Date 25 Apr 2021

Smooth Parametrization or Smooth Function / Curve:-

The curve represent by $r(t)$ is smoothly parametrization by $r(t)$ or The $r(t)$ is a smooth function of t if $r'(t)$ is continuous and $r'(t) \neq 0$, for any allowable value of t.

Example:-

Determine whether the following vector valued function are smooth or not.

① $r(t) = t^3 i + (3t^2 - 2t) j + t^2 k$

Solution:-

we check that the smoothness of $r(t)$

$\Rightarrow$ $r(t)$ is continuous for all value of t,

$$\Rightarrow r'(t) = 3t^2 i + (6t - 2) j + 2t k$$

$\Rightarrow$ $r'(t) \neq 0$, for any value of t

$\Rightarrow$ $r(t)$ is a smooth function

② $r(t) = \sin \pi t i + (2t - \ln t) j + (t^2 - t) k$

$$\Rightarrow r(t) = \sin \pi t i - (2t - \ln t) j + (t^2 - t) k$$

$$\Rightarrow r'(t) = \pi \cos(\pi t) - (2 - 1/t) j + (2t - 1) k$$

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not that

$$r'(t) = \pi\cos(\pi t)\,i - (2 - 1/t)\,j + (2t-1)\,k$$

if $t = 1/2$

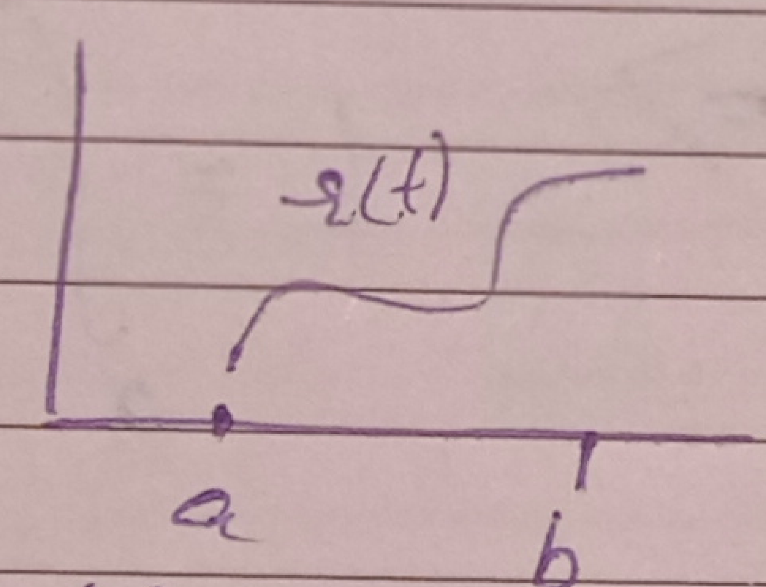
$\Rightarrow r'(1/2) = \pi\cos(\pi/2)\,i - \left(2 - \frac{1}{1/2}\right)j + \left(2\left(\frac{1}{2}\right) - 1\right)k$

$\Rightarrow r'(1/2) = \pi(0) - (2-2)\,j + (1-1)\,k$

$\Rightarrow r'(1/2) = 0$

$\Rightarrow$ $r(t)$ is not smooth function.

Arc length :-



if C is the graph in 2 space or 3-space of a smooth vector-valued function $r(t)$, then its arc length l from $t=a$ to $t=b$ is. (length of curve a to b).

if $r(t)$ in 2-space

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

if $r(t)$ in 3-space

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

or $\quad L = \int_a^b \left\| \frac{dr}{dt} \right\| dt$

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Question:-

Find the length of the Parametric curve.

$\Rightarrow$ $x = \cos^3 t$, $y = \sin^3 t$, $z = 2$

$0 \le t \le \pi/2$

The vector valued function of parametric curve is

$\vec{r}(t) = \cos^3 t\,\hat{i} + \sin^3 t\,\hat{j} + 2\hat{k}$.

