

A Proof of Vertex-disjoint Menger's Theorem by Bipartite Matching and Contraction

SHOICHIRO YAMANISHI

August 12, 2020

Abstract

A proof of vertex-disjoint Menger's theorem between two distinctive vertices s and t in G is proposed. Starting from a minimum separator X and the component G_t of $G - X$ to which t belongs, $|X|$ vertex-disjoint s - X paths are found in $G - V(G_t)$ by recursively applying contraction to bipartite matchings of X . Similarly, $|X|$ vertex-disjoint X - t paths are found. Concatenating two paths at each vertex in X yields $|X|$ vertex-disjoint s - t paths. The contraction of the bipartite matchings must not decrease the connectivity. Existence of such bipartite matchings are proven by induction on $|X|$.

1 Introduction

Menger's theorem is one of the early fundamental discoveries in graph theory. Since the original theorem was proposed by Menger [7], some variants have been proposed, which are roughly divided into vertex-disjoint ones, e.g. Whitney [9], and edge-disjoint one by Ford and Fulkerson [4]. For the vertex-disjoint theorems, several proofs have been proposed by Dirac [3]; Böhme, Göring, and Harant [1]; Pym [8]; and Grünwald (later Gallai) [5]. Also, the edge-disjoint theorem is proven by the min-cut/max-flow theorem [4]. Of all the variants, we prove the following vertex disjoint theorem.

Theorem 1 (Menger's Theorem)

Given a graph $G = (V, E)$, let s and t be two distinct non-adjacent vertices in G . The size of the minimum s - t separators is equal to the maximum number of internally vertex-disjoint s - t paths.

In this article, we call the minimum number of separating vertices between s and t s - t connectivity of G and denote it by $\kappa_G(s, t)$ hereinafter.

The purpose of this article is to propose a new proof by recursively applying contraction of a maximum bipartite matching of a minimum s - t separator in a graph. We also briefly discuss the possibility of finding algorithms for determining s - t connectivity and maximum number of vertex-disjoint s - t paths.

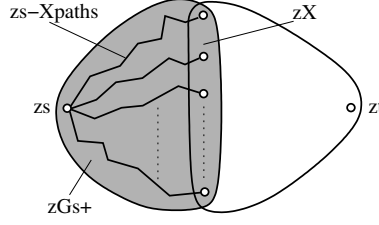


Figure 1: X , G_s^+ , and s - X paths.

It is easy to see the maximum number of disjoint s - t paths does not exceed $\kappa_G(s, t)$. We prove that we can actually construct $\kappa_G(s, t)$ disjoint s - t paths in G . It is also trivial to prove the case for $\kappa_G(s, t) = 0$, so the following discussion assumes $\kappa_G(s, t) > 0$.

2 Proof

We follow the notational conventions in Diestel's [2]. We treat a matching M as a set of edges in a graph G . We denote the graph obtained from G by contracting all the edges in M by G/M , and the set of vertices in G/M into which the edges in M are contracted by V_M .

Let $k := \kappa_G(s, t)$. Let X be a minimum s - t separator ($X \cap \{s, t\} = \emptyset$). Let G_s, G_t be the two components of $G - X$ to which s and t belong respectively. Let G_s^+, G_t^+ be $G[V(G_s) \cup X]$, $G[V(G_t) \cup X]$ respectively. We prove G_s^+ has k vertex-disjoint s - X paths. Similarly, G_t^+ has k vertex-disjoint X - t paths by symmetry. Concatenating the two paths at each vertex in X from each of s - X paths and X - t paths in G yields k vertex-disjoint s - t paths (Fig. 1).

The rest of the proof is for finding k s - X paths in G_s^+ . Our strategy is as follows. Let $G_1 := G$ and $X_1^* := X$. Split X_1^* into two subsets X_1^a and X_1^n such that $X_1^a = \{x \in X_1^* \mid \{s, x\} \in E(G_s^+)\}$ and $X_1^n = \{x \in X_1^* \mid \{s, x\} \notin E(G_s^+)\}$ (Fig. 2(a)). We have already found $|X_1^a|$ vertex-disjoint s - X_1^a paths between s and X_1^a in G_s^+ , each of which is merely an edge. Please observe that X_1^n is a minimum separator of $G_1 - X_1^a$ (Fig. 2(b)). It is easy to see X_1^n is a separator of $G_1 - X_1^a$. Suppose it were not minimum. Let X' be a minimum separator of $G_1 - X_1^a$ such that $|X'| < |X_1^n|$. Then $X' \cup X_1^a$ would give a minimum separator of G_1 , which contradicts the minimality of X_1^* .

