A Proof of Vertex-disjoint Menger's Theorem by Bipartite Matching and Contraction

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Abstract

A proof of vertex-disjoint Menger's theorem between two distinctive vertices s and t in G is proposed. Starting from a minimum separator X and the component G_t of $G-X$ to which t belongs, $|X|$ vertex-disjoint s-X paths are found in $G-V(G_t)$ by recursively applying contraction to bipartite matchings of X. Similarly, $|X|$ vertex-disjoint X-t paths are found. Concatenating two paths at each vertex in X yields $|X|$ vertex-disjoint s-t paths. The contraction of the bipartite matchings must not decrease the connectivity. Existence of such bipartite matchings are proven by induction on $|X|$.

1 Introduction

Menger's theorem is one of the early fundamental discoveries in graph theory. Since the original theorem was proposed by Menger [7], some variants have been proposed, which are roughly divided into vertex-disjoint ones, e.g. Whitney [9], and edge-disjoint one by Ford and Fulkerson [4]. For the vertex-disjoint theorems, several proofs have been proposed by Dirac [3]; Böhme, Göring, and Harant [1]; Pym [8]; and Grünwald (later Gallai) [5]. Also, the edge-disjoint theorem is proven by the min-cut/max-flow theorem [4]. Of all the variants, we prove the following vertex disjoint theorem.

Theorem 1 (Menger's Theorem)
Given a graph $G = (V, E)$, let s and t be two distinct non-adjacent vertices in G. The size of the minimum s-t separators is equal to the maximum number of internally vertex-disjoint s-t paths.

In this article, we call the minimum number of separating vertices between s and t *s-t connectivity* of G and denote it by $\kappa_G(s, t)$ hereinafter.

The purpose of this article is to propose a new proof by recursively applying contraction of a maximum bipartite matching of a minimum s-t separator in a graph.

It is easy to see the maximum number of disjoint s-t paths does not exceed $\kappa_G(s, t)$. We prove that we can actually construct $\kappa_G(s, t)$ disjoint s-t paths in

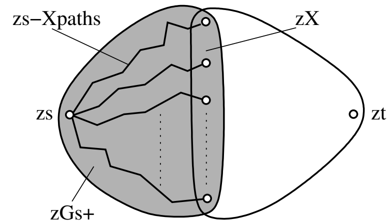

Figure 1: X, G_s^+, and s-X paths.

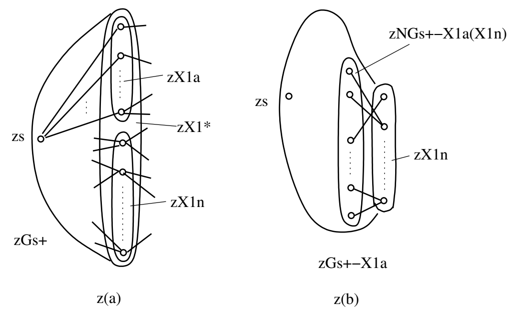

Figure 2: X_1^a, X_1^n, and $G_s^+ - X_a$.

G. It is also trivial to prove the case for $\kappa_G(s,t) = 0$, so the following discussion assumes $\kappa_G(s,t) > 0$.

2 Proof

We follow the notational conventions in Diestel's [2]. We treat a matching M as a set of edges in a graph G. We denote the graph obtained from G by contracting all the edges in M by G/M, and the set of vertices in G/M into which the edges in M are contracted by V_M.

Let $k := \kappa_G(s,t)$. Let X be a minimum s-t separator ($X \cap \{s,t\} = \emptyset$). Let G_s, G_t be the two components of $G - X$ to which s and t belong respectively. Let G_s^+, G_t^+ be $G[V(G_s) \cup X]$, $G[V(G_t) \cup X]$ respectively. We prove G_s^+ has k vertex-disjoint s-X paths. Similarly, G_t^+ has k vertex-disjoint X-t paths by symmetry. Concatenating the two paths at each vertex in X from each of s-X paths and X-t paths in G yields k vertex-disjoint s-t paths (Fig. 1).

The rest of the proof is for finding k s-X paths in G_s^+. Our strategy is as follows. Let $G_1 := G$ and $X_1^* := X$. Split X_1^* into two subsets X_1^a and X_1^n such that $X_1^a = \{x \in X_1^* \mid \{s,x\} \in E(G_s^+)\}$ and $X_1^n = \{x \in X_1^* \mid \{s,x\} \notin E(G_s^+)\}$ (Fig. 2(a)). We have already found $|X_1^a|$ vertex-disjoint s-X_1^* paths between s and X_1^a in G_s^+, each of which is merely an edge. Please observe that X_1^n is a

