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5. a) if $|n_1\rangle$ and $|n_2\rangle$ be eigenstates with energy $n\hbar\omega + \hbar\omega/2$

$\Rightarrow$ $\hat{a}|n_1\rangle$ and $\hat{a}|n_2\rangle$ are distinct since if these were same, $|n_1\rangle$ and $|n_2\rangle$ would have been same

$\Rightarrow$ $\hat{a}^2|n_1\rangle$ and $\hat{a}^2|n_2\rangle$ are also distinct

doing this n times, we conclude that ground states must be different

b) $[a, a^+] = 1$

$$[(a^+)^n, a] = [a^+, a]\, n(a^+)^{n-1}$$

$$\Rightarrow -[(a^+)^n, a] = -[a^+, a]\, n(a^+)^{n-1}$$

$$\Rightarrow \boxed{[a, (a^+)^n] = n(a^+)^{n-1}}$$

c) $\langle m|\hat{a}|n\rangle = \langle a^{+} m|n\rangle = \sqrt{m+1}\,\langle m+1|n\rangle$

$= \cancel{\sqrt{m+1}\,\delta_{m,n-1}}\ \sqrt{m+1}\,\delta_{m+1,n}$

$\langle m|\hat{a}^{+}|n\rangle = \sqrt{n+1}\,\langle m|n+1\rangle = \sqrt{n+1}\,\delta_{m,n+1}$

$\langle m|\hat{x}|n\rangle = \sqrt{\dfrac{\hbar}{2m\omega}}\,\langle m|\hat{a}+\hat{a}^{+}|n\rangle$

$= \sqrt{m+1}\,\delta_{m+1,n} + \sqrt{n+1}\,\delta_{m,n+1}$

$= \sqrt{m+1}\,\delta_{m+1,n} + \sqrt{m}\,\delta_{m-1,n}$

$\langle m|\hat{p}|n\rangle = \sqrt{\dfrac{\hbar m\omega}{2}}\,\langle m|\hat{a}^{+}-\hat{a}|n\rangle$

$= -\sqrt{m+1}\,\delta_{m+1,n} + \sqrt{m}\,\delta_{m-1,n}$

$\langle m|\hat{N}|n\rangle = \langle m|a^{+}a|n\rangle = n\,\delta_{n,m}$

$\langle m|\hat{x}|n\rangle = \sqrt{m+1}\,\delta_{m+1,n} + \sqrt{m}\,\delta_{m-1,n}$

$= \sqrt{n}\,\delta_{m,n-1} + \sqrt{n+1}\,\delta_{m,n+1}$

$\Rightarrow \langle m|\hat{x}^{2}|n\rangle = \sum_{i} \langle m|\hat{x}|i\rangle\langle i|\hat{x}|m\rangle$

$= \left(\sqrt{m+1}\,\delta_{m+1,i} + \sqrt{m}\,\delta_{m-1,i}\right)$

$\left(\sqrt{n}\,\delta_{i,n-1} + \sqrt{n+1}\,\delta_{i,n+1}\right)$

$= \Big(\sqrt{m+1}\cdot\sqrt{n}\,\delta_{m+1,n-1} + \sqrt{mn}\,\delta_{m,n} + \sqrt{n+1}\sqrt{m+1}\,\delta_{m,n}$

$+ \sqrt{n+1}\sqrt{m}\,\delta_{m-1,n+1}\Big) \times \dfrac{\hbar}{2m\omega}$

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Similarly, $\langle m|\hat p^2|n\rangle = \left(\sqrt{m+1}\,\delta_{m+1,i} - \sqrt{m}\,\delta_{m-1,i}\right)\left(-\sqrt{n}\,\delta_{i,n-1} + \sqrt{n+1}\,\delta_{i,n+1}\right)$

$$= -\sqrt{n(m+1)}\,\delta_{m+1,n-1} + \sqrt{n}\sqrt{m}\,\delta_{m,n} + \sqrt{n+1}\sqrt{m+1}\,\delta_{m,n} - \delta_{m-1,n+1}\sqrt{m(n+1)}$$

$$\hat a = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt2 & 0 \\ 0 & 0 & 0 & \sqrt3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \hat a^{+} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \sqrt2 & 0 & 0 \\ 0 & 0 & \sqrt3 & 0 \end{bmatrix}$$

$$\hat x = \sqrt{\frac{\hbar}{2m\omega}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \sqrt2 & 0 \\ 0 & \sqrt2 & 0 & \sqrt3 \\ 0 & 0 & \sqrt3 & 0 \end{bmatrix} \qquad \hat p = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -\sqrt2 & 0 \\ 0 & \sqrt2 & 0 & -\sqrt3 \\ 0 & 0 & \sqrt3 & 0 \end{bmatrix} \times i\sqrt{\frac{\hbar m\omega}{2}}$$

$$\hat N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\hat x^2 = \frac{\hbar}{2m\omega} \begin{bmatrix} 1 & 0 & \sqrt2 & 0 \\ 0 & 3 & 0 & \sqrt6 \\ \sqrt2 & 0 & 5 & 0 \\ 0 & \sqrt6 & 0 & 7 \end{bmatrix} \qquad \hat p^2 = \frac{\hbar m\omega}{2} \begin{bmatrix} 1 & 0 & -\sqrt2 & 0 \\ 0 & 3 & 0 & -\sqrt6 \\ -\sqrt2 & 0 & 5 & 0 \\ 0 & -\sqrt6 & 0 & 7 \end{bmatrix}$$

d) $[\hat x\ \hat p] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & - \end{bmatrix}$

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d) $\hat{x}\hat{p} - \hat{p}\hat{x}$

$$\frac{i\hbar}{2}\begin{bmatrix} 1 & 0 & -\sqrt{2} & 0 \\ 0 & 1 & 0 & -\sqrt{6} \\ \sqrt{2} & 0 & 1 & 0 \\ 0 & \sqrt{6} & 0 & -3 \end{bmatrix} - \frac{i\hbar}{2}\begin{bmatrix} -1 & 0 & -\sqrt{2} & 0 \\ 0 & -1 & 0 & -\sqrt{6} \\ \sqrt{2} & 0 & -1 & 0 \\ 0 & \sqrt{6} & 0 & 3 \end{bmatrix}$$

$$= i\hbar \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

It is not $i\hbar I$ since truncation causes error

e) indeed, $\langle n|\hat{x}^2|n\rangle = \frac{\hbar}{m\omega}(n+\frac{1}{2})$

$$\langle n|\hat{p}^2|n\rangle = m\hbar\omega(n+\tfrac{1}{2})$$

$$\Delta x = \sqrt{\langle n|\hat{x}^2|n\rangle}$$
$$\Delta p = \sqrt{\langle n|\hat{p}^2|n\rangle}$$
since $\langle n|\hat{x}|n\rangle = \langle n|\hat{p}|n\rangle = 0$

x_{max} : $\frac{1}{2}m\omega^2 x_{max}^2 = \hbar\omega(n+\frac{1}{2})$

$$\Rightarrow \quad x_{max} = \sqrt{\frac{\hbar}{m\omega}(2n+1)}$$

while $\Delta x = \sqrt{\frac{\hbar}{m\omega}(n+\frac{1}{2})}$