PSEI-6

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1. a) Using Cauchy-Schwarz inequality ($(|\langle f|g\rangle|)^2 \leq \langle f|f\rangle \langle g|g\rangle$)

$$|\langle\psi(0)|\psi(t)\rangle|^2 \leq \langle\psi(0)|\psi(0)\rangle\langle\psi(t)|\psi(t)\rangle$$

Given that $\langle\psi(0)|\psi(0)\rangle = 1$ and we know once normalized $\langle\psi(t)|\psi(t)\rangle = 1 \quad \forall\ t \geq 0$

$$\Rightarrow \boxed{|\langle\psi(0)|\psi(t)\rangle|^2 \leq 1}$$

if $\langle\psi(0)|$ is eigenket of $\hat{H}$, $|\langle\psi(0)|\psi(t)\rangle|^2 = 1$ and inequality is saturated

b) Time-independent $\hat{H} \Rightarrow \boxed{|\psi(t)\rangle = e^{-i\hat{H}(t-t_0)/\hbar}|\psi(t_0)\rangle}$

$$|\langle\psi(0)|\psi(t)\rangle|^2 = \langle\psi(0)|\psi(t)\rangle\langle\psi^*(0)|\psi^*(t)\rangle$$

$$= \langle\psi(0)|\psi(t)\rangle\langle\psi(t)|\psi(0)\rangle$$

for small t $(t_0 = 0)$:

$$|\psi(t)\rangle = \left(1 - \frac{i\hat{H}t}{\hbar} - \frac{\hat{H}^2t^2}{2\hbar^2}\right)|\psi(0)\rangle$$

$$\Rightarrow \langle\psi(t)| = \left(1 + \frac{i\hat{H}t}{\hbar} - \frac{\hat{H}^2t^2}{2\hbar^2}\right)\langle\psi(0)| \quad (\text{since } \hat{H}^\dagger = \hat{H})$$

$$\Rightarrow \langle\psi(0)|\psi(t)\rangle = \langle\psi(0)|\, 1 - \frac{i\hat{H}t}{\hbar} - \frac{\hat{H}^2t^2}{2\hbar^2}\,|\psi(0)\rangle$$

$$= \langle\psi(0)|\psi(0)\rangle - \frac{it}{\hbar}\langle\psi(0)|\hat{H}|\psi(0)\rangle - \frac{t^2}{2\hbar^2}\langle\psi(0)|\hat{H}^2|\psi(0)\rangle$$

$$= 1 - \frac{it}{\hbar}\langle\hat{H}\rangle - \frac{t^2}{2\hbar^2}\langle\hat{H}^2\rangle + O(t^3)$$

(since $\langle\hat{H}\rangle$ is conserved, we drop time)

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similarly $\langle \psi(t) | \psi(0) \rangle = 1 + \frac{it}{\hbar}\langle \hat{H} \rangle - \frac{t^2}{2\hbar^2}\langle \hat{H}^2 \rangle + O(t^3)$

so $|\langle \psi(t) | \psi(0) \rangle|^2 = \left(1 + \frac{it}{\hbar}\langle \hat{H} \rangle - \frac{t^2}{2\hbar^2}\langle \hat{H}^2 \rangle + O(t^3)\right)\left(1 - \frac{it}{\hbar}\langle \hat{H} \rangle - \frac{t^2}{2\hbar^2}\langle \hat{H}^2 \rangle + O(t^3)\right)$

$= 1 - \frac{t^2}{\hbar^2}\langle \hat{H}^2 \rangle + \frac{t^2}{\hbar^2}\left(\langle \hat{H} \rangle\right)^2 + O(t^3)$

since $(\Delta H)^2 = \langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2$

$\Rightarrow$ $\boxed{|\langle \psi(t) | \psi(0) \rangle|^2 = 1 - \frac{t^2(\Delta H)^2}{\hbar^2} + O(t^3) \quad \text{for small } t}$