In Linear Algebra we show up with a resume of basic skills

- Arithmetic

$\underbrace{+ \quad -}_{\text{addition}}$ $\quad$ $\underbrace{\times \quad \div}_{\text{multiplication}}$

$a - b = a + (-b)$ $\quad$ $\frac{a}{b} = a \times b^{-1}$

- Basic algebra

$$ax + by = c$$

example : solve for y

$$by = c - ax$$

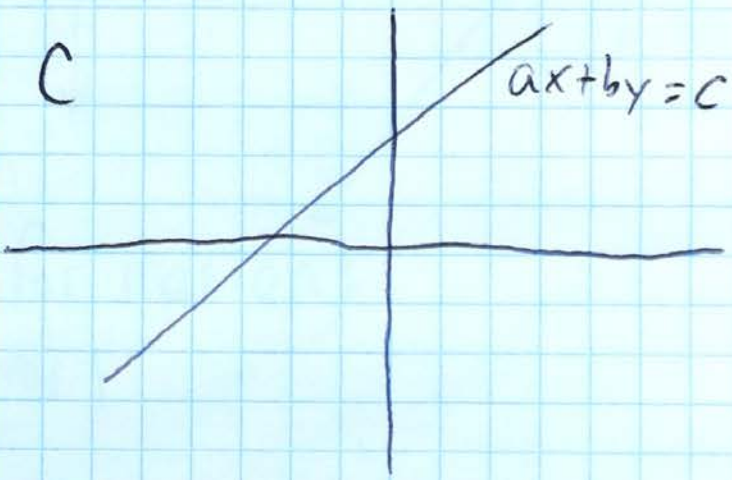
$$y = \frac{1}{b}(c - ax)$$

We emphasize the <u>linear</u> in linear algebra

x^2, x^3, $\sqrt{x}$, $\ln x$, e^x, $\cos x$

Not linear

$a \cdot x + b \cdot y = c$



The challenge in linear algebra is the <u>concepts</u> involved and their <u>interrelationships</u>.

Systems of Linear Equations

Matrices

Geometry
(Linear)

Transformations
(Linear)

- We have different ways of looking at the same thing.
- Ideas in one domain may be translated into another domain and vice versa.

