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3. a) $\hat{R}_n(\alpha) = \exp\left(-\frac{i\alpha}{2} n\cdot\sigma\right)$

$$= \sum_{i=0}^{\infty} \left(-\frac{i\alpha}{2}\right)^i \frac{(n\cdot\sigma)^n}{i!}$$

now, $(n\cdot\sigma)^2 = n_i \sigma_i n_j \sigma_j$

since $\{\sigma_i, \sigma_j\} = 0$ and $\sigma_i^2 = \mathbb{1}$

$\Rightarrow (n\cdot\sigma)^2 = n_x^2 + n_y^2 + n_z^2 = 1 \Rightarrow (n\cdot\sigma)^2 = \mathbb{1}$

$$\Rightarrow \hat{R}_n(\alpha) = (n\cdot\sigma)\sum_{i=0}^{\infty}\left(-\frac{i\alpha}{2}\right)^{2i+1}\frac{1}{(2i+1)!} + \sum_{i=0}^{\infty}\left(-\frac{i\alpha}{2}\right)^{2i}\frac{1}{(2i)!}\mathbb{1}$$

$$\Rightarrow \boxed{\hat{R}_n(\alpha) = I\cos\frac{\alpha}{2} - i(\sigma\cdot n)\sin\frac{\alpha}{2}}$$

$$\hat{R}_n^\dagger(\alpha)\hat{R}_n(\alpha) = \left(I\cos\frac{\alpha}{2} + i(\sigma\cdot n)\sin\frac{\alpha}{2}\right)\left(I\cos\frac{\alpha}{2} - i\sigma\cdot n\sin\frac{\alpha}{2}\right)$$

$$= I\cos^2\frac{\alpha}{2} + i\cdot(-i)(\sigma\cdot n)^2\sin^2\frac{\alpha}{2}$$

$$= I \qquad (\text{since } (\sigma\cdot n)^2 = I)$$

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b) $\hat{R}_y(\alpha)\,\hat{S}_z\,\hat{R}_y^{\dagger}(\alpha) = \frac{\hbar}{2}\left(I\cos\frac{\alpha}{2} - i\sigma_2\sin\frac{\alpha}{2}\right)\sigma_3\left(I\cos\frac{\alpha}{2} + i\sigma_2\sin\frac{\alpha}{2}\right)$

$\Rightarrow \quad \sigma_3\cos^2\frac{\alpha}{2} + \sigma_2\sigma_3\sigma_2\sin^2\frac{\alpha}{2} + i\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\left(\sigma_3\sigma_2 - \sigma_2\sigma_3\right)$

using $\sigma_2\sigma_3 = -\sigma_3\sigma_2 \Rightarrow \sigma_2\sigma_3\sigma_2 = -\sigma_3\sigma_2^2 = -\sigma_3$

$\sigma_3\sigma_2 - \sigma_2\sigma_3 = -2i\sigma_1$

$\Rightarrow \hat{R}_y(\alpha)\,\hat{S}_z\,\hat{R}_y^{\dagger}(\alpha) = \frac{\hbar}{2}\left(\sigma_3\cos\alpha + \sigma_1\sin\alpha\right)$

$= \hat{S}_z\cos\alpha + \hat{S}_x\sin\alpha$

c) $R_y(\alpha)\,|+\rangle = \left(I\cos\frac{\alpha}{2} - i\sigma_2\sin\frac{\alpha}{2}\right)\begin{pmatrix}+1\\0\end{pmatrix}$

$= |z,+\rangle\cos\frac{\alpha}{2} + |z,-\rangle\sin\frac{\alpha}{2}$

This state is eigenvector for filter at an angle $\phi = 0$, $\theta = \alpha$ is z-x plane, $\hat{R}_y(\alpha)\,\hat{S}_z\,\hat{R}_y^{\dagger}(\alpha)$, so we can think of $R_n(\alpha)$ as rotation operator