-PSET-8
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3. $|0_r\rangle = S(r)|0\rangle \qquad S(r) = \exp\left(-\frac{r}{2}(a^\dagger a^\dagger - aa)\right)$

$|\alpha\rangle = D(\alpha)|0\rangle \qquad D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$

$|\alpha, r\rangle = D(\alpha)S(r)|0\rangle$

a) let $A = \frac{r}{2}(a^\dagger a^\dagger - aa)$, $B = a$

$\Rightarrow a(r) = S^\dagger(r)\, a\, S(r) = e^{A} B e^{-A}$

We can expand it using Hadamard Lemma:

$[A, B] = \frac{r}{2}[a^\dagger a^\dagger - aa, a] = \frac{r}{2}[a^\dagger a^\dagger, a] = -r a^\dagger$

$[A,[A,B]] = \frac{r}{2}[a^\dagger a^\dagger - aa, -r a^\dagger] = +\frac{r^2}{2}[-aa, -a^\dagger] = +r^2 a$

$\Rightarrow [A,[A,B]] = +r^2 B$

$\Rightarrow [A,[A,[A,B]]] = +r^2[A,B] = -r^3 a^\dagger$

$\Rightarrow a(r) = a - r a^\dagger + \frac{r^2 a}{2!} - \frac{r^3 a^\dagger}{3!} + \dots$

$\Rightarrow a(r) = a\cosh(r) - a^\dagger \sinh(r)$

$a^\dagger(r) = S^\dagger(r) a^\dagger S(r) = a^\dagger \cosh(r) - a \sinh(r)$

$[a(r), a^\dagger(r)] = \cosh^2(r)[a, a^\dagger] + \sinh^2(r)[a^\dagger, a]$

$= 1 \qquad (\text{since } \cosh^2(r) - \sinh^2(r) = 1)$

b) $\langle N \rangle = \langle 0_r | a^\dagger a | 0_r \rangle$

$= \langle 0 | S^\dagger(r)\, a^\dagger a\, S(r) | 0 \rangle$

$= \langle 0 | S^\dagger(r) a^\dagger S(r) S^\dagger(r) a S(r) | 0 \rangle$ (Since $S(r)$ is unitary)

$= \langle 0 | a^\dagger(r)\, a(r) | 0 \rangle$

$= \langle 0 | [a^\dagger \cosh(r) - a \sinh(r)][a \cosh(r) - a^\dagger \sinh(r)] | 0 \rangle$

$= \langle 0 | a a^\dagger \sinh^2(r) | 0 \rangle$ (Other terms give 0 overlap)

$= \sinh^2(r) \langle a^\dagger 0 | a^\dagger 0 \rangle$

$= \sinh^2(r)$

$\langle N^2 \rangle = \langle 0_r | a^\dagger a a^\dagger a | 0_r \rangle$

$= \langle 0 | S^\dagger a^\dagger S S^\dagger a S S^\dagger a^\dagger S S^\dagger a S | 0 \rangle$

$= \langle 0 | a^\dagger(r) a(r) a^\dagger(r) a(r) | 0 \rangle$

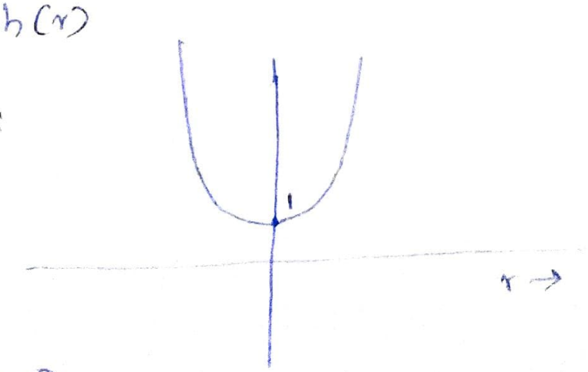
$= \langle 0 | a^2 a^{\dagger 2} \cosh^2(r) \sinh^2(r) + a a^\dagger a a^\dagger (\sinh^2(r))^2 | 0 \rangle$

$= \cosh^2(r) \sinh^2(r) \cdot 2 \langle 2 | 2 \rangle + \sinh^4(r) \langle 1 | 1 \rangle$