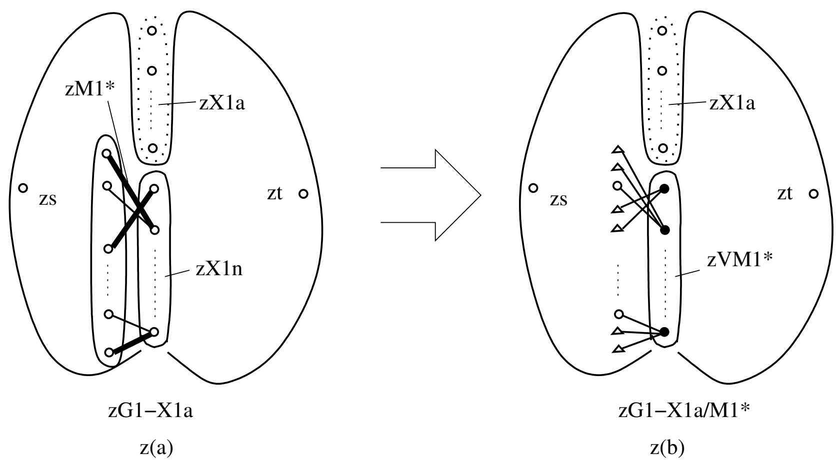

Figure 3: $G_1 - X_1^a$, and $(G_1 - X_1^a)/M_1^*$.

minimum separator of $G_1 - X_1^a$ (Fig. 2(b)). It is easy to see X_1^n is a separator of $G_1 - X_1^a$. Suppose it were not minimum. Let X' be a minimum separator of $G_1 - X_1^a$ such that $|X'| < |X_1^n|$. Then $X' \cup X_1^a$ would give a minimum separator of G_1, which contradicts the minimality of X_1^*.

We prove there is a bipartite matching M_1^* of X_1^n to $N_{G_s^+ - X_1^a}(X_1^n)$ such that $|M_1^*| = |X_1^n|$, and contracting all the edges in M_1^* does not decrease the s-t connectivity of $(G_1 - X_1^a)/M_1^*$, i.e. $\kappa_{G_1 - X_1^a}(s,t) \leq \kappa_{(G_1 - X_1^a)/M_1^*}(s,t)$ (Fig. 3(a)). We eventually obtain the equality here since $V_{M_1^*} \cup X_1^a$ form an s-t separator in G_1/M_1^*, which indicates they are again a minimum separator X_2^* in G_1/M_1^* and $\kappa_{G_1}(s,t) = \kappa_{G_1/M_1^*}(s,t)$ (Fig. 3(b)). Let $G_2 = G_1/M_1^*$.

We can recursively apply this process of finding a matching and contracting all the edges in it r times until all the vertices in the minimum s-t separator X_r^* in G_r are adjacent to s (Fig. 4 (a)). This process is guaranteed to terminate as G is finite, and at each iteration at least one edge is contracted. If we "unfold" the edge $\{s, x\}$ and the vertex x in X_r^*, to which some incident edges have been contracted, we obtain $|X|$ trees in G_s^+, which are mutually vertex-disjoint except at s. In each tree we can find a unique s-x' path for each $x' \in X_1^*$. Those paths form a set of k vertex-disjoint s-X paths (Fig. 4 (b)).

The rest of the proof is dedicated to prove existence of a bipartite matching M of X_1^n to $N_{G_s^+ - X_1^a}(X_1^n)$ such that contraction of all the edges in M does not decrease the s-t connectivity of in $G - X_1^a$. In the following discussion, we assume $X_1^a = \emptyset$, i.e. $X = X_1^n$. If $X_1^a \neq \emptyset$, consider the graph $G - X_1^a$ instead of G for its connectivity $|X_1^n|$. We later add the edges (disjoint paths) in X_1^a, after we find vertex-disjoint s-X_1^n paths.

First we prove existence of a bipartite matching of X. Let $Y := N_{G_s^+}(X)$. Since we assume X_1^a is empty, s is not in Y. Consider the bipartite graph

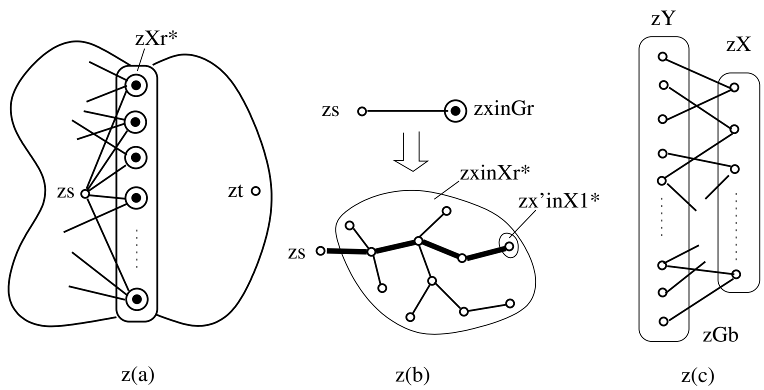

Figure 4: $G_r, X_r^*, and G_\beta$.

G_β with the bipartition $\{X, Y\}$ (Fig. 4 (c)). Formally $G_\beta = G[X \cup Y] - (E(G[X]) \cup E(G[Y]))$. We can easily check that G_β contains a bipartite matching of X. Indeed, for all $S \subseteq X$, $d_{G_\beta}(S) \geq |S|$. Suppose $\exists S \subseteq X$ such that $d_{G_\beta}(S) < |S|$, then we can replace S with $N_{G_\beta}(S)$ in X, which would be an s-t separator in G whose cardinality is less than $|X|$, which contradicts the minimality of X. By the marriage theorem by Hall [6], G_β contains a bipartite matching of X.

