Ph-12c

P-Set-1

1. Sterling's approximation

$$n! = (2\pi n)^{1/2} n^n e^{-n}\left(1 + \frac{1}{12n} + O\left(\frac{1}{n^2}\right)\right)$$

a) $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx$

Clearly, $\Gamma(1) = \int_0^\infty e^{-x}\, dx = 1 = 0!$

now, $\Gamma(n+1) = \int_0^\infty x \cdot x^{n-1} e^{-x}\, dx = -x^n e^{-x}\Big|_0^\infty + n\int_0^\infty x^n e^{-x} dx$

(here we took $u = x^n$, $dv = e^{-x}$)

$\Rightarrow \Gamma(n+1) = n\,\Gamma(n)$

b) $x^n e^{-x} = \exp(n\ln x - x)$ has maxima at $x_0 = n$

around x_0:

$$n\ln x - x = a_0 - a_2 y^2 + a_3 y^3 + a_4 y^4 + a_5 y^5 + a_6 y^6 + \ldots$$

$y = x - x_0$; $a_i = \frac{1}{i}\frac{1}{n^{i-1}}(-1)^{i-1}$; $i \geq 3$ $\quad a_2 = \frac{1}{2n}$ $\quad a_0 = n\ln n - n$

By ratio test, we see that this power series converges in $x \in [0, 2n]$

Now, we need to establish that for large n, we can work in $[0, 2n]$ without much loss, so we want:

$$\frac{\int_{2n}^\infty x^n e^{-x}\, dx}{\int_0^{2n} x^n e^{-x}\, dx} \to O\left(\frac{1}{n^2}\right) \quad \text{as } n \to \infty$$

since we need accurate answer upto $O\left(\frac{1}{n}\right)$

You may think it is shameful, yet I couldn't establish this relation, fortunately, I found help on stack exchange by Daniel Fischer, he showed:

$$\frac{\int_{2n}^{\infty} x^n e^{-x}\, dx}{\int_0^{2n} x^n e^{-x}\, dx} \sim \left(\frac{2}{e}\right)^n$$

so it decays exponentially

$$\Rightarrow T(n+1) \approx \int_{-x_0}^{x_0} dy \exp\left(a_0 - a_2 y^2 + a_3 y^3 + a_4 y^4 + a_5 y^5 + a_6 y^6 \cdots\right)$$

c) now when $a_2 y^2 = O(1) \Rightarrow y = O(n^{1/2})$

$$a_0 - a_2 y^2 + a_3 y^3 + \cdots = a_0 - O(1) + O(n^{-1/2}) + \cdots$$

while if $y = O(n) \Rightarrow a_0 - a_2 y^2 + a_3 y^3 = a_0 - O(n) + O(n) + \cdots$

$\Rightarrow$ we can evalute only in region where $y = O(n^{1/2})$ since argument is sizable only here

since power-series is convergent in $[0, 2n]$, we can do expansion freely:

$$T(n+1) \approx \int_{-x_0}^{x_0} dy\, e^{a_0 - a_2 y^2}\left(1 + a_3 y^3 + a_4 y^4 + a_5 y^5 + \left(a_6 + \frac{a_3^2}{2}\right) y^6 + \cdots\right)$$

d) now we have a series of Gaussian integrals integrated from $-x_0$ to x_0 however in part c) we showed that these integrands are not sizeable outside $[-x_0, x_0]$ since they become $O(e^{-n})$ near boundary $(\pm x_0)$

$$\Rightarrow \int_{-x_0}^{x_0} \rightarrow \int_{-\infty}^{\infty}$$ will be accurate for large x_0

$$\Rightarrow \Gamma(n+1) = \int_{-\infty}^{\infty} e^{a_0 - a_2 y^2} \left(1 + a_3 y^3 + a_4 y^4 + a_5 y^5 + \left(a_6 + \frac{a_3^2}{2}\right) y^6\right) dy$$

$$= n^n e^{-n} \int_{-\infty}^{\infty} e^{-y^2/2n} \left(1 + \frac{y^4}{4n^3} + \frac{y^6}{18n^4}\right) dy$$

$$= n^n e^{-n} \left[1 - \frac{3}{24n}\sqrt{2\pi n} + \frac{1}{2}\frac{5}{3n}\sqrt{2\pi n}\right]$$

$$= \sqrt{2\pi n}\, n^n e^{-n} \left[1 + \left(\frac{5}{6} - \frac{3}{4}\right)\frac{1}{n}\right] = \sqrt{2\pi n}\, n^n e^{-n} \left[1 + \frac{1}{12n}\right]$$

$$\Rightarrow \boxed{n! = \sqrt{2\pi n}\, n^n e^{-n} \left(1 + \frac{1}{12n} + O(n^2)\right)}$$

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