$\Rightarrow$ $\vec{r}(t)$ in 3-space the

arc length of $r(t)$ is given by

$$\Rightarrow \quad L = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \quad \rightarrow (A)$$

$$\Rightarrow \quad x = \cos^3 t \Rightarrow \frac{dx}{dt} = -3\cos^2(t)\sin(t).$$

$$\Rightarrow \quad y = \sin^3 t \Rightarrow \frac{dy}{dt} = 3\sin^2(t)\cos t$$

$$\Rightarrow \quad z = 2 \Rightarrow \frac{dz}{dt} = 0$$

$$\Rightarrow \quad L = \int_0^{\pi/2} \sqrt{\left(-3\cos^2(t)\sin(t)\right)^2 + \left(3\sin^2(t)\cos t\right)^2 + (0)^2}\,dt$$

$\because \sqrt{} = 1/2$

$$\Rightarrow \quad L = \int_0^{\pi/2} \left[\left(9\cos^4(t)\sin^2 t + 9\sin^4(t)\cos^2 t\right)\right]^{1/2} dt$$

$$\Rightarrow L = \int_0^{\pi/2} \left[9 \cos^2(t) \cdot \cos^2(t) \cdot \sin^2(t) + 9 \sin^2(t) \sin^2(t) \cos^2(t) \right]^{1/2} dt$$

$$\Rightarrow L = \int_0^{\pi/2} \left[9 \cos^2 t \cdot \sin^2 t \left[\underbrace{\cos^2(t) + \sin^2(t)}_{=1} \right] \right]^{1/2} dt$$

$$\Rightarrow L = \int_0^{\pi/2} \left[9 \cos^2 t \cdot \sin^2 t \; (1) \right]^{1/2} dt$$

$$\Rightarrow L = \int_0^{\pi/2} \left[9 \cos^2 t \cdot \sin^2 t \right]^{1/2} dt$$

$$\Rightarrow L = \int_0^{\pi/2} \sqrt{9 \cos^2 t \cdot \sin^2 t} \; dt$$

$$\Rightarrow L = \int_0^{\pi/2} 3 \cos t \cdot \sin t \; dt$$

$$\Rightarrow L = 3 \int_0^{\pi/2} \sin t \cdot \cos t$$

using $\int [f(x)]^n f'(x) dx = \frac{f(x)^{n+1}}{n+1} \quad n \neq -1$

$$\Rightarrow L = 3 \left[\frac{\sin^2(t)}{2} \right]_0^{\pi/2} \Rightarrow \frac{3}{2} \left[\sin^2(t) \right]_0^{\pi/2}$$

$$\Rightarrow L = \frac{3}{2} \left[\sin^2(\pi/2) - \sin^2(0) \right]$$

$$\Rightarrow L = \frac{3}{2} \left[(1)^2 - (0)^2 \right] \Rightarrow L = 3/2 \Rightarrow \text{Ans}$$

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(ii) $x = \frac{t}{2}, \quad y = \frac{1}{3}(1-t)^{3/2}, \quad z = \frac{1}{3}(1+t)^{3/2}$

$-1 \le t \le 1$

$\Rightarrow \vec{r}(t) = \frac{t}{2}\hat{i} + \frac{1}{3}(1-t)^{3/2}\hat{j} + \frac{1}{3}(1+t)^{3/2}\hat{k}$

$\Rightarrow L = \int_{-1}^{1} \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \right]^{1/2} dt$

$\Rightarrow L = \int_{-1}^{1} \left[\text{''} + \text{''} + \text{''} \right]^{1/2} dt \longrightarrow (A)$

$\Rightarrow x = \frac{t}{2} \Rightarrow \frac{dx}{dt} = 1/2.$

$y = \frac{1}{3}(1-t)^{3/2} \Rightarrow \frac{dy}{dt} = \frac{1}{3}\left(\frac{3}{2}\right)(1-t)^{3/2-1} \frac{d}{dt}(1-t)$

$\Rightarrow \frac{dy}{dt} = \frac{1}{2}(1-t)^{1/2}(-1).$

$\Rightarrow \frac{dy}{dt} = -\frac{1}{2}(1-t)^{1/2}$

$\Rightarrow z = \frac{1}{3}(1+t)^{3/2} \Rightarrow \frac{dz}{dt} = \frac{1}{3}\left(\frac{3}{2}\right)(1+t)^{3/2-1} \frac{d}{dt}(1+t)$

