2. a) $\cancel{\alpha}\, \hat{a}|\alpha\rangle = \hat{a}\, e^{\alpha \hat{a}^{\dagger} - \alpha^{*} a} |0\rangle$

since $e^{\alpha \hat{a}^{\dagger} - \alpha^{*} a}\, \hat{a}|0\rangle = 0$

$\Rightarrow \hat{a}|\alpha\rangle = [\hat{a}, e^{\alpha \hat{a}^{\dagger} - \alpha^{*} a}] |0\rangle$

$= \alpha\, e^{\alpha \hat{a}^{\dagger} - \alpha^{*} a} |0\rangle$

$\Rightarrow \hat{a}|\alpha\rangle = \alpha |\alpha\rangle$

b) $|\alpha\rangle = e^{\alpha \hat{a}^{\dagger} - \alpha^{*} a} |0\rangle$

$= e^{\alpha \hat{a}^{\dagger}}\, e^{-\alpha^{*} a}\, e^{-|\alpha|^2/2} |0\rangle$

$= e^{\alpha \hat{a}^{\dagger}}\, e^{-|\alpha|^2/2} |0\rangle$

$= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n (\hat{a}^{\dagger})^n}{n!} |0\rangle$

$= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$

$\Rightarrow |\alpha\rangle = \cancel{e^{-|\alpha|^2/2}} \sum_{n=0}^{\infty} c_n |n\rangle \qquad c_n = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}$

$\Rightarrow$ prob. of finding $|\alpha\rangle$ in E_n is $|c_n|^2 = \frac{e^{-|\alpha|^2} |\alpha|^{2n}}{n!}$

c) $\langle \beta | \alpha \rangle = \langle e^{\beta \hat{a}^\dagger - \beta^* \hat{a}} 0 | e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} | 0 \rangle$

$= \langle 0 | e^{\beta^* \hat{a} - \beta \hat{a}^\dagger} e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} | 0 \rangle$

$= \langle 0 | e^{-|\beta|^2/2} e^{-\beta \hat{a}^\dagger} e^{\beta^* \hat{a}} e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} e^{-|\alpha|^2/2} | 0 \rangle$

$= e^{-(|\beta|^2 + |\alpha|^2)/2} \langle 0 | e^{\beta^* \hat{a}} e^{\alpha \hat{a}^\dagger} | 0 \rangle$

Using $e^A e^B e^{[B,A]/2} = e^B e^A e^{[A,B]/2}$ if $[A,B]$ is c-number

$B = \alpha \hat{a}^\dagger$, $A = \beta^* \hat{a}$ $\Rightarrow$ $[B,A] = -\alpha \beta^*$

$\Rightarrow e^{\beta^* \hat{a}} e^{\alpha \hat{a}^\dagger} = e^{\alpha \hat{a}^\dagger} e^{\beta^* \hat{a}} e^{\alpha \beta^*}$

$\Rightarrow \langle \beta | \alpha \rangle = e^{-(|\beta|^2 + |\alpha|^2)/2} \langle 0 | e^{\alpha \hat{a}^\dagger} e^{\beta^* \hat{a}} | 0 \rangle e^{\alpha \beta^*}$

$\Rightarrow \langle \beta | \alpha \rangle = e^{-(|\beta|^2 + |\alpha|^2 - 2\alpha \beta^*)/2}$

Clearly $\langle \beta | \alpha \rangle \neq 0$ although $\langle \alpha | \alpha \rangle = 1$

d) $\langle H \rangle = \hbar\omega \langle \hat{N} + \frac{1}{2} \rangle = \hbar\omega|\alpha|^2 + \frac{\hbar\omega}{2}$

since $\langle \hat{N} \rangle = \langle 0| D^{\dagger}(\alpha) \hat{N} D(\alpha)|0\rangle$

$= \langle 0| D(\alpha^*) a^{\dagger} a D(\alpha)|0\rangle$

$= \langle 0| \alpha^* \alpha |0\rangle = |\alpha|^2$

$\langle H^2 \rangle = \hbar^2\omega^2 \langle \hat{N}^2 + \hat{N} + \frac{1}{4} \rangle$

$= \hbar\omega^2 \langle \hat{N}^2 \rangle + \hbar^2\omega^2 \langle \hat{N} \rangle + \frac{\hbar^2\omega^2}{4}$

