S-Pset-S
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4.a) for any operator $T \in \mathcal{L}(V)$,

$\dim V = \dim(\text{null } T) + \dim(\text{range } T)$

let $u \in \text{null } P \Rightarrow Pu = 0$, $u \neq 0$

let $u \in \text{rang } T$ too

$\Rightarrow u = Pv$ for some $v \in V$

$\Rightarrow Pu = P^2 v = Pv = u \neq 0$

$\Rightarrow Pu \neq 0$, a contradiction

$\Rightarrow \text{null } T \cap \text{range } T = \emptyset$

But $\dim V = \dim(\text{null } T) + \dim(\text{range } T)$

$\Rightarrow$ if V has $1, \ldots, n, n+1, \ldots m$ basis,

null T will have $1, \ldots, n$ (say),

range T will have $n+1, \ldots, m$ (say)

and these will be linearly independent vectors

since $\text{null } T \cap \text{range } T = \emptyset$

$\Rightarrow V = \text{null } T \oplus \text{range } T$

(Because null $T \oplus$ rang T will have $1, \ldots, n, n+1, \ldots, m$

LI basis vectors, they should span V

b) $P^2 = P$

$\Rightarrow$ in diagonlazed basis,

$\lambda_i^2 = \lambda_i \quad \Rightarrow \lambda_i = 0, 1$, λ_i are eigenvalues of P

$\Rightarrow$ P will look like:

$$\begin{pmatrix} 0 \ddots & & & \\ & 1 \cdot 0 \ldots & & \\ & & 1_0 \ldots & \\ & & & 1 \ldots \end{pmatrix}$$

only zeroes and 1 on diagon

$\Rightarrow$ P is orthogonal projector

$|u| \leq |u + av| \Rightarrow$ $|u|^2 \leq |u|^2 + |a|^2|v|^2 + 2Re(avu^*)$

$\Rightarrow$ $-2Re(avu^*) \leq |a|^2|v|^2$

for any given v, u, this can not hold for any $a \in F$ $\Rightarrow$ $vu^* = 0$

$\Rightarrow$ $\langle v | u \rangle = 0$

$\Rightarrow$ let $u = Pu + u - Pu$,

$\Rightarrow$ $|Pv| \leq |v|$ $\Rightarrow$ $|Pv| \leq |Pv + v - Pv|$

$\Rightarrow$ $\langle Pv | v - Pv \rangle = 0$

$\Rightarrow$ $\langle v | P^\dagger (I - P) | v \rangle = 0 \quad \forall v$

$\Rightarrow$ $P^\dagger (I - P) = 0$ using result from (3)

$\Rightarrow$ $P^\dagger I - P^\dagger P = 0$

$\Rightarrow$ $P^\dagger - P - P^\dagger P + P^2 = 0$ (since $P^2 - P = 0$)

$\Rightarrow$ $(P^\dagger - P) - (P^\dagger - P)P = 0$

$\Rightarrow$ $(P^\dagger - P)(I - P) = 0$, if $I = P$, P is orthognal projector

ow $P^\dagger = P$ $\Rightarrow$ P is hermitian

$\Rightarrow$ P is orthogonal projector from previous proof