Enhancing PQ-tree Planarity Algorithms for non-adjacent s and t

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Abstract

The PQ-tree based planarity testing algorithm presented by Booth and Lueker in [1] requires an st-ordering where $\{s, t\}$ must exist. We propose an enhancement to the algorithm that permits st-ordering of any vertex pair. The enhancement is made by introducing a type of circular consecutiveness of pertinent leaves denoted by *complementarily partial*, where the pertinent leaves can be consecutively arranged only at both ends of the frontier with one or more non-pertinent leaves in between. The implementation is enhanced with 4 additional templates P7, P8, Q4, and Q5. The correctness of the new algorithm is given following the proof in [2] and [7] on the equivalence of PQ-tree and its new reduction operations to the corresponding bush form. The complexity of the new algorithm stays $O(|N|)$.

1 Introduction

PQ-tree data structure and the reduction algorithm were first proposed by Booth and Lueker [1] in 1976. PQ-tree is a rooted ordered tree with three node types: L-node represents a leaf with no children, P-node permits any permutation of its children, Q-node has an ordering on its children but it can be reversed. PQ-tree is used to represent a set of permissible permutations of elements in S, and the reduction operation tries to find a consecutive arrangement of subset U of S. It has many applications including the planarity testing of an undirected biconnected graph $G(V, E)$. Booth and Lueker proposed a planarity testing algorithm in their original paper [1]. It has $O(|N|)$ time complexity, and it depends on a particular ordering on graph vertices called st-ordering, where the two terminal nodes s and t must be adjacent as in $\{s, t\}$. An st-ordering can be found in $O(|N|)$ time with an algorithm such as [10], assuming $|E| \leq 3|V| - 6$.

The algorithm falls in a category called vertex addition. Each graph vertex v is processed in an iteration according to the st-ordering. Conceptually each graph vertex is added to the bush form [7],[2], and the PQ-tree evolves reflecting the state of the bush form. Each iteration of the algorithm transforms the PQ-tree for v and its incident edges. In the beginning of an iteration, the tree leaves that correspond to the incoming graph edges incident to v, or *pertinent leaves*, are gathered consecutively in the tree by transforming the tree in a series of template applications. The minimum connected subtree for all the pertinent leaves, or *pertinent tree*, is removed from the tree, and then new leaves that corresponds to the outgoing graph edges incident to v are added to the tree node that was the pertinent tree root. The st-ordering ensures there is at least one incoming graph edge and one out going graph edge at each iteration except for the first and the last. At the first iteration the leaves for the outgoing graph edges of the first graph vertex in the st-ordering are attached to the initial P-node, and the PQ-tree evolves form there over the iterations. At an iteration, if the pertinent leaves can not be arranged consecutively, the algorithm aborts and declares it non-planar. At the second to last iteration, if all the pertinent leaves are consecutively arranged, the graph is declared planar.

The PQ-tree itself is an elegant data structure to represent a set of permissible permutations of elements, but its reduction algorithm involves 10 rather cumbersome tree transformation operations called templates, though each of them is straightforward and intuitive to understand. Since the original algorithm was published, many graph planarity-related algorithms have been proposed, but some of them were later proven to have some issues in them [8] [4] [6]. For example, Jünger, Leipert, and Mutzel [5] discuss the pitfalls and difficulties of using PQ-trees for maximal planarization graphs. It seems to take a very careful attention when we apply PQ-tree to graph planarity algorithms despite its apparent straight-forwardness and intuitiveness.

Another data structure called PC-tree was proposed by Shih and Hsu [9]. It is a rootless tree to express permissible 'circular' permutations. Hsu [9] proposes a planarity test algorithm using PC-tree with iterative vertex addition along a DFS exploration of graph vertices. Hsu [3] compares PQ-tree and PC-tree in terms of planarity testing. Hsu[3] mentions testing 'circular ones' property with PC-tree and 'consecutive ones' property with PQ-tree.

In this paper, we will expand the PQ-tree planarity algorithm for any (not necessarily adjacent) s,t pair. A non-adjacent st-ordering will introduce a new type of consecutiveness through the course of the algorithm on the PQ-tree. It is a type of 'circular' permutation. We define the type of circular permutations that the original algorithm can not handle as *complementarily partial.* We show the insufficiency of the set of the original templates in Booth and Lueker [1] with a specific example, and propose 4 new templates to handle complementarily partial nodes. Then we prove the correctness of the new algorithm following Lempel, Even, and Cederbaum [7] and Even [2] in equivalence to the corresponding bush form. We show the time complexity stays $O(|N|)$. We then discuss some implementation details. The algorithm mainly consists of two parts called BUBBLE() and REDUCE() in [1]. We show that no change is required for BUBBLE(), but the 4 new templates have to be added to REDUCE().

PQ-tree and its reduction technique are also used for some planarization algorithms such as [8], [4], and [6]. In those algorithms, the costs are calculated per tree node, and some tree leaves that have associated graph edges are removed based on the costs to maintain planarity. The costs are basically the number of descendant tree leaves that would have to be removed to make the tree node of certain pertinent and non-pertinent type. We briefly propose an improvement on the cost values. Without the improvement, some graph edges can be removed unnecessarily as a consequence of lack of handling the *complementarily partial* nodes defined below.

2 Circular Consecutiveness

The insufficiency of the existing algorithm is best shown by a real example. Please see figure 1 and 2. They are taken from a snapshot of the original algorithm for the graph and the st-numbering shown in Appendix A. Figure 1 is the PQ-tree at the 23rd iteration after BUBBLE(), and figure 2 is the corresponding bush form. In this iteration the graph vertex 2 is to be added. The pertinent leaves that correspond to $\{8, 2\}$,$\{3, 2\}$, and $\{7, 2\}$ are about to be consecutively arranged in REDUCE(). Those edges can be consecutively arranged in a circular manner. However, the reduction fails at Q-node $Q4$ as none of the templates Q1,Q2, or Q3 can handle this arrangement. In $Q4$, the pertinent leaves can only be arranged consecutively at both ends with one or more non-pertinent leaves in between.

As shown above, the original algorithm is not capable of handling this type of arrangements of the pertinent leaves with pertinent leaves at both ends. This is a result of using a rooted tree structure to handle circular consecutiveness around the outer face of the corresponding bush form. Once a Q-node has formed in the PQ-tree, at a later iteration if there is a pertinent node of complementarily partial, there will be no way to arrange the pertinent nodes consecutively using the original set of templates, even if the corresponding bush form permits circular consecutiveness.

