Curvature :-

basically the curvature measure how fast a curve is changing direction at a given point.

(rate of bending curve 8.6)

There are several formulas for determing The curvature for a curve but we use Under formula.

⇒ **Theorem :-**

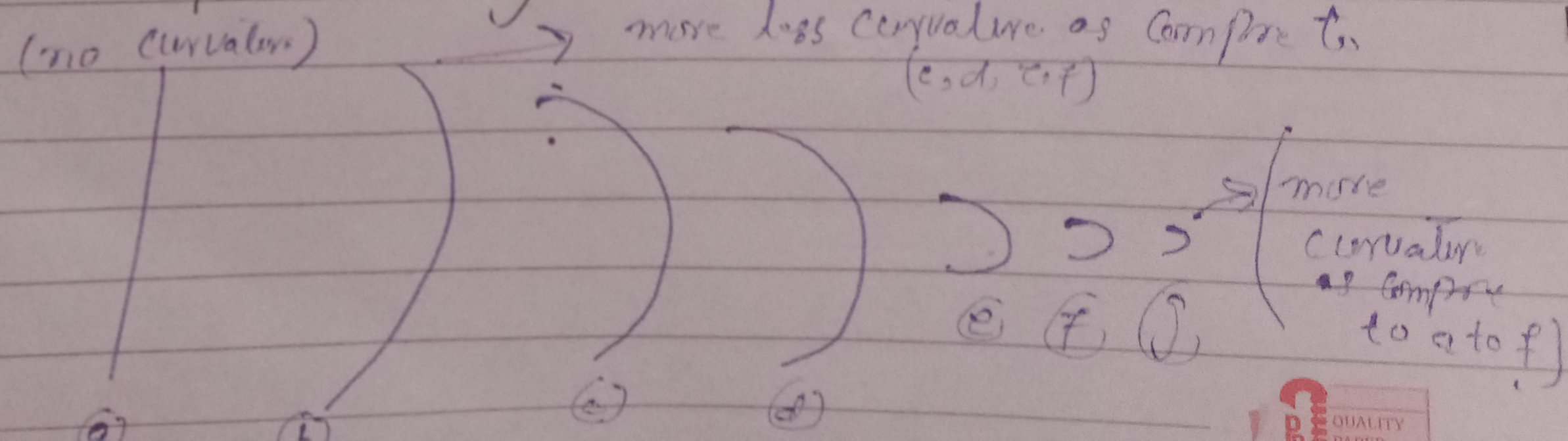
if $r(t)$ is a smooth vector-valued function in 2space or 3-space Then for each value of t at which $T'(t)$ and $r''(t)$ exist exist, The curvature k can be expressed as.

ⓐ $$k(t) = \frac{\|T'(t)\|}{\|r'(t)\|}$$

ⓑ $$k(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

read (kappa) Greek letter

Geometrically:



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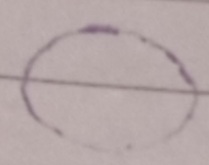
Remark:

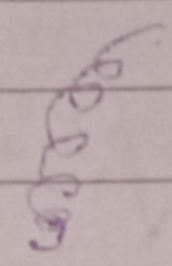
$r(t) = a\cos t\, i + a\sin t\, j$

(a) Curvature of a Circle is Constant.



(یہ Constant rate of bending curve ہے Circle ہوگا)

(b) Curvature of helix (سپرنگ) is also Constant



$\Rightarrow r(t) = a\cos t\, i + a\sin t\, j + ct\, k.$

(c) Straight line have zero curvature at every point

Q Show that the $K(t)$ for the Circular helix is Constant.

