# 1 Introduction

In nearly all areas of high-dimensional statistics distinguishing between the important and unimportant predictors is of intrest. In genetics many candidate snips and their association with some outcome is of interest. Many studies take measurements on a large number of biomarkers to predict medical outcomes such as the presence or progression of a disease. While there has been much focus on finding important predictors, there has been less interest in knowing if any predictors are actually important. The vast number of potential predictors creates new statistical challenges to answering this less obvious question.

In the univariate case, once a measure of association is selected, standard approaches exist to construct asymptotically valid (and sometimes optimal) tests [Neyman Jerzy et al.]. In this setting, a test is considered optimal if it achieves the largest power for every alternative compared to every other test with the same type one error. However, scientists are often interested if any of a multitude of covariate are associated with the outcome of interest. Even in the case with two covariates, a test with the above optimal property no longer exists and the alternative plays a larger role in which tests are more or less powerful.

To illustrate this, consider two tests with equal type one error rates. Both test the null that both of two association measures $\psi_1$, $\psi_2$ are zero. The first test focuses only on the $\psi_1$ and the second test considers both $\psi_1$ and $\psi_2$. When the $\psi_1$ is non-zero $\psi_2$ is zero the first test can outperform all other tests. Conversely when only $\psi_2$ is non-zero the first test will be outperformed by the second test. Additionally in this setting, the power of these tests will be tied to the covariance between the covariates (a problem that did not exist for a single covariate). These two factors make it difficult to define a test that performs well across all possible alternatives.

Work with Simultaneous hypothesis testing began with Tukey in 1953 [Miller]. Previous work by Bonferroni was used by [Dunn, a,b] to come up with some of the first multiple hypothesis testing procedures. Further improvements were proposed by [Hochberg, Holm, S. Holland and DiPonzio Copenhaver]. Bonferroni-based correction procedures have the advantage of being easy to apply to already existing tests, while guaranteeing family-wise error control. However, these tests suffer from low power, especially in cases where the probability of rejecting each hypothesis is highly correlated.

Newer procedures [Donoho and Jin] and add other papers here attain improved power compared to Bonferroni-based methods, but often rely on asymptotics to obtain these results, and don't account for the irregularity of the estimator on which their test is based. Subsequent work [McKeague and Qian, Pan et al., Xu et al.] addressed these concerns by accounting for the adaptive nature of the considered tests. Still, these newer tests are restricted to testing a particular hypothesis, must make assumptions about the data-generating mechanism to obtain theoretical guarantees.

In this article a test is proposed that works across a wide variety of data generating mechanisms and parameters of interest, but also achieves comparable power to tailor made procedures. Section 2 describes the data generating mechanisms that are considered to evaluate the performance of the test and the competing test made for the given data generating mechanism. Section 3 proposes the testing procedure. Add other sections here

# 2 Working Examples

Let $X_1, \ldots, X_n$ be independent identically distributed draws from some distribution $P$, and let $\boldsymbol{X} = \{X_1, \ldots, X_n\}$. Let $X_i = (Y_i, W_{1i}, \ldots, W_{di}), i \in \{1, \ldots n\}$ where $Y$ is the outcome of interest, and each $W$ is a covariate. Let $\psi_1 = \Psi_1(P), \ldots, \psi_a = \Psi_a(P)$ be measures of association between $Y$ and some combination of the $W_j$'s. While the results found in this article are valid for any integer $a$, for the remainder of this article assume $a = d$, the number of covariates. Also, let $\psi_j$ correspond to a measure of association between $Y$ and $W_j$. The null hypothesis for our test will be the strong null:

$$H_0 : \psi_1 = \psi_2 = \cdots = \psi_a = 0 \ \text{ versus } \ H_1 : \psi_j \neq 0 \text{ for some } j \in \{1, \ldots, a\}.$$

Last, let $\mathcal{M}$ denote the set of all possible distributions. Define $\mathcal{M}_0 = \{P \in \mathcal{M} : H_0 \text{ holds}\}$

## 2.1 Correlation Parameter

We will compare our method to both a simple bonferroni correction method, and the method described in [Zhang and Laber]. The settings considered will be the same as the firs setting in [McKeague and Qian]. The parameter of interest, $\psi_j(P)$ will be the correlation between the outcome of interest and the $j$'th covariate.

The vector of covariates in this setting will be generated from a normal distribution with mean zero and a variance covariance of $\Sigma$ with $\Sigma_{ij}$ equal to $\rho$ when $i \neq j$ and equal to 1 when $i = j$, and . Three different models are considered. Three different models for the outcome of interest ($Y$) will be considered. Letting $\varepsilon \sim N(0,1)$ and be independent of $X$, the first model let $Y = \varepsilon$, the second has $Y = X_1/4$, and third has $Y = \sum_{k=1}^{1} 0\beta_k X_k + \varepsilon$ where $\beta_k = 0.15$ for $k = \{1, \ldots, 5\}$, and $\beta_k = -0.1$ for $k = \{6 \ldots 10\}$. in which Sample sizes of 100 and 200, dimensions of 10, 50, 100, 150, and 200, and $\rho$ of $0, 0.5$, or $0.8$ will be considered,