$\Rightarrow \frac{dz}{dt} = \frac{1}{2}(1+t)^{1/2}$

$\Rightarrow (A)$

$\Rightarrow L = \int_{-1}^{1} \left[\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}(1-t)^{1/2}\right)^2 + \left(\frac{1}{2}(1+t)^{1/2}\right)^2 \right]^{1/2} dt$

$$\Rightarrow L = \int_{-1}^{1} \left[\frac{1}{4} + \frac{1}{4}(1-t) + \frac{1}{4}(1+t) \right]^{1/2} dt$$

$$\Rightarrow L = \int_{-1}^{1} \sqrt{\frac{1}{4} \left[1 + 1 - \cancel{t} + 1 + \cancel{t} \right]} \, dt$$

$$\Rightarrow L = \int_{-1}^{1} \sqrt{\frac{1}{4} \left[3 \right]} \, dt$$

$$\Rightarrow L = \int_{-1}^{1} \frac{\sqrt{3}}{2} dt \Rightarrow L = \left[\frac{\sqrt{3}t}{2} \right]_{-1}^{1}$$

$$\Rightarrow L = \frac{\sqrt{3}}{2} \left[1 - (-1) \right] \Rightarrow L = \frac{\sqrt{3}}{\cancel{2}} (\cancel{2})$$

$$\Rightarrow L = \sqrt{3} \Rightarrow \text{Ans.}$$

Practise question

Ex 12.3.

Q, 1, 2, 3, 4, 5, 6, 7, 8.

$\underbrace{1, 2, 3, 4}_{\Downarrow \text{ Smoothness.}}$ $\underbrace{5, 6, 7, 8}_{\Downarrow \text{ arc lengths}}$