We prove there is a bipartite matching M_1^* of X_1^n to $N_{G_s^+ - X_1^a}(X_1^n)$ such that $|M_1^*| = |X_1^n|$, and contracting all the edges in M_1^* does not decrease the s - t connectivity of $(G_1 - X_1^a)/M_1^*$, i.e. $\kappa_{G_1 - X_1^a}(s, t) \leq \kappa_{(G_1 - X_1^a)/M_1^*}(s, t)$ (Fig. 3(a)). We eventually obtain the equality here since $V_{M_1^*} \cup X_1^a$ form an s - t separator in G_1/M_1^* , which indicates they are again a minimum separator X_2^* in G_1/M_1^* and $\kappa_{G_1}(s, t) = \kappa_{G_1/M_1^*}(s, t)$ (Fig. 3(b)). Let $G_2 = G_1/M_1^*$.

We can recursively apply this process of finding a matching and contracting

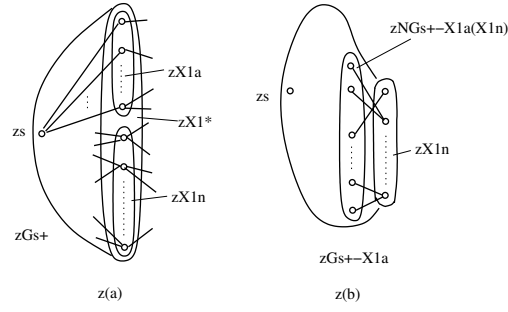


Figure 2: X_1^a , X_1^n , and $G_s^+ - X_a$.

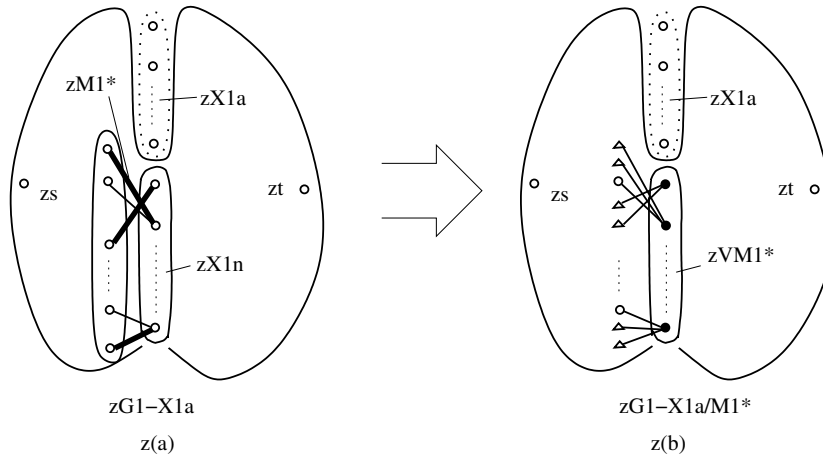


Figure 3: $G_1 - X_1^a$, and $(G_1 - X_1^a)/M_1^*$.

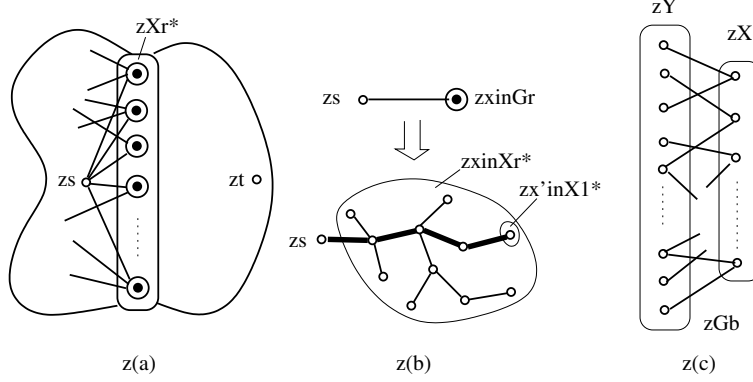


Figure 4: G_r, X_r^* , and G_β .

all the edges in it r times until all the vertices in the minimum s - t separator X_r^* in G_r are adjacent to s (Fig. 4 (a)). This process is guaranteed to terminate as G is finite, and at each iteration at least one edge is contracted. If we “unfold” the edge $\{s, x\}$ and the vertex x in X_r^* , to which some incident edges have been contracted, we obtain $|X|$ trees in G_s^+ , which are mutually vertex-disjoint except at s . In each tree we can find a unique s - x' path for each $x' \in X_1^*$. Those paths form a set of k vertex-disjoint s - X paths (Fig. 4 (b)).

