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8.05 - 2013
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2. A and B operators : $[A,B] \neq 0$

a) To show : $\frac{d}{dt} e^{t(A+B)} = (A+B)\, e^{t(A+B)} = e^{t(A+B)}(A+B)$

I assume $\frac{d}{dt}A = \frac{d}{dt}B = 0$

$$\dot{e}^{t(A+B)} = (A+B)\, e^{t(A+B)} = e^{t(A+B)}(A+B)$$

$$e^{t(A+B)} = \sum_{n=0}^{\infty} \frac{t^n}{n!}(A+B)^n$$

$$\downarrow$$

$$\frac{d}{dt} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!}(A+B)^{n-1}\cdot(A+B) = (A+B)\sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!}(A+B)^{n-1}$$

$$= e^{t(A+B)}(A+B) = (A+B)\, e^{t(A+B)}$$

b) $F(t) = e^{tA} B e^{-tA}$

$$\frac{dF(t)}{dt} = A e^{tA} B e^{-tA} + e^{tA} B e^{-tA}(-A)$$

$$= e^{tA}(AB - BA)\, e^{-tA} \qquad (\because [A, f(A)] = 0)$$

$$= e^{tA}\, c\, e^{-tA} = c\mathbb{1}$$

$$\Rightarrow \boxed{F(t) = B + ct\mathbb{1}} \qquad (\text{Since } F(0) = B)$$

At $t = 1$:

$$\boxed{e^{A} B e^{-A} = B + c\mathbb{1}}$$

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c) $\hat{T}^\dagger(a)\hat{x}\hat{T}(a) = e^{ia\hat{p}/\hbar}\,\hat{x}\,e^{-ia\hat{p}/\hbar}$

since $[\frac{ia\hat{p}}{\hbar}, \hat{x}] = a$

$\Rightarrow$ from b), $\boxed{\hat{T}^\dagger(a)\hat{x}\hat{T}(a) = \hat{x} + a\mathbb{1}}$

d) First we try to show $\hat{T}(a)|x\rangle = |x+a\rangle$

since $\hat{T}(a)$ is unitary, we may expect that $\hat{T}^\dagger(a)\hat{x}\hat{T}(a)$ is just the transformation of $\hat{x}$ when $|x\rangle \rightarrow \hat{T}^\dagger(a)|x\rangle$

now, in our normal $|x\rangle$ basis, $|x\rangle$ is an eigenket of $\hat{x}$ with eigenvalue x, upon transforming the basis, $|x\rangle \rightarrow \hat{T}^\dagger(a)|x\rangle$ such that eigenvalue of eigenket $\hat{T}^\dagger(a)|x\rangle = |y\rangle$ is still x

$\Rightarrow \quad \hat{T}^\dagger(a)\hat{x}\hat{T}(a)|y\rangle = (\hat{x} + a\mathbb{1})|y\rangle = x|y\rangle$

$\Rightarrow \quad |y\rangle = |x-a\rangle$

$\Rightarrow \quad \boxed{\hat{T}^\dagger(a)|x\rangle = |x-a\rangle} \quad \Rightarrow \boxed{\hat{T}(a)|x\rangle = |x+a\rangle}$

now $\langle x|\hat{T}(a)|\psi\rangle = \langle \hat{T}^\dagger(a)x|\psi\rangle$

$= \langle x-a|\psi\rangle = \psi(x-a)$

Thus $\boxed{\hat{T}(a)|\psi\rangle \text{ described by } \psi(x-a)}$