## 2.2 Missing Data Example

In the second example, $Y$ is binary, $\Delta$ is a missingness indicator, and each $W_j$ is a covariate of interest. When $\Delta = 0$ we don't observe $Y$. The identifying assumption is $\Delta \perp\!\!\!\perp Y | W$. The parameter of interest is the risk ratio,

$$\Psi_j\left(P^{\text{full}}\right) = \frac{\text{Cov}\left(\log\left(Pr\left(Y = 1|W_j\right)\right), W_j\right)}{\text{Var}(W_j)}.$$

Using the identifying assumption, the observed data parameter is:

$$\tilde{\Psi}_j\left(P^{\text{obs}}\right) = \frac{\text{Cov}\left(\log\left(E\left[Pr\left(\Delta Y = 1|\Delta = 1, W = W\right)|W_j\right]\right), W_j\right)}{\text{Var}(W_j)}$$

Right now, it seems that the time for a single test is quite a bit larger for this test than it is for the correlation test (which is reasonable), though it may speed up if Superlearner can be improved. I am thinking of using similar simulation settings as was used before. There will be a single mechanism for the missingness, but it will be variable if the mechanism is known. There will be three settings for between $W$ correlation. All $W$ will be equally correlated, with $\rho = 0, 0.3$, or $0.7$. Dimension will (depending on computation time) be $10, 50, 100$ and possibly $200$.

There will be three settings for the $\beta$'s in the data generating model:

$$\log(Pr(Y = 1|W)) = \beta_0 + W^\top \beta$$

In one setting $\beta = 0$. In the second only one or two entries of $\beta$ will be non-zero. In the last setting, many of the entries of $\beta$ will be non-zero (possibly 70 or 80 percent of the entries).

## 2.3 Marginal Structural Model

For this marginal structural model, we are interested in if the average treatment is modified by any covariates. The marginal structural model for each $W_j$ is defined by

$$\text{logit}\left(Pr(Y^{(a)} = 1|w)\right) = \beta_0 + \beta_1 a + \beta_2 w_j + \beta_3 w_j a$$

$$(\beta_0^*, \beta_1^*, \beta_2^*, \beta_3^*) = \text{argmin}_{\beta_0, \beta_1, \beta_2, \beta_3} \int \left(\text{logit}\left(Pr(Y^{(a)} = 1|w) - (\beta_0 + \beta_1 a + \beta_2 w_j + \beta_3 w_j a)\right)\right)^2 dP(w, a)$$

The parameter of interest is $\beta_3^*$ The other $W_j$'s are included in the analysis and are marginalized over, but are not part of the defined model. I am planning on using the same simulation settings as above, but also including non-changing effects for $a$, but am open to other ideas.

# 3 Proposed testing procedure

Denote the estimator of $\psi$ by $\hat{\Psi}(X)$ and let $\hat{\psi} \equiv \hat{\Psi}(x)$ be an estimate of $\psi$. Suppose that $\sqrt{n}\hat{\psi}$ converges in law to some distribution $Q(P)$ when $P \in \mathcal{M}_0$. Any test of $H_0 : P \in \mathcal{M}_0$ versus $H_1 : P \notin \mathcal{M}_0$ can be characterized by an acceptance region $\Theta_0(P) \subset \mathbb{R}^d$. Letting $Z$ be distributed $Q(P)$, the region $\Theta_0(P)$ can be chosen so the probability of rejection under the null is controlled asymptotically:

$$Pr_Q\{Z \notin \Theta_0(P)\} = 1 - \alpha \text{ for every } P \in \mathcal{M}_0. \tag{1}$$

While there are infinitely many regions satisfying (1), we focus on a particular class of regions defined using $\ell_p$ norms. For simplicity, first consider regions defined using an $\ell_2$ norm:

$$\Theta_0(r) = \{\omega : ||\omega||_2 \leq r\}.$$

A region satisfying (1) has a radius defined by:

$$r_\alpha(P) = \min\{r : Pr_Q(||Z||_2 \leq r) \geq 1 - \alpha\}.$$

By constraining the possible regions we consider, we can now define our test in a simple algebraic form:

