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3. $H = -\gamma B \hat{S}_z = -\gamma B \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

since $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^2 = I$ and in PSET 2 we showed that:

$$e^{iM\theta} = \cos\theta \mathbb{1} + i\sin\theta M \qquad \text{if } M^2 = I$$

$$\Rightarrow e^{-i\hat{H}t/\hbar} = \cos\left(\frac{-\gamma B\hbar t}{2\hbar}\right)\mathbb{1} + i\sin\left(\frac{-\gamma B\hbar t}{2\hbar}\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$= \cos\left(\frac{\gamma B\hbar t}{2\hbar}\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - i\sin\left(\frac{\gamma B\hbar t}{2\hbar}\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

Given $|\psi(0)\rangle = |x;+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$$\Rightarrow |\psi(t)\rangle = U(t,0)|\psi(0)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle$$

$$= \left[\cos\left(\frac{\gamma B\hbar t}{2\hbar}\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - i\sin\left(\frac{\gamma B\hbar t}{2\hbar}\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1}{\sqrt{2}}\cos\left(\frac{\gamma B\hbar t}{2\hbar}\right) - \frac{i}{\sqrt{2}}\sin\left(\frac{\gamma B\hbar t}{2\hbar}\right)\\ \frac{1}{\sqrt{2}}\cos\left(\frac{\gamma B\hbar t}{2\hbar}\right) + \frac{i}{\sqrt{2}}\sin\left(\frac{\gamma B\hbar t}{2\hbar}\right)\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\theta t}\\ e^{i\theta t}\end{pmatrix}; \theta = \frac{\gamma B\hbar}{2\hbar}$$

$$= \cos\theta t|x;+\rangle - i\sin\theta t|x;-\rangle$$

So direcⁿ is along $\pm x$-axis always
or along $\pm z$-axis always
or along $\pm y$-axis always
since direcⁿ is ascertained by act of measurement
but we can say, measurement along z-axis will
yield equally divided samples, so will y-axis
But along x-axis, $\cos^2(\theta t)$ will give $|x;+\rangle$
(proportion of no. of measurement)

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b) Measurement of spin along z-axis gives $+\frac{\hbar}{2}$ with prob. $\frac{1}{2}$

$\Rightarrow \langle H \rangle = \frac{1}{2} \cdot -\frac{\gamma B \hbar}{2} + \frac{1}{2} \frac{\gamma B \hbar}{2} = 0$

$\langle H^2 \rangle = \frac{1}{2} \frac{\gamma^2 B^2 \hbar^2}{4} + \frac{1}{2} \frac{\gamma^2 B^2 \hbar^2}{4} = \frac{\gamma^2 B^2 \hbar^2}{4}$

(same value is obtained by $\langle \psi(0) | H | \psi(0) \rangle$)

$\Rightarrow \Delta H = \frac{\gamma B h}{4\pi}$

while $\cos^2(\theta(t)) = |\langle \psi(0) | \psi(t) \rangle|^2$

$= \left[\cos\left(\frac{\gamma B \hbar t}{2\hbar}\right)\right]^2$, for orthogonality, $\frac{\gamma B \cancel{\hbar} t}{2\cancel{\hbar}} = \frac{\pi}{2}$

$\Rightarrow \Delta t_\perp = \frac{\pi}{\gamma B}$

$\Rightarrow \boxed{\Delta H \Delta t_\perp = \frac{h}{4}}$

Inequality is saturated thus