(I) Systems of Linear Equations

$E_1: \quad ax + by = e$

$E_2: \quad cx + dy = f$

2×2 system

2 equation E_1

E_2

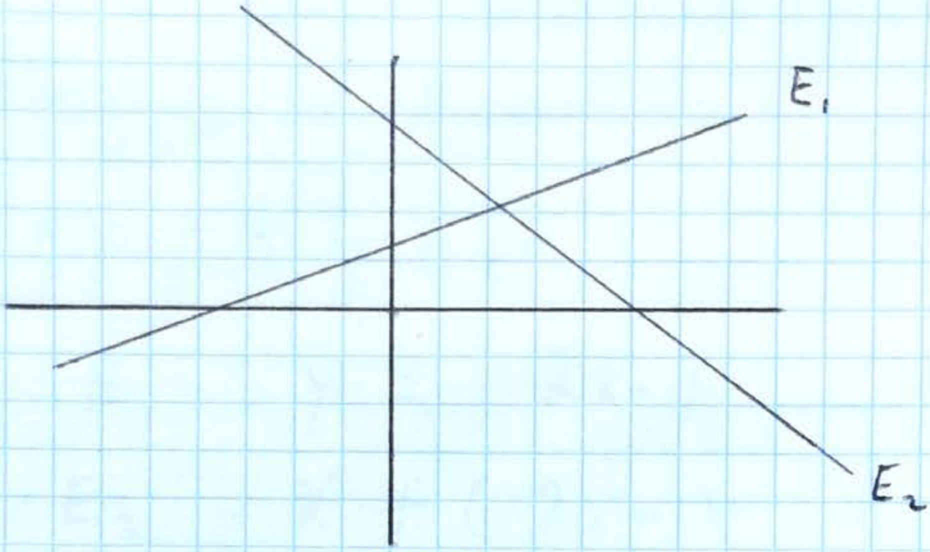
2 unknowns / variables

x, y

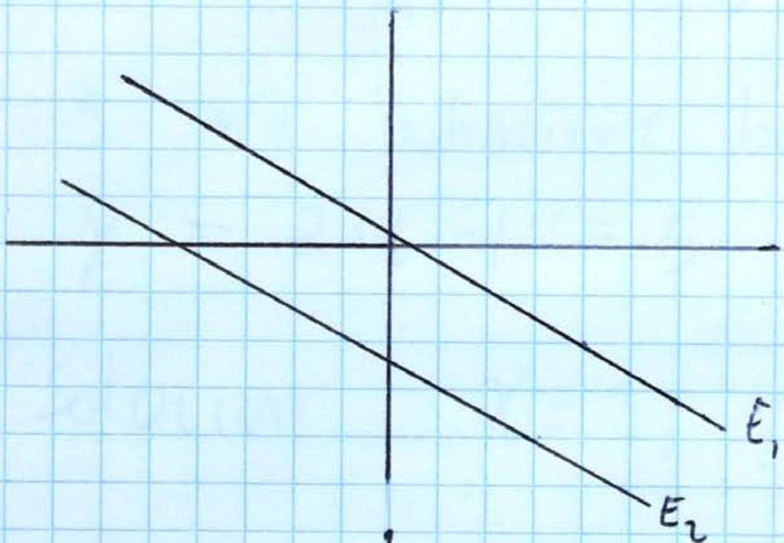
3 Types of Solutions

qualitatively Speaking

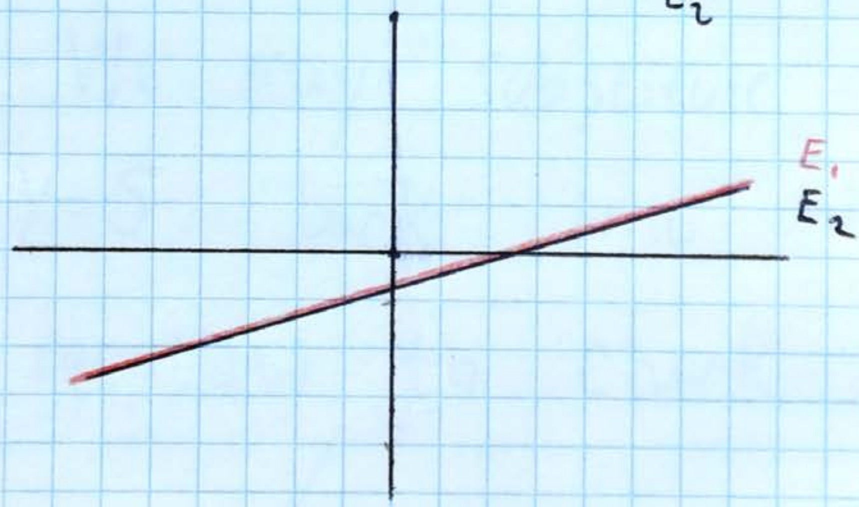
Uniques Solution



No solution



Many solutions



Examples

① $\begin{cases} 2x - y = 1 & E_1 \\ x + y = 8 & E_2 \end{cases}$

From E_1 $y = 2x - 1$

From E_2 $x + (2x-1) = 8$

$3x - 1 = 8 \Rightarrow \underline{\underline{x = 3}}$

Since $x = 3$ we substitute back

$y = 2 \times 3 - 1 = 5$

Unique Solution $x = 3$, $y = 5$

Note : We may substitute $x = 3$, $y = 5$ back into the original equations to check our work.

✓

(2) $\begin{cases} x - y = 1 \\ 3x - 3y = 2 \end{cases}$ $\begin{matrix} E_1 \\ E_2 \end{matrix}$

From E_1 $\quad y = x - 1$

From E_2 $\quad 3x - 3(x-1) = 2$

$$\underbrace{3x - 3x} + 3 = 2$$

$$3 = 2 \quad ?$$

This is a contradiction

$\therefore$ No solution

Note in E_2

$$3x - 3y = 2$$

multiply both sides by $1/3$

E_2 becomes $\quad x - y = 2/3$

Now the contradiction is more obvious

3 $\begin{cases} x - y = 1 & E_1 \\ 3x - 3y = 3 & E_2 \end{cases}$

From E_1 $\quad y = x - 1$

From E_2 $\quad 3x - 3(x-1) = 3$

$3x - 3x + 3 = 3$

$3 = 3$ ✓

We notice that for any choice of x, if $y = x - 1$, then the pair x, y is a solution.

