PSET-7
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8.05

7. $|\alpha\rangle = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})|0\rangle$

$$\alpha = \frac{x_0}{\sqrt{2}d} + \frac{ip_0 d}{\sqrt{2}\hbar}, \quad d = \sqrt{\frac{\hbar}{m\omega}}, \quad x_0, p_0 \in \mathbb{R}$$

$$\alpha\hat{a}^\dagger - \alpha^*\hat{a} = \frac{ix_0}{\sqrt{2}d}(\hat{a}^\dagger - \hat{a}) + \frac{ip_0 d}{\sqrt{2}\hbar}(\hat{a}^\dagger + \hat{a})$$

$$\Rightarrow \alpha\hat{a}^\dagger - \alpha^*\hat{a} = -\frac{ix_0}{\hbar}\hat{p} + \frac{ip_0}{\hbar}\hat{x}$$

$$\Rightarrow \exp\left(-\frac{ix_0}{\hbar}\hat{p} + \frac{ip_0\hat{x}}{\hbar}\right)|0\rangle = |\alpha\rangle$$

since $e^{A+B} = e^B e^A e^{[A,B]/2}$

$$\Rightarrow \exp\left(-\frac{ix_0}{\hbar}\hat{p} + \frac{ip_0\hat{x}}{\hbar}\right) = \exp\left(\frac{ip_0\hat{x}}{\hbar}\right)\exp\left(-\frac{ix_0\hat{p}}{\hbar}\right)\exp\left(-\frac{ix_0p_0}{\hbar}\right)$$

$$\Rightarrow \psi(x) = \langle x|0\rangle$$

$$= \langle x|\exp\left(\frac{ip\hat{x}}{\hbar}\right)\exp\left(-\frac{ix_0\hat{p}}{\hbar}\right)\exp\left(-\frac{ix_0p_0}{\hbar}\right)|\psi_0\rangle$$

$\exp\left(-\frac{ix_0p_0}{\hbar}\right)$ introduced overall phase shift and is not physically significant

$$\exp\left(-\frac{ix_0\hat{p}}{\hbar}\right)|\psi_0\rangle = |\psi_0(x-x_0)\rangle$$

$$\Rightarrow \langle x|\exp\left(\frac{ip\hat{x}}{\hbar}\right)|\exp\left(-\frac{ix_0p_0}{\hbar}\right)\psi_0(x-x_0)\rangle$$

$$\Rightarrow \psi(x) = \langle \exp\left(-\frac{ip\hat{x}}{\hbar}\right)x|\exp\left(-\frac{ix_0p_0}{\hbar}\right)\psi_0(x-x_0)\rangle$$

$$\Rightarrow \psi(x) = \langle \exp\left(-\frac{ip_0 x}{\hbar}\right) x \,|\, \exp\left(-\frac{ip_0 x_0}{\hbar}\right) \psi_0(x-x_0) \rangle$$

$$\Rightarrow \psi_\alpha(x) = \exp\left(+\frac{ip_0 x}{\hbar}\right) \exp\left(-\frac{ip_0 x_0}{\hbar}\right) \psi_0(x-x_0)$$

This makes sense since is space this sol^n is ground state shifted to $x = x_0$ while in frequency spectrum, it is shifted to $p = p_0$

Note: $\langle \exp\left(-\frac{ip_0 x}{\hbar}\right) x | = \exp\left(\frac{ip_0 x}{\hbar}\right) \langle x |$

since $\langle c\alpha | = c^* \langle \alpha |$