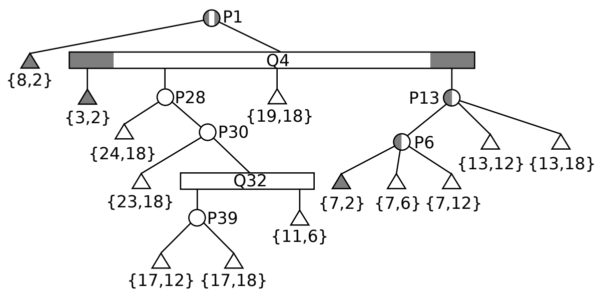


Figure 1: PQ-tree

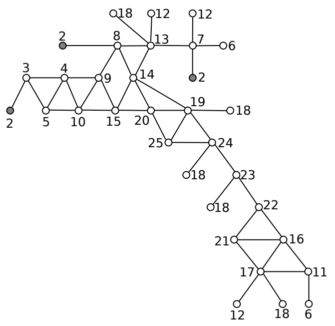


Figure 2: Bush form

3 Enhancement with New Templates

In the previous section we showed a type of node arrangement in PQ-tree that the original algorithm can not handle. In this section we first formulate this condition by defining a new pertinent node type *complementarily partial*, and then introduce 4 new templates. Then we discuss other changes required in BUBBLE() and REDUCE().

Definition A *P-node* is *complementarily partial*, if either of the following holds:

1. It is not a pertinent root, and it satisfies the condition for template $P6$ on the arrangement of child nodes, i.e., if there are exactly two singly partial children. (Figure 3)

2. There is exactly one complementarily partial child, and all the children are full. (Figure 4)

Definition A *Q-node* is *complementarily partial*, if either of the following holds:

1. The children are arranged as the complement of permissible arrangement for template $Q3$, i.e., if the descendant pertinent leaves can be arranged consecutively at both ends with one or more non-pertinent leaves in between. (Figure 5)

2. There is exactly one complementarily partial child, and all the children are full (Figure 6)

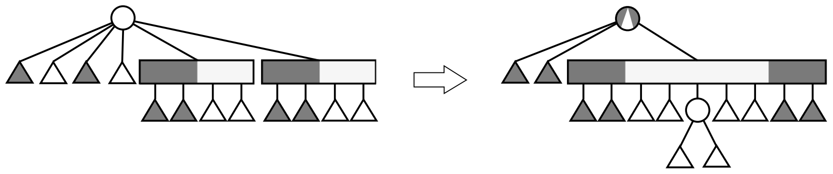


Figure 3: P-node Condition 1, Template P7

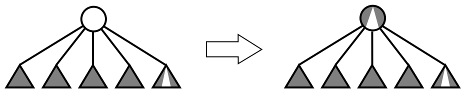


Figure 4: P-node Condition 2, Template P8

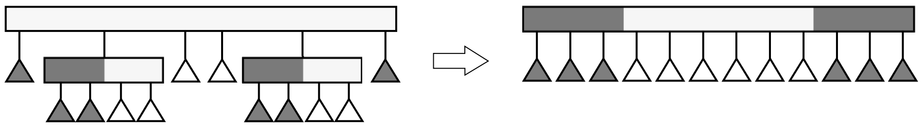


Figure 5: Q-node Condition 1, Template Q4

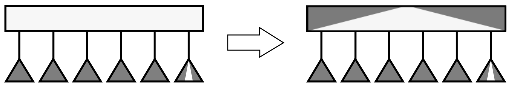


Figure 6: Q-node Condition 2, Template Q5

The first condition for Q-node is formerly defined using a regular expression for the children as:

$$F + ((S(D?|E + D))|(E + D))|SE * D$$

where F denotes a full child, S a singly partial child, E a non-pertinent child, and $D := ((F|S)F*)$ for notational convenience.

First we show there is no need to change BUBBLE() to handle the complementarily consecutive cases. If the PQ-tree permits complementarily partial arrangement, then the tree node will be the pertinent tree node. During BUBBLE(), the parent of all the pertinent nodes will be eventually found, and there will be no need for a surrogate parent, or *pseudo node* in [1].

Next, we introduce 4 new templates for each of 4 conditions shown in the definitions above. Basically Template P7 is a complement version of Template P6, and Template Q4 is a complementary version of Q3. Template P8 and Q5 are for trivial recursive cases.

Finally we show the updated REDUCE().

4 Correctness of the New Algorithm

We prove the correctness following the series of lemmas, theorems, and a corollary given in Section 8.4 of [2] by Even. In their book, the equivalence between the transformations on the bush form and the reductions on PQ-tree is left for the readers on pp 190. We fill the missing part with the following proof.

In a similar case, Hsu [3] tries to prove the equivalence between PQ-tree and PC-tree in its Theorem 5, but it is not sufficient. It proves the equivalence from PQ-tree to PC-tree for each of the templates. However the sufficiency of the PQ-tree templates for all the possible PC-tree transformations is not given.

The proof presented here is relatively long involving many notations and concepts. However, it may be a useful guide to study the details of the PQ-tree behavior.

Figure 7 shows the overview of the proof. It proves the equivalence between bush form with the operations on it and the corresponding PQ-tree with the operations on it. The proof uses two intermediate representations: Marked bush form and its underlying rooted embedded bc-tree. A marked bush form is based on a bush form by placing a root marker on a cut vertex or an edge on the outer face of a block. Such a marker splits the circular arrangement of virtual edges around the bush form into one of three types of a linear consecutive arrangement with two designated end points at the marker. The root marker also introduces the root-to-descendants orientation to the bush form.

We prove in Lemma 1 that a circular consecutive arrangement of virtual edges by arbitrary reorderings of incident edges around cut vertices and flippings of blocks in a bush form is equivalent to a linear consecutive arrangement by reordering of cut vertices and flipping of blocks from the leave components toward the root. In the proof we also show that the incident components around each cut vertex or a block will be arranged in one of 5 types of orderings.