Examples $\quad x = 1 \quad y = 0$

$x = 2 \quad y = 1$

$x = -10 \quad y = -11$

are all solutions

Higher dimensions

$ax + by = c$ is a line

$ax + by + cz = d$ is a plane

Solve a $\underset{\text{equations}\quad\text{variables}}{3 \times 3}$ system

Example (First Attempt)

$$x + y + z = 7 \quad : E_1$$

$$2x - y + 3z = 12 \quad : E_2$$

$$-x + 3y + z = 9 \quad : E_3$$

E_1 $\quad x = -y - z + 7$

E_2 $\quad 2(\overbrace{-y - z + 7}^{x}) - y + 3z = 12$

$\quad\quad -2y - 2z + 14 - y + 3z = 12$

$\quad\quad -3y + z = -2$

$\quad\quad z = 3y - 2$

$$E_3 \quad -\underbrace{(-y - z + 7)}_{x} + 3y + \underbrace{(3y - 2)}_{z} = 9$$

$$\hookrightarrow \quad y + z - 7 + 3y + 3y - 2 = 9$$

$$\hookrightarrow 7y + z = 18$$

$$\hookrightarrow 7y + (3y - 2) = 18$$

$$\hookrightarrow 10y - 2 = 18$$

$$\hookrightarrow 10y = 20 \implies y = 2$$

$$z = 3y - 2 = 4$$

$$x = -2 - 4 + 7 = 1$$

Solution $x = 1$, $y = 2$, $z = 4$

Not very efficient !

Example 1 (Second Attempt)

$$x + y + z = 7$$

$$2x - y + 3z = 12 \qquad E_2 \to -2E_1 + E_2$$

$$-x + 3y + x = 9 \qquad E_3 \to E_1 + E_3$$

$$x + y + z = 7$$

$$-3y + z = -2 \qquad -\tfrac{1}{3} E_2$$

$$4y + 2z = 16$$

$$x + y + z = 7$$

$$y - \tfrac{1}{3} z = \tfrac{2}{3}$$

$$4y + 2z = 16 \qquad E_3 \to -4E_2 + E_3$$

$$X + Y + Z = 7$$

$$Y - \tfrac{1}{3}Z = \tfrac{2}{3}$$

$$\frac{10}{3}Z = \frac{40}{3} \qquad \cdot \frac{3}{10} E_3$$

$$\left.\begin{aligned} X + Y + Z &= 7 \\ Y - \tfrac{1}{3}Z &= \tfrac{2}{3} \\ Z &= 4 \end{aligned}\right\} \text{Two Paths}$$

Path I : Back Substitution

E_2 : $Y = \tfrac{2}{3} + \tfrac{1}{3}Z = \tfrac{2}{3} + \tfrac{4}{3} = 2$

E_1 : $x = 7 - y - Z = 7 - 2 - 4 = 1$

$\therefore X = 1, \quad Y = 2, \quad Z = 4$

Path II -

$$x + y + z = 7 \qquad -E_3 + E_1 \to E_1$$

$$y - \tfrac{1}{3} z = 2/3 \qquad \tfrac{1}{3} E_3 + E_2 \to E_2$$

$$z = 4$$

. . . .

$$x + y = 3 \qquad -E_2 + E_1 \longrightarrow E_1$$

$$y = 2$$

$$z = 4$$

. . . .

$$x = 1$$

$$y = 1$$

$$z = 1$$

Path I . Gauss Elimination with Back Substitution

Path II Gauss- Jordan Elimination

Idea :

* Transform a complex system of equations into a simple system of equations

* Each step of the transformation preserves the solutions of the two systems — the original system and the transformed system have exactly the same solutions

$$\begin{aligned} x \quad y + z &= 7 \\ 2x - y + 3z &= 12 \\ -x + 3y + z &= 9 \end{aligned} \longrightarrow \begin{aligned} x \quad\quad\quad &= 1 \\ y \quad\quad &= 2 \\ z &= 4 \end{aligned}$$

These 3 operations (transformations) on the equations of a system do not change the solutions

I Switch the order of two equations

II Multiply an equation by a non-zero constant

III Add a multiple of one equation to another

In Gauss Elimination we apply these steps one at a time with the intention of simplifying the system of equations

Example 1 Third attempt

$$\begin{aligned} x + y + z &= 7 \\ 2x - y + 3z &= 12 \\ -x + 3y + z &= 9 \end{aligned} \Longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 2 & -1 & 3 & 12 \\ -1 & 3 & 1 & 9 \end{array}\right)$$