The rest of the proof is dedicated to prove existence of a bipartite matching M of X_1^n to $N_{G_s^+ - X_1^a}(X_1^n)$ such that contraction of all the edges in M does not decrease the s - t connectivity of in $G - X_1^a$. In the following discussion, we assume $X_1^a = \emptyset$, i.e. $X = X_1^n$. If $X_1^a \neq \emptyset$, consider the graph $G - X_1^a$ instead of G for its connectivity $|X_1^n|$. We later add the edges (disjoint paths) in X_1^a , after we find vertex-disjoint s - X_1^n paths.

First we prove existence of a bipartite matching of X . Let $Y := N_{G_s^+}(X)$. Since we assume X_1^a is empty, s is not in Y . Consider the bipartite graph G_β with the bipartition $\{X, Y\}$ (Fig. 4 (c)). Formally $G_\beta = G[X \cup Y] - (E(G[X]) \cup E(G[Y]))$. We can easily check that G_β contains a bipartite matching of X . Indeed, for all $S \subseteq X$, $d_{G_\beta}(S) \geq |S|$. Suppose $\exists S \subseteq X$ such that $d_{G_\beta}(S) < |S|$, then we can replace S with $N_{G_\beta}(S)$ in X , which would be an s - t separator in G whose cardinality is less than $|X|$, which contradicts the minimality of X . By the marriage theorem by Hall [6], G_β contains a bipartite matching of X .

Next we prove existence of a bipartite matching M of X which does not reduce the s - t connectivity when contraction of M is applied to G . We prove this by induction on $\kappa_G(s, t)$, or on $|X|$. If $|X| = 1$, it is trivial to prove contraction of any bipartite matching M does not reduce the s - t connectivity. So the induction starts.

Let $X_0 := X$, $Y_0 := Y$, and M_0 be any bipartite matching of X_0 in N_{G_β} (Fig. 5(a)). If it preserves the s - t connectivity in G/M_0 , we are done. So we

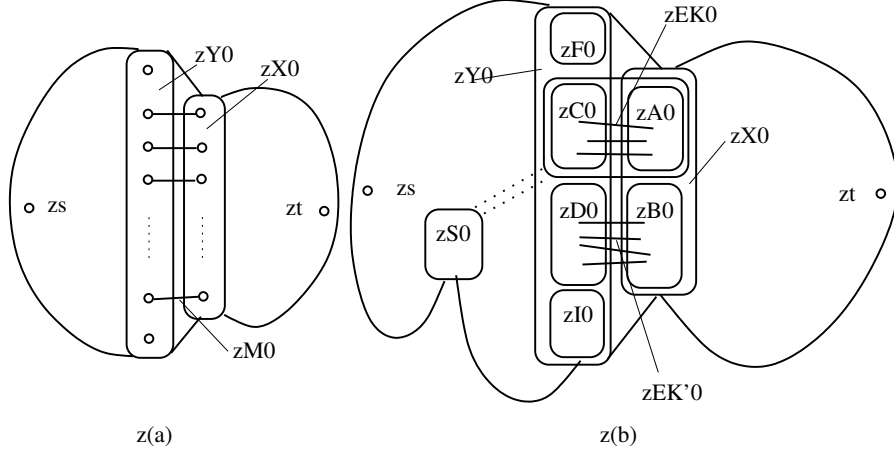


Figure 5: X , Y , and the subsets of V .

assume $\kappa_{G/M_0}(s, t) < \kappa_G(s, t)$. G/M_0 has an s - t separator W_0 in G_s^+ such that $|W_0| < |X_0|$. W_0 contains at least one vertex in V_{M_0} , otherwise W_0 would also be an s - t separator in G whose cardinality is less than $|X|$, a contradiction. So $W_0 \cap V_{M_0} \neq \emptyset$. Let $K_0 = W_0 \cap V_{M_0}$ and $S_0 = W_0 \setminus K_0$.