Algorithm 1 Template P7

```
procedure TEMPLATEP7(X: reference to a node object)
    if $X.type \neq P$ then return false
    if Number of singly partial children $\neq 2$ then return false
    $newX \leftarrow$ CreateNewPNode()
    Move all the full children of $X$ to $newX$
    $C_1 \leftarrow$ Singly partial child$_1$
    $C_2 \leftarrow$ Singly partial child$_2$
    Remove links of $C_1$ and $C_2$ from $X$
    ▷ At this point $X$ contains zero or more empty children only.
    Save the location of $X$ in the PQ-tree to $L_X$
    Unlink $X$ from the PQ-tree
    if Number of empty children of $X > 1$ then
        Put $X$ to the empty side of sibling list of $C_1$
    else if Number of empty children of $X = 1$ then
        Put the empty child to the empty side of sibling list of $C_1$
        Discard $X$
    else
        Discard $X$
    Concatenate the children list of $C_2$ to $C_1$'s on the empty sides
    Discard $C_2$
    $C_1.pertinentType \leftarrow ComplementarilyPartial$
    if Number of full children of $newX \geq 1$ then
        Put $C_1$ under $newX$
        Link $newX$ at $L_X$ in the PQ-tree
        $newX.pertinentType \leftarrow ComplementarilyPartial$
    else
        Link $C_1$ at $L_X$ in the PQ-tree
    return true
```

Algorithm 2 Template P8

```
procedure TEMPLATEP8(X: reference to a node object)
    if $X.type \neq P$ then return false
    $|F| \leftarrow$ Number of full children of $X$
    $|C| \leftarrow$ Number of children of $X$
    if $|F| + 1 \neq |C|$ then return false
    $C_{cp} \leftarrow$the non-full child of $X$
    if $C_{cp}.pertinentType \neq ComplementarilyPartial$ then return false
    $X.pertinentType \leftarrow ComplementarilyPartial$
    return true
```

Algorithm 3 Template Q4

```
1: procedure TemplateQ4(X: reference to a node object)
2:     if $X.type \neq Q$ then return false
3:     if Children of $X$ are not ordered according to the condition for Q4 then
   return false
4:     for each $C_{sp}$ of singly partial children do
5:         Flatten $C_{sp}$ into $X$ such that the full side of the children list of $C_{sp}$
   is concatenated to the full immediate sibling of $C_{sp}$
6:     $X.pertinentType \leftarrow ComplementarilyPartial$
7:     return true
```

Algorithm 4 Template Q5

```
1: procedure TemplateQ5(X: reference to a node object)
2:     if $X.type \neq Q$ then return false
3:     $|F| \leftarrow$ Number of full children of $X$
4:     $|C| \leftarrow$ Number of children of $X$
5:     if $|F| + 1 \neq |C|$ then return false
6:     ▷ The check above can be made wihtout calculating $|C|$
7:     $C_{cp} \leftarrow$the non-full child of $X$
8:     if $C_{cp}.pertinentType \neq ComplementarilyPartial$ then return false
9:     $X.pertinentType \leftarrow ComplementarilyPartial$
10:    return true
```

Algorithm 5 REDUCE

```
 1: procedure REDUCE(T, S)
 2:     ▷ $PERTINENT\_LEAF\_COUNT$ is shortened to $PLC$.
 3:     ▷ $PERTINENT\_CHILD\_COUNT$ is shortened to $PCC$.
 4:     $QUEUE \leftarrow$ empty list
 5:     for each leaf $X \in S$ do
 6:         place $X$ to the back of $QUEUE$
 7:         $X.PLC \leftarrow 1$
 8:     while $|QUEUE| > 0$ do
 9:         remove $X$ from the front of $QUEUE$
10:         if $X.PLC < |S|$ then                                  ▷ $X$ is not $ROOT(T, S)$
11:             $Y \leftarrow X.PARENT$
12:             $X.PLC \leftarrow X.PLC + Y.PLC$
13:             $X.PCC \leftarrow X.PCC - 1$
14:             if $X.PCC = 0$ then
15:                 place $Y$ to the back of $QUEUE$
16:             if not TEMPLATE_L1(X) then
17:                 if not TEMPLATE_P1(X) then
18:                     if not TEMPLATE_P3(X) then
19:                         if not TEMPLATE_P5(X) then
20:                             if not TEMPLATE_P7(X) then
21:                                 if not TEMPLATE_P8(X) then
22:                                     if not TEMPLATE_Q1(X) then
23:                                         if not TEMPLATE_Q2(X) then
24:                                             if not TEMPLATE_Q4(X) then
25:                                                 if not TEMPLATE_Q5(X) then
26:                                                     $T \leftarrow T(\emptyset, \emptyset)$
27:                                                     exit from do
28:         else                                                   ▷ $X$ is not $ROOT(T, S)$
29:             if not TEMPLATE_L1(X) then
30:                 if not TEMPLATE_P2(X) then
31:                     if not TEMPLATE_P4(X) then
32:                         if not TEMPLATE_P6(X) then
33:                             if not TEMPLATE_P8(X) then
34:                                 if not TEMPLATE_Q1(X) then
35:                                     if not TEMPLATE_Q2(X) then
36:                                         if not TEMPLATE_Q3(X) then
37:                                             if not TEMPLATE_Q4(X) then
38:                                                 if not EMPLATE_Q5(X) then
39:                                                     $T \leftarrow T(\emptyset, \emptyset)$
40:                                                     exit from do
        return T
```

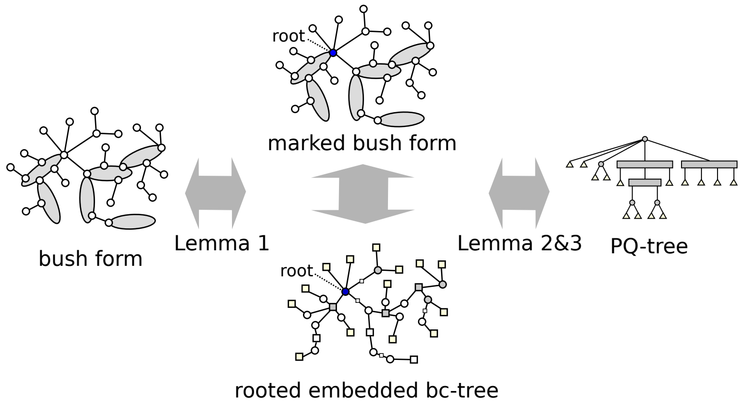


Figure 7: Overview of the Proof of the Correctness of the Proposed Algorithm

We then introduce the underlying block-cut tree of the marked bush form, called rooted embedded bc-tree, and prove equivalence between the rooted embedded bc-tree and the PQ-tree with their operations in Lemma 2 and 3.

First, we introduce some concepts, operations, and notations required for the following discussions.

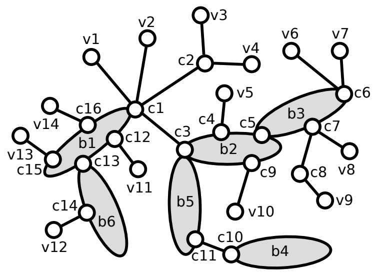


Figure 8: A Bush Form

Definition Type of nodes and edges along the outer face of a bush form

- ***virtual node*** of a bush form is a node of degree 1 that represents a copy of a node in the original graph [2]. In figure 8, $v1...v14$ are virtual nodes.
- ***virtual edge*** of a bush form is an edge incident to a virtual node.
- ***pertinent virtual node*** is a virtual node that corresponds to a pertinent leaf in PQ-tree. I.e., a virtual node to be merged.
- ***pertinent virtual edge*** is a virtual edge incident to a pertinent virtual node.