$(\Delta N)^2 = 2\cosh^2(r) \sinh^2(r) + (\sinh^2(r))^2 - (\sinh^2(r))^2$

$\Rightarrow \Delta N = \sqrt{2} \cosh(r) \sinh(r)$

$\dfrac{\Delta N}{\langle N \rangle} = \sqrt{2} \cosh(r)$ :




smallest when $r = 0$

Ground-State of Oscillator

c) $\langle N \rangle = \langle \alpha, r | a^\dagger a | \alpha, r \rangle$

$|\alpha, r\rangle = D(\alpha) S(r) |0\rangle$

$\Rightarrow \langle N \rangle = \langle 0 | S^\dagger(r) D^\dagger(\alpha) a^\dagger a D(\alpha) S(r) |0\rangle$

since $D(\alpha)$ is unitary :

$\langle N \rangle = \langle 0 | S^\dagger(r) D^\dagger(\alpha) a^\dagger D(\alpha) D^\dagger(\alpha) a D(\alpha) S(r) |0\rangle$

let $\alpha^* a - \alpha a^\dagger = A$,

$\Rightarrow D^\dagger(\alpha) a D(\alpha) = e^{A} a e^{-A}$

$[A, a] = \alpha$

$\Rightarrow D^\dagger(\alpha) a D(\alpha) = a + \alpha$

similarly $D^\dagger(\alpha) a^\dagger D(\alpha) = \alpha^* + a^\dagger$

$\Rightarrow \langle N \rangle = \langle 0 | S^\dagger(r) (a^\dagger + \alpha^*)(a + \alpha) S(r) |0\rangle$

$= \langle 0 | S^\dagger(r) a^\dagger a S(r) |0\rangle + |\alpha|^2 + 0$

$= \sinh^2(r) + |\alpha|^2$

$\Rightarrow \boxed{\langle N \rangle = \sinh^2(r) + |\alpha|^2}$

(d) $\langle \hat{E}(t)\rangle = \langle \alpha, r| \mathcal{E}_0 (a e^{-i\omega t} + a^\dagger e^{i\omega t}) |\alpha, r\rangle$

$= \langle r| \mathcal{E}_0 (a e^{-i\omega t} + \alpha e^{-i\omega t} + a^\dagger e^{i\omega t} + \alpha^* e^{i\omega t} | r\rangle$

$= \underbrace{\langle r|\mathcal{E}_0 (a e^{-i\omega t} + a^\dagger e^{i\omega t})|r\rangle}_{0} + 2\mathrm{Re}(\alpha e^{-i\omega t})\langle r|r\rangle$

$\Rightarrow \langle \hat{E}(t)\rangle = 2\mathrm{Re}(\alpha e^{-i\omega t})$

$\langle \hat{E}^2(t)\rangle = \langle r| \mathcal{E}_0 (a e^{-i\omega t} + a^\dagger e^{i\omega t} + 2\mathrm{Re}(\alpha e^{-i\omega t}))^2 |r\rangle$

$= \langle r| \mathcal{E}_0^2 (a e^{-i\omega t} + a^\dagger e^{i\omega t})^2 |r\rangle + \langle r|[2\mathrm{Re}(\alpha e^{-i\omega t})]^2|r\rangle$

$+ 2\langle r| (a e^{-i\omega t} + a^\dagger e^{i\omega t}) \cdot 2\mathrm{Re}(\alpha e^{-i\omega t})|r\rangle$

$= \langle r| \mathcal{E}_0^2 (a e^{-i\omega t} + a^\dagger e^{i\omega t})^2 |r\rangle + (\langle \hat{E}(t)\rangle)^2$

$= \langle 0| \mathcal{E}_0^2 \left( a(\cosh(r) e^{-i\omega t} - \sinh(r) e^{i\omega t}) + a^\dagger(\cosh(r) e^{i\omega t} - \sinh(r) e^{-i\omega t})\right)^2 |0\rangle$

$= \mathcal{E}_0^2 \langle 0| a a^\dagger (\cosh(r) e^{-i\omega t} - \sinh(r) e^{i\omega t})(\cosh(r) e^{i\omega t} - \sinh(r) e^{-i\omega t}) |0\rangle$

$= \mathcal{E}_0^2 (\cosh^2(r) + \sinh^2(r) - 2\sinh(r)\cosh(r)\cos 2\omega t)$
$+ (\langle \hat{E}(t)\rangle)^2$

$\Rightarrow \Delta E(t) = \mathcal{E}_0 \left(\cosh^2(r) + \sinh^2(r) - 2\sinh(r)\cosh(r)\cos 2\omega t\right)^{1/2}$