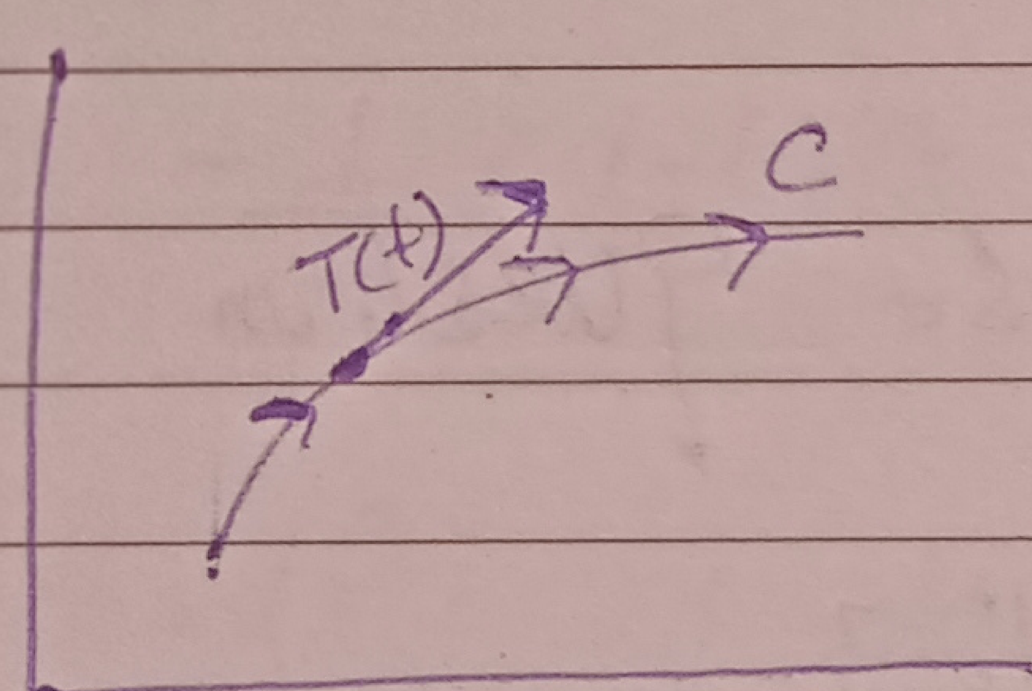
Unit Tangent, Normal, vectors:-

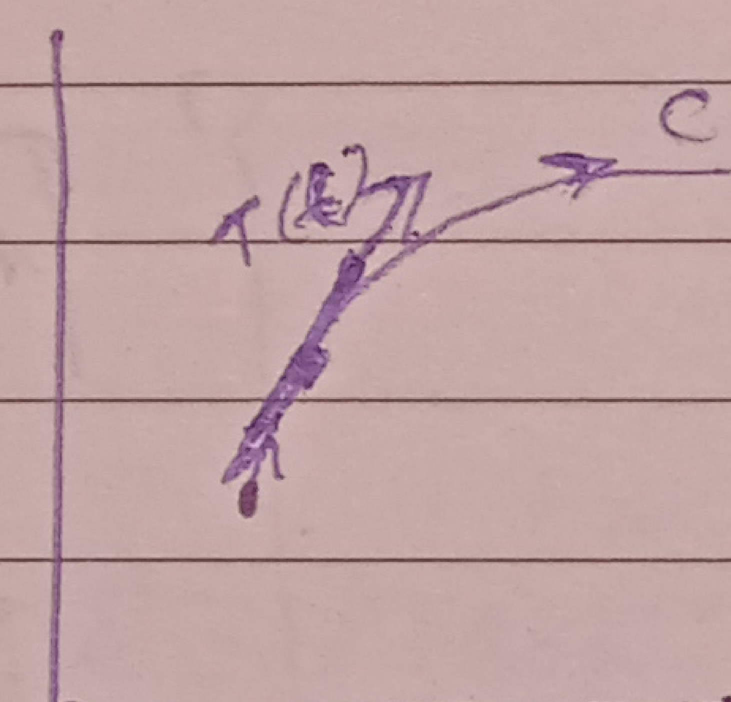
Unit Tangent:

if C is the graph of a Smooth vector-valued function $r(t)$ in 2-space or 3-space then the vector $r'(t)$ is nonzero, tangent to C, and point in the direction of increasing prameter. Thus by normalization $r'(t)$ we obtained a unit vector.

$$\boxed{T(t) = \frac{r'(t)}{\|r'(t)\|}}$$

thate is tangent to C and point in the direction of increassing porameter we call T(t) the unit tangent vector to C at t.





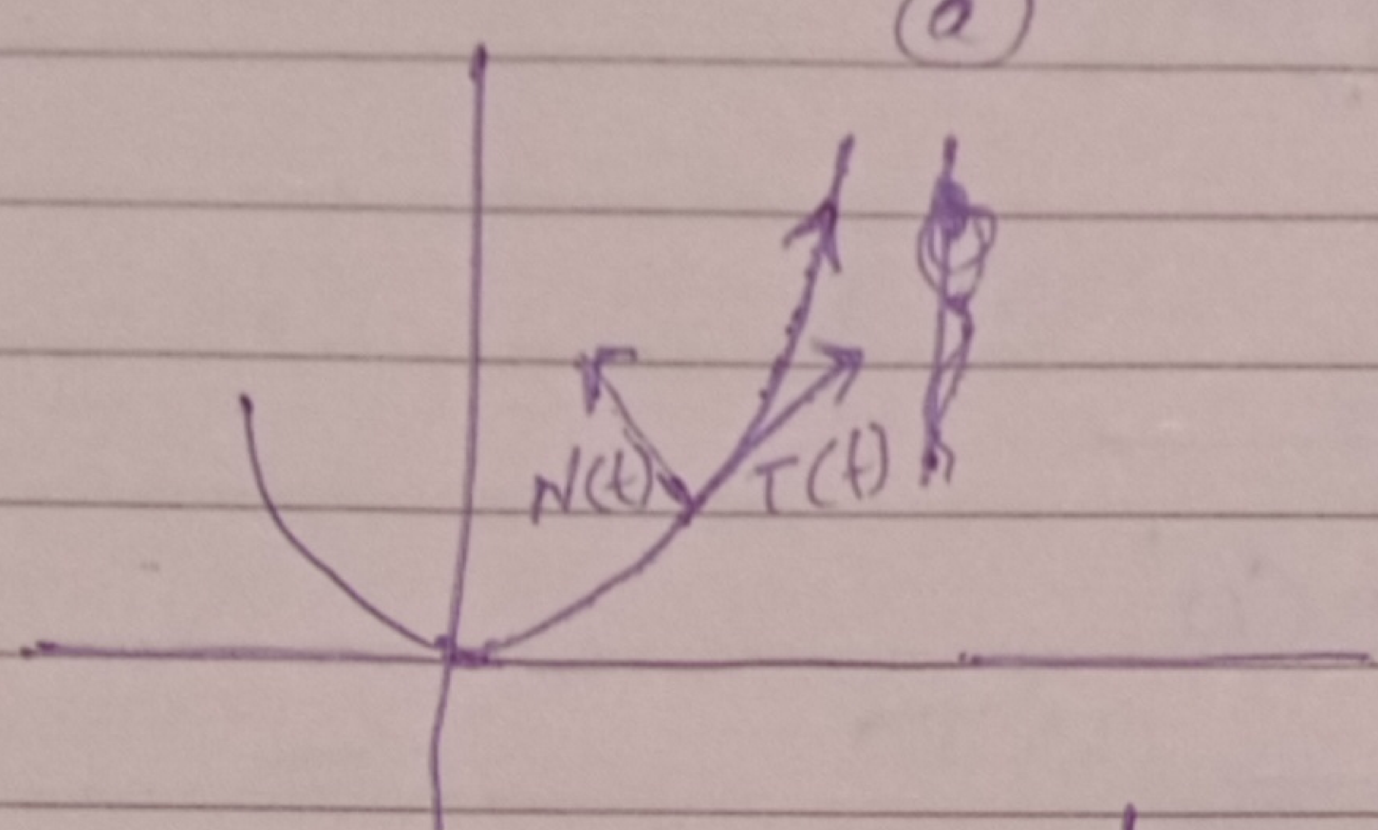
Unit Normal vector:

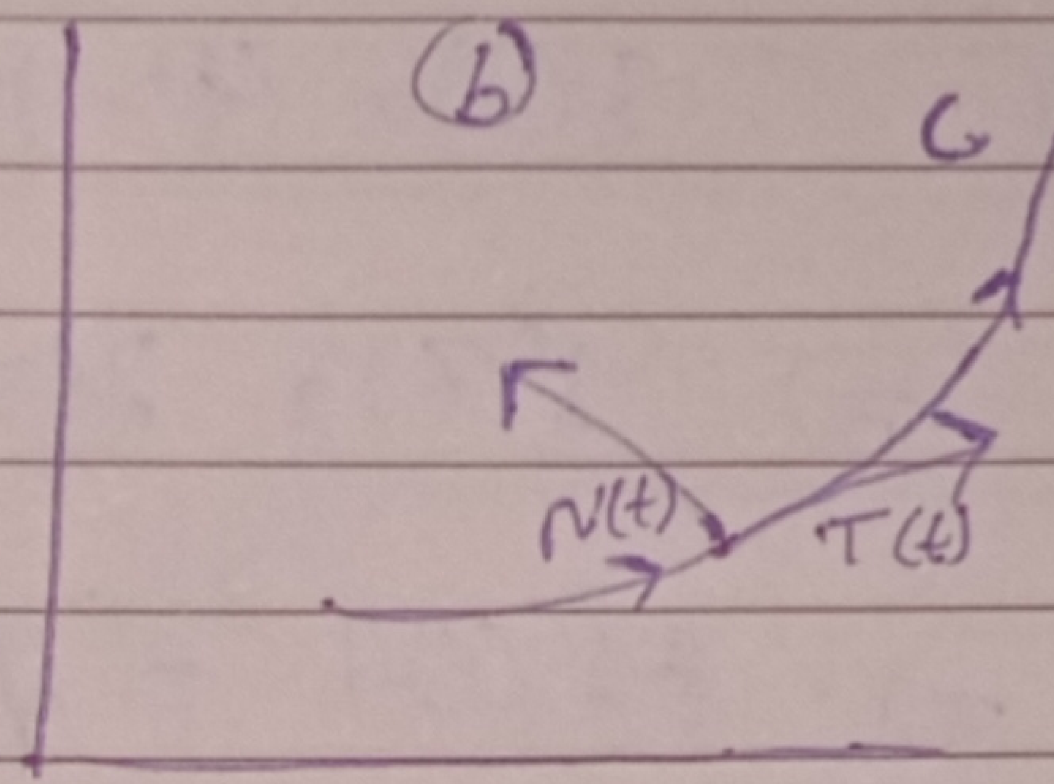
The Unit Normal vector is define as.