Definition Operations on a bush form and a marked bush form.

- ***attach*** is an operation to attach new virtual edges to a vertex v in the bush form. As a result it makes v a cut vertex in the bush form.
- ***reorder*** of a cut vertex in a bush form is the operation to rearrange the circular ordering of the incident edges around the cut vertex.
- ***flip*** of a block in a bush form is reversing the combinatorial embedding of the block. As a result, the ordering the outer face of the block is reversed.
- ***merge*** is an operation to merge virtual nodes into single real node in the bush form. As a result a new block forms in the bush form.

Definition If a bush form is decomposed to maximal connected components by removing a cut vertex c, or a block B, an ***active component*** of a cut vertex c or B is a maximal connected component incident to c or B that has at least one virtual node in it. In figure 8, $c3$ has three maximally connected components. The component that includes $b5$, $c11$, $c10$, and $b4$ is not an active component. The other two are.

Definition An ***orienting*** cut vertex or a block is a cut vertex or a block in the bush form that has at least 3 incident active components if the corresponding node in the PQ-tree is not the root. If the corresponding PQ-tree is the root, then it is a cut vertex or a block that has at least 2 incident active components. Such a correspondence is proved in Lemma 3. In figure 8, assuming $c1$ corresponds to the root of the PQ-tree, $c1$, $b1$, and $c7$ are orienting. The components $c3$, $b5$, $c5$, and $c8$ are not.

Definition Additional operations on a marked bush form. These are used in the proof of Lemma 2 and 3.

- ***interlock*** is an operation to fix the orientation of one block relative to another. As a result, flipping one of them will flip the other.
- ***split*** is an operation to change a node in the bush form to k_2, k_3, or C_4, and distribute the incident edges among them. If the split is for a k_3 or C_4, a new block will result in the (marked) bush form.

The following definitions are for the marked bush forms. If we place a root marker on a cut vertex, or an edge of an outer face of a block, it will split the circular consecutive arrangement around the bush form into one of 3 types. In figures 9 and 10, the dots and line segments in navy blue indicate the marked vertex or edge. The dots in light blue are pertinent virtual nodes. The dots in light yellow are non-pertinent virtual nodes.

Definition Types of linear consecutive arrangements on the marked bush form.

- *singly partially consecutive*: the pertinent virtual nodes are arranged on either side of the linear order. In figure 9 left, ($v1$, $v3$, $v4$, $v14$, $v13$, $v12$, $v11$, $v5$, $v7$, $v6$, $v8$, $v9$, $v10$, $v2$) is such an arrangement. In figure 10 left, ($v10$, $v1$, $v2$, $v14$, $v13$, $v12$, $v11$, $v3$, $v4$, $v5$, $v6$, $v7$, $v9$, $v8$) is such an arrangement.

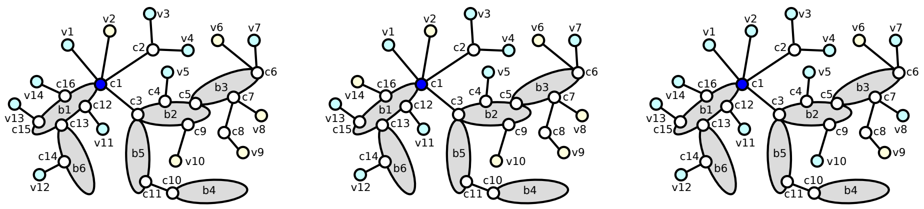


Figure 9: Bush forms with the root marker on a cut vertex

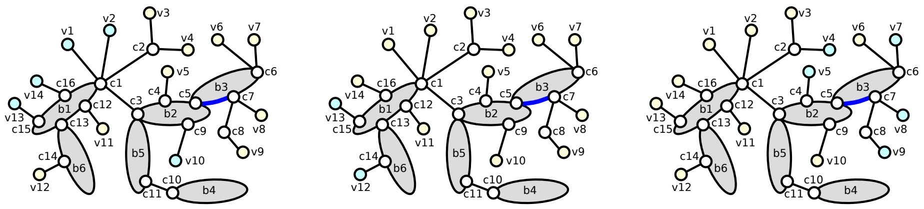


Figure 10: Bush forms with the root marker on a block

- *doubly partially consecutive*: the pertinent virtual nodes are arranged in the middle of the linear order. In figure 9 center, ($v2$, $v14$, $v13$, $v12$, $v11$, $v1$, $v3$, $v4$, $v5$, $v7$, $v6$, $v8$, $v9$, $v10$) is such an arrangement. In figure 10 center, ($v10$, $v11$, $v12$, $v13$, $v14$, $v1$, $v2$, $v3$, $v4$, $v5$, $v6$, $v7$, $v8$, $v9$) is such an arrangement.

- *complementarily partially consecutive*: the pertinent virtual nodes are arranged on both side of the linear order with one or more non-pertinent nodes in the middle. In figure 9 right, ($v1$, $v2$, $v3$, $v4$, $v5$, $v7$, $v6$, $v9$, $v8$, $v10$, $v11$, $v12$, $v13$, $v14$) is such an arrangement. In figure 10 right, ($v5$, $v4$, $v3$, $v2$, $v1$, $v14$, $v13$, $v12$, $v11$, $v10$, $v6$, $v7$, $v8$, $v9$) is such an arrangement.

Definition ***rooted embedded bc-tree*** is the underlying block-cut vertex tree of a marked bush form. The root marker on the bush form gives a natural root-to-descendants orientation on the underlying block-cut tree, and the embedding in the bush form determines an embedding of the block-cut tree and the embeddings of the blocks.

Definition ***pertinent root*** of a rooted embedded bc-tree is the highest (the closest to the root) node in the minimal connected subtree that spans the nodes for all the pertinent virtual nodes.

The left-to-right ordering of the children of the root node is determined as follows. If the root is for a cut vertex, then pick an arbitrary incident component of the root of the marked bush form, and arrange the incident components in the counter-clock wise ordering. If the root is for a block, then pick the incident component immediately after the marked edge in the counter-clock wise ordering, and arrange the incident components accordingly.

Definition *Pertinent Types* of the nodes in a rooted embedded bc-tree are recursively defined as follows.