Next we prove existence of a bipartite matching M of X which does not reduce the s-t connectivity when contraction of M is applied to G. We prove this by induction on $\kappa_G(s,t)$, or on $|X|$. If $|X| = 1$, it is trivial to prove contraction of any bipartite matching M does not reduce the s-t connectivity. So the induction starts.

Let $X_0 := X$, $Y_0 := Y$, and M_0 be any bipartite matching of X_0 in N_{G_β} (Fig. 5(a)). If it preserves the s-t connectivity in G/M_0, we are done. So we assume $\kappa_{G/M_0}(s,t) < \kappa_G(s,t)$. G/M_0 has an s-t separator W_0 in G_s^+ such that $|W_0| < |X_0|$. W_0 contains at least one vertex in V_{M_0}, otherwise W_0 would also be an s-t separator in G whose cardinality is less than $|X|$, a contradiction. So $W_0 \cap V_{M_0} \neq \emptyset$. Let $K_0 = W_0 \cap V_{M_0}$ and $S_0 = W_0 \backslash K_0$.

Here we categorize the edges and vertices in G for the following discussion (Fig. 5(b)).

- Let E_{K_0} be the set of edges in G which corresponds to K_0 in G/M_0.
- Let $E_{K_0'}$ be $M_0 \backslash E_{K_0}$.
- Let A_0 and C_0 be the sets of vertices in X_0 and Y_0 respectively, which are incident to the edges in E_{K_0}.
- Let B_0 and D_0 be the sets of vertices in X_0 and Y_0 respectively, which are incident to the edges in $E_{K_0'}$.

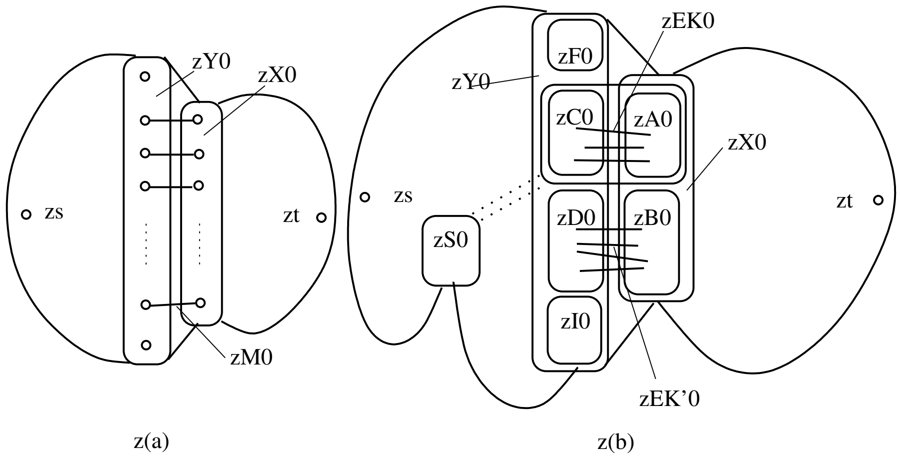

Figure 5: X, Y, and the subsets of V.

- Let F_0 be the set of vertices in $Y_0\backslash(C_0 \cup D_0)$ which are not incident to any vertices in B_0.
- Let I_0 be the set of vertices in $Y_0\backslash(C_0 \cup D_0)$ which are incident to any vertices in B_0.

Please observe that $A_0 \cup C_0 \cup S_0$ separates s from t in G, since $K_0 \cup S_0$ separates s from t in G/M_0. Please also note that S_0 can be $\emptyset$, but neither B_0 nor D_0 can be empty.

Here we further define six subgraphs of G and one new graph as follows.

- Let H_s^0 and H_t^0 be the two components of $G - (A_0 \cup C_0 \cup S_0)$ to which s and t belong respectively.
- Let H_{s+}^0, H_{t+}^0 be $G[V(H_s^0) \cup A_0 \cup C_0 \cup S_0]$, $G[V(H_t^0) \cup A_0 \cup C_0 \cup S_0]$ respectively.
- Let J_-^0 be a subgraph of G induced by $V(H_t^0) \cap V(G_s^+)$ (Fig. 6(a)).
- Let J_+^0 be a subgraph of G induced by $\big(V(H_t^0) \cap V(G_s^+)\big) \cup S_0$, i.e. $J_+^0 = G[V(J_-^0) \cup S_0]$.
- Let L^0 be a graph based on $G_s^+ - J_-^0$ augmented by a new vertex t_0, and new edges between t_0 and all the vertices in $A_0 \cup S_0$ (Fig. 6(b)).

Please note H_s^0 is a subgraph of G_s^+, since $W_0 \subseteq V(G_s^+/M_0)$ and $M_0 \subseteq E(G_s^+)$. Also please note that F_0 belongs to L^0 and I_0 belongs to J_-^0.