$$\text{reject } H_0 \text{ if } ||\sqrt{n}\hat{\psi}||_2 \geq r_\alpha.$$
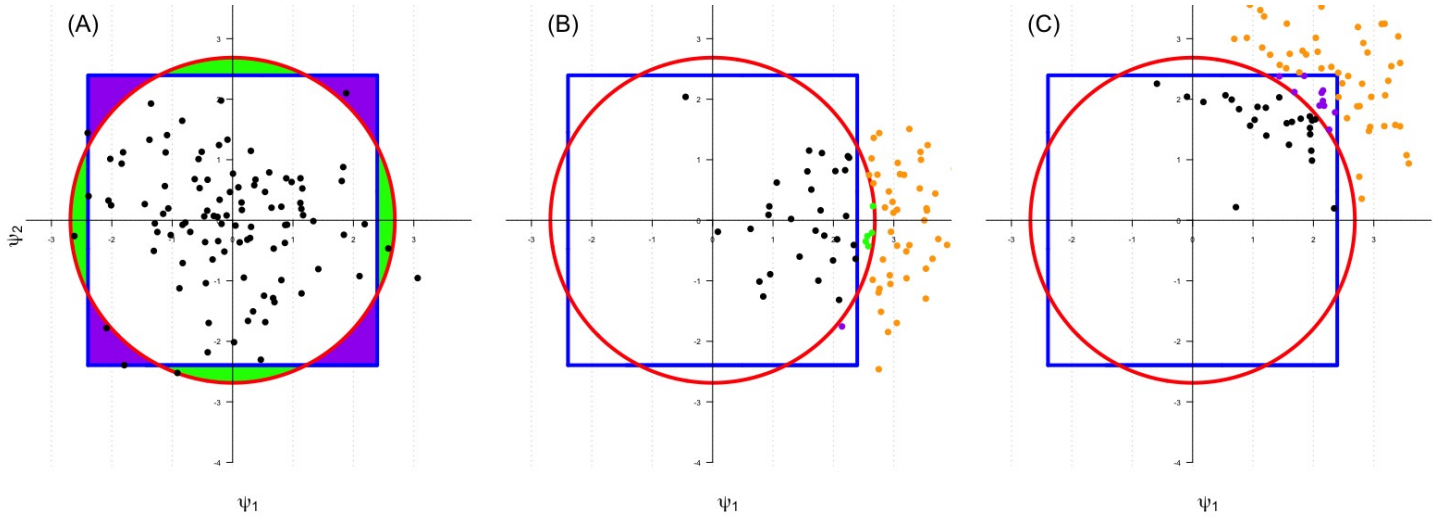
Figure 1: Plots of 100 observations from a limiting distribution of a hypothetical vector of parameter estimators in $\mathbb{R}^2$ (A) under the null, (B) under an alternative with $\psi_1 = 0, \psi_2 \neq 0$, and (C) under an alternative with $\psi_1, \psi_2 \neq 0$. The 95% quantiles for the data based on the max (blue) and $\ell_2$ (red) norms under the null are given in all three panels. If a test statistic fell within the purple regions the test would fail to reject $H_0$ if the $\ell_\infty$ norm was used, but would reject $H_0$ if the $\ell_2$ norm was used. The converse is true for the green regions. Depending on the alternative, the $\ell_\infty$ norm (B) or the $\ell_2$ norm(C) will achieve higher power.

Similarly, p-values can now be simply defined as $Pr_Q(||Z||_2 \geq ||\sqrt{n}\hat{\psi}||_2)$.

Above, we have used a norm to transform both the distribution of $Z$ and $\sqrt{n}\hat{\psi}$ into $\mathbb{R}$. This allows for creating tests and calculating p-values in a similar manner to the univariate case. However, this approach also works with more complicated functions. For every distribution $Q$ with support conntained in $\mathbb{R}^d$, let $\Gamma_Q$ be a function mapping from $\mathbb{R}^d$ to $\mathbb{R}$. The distribution of $\Gamma_Q(Z)$ (where $Z \sim Q$) can be compared to $\Gamma_Q(\sqrt{n}\hat{\psi})$ to obtain p-values and define a test. An example of $\Gamma_Q$ is given below:

$$\Gamma_Q(x) = Pr_Q(||Z^* + x|| > c_{0.8}) \text{ where } c_{Q,0.8} \equiv \min_c\{c : Pr_Q(||Z^*|| < c) \geq 0.8\} \text{ and } Z^* \stackrel{d}{=} Z. \tag{2}$$

While this method for defining tests allows for many different specifications, it often requires choosing a norm as part of the definition of $\Gamma$. Figure 1 shows that the power a test achieves can depend on the norm chosen. Consider the case when $d = 2$ and 100 draws are taken from $Z$. Cutoffs can be obtained by transforming these data (using the $\ell_p$ norm) and taking the 95% quantile of this transformed data. The perimeter of the acceptance region when using the $\ell_2$ and $\ell_\infty$ norm are given in red and blue respectively. If $\sqrt{n}\hat{\psi}$ falls within any of the purple regions, using the $\ell_\infty$ norm would result in failing to reject $H_0$ and using the $\ell_2$ norm would result in rejecting $H_0$. The converse is true for the green regions.

## 3.1 Adaptive selection of a norm

In the previous section it was shown that a test can be defined by choosing a summarizing function and norm. However, the choice of norm influences the power of the test. In most scenarios it will not be clear a priori which norm will have maximal power for the true alternative. The procedure proposed in this section adaptively selects a norm and it will be shown that this procedure achieves greater power than a test with a fixed norm, while maintaining type 1 error control.

To achieve this, first define $\Gamma_{1,Q}(x), \ldots, \Gamma_{p,Q}(x)$ as collection of functions in which the functions only differ by the norm used in their definition. Next, define

$$\Gamma_Q^*(x) = \max\left\{\Gamma_{1,Q}(x), \Gamma_{2,Q}(x), \ldots, \Gamma_{p,Q}(x)\right\}.$$

Change * so that they are not suppersetting both the rv and functions This could be the min if smaller values indicate larger distances away from the norm (such as p-values).

While this function is more complicated than before, $\Gamma_Q^*(Z))$ can still be compared to $\Gamma_Q^*(\sqrt{n}\hat{\psi})$ to obtain a p-value. Also, while it may be difficult to obtain the exact distribution of $\Gamma_Q^*(Z)$, the distribution is a function of $Q$, so obtaining very good approximations of $\Gamma_Q^*(Z)$ is possible by taking many draws from $Z$. This technique can be used both to estimate the function $\Gamma$ and to estimate $\tilde{\Gamma}_Q^*(Z)$ .