Here we categorize the edges and vertices in G for the following discussion (Fig. 5(b)).

- Let E_{K_0} be the set of edges in G which corresponds to K_0 in G/M_0 .
- Let $E_{K'_0}$ be $M_0 \setminus E_{K_0}$.
- Let A_0 and C_0 be the sets of vertices in X_0 and Y_0 respectively, which are incident to the edges in E_{K_0} .
- Let B_0 and D_0 be the sets of vertices in X_0 and Y_0 respectively, which are incident to the edges in $E_{K'_0}$.
- Let F_0 be the set of vertices in $Y_0 \setminus (C_0 \cup D_0)$ which are not incident to any vertices in B_0 .
- Let I_0 be the set of vertices in $Y_0 \setminus (C_0 \cup D_0)$ which are incident to any vertices in B_0 .

Please observe that $A_0 \cup C_0 \cup S_0$ separates s from t in G , since $K_0 \cup S_0$ separates s from t in G/M_0 . Please also note that S_0 can be \emptyset , but neither B_0 nor D_0 can be empty.

Here we further define six subgraphs of G and one new graph as follows.

- Let H_s^0 and H_t^0 be the two components of $G - (A_0 \cup C_0 \cup S_0)$ to which s and t belong respectively.

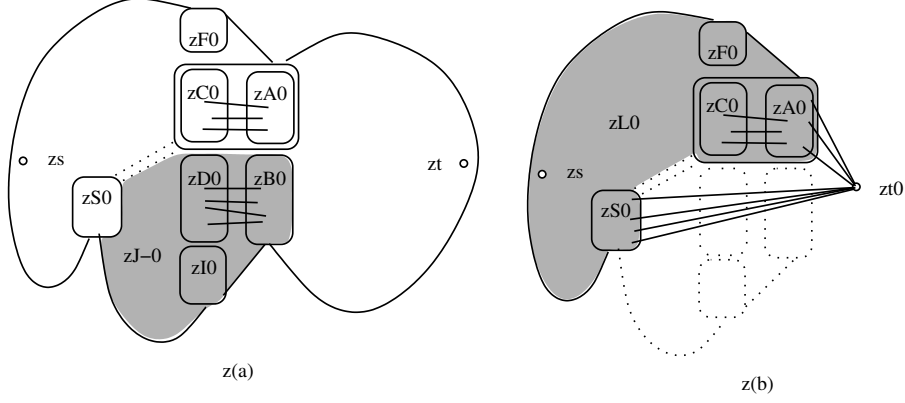


Figure 6: J_-^0 and L^0 .

- Let H_{s+}^0, H_{t+}^0 be $G[V(H_s^0) \cup A_0 \cup C_0 \cup S_0]$, $G[V(H_t^0) \cup A_0 \cup C_0 \cup S_0]$ respectively.
- Let J_-^0 be a subgraph of G induced by $V(H_t^0) \cap V(G_s^+)$ (Fig. 6(a)).
- Let J_+^0 be a subgraph of G induced by $(V(H_t^0) \cap V(G_s^+)) \cup S_0$, i.e. $J_+^0 = G[V(J_-^0) \cup S_0]$.
- Let L^0 be a graph based on $G_s^+ - J_-^0$ augmented by a new vertex t_0 , and new edges between t_0 and all the vertices in $A_0 \cup S_0$ (Fig. 6(b)).

Please note H_s^0 is a subgraph of G_s^+ , since $W_0 \subseteq V(G_s^+/M_0)$ and $M_0 \subseteq E(G_s^+)$. Also please note that F_0 belongs to L^0 and I_0 belongs to J_-^0 .

As $A_0 \cup S_0$ does not separate s from t in G , there are some vertices in C_0 which are incident to edges into J_-^0 .

- Let $Q_0 := \{q \in C_0 | q \text{ is adjacent to at least one vertex in } J_-^0\}$.
- Let $Q'_0 := \{q' \in V(J_-^0) | q' \text{ is adjacent to at least one vertex in } Q_0\}$.

From the s - t connectivity of G , we claim $|Q_0| \geq |B_0| - |S_0|$, and $|Q'_0| \geq |B_0| - |S_0|$. Please observe that both $S_0 \cup Q_0 \cup A_0$ and $S_0 \cup Q'_0 \cup A_0$ form s - t separators in G (Fig. 7(a)).