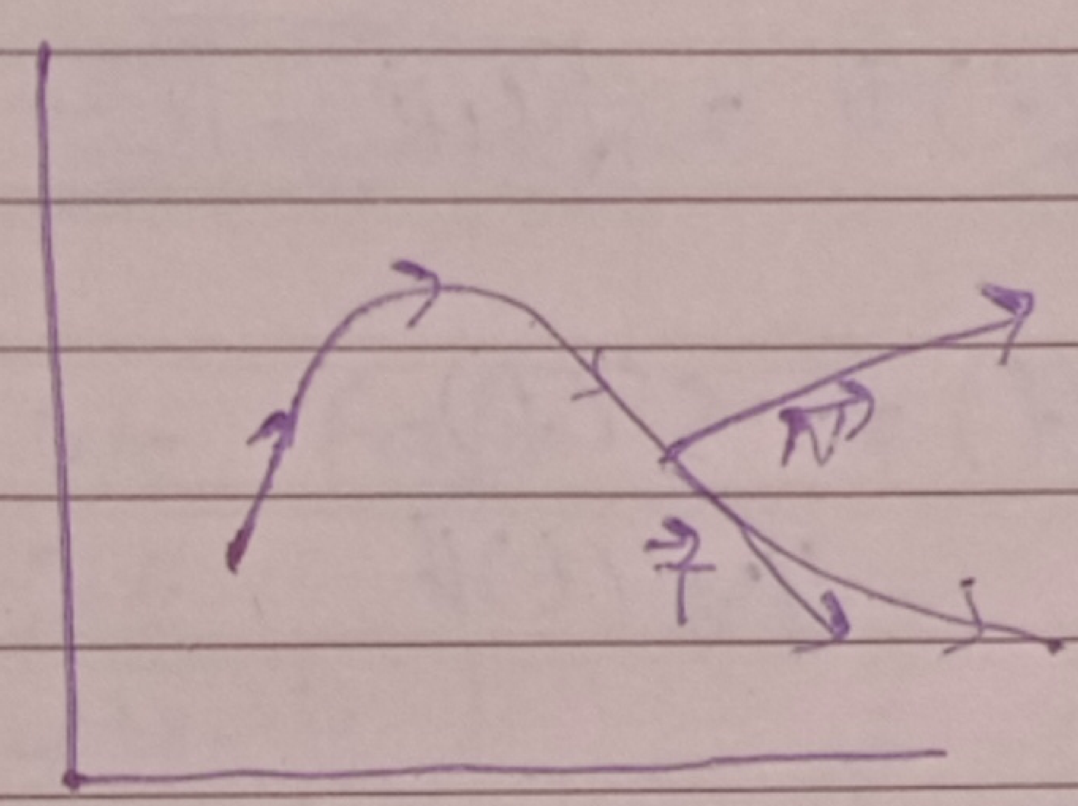
$$\boxed{N(t) = \frac{T'(t)}{\|T'(t)\|}}$$

The Unit normal is orthogonal (or normal, or perpendicular) to the tangent vector.

Consider a Curve







Question (Ex 12.4).

Find T(t) and N(t) at the given point

$\Rightarrow$ $r(t) = (t^2-1)i + tj$; $t = 1$

Sol: we find out Tangent & Normal vector of The curve $r(t)$ at point $t = 1$

As we know that

The Unit tangent vector for , at a Curve is

$\Rightarrow$ given by.

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

$\Rightarrow$ $r(t) = (t^2-1)i + tj$

$\Rightarrow$ $r'(t) = 2ti + j$

$\Rightarrow$ $\|r'(t)\| = \sqrt{(2t)^2 + (1)^2}$

$\Rightarrow$ $\|r'(t)\| = \sqrt{4t^2+1}$

$\Rightarrow$ $T(t) = \frac{r'(t)}{\|r'(t)\|}$

$\Rightarrow$ $T(t) = \frac{2ti + j}{\sqrt{4t^2+1}}$

at Point $t = 1$.

$$T(1) = \frac{2(1)i + j}{\sqrt{4(1)^2+1}}$$

$$T(1) = \frac{2i + j}{\sqrt{5}}$$

$$T(1) = \frac{2}{\sqrt{5}}i + \frac{j}{\sqrt{5}}$$ $\Rightarrow$ Ans.