As $A_0 \cup S_0$ does not separate s from t in G, there are some vertices in C_0 which are incident to edges into J_-^0.

- Let $Q_0 := \{q \in C_0 | q \text{ is adjacent to at least one vertex in } J_-^0\ \}$.

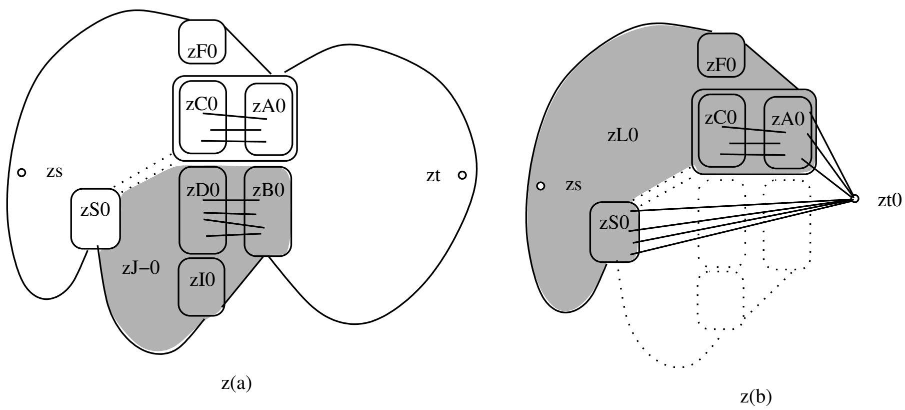

Figure 6: J_-^0 and L^0.

- Let $Q_0' := \{q' \in V(J_-^0) | q'$ is adjacent to at least one vertex in Q_0 $\}$.

From the s-t connectivity of G, we claim $|Q_0| \geq |B_0| - |S_0|$, and $|Q_0'| \geq |B_0| - |S_0|$. Please observe that both $S_0 \cup Q_0 \cup A_0$ and $S_0 \cup Q_0' \cup A_0$ form s-t separators in G (Fig. 7(a)).

For the induction process, we consider two graphs L_R^n and J_R^n as follows, where n indicates a number of iterations. First, let J_Q^0 be a graph based on the subgraph of G induced by $V(J_+^0) \cup Q_0$, and augmented by two new vertices u_0 and w, and new edges between u_0 and all the vertices in $S_0 \cup Q_0$, and new edges between all the vertices in B_0 and w (Fig. 7(b)). We claim $\kappa_{J_Q^0}(u_0, w) \geq |B_0|$. Suppose not. Then there would be a minimum u_0-w separator U_0 such that $|U_0| < |B_0|$. However, $U_0 \cup A_0$ would form an s-t separator in G, which contradicts the minimality of X. Please note that there is no edge between $L^0 - (S_0 \cup A_0)$ and J_-^0 except $E(Q_0, Q_0')$.

By Proposition 2, we can find at least one subset R_0 of Q_0 such that $|R_0| = |B_0| - |S_0|$ and $\kappa_{J_Q^0 - (Q_0 \backslash R_0)}(u_0, w) = |B_0|$. Let the family of such sets be $\mathcal{R}_0$, and pick any set R_0 in $\mathcal{R}_0$. Let $J_R^0 := J_Q^0 - (Q_0 \backslash R_0)$, and let L_R^0 be a graph based on L^0 augmented by new edges between all the vertices in R_0 and t_0 (Fig. 8). It is easy to see $\kappa_{L_R^0}(s, t_0) \geq |X|$, since existence of an s-t_0 separator whose cardinality is less than $|X|$ in L_R^0 implies that such a separator would also be an s-t separator in G. Also, that $A_0 \cup R_0 \cup S_0$ form an s-t_0 separator implies $\kappa_{L_R^0}(s, t_0) = |X|$, and $A_0 \cup R_0 \cup S_0$ form a minimum s-t_0 separator.

Let $X_1 := A_0$ and $Y_1 := F_0 \cup (C_0 \backslash R_0)$. Let M_1 be any bipartite matching of the bipartition $\{Y_1,\ X_1\}$. Existence of the matching is proven using the Hall's marriage theorem similar to the proof of existence of M_1.

If $\kappa_{L_R^0/M_1}(s, t_0) \geq |X|$, we set $L_R^n := L_R^0$, $J_R^n := J_R^0$, and we are done. So we assume $\kappa_{L_R^0/M_1}(s, t_0) < |X|$.

