Improved bounds for the sunflower lemma

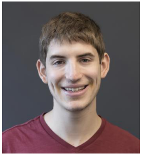
Kewen Wu
Peking University

Joint work with

Ryan Alweiss
Princeton

Shachar Lovett
UCSD

Jiapeng Zhang
Harvard



Definitions

Definition (w-set system and r-sunflower)

A w-set system is a family of sets of size at most w.
An r-sunflower is r sets $S_1, \ldots, S_r$ where

- **Kernel**: $Y = S_1 \cap \cdots \cap S_r$;
- **Petals**: $S_1 \setminus Y, \ldots, S_r \setminus Y$ are pairwise disjoint.

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Example

$\{\{1,2\}, \{1,3,4,6\}, \{1,5\}, \{2,3\}\}$ is a 4-set system of size 4.
It has a 3-sunflower $\{\{1,2\}, \{1,3,4,6\}, \{1,5\}\}$ with kernel $\{1\}$ and petals $\{2\}, \{3,4,6\}, \{5\}$.

Main result

Theorem (Erdős-Rado sunflower)

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Let's focus on $r = 3$.

- Erdős and Rado 1960: $s = w! \cdot 2^w \approx w^w$.
- Kostochka 2000: $s \approx (w \log\log\log w / \log\log w)^w$.
- Fukuyama 2018: $s \approx w^{0.75w}$.
- Now: $s \approx (\log w)^w$ and this is tight for our approach.

Actual bound and further refinement

Theorem (Improved sunflower lemma)

For some constant C, any w-set system of size s has an r-sunflower, where

$$s = \left(Cr^2 \cdot \left(\log w \log\log w + (\log r)^2\right)\right)^w .$$

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Recently, Anup Rao improved it to

$$s = \left(Cr(\log w + \log r)\right)^w .$$

Applications – Theoretical computer science

- Circuit lower bounds
- Data structure lower bounds
- Matrix multiplication
- Pseudorandomness
- Cryptography
- Property testing
- Fixed parameter complexity
- Communication complexity
- ...

Applications – Combinatorics

- Erdős-Szemerédi sunflower lemma
- Intersecting set systems
- Packing Kneser graphs
- Alon-Jaeger-Tarsi nowhere-zero conjecture
- Thersholds in random graphs
- ...

Section 3

Proof overview

Make it robust

Assume $\mathcal{F} = \{S_1, \ldots, S_m\}$ is a w-set system.
Define a width-w DNF $f_{\mathcal{F}}$ as $f_{\mathcal{F}} = \bigvee_{i=1}^{m} \bigwedge_{j \in S_i} x_j$.

Example

If $\mathcal{F} = \{\{1,2\}, \{1,3,4,6\}, \{1,5\}, \{2,3\}\}$, then
$f_{\mathcal{F}} = (x_1 \wedge x_2) \vee (x_1 \wedge x_3 \wedge x_4 \wedge x_6) \vee (x_1 \wedge x_5) \vee (x_2 \wedge x_3)$.

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Definition (Satisfying system)

$\mathcal{F}$ is satisfying if $\Pr[f_{\mathcal{F}}(x) = 0] < 1/3$ with $\Pr[x_i = 1] = 1/3$,
i.e., $\Pr[\forall i \in [m], S_i \not\subset S] < 1/3$ with $\Pr[x_i \in S] = 1/3$.

Satisfyingness implies sunflower

Assume $\mathcal{F}$ is a set system on ground set $\{x_1, \ldots, x_n\}$.

Lemma

If $\mathcal{F}$ is satisfying, then it has 3 pairwise disjoint sets.

In particular, 3 pairwise disjoint sets is a 3-sunflower.

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Color $x_1, \ldots, x_n$ to red, green, blue uniformly and independenty.

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Proof.

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By definition, $\mathcal{F}$ contains a purely red (green/blue) set w.p $> 2/3$.
By union bound, $\mathcal{F}$ contains one purely red set, one purely green set, and one purely blue set w.p > 0. $\square$

Structure vs pseudorandomness

Assume $\mathcal{F} = \{S_1, \ldots, S_m\}, m > \kappa^w$ is a w-set system. Define link $\mathcal{F}_Y = \{S_i \backslash Y \mid Y \subset S_i\}$, which is a $(w - |Y|)$-set system.

Example

If $\mathcal{F} = \{\{1,2\}, \{1,3,4\}, \{1,5\}, \{2,3\}\}$, then $\mathcal{F}_{\{2\}} = \{\{1\}, \{3\}\}$.

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If there exists Y such that $|\mathcal{F}_Y| \geq m/\kappa^{|Y|} > \kappa^{w-|Y|}$, then we can apply induction and find 3-sunflower in $\mathcal{F}_Y$.

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If there exists Y such that $|\mathcal{F}_Y| \geq m/\kappa^{|Y|} > \kappa^{w-|Y|}$, then we can apply induction and find 3-sunflower in $\mathcal{F}_Y$.