- ***empty***: Each child of the node n is either non-pertinent virtual edge or an empty node.
- ***singly partial***: The node n meets one of the following conditions.
  - n is for a cut vertex, there is one singly partial child, and there is no complementarily partial child.
  - n is for a cut vertex, there is no singly partial child, at least one empty child, at least one full child, and no complementarily partial child.
  - n is for a block, there is no complementarily partial child, there is at least one full child, all the full children are consecutively embedded on the outer face on either side of the parent with possibly at most one singly partial child at the boundary between full children and the empty ones.
  - n is for a block, there is no complementarily partial child, there is no full child, and there is exactly one singly partial child immediately next to the parent.
- ***doubly partial***: The node n meets one of the following conditions.
  - n is the pertinent root, n is for a cut vertex, and there are exactly two singly partial children.
  - n is the pertinent root, n is for a block, at least one full child, all the full children are consecutively embedded in the middle of the outer face away from the parent, and possibly a singly partial child at each of the two boundaries between full and empty children.
  - n is the pertinent root, n is for a block, there is no full child, and there are exactly two consecutively embedded singly partial children.
- ***complementarily partial***: The node n meets one of the following conditions.
  - n is not the pertinent root, n is for a cut vertex, and there are exactly two singly partial children.
  - n is for a cut vertex and the children are all full except for one that is complementarily partial.
  - n is for a block, and the children are arranged as the complement of the arrangement for doubly partial.
  - n is for a block and the children are all full except for one that is complementarily partial.
- ***full***: If all the children of n are either pertinent virtual edge or full.

We have defined all the necessary types and operations. Next we prove the equivalence of a PQ-tree to its bush form.

Lemma 1 *If and only if the pertinent virtual nodes in a bush form are arranged consecutively by arbitrary reorder and flip operations, then there is a sequence of reorder and flip operations on any rooted embedded bc-tree from the leaf nodes toward the root to arrange the pertinent virtual edges in one of the linear consecutive arrangements. At each node of the rooted embedded bc-tree, the operation is such that its children are arranged in one of 5 pertinent types.*

Proof: The details is given in Appendix B. The 'only if' part is trivial. The 'if' part is by induction on $|T_{BF}|$ of the rooted embedded bc-tree T_{BF} of a marked bush form BF. □

We prove the equivalence between the marked bush & its underlying rooted embedded bc-tree, and the PQ-tree with Lemma 2 & 3 in the same induction step on the number of iterations of the algorithm.

Lemma 2 *Following holds for PQ-tree and its bush form:*

- *There is a one-to-one mapping between a P-node and an orienting cut vertex in the marked bush form.*
- *There is a one-to-one mapping between a Q-node and an orienting block in the marked bush form.*

Lemma 2 gives the location of the marker on the marked bush form. If the root of the PQ-tree is a P-node, then there is a corresponding orienting cut vertex in the bush form, and we place the marker on it. If the root is a Q-node, there is a corresponding block B in the bush form. We place the marker on an edge e of B on the outer face. The edge e is determined as follows. The children of the root Q-node in the left-to-right ordering in the PQ-tree corresponds to a counter-clock wise ordering of the orienting active components around B in the bush form. Proceed from the cut vertex that corresponds to the right most child of the Q-node in the counter-clock wise orientation, find the first edge e on the outer face of B.

Lemma 3 *If and only if the pertinent virtual nodes in the marked bush form can be consecutively arranged into one of 5 types using a series of reorder and flip operations from the leaves to the root, then there is an equivalent series of transformations by templates in REDUCE() for the PQ-tree to arrange the corresponding pertinent leaves to the same consecutive type.*

Proof: The details of a proof of Lemma 2 & 3 is given in Appendix C. The 'only if' part is trivial. The 'if' part is by induction on the number of iterations of the algorithm Lemma 3 is proved by examining exhaustively all the cases of operations on the marked bush form and finding equivalent templates of PQ-tree. Lemma 2 is proved by examining all the cases for the births and the deaths of the orienting cut vertices and blocks, and find equivalent P-nodes and Q-nodes on PQ-tree. □

Theorem 1 *If and only if the pertinent virtual edges in the bush form can be circularly consecutively arranged, then REDUCE() can transform the corresponding PQ-tree such that the pertinent leaves are arranged consecutively in one of 5 pertinent types.*

Proof: This is a direct application of Lemma 1, 2, and 3. □

This concludes the correctness of the new algorithm.

5 Time Complexity

Theorem 2 *The time complexity of the new algorithm stays $O(|N|)$.*

Proof: We follow the discussion given in the original [1] around Theorem 5, which depends on Lemma 2, 3 and 4 in [1]. Lemma 2 in [1] holds for the new algorithm as there is no change in BUBBLE().

Lemma 3 in [1] holds for the updated REDUCE() with new templates as follows. It's easy to see the required time for P8, Q4, and Q5 are on the order of the number of pertinent children. As for P7, just like P5, the empty children can implicitly stay in the original P-node, and it will be put under the new Q-node. This way, P7 runs in the order of the number of pertinent children.

Lemma 4 holds for the updated REDUCE() as there are only $|S|$ leaves and at most $O(|S|)$ nonunary nodes in the PQ-tree.

Theorem 5 holds as the Lemma 2, 3, and 4 hold, and also the new templates P7, P8, Q4, and Q5 do not apply to the unary nodes.

In theorem 5, We substitute $m = |E|$, $n = |V|$, and $SIZE(\mathbb{S}) = |E|$, and hence the algorithm runs in $O(|E|+|V|+|E|) = O(|V|)$, assuming $|E| \leq 3|V|-6$. □

6 Implementation Issues

In this section we discuss two topics regarding implementation. First we discuss the changes required to remove the pertinent tree of the complementarily partial type. Second we propose an improvement to Ozawa-type planarization algorithms with a new cost calculation technique. A reference implementation is found in github:yamanishi/colameco.

Removal of a complementarily partial pertinent sub-tree and emitting new leaves from a new P-node is different from the other cases. A complementarily partial pertinent subtree originates from a Q-node in template P7, or Q4. All the other nodes above this lowest Q-node including the tree root are pertinent nodes. They will be removed and the Q-node will become the new tree root. The P-node where the new leaves are attached will be put under the Q-node on either end. See figure 11.