L_R^0/M_1 has an s-t_0 separator W_1 such that $|W_1| < |X|$. W_1 contains at

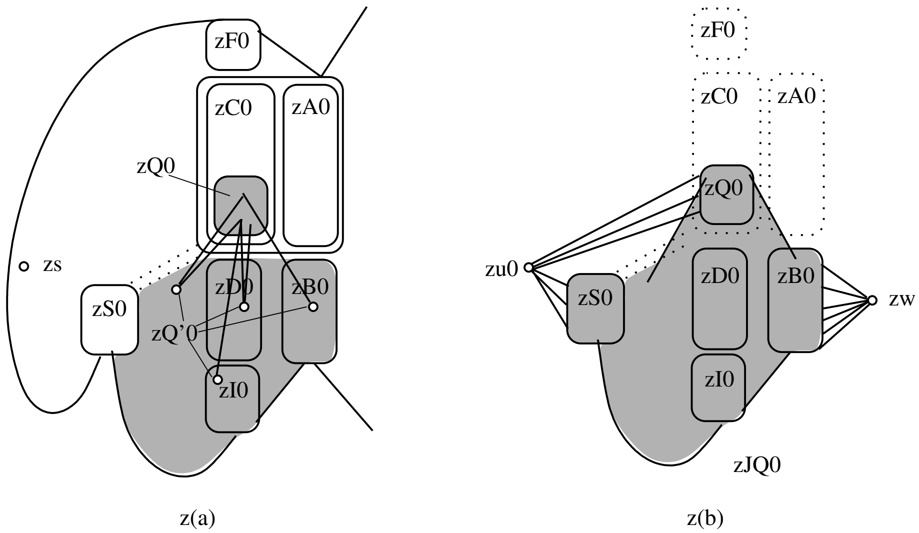

Figure 7: Q_0, Q_0', and J_Q^0.

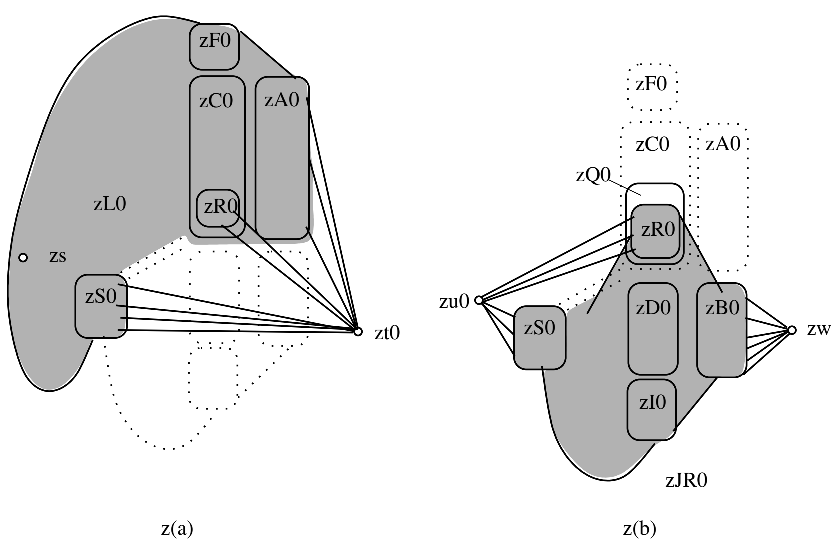

Figure 8: J_R^0 and L_R^0.

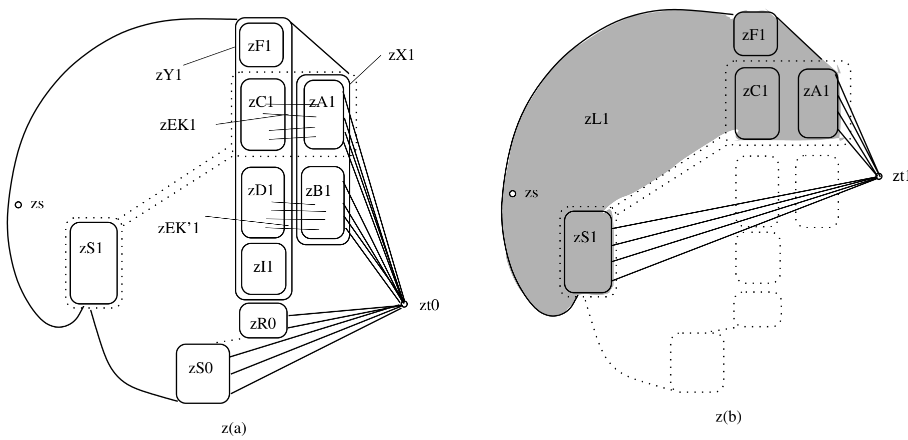

Figure 9: Subsets of L^0_R and L^1.

least one vertex in V_{M_1}, otherwise W_1 would also be an s-t separator in G whose cardinality is less than $|X|$, a contradiction. So $W_1 \cap V_{M_1} \neq \emptyset$. Let $K_1 = W_1 \cap V_{M_1}$ and $S_1 = W_1 \backslash K_1$.

Here we categorize the edges and vertices in L^0_R according to W_1 similar to the discussion above when we found W_0 (Fig 9(a)).

- Let E_{K_1} be the set of edges in L^0_R which corresponds to K_1 in L^0_R/M_1.
- Let $E_{K'_1}$ be $M_1 \backslash E_{K_1}$.
- Let A_1 and C_1 be the sets of vertices in X_1 and Y_1 respectively, which are incident to the edges in E_{K_1}.
- Let B_1 and D_1 be the sets of vertices in X_1 and Y_1 respectively, which are incident to the edges in $E_{K'_1}$.
- Let F_1 be the set of vertices in $Y_1 \backslash (C_1 \cup D_1)$ which are not incident to any vertices in B_1.
- Let I_1 be the set of vertices in $Y_1 \backslash (C_1 \cup D_1)$ which are incident to any vertices in B_1.

