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4. a) $f(\lambda) \geq 0 \Rightarrow |a|^2 + \lambda^2 |b|^2 - 2\lambda(a\cdot b) \geq 0$

minima occurs at $\frac{\partial f(\lambda)}{\partial \lambda} = 0$

$\Rightarrow \lambda = \frac{(a\cdot b)}{|b|^2}$

$\Rightarrow \min_{\lambda \in R} f(\lambda) = f(a\cdot b/|b|^2) = |a|^2 - \frac{(a\cdot b)^2}{|b|^2} \geq 0$

$\Rightarrow \boxed{|a\cdot b| \leq |a|\,|b|}$

b) $f(\lambda) = (\langle a| - \lambda^* \langle b|)(|a\rangle - \lambda |b\rangle) \geq 0$

$\Rightarrow |a|^2 + \lambda\bar{\lambda}|b|^2 - \bar{\lambda}\langle b|a\rangle - \lambda\langle a|b\rangle \geq 0$

Treating λ and $\bar{\lambda}$ as independent variables:

$\frac{\partial f(\lambda)}{\partial \lambda} = \frac{\partial f(\lambda)}{\partial \lambda^*} = 0$

$\Rightarrow \lambda = \frac{\langle b|a\rangle}{|b|2}$

$\Rightarrow \min_{\lambda \in \mathbb{C}} f(\lambda) = |a|^2 - |\langle b|a\rangle|^2 / |b|^2 \geq 0$

$\Rightarrow \boxed{|\langle a|b\rangle|^2 \leq \langle a|a\rangle \langle b|b\rangle}$

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c) $|a+b| \leq |a| + |b|$

$\Rightarrow |a+b|^2 \leq |a|^2 + |b|^2 + 2|a||b|$

$\Rightarrow \cancel{|a|^2} + \cancel{|b|^2} + \langle b|a\rangle + \langle a|b\rangle \leq \cancel{|a|^2} + \cancel{|b|^2} + 2|a||b|$

$\langle b|a\rangle + \langle a|b\rangle = 2\,\mathrm{Real}(\langle a|b\rangle) \leq 2|\langle a|b\rangle|$

$\Rightarrow 2|\langle a|b\rangle| \leq 2|a||b|$

$\Rightarrow$ writing in reverse order, we get inequality of triangle