On deriving the relativistic lagrangian for a free particle

Shlok Vaibhav Singh

2020/05/19

It is demonstrated in Landau-Lifshitz Analytical mechanics that using the Galilean relativity and Hamilton's principle of stationary action along with the uncertainity in lagrangian upto addition of total time derivative and multiplication by a constant, it is possible to derive the Lagrangian of a free particle in a non-relativistic setting, i.e. $\frac{1}{2}mv^2$. In this short exercise, which I will pretend as if a great endeavor, I shall try to derive the relativistic analog, i.e. $-\gamma^{-1}m_0c^2$.

1 The setting

The stage: Two frames K and K', with there x,y and z-axes aligned with K' moving at a velocity $\epsilon\hat{i}$. A point particle moves with a constant velocity (v_x, v_y) in K frame following the trajectory $(v_xt, v_yt, 0)$ in K frame and $(v'_xt^{'}, v'_yt^{'}, 0)$ where

$$\begin{aligned} t' &= \gamma_\epsilon(t - t\frac{\epsilon v_x}{c^2}) \\ v'_x &= \frac{v_x - \epsilon}{1 - v_x\epsilon/c^2} \\ v'_y &= \frac{1}{\gamma_\epsilon}\frac{v_y}{1 - v_x\epsilon/c^2} \end{aligned} \tag{1}$$

Declaring the Hamilton's principle, the motion of a free particle is such that $S = \int_{\overrightarrow{r_1},t_1}^{\overrightarrow{r_2},t_2} \mathcal{L}(v^2)dt$, where $v^2 = v_x^2 + v_y^2$ takes an extremum. Since dynamical equations must take same form in all frames, in K' frame, the Hamilton principle states $S' = \int_{\vec{r'_1},t'_1}^{\vec{r'_2},t'_2} \mathcal{L}(v^{'2})dt'$, where $v^{'2} = v_x^{'2} + v_y^{'2}$ takes an extremum where $v_x^{'}, v_y^{'}$ are defined as in (1) thus.

2 Derivation

Now let use lorentz transformation equations to transform the integral in $K^{'}$ frame, giving:

$$S' = \gamma_\epsilon(1 - \frac{\epsilon v_x}{c^2}) \int_{\vec{r_1},t_1}^{\vec{r_2},t_2} \mathcal{L}(v^{'2})dt \tag{2}$$

here the limits of integration have been changed to those of K frame by application of our transformation thus.

Now, we shall use the technique described in Landau-Lifshitz Analytical Mechanics, we have two actions S and $S^{'}$ which describe the same physics in same frame, definitely we cannot have two candidates for Lagrangian, the only relation, therefore between the two integrands is addition of a total derivative function of positional coordinates and time and multiplication by a constant,we further bring ϵ into constant and total derivative giving the most general relation thus:

$$\gamma_\epsilon(1 - \frac{\epsilon v_x}{c^2})\mathcal{L}(v^{'2}) = \alpha(\epsilon)\mathcal{L}(v^2) + \frac{df(\epsilon, x, y, z, t)}{dt} \tag{3}$$

Now, let ϵ be very small so that we can do the approximation:

$$\mathcal{L}(v^{'2}) = \mathcal{L}(v^2) + \frac{d\mathcal{L}(v^2)}{dv^2}(v^{'2} - v^2) \tag{4}$$

Computing the difference $v^{'2} - v^2$ upto first order in ϵ which is explicitly calculated in the appendix:

$$\mathcal{L}(v^{'2}) = \mathcal{L}(v^2) - 2\frac{v_x\epsilon}{\gamma_v^2}\frac{d\mathcal{L}(v^2)}{dv^2} + \mathcal{O}(\epsilon^2) \tag{5}$$

Putting this in (3) gives:

$$\gamma_\epsilon(1 - \frac{\epsilon v_x}{c^2})(\mathcal{L}(v^2) - 2\frac{v_x\epsilon}{\gamma_v^2}\frac{d\mathcal{L}(v^2)}{dv^2}) = \alpha(\epsilon)\mathcal{L}(v^2) + \frac{df(\epsilon, x, y, z, t)}{dt} \tag{6}$$

Let us now attack $\alpha(\epsilon)$, this factor must be a consequence of the relativity principle and lorentz transformation and is not related, in any way, to any property of the particle itself except for the fact that it is free particle. So, if another particle is travelling with velocity $-v$ in K, and we relate its motion in K and K'' using Hamilton's principle, then we should really expect that $\alpha(\epsilon) = \alpha(-\epsilon)$, just because two motions are mirror images of each other, we should not expect the multiplication to be different, a similar argument can be said for f but there x-coordintes will also flip sign, since it is a multi-variable function, so its better to keep quiet on it.

Using our new insight,we can collect the coefficients of ϵ on LHS:

$$-2\gamma_\epsilon \frac{\epsilon v_x}{\gamma_v^2} \frac{d\mathcal{L}(v^2)}{dv^2} - \gamma_\epsilon \frac{\epsilon v_x}{c^2} \mathcal{L}(v^2) \tag{7}$$

we expand γ_ϵ in power series as $1 + \dfrac{\epsilon^2}{2c^2} + \mathcal{O}(\epsilon^4)$ and then the coefficient becomes:

$$-2\frac{\epsilon v_x}{\gamma_v^2} \frac{d\mathcal{L}(v^2)}{dv^2} - \frac{\epsilon v_x}{c^2} \mathcal{L}(v^2) \tag{8}$$

That is:

$$-\epsilon \frac{dx}{dt} \left(\frac{1}{\gamma_v^2} \frac{d\mathcal{L}(v^2)}{dv^2} + \frac{\mathcal{L}(v^2)}{c^2} \right) \tag{9}$$

This term can be seen as the linear term wrt ϵ in power series expansion of f, this must be equal to a total drivative, but we see it is already a total derivative multiplied by a term

$$\frac{1}{\gamma_v^2} \frac{d\mathcal{L}(v^2)}{dv^2} + \frac{\mathcal{L}(v^2)}{c^2} \tag{10}$$

This must be equal to a constant, say, d, so finally we have a differential equation in $\mathcal{L}$:

$$\frac{1}{\gamma_v^2} \frac{d\mathcal{L}(v^2)}{dv^2} + \frac{\mathcal{L}(v^2)}{c^2} = d \tag{11}$$

The solution is clearly $\frac{1}{\gamma_v} + d$, discarding the constant , the Lagrangian is proportional to $\sqrt{1 - \frac{v^2}{c^2}}$ thus. We can choose a suitable form by approximating our function for $v/c << 1$ with the classical lagrangian $\dfrac{1}{2}mv^2$. $-m_0c^2\sqrt{1 - \dfrac{v^2}{c^2}}$ is a suitable choice which yields classical lagrangian for low velocities thus.