Please observe that $A_1 \cup C_1 \cup S_1$ separates s from t_0 in L^0_R, since $K_1 \cup S_1$ separates s from t_0 in L^0_R/M_1. Please also note that S_1 can be $\emptyset$, but neither B_1 nor D_1 can be empty.

Here we further define two subgraphs of L^0_R and one new graph as follows.

- Let J^1_- be the component of $L^0_R - (A_1 \cup C_1 \cup S_1)$ to which t_0 belongs.
- Let J^1_+ be a subgraph of L^0_R induced by $V(J^1_-) \cup S_1$.

- Let L^1 be a graph based on $L_R^0 - J_-^1$ augmented by a new vertex t_1, and new edges between t_1 and all the vertices in $A_1 \cup S_1$ (Fig 9(b)).

Also please note that F_1 belongs to L^1 and I_1 belongs to J_-^1.

As $A_1 \cup S_1$ does not separate s from t_0 in L_R^0, there are some vertices in C_1 which are incident to edges into J_-^1.

- Let $Q_1 := \{q \in C_1 | q$ is adjacent to at least one vertex in J_-^1 $\}$.
- Let $Q_1' := \{q' \in V(J_-^1) | q'$ is adjacent to at least one vertex in Q_1 $\}$.

From the s-t_0 connectivity of L_R^0, we claim $|Q_1| \geq |B_1| + |R_0| + |S_0| - |S_1|$, and $|Q_1'| \geq |B_1| + |R_0| + |S_0| - |S_1|$. Please observe that both $S_1 \cup Q_1 \cup A_1$ and $S_1 \cup Q_1' \cup A_1$ form s-t_0 separators in L_R^0.

Let J_Q^1 be a graph based on the subgraph of G induced by $V(J_+^1) \cup Q_1$, and augmented by two new vertices u_1 and w_1, and new edges between u_1 and all the vertices in $S_1 \cup Q_1$, and new edges between all the vertices in $B_1 \cup R_0 \cup S_0$ and w_1. We claim $\kappa_{J_Q^1}(u_1, w_1) \geq |B_0| + |B_1|$ similar to the reasoning for $\kappa_{J_Q^0}(u_0, w) \geq |B_0|$. Please note that $|B_1| = |R_0| + |S_0|$. By Proposition 2, we can find at least one subset R_1 of Q_1 such that $|R_1| = |B_0| + |B_1| - |S_1|$ and $\kappa_{J_Q^1 - (Q_1 \backslash R_1)}(u_1, w_1) = |B_0| + |B_1|$. Let the family of such sets be $\mathcal{R}_1$ and pick any R_1 in $\mathcal{R}_1$. Let J_R^1 be a graph based on $J_R^0 - u_0$ and $J_Q^1 - (Q_1 \backslash R_1) - w_1$, pasted along $S_0 \cup R_0$, and add new edges between B_1 and w (Fig 10(a)). Please note that $J_R^1 - \{u_1, w\}$ is an induced subgraph of G.

Let L_R^1 be a graph based on L^1 augmented by new edges between all the vertices in R_1 and t_1 (Fig 10(b)). It is easy to see $\kappa_{L_R^1}(s, t_1) = |X|$ due to the similar reasoning for $\kappa_{L_R^0}(s, t_0) = |X|$, and $A_1 \cup R_1 \cup S_1$ form a minimum s-t_1 separator in L_R^1.

We repeat this process n times until we obtain L_R^n, J_R^n, such that there is a bipartite matching M_n for L_R^n such that $\kappa_{L_R^n/M_n}(s, t_n) = |X|$ (Fig 11).

This process is guaranteed to terminate at the n-th iteration due to the following reason. Assume we are at the i-th iteration. Let $\alpha_i = \kappa_{L_R^i}(s, t_i) - \kappa_{L_R^i/M_i}(s, t_i)$. The discussion above implies the necessary condition for α_i to be any positive integer is existence of R_i in $F_i \cup C_i$ such that $|R_i| = \alpha_i$, and $|A_i| \leq |F_i \cup (C_i \backslash R_i)|$. Thus, taking the contraposition, α_i has to be less than or equal to $|Y_i| - |X_i|$. However, $|Y_{i-1}| - |X_{i-1}| > |Y_i| - |X_i|$ due to existence of R_{i-1}. This implies that α_i is strictly monotone decreasing, and there exists $n \in \mathbf{N}$ such that at the n-th iteration, $\alpha_n = 0$.