$\Rightarrow \quad x = a\cos t \qquad y = a\sin t \qquad z = ct$

Solution:

The radius vector for the helix is

$r(t) = a\cos t\, i + a\sin t\, j + ct\, k.$

The Curvature of the $r(t)$ is given by

$$\Rightarrow \quad k(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} \longrightarrow (1)$$

$\Rightarrow \quad r(t) = a\cos t\, i + a\sin t\, j + ct\, k$

$\Rightarrow \quad r'(t) = -a\sin t\, i + a\cos t\, j + ck.$

$\Rightarrow \quad r''(t) = -a\cos t\, i - a\sin t\, j.$

$$\Rightarrow \quad \|r'(t) \times r''(t)\| = \begin{vmatrix} i & j & k \\ -a\sin t & a\cos t & c \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

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$$\Rightarrow \quad = (ac\sin t)i - (ac\cos t)j + a^2 k$$

$$\Rightarrow \quad \|r'(t)\| = \sqrt{(-a\sin t)^2 + (-a\cos t)^2 + c^2}$$

$$= \sqrt{a^2\sin^2 t + a^2\cos^2 t + c^2}$$

$$= \sqrt{a^2 + c^2}$$

$$\Rightarrow \quad \|r'(t) \times r''(t)\| = \sqrt{(ac\sin t)^2 + (-ac\cos t)^2 + (a^2)^2}$$

$$\Rightarrow \quad = \sqrt{a^2c^2\sin^2 t + a^2c^2\cos^2 t + a^4}$$

$$= \sqrt{a^2c^2(\sin^2 t + \cos^2 t) + a^4}$$

$$= \sqrt{a^2c^2 + a^4} \Rightarrow a\sqrt{c^2 + a^2}$$

① $\Rightarrow$

$$k(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

$$\Rightarrow \quad k(t) = \frac{a\sqrt{a^2+c^2}}{\left(\sqrt{a^2+c^2}\right)^3} = \frac{a}{\sqrt{a^2+c^2}}$$

$\Rightarrow$ Ans

Show The curvature is constant not t appears.

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Ex 12.5

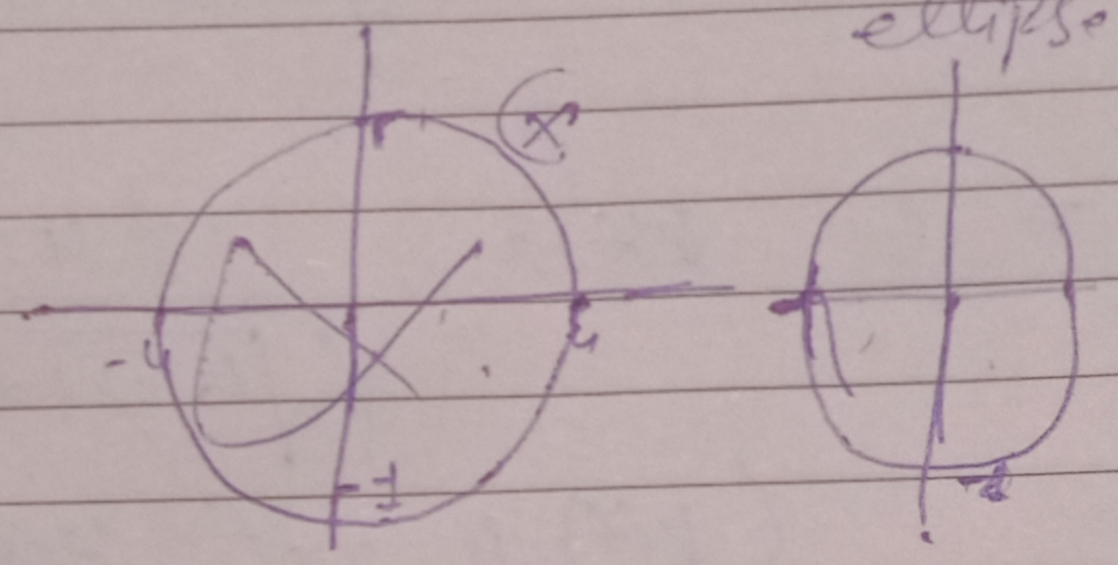
Q1(b) Find the $k(t)$ of ellipse having a radius function is

$$r(t) = 4\cos t\, i + \sin t\, j$$

Solution:-

we know that curvature of ellipse is not constant. The above ellipse given with radius 4 unit along x axis & 1 unit along y axis

Geometrically:



We know that the curvature is given by:

$$\Rightarrow k(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} \longrightarrow (A)$$

$$