* We don't need the variables
* We use matrix notation
* The rows of the matrix correspond to the equations
* We perform the 3 row operations

I) Switch 2 rows

II) Multiply a row by a non-zero constant

III) add a multiple of one row to another

$$\left(\begin{array}{ccc:c} 1 & 1 & 1 & 7 \\ 2 & -1 & 3 & 12 \\ -1 & 3 & 1 & 9 \end{array}\right) \begin{array}{l} -2R_1+R_2 \\ R_1+R_3 \end{array}$$

$$\begin{aligned} x + y + z &= 7 \\ 2x - y + 3z &= 12 \\ -x + 3y + z &= 9 \end{aligned}$$

$$\left(\begin{array}{ccc:c} 1 & 1 & 1 & 7 \\ 0 & -3 & 1 & -2 \\ 0 & 4 & 2 & 16 \end{array}\right) -\tfrac{1}{3}R_2$$

$$\begin{aligned} x + y + z &= 7 \\ -3y + z &= -2 \\ 4y + 2z &= 16 \end{aligned}$$

$$\left(\begin{array}{ccc:c} 1 & 1 & 1 & 7 \\ 0 & 1 & -1/3 & -2/3 \\ 0 & 4 & 2 & 16 \end{array}\right) -4R_2+R_3$$

$$\begin{aligned} x + y + z &= 7 \\ y - \tfrac{1}{3}z &= -\tfrac{2}{3} \\ 4y + 2z &= 16 \end{aligned}$$

$$\left(\begin{array}{ccc:c} 1 & 1 & 1 & 7 \\ 0 & 1 & -1/3 & -2/3 \\ 0 & 0 & \frac{10}{3} & \frac{40}{3} \end{array}\right) \tfrac{3}{10}R_3$$

$$\begin{aligned} x + y + z &= 7 \\ y - \tfrac{1}{3}z &= -\tfrac{2}{3} \\ \tfrac{10}{3}z &= \tfrac{40}{3} \end{aligned}$$

$$\left(\begin{array}{ccc:c} 1 & 1 & 1 & 7 \\ 0 & 1 & -1/3 & -2/3 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

$$\begin{aligned} x + y + z &= 7 \\ y - \tfrac{1}{3}z &= -\tfrac{2}{3} \\ z &= 4 \end{aligned}$$

$$\left(\begin{array}{ccc:c} 1 & 1 & 1 & 7 \\ 0 & 1 & -1/3 & -2/3 \\ 0 & 0 & 1 & 4 \end{array}\right) \quad \begin{array}{l} -R_3 + R_1 \\ \frac{1}{3}R_3 + R_2 \end{array}$$

$$\begin{aligned} x + y + z &= 7 \\ y - \tfrac{1}{3}z &= -\tfrac{2}{3} \\ z &= 4 \end{aligned}$$

$$\left(\begin{array}{ccc:c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right) \quad -R_2 + R_1$$

$$\begin{aligned} x + y &= 3 \\ y &= 2 \\ z &= 4 \end{aligned}$$

$$\left(\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 4 \end{aligned}$$

The (unique) solution is

$x = 1$

$y = 2$

$z = 4$

- We will solve $m \times n$ systems of linear equations
- We will learn the techniques of Gauss Elimination with Back Substitution / Gauss-Jordan Elimination
- We solve our equations in matrix form
- We will learn the various concepts that relate these topics

$m \times n$ System of linear equations

m - number of equations.

n - number of variables.

- We use subscripts to denote our variables

$$x_1, \; x_2, \; x_3, \; \ldots, \; x_n$$

- We use double subscripts to denote our coefficients

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{matrix}$$

$$a_{ij}$$

row index

column index

Example : Generic 2×2 system

$$a_{11} x_1 + a_{12} x_2 = b_1$$
$$a_{21} x_1 + a_{22} x_2 = b_2$$

Generic 3×3 system

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$$
$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$$
$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3$$

Generic $m \times n$ system

$$\begin{array}{ccccccccc} a_{11} x_1 & + & a_{12} x_2 & + & \cdots & + & a_{1n} x_n & = & b_1 \\ a_{21} x_1 & + & a_{22} x_2 & + & \cdots & + & a_{2n} x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1} x_1 & + & a_{m2} x_2 & + & \cdots & + & a_{mn} x_n & = & b_n \end{array}$$