$\Rightarrow$ So The Tangent vector at point $t = 1$ is $\frac{2}{\sqrt{5}}i + \frac{j}{\sqrt{5}}$

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Now we also we find The Normal vector at point $t=1$.

$\Rightarrow$ $$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

$\Rightarrow$ $$T(t) = \frac{2t\,i}{\sqrt{4t^2+1}} + \frac{j}{\sqrt{4t^2+1}}$$

$\Rightarrow$ $$T'(t) = \frac{d}{dt}\left(\frac{2t}{(4t^2+1)^{1/2}}\right) + \frac{d}{dt}\left(\frac{j}{\sqrt{4t^2+1}}\right)$$

$\Rightarrow$ First we take

$$\frac{d}{dt}\left(\frac{2t}{(4t^2+1)^{1/2}}\right)$$ using $$\frac{d}{dt}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$\Rightarrow$ $$= \frac{\sqrt{(4t^2+1)}\,(2) - \frac{1}{2}(4t^2+1)^{\frac{1}{2}-1}\frac{d}{dt}(4t^2+1)\,2t}{\left(\sqrt{(4t^2+1)}\right)^2}$$

$$= \frac{2\sqrt{4t^2+1} - \frac{(\overset{4}{\cancel{8}}t)(2t)}{\cancel{2}\sqrt{4t^2+1}}}{(4t^2+1)}$$

$$= \frac{\frac{2(4t^2+1) - 8t^2}{(4t^2+1)^{1/2}}}{4t^2+1} = \frac{\cancel{8t^2} - \cancel{8t^2} + 2}{(4t^2+1)^{3/2}}$$

$\Rightarrow$ $2\ i$

Similary. $\frac{d}{dt}\left(\frac{1}{(4t^2+1)^{1/2}}\right)$

$\Rightarrow$

$-\frac{1}{2}(4t^2+1)^{-\frac{1}{2}-1} \Rightarrow \frac{-\cancel{8}t}{\cancel{2}(4t^2+1)^{+3/2}}$

$\Rightarrow \frac{-4t}{(4t^2+1)^{3/2}}$

$\Rightarrow T'(t) = \frac{2}{\sqrt[3]{4t^2+1}} i - \frac{4t}{(4t^2+1)^{3/2}} j$

$\Rightarrow$ Now

$\|T'(t)\| = \sqrt{\left(\frac{2}{(4t^2+1)^{3/2}}\right)^2 + \left(\frac{-4t}{(4t^2+1)^{3/2}}\right)^2}$

$\Rightarrow \|T'(t)\| = \sqrt{\frac{4}{(4t^2+1)^3} + \frac{16t^2}{(4t^2+1)^3}}$

$\Rightarrow \|T'(t)\| = \sqrt{\frac{4+16t^2}{(4t^2+1)^3}}$

Now

$N(t) = \frac{T'(t)}{\|T'(t)\|}$

$N(t) = \frac{\frac{2}{(4t^2+1)^{3/2}} i - \frac{4t}{(4t^2+1)^{3/2}}}{\sqrt{\frac{4+16t^2}{(4t^2+1)^3}}}$

at $t=1$

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$$\Rightarrow N(1) = \frac{\frac{2}{(4+1)^{3/2}}\hat{i} - \frac{4}{(4+1)^{3/2}}\hat{j}}{\sqrt{\frac{4+16}{(4+1)^3}}} \Rightarrow = \frac{20}{(5)^3} = \frac{2\cancel{0}}{1\cancel{2}5} = \frac{4}{25}$$

$$\Rightarrow N(1) = \frac{\frac{2}{(5)^{3/2}}\hat{i} - \frac{4}{(5)^{3/2}}\hat{j}}{\sqrt{4/25}}$$

$$\Rightarrow N(1) = \frac{\frac{2}{(5)^{3/2}}\hat{i} - \frac{4}{(5)^{3/2}}\hat{j}}{\frac{2}{5}}$$

$$\Rightarrow N(1) = \frac{\frac{\cancel{2}}{(5)^{3/2}}}{\cancel{2}/5}\hat{i} + \frac{\frac{-\cancel{4}}{(5)^{3/2}}}{\cancel{2}/5}\hat{j}$$

