PSET-8

1. a) $|x\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right) \exp\left(\sqrt{\frac{2m\omega}{\hbar}}\, x a^{+} - \frac{1}{2} a^{+}a^{+}\right)|0\rangle$

$\Rightarrow \frac{N}{\sqrt{2}} \langle 0| \exp\left(\sqrt{\frac{2m\omega}{\hbar}}\, x a - \frac{1}{2} a a\right) a^{+}a^{+}|0\rangle$

$\Rightarrow \frac{N}{\sqrt{2}} \langle 0| \exp\left(\sqrt{\frac{2m\omega}{\hbar}}\, x a\right) \underbrace{\exp\left(-\frac{1}{2} a a\right)}_{\left(1-\frac{1}{2}aa\right)} a^{+}a^{+}|0\rangle$

(further terms annihilate $a^{+}a^{+}|0\rangle$)

$\left(1-\frac{1}{2}aa\right)a^{+}a^{+}|0\rangle = a^{+}a^{+}|0\rangle - |0\rangle$

since $a^{+}a^{+}|0\rangle = \sqrt{2}|2\rangle$, $aa\sqrt{2}|2\rangle = 2|0\rangle$

$\Rightarrow \frac{N}{\sqrt{2}} \langle 0| \left(1 + \sqrt{\frac{2m\omega}{\hbar}}\, x a + \left(\sqrt{\frac{2m\omega}{\hbar}}\, x\right)^2 \frac{aa}{2!}\right)\left(a^{+}a^{+} - 1\right)|0\rangle$

writing only terms which give non-zero expectation

$\Rightarrow \frac{N}{\sqrt{2}} \langle 0| \left(\sqrt{\frac{2m\omega}{\hbar}}\, x\right)^2 \frac{aaa^{+}a^{+}}{2} - 1|0\rangle$

$\Rightarrow \frac{N}{2\sqrt{2}} \left(4\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right)^2 - 2\right)$

$\Rightarrow |2\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right) \frac{1}{\sqrt{2^2 2!}} \left(4\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right)^2 - 2\right)$

$= \frac{1}{\sqrt{2^2 2!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right) H_2\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right)$

H_2 is second-order Hermite polynomial

b) clearly, $|p\rangle = N\exp\left(i\sqrt{\frac{2}{\hbar m\omega}}\, p_0 a^\dagger + \frac{1}{2} a^\dagger a^\dagger\right)|0\rangle$, ignore N

since $\hat{p}|p\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\,(a^\dagger - a)\exp\left(i\sqrt{\frac{2}{\hbar m\omega}}\, p_0 a^\dagger + \frac{1}{2} a^\dagger a^\dagger\right)|0\rangle$

since $\exp\left(i\sqrt{\frac{2}{\hbar m\omega}}\, p_0 a^\dagger + \frac{1}{2} a^\dagger a^\dagger\right) a|0\rangle = 0$, as always

$\hat{p}|p\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\,\left(a^\dagger(\;) + \left[\exp\left(i\sqrt{\frac{2}{\hbar m\omega}}\, p_0 a^\dagger + \frac{1}{2} a^\dagger a^\dagger\right), a\right]\right)|0\rangle$

Using results from Pset-4:

$\hat{p}|p\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\,\left(a^\dagger - i\sqrt{\frac{2}{\hbar m\omega}}\, p_0 - a^\dagger\right)\exp\left(i\sqrt{\frac{2}{\hbar m\omega}}\, p_0 a^\dagger + \frac{1}{2} a^\dagger a^\dagger\right)|0\rangle$

$= p_0|p\rangle$

Thus $\hat{p}|p\rangle = p_0|p\rangle$

As for normalisation, $\langle p|\psi_0\rangle = \left(\frac{1}{m\pi\hbar\omega}\right)^{1/4} e^{-p^2/2m\hbar\omega}$

$\Rightarrow N\langle 0|\exp\left(-i\sqrt{\frac{2}{\hbar m\omega}}\, p_0 a + \frac{1}{2} aa\right)|0\rangle =$

$\Rightarrow N =$

$$\Rightarrow \boxed{|p\rangle = \left(\frac{1}{m\pi\hbar\omega}\right)^{1/4} e^{-p^2/2m\hbar\omega} \exp\left(i\sqrt{\frac{2}{\hbar m\omega}}\, p_0 a^\dagger + \frac{1}{2} a^\dagger a^\dagger\right)}$$