## 3.2 Obtaining the null distribution

The above procedure requires knowledge of the limiting distribution of $\sqrt{n}\hat{\psi}$ when $P \in \mathcal{M}_0$. To obtain an estimate of this limiting distribution, assume the vector of parameter estimates $\hat{\psi}$, converges to a normal distribution with an estimable variance covariance matrix when properly centered and normalized. Also assume each of the estimators $\hat{\psi}_1, \ldots, \hat{\psi}_d$ of $\psi_1, \ldots, \psi_d$ is asymptotically linear. That is for each $j \in \{1, \ldots, d\}$:

$$\hat{\psi}_j = \psi_j + \frac{1}{n}\sum_{i=1}^{n} D_j(\boldsymbol{x}_i) + o_p(1/\sqrt{n}) \text{ for some function } D_j$$

When there is a fixed number of covariates, the Cramer-Wold device can be used to show that the vector of parameter estimates is asymptotically normal with mean zero, and variance covariance matrix given by $\Sigma = E_{P_0}\left[D(X)D(X)^\top\right]$:

$$\sqrt{n}\left(\hat{\boldsymbol{\psi}} - \boldsymbol{\psi}\right) \xrightarrow{d} Z \sim N\left(0, \Sigma\right)$$

Under $H_0$, $\sqrt{n}\hat{\psi}$ converges to $Z$, and $\Sigma$ can be approximates with $\widehat{\Sigma} = \frac{1}{n}\sum_{i=1}^{n} D(\boldsymbol{x}_i)D(\boldsymbol{x}_i)^\top$. Thus, in practice we will use an estimate of $Q$, $\hat{Q}$ which is distributed $N(0, \widehat{\Sigma})$ in place of $Q$, and the random variable $\hat{Z}$ with distribution $\hat{Q}$ in place of $Z$. Thus, the test statistic will be $\Gamma_{\hat{Q}}(\sqrt{n}\hat{\psi})$ and will be compared to $\Gamma_{\hat{Q}}(\hat{Z})$.

## 3.3 Using a permutation test for the test statistic

While the above approach works asymptotically, there can be issues for small sample sizes. To avoid inflated type one error, a permutation based test can be used. Here, $\hat{Q}^{\#}$ is used to define $\Gamma$, and the test will compare $\Gamma_{\hat{Q}}(\sqrt{n}\hat{\psi})$ to $\Gamma_{\hat{Q}}(Z^{\#})$. To determine $\hat{Q}^{\#}$, the $Y$'s from the observed data are permuted before calculating $\widehat{\Sigma}$. Draws from $Z^{\#}$ are taken by permuting all of the $Y$'s of the observed data. There are a few more complications here that need to be figured out.

