9- Pset-1
M27 OCW
8.05-2013

4. a) If well is very deep,

highest mode will have, say n roots

these are equally spaced, since n is

expected to be large, we can assume

$\frac{\lambda n}{2} \approx a$, $\frac{\lambda}{2}$ is distance between any

two nodes

$$\lambda = \frac{2a}{n} \Rightarrow k = \frac{2\pi}{\lambda} = \frac{\pi n}{a}$$

we know for $|x| < a$,

$$k = \sqrt{\frac{2m}{\hbar^2}(E_n + V_0)}$$

for highest state $E_n \approx 0$

$$\Rightarrow k = \frac{1}{\hbar}\sqrt{2mV_0}$$

$$\Rightarrow n = \frac{1}{\pi}\frac{a}{\hbar}\sqrt{2mV_0}$$

$$\Rightarrow \boxed{n = \frac{z_0}{\pi}}$$

This is also seen from fig 2.18, as z_0 gets larger, successive roots get closer to $n\pi$

Thus nearly $\frac{z_0}{\pi}$ roots are found

b) If z_0 is very small, $\tan z$ will also be small

$\Rightarrow$ $z \tan z = \sqrt{z_0^2 - z^2}$

$\tan z \approx z$

$\Rightarrow$ $z^2 = \sqrt{z_0^2 - z^2}$

$\Rightarrow$ $z^4 = z_0^2 - z^2$

$\Rightarrow$ $z^4 + z^2 - z_0^2 = 0$

$\Rightarrow$ $z^2 = -\frac{1}{2} \pm \frac{\sqrt{1 + 4z_0^2}}{2}$

$\Rightarrow$ $z^2 = -\frac{1}{2} + \frac{\sqrt{1+4z_0^2}}{2}$

z_0 is small

$\Rightarrow$ $z^2 \approx -\frac{1}{2} + \frac{1}{2}(1 + 2z_0^2 - 2z_0^4)$

$z \approx \sqrt{z_0^2 - z_0^4}$

$\Rightarrow$ $l \approx \frac{\sqrt{z_0^2 - z_0^4}}{a}$

$\Rightarrow$ as $l = \sqrt{\frac{2m(E+V_0)}{\hbar}}$

$\frac{z_0^2 - z_0^4}{a^2} = \frac{2m(E+V_0)}{\hbar}$

$\Rightarrow$ $\boxed{\frac{\hbar}{2m} \frac{(z_0^2 - z_0^4)}{a^2} - V_0 \approx E}$