There has not been a correct $O(|N|^2)$ linear time maximal planarization algorithm using PQ-tree as shown in [5], except for $O(|N||E|)$ add-edge-and-test type that calls an $O(|N|)$ planarity test multiple times. However those

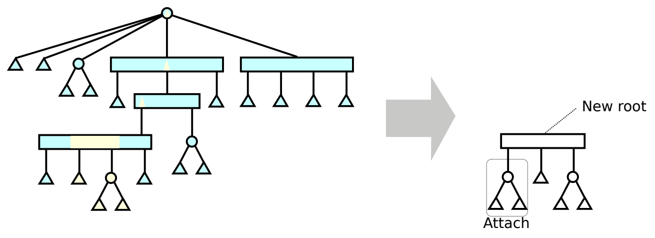


Figure 11: Removal of a Complementarily Partial Pertinet Tree

algorithms such as the first stage of [4] called $PLANALIZE(G)$ generates a planar spanning connected subgraph and it can be used as a base graph for further maximal planarization. Here we base our discussion on [4] and propose an improvement with additional node type and a cost value. [4] defines 4 node types: W, B, H, and A which correspond to empty, full, singly partial, and doubly partial respectively. We propose a new type C which corresponds to 'complementarily partial', and it associated cost value c for each node. The value for c in $COMPUTE1()$ in [4] is calculated as follows.

1. X is a pertinent leaf, $c = 0$.

2. X is a full node, $c = 0$.

3. X is a partial P-node, $c = a$.

4. X is a partial Q-node, $c = min\{\gamma_1, \gamma_2\}$.

$$
\begin{aligned}
\gamma_1 &= \sum_{i \in P(X)} (w_i) - max_{i \in P(X)}\{(w_i - c_i)\} \\
\gamma_2 &= \sum_{i \in P(X)} (w_i) - (max_{i \in P_L(X)}\{(w_i - h_i)\} + max_{i \in P_R(X)}\{(w_i - h_i))\}
\end{aligned}
\tag{1}
$$

where $P_L(X)$ means the maximal consecutive sequence of pertinent children of X from the left end such that all the nodes except the right most one are full. The right most one may be either full or singly partial. $P_R(X)$ is defined similarly from the right end.

After the cost calculation in the bottom-up manner, the type of the nodes can be determined top-down from the tree root using the new type C. In this way, the algorithm is capable of handling the complementarily partial situations, and would be able to reduce the number of edges removed.

7 Experiments

The execution time of the new algorithm is reported in Figure 12. X-axis is the time to process one graph in micro seconds taken from clock() function available on C++ standard. Y-axis indicates the number of vertices in the given biconnected planar graph. The test program was run on Mac 2.8GHz Corei7. The program was compiled by Apple LLVM 8.0.0. The plot shows the linearity of the algorithm.

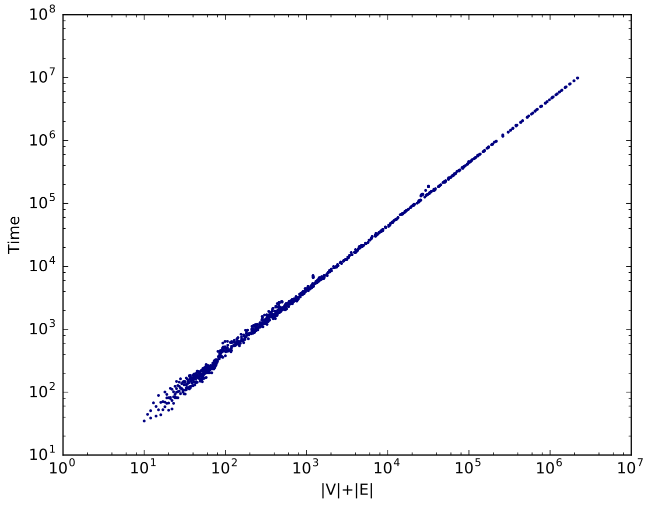


Figure 12: Processing Time of the New Algorithm

For all the test cases the new algorithm detected planarity. Here 'ramdom'ness is not rigorous in the mathematical sense. It is handled by the pseudorandom number generator library std::rand() and std::srand() in C++. The code and the data used for this experiments are found in github:yamanishi/colameco.

8 Conclusion

We have shown an enhancement to the original PQ-tree planarity algorithm proposed by [1] and proved the correctness. The time complexity stays in $O(|N|)$. The enhancement applies not only to the planarity test for the graphs, but also for anything involving circular consecutive ones. As far as the author knows, it seems there is no other applications than planarity testing.

9 Acknowledgements

The author wishes to thank whoever has encouraged and supported him for this article.

References

Appendices

A A Planar Biconnected Graph and a ST-Numbering

Following is the planar biconnected graph and the st-numbering used to produce the PQ-tree and the bush form in figure 1 and 2.

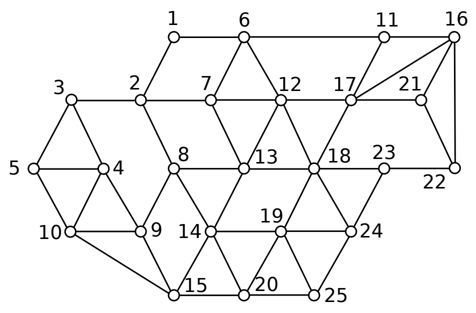


Figure 13: Biconnected Planar Graph Used for Figure 1 and 2

The st-numbering:
(8, 13, 7, 14, 19, 20, 25, 24, 23, 22, 21, 16, 11, 17, 15, 9, 10, 5, 4, 3, 2, 1, 6, 12, 18)

B Proof of Lemma 1

The 'only if' part is trivial. We prove the 'if' part by induction on $|T_{BF}|$ of the rooted embedded bc-tree T_{BF} of a marked bush form BF. Assume after some arbitrary reorder and flip operations, the pertinent virtual edges have been arranged consecutively around the bush form. This is equivalent to arranging the pertinent virtual edges into one of 5 types of linear consecutive arrangements in a marked bush form BF. If $|T_{BF}| = 1$, the lemma trivially holds as the bush form consists of one pertinent or non-pertinent virtual node. If $|T_{BF}| = 2$, the lemma trivially holds as the bush form consists of k_2 whose incident nodes are a cut vertex and a virtual node. Technically the node in k_2 is not a cut vertex but we consider it a cut vertex for the sake of the arguments.

Assume $|T_{BF}| \geq 3$. We split T_{BF} at the root node by splitting one connected component C from the rest of T_{BF}. If the marker in BF is on a cut vertex c, then we split one connected active component C of T_{BF}. If the marker is on an edge e of a block B in BF, then we pick the closest cut vertex c after e in the counter-clock wise ordering around B such that the maximal connected component after removing the block from c is still active. Let n_c be the corresponding node of c in T_{BF}. C is the maximal connected component of $T_{BF} \setminus B$ that contains n_c.