## 3.4 Summary of how simulations are run

# 4  Simulation Study
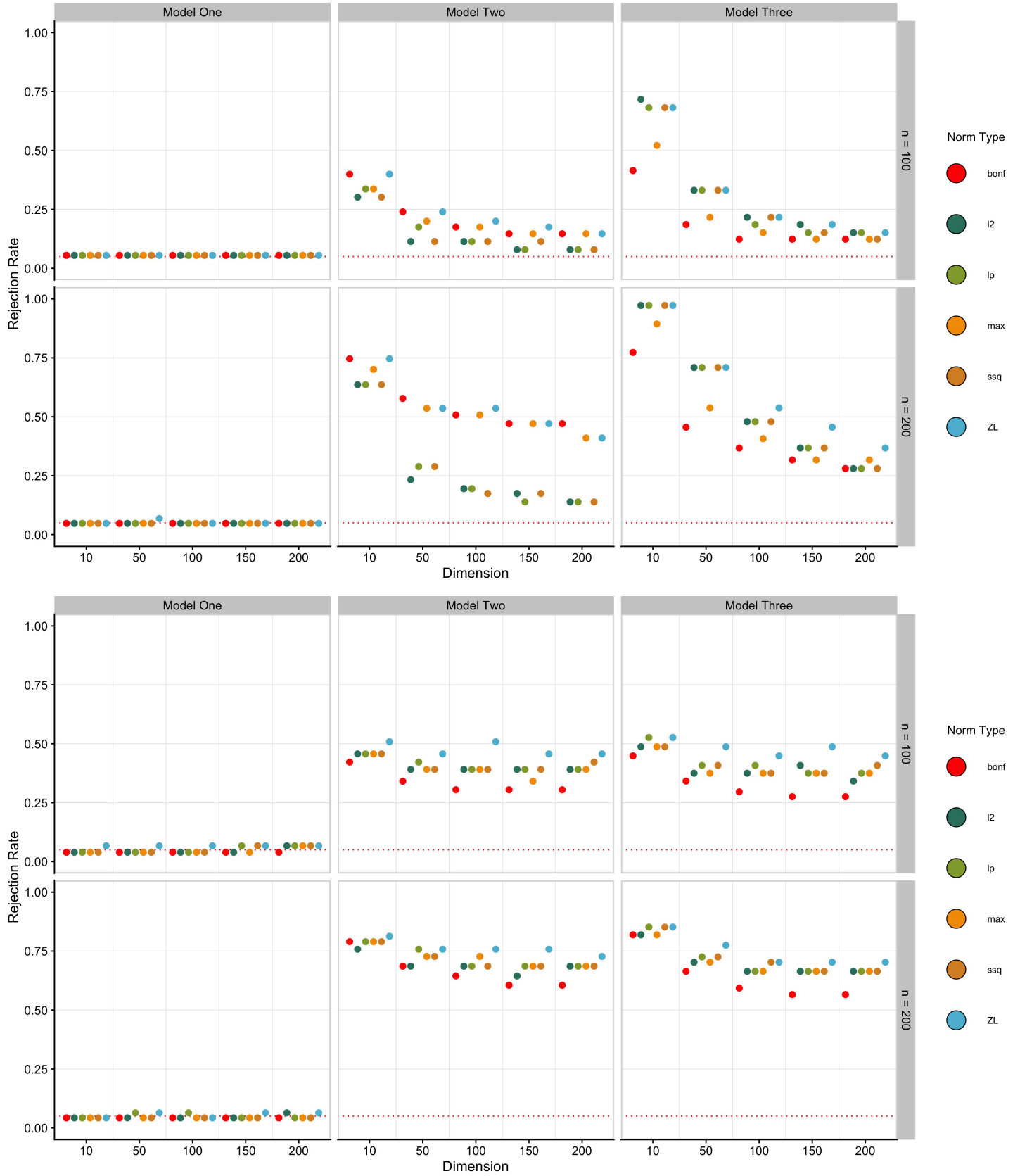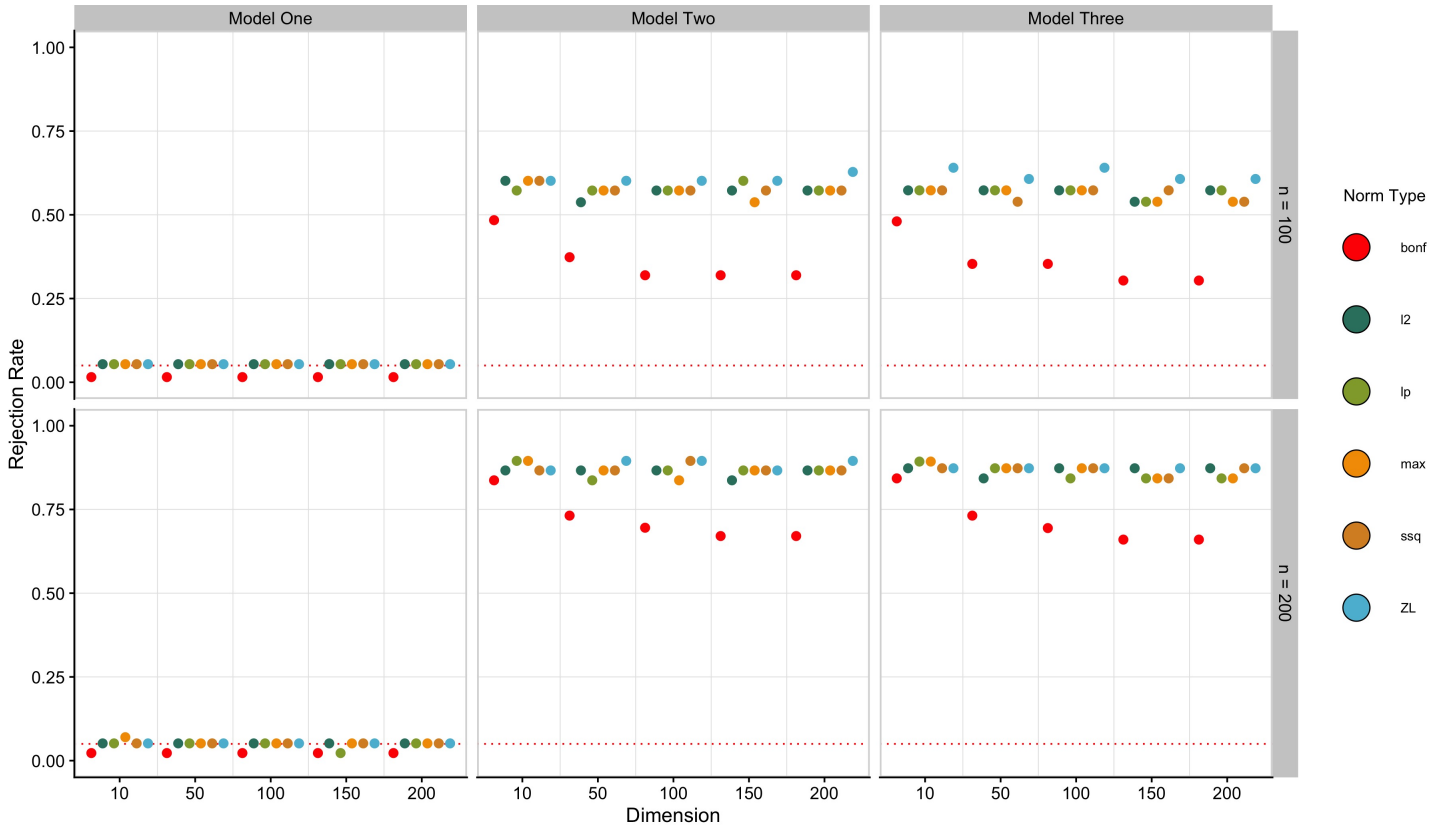
## 4.1  Correlation

2. Marginal structural model example.

All three examples will come with simulation results in the form of figures. Not sure how many figures to use or what sample size(s) to use.

