6-PSET-4
8.05 - 2013
MIT OCW

4. $A, B : \quad [A, B] = cI \qquad c \in \mathbb{C}$

To prove: $e^{A+B} = e^{B} e^{A} e^{c/2} = e^{A} e^{B} e^{-c/2}$

let $G(t) = e^{t(A+B)} e^{-tA}$ — (0)

$\Rightarrow G^{-1}(t) = e^{tA} e^{-t(A+B)}$ (Due to associative matmul)

$$\frac{dG(t)}{dt} = e^{t(A+B)}(A+B)e^{-tA} - e^{t(A+B)} A e^{-tA}$$

$$\Rightarrow \frac{dG(t)}{dt} = e^{t(A+B)} B e^{-tA}$$

$$\Rightarrow G^{-1}(t)\frac{dG(t)}{dt} = e^{tA} B e^{-tA}$$

$$\Rightarrow \frac{dG(t)}{dt} = G(t)\left(e^{tA} B e^{-tA}\right)$$

But from 3 b), $e^{tA} B e^{-tA} = B + [A,B]\,t$

$$\Rightarrow \frac{dG(t)}{dt} = G(t)(B + cIt) \quad —(1)$$

if $G(t) = G(0)\, e^{tB} e^{\frac{1}{2}ct^2}$ — (2)

$$\frac{dG(t)}{dt} = G(0)\, e^{tB} B e^{1/2 ct^2} + G(0)\, e^{tB} e^{1/2 ct^2} ctI$$

$\Rightarrow G(t) = G(0)\, e^{tB} e^{1/2 ct^2}$ satisfies (1)

$G(1) = e^{A+B} e^{-A} = e^{B} e^{1/2 c}$ (from (2) and (0))

$$\Rightarrow \boxed{e^{A+B} = e^{B} e^{A} e^{c/2}}$$

swapping A and B and changing $c \to -c$

$$\boxed{e^{A+B} = e^{A} e^{B} e^{-c/2}}$$

QED