For the induction process, we consider two graphs L_R^n and J_R^n as follows, where n indicates a number of iterations. First, let J_Q^0 be a graph based on the subgraph of G induced by $V(J_+^0) \cup Q_0$, and augmented by two new vertices u_0 and w , and new edges between u_0 and all the vertices in $S_0 \cup Q_0$, and new edges between all the vertices in B_0 and w (Fig. 7(b)). We claim $\kappa_{J_Q^0}(u_0, w) \geq |B_0|$. Suppose not. Then there would be a minimum u_0 - w separator U_0 such that $|U_0| < |B_0|$. However, $U_0 \cup A_0$ would form an s - t separator in G , which

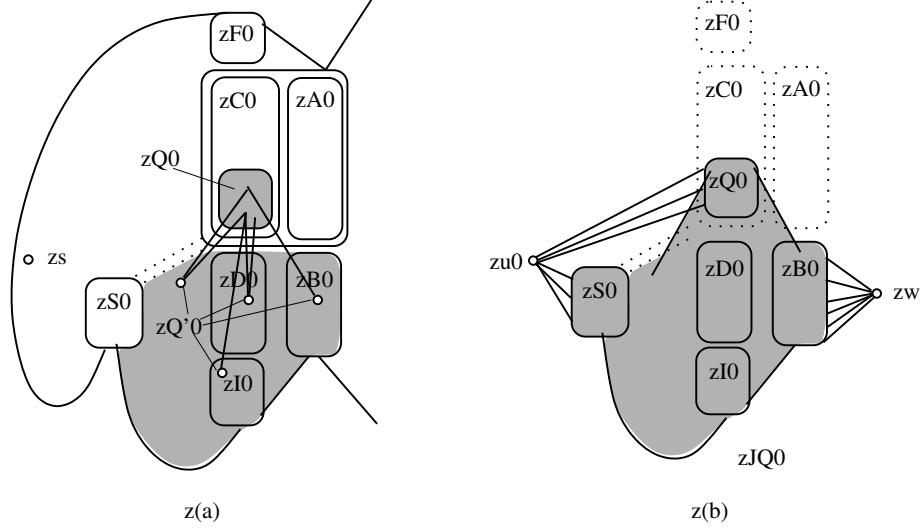


Figure 7: Q_0 , Q'_0 , and J_Q^0 .

contradicts the minimality of X . Please note that there is no edge between $L^0 - (S_0 \cup A_0)$ and J_-^0 except $E(Q_0, Q'_0)$.

By Proposition ??, we can find at least one subset R_0 of Q_0 such that $|R_0| = |B_0| - |S_0|$ and $\kappa_{J_Q^0 - (Q_0 \setminus R_0)}(u_0, w) = |B_0|$. Let the family of such sets be \mathcal{R}_0 , and pick any set R_0 in \mathcal{R}_0 . Let $J_R^0 := J_Q^0 - (Q_0 \setminus R_0)$, and let L_R^0 be a graph based on L^0 augmented by new edges between all the vertices in R_0 and t_0 (Fig. 8). It is easy to see $\kappa_{L_R^0}(s, t_0) \geq |X|$, since existence of an s - t_0 separator whose cardinality is less than $|X|$ in L_R^0 implies that such a separator would also be an s - t separator in G . Also, that $A_0 \cup R_0 \cup S_0$ form an s - t_0 separator implies $\kappa_{L_R^0}(s, t_0) = |X|$, and $A_0 \cup R_0 \cup S_0$ form a minimum s - t_0 separator.

Let $X_1 := A_0$ and $Y_1 := F_0 \cup (C_0 \setminus R_0)$. Let M_1 be any bipartite matching of the bipartition $\{Y_1, X_1\}$. Existence of the matching is proven using the Hall's marriage theorem similar to the proof of existence of M_1 .

If $\kappa_{L_R^0/M_1}(s, t_0) \geq |X|$, we set $L_R^n := L_R^0$, $J_R^n := J_R^0$, and we are done. So we assume $\kappa_{L_R^0/M_1}(s, t_0) < |X|$.

L_R^0/M_1 has an s - t_0 separator W_1 such that $|W_1| < |X|$. W_1 contains at least one vertex in V_{M_1} , otherwise W_1 would also be an s - t separator in G whose cardinality is less than $|X|$, a contradiction. So $W_1 \cap V_{M_1} \neq \emptyset$. Let $K_1 = W_1 \cap V_{M_1}$ and $S_1 = W_1 \setminus K_1$.