# 5 Data Application

1. Data from Peter Gilbert

# 6 Discussion

# 7 Appendix

## 7.1 Test Consistency

**Theorem 1.** *Assume the performance metric of choice is $\hat{r}_{\alpha,p}(a)$, and each the norms $g_a$ considered have the following properties:*

$$for\ x \in \mathbb{R}^d,\ and\ s,l > 0, s \cdot \max(\boldsymbol{x}) \leq g_a(\boldsymbol{x}) \leq l \cdot d \cdot \max(\boldsymbol{x}) \tag{3}$$

$$for\ s \leq 1, l \geq 1, g_a(s \cdot \boldsymbol{x}) \leq g_a(\boldsymbol{x}) \leq g_a(l \cdot \boldsymbol{x}) \tag{4}$$

*Then for all $P \notin \mathfrak{M}_0$*

$$Pr\left(\frac{1}{B}\sum_{k=1}^{B} I\left\{\hat{T} \leq \hat{T}_k^{\#}\right\} < 0.05\right) \xrightarrow{p} 1\ as\ n \to \infty$$

*or Equivalently*

$$Pr\left(\hat{T} \leq F_{\hat{T}^{\#}}^{-1}(0.05)\right) \xrightarrow{p} 1\ as\ n \to \infty$$

*Proof.* This proof will consist of three parts. We will first show that as $n \to \infty, \hat{T}_a^{\#}$ becomes bounded away from 0 for any valid norm. Next we will show $\hat{T}_a$ converges to 0 in probability for any valid norm. Last we will show the two previous findings imply theorem 1.

Let $P_X^{\#}$ denote the distribution of the randomly permuted observations. Because $Pr(Y_i^{\#} \perp\!\!\!\perp W_i^{\#}) \to 0, \mathbf{\Psi}(P_X^{\#}) = \mathbf{0}$. Additionally, $\sqrt{n}\left(\hat{\boldsymbol{\psi}}^{\#} - \boldsymbol{\psi}^{\#}\right) \xrightarrow{d} Z^{\#} \sim N\left(\mathbf{0}, \Sigma_{\text{perm}}\right)$. Define $\hat{T}_a^{\#} = \min_s\{s : g_a(s \cdot \sqrt{n}\hat{\boldsymbol{\psi}}^{\#}) \geq C_{0.95,a}^{\#}\}$ and $C_{0.95,a}^{\#}$ is $F_{Z^{\#}}^{-1}(0.95)$. Since we know that $\boldsymbol{\psi}^{\#} = \mathbf{0}$, it follows that $\sqrt{n}\hat{\boldsymbol{\psi}}^{\#} \overset{d}{\approx} Z^{\#}$.

Now, consider:

$$
\begin{aligned}
\Pr\left(\hat{T}_a^{\#} > \epsilon\right) &= \Pr\left(g_a\left(\epsilon \cdot \sqrt{n}\hat{\boldsymbol{\psi}}^{\#}\right) \leq C_{0.95,a}\right) \\
&\geq \Pr\left(\epsilon \cdot d \cdot \max\left(\sqrt{n}\hat{\boldsymbol{\psi}}^{\#}\right) \leq C_{0.95,a}\right) \\
&= \Pr\left(\max\left(\sqrt{n}\hat{\boldsymbol{\psi}}^{\#}\right) \leq C_{0.95,a}/(\epsilon \cdot d)\right)
\end{aligned}
$$

Because $\max\left(\left|\sqrt{n}\hat{\boldsymbol{\psi}}^{\#}\right|\right)$ converges to a well defined, positive distribution as a result of the continuous mapping theorem, for each constant $c < 1$, we know there exists an $\epsilon_c$ such that $\Pr\left(\hat{T}_a^{\#} > \epsilon_c\right) \geq c$.

Now, shifting our focus to $\hat{T}_a$, under alternatives, $\boldsymbol{\psi} \neq \mathbf{0}$. Define $\psi_{\max} = \max(\psi_1, \ldots, \psi_d)$. Using this knowledge, and (3), note that

$$
\begin{aligned}
\Pr\left(\hat{T}_a < \epsilon\right) &= \Pr\left(g_a\left(\epsilon \cdot \sqrt{n}\hat{\boldsymbol{\psi}}\right) \geq C_{0.95,a}\right) \\
&\geq \Pr\left(\epsilon \cdot \sqrt{n}\max(\hat{\psi}_1, \ldots, \hat{\psi}_d) \geq C_{0.95,a}\right) \\
&= \Pr\left(\max(\hat{\psi}_1, \ldots, \hat{\psi}_d) \geq C_{0.95,a}/\left(\epsilon \cdot \sqrt{n}\right)\right) \\
&\geq \Pr\left(\max(\hat{\psi}_1, \ldots, \hat{\psi}_d) \geq \psi_{\max}/2\right) \Pr\left(\psi_{\max}/2 \geq C_{0.95,a}/\left(\epsilon \cdot \sqrt{n}\right)\right)
\end{aligned} \tag{5}
$$

The first factor of the product in (5) will converge to 1 as $n \to \infty$ from the consistency of $\hat{\boldsymbol{\psi}}$. The second quantity will be equal to 1 for sufficiently large $n$. Thus $\hat{T}_a \xrightarrow{p} 0$ under any alternative.