Also, $V(L_R^n) \cap |X| \neq \emptyset$. So $V(J_R^n) \cap |X|$ is a proper subset of X, and $\kappa_{J_R^n}(u_n, w) < |X|$. By the induction hypothesis, J_R^n has a bipartite matching M_n' of $V(J_R^n) \cap |X|$ such that $\kappa_{J_R^n/M_n'}(u_n, w) = \kappa_{J_R^n}(u_n, w)$. Finally we obtain a bipartite matching $M_n \cup M_n'$ of X in G_s^+, contraction of which does not decrease the s-t connectivity of G. **Q.E.D.**

Proposition 2 *Let $G = (V, E)$ be a graph and s and t be two distinct vertices in G such that $\{s, t\} \notin E \wedge d_G(s) = \kappa_G(s, t)$. Let m and n be any positive*

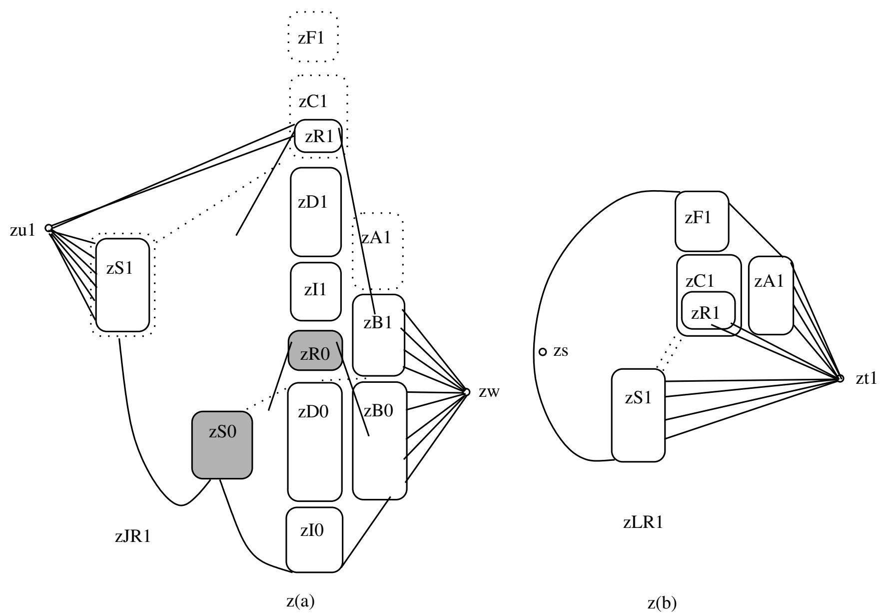

Figure 10: Subsets of J_R^1 and L_R^1.

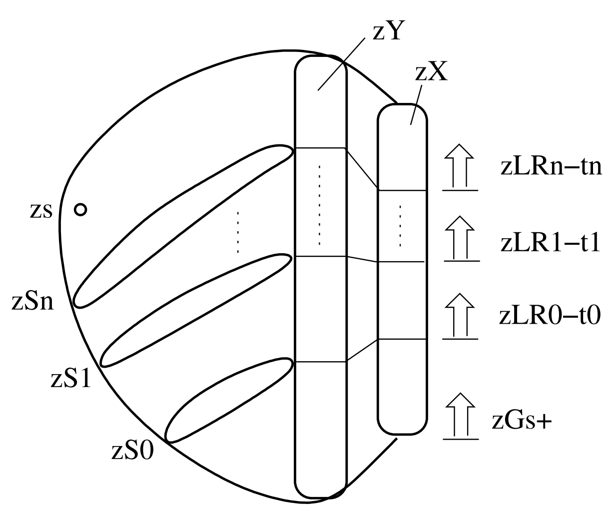

Figure 11: $G_s^+, L_R^0, \ldots, L_R^n$.

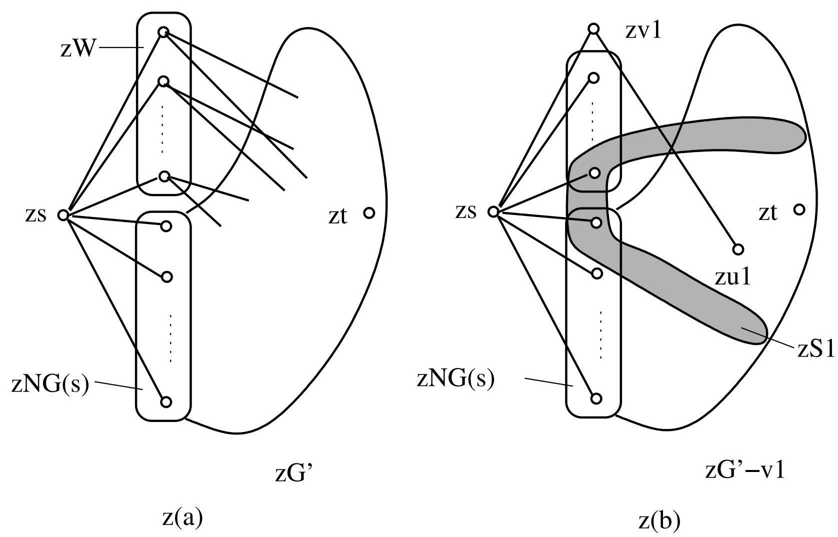

Figure 12: G' and $G' - v_1$.

integers such that $m \geq n$. We add to G m new vertices W, and for each $v \in W$ add an edge $\{s, v\}$. We also add some new edges between W and $V \backslash N_G(s)$, and denote the resultant graph by G'. If $\kappa_{G'}(s,t) \geq \kappa_G(s,t) + n$, then we can find a subset X of W such that $|X| = n$, and $\kappa_{G'[V(G) \cup X]}(s,t) = \kappa_G(s,t) + n$ (Fig. 12(a)).

