3- Part 2
m2'10rw
8-05-2013

2. $\psi'' + (\mathcal{E} - U(x))\psi = 0$

$\psi_n \; : \; \mathcal{E}_n$

$\psi_{n+1} \; : \; \mathcal{E}_{n+1} \quad , \quad \mathcal{E}_{n+1} > \mathcal{E}_n$

a) $\int_a^b \psi_n \hat{H} \psi_{n+1} \, dx = \mathcal{E}_{n+1} \int_a^b \psi_n \psi_{n+1} \, dx \quad -(1)$

$\int_a^b \psi_{n+1} \hat{H} \psi_n \, dx = \mathcal{E}_n \int_a^b \psi_n \psi_{n+1} \, dx \quad -(2)$

(1) − (2) :

$$\int_a^b (\psi_n \hat{H} \psi_{n+1} - \psi_{n+1} \hat{H} \psi_n) dx = (\mathcal{E}_{n+1} - \mathcal{E}_n) \int_a^b dx \, \psi_n \psi_{n+1}$$

$\Downarrow$

$$= \int_a^b -\frac{\hbar^2}{2m}\psi_n \psi''_{n+1} + \frac{\hbar^2}{2m} \psi_{n+1} \psi''_n + V(\psi_n \psi_{n+1} - \psi_{n+1}\psi_n) \, dx$$

$$= \frac{\hbar^2}{2m} \int_a^b (\psi_{n+1} \psi''_n - \psi_n \psi''_{n+1}) \, dx$$

By parts

$$= \frac{\hbar^2}{2m} \left[ \psi_{n+1} \psi'_n \Big|_a^b - \cancel{\int_a^b \psi'_{n+1} \psi'_n} - \psi_n \psi'_{n+1} + \cancel{\int_a^b \psi'_{n+1} \psi'_n \, dx} \right]$$

$$\Rightarrow \frac{\hbar^2}{2m} (\psi_{n+1} \psi'_n - \psi_n \psi'_{n+1}) \Big|_a^b = (\mathcal{E}_{n+1} - \mathcal{E}_n) \int_a^b dx \, \psi_n \psi_{n+1} \quad -(3)$$

This is missing in PSET

Thus shown!

b) $\psi_k(x) > 0 \quad ; \; a < x < b$

Assume $\psi_{k+1}(x) > 0$ for $a < x < b$ WLOG

Then RHS of (3) $> 0$

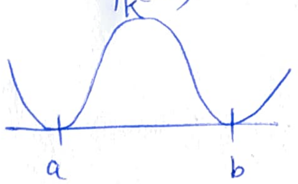
The LHS is: $\psi_{k+1}\psi_k' - \psi_k \psi_{k+1}' \Big|_a^b$

$\psi_k(b) = \psi_k(a) = 0$

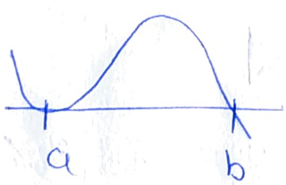
$\Rightarrow$ LHS is $\psi_{k+1}(b)\,\psi_k'(b) - \psi_{k+1}(a)\,\psi_k'(a) > 0$ (since RHS $> 0$)

let's see possible values $\psi_k'(a, b)$ can ~~ever~~ take:

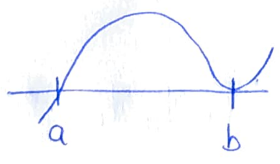
$\psi_k(x)$ $\qquad$ $\psi_k'(a)$ $\quad$ $\psi_k'(b)$



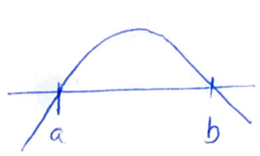

$0 \qquad 0 \qquad$ LHS zero




$0 \qquad < 0 \qquad$ LHS $\leq 0$




$> 0 \qquad 0 \qquad$ LHS $\leq 0$




$> 0 \qquad < 0 \qquad$ LHS $\leq 0$

So if we assume $\psi_{k+1} \geq 0 \;\forall x \in [a,b]$, we reach a contradiction $\Rightarrow$ $\psi_{k+1}$ must ~~change sign once at a and b~~ have opposite signs at a and b

$\psi_{k+1} \leq 0$ is similarly dealt, we can multiply it by $-1$ and claim $-\psi_{k+1}$ cannot be monosigned over $x \in [a,b]$