It was shown that for each $a$ that $\hat{T}_a \xrightarrow{p} 0$. This means that our adaptive estimator $\hat{T} \xrightarrow{p} 0$ as well. Now, let $c = 0.05/k$ and $\epsilon_c$ be small enough that $\Pr\left(\hat{T}_a^{\#} > \epsilon_c\right) \geq 1 - (0.05/k)$. The permutation version of the adaptive estimator $\hat{T}^{\#}$ has the property that

$$
\Pr\left(\hat{T}^{\#} < \epsilon_c\right) \leq \Pr(\hat{T}_1^{\#} < \epsilon_c) + \cdots + \Pr(\hat{T}_k^{\#} < \epsilon_c) \leq 0.05,
$$

and the theorem's conclusion follows. $\qquad\square$

## 7.2 Unbiasedness at local alternatives

Consider a local alternative in which the true value of $\psi$ is shrinking towards zero at a root $n$ rate: $\psi = \underline{h}/\sqrt{n}$. We assume that each potential norm is convex. This assumption can be relaxed to what was described by Eaton and Perlman (1991) (Concentration inequalitites for multivariate distributions: mv normal). Under this local alternative, we will have $\sqrt{n}\hat{\psi} \xrightarrow{d} N(\underline{h}, \Sigma)$. Show that the test will reject the null with a probability greater than $\alpha$ for an $\alpha$ level test

**Theorem 2.** *Under local alternatives described above,*

$$
Pr_P\left(\Gamma_{\hat{Q}}^*(\sqrt{n}\hat{\psi}) \leq F_{\Gamma_{\hat{Q}}^*(\hat{Z})}^{-1}(\alpha)\right) \geq \alpha,
$$

*Where $\hat{Z} \sim \hat{Q}$ and $Z \sim Q$. Here (unlike other parts of the paper) small values of $\Gamma$ provide evidence against the null. Values of $\Gamma$ can be thought of as similar to p-values.*

**Lemma 3.** *plus h?*

$$
\Gamma_{\hat{Q}}^*(\sqrt{n}\hat{\psi}) \xrightarrow{d} \Gamma_Q^*(Z + \underline{h}) \text{ and } F_{\Gamma_{\hat{Q}}^*(\hat{Z})}^{-1}(\alpha) \to F_{\Gamma_Q^*(Z)}^{-1}(\alpha)
$$

*The proofs of these will be similar, but I am not sure quite how to do it. While the following expansion is useful for me, I am not sure how to deal with each part:*

$$Pr_P(\Gamma^*_{\hat{Q}}(\sqrt{n}\hat{\psi}) > t) - Pr_P(\Gamma^*_Q(Q) > t) = Pr_P(\Gamma^*_{\hat{Q}}(\sqrt{n}\hat{\psi}) > t) - Pr_P(\Gamma^*_{\hat{Q}}(Q) > t) - \left( Pr_P(\Gamma^*_Q(Q) > t) - Pr_P(\Gamma^*_{\hat{Q}}(Q) > t) \right)$$

*We can make arguments that the second two terms will cancel because of an assumed continuity of our function $\Gamma^*$ with respect to $Q$. We can also think of $\Gamma^*$ as being indexed by $Q$'s covariance matrix instead of its entire distribution to make more simple continuity arguments.*

*However, I am not exactly sure how to argue that the first two terms cancel. Initially I want to use the continuous mapping theorem. Because the function itself changes with n and is random, I am not sure what to do. If we again use the continuity of $\Gamma$ could we argue that the values of the random functions will be close to the values of the non-random funcitons (where $\hat{Q}$ is replaced by $Q$)*

*My other idea was to assume that the function class of $\Gamma*$ is P-Donsker, but I think this may be overly complicated, and not necessary.*

*For fixed every t,*

$$\Gamma^*_{\hat{Q}}(t) \to \Gamma^*_Q(t) \ if \ \Gamma_{i,\hat{Q}}(t) \to \Gamma_{i,Q}(t) \ for \ every \ i \in \{1,2,\dots,d\}$$

*The function defined in (2) satisfy this condition*

**Lemma 4.** *Let t be fixed. By the consistency of our variance estimator, we know that $\hat{Z} \xrightarrow{d} Z$, and by continuous mapping theorem $||\hat{Z}|| \xrightarrow{d} ||Z||$. Thus it can be shown that $C_{\hat{Q},0.8} \to C_{Q,0.8}$ because the distribution functions for both $\hat{Z}$ and $Z$ are continuous.*

*The above findings and Slutsky's theorem imply that $||\hat{Z} + \underline{t}|| - C_{\hat{Q},0.8} \xrightarrow{d} ||Z - \underline{t}|| + C_{Q,0.8}$ so:*

$$\Gamma_{i,\hat{Q}}(t) = Pr_{\hat{Q}}|| \left( \hat{Z} + \underline{t}|| > C_{\hat{Q},0.8} \right) = Pr_{\hat{Q}} \left( ||\hat{Z} + \underline{t}|| - C_{\hat{Q},0.8} > 0 \right) \to$$
$$Pr_Q \left( ||Z + \underline{t}|| - C_{Q,0.8} > 0 \right) = Pr_Q \left( ||Z + \underline{t}|| > C_{Q,0.8} \right) = \Gamma_{i,Q}(t)$$

