B|^2 + \underbrace{\underbrace{\left(B^*A\, e^{2ipx/\hbar} - A^*B\, e^{-2ipx/\hbar}\right)}_{\text{Purely imaginary}} \frac{ip}{\hbar}}_{\text{Product real}}$$

e) $$J = \frac{\hbar}{m} \operatorname{Im}\left(\psi^* \frac{d\psi}{dx}\right)$$

$$= \frac{p}{m}\left(|A|^2 - |B|^2\right)$$

No, as we see cross-terms are real and don't appear in J