1.

$$\vec{B} = B_0 \cos(\omega t)\, \hat{e}_z$$

$$\Rightarrow \quad H(t) = -(\vec{B}\cdot\hat{S}_z)\gamma$$

where γ related spin to magnetic moment

Since $[H(t), H(t')] = 0$

$$\Rightarrow \quad U(t,t_0) = \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} H(t')\,dt'\right]$$

$$\int_{t_0}^{t} H(t')\,dt' = -\gamma B \frac{\sin\omega t}{\omega}\hat{S}_z \qquad (t_0 = 0)$$

$$\Rightarrow \quad U(t,t_0) = \exp\left[\frac{i}{\hbar}\gamma B \frac{\sin\omega t}{\omega}\hat{S}_z\right]$$

From last Pset,

a) $$\boxed{U(t,t_0) = \cos\left(\frac{\gamma B}{2}\frac{\sin\omega t}{\omega}\right)\mathbb{1} + i\sin\left(\frac{\gamma B}{2}\frac{\sin\omega t}{\omega}\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}$$

$$|\psi, t\rangle = U(t,0)\,|\psi, 0\rangle \qquad \text{(Spin-component)}$$

$$\Rightarrow |\psi,t\rangle = \left[\cos\left(\frac{\gamma B}{2}\frac{\sin\omega t}{\omega}\right) + i\sin\left(\frac{\gamma B}{2}\frac{\sin\omega t}{\omega}\right)\right]|+\rangle$$
$$+ \left[\cos\left(\frac{\gamma B}{2}\frac{\sin\omega t}{\omega}\right) - i\sin\left(\frac{\gamma B}{2}\frac{\sin\omega t}{\omega}\right)\right]|-\rangle$$

let $\gamma B \frac{\sin\omega t}{\omega} = \phi(t)$

$$\Rightarrow |\psi,t\rangle = e^{i\phi(t)/2}|+\rangle + e^{-i\phi(t)/2}|-\rangle$$

$$= \frac{1}{\sqrt{2}}|+\rangle + \frac{e^{-i\phi(t)}}{\sqrt{2}}|-\rangle \qquad \text{(Since common phase is arbitrary) (and normalizing)}$$

$$\Rightarrow \boxed{\theta(t) = \frac{\pi}{2}}$$

$$\boxed{\phi(t) = -\frac{\gamma B \sin \omega t}{\omega}}$$

c) Probability of find $S_x = -\frac{\hbar}{2}$

$$= |\langle x; - | \psi, t \rangle|^2$$

$$= \left(\frac{1}{\sqrt{2}}\langle +| - \frac{1}{\sqrt{2}}\langle -|\right)\left(\frac{e^{i\phi/2}}{\sqrt{2}}|+\rangle + \frac{e^{-i\phi/2}}{\sqrt{2}}|-\rangle\right)$$

$$= \left|\frac{1}{2}e^{i\phi/2} - \frac{1}{2}e^{-i\phi/2}\right|^2$$

$$= \sin^2\left(\frac{\gamma B}{2}\frac{\sin \omega t}{\omega}\right)$$

d) for full fillipng ; $\frac{\gamma B}{2}\frac{\sin \omega t}{\omega} = (2n+1)\frac{\pi}{2}$

or atleast $\frac{\gamma B}{2}\frac{|\sin \omega t|}{\omega} \geq \frac{\pi}{2}$

$$\Rightarrow \frac{\gamma B}{2}\frac{1}{\omega} \geq \frac{\pi}{2}$$

$$\Rightarrow \boxed{\frac{\gamma B}{\pi} \geq \omega}$$