*A result of [Anderson] is that for two centrally symmetric, unimodal functions, $f_1(x)$ and $f_2(x)$, the convolution,*

$$g(\theta) = \int f_1(\theta) f_2(x - \theta)$$

*is centrally symmetric and ray decreasing. A function $f(x)$ on $\mathbb{R}^p$ is centrally symmetric if $f(-x) = f(x)$ for every $x$. A function is unimodal if for every $k$, the set $\{x : f(x) \geq k\}$ is convex. A function $f(x)$ on $\mathbb{R}^p$ is ray decreasing if for every $x$ in $\mathbb{R}^p$, the function $g(\beta) = f(\beta x), \beta \in \mathbb{R}$ is a decreasing function of $\beta$.*

*For now, we will assume that each of our $\Gamma_Q$ is a unimodal, centrally symmetric function. In certain cases, $\Gamma_Q$ is increasing x moves away from the origin. In these cases, for the purposes of this proof, instead use $\Gamma_Q^{-1}(x) = (\Gamma_Q(x))^{-1}$.*

**Lemma 5.** *Let $\Gamma_1, \dots, \Gamma_d$ all be centrally symmetric, unimodal functions. Then $\Gamma^* = \min\{\Gamma_1, \dots, \Gamma_d\}$ is also centrally symmetric and unimodal.*

*Centrally symmetric:*

$$\Gamma^*(x) = \min\{\Gamma_1(x), \dots, \Gamma_d(x)\} = \min\{\Gamma_1(-x), \dots, \Gamma_d(-x)\} = \Gamma^*(-x)$$

*Unimodal:*

$$\{x : \Gamma^*(x) \geq k\} = \{x : \Gamma_1(x) \geq k, \dots, \Gamma_d(x) \geq k\} = \bigcap_{i=1}^{d} \{x : \Gamma_i(x) \geq k\}$$

*The intersection of multiple convex sets is also convex, so $\Gamma^*(x)$ is unimodal.*

**Lemma 6.** *Let $f(x)$ be a centrally symmetric, unimodal function. Then $g(x) = I\{f(x) \geq c\}$ where $c \in \mathbb{R}$ is also centrally symmetric and unimodal.*

*Centrally symmetric:*

$$g(x) = I\{f(x) \geq c\} = I\{f(-x) \geq c\} = g(-x)$$

*Unimodal: If $k < 0$ or $k > 1$, then $\{x : g(x) \geq k\}$ will be the empty set or all of $\mathbb{R}^d$ respectively, and both sets are convex. Otherwise,*

$$\{x : g(x) \geq k\} = \{x : I\{f(x) \geq c\} \geq k\} = \{x : f(x) \geq c\}$$

*which is convex by definition.*

*A consequence of lemmas 5 and 6, the function $I\left\{\Gamma_Q^*(x) \geq c_{0.95}\right\}$ is centrally symmetric and unimodal.*

$$\int I\left\{\Gamma_Q^*(x) \leq c_{0.95}\right\}\phi(x-\mu)dx = \int \left(1 - I\left\{\Gamma_Q^*(x) \geq c_{0.95}\right\}\right)\phi(x-\mu)dx$$
$$= 1 - \int I\left\{\Gamma_Q^*(x) \geq c_{0.95}\right\}\phi(x-\mu)dx$$

*Since the subtracted quantity is decreasing by [Anderson], the quantity as a whole will be increasing. Thus local power is obtained.*

Proof Outline:

- Under local alternatives, $\sqrt{n}\hat{\psi} \overset{d}{\to} N(\boldsymbol{c}, \Sigma)$

- With a large enough sample size, and enough MC draws, we have that for each norm:

- Think of Gamma as a funciton indexed by $\Sigma$ to allow it to be similar. Also make some assumptions about how smooth $\Gamma$ is with respect to this parameters. For any given value of the input $(t)$ think about

- Right now we have given up on proving things for the permutation test, but we may try to do it again at some point. The permutation test statistic will converge in distribution to a standard normal. This paper should help: [Omelka and Pauly]

- For each norm selected, [Gupta et al.] states that the power will be non-decreasing as long as the rejection region is convex (This should be true most of our rejection regions), and the probability density is decreasing away from the mean (which is true of a normal distribution).

- Show that for $n$ large, the norm is selected to give the best power. Thus since each norm obtains local power, the adaptive test will also obtain local power for $n$ large.

$$\mathcal{L}\left(\Gamma_{\hat{Q}}\left(\sqrt{n}\hat{\boldsymbol{\Psi}}(\boldsymbol{X})\right), \Gamma_Q\left(\sqrt{n}\hat{\boldsymbol{\Psi}}(\boldsymbol{X})\right)\right) \to 0$$

## 7.3 Consistency of norm selection

This proof would likely be difficult and require some slower than $\sqrt{n}$ covergence rates of the local alternative considered. Under fixed alternatives, all norms perform equally perfectly

## 7.4 Type 1 error control

Proof is so short, I am not sure if it is worth including Under the null, $Y \perp\!\!\!\perp X$. Thus the test statistic will be taken from the same distribution as all of the permutation based test statistics used to estimate the distirbution of the test statistic under the null. Therefore, as $B$ grows, $Pr(T_n \geq F_{T_{k,n}^{\#}|X_n}(0.95)) \Rightarrow 0.05$

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