Here we categorize the edges and vertices in L_R^0 according to W_1 similar to the discussion above when we found W_0 (Fig 9(a)).

- Let E_{K_1} be the set of edges in L_R^0 which corresponds to K_1 in L_R^0/M_1 .
- Let $E_{K'_1}$ be $M_1 \setminus E_{K_1}$.

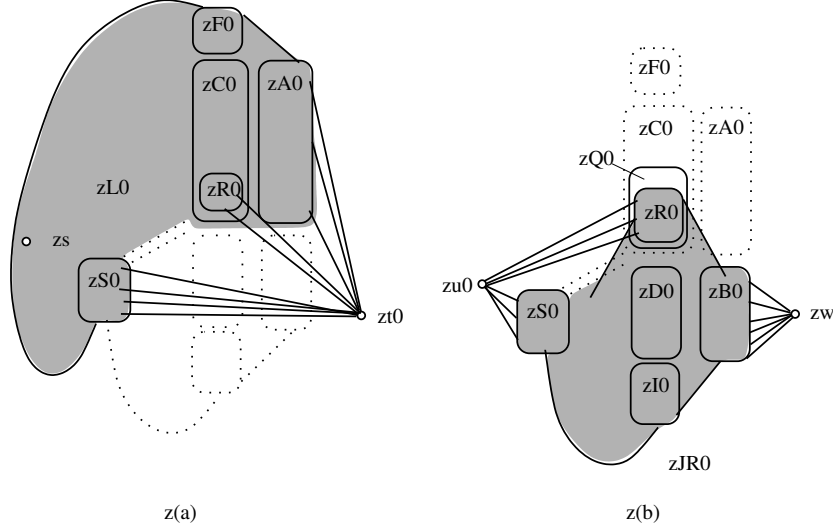


Figure 8: J_R^0 and L_R^0 .

- Let A_1 and C_1 be the sets of vertices in X_1 and Y_1 respectively, which are incident to the edges in E_{K_1} .
- Let B_1 and D_1 be the sets of vertices in X_1 and Y_1 respectively, which are incident to the edges in $E_{K'_1}$.
- Let F_1 be the set of vertices in $Y_1 \setminus (C_1 \cup D_1)$ which are not incident to any vertices in B_1 .
- Let I_1 be the set of vertices in $Y_1 \setminus (C_1 \cup D_1)$ which are incident to any vertices in B_1 .

Please observe that $A_1 \cup C_1 \cup S_1$ separates s from t_0 in L_R^0 , since $K_1 \cup S_1$ separates s from t_0 in L_R^0/M_1 . Please also note that S_1 can be \emptyset , but neither B_1 nor D_1 can be empty.

Here we further define two subgraphs of L_R^0 and one new graph as follows.

- Let J_-^1 be the component of $L_R^0 - (A_1 \cup C_1 \cup S_1)$ to which t_0 belongs.
- Let J_+^1 be a subgraph of L_R^0 induced by $V(J_-^1) \cup S_1$.
- Let L^1 be a graph based on $L_R^0 - J_-^1$ augmented by a new vertex t_1 , and new edges between t_1 and all the vertices in $A_1 \cup S_1$ (Fig 9(b)).

Also please note that F_1 belongs to L^1 and I_1 belongs to J_-^1 .

As $A_1 \cup S_1$ does not separate s from t_0 in L_R^0 , there are some vertices in C_1 which are incident to edges into J_-^1 .

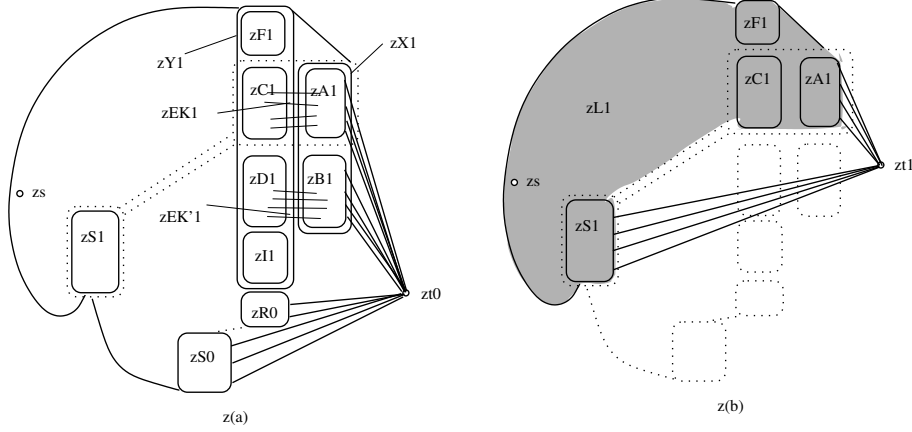


Figure 9: Subsets of L_R^0 and L^1 .

