3. $V(x) = -V_0 a \sum_{n=-1}^{1} \delta(x-na)$; $V_0 > a$, $a>0$

a) $\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(V(x) - E)\psi$

$$\Rightarrow \int_{0^-}^{0^+} \frac{d^2\psi}{dx^2}\,dx = \frac{2m}{\hbar^2}\int_{0^-}^{0^+} V(x)\psi(x)\,dx - \frac{2mE}{\hbar^2}\int_{0^-}^{0^+}\psi(x)\,dx$$

$$\Rightarrow \left.\frac{d\psi}{dx}\right|_{0^+} - \left.\frac{d\psi}{dx}\right|_{0^-} = -\frac{2mV_0\alpha}{\hbar^2}\int_{0^-}^{0^+}\sum_{n=-1}^{1}\delta(x-na)\psi(x)\,dx + 0$$

(since $\psi(x)$ is cont. for δ-potential)

$$\Rightarrow \boxed{\left.\frac{d\psi}{dx}\right|_{0^+} - \left.\frac{d\psi}{dx}\right|_{0^-} = -\frac{2mV_0\alpha}{\hbar^2}\psi(0)}$$ ($\delta(x)$ picked $\psi(0)$)

$$\left.\frac{d\psi}{dx}\right|_{\pm a^-}^{\pm a^+} = -\frac{2mV_0 a}{\hbar^2}\psi(\pm a)$$ Similarly — (1)

b) i) No node (since $\psi \sim e^{-kx}$, $k>0$)

ii) One node (since $\psi \sim (\ )e^{-kx} + (\ )e^{kx}$, $k>0$)
(gives unique zero)

iii) No, if $\psi(a) = 0 \Rightarrow \psi(x) = 0 \quad \forall\ x > a$
(since $\psi \sim e^{-kx}\,\psi(a)$)

$\Rightarrow \psi'(a^+) = 0$

Using (1) $\Rightarrow \psi'(a^-) = 0$

$\Rightarrow \psi(x) = 0 \quad \forall\ x \in [0, a]$

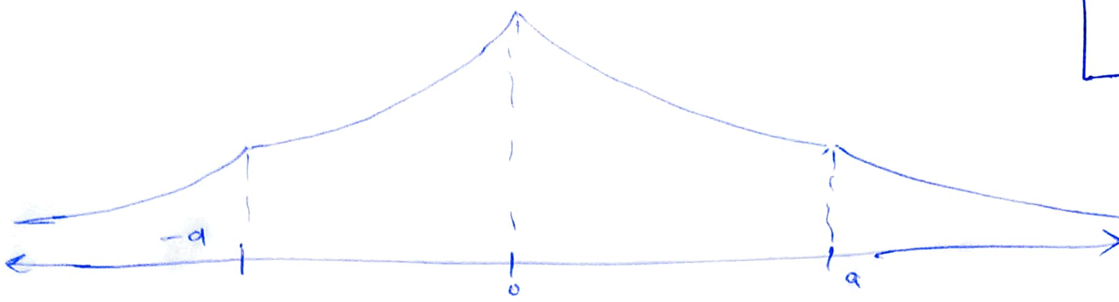
iteratively, in all region $\psi = 0 \ \forall x$

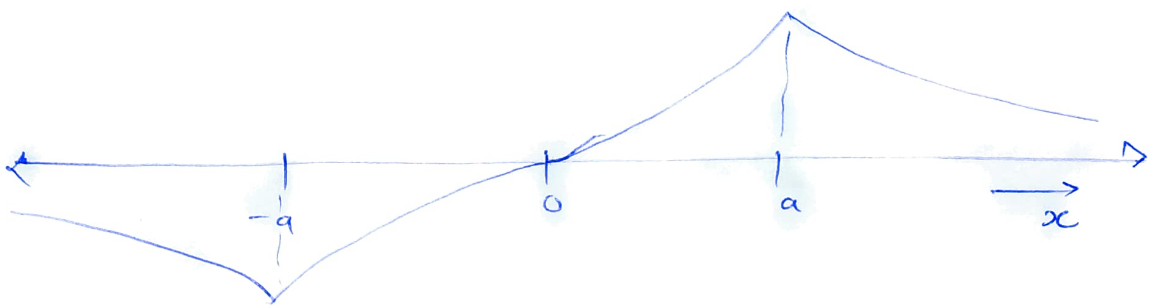
iv) Yes, $\psi'(0^-) = \psi'(0^+)$ so we get odd states