Proof

This is obvious, but we include a proof for formality. Let $\kappa_G(s,t) = k$, $\kappa_{G'}(s,t) = l$. First, we prove we can remove $k + m - l$ vertices in W without decreasing the s-t connectivity below l.

If $k = l$, then we can remove all the vertices in W so we assume $k < l$. Pick any vertex v_1 in W whose removal decreases the s-t connectivity in G', i.e. $\kappa_{G'}(s,t) - 1 = \kappa_{G'-v_1}(s,t)$. We can find such a vertex in W otherwise $\kappa_G(s,t)$ would be greater than k. Let G^1 be $G' - v_1$, and S_1 be a minimum s-t separator in G^1. Let G^1_t be the component of $G^1 - S_1$ to which t belongs, and G^1_s be $G^1 - G^1_t$. Please note G^1_s contains S_1. As $S_1 \cup \{v_1\}$ form a minimum s-t separator in G', v_1 is adjacent to at least one vertex u_1 in G^1_t (Fig. 12(b)).

If $l - k = 1$, removal of any vertex in $W \backslash \{v_1\}$ does not decrease s-t connectivity of G^1, otherwise $\kappa_G(s,t)$ would be less than k. So we can remove all the vertices in $W \backslash \{v_1\}$ from G' and obtain a graph G^* such that $\kappa_{G^*}(s,t) = l$.

If $l - k > 1$, let G^1_+ be a graph based on G^1_s augmented by a new vertex t_1 and new edges between all the vertices in S_1 and t_1. Please note that $\kappa_{G^1_+}(s,t_1) = l-1$. Pick any vertex v_2 in $W \backslash \{v_1\}$ whose removal decreases the s-t connectivity in G^1_+, i.e. $\kappa_{G^1_+}(s,t_1) - 1 = \kappa_{G^1_+ - v_2}(s,t_1)$. We can find such a vertex in $W \backslash \{v_1\}$ otherwise $\kappa_G(s,t)$ would be greater than k. Let G^2 be $G^1_+ - v_2$, and S_2 be a

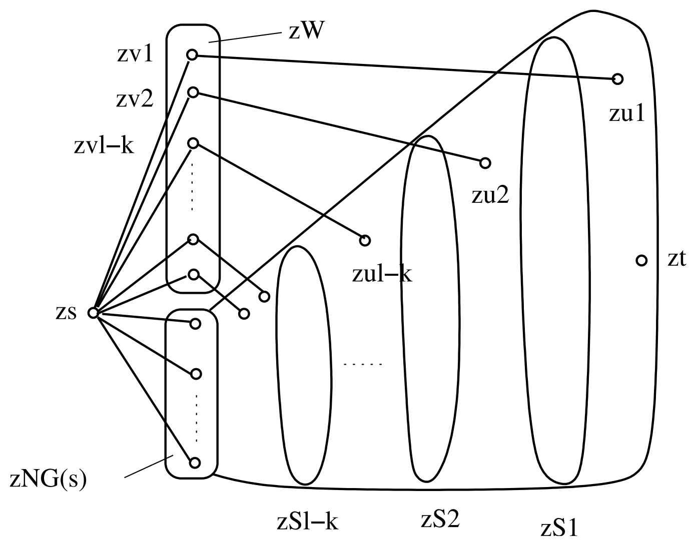

Figure 13: $G^1, \ldots, G^{l-k}$.

minimum s-t_1 separator in G^2. Let G^2_t be the component of $G^2 - S_2$ to which t belongs, and G^2_s be $G^2 - G^2_t$. Please note G^2_s contains S_2. As $S_2 \cup \{v_2\}$ form a minimum s-t_1 separator in G^1_+, v_2 is adjacent to at least one vertex u_2 in G^2_t.

We can continue this process up to $l-k$ times and obtain v_{l-k}, S_{l-k}, and G^{l-k}_+. Now $\kappa_{G^{l-k}_+}(s, t_{l-k}) = l - (l-k) = k$, and removal of any vertex in $W \backslash \{v_1, \ldots, v_{l_k}\}$ does not decrease the s-t_{l-k} connectivity of G^{l-k}_+, otherwise $\kappa_G(s,t)$ would be less than k. So we can remove all the vertices in $W \backslash \{v_1, \ldots, v_{l-k}\}$ from G', and obtain a graph G^* such that $\kappa_{G^*}(s,t) = l$ (Fig. 13).

Let $W' := \{v_1, \ldots, v_{l-k}\}$. Now $N_G(s) \cup W'$ is a minimum s-t separator of G^*. We can pick any n vertices in W' and remove the rest from G^*. The s-t connectivity of the resultant graph is exactly $n+k$. **Q.E.D.**

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