$\downarrow$

$= \hbar^2\omega^2 \langle 0| D(\alpha^*)\, a^{\dagger} a a^{\dagger} a D(\alpha) |0\rangle$

$= \hbar^2\omega^2 \langle 0|\, |\alpha|^2 D(\alpha^*)\, a a^{\dagger} D(\alpha)|0\rangle$

$= |\alpha|^2 \hbar^2\omega^2 \langle 0| D(\alpha^*)(1 + a^{\dagger} a) D(\alpha)|0\rangle$

$\langle H^2 \rangle = |\alpha|^4 \hbar^2\omega^2 + |\alpha|^2 \hbar^2\omega^2 + \frac{\hbar^2\omega^2}{4} + |\alpha|^2 \hbar\omega^2$

$(\Delta H)^2 = \langle H^2 \rangle - (\langle H \rangle)^2 = |\alpha|^2 \hbar^2\omega^2$

$\rightarrow \frac{\Delta H}{\langle H \rangle} = \frac{|\alpha| \hbar\omega}{(|\alpha|^2 + \frac{1}{2})\hbar\omega} = \frac{|\alpha|}{|\alpha|^2 + 1/2}$

Clearly for $|\alpha| > 1/\sqrt{2}$, it decreases as $|\alpha| \uparrow$

e) $\langle\alpha|\hat{x}|\alpha\rangle = \langle\alpha|\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^{+})|\alpha\rangle$

$$= \left(\langle\alpha|\hat{a}|\alpha\rangle + \langle\alpha|\hat{a}^{+}|\alpha\rangle\right)\sqrt{\frac{\hbar}{2m\omega}}$$

$$= (\alpha+\alpha^{*})\sqrt{\frac{\hbar}{2m\omega}} = 2Re(\alpha)\sqrt{\frac{\hbar}{2m\omega}}$$

$$\langle\alpha|\hat{p}|\alpha\rangle = -i\sqrt{\frac{m\omega\hbar}{2}}\langle\alpha|\hat{a}-\hat{a}^{+}|\alpha\rangle$$

$$= -i\sqrt{\frac{m\omega\hbar}{2}}(\alpha-\alpha^{*})$$

$$= 2\,Im(\alpha)\sqrt{\frac{m\omega\hbar}{2}}$$

$$\langle\alpha|\hat{x}^{2}|\alpha\rangle = \frac{\hbar}{2m\omega}\langle\alpha|(\hat{a}+\hat{a}^{+})^{2}|\alpha\rangle$$

$$= \frac{\hbar}{2m\omega}\langle\alpha|\hat{a}^{2}+\hat{a}^{+2}+\hat{a}\hat{a}^{+}+\hat{a}^{+}a|\alpha\rangle$$

$$= \frac{\hbar}{2m\omega}\alpha^{2}+\alpha^{*2}+\underbrace{1+\alpha\alpha^{*}}_{\hat{a}\hat{a}^{+}=1+\hat{a}^{+}\hat{a}}+\alpha\alpha^{*}$$

$$= \frac{\hbar}{2m\omega}(1+(2Re(\alpha))^{2}$$

$$\Rightarrow \Delta x^{2} = \langle\alpha|\hat{x}^{2}|\alpha\rangle - (\langle\alpha|\hat{x}|\alpha\rangle)^{2} = \frac{\hbar}{2m\omega}$$

Similarly

$$\Delta p^{2} = \frac{m\omega\hbar}{2} \Rightarrow \Delta x\,\Delta p = \frac{\hbar}{2}$$

Thus minimum - energy state

f) Shown in lecture: $|\alpha, t\rangle = e^{-i\omega t/2} |e^{-i\omega t}\alpha\rangle$

$\Rightarrow \alpha(t) = e^{-i\omega t}\alpha$

g) since from (f), $\alpha(t) = e^{-i\omega t}\alpha$

$$\Rightarrow \langle x\rangle = 2Re(\alpha)\sqrt{\frac{\hbar}{2m\omega}}$$

$$= 2\alpha_0 \cos\omega t \sqrt{\frac{\hbar}{2m\omega}}$$

$$\langle y\rangle = 2Im(\alpha)\sqrt{\frac{\hbar}{2m\omega}}$$

$$\Rightarrow \langle y\rangle = 2\alpha_0 \sin\omega t \sqrt{\frac{\hbar}{2m\omega}}$$

Clearly, $\langle \hat{H}\rangle$, Δx, Δp are time-independent and are the same as derived in d) and e)