- Let $Q_1 := \{q \in C_1 \mid q \text{ is adjacent to at least one vertex in } J_-^1\}$.
- Let $Q'_1 := \{q' \in V(J_-^1) \mid q' \text{ is adjacent to at least one vertex in } Q_1\}$.

From the s - t_0 connectivity of L_R^0 , we claim $|Q_1| \geq |B_1| + |R_0| + |S_0| - |S_1|$, and $|Q'_1| \geq |B_1| + |R_0| + |S_0| - |S_1|$. Please observe that both $S_1 \cup Q_1 \cup A_1$ and $S_1 \cup Q'_1 \cup A_1$ form s - t_0 separators in L_R^0 .

Let J_Q^1 be a graph based on the subgraph of G induced by $V(J_+^1) \cup Q_1$, and augmented by two new vertices u_1 and w_1 , and new edges between u_1 and all the vertices in $S_1 \cup Q_1$, and new edges between all the vertices in $B_1 \cup R_0 \cup S_0$ and w_1 . We claim $\kappa_{J_Q^1}(u_1, w_1) \geq |B_0| + |B_1|$ similar to the reasoning for $\kappa_{J_Q^0}(u_0, w) \geq |B_0|$. Please note that $|B_1| = |R_0| + |S_0|$. By Proposition ??, we can find at least one subset R_1 of Q_1 such that $|R_1| = |B_0| + |B_1| - |S_1|$ and $\kappa_{J_Q^1 - (Q_1 \setminus R_1)}(u_1, w_1) = |B_0| + |B_1|$. Let the family of such sets be \mathcal{R}_1 and pick any R_1 in \mathcal{R}_1 . Let J_R^1 be a graph based on $J_R^0 - u_0$ and $J_Q^1 - (Q_1 \setminus R_1) - w_1$, pasted along $S_0 \cup R_0$, and add new edges between B_1 and w (Fig 10(a)). Please note that $J_R^1 - \{u_1, w\}$ is an induced subgraph of G .

Let L_R^1 be a graph based on L^1 augmented by new edges between all the vertices in R_1 and t_1 (Fig 10(b)). It is easy to see $\kappa_{L_R^1}(s, t_1) = |X|$ due to the similar reasoning for $\kappa_{L_R^0}(s, t_0) = |X|$, and $A_1 \cup R_1 \cup S_1$ form a minimum s - t_1 separator in L_R^1 .

We repeat this process n times until we obtain L_R^n, J_R^n , such that there is a bipartite matching M_n for L_R^n such that $\kappa_{L_R^n/M_n}(s, t_n) = |X|$ (Fig 11).

This process is guaranteed to terminate at the n -th iteration due to the following reason. Assume we are at the i -th iteration. Let $\alpha_i = \kappa_{L_R^i}(s, t_i) - \kappa_{L_R^i/M_i}(s, t_i)$. The discussion above implies the necessary condition for α_i to be any positive integer is existence of R_i in $F_i \cup C_i$ such that $|R_i| = \alpha_i$, and