So induction starts at such $\mathcal{F}$, that $|\mathcal{F}_Y| < m/\kappa^{|Y|}$ holds for any Y.

Lemma

Let $\kappa \geq (\log w)^{O(1)}$. If $|\mathcal{F}_Y| < m/\kappa^{|Y|}$ holds for any Y, then $\mathcal{F}$ is satisfying, which means $\mathcal{F}$ has 3 pairwise disjoint sets.

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Take $\approx 1/\sqrt{\kappa}$-fraction of the ground set as W,
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  E.g., $S_j \backslash W = \{1\}, S_i \backslash W = \{1, 2, 3, 4, 5\}$.
- Bad: otherwise, we do nothing for S_i.

Example

If $\mathcal{F} = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}, \{4, 5, 6, 7\}\}$ and $w = 4, W = \{1\}$,
then $\mathcal{F}' = \{\{1,2\}, \{1,3\}, \{2, 3, 4\}, \{4, 5, 6, 7\}\}$.

One reduction step

Then $|\mathcal{F}'_Y| \leq |\mathcal{F}_Y|$ and $|\mathcal{F}'| \approx |\mathcal{F}|$. $\Leftarrow \mathcal{F}'$ is also pseudorandom

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$\mathcal{F}' = \{\{1,2\}, \{2,3,4\}, \{1,4,5,6\}, \{4,5,6,7\}\}, W = \{1\}, i = 4.$
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- $k = 2$ S_i ranks 2 in $\mathcal{F}_{\{4,5,6\}}$, we recover $i = 4$

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- $k = 2$ S_i ranks 2 in $\mathcal{F}_{\{4,5,6\}}$, we recover $i = 4$
- aux$_2$ $= \$\$\$\$$ we recover $W = W' \backslash \{4,5,6,7\}$

Reductions

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Split S to several steps,

- $\Pr[x_i \in S] = 1/3$
  $\approx$ take $1/3$-fraction of the ground set as S
  $\approx$ view S as $W_1, W_2, \ldots, W_{\log w}$, each of $\approx 1/\sqrt{\kappa}$-fraction

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Then we iteratively apply reductions,

$$\mathcal{F} \xrightarrow{W_1} \mathcal{F}' \xrightarrow{W_2} \mathcal{F}'' \xrightarrow{W_3} \cdots \xrightarrow{W_{\log w}} \mathcal{F}^{\text{last}}.$$

Final step

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- or, $\mathcal{F}^{\text{last}}$ is a width-0 (multi-)set system of size $\approx m > \kappa^w$,
  and $\left|\mathcal{F}^{\text{last}}_Y\right| \lessapprox \left|\mathcal{F}^{\text{last}}\right| / \kappa^{|Y|}$ still holds for any Y.
  $\Rightarrow$ Impossible

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$$\underbrace{\mathcal{F}}_{\text{width-}w} \xrightarrow{W_1} \underbrace{\mathcal{F}'}_{\text{width-}w/2} \xrightarrow{W_2} \underbrace{\mathcal{F}''}_{\text{width-}w/4} \xrightarrow{W_3} \cdots \xrightarrow{W_{\log w}} \underbrace{\mathcal{F}^{\text{last}}}_{\text{width-}0}.$$

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- or, $\mathcal{F}^{\text{last}}$ is a width-0 (multi-)set system of size $\approx m > \kappa^w$, and $\left|\mathcal{F}^{\text{last}}_Y\right| \lessapprox \left|\mathcal{F}^{\text{last}}\right| / \kappa^{|Y|}$ still holds for any Y. $\Rightarrow$ Impossible

Thus, (informally) we proved such $\mathcal{F}$ is satisfying, which means $\mathcal{F}$ has 3-sunflower (3 pairwise disjoint sets).

Section 4

Open problems

Erdős-Rado sunflower

Problem (Erdős-Rado sunflower conjecture)

Any w-set system of size $O_r(1)^w$ has r-sunflower.

- Our approach cannot go beyond $(\log w)^{(1-o(1))w}$.
  We need new ideas.
- Lift the sunflower size?
  $r = 3 \implies r = 4$.
- Is $(\log w)^{(1-o(1))w}$ actually tight? Counterexamples?

Erdős-Szemerédi sunflower

Assume $\mathcal{F} = \{S_1, \ldots, S_m\}$ and $S_i \subset \{1, 2. \ldots, n\}$.

Problem (Erdős-Szemerédi sunflower conjecture)

There exists function $\varepsilon = \varepsilon(r) > 0$, such that, if $m > 2^{n(1-\varepsilon)}$, then $\mathcal{F}$ has r-sunflower.

- Now:
  - general r: $\varepsilon = O_r\left(1/\log n\right)$ from ER sunflower.
  - $r = 3$: Naslund proved it using polynomial method.
- ER sunflower conjecture $\implies$ ES sunflower conjecture.

Section 5

Thanks