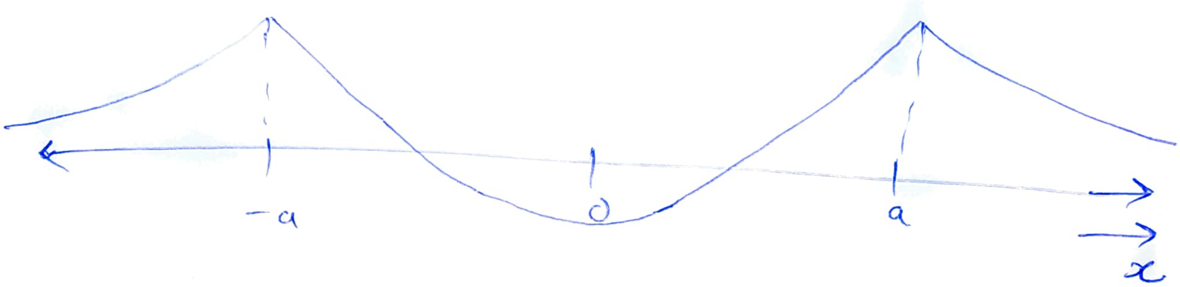
(Unlike iii), here we have 2 coefficients on each side, so ψ will exist

S-PSet-1
M170CW
2013-8-05

c)



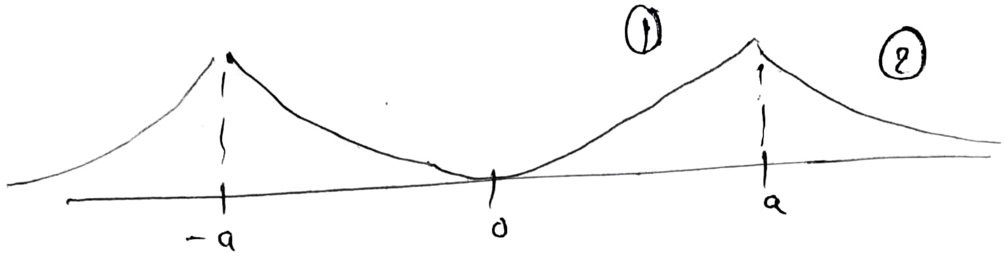




These are only possible states

7 – pset-1
m270cw
8-05-2013

d)



①: $Ae^{kx} + Be^{-kx}$ $K = \sqrt{\frac{2m|E|}{\hbar^2}}$

②: Ce^{-kx}

Continuity at $x = a$ $\Rightarrow$ ~~A+B=C~~ $Ae^{ka} + Be^{-ka} = Ce^{-ka}$

$\psi(0) = 0$ $\Rightarrow$ $A = -B$

$\Rightarrow C = 2A\sinh(ka)\, e^{ka}$

$$\frac{d\psi}{dx}\Big|_{a_-}^{a_+} = -\frac{2mV_0 a}{\hbar^2}\psi(a)$$

$\Rightarrow -2Ak\sinh(ka)e^{ka-ka} - 2Ak\cosh(ka) = -\frac{2mV_0 a A\cdot 2\sinh(ka)}{\hbar^2}$

$\downarrow$

$\Rightarrow (-kCe^{-ka})$

$\Rightarrow k\sinh(ka) + k\cosh(ka) = \frac{mV_0 a}{\hbar^2}\sinh(ka)$

$\Rightarrow k(1+\coth(ka)) = \frac{mV_0 a}{\hbar^2}$ — (1)

$k\coth(ka)$ can be defined at $x=0$;

derivative of same is: $1+\coth(ka) - k\,\mathrm{cosech}^2(ka)$

$= 1+\coth(ka) - k(1-\coth(ka))(1+\coth(ka))$

$= [1+\coth(ka)](1-k(1-\coth(ka)))$

$\uparrow$ Always > 0 $\downarrow$ Numerator ambiguous, Denominator > 0

So looking at numerator of $1-k(1-\coth(ka))$

$$= 1-k\left(1-\frac{e^{ka}+e^{-ka}}{e^{ka}-e^{-ka}}\right)$$

$$= 1+\frac{2k\,e^{-ka}}{e^{ka}-e^{-ka}}$$

$$= e^{ka}-e^{-ka}+2ke^{-ka} > 0$$

So that derivative of LHS of (1) is always ≥ 0

$\Rightarrow$ minima at $x=0$ (k can't be <0)

so that $V_{min} = \lim_{k\to 0}\left(k+k\coth(ka)\right)\times\frac{\hbar^2}{ma}$

$$= \frac{\hbar^2}{ma^2}$$

$$\Rightarrow \boxed{V_{min} = \frac{\hbar^2}{ma^2}}$$