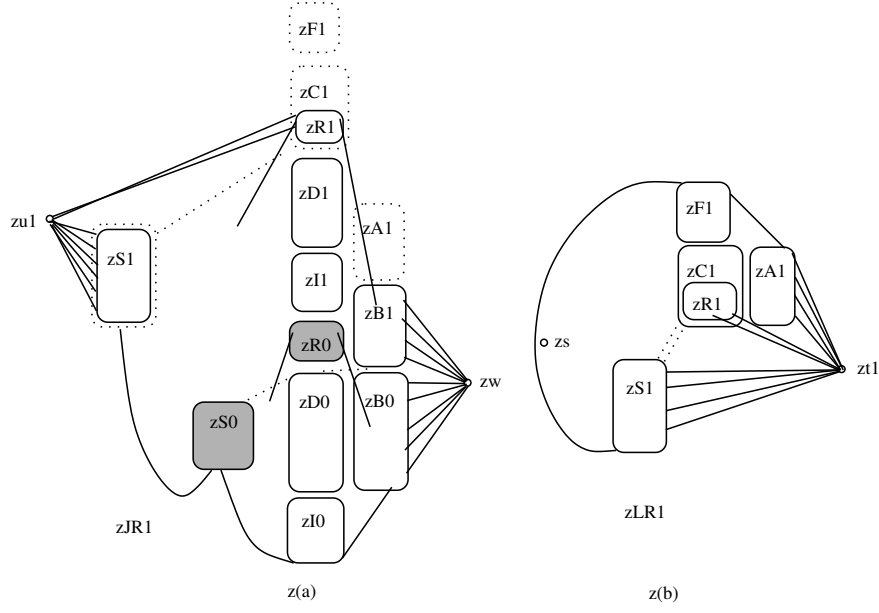


Figure 10: Subsets of J_R^1 and L_R^1 .

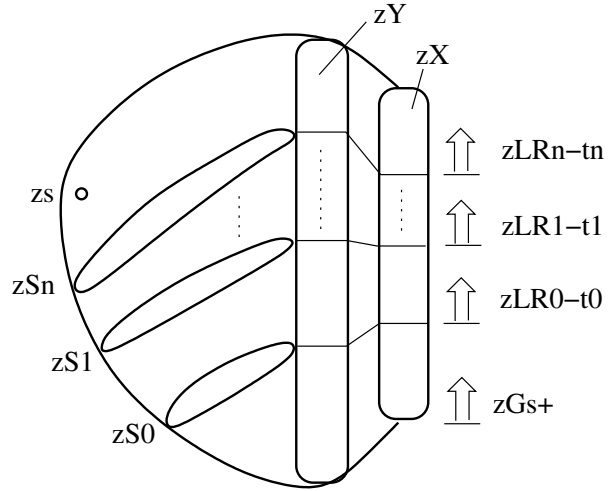


Figure 11: $G_s^+, L_R^0, \dots, L_R^n$.

$|A_i| \leq |F_i \cup (C_i \setminus R_i)|$. Thus, taking the contraposition, α_i has to be less than or equal to $|Y_i| - |X_i|$. However, $|Y_{i-1}| - |X_{i-1}| > |Y_i| - |X_i|$ due to existence of R_{i-1} . This implies that α_i is strictly monotone decreasing, and there exists $n \in \mathbf{N}$ such that at the n -th iteration, $\alpha_n = 0$.

Also, $V(L_R^n) \cap |X| \neq \emptyset$. So $V(J_R^n) \cap |X|$ is a proper subset of X , and $\kappa_{J_R^n}(u_n, w) < |X|$. By the induction hypothesis, J_R^n has a bipartite matching M'_n of $V(J_R^n) \cap |X|$ such that $\kappa_{J_R^n/M'_n}(u_n, w) = \kappa_{J_R^n}(u_n, w)$. Finally we obtain a bipartite matching $M_n \cup M'_n$ of X in G_s^+ , contraction of which does not decrease the s - t connectivity of G . **Q.E.D.**

References

- [1] T. Böhme, F. Göring, and J. Harant. Menger's theorem. *Journal of Graph Theory*, 37:35–36, 2001.
- [2] R. Diestel. *Graph Theory*. Springer-Verlag, 1997.
- [3] G.A. Dirac. Short proof of menger's graph theorem. *Mathematika*, 13:42–44, 1966.
- [4] L.R. Ford and D.R. Fulkerson. Maximal flow through a network. *Canad. J. Math.*, 8:399–404, 1956.
- [5] T. Grünwald. Ein neuer beweis eines mengerschen satzes. *J. London Math. Soc.*, 13:188–192, 1938.
- [6] P. Hall. On representations of subsets. *J. London Math. Soc.*, 10:26–30, 1935.
- [7] K. Menger. Zur allgemeinen kurventheorie. *Fund. Math.*, 10:96–115, 1927.
- [8] J.S. Pym. A proof of menger's theorem. *Monashefte Math*, 73:81–88, 1969.
- [9] H. Whitney. Congruent graphs and the connectivity of graphs. *Amer. J. Math.*, 54:150–168, 1932.