Ph12C Statistical mechanics 2019
Pset1 P1
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1. Sterling's approximation:

$$n! = (2\pi n)^{\frac{1}{2}} n^n e^{-n}\left(1 + \frac{1}{12}n + O\left(n^2\right)\right)$$

a) $\Gamma(n) = \int_0^{\infty} x^{n-1}e^{-x}dx$

clearly, $\Gamma(1) = \int_0^{\infty} e^{-x}dx = 1 = 0!$

now $\Gamma(n+1) = \int_0^{\infty} x^n e^{-x}dx = \int_0^{\infty} x^n \cdot e^{-x}dx = -x^n e^{-x}\Big|_0^{\infty} + n\int_0^{\infty} x^n e^{-x}dx$

$\Rightarrow \Gamma(n+1) = n\Gamma(n)$

b) $x^n e^{-x} = exp(n\ln(x) - x)$ has maxima at x_0=n, around x_0:

$n\ln x - x = a_0 - a_2y^2 + a_3y^3 + a_4y^4 + a_5y^5 + a_6y^6 + \ldots$

where, $y = x - x_0,\ a_i = \frac{1}{i}\frac{1}{n^{i-1}}(-1)^{i-1},\ i \geq 3,\ a_2 = \frac{1}{2n},\ a_1 = n\ln n - n$

By ratio test we see that this series converges for $x \in [0, 2n]$, so we need to establish that we can work in $[0,\ 2n]$ without incurring any loss for large n. Since we need an expression accurate upto $O\left(\frac{1}{n}\right)$ in multiplication, we need to show that $\frac{\int_{2n}^{\infty} x^n e^{-x}dx}{\int_0^{2n} x^n e^{-x}dx}$ is atmost $O\left(\frac{1}{n^2}\right)$

I should shamefully admit that I was unable to establish this relation, so I took help from Stack Exchange, Daniel Fischer has established that $\frac{\int_{2n}^{\infty} x^n e^{-x}dx}{\int_0^{2n} x^n e^{-x}dx}$=$O\left(\left(\frac{e}{2}\right)^{-n}\right)$ $as\ n \to \infty$,

his derivation is accesible [here](). This is much better than what we needed!

$$\Rightarrow \boxed{\Gamma(n+1) \approx \int_{-n}^{n} dy\ exp\left(a_0 - a_2y^2 + a_3y^3 + a_4y^4 + a_5y^5 + a_6y^6 + \ldots\right)}$$

c) Now we can certainly expand as:

$$\Gamma(n+1) \approx \int_{-n}^{n} dy\ \exp\left(a_0 - a_2y^2\right)\left(1 + a_3y^3 + a_4y^4 + a_5y^5 + \left(a_6 + \frac{a_3^2}{2}y^6\right) + \ldots\right)$$

were we have expanded all but first two terms in form of power series of exponential.

Now, let $u = \dfrac{y^2}{2n} \implies dy = \sqrt{\dfrac{n}{2u}} du$

$\implies \Gamma(n+1) \approx \sqrt{\dfrac{n}{2}} \exp(a_0) \int_{-n/2}^{+n/2} du \exp(-u) u^{-1/2} (1 + ... \ )$

Exploring the "...", we note that each basic term of the form $a_i y^i$ is written as (ignoring minus signs):

$$\frac{\sqrt{2nu}^i}{i \cdot n^{i-1}} = \frac{2^{i/2} n^{i/2} u^{i/2}}{i \cdot n^{i-1}}$$

We see that $i/2 \geq i-1 \implies i \leq 2$, but all our terms have index $i \geq 3$, so all the terms will be of the form $O\left(n^{-j/2}\right),\ j \in \mathbb{N}$, and so will be their linear combination while the constant will be the Gamma integrals of the half-integers. Thus we can approximate as:

$\Gamma(n+1) \approx \int_{-n}^{n} dy \exp\left(a_0 - a_2 y^2\right) \left(1 + a_3 y^3 + a_4 y^4 + a_5 y^5 + \left(a_6 + \dfrac{a_3^2}{2} y^6\right)\right)$

here we have ignored the higher terms denoted by "..."

Now, since we know that each term between two + signs will be a gamma integral which decays quickly as a function of the independent variable, so we can extend the integral limits to ∞, further, we are ignoring odd integrands which will integrate to 0, our final equation is:

$$\Gamma(n+1) = \int_{-\infty}^{\infty} dy \exp\left(a_0 - a_2 y^2\right) \left(1 + a_4 y^4 + \frac{a_3^2}{2} y^6 + O\left(\frac{1}{n^2}\right)\right)$$

here, next significant terms will be $O\left(\dfrac{1}{n^2}\right)$ and can be verified easily.

d) Here, we have been asked to evaluate the integral we arrived at in part c):

$\Rightarrow \Gamma(n+1) = \int_{-\infty}^{\infty} dy \exp\left(n \ln n - \dfrac{y^2}{2n}\right) \left(1 - \dfrac{1}{4n^3} y^4 + \dfrac{1}{18n^4} y^6 + O\left(\dfrac{1}{n^2}\right)\right)$

again let $u = \dfrac{y^2}{2n} \implies dy = \sqrt{\dfrac{n}{2u}} du$

$\Rightarrow \Gamma(n+1) = 2\sqrt{\dfrac{n}{2}} \exp(n \ln n - n) \int_0^{+\infty} du \left(u^{-1/2} e^{-u} - \dfrac{u^{3/2} e^{-u}}{4n} + \dfrac{4u^{5/2} e^{-u}}{9n}\right)$

$$\Rightarrow \Gamma(n+1) = \sqrt{2n}n^n e^{-n}\left(\Gamma(1/2) - \frac{\Gamma(5/2)}{n} + \frac{4\Gamma(7/2)}{9n} + O\left(\frac{1}{n^2}\right)\right)$$

$$\Rightarrow \Gamma(n+1) = \sqrt{2n}\, n^n e^{-n}\left(\sqrt{\pi} - \frac{3\sqrt{\pi}}{4n} + \frac{5\sqrt{\pi}}{6n} + O\left(\frac{1}{n^2}\right)\right)$$

$$\Rightarrow \boxed{\Gamma(n+1) = (2n\pi)^{1/2} n^n e^{-n}\left(1 + \frac{1}{12n} + O\left(\frac{1}{n^2}\right)\right)}$$