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40. $\hat{H} = \frac{1}{2m}\hat{p}^2 + V(\hat{x})$

$H|a\rangle = E_a|a\rangle$

a) $[[\hat{x},\hat{H}],\hat{x}] = 2\hat{x}\hat{H}\hat{x} - \hat{H}\hat{x}^2 - \hat{x}^2\hat{H}$

$\hat{H}$ contains $V(\hat{x})$ which will commute with $\hat{x}$

$\Rightarrow 2\hat{x}\frac{\hat{p}^2}{2m}\hat{x} - \frac{\hat{p}^2}{2m}\hat{x}^2 - \frac{\hat{x}^2\hat{p}^2}{2m}$

$[\hat{x},\hat{p}^2]\frac{\hat{x}}{2m} - \frac{\hat{x}}{2m}[\hat{x},\hat{p}^2]$

$= \frac{i\hbar}{m}[\hat{p}\hat{x} - \hat{x}\hat{p}] = \frac{\hbar^2}{m}$

$\Rightarrow (2\hat{x}\hat{H}\hat{x} - \hat{H}\hat{x}^2 - \hat{x}^2\hat{H})|a\rangle = \frac{\hbar^2}{m}|a\rangle$

$\Rightarrow \langle a| 2\hat{x}\hat{H}\hat{x} - \hat{H}\hat{x}^2 - \hat{x}^2\hat{H}|a\rangle = \frac{\hbar^2}{m}$ (Assuming $\langle a|a\rangle = 1$)

$\cancel{\Rightarrow 2\langle a|\hat{x}|a'\rangle\langle a|\hat{H}\hat{x}|a\rangle - \langle a|\hat{H}|a'\rangle\langle a'|\hat{x}^2|a\rangle - \langle a|\hat{x}^2|a'\rangle}$

The first term is $\sum_{a'} 2\langle a|\hat{x}|a'\rangle\langle a'|\hat{H}|\hat{x}|a\rangle$

since By completeness $\sum_{a'}|a'\rangle\langle a'| = I$

now $\langle a'|\hat{H} = E_{a'}\langle a'|$

$\Rightarrow$ first term is $\sum_{a'} 2\langle a|\hat{x}|a'\rangle\langle a'|\hat{x}|a\rangle E_{a'}$

since $\hat{x}$ is hermitian $\Rightarrow \langle a|\hat{x}|a'\rangle = \langle a'|\hat{x}|a\rangle^*$

$\Rightarrow$ first term is: $\sum_{a'} 2E_{a'}|\langle a|\hat{x}|a'\rangle|^2$

second term is: $-\langle a|\hat{H}|\hat{x}^2|a\rangle = -E_a\langle a|\hat{x}^2|a\rangle$

$= -E_a\sum_{a'}\langle a|\hat{x}|a'\rangle\langle a'|\hat{x}|a\rangle = -E_a\sum_{a'}|\langle a|\hat{x}|a'\rangle|^2$

third term is also same

thus $\sum_{a'}|\langle a|\hat{x}|a'\rangle|^2(E_{a'} - E_a) = \frac{\hbar^2}{2m}$

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b) $\langle a|\hat{p}|a'\rangle = \frac{im}{\hbar}(E_a - E_{a'})\langle a|\hat{x}|a'\rangle$

$$[\hat{H},\hat{x}] = \hat{H}\hat{x} - \hat{x}\hat{H}$$

$$= \frac{\hat{p}^2\hat{x}}{2m} - \frac{\hat{x}\hat{p}^2}{2m} = -\frac{i\hbar\hat{p}}{m}$$

$$\Rightarrow \langle a|\hat{H}\hat{x} - \hat{x}\hat{H}|a'\rangle = -\frac{i\hbar}{m}\langle a|\hat{p}|a'\rangle$$

$$\Rightarrow (E_a - E_{a'})\langle a|\hat{x}|a'\rangle = -\frac{i\hbar}{m}\langle a|\hat{p}|a'\rangle$$

$$\Rightarrow \langle a|\hat{p}|a'\rangle = \frac{im}{\hbar}(E_a - E_{a'})\langle a|\hat{x}|a'\rangle$$

or $\langle a'|\hat{p}|a\rangle = -\frac{im}{\hbar}(E_a - E_{a'}\langle a'|\hat{x}|a\rangle$ (since $\hat{x},\hat{p}$ hermitian)

multiplying both:

$$\langle a|\hat{p}|a'\rangle\langle a'|\hat{p}|a\rangle = \frac{m^2}{\hbar^2}(E_a - E_{a'})^2\langle a|\hat{x}|a'\rangle\langle a'|\hat{x}|a\rangle$$

Summing over all a':

$$\sum_{a'}\langle a|\hat{p}|a'\rangle\langle a'|\hat{p}|a\rangle = \frac{m^2}{\hbar^2}\sum_{a'}(E_a - E_{a'})^2\langle a|\hat{x}|a'\rangle\langle a'|\hat{x}|a\rangle$$

$$\Rightarrow \boxed{\langle a|\hat{p}^2|a\rangle = \frac{m^2}{\hbar^2}\sum_{a'}(E_a - E_{a'})^2|\langle a|\hat{x}|a'\rangle|^2}$$

(since $\sum_{a'}|a'\rangle\langle a'| = \mathbb{1}$)

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c) Virial's Theoram:

$[\hat{x}\hat{p}, \hat{H}] = \hat{x}\hat{p}\hat{H} - \hat{H}\hat{x}\hat{p}$

$\langle a| \hat{x}\hat{p}\hat{H} - \hat{H}\hat{x}\hat{p} |a\rangle = E\langle a| \hat{x}\hat{p} - \hat{x}\hat{p} |a\rangle = 0$

$\Rightarrow \hat{x}\hat{p}\hat{H} - \hat{H}\hat{x}\hat{p} = \frac{\hat{x}\hat{p}^3}{2m} + \hat{x}\hat{p}V(\hat{x}) - \frac{\hat{p}^2}{2m}\hat{x}\hat{p} - V(\hat{x})\hat{x}\hat{p}$

$= \frac{i\hbar\hat{p}^2}{m} + \hat{x}\hat{p}V(\hat{x}) - V(\hat{x})\hat{x}\hat{p}$

in x-basis, $[\hat{p}, V(\hat{x})]$ is

$= \hat{p}V(\hat{x})f - V(\hat{x})\hat{p}f$

$= -i\hbar\frac{\partial}{\partial x}V(x)f$

$\Rightarrow \hat{x}\hat{p}V(\hat{x}) - V(\hat{x})\hat{x}\hat{p} = -i\hbar\,\hat{x}\frac{\partial V(\hat{x})}{\partial x}$

$\Rightarrow \langle a|\frac{\hat{p}^2}{m}|a\rangle - \langle a|\hat{x}\frac{\partial V(\hat{x})}{\partial x}|a\rangle = 0$

or $\boxed{\langle a|\frac{\hat{p}^2}{2m}|a\rangle = \frac{1}{2}\langle a|\hat{x}\,\partial_{\hat{x}}V(\hat{x})|a\rangle}$

Since this relation hold for each eigenket, it must hold for any superposition

$\Rightarrow \langle\psi|\frac{\hat{p}^2}{2m}|a\rangle = \frac{1}{2}\langle\psi|\hat{x}\,\partial_{\hat{x}}V(\hat{x})|\psi\rangle$

$\Rightarrow \boxed{2\langle T\rangle = \langle x\frac{dV}{dx}\rangle}$

For $V = \alpha x^n$, $2\langle T\rangle = \alpha\langle V\rangle$