PSET-6 P1

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1. a) Using Cauchy-Schwarz inequality ($|\langle f|g\rangle|^2 \leq \langle f|f\rangle \langle g|g\rangle$):

$$|\langle \Psi(0)|\Psi(t)\rangle|^2 \leq 1 \tag{1}$$

if $|\Psi\rangle$ is an eigenket of $\hat{H}$, $|\langle \Psi(0)|\Psi(t)\rangle|^2 = 1$ and inequality is saturated.

b) Time independent $\hat{H} \Rightarrow |\Psi, t\rangle = e^{-iH(t-t_0)/\hbar} |\Psi, t_0\rangle$

$$|\langle \Psi, 0|\Psi, t\rangle|^2 = \langle \Psi, 0|\Psi, t\rangle \overline{\langle \Psi, 0|\Psi, t\rangle} \tag{2}$$

Since inner product shows complex conjugation upon switching order of the vectors:

$$|\langle \Psi, 0|\Psi, t\rangle|^2 = \langle \Psi, 0|\Psi, t\rangle \langle \Psi, t|\Psi, 0\rangle \tag{3}$$

let $t_0 = 0$, then for small t:

$$|\Psi, t\rangle = \left(1 - \frac{i\hat{H}t}{\hbar} - \frac{\hat{H}^2 t^2}{2\hbar^2}\right) |\Psi, 0\rangle \tag{4}$$

Since $\hat{H}$ is hermitian:

$$\Rightarrow \langle \Psi, t| = \langle \Psi, 0| \left(1 + \frac{i\hat{H}t}{\hbar} - \frac{\hat{H}^2 t^2}{2\hbar^2}\right) \tag{5}$$

$$\Rightarrow \langle \Psi, 0|\Psi, t\rangle = \langle \Psi, 0| 1 - \frac{i\hat{H}t}{\hbar} - \frac{\hat{H}^2 t^2}{2\hbar^2} |\Psi, 0\rangle \tag{6}$$

$$= \langle \Psi, 0|\Psi, 0\rangle - \frac{it}{\hbar} \langle \Psi, 0| \hat{H} |\Psi, 0\rangle 1 - \frac{t^2}{2\hbar^2} \langle \Psi, 0| \hat{H}^2 |\Psi, 0\rangle \tag{7}$$

$$\Rightarrow \langle \Psi, 0|\Psi, t\rangle = 1 - \frac{it}{\hbar}\langle \hat{H}\rangle - \frac{t^2}{2\hbar^2}\langle \hat{H}^2\rangle + \mathcal{O}(t^3) \tag{8}$$

(since expectations of the Hamiltonian are conserved, we drop any reference of time in the expectation notations). Similarly,

$$\Rightarrow \langle \Psi, t|\Psi, 0\rangle = 1 + \frac{it}{\hbar}\langle \hat{H}\rangle - \frac{t^2}{2\hbar^2}\langle \hat{H}^2\rangle + \mathcal{O}(t^3) \tag{9}$$

Multiplying both equations :

$$|\langle \Psi, 0|\Psi, t\rangle\,|^2 = \left(1 - \frac{it}{\hbar}\langle \hat{H}\rangle - \frac{t^2}{2\hbar^2}\langle \hat{H}^2\rangle + \mathcal{O}(t^3)\right)\left(1 + \frac{it}{\hbar}\langle \hat{H}\rangle - \frac{t^2}{2\hbar^2}\langle \hat{H}^2\rangle + \mathcal{O}(t^3)\right) \tag{10}$$

$$|\langle \Psi, 0|\Psi, t\rangle\,|^2 = 1 - \frac{t^2}{\hbar^2}\langle \hat{H}^2\rangle + \frac{t^2}{\hbar^2}\langle \hat{H}\rangle^2 + \mathcal{O}(t^3) \tag{11}$$

since $(\Delta H)^2 = \langle \hat{H}^2\rangle - \langle \hat{H}\rangle^2$:

$$\boxed{|\langle \Psi, 0|\Psi, t\rangle\,|^2 = 1 - \frac{t^2}{\hbar^2}\Delta(\hat{H})^2 + \mathcal{O}(t^3)} \tag{12}$$