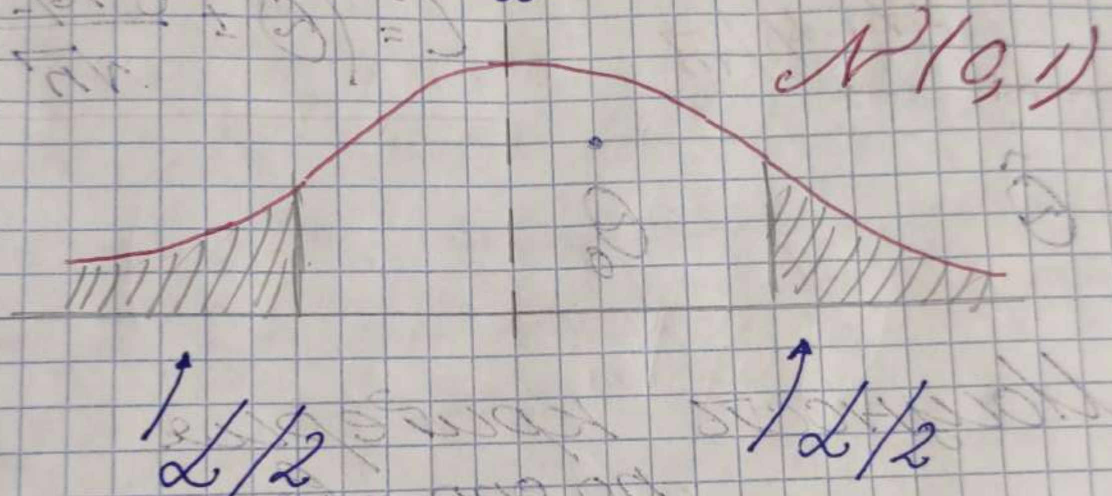
Задача 1.

– Двухсторонняя альтернатива

$$S = \{ |W| > C_\alpha \} \quad (*)$$



$\Downarrow$

Квантиль $z_{1-\frac{\alpha}{2}} = C_\alpha$

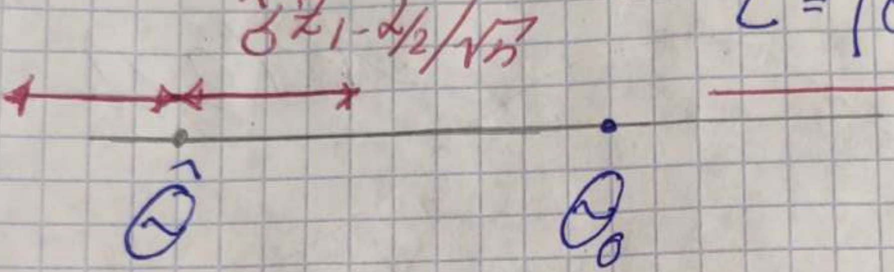
$\Downarrow$

$$(*) \Leftrightarrow \left| \sqrt{n} \, \frac{\hat{\theta} - \theta_0}{\hat{\sigma}} \right| > z_{1-\frac{\alpha}{2}}$$

$$|\hat{\Theta} - \Theta_0| > \frac{\hat{\sigma} z_{1-\alpha/2}}{\sqrt{n}}$$

$\Updownarrow$ „доверительный интервал"

$$C = \left(\hat{\Theta} \pm \frac{\hat{\sigma} z_{1-\alpha/2}}{\sqrt{n}}\right)$$



Мощность критерия

по опр. – вер-ть откл. от основной гипотезы, если она отвергнута

$$\beta_S(\Theta) = P(\cancel{\hat{\Theta}} \in S)$$

$\Theta \in \Theta_1$

$$= P_{\Theta}\left(\sqrt{n}\,\frac{|\hat{\Theta} - \Theta_0|}{\hat{\sigma}} > z_{1-\frac{\alpha}{2}}\right) =$$

$=$

$$= P_\theta\left(\sqrt{n}\,\frac{\hat{\theta}-\theta_0}{\hat{\sigma}} < -z_{1-\frac{\alpha}{2}}\right) +$$

$$P_\theta\left(\sqrt{n}\,\frac{\hat{\theta}-\theta_0}{\hat{\sigma}} > z_{1-\alpha/2}\right)$$

Найдем вероятности:

Знаем, что $\sqrt{n}\,\frac{\hat{\theta}-\theta}{\hat{\sigma}}$ нормально распределена в силу ассимптотической нормальности $\hat{\theta}$

$$\Downarrow$$

$P_\theta\left(\sqrt{n}\,\frac{\hat{\theta}-\theta_0}{\hat{\sigma}} < \ldots\right)$ находится из функции распределения

$$P_{\theta}\left(\left(\sqrt{n}\,\frac{\hat\theta-\theta_0}{\tilde\sigma}+\sqrt{n}\,\frac{\hat\theta-\theta}{\tilde\sigma}-\right.\right.$$

$$\left.\left.-\sqrt{n}\,\frac{\hat\theta-\theta}{\tilde\sigma}\right)<-z_{1-\frac{\alpha}{2}}\right)=$$

$$=P_{\theta}\left(\sqrt{n}\,\frac{\hat\theta-\theta}{\tilde\sigma}<+z_{1-\frac{\alpha}{2}}-\right.$$

$$\left.-\sqrt{n}\,\frac{\hat\theta-\theta}{\tilde\sigma}+\sqrt{n}\,\frac{\hat\theta-\theta_0}{\tilde\sigma}\right)=$$

$$=\Phi\left(-z_{1-\frac{\alpha}{2}}-\sqrt{n}\,\frac{\theta-\theta_0}{\tilde\sigma}\right)$$

Аналогично вторая вероятность: $1-\Phi\left(z_{1-\frac{\alpha}{2}}-\right.$

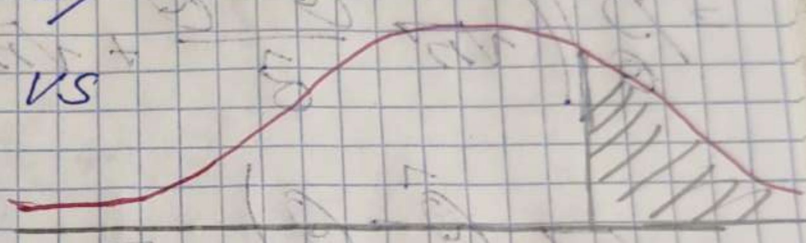
$$\left.-\sqrt{n}\,\frac{\theta-\theta_0}{\tilde\sigma}\right)$$

$$\Rightarrow \beta_S = \boxed{1 - \Phi\left(z_{1-\frac{\alpha}{2}} - \sqrt{n}\,\frac{\theta - \theta_0}{\tilde{\sigma}}\right) + \Phi\left(-z_{1-\frac{\alpha}{2}} - \sqrt{n}\,\frac{\theta - \theta_0}{\tilde{\sigma}}\right)}$$

– Правосторонняя

$H_0: \theta = \theta_0$ vs

$H_1: \theta > \theta_0$



$z_{1-\alpha}$

Множество - критерий:

$$S = \{W > z_{1-\alpha}\}$$

$$\hat{\theta} - \theta_0 > z_{1-\alpha}\,\frac{\tilde{\sigma}}{\sqrt{n}}$$

$$\sqrt{n}\,\frac{\hat{\theta} - \theta_0}{\tilde{\sigma}} > z_{1-\alpha}$$

$\Downarrow$ доверительный интервал

$$\theta_0 < \left(\hat{\theta} - z_{1-\alpha}\,\tilde{\sigma}/\sqrt{n}\right)$$

$$\beta_S = P_\theta\left(W > z_{1-\alpha}\right) =$$

$$= P_\theta\left(\sqrt{n}\,\frac{\hat\theta-\theta_0}{\hat\sigma} > z_{1-\alpha}\right) =$$

$$= P_\theta\left(\left(\sqrt{n}\,\frac{\hat\theta-\theta}{\hat\sigma} + \sqrt{n}\,\frac{\hat\theta-\theta_0}{\hat\sigma} - \right.\right.$$

$$\left.\left. - \sqrt{n}\,\frac{\hat\theta-\theta}{\hat\sigma}\right) > z_{1-\alpha}\right) =$$

$$= P_\theta\left(\sqrt{n}\,\frac{\hat\theta-\theta}{\hat\sigma} > z_{1-\alpha} - \sqrt{n}\cdot\right.$$

$$\left.\cdot\frac{\hat\theta-\theta_0}{\hat\sigma} + \sqrt{n}\,\frac{\hat\theta-\theta}{\hat\sigma}\right) =$$

$$= 1 - \Phi\left(z_{1-\alpha} - w(\theta)\right)$$

Левосторонняя альтернатива

$$S = \{ W < C_\alpha \}$$

$$\sqrt{n}\,\frac{\hat{\theta} - \theta_0}{\tilde{\sigma}} < z_\alpha$$

$$\theta_0 > -\frac{\hat{\sigma} z}{\sqrt{n}} + \hat{\theta}$$

Аналогично $\beta_S = \Phi(z_\alpha - w(\theta)$