Now T_{BF} is decomposed into two components $D = T_{BF} \setminus C$ and C at n_c, and BF is decomposed into two parts BF_D and BF_C at c. BF_D and BF_C can be considered two marked bush form with the marker placed on c in BF_C.

We examine each of the 5 types of linear consecutive arrangements around BF, and find all the possible combinations of pertinent types of BF_D and BF_C. We observe table 1 enumerates all the possible combinations. We observe for each linear consecutive arrangement around BF, any other arrangements of $BF_{T\setminus C}$ or BF_C not shown in table 1 would lead to non-nonsecutive arrangement of the pertinent virtual nodes around BF.

By the induction hypothesis, the condition on the pertinent types of the children hold for both BF_D and BF_C. For each of 14 cases in Table 1, if we re-compose BF_D and BF_C at c into BF, then we can see the condition on the corresponding pertinent type for the case holds for BF. For example, if and only if BF is singly partial, then there will be 6 permissible combinations of the pertinent types of BF_D and BF_C. We observe any other combination would leat to a different pertinent type of BF or to a non-planar arrangement. If BF_D ends up in full, and BF_C in singly partial after some operations from the leaves to the root by the induction hypothesis, then it is easy to see that the BF will end up in singly partial as desired, possibly after flipping BF_C and reordering all the incident component around c or B. The other cases can be proved in the same manner.

Type on BF	**Type on** BF_D	**Type on** BF_C
Empty	Empty	Empty
Singly Partial	Empty	Singly Partial
Singly Partial	Empty	Full
Singly Partial	Singly Partial ^{*1}	Empty
Singly Partial	Singly Partial ^{*2}	Full
Singly Partial	Full	Empty
Singly Partial	Full	Singly Partial
Doubly Partial	Empty	Doubly Partial
Doubly Partial	Singly Partial ^{*3}	Empty
Doubly Partial	Singly Partial ^{*2}	Singly Partial
Doubly Partial	Doubly Partial	Empty
Complementarily Partial	Singly Partial ^{*4}	Singly Partial
Complementarily Partial	Full	Complementarily Partial
Full	Full	Full

^{*1}If the root is a block, then the pertinent virtual edges are on e side, not on c side.

^{*2}If the root is a block, then the pertinent virtual edges are on c side, not on e side.

^{*3}If the root is a block, then the pertinent virtual edges are on c. If the root is a cut vertex, this row does not apply.

^{*4}If the root is a block, then the pertinent virtual edges are on e. If the root is a cut vertex, this row does not apply.

Table 1: **All the arrangements of pertinent types of $BF_{T\setminus C}$ and BF_C.**

C Proof of Lemma 2 and 3

At the first iteration, Lemma 2 and 3 trivially hold. The only operation is an attach operation for the initial virtual edges to an initial cut vertex c in the bush form, and the corresponding operation on PQ-tree is creation of a new P-node and attaching pertinent leaves to it. The root marker will be placed on c.

By the induction hypothesis, Lemma 2, and 3, hold up to i-th iteration. By Lemma 1 and 2, there is a marked bush form BF whose marker is placed on a cut vertex c if the root of the PQ-tree is P-node, or on an edge of a block B if the root is a Q-node. Assume we have arranged the pertinent nodes for the $i+1$-th iteration by an arbitrary set of reorder and flip operations on the bush form. Then by Lemma 1, we can arrange the child components of each node of the rooted embedded bc-tree T_{BF} into one of 5 pertinent types from the descendants toward the root in T_{BF}.

Now we examine each pertinent type for a cut vertex and a block in a marked bush form for their equivalence to their corresponding nodes in the PQ-tree. We introduce two new operations to the marked bush form, which are interlock and split. Table 2 shows all the operations on an orienting cut vertex and their equivalent templates on PQ-tree. Table 3 is for an orienting block.

Following explains the symbols used in those tables.

- a square is a component
- a circle is a cut vertex
- sky blue indicates pertinent component.
- yellow indicates non-pertinent component.
- a square enclosing 'p' is a parent bock component. This does not apply if the cut vertex is the root.
- a circle enclosing 'p' is a parent cut vertex component. This does not apply if the block is the tree root.
- a square in sky blue is a full component.
- a square in yellow is a non-pertinent component.
- a rectangle enclosing yellow and sky blue squares is a singly partial component.
- a circle is in sky blue with a yellow wedge is a complementarily partial component.
- a polygon is a block.
- a grey triangle or a quadrilateral is a block induced by a split operation at a cut vertex.
- a dashed broken line indicates interlocking of two blocks.

We observe the rows of those tables cover exhaustively all the cases of 4 pertinent types. (Empty type is not considered.) Then we can prove the equivalence for each case with the aid of interlock and split operations as well as reorder and flip. For example, take the case for 'Singly Partial 2' for an orienting cut vertex c. Originally, c has 3 full child components, 3 non-pertinent child components, and a singly partial child component. To make it singly partial, we first reorder the incident ordering of those children so that the full children are arranged consecutively on one side, the non-pertinent children on the other, and the singly partial child between those two oriented such that the full side of the singly partial child is on the full side of children. Please note it is not a circular ordering due to the presence of the parent component. To make those changes permanent, we split c to a triangle k_3 or a qiadrilateral C_4 and fix the orientation of the singly partial child relative to them with the interlock operation. The 3rd column in Table 2 is for the case if c is a root component, and the 5th column is for the case if it is not. We can see those are in fact equivalent to the template P4 and P5 respectively for the case in which the are both full and non-pertinent components. In the 3rd column, the interlocked k_3 and the singly partial child together correspond to the resultant Q-node in P4 with the orientation of the children preserved. If there are more than 1 full child component, the full vertex on k_3 becomes a new orienting cut vertex, which corresponds to the new full P-node under the Q-node in P4. Similarly, in the 5th column, the interlocked C_4 and the singly partial child together correspond to the resultant Q-node in P5. The other cases can be examined in the same way. We see that in fact, those tables cover all the cases for all the templates for PQ-tree. In case of 'Doubly Partial 4', there are no corresponding PQ-tree templates, as those nodes are above the pertinent root. This concludes the proof of Lemma 3.

As for Lemma 2, we prove the equivalence in the birth and the death of the orienting cut vertex and the corresponding P-node, and the equivalence of an orienting block to Q-node.

An orienting cut vertex is born only at the following location.

- an attach operation with more than 1 virtual edge to be attached.

Please note that a split operation can produce k_3 or C_4, which means 2 or 3 new cut vertices are created. However, the one incident to the parent is not orienting as it has just 2 active components. The one incident to the interlocked singly partial child is not orienting either. The one on the full side is a temporary cut vertex which will be absorbed inside a newly created block on a merge operation later. The remaining one on the non-pertinent side is considered to have been transferred from the original cut vertex. So, effectively a split operation does not create new orienting cut vertex.

An orienting cut vertex will die at the following locations.

- Becoming full. In this case the children of the cut vertex will be merged into a new cut vertex, and eventually it will have two active components and will become non-orienting.

- Becoming complementarily partial, and it has a complementarily partial child. In this case all the full children together with the parent will be merged into a new cut vertex, and eventually it will have two active components and will become non-orienting.
- Singly Partial 1, non-pertinent root, and there is one non-pertinent child. In this case the new cut vertex on the non-pertinent side of k_3 will have only one child, and it has two active components and it will become non-orienting.
- Singly Partial 3, non-pertinent root, and there is one non-pertinent child. In this case the new cut vertex on the non-pertinent side of k_3 will have only one child, and it has two active components and it will become non-orienting.
- Singly Partial 4, pertinent root and non-pertinent root, and there is no non-pertinent child. In this case, the k_3 by the split operation does not produce any cut vertex for non-pertinent children and hence the cut vertex vanish.
- Doubly Partial or Complementarily Partial 3, Root and Non-root, and there is no non-pertinent child. In this case, the C_4 by the split operation does not produce any cut vertex for non-pertinent children and hence the cut vertex vanish.

An orienting block is born at the following conditions.

- k_3 or C_4 is generated by a split operation

An orienting block dies at the following locations.

- Where a singly partial child is interlocked to the parent. In this case there is a singly partial block in the component specified by the singly partial child, and the parent.
- Becoming full. In this case the children of the block will be merged into a new cut vertex, and eventually it will have two active components and will become non-orienting.
- Becoming complementarily partial, and it has a complementarily partial child. In this case all the full children together with the parent will be merged into a new cut vertex, and eventually it will have two active components and will become non-orienting.

For each of the birth and death cases above, there is a corresponding creation or destruction of P-node or Q-node. For example, take the 3rd death case of an orienting cut vertex that is a singly partial and non-root and that has one non-pertinent child. This corresponds to template P3. Since there is one non-pertinent child, it will be directly placed as one of two children of the Q-node. Eventually, the original P-node is considered removed from the PQ-tree. We can show the equivalence for the other cases. We can also show those cases cover all the locations in the templates, attach and removal of the pertinent subtree in which new P-nodes and Q-nodes are created and destructed. This concludes the proof of Lemma 2.

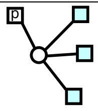
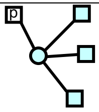
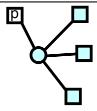
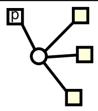
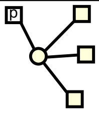
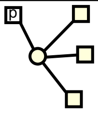
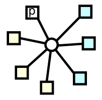
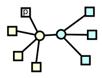
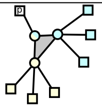
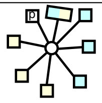
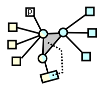
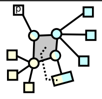
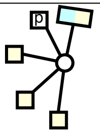
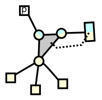
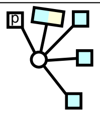
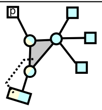
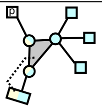
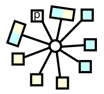
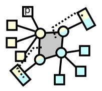
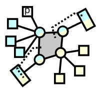
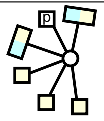
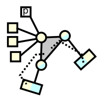
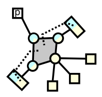
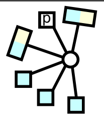
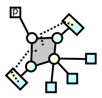
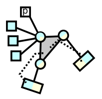
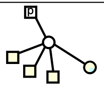
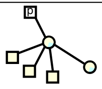
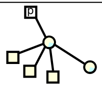
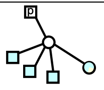
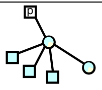
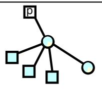
Pertinent type	Original State	Operation for Pert Root	PQ Op.	Operation for Non-Pert Root	PQ Op.
Full			P1		P1
Empty			-		-
Singly Partial 1			P2		P3
Singly Partial 2			P4		P5
Singly Partial 3		N/A	-		P5
Singly Partial 4			P4		P5
Doubly Partial or Complementarily Partial 1			P6		P7
Doubly Partial or Complementarily Partial 2			P6		P7
Doubly Partial or Complementarily Partial 3			P6		P7
Doubly Partial 4			-		-
Complementarily Partial			P8		P8

Table 2: Operations on an orienting cut vertex and equivalent PQ-tree templates

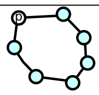
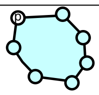
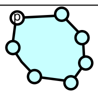
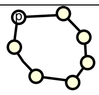
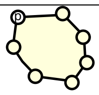
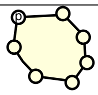
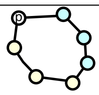
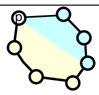
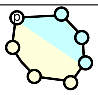
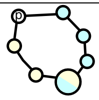
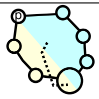
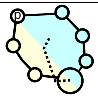
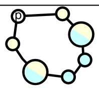
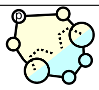
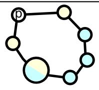
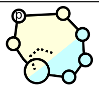
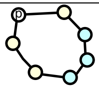
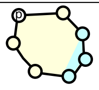
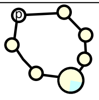
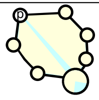
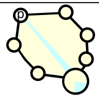
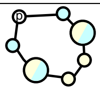
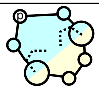
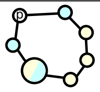
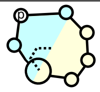
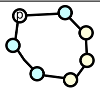
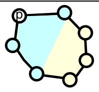
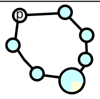
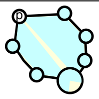
Pertinent type	Original State	Operation for Pert Root	PQ Op.	Operation for Non-Pert Root	PQ Op.
Full			Q1		Q1
Empty			-		-
Singly Partial 1			Q2		Q2
Singly Partial 2			Q2		Q2
Doubly Partial 1			Q3	N/A	-
Doubly Partial 2			Q3	N/A	-
Doubly Partial 3			Q3	N/A	-
Doubly Partial 4			-		-
Complementarily Partial 1		N/A	-		Q4
Complementarily Partial 2		N/A	-		Q4
Complementarily Partial 3		N/A	-		Q4
Complementarily Partial 4		N/A	-		Q5

Table 3: Operations on an orienting block and equivalent PQ-tree templates