$$\Rightarrow N(1) = \frac{5}{(5)^{3/2}}\hat{j} + -\frac{10}{(5)^{3/2}}\hat{j}$$

$$N(1) = 5^{1-3/2}\hat{i} + -2\times 5^{1-3/2}\hat{j}$$

$$N(1) = 5^{-1/2}\hat{i} - 2\times 5^{-1/2}\hat{j}$$

$$N(1) = \frac{\hat{i}}{\sqrt{5}} - \frac{2\hat{j}}{\sqrt{5}}$$

$$\Rightarrow \boxed{N(1) = \frac{\hat{i}}{\sqrt{5}} - \frac{2\hat{j}}{\sqrt{5}}} \Rightarrow \text{Ans.}$$

Q.No 2 Find $T(t)$ and $N(t)$ at $t = \pi/2$

$\Rightarrow$

$r(t) = 4\cos t\, i + 4\sin t\, j + tk \; ; \; t = \pi/2$

So,

$\Rightarrow$

$$T(t) = \frac{r'(t)}{\|r'(t)\|} \rightarrow (A)$$

$\Rightarrow r(t) = 4\cos t\, i + 4\sin t\, j + tk$

$\Rightarrow r'(t) = -4\sin t\, i + 4\cos t\, j + k$

$\Rightarrow \|r'(t)\| = \sqrt{(-4\sin t)^2 + (4\cos t)^2 + (1)^2}$

$\Rightarrow \|r'(t)\| = \sqrt{16\sin^2 t + 16\cos^2 t + 1}$

$\Rightarrow \|r'(t)\| = (16(\underbrace{\sin^2 t + \cos^2 t}_{1}) + 1)^{1/2}$

$= (16+1)^{1/2}$

$\Rightarrow \|r'(t)\| = \sqrt{17}$

$\Rightarrow (A)$

$$\Rightarrow T(t) = \frac{-4\sin t\, i + 4\cos t\, j + k}{\sqrt{17}}$$

at $t = \pi/2$

$\Rightarrow$

$$T(\pi/2) = \frac{-4\cancel{\sin(\pi/2)}^{1}\, i + 4\cancel{\cos(\pi/2)}^{0}\, j + k}{\sqrt{17}}$$

$$\Rightarrow T(\pi/2) = \frac{-4i + k}{\sqrt{17}}$$

$$\Rightarrow T(\pi/2) = \frac{-4}{\sqrt{17}} i + \frac{k}{\sqrt{17}} \Rightarrow \text{Ans.}$$

Now,

We again find the Normal Vector at $t = \pi/2$.

$$N(t) = \frac{T'(t)}{\|T'(t)\|} \longrightarrow \text{(B)}$$

$$T(t) = \frac{-4\sin t\, i}{\sqrt{17}} + \frac{4\cos t\, j}{\sqrt{17}} + \frac{k}{\sqrt{17}}$$

$$T'(t) = \frac{-4\cos t\, i}{\sqrt{17}} - \frac{4\sin t\, j}{\sqrt{17}} + 0$$

$$\Rightarrow \|T'(t)\| = \left[\left(-\frac{4\cos t}{\sqrt{17}}\right)^2 + \left(\frac{-4\sin t}{\sqrt{17}}\right)^2\right]^{1/2}$$

$$\Rightarrow \|T'(t)\| = \sqrt{\frac{16\cos^2 t + 16\sin^2 t}{17}}$$

$$\Rightarrow \|T'(t)\| = \frac{4}{\sqrt{17}}$$

at $t = \pi/2$

$$\Rightarrow N(\pi/2) = \frac{\frac{-4\overset{0}{\cancel{\cos(\pi/2)}}\, i}{\sqrt{17}} - \frac{4\overset{1}{\cancel{\sin(\pi/2)}}\, j}{\sqrt{17}}}{4/\sqrt{17}}$$

$$\Rightarrow N(\pi/2) = \frac{-\cancel{\frac{4}{\sqrt{17}}}\, j}{\cancel{4/\sqrt{17}}}$$

$$\Rightarrow N(\pi/2) = -j \Rightarrow \text{Ans}$$

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Practise question (Tangent & Normal vector)

Ex. 12.4

Question 5 to 11