Example 2

$$4x_1 + 4x_2 + x_3 = 9$$

$$2x_1 + 2x_2 + 2x_3 = 6$$

$$3x_1 + 2x_2 - x_3 = 4$$

Write in Matrix Form

$$\left(\begin{array}{ccc|c} 4 & 4 & 1 & 9 \\ 2 & 2 & 2 & 6 \\ 3 & 2 & -1 & 4 \end{array}\right) \quad R_1 \longleftrightarrow R_2$$

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 6 \\ 4 & 4 & 1 & 9 \\ 3 & 2 & -1 & 4 \end{array}\right) \quad \tfrac{1}{2} R_1$$

We want a coefficient of 1 in the <u>leading</u> <u>position</u> - <u>the pivot</u>

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 4 & 4 & 1 & 9 \\ 3 & 2 & -1 & 4 \end{array}\right) \quad \left.\begin{array}{l} -4R_1 + R_2 \\ -3R_1 + R_3 \end{array}\right\}$$

Use the leading 1 to clear the first column

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & -1 & -4 & -5 \end{array}\right) \quad R_2 \longleftrightarrow R_3$$

We need a leading 1 in the proper place

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & -4 & -5 \\ 0 & 0 & -3 & -3 \end{array}\right) \quad \left.\begin{array}{l} -R_2 \\ -1/3 R_3 \end{array}\right\}$$

Two steps at once. Leading 1's in the right places.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

This matrix is in <u>Row</u> <u>echelon</u> <u>form</u>

We may use back substitution

I) All zero-rows are on the bottom

II) Each <u>leading</u> <u>entry</u> of a row is in a column to the right of leading entries of any row above it

III) Each leading entry of a row is equal to 1.

We put the matrix in <u>reduced</u> <u>row</u> <u>echelon</u> <u>form</u> (rref)

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad \left.\begin{array}{l} -R_3 + R_1 \\ \\ -4R_3 + R_2 \end{array}\right\} \text{ Clear Column 3}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad \left.\begin{array}{l} -R_2 + R_3 \\ \\ \end{array}\right\} \text{ Clear Column 2}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad \begin{array}{l} x_1 = 1 \\ x_2 = 1 \\ x_3 = 1 \end{array}$$

This last matrix is in rref

rref = reduced echelon form +

IV) All entries in a column above and below a leading entry are zero

Solve an $m \times n$ system of linear equations

- Write the system in matrix form.
- Use the 3 row operations on this matrix
- Put the matrix in REF and use back-substitution to solve the system (Gauss Elimination with Back Substitution)

or

- Put the matrix into RREF (Gauss-Jordan Elimination)

Some Formalities

A <u>matrix</u> is a rectangular array of numbers arranged in rows and columns:

$m \times n$ matrix

number of rows — number of columns

$$A = \begin{pmatrix} 2 & 0 & 1 & 3 \\ 1 & 0 & 4 & 5 \end{pmatrix}$$

2×4 matrix

A_{ij} - number in row i, column j

$$A_{23} = 4$$

$$A_{14} = 3$$

$$A_{12} = 0$$

Put a system into matrix form

$$\left.\begin{aligned} x_1 + 3x_2 &= 9 \\ 4x_3 - x_2 &= -2 \\ x_1 + 9x_3 &= 0 \end{aligned}\right\} \quad 3 \times 3 \text{ system}$$

Be careful. Line up the variables

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & & & = & 9 \\ & & -x_2 & + & 4x_3 & = & -2 \\ x_1 & & & + & 9x_3 & = & 0 \end{array}$$

Left-hand side = Right-hand side

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 9 \\ 0 & -1 & 4 & -2 \\ 1 & 0 & 9 & 0 \end{array}\right)$$

Coefficient matrix

Augmented matrix

We write a system as an <u>Augmented Matrix</u>

Use the row operations

- Two $m \times n$ systems of equations are called equivalent if the have exactely the same solutions.

Observation : The row operations (applied to an augmented matrix of a system) preserve the solutions.

$$\left(\begin{array}{ccc:c} 1 & 3 & 0 & 9 \\ 0 & -1 & 4 & -2 \\ 1 & 0 & 9 & 0 \end{array}\right) -R_1+R_3 \qquad \begin{aligned} x_1 + 3x_2 \quad\quad &= 9 \\ -x_2 + 4x_3 &= -2 \\ x_1 \quad\quad + 9x_3 &= 0 \end{aligned}$$

$\updownarrow$ Same $\downarrow$ Solutions

$$\left(\begin{array}{ccc:c} 1 & 3 & 0 & 9 \\ 0 & -1 & 4 & -2 \\ 0 & -3 & 9 & -9 \end{array}\right) \qquad \begin{aligned} x_1 + 3x_2 \quad\quad &= 9 \\ -x_2 + 4x_3 &= -2 \\ -3x_3 + 9x_3 &= -9 \end{aligned}$$

$$\left(\begin{array}{ccc:c} 1 & 3 & 0 & 9 \\ 0 & -1 & 4 & -2 \\ 0 & -3 & 9 & -9 \end{array}\right) \begin{array}{l} \\ -R_2 \\ -\tfrac{1}{3} R_3 \end{array} \qquad \begin{aligned} x_1 + 3x_2 \quad\quad &= 9 \\ -x_2 + 4x_3 &= -2 \\ -3x_2 + 9x_3 &= -9 \end{aligned}$$

We apply ↕ any row operations we choose

Same ↕ Solutions

$$\left(\begin{array}{ccc:c} 1 & 3 & 0 & 9 \\ 0 & 1 & -4 & 2 \\ 0 & 1 & -3 & 3 \end{array}\right) \begin{array}{l} \\ \\ -R_2 + R_3 \end{array} \qquad \begin{aligned} x_1 + 3x_2 \quad\quad &= 9 \\ x_2 - 4x_3 &= 2 \\ x_3 + 3x_3 &= 3 \end{aligned}$$

In ↕ REF

Same ↕ Solutions

$$\left(\begin{array}{ccc:c} 1 & 3 & 0 & 9 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right) \qquad \begin{aligned} x_1 + 3x_2 \quad\quad &= 9 \\ x_2 - 4x_3 &= 2 \\ x_3 &= 1 \end{aligned}$$

We may use back substitution

Put the Augmented Matrix into RREF

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 9 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad 4R_3 + R_2$$

$$\begin{aligned} x_1 + 3x_2 \quad &= 9 \\ x_2 - 4x_3 &= 2 \\ x_3 &= 1 \end{aligned}$$

Clear $\updownarrow$ column 3

Same $\updownarrow$ solutions

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 9 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad -3R_2 + R_1$$

$$\begin{aligned} x_1 + 3x_2 &= 9 \\ x_2 &= 6 \\ x_3 &= 1 \end{aligned}$$

Clear $\updownarrow$ column 2

Same $\updownarrow$ Solutions

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\begin{aligned} x_1 &= -9 \\ x_2 &= 6 \\ x_3 &= 1 \end{aligned}$$

Solve an $m \times n$ system of Equations

Qualitatively, 3 types of Solutions

Consistent $\begin{cases} \text{Many} \\ \text{Unique} \end{cases}$

Unique, None $\Big\}$ at most one solution

- Ideally we desire a unique solution
- Many solutions = infinitely many solutions
- We learn how to distinguish these three types of solutions.

$m \times n$ system

What role do the dimensions play

Example: Many Solutions

$$\left.\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + 4x_3 &= 10 \end{aligned}\right\} \quad 2 \times 3 \text{ system}$$

Put into an augmented matrix

$$\left(\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 3 & 2 & 4 & 10 \end{array}\right) \quad -3R_1 + R_2$$

Do the row operations

$$\left(\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 0 & -1 & 1 & -2 \end{array}\right) \quad -R_2$$

Put into REF

$$\left(\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & 2 \end{array}\right) \quad -R_2 + R_1$$

Put into RREF

$$\left(\begin{array}{ccc:c} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

Write as a system of Equations

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & 2 \end{array}\right) \longrightarrow \begin{array}{l} x_1 \quad\quad + 2x_3 = 2 \\ \quad\quad x_2 - x_3 = 2 \end{array}$$

Pivot Position

Leading variables
or Basic Variables

* Solve for the leading variables x_1, x_2
* in terms of the free variable x_3

$$x_1 = 2 - 2x_3$$

$$x_2 = 2 + x_3$$

For each choice of the free variable x_3 we have a solution to the system

$x_3 = 0 \Rightarrow x_1 = 2 \quad x_2 = 2$

$x_3 = 1 \Rightarrow x_1 = 0 \quad x_2 = 3$

$x_3 = -1 \Rightarrow x_1 = 4 \quad x_2 = 1$

Parameter form of the solutions

$$x_1 = 2 - 2x_3$$
$$x_2 = 2 + x_3$$

We let x_3 be any number : $x_3 = t$

All solutions are of the form

$$x_1 = 2 - 2t$$
$$x_2 = 2 + t$$
$$x_3 = t$$

The variable t is called a parameter

Example (3x4 system)

$$x_1 + 2x_2 - x_3 + x_4 = 3$$
$$x_1 + x_2 - x_3 + x_4 = 1$$
$$x_1 + 3x_2 - x_3 + x_4 = 5$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 3 & -1 & 1 & 5 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 3 & -1 & 1 & 5 \end{array}\right) \begin{array}{l} \\ -R_1+R_2 \\ -R_1+R_3 \end{array} \longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 2 \end{array}\right) \begin{array}{l} \\ -R_2 \\ R_2+R_3 \end{array} \longrightarrow$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} -2R_2+R_1 \\ \\ \\ \end{array} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- Note: Last row says $0x_1 + 0x_2 + 0x_3 + 0x_4 = 0$

$$x_1 \quad - x_3 + x_4 = -1$$
$$x_2 \quad = 2$$

Leading variables x_1, x_2

Free variables $x_3 \; x_4$

$$x_1 = -1 + x_3 - x_4$$
$$x_2 = 2$$

Parameter form $\quad x_3 = t \quad x_4 = s$

$$x_1 = -1 + t - s$$
$$x_2 = 2$$
$$x_3 = t$$
$$x_4 = s$$

Many Solutions

- We may get a row of all zeros
  no problem
- We have leading / basic variables
  These correspond to the pivots
- We have free variables
  These are the remaining variables
- We solve for the leading variables
  in terms of the free variables
- The free variables are assigned
  parameters
- One solution for each choice
  of parameters.
- Therefore, many solutions

No Solutions

$$\left.\begin{aligned} 3x_1 + 2x_2 &= 5 \\ x_1 + x_2 &= 2 \\ 2x_1 - x_2 &= 2 \end{aligned}\right\} \quad 3\times 2 \text{ system}$$

$$\left(\begin{array}{cc|c} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 2 & -1 & 2 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 2 & 5 \\ 2 & -1 & 2 \end{array}\right) \xrightarrow[-2R_1 + R_3]{-3R_1 + R_2}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & -3 & -2 \end{array}\right) \xrightarrow{-R_2} \left(\begin{array}{ccc} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -3 & -2 \end{array}\right) \quad 3R_2 + R_3 \quad \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right)$$

The 3rd row says $0x_1 + 0x_2 = 1$, which is impossible.

This system has no solutions

Example 4x3 system

$$\begin{array}{c} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ -1 & 3 & 4 & 10 \\ 2 & 2 & -5 & -6 \\ 6 & 0 & 3 & 3 \end{array}\right) \end{array} \begin{array}{l} \\ R_1 + R_2 \\ -2R_1 + R_3 \\ -6R_1 + R_4 \end{array} \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 2 & 6 & 14 \\ 0 & 4 & -9 & -14 \\ 0 & 6 & -9 & -21 \end{array}\right) \begin{array}{l} \\ \\ -2R_2 + R_3 \\ -3R_2 + R_4 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 2 & 6 & 14 \\ 0 & 0 & -21 & -42 \\ 0 & 0 & -27 & -63 \end{array}\right) \begin{array}{l} \\ \\ -\frac{1}{21} R_3 \\ -\frac{1}{9} R_4 \end{array} \qquad \left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 2 & 6 & 14 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 7 \end{array}\right) \begin{array}{l} \\ \\ \\ -3R_3 + R_4 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 2 & 6 & 14 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Not in REF or RREF

However, the last row implies no solutions

Solve an $m \times n$ system

- Write the system as an augmented matrix.
- Perform the elementary row operations on this matrix
- Do this until the transformed matrix is in RREF
- From the RREF we recognize the type of solution
  - many
  - unique
  - none

Q: Does this always work ?

A: Yes

Some Theory

* Two matrices of the same dimension are called equivalent if we may get from one to the other by a series of elementary row operations

Theorem 1 The two linear systems of equations corresponding to two equivalent augmented matrices have exactly the same solutions

Theorem 2 Every matrix is equivalent to a unique matrix in RREF.