11-PSet-1
M27OC(W)
2013-8-05

5. a) $\langle p \rangle = \int_{-\infty}^{\infty} \psi^*(x)\, \hat{p}\, \psi(x)\, dx$

$= \int_{-\infty}^{\infty} \phi(x) e^{-i\alpha} e^{i\alpha} \hat{p} \phi(x) dx$

$= \int_{-\infty}^{\infty} \phi(x) \, -i\hbar \frac{\partial \phi(x)}{\partial x} dx$

lets ignore $-i\hbar$ and focus on $\int_{-\infty}^{\infty} \phi(x) \frac{\partial \phi}{\partial x} dx$

$\int_{-\infty}^{\infty} \phi \phi' dx = \phi\phi\Big|_{-\infty}^{\infty} - \int \phi' \phi \, dx$

$\left(\int u\, dv = uv\Big| - \int v\, du \right)$

for square integrability $\lim_{|x|\to\infty} \phi^2 = 0$

$\Rightarrow \int_{-\infty}^{\infty} \phi \phi' dx = - \int \phi \phi' dx$

$\Rightarrow \int_{-\infty}^{\infty} \phi \phi' dx \Rightarrow \boxed{\langle p \rangle = 0}$

(Follows also from Ehrenfest Theoram)

b) $\psi(x) = \phi_1 + \phi_2 e^{i\alpha}$

$$\langle \hat{p} \rangle = \int_{-\infty}^{\infty} (\phi_1 + \phi_2 e^{-i\alpha}) \hat{p} (\phi_1 + \phi_2 e^{i\alpha}) dx$$

$$= \cancel{\int_{-\infty}^{\infty} \phi_1 \hat{p} \phi_1} + \cancel{\int_{-\infty}^{\infty} \phi_2 \hat{p} \phi_2 dx} + e^{i\alpha} \int_{-\infty}^{\infty} \phi_1 \hat{p} \phi_2 dx + e^{-i\alpha} \int_{-\infty}^{\infty} \phi_2 \hat{p} \phi_1 dx$$

$\downarrow$

0 (from (a))

now again $-i\hbar \int_{-\infty}^{\infty} \phi_2 \frac{d\phi_1}{dx} dx = i\hbar \int_{-\infty}^{\infty} \phi_1 \frac{d\phi_2}{dx}$

(follows also from Hermiticity of $\hat{p}$:

$$\int \bar{\phi}_1 \hat{p} \phi_2 dx = \int \overline{\phi_2 \hat{p}_2} \phi_1 dx$$

$$\Rightarrow \langle \hat{p} \rangle = e^{i\alpha} \left[-i\hbar \int_{-\infty}^{\infty} \phi_1 \frac{d\phi_2}{dx} dx \right] + e^{-i\alpha} i\hbar \int_{-\infty}^{\infty} \phi_1 \frac{d\phi_2}{dx} dx$$

$$= 2 \times \text{real} \left[-e^{i\alpha} i\hbar \int_{-\infty}^{\infty} \frac{d\phi_2}{dx} dx \right]$$

$\alpha = \pm \frac{\pi}{2}, 0$ ensures $\langle \hat{p} \rangle = 0$

13- Bset -1
MZ7OCW
8-05-2023

c) $\psi(x) = e^{ikx}\phi(x)$

$$\langle\hat{p}\rangle = -i\hbar\int_{-\infty}^{\infty} e^{-ikx}\phi\frac{d}{dx}\left(e^{ikx}\phi(x)\right)dx$$

$$= -i\hbar\int_{-\infty}^{\infty} \cancel{e^{-ikx}}\phi\,\cancel{e^{ikx}}\phi'\,dx - i\hbar\int_{-\infty}^{\infty}\cancel{e^{-ikx}}\phi\,(ik)\cancel{e^{ikx}}\phi(x)\,dx$$

$\downarrow$

0

$$= \hbar k\int_{-\infty}^{\infty}\phi^2(x)\,dx$$

But $\int_{-\infty}^{\infty}\phi^2(x)\,dx = 1$ since $\psi(x)$ most be normolized

$$\Rightarrow \quad \langle\hat{p}\rangle = \hbar k$$