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~~c) $\hat{a}^\dagger_x$~~

6. a) $\hat{H} = \frac{1}{2m}\hat{p}_x^2 + \frac{1}{2m}\hat{p}_y^2 + \frac{1}{2}m\omega_x^2\hat{x}^2 + \frac{1}{2}m\omega_y^2\hat{y}^2$

It is reasonable to introduce:

$$\hat{a}_x = \frac{1}{\sqrt{2\hbar m\omega_x}}(+i\hat{p}_x + m\omega_x\hat{x})$$

$$\hat{a}_y = \frac{1}{\sqrt{2\hbar m\omega_x}}(+i\hat{p}_y + m\omega_y\hat{y})$$

$\Rightarrow \hat{H} = \hbar\omega_x(\hat{N}_x + \frac{1}{2}) + \hbar\omega_y(\hat{N}_y + \frac{1}{2})$

Clearly $E_{n,m} = \hbar\omega_x(n+\frac{1}{2}) + \hbar\omega_y(m+\frac{1}{2})$

where $|n,m\rangle = \frac{1}{\sqrt{n!}}\frac{1}{\sqrt{m!}}[\hat{a}^\dagger_x]^n[\hat{a}^\dagger_y]^m|0\rangle$

b) Energy-level diagram, $(\omega_x \gtrsim \omega_y)$

$\frac{5}{2}\hbar\omega_x + \frac{\hbar\omega_y}{2}$ ——— $(2,0)$

$\frac{3}{2}\hbar\omega_x + \frac{3}{2}\hbar\omega_y$ ——— $(1,1)$

$\frac{5}{2}\hbar\omega_y + \frac{\hbar\omega_x}{2}$ ——— $(0,2)$

$\frac{\hbar\omega_y}{2} + \frac{3\hbar\omega_x}{2}$ ——— $(1,0)$

——— $(0,1)$ $\frac{3\hbar\omega_y}{2} + \frac{\hbar\omega_x}{2}$

——— $(0,0)$ $\frac{\hbar\omega_x}{2} + \frac{\hbar\omega_y}{2}$

c) $\hat{N}_x = \frac{\hat{N}+\hat{n}}{2}$, $\hat{N}_y = \frac{\hat{N}-\hat{n}}{2}$

$\Rightarrow E_{n,m} = \hbar\omega_x\left(\frac{N+n}{2}+\frac{1}{2}\right) + \hbar\omega_y\left(\frac{N-n}{2}+\frac{1}{2}\right)$

$\{\hat{N},\hat{n}\}$ and $\{\hat{N}_x,\hat{N}_y\}$ are ~~commuting variables~~ complete sets

if ω_x/ω_y is rational

else any one of four set suffices

d) $\hat{l} = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$

$$\hat{l} = i\sqrt{\frac{\hbar}{2m\omega_x}}\sqrt{\frac{\hbar m\omega_y}{2}}(\hat{a}_x+\hat{a}_x^\dagger)(\hat{a}_y-\hat{a}_y^\dagger)$$

$$-i\sqrt{\frac{\hbar m\omega_x}{2}}\sqrt{\frac{\hbar}{2m\omega_y}}(\hat{a}_y+\hat{a}_y^\dagger)(\hat{a}_x-\hat{a}_x^\dagger)$$

$$= \frac{\hbar}{2}i(2\hat{a}_x^\dagger\hat{a}_y - 2\hat{a}_x\hat{a}_y^\dagger) = i\hbar(\hat{a}_x^\dagger\hat{a}_y - \hat{a}_x\hat{a}_y^\dagger)$$

$$[\hat{H},\hat{l}] = [a_x^\dagger a_x + a_y^\dagger a_y,\ a_x^\dagger a_y - a_x a_y^\dagger]$$

$$= a_x^\dagger a_x a_x^\dagger a_y + a_y^\dagger a_y a_x^\dagger a_y - a_x^\dagger a_x a_x a_y^\dagger - a_y^\dagger a_y a_x a_y^\dagger$$

$$- a_x^\dagger a_y a_x^\dagger a_x - a_x^\dagger a_y a_y^\dagger a_y + a_x a_y^\dagger a_x^\dagger a_x + a_x a_y^\dagger a_y^\dagger a_y$$

$$a_x^\dagger[a_x,a_x^\dagger]a_y + a_x^\dagger[a_y^\dagger,a_y]a_y + [a_x,a_x^\dagger]a_y^\dagger a_x$$

$$[a_y^\dagger,a_y]\,a_y^\dagger a_x$$

$= 0$

Thus $\hat{H}$ and $\hat{l}$ commute

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e) $\hat{a}_L = \frac{1}{\sqrt{2}}(\hat{a}_x + i\hat{a}_y) \qquad \hat{a}_R = \frac{1}{\sqrt{2}}(\hat{a}_x - i\hat{a}_y)$

$$\hat{N}_L = \hat{a}_L^\dagger a_L = \frac{1}{\sqrt{2}}(\hat{a}_x^\dagger - i\hat{a}_y^\dagger)\frac{1}{\sqrt{2}}(\hat{a}_x + i\hat{a}_y)$$

$$= \frac{1}{2}(\hat{a}_x^\dagger \hat{a}_x + \hat{a}_y^\dagger \hat{a}_y + i\hat{a}_x^\dagger \hat{a}_y - i\hat{a}_y^\dagger \hat{a}_x)$$

$$\hat{N}_R = \hat{a}_R^\dagger \hat{a}_R = \frac{1}{\sqrt{2}}(\hat{a}_x^\dagger + i\hat{a}_y^\dagger)\frac{1}{\sqrt{2}}(\hat{a}_x - i\hat{a}_y)$$

$$= \frac{1}{2}(\hat{a}_x^\dagger a_x + \hat{a}_y^\dagger a_y + i\hat{a}_y^\dagger \hat{a}_x - i\hat{a}_x^\dagger \hat{a}_y)$$

$$\hat{H} = \hbar\omega(\hat{N}_L + \hat{N}_R)$$

$$\vec{J} = \hbar(\hat{N}_L - \hat{N}_R)$$

Claim: If $(\hat{a}_R^\dagger)^n |0,0\rangle$ is eigenstate of $\hat{N}_i$ with eigenvalue n, $(a_R^\dagger)^{n+1}|0,0\rangle$ is also an eigenstate with value $n+1$:

$$\hat{N}_R(a_R^\dagger)^n|0,0\rangle = n(a_R^\dagger)^n|0,0\rangle$$

$$\hat{N}_R(a_R^\dagger)^{n+1}|0,0\rangle = a_R^\dagger \hat{a}_R a_R^\dagger (a_R^\dagger)^n |0,0\rangle$$

$$= a_R^\dagger[1 + a_R^\dagger a_R](a_R^\dagger)^n|0,0\rangle$$

Since $[a_R, a_R^\dagger] = 1$

$$= (a_R^\dagger)^{n+1}|0,0\rangle + a_R^\dagger n (a_R^\dagger)^n|0,0\rangle$$

$$= (n+1)(a_R^\dagger)^{n+1}|0,0\rangle$$

Similar ~~an~~ result for $\hat{N}_L$ and $\hat{a}_L$

$$\hat{N}_R|0,0\rangle = 0 \Rightarrow \hat{N}_R(a_R^\dagger)^n|0,0\rangle = n(a_R^\dagger)^n|0,0\rangle$$

$$\hat{N}_L(a_L^\dagger)^n|0,0\rangle = n(a_L^\dagger)^n|0,0\rangle$$

Similar to harmonic oscillator case,

$\frac{(a_R^\dagger)^n}{\sqrt{n!}}|0,0\rangle$ and $\frac{(a_L^\dagger)^n}{\sqrt{n!}}|0,0\rangle$ are normalized

eigenkets of $\hat{N}_R$ and $\hat{N}_L$ respectively with

eigenvalue n

we note $[\hat{a}_R, \hat{a}_L] = [\hat{a}_R^\dagger, \hat{a}_L] = 0$

$$[\hat{a}_R, \hat{a}_R^\dagger] = [\hat{a}_L, \hat{a}_L^\dagger] = 0$$

$$\Rightarrow \hbar(\hat{N}_R - \hat{N}_L)\frac{(\hat{a}_R^\dagger)^n}{\sqrt{n!}}\frac{(\hat{a}_L^\dagger)^m}{\sqrt{m!}}|0,0\rangle$$

$$= \hbar(n-m)\left(\frac{\hat{a}_R^\dagger}{\sqrt{n!}}\right)^n\frac{(\hat{a}_L^\dagger)^m}{\sqrt{m!}}|0,0\rangle$$

$\Rightarrow \frac{(\hat{a}_R^\dagger)^n}{\sqrt{n!}}\frac{(\hat{a}_L^\dagger)^m}{\sqrt{m!}}|0,0\rangle$ is an eigenvector of $\vec{l}$ with

eigenvalue $\hbar(n+m)$

now, each time we operate $\hat{a}_R^\dagger$ or $\hat{a}_L^\dagger$ on a state with energy $n\hbar\omega$, we get a superposition of states with energy $(n+1)\hbar\omega$

Thus, for degenerate state $|n,m\rangle$, $n+m=N$

$\vec{l}$ gives eigen values $n-m = N, N-2, \dots, -N+2, -N$

$\Rightarrow$ By operating $\hat{H}$ and $\hat{l}$ on $|n,m\rangle$, n and m can be determined

$\Rightarrow \hat{H}$ and $\hat{l}$ together constitute a complete sets of commuting variables of entire hilbert space