MIT OCW 8.05 - 2013
Pset-3 P5
Shlok Vaibhav Singh
2020/06/24

a) Given $V = U_1 \oplus W$ and $V = U_2 \oplus W$

So we know that $U_1 \cap W = \{0\}$ and $U_2 \cap W = \{0\}$, this means $(U_1 - U_2) \cap W = \phi$ (Difference is distributive over intersection). This means if U_1 has an element a which is not in U_2, it will not be an element of W as well, this means it will not be in the direct sum of U_1 and W as well, this means it is not an element of V, but this is a contradiction, this means $U_1 - U_2$=$\phi \Rightarrow U_2 \subseteq U_1$. Similarly it can be shown that $U_1 \subseteq U_2$, this means that $U_1 = U_2$.

b) In a finite-dimensional vector space, the length of any spanning list must be larger than or equal to the length of any list of linearly independent vectors. Let the dimensionality of F^∞ be k, where elements of F^∞ be represented by the sequence $\{a_n\}$, then can choose a set of linearly independent vectors as sequences : $v_i : \{a_n = 1 \; if \; n = i, \; else \; a_n = 0\}, 1 < i \leq k$, clearly the vector v_{k+1} is linearly independent to each element of our chosen set, this implies the dimensionality of F^∞ is atleast k+1, thus our assumption about k being an finite number is wrong.

c) Let T be injective, this means there is only one element, $a \in T : null(T) = \{a\}$ since two elements cannot map to 0, now it is shown in lecture notes that T(0)=0 $\Rightarrow null(T) = \{0\}$, now let T be a linear map such that $null(T) = \{0\}$, now assume $u, \; v \; \in T: \; T(v) = T(u) \Rightarrow T(v-u) = T(v) - T(u) = 0 \Rightarrow v - u \; \in null(T) \Rightarrow v = u$. Thus T is injective. And we have shown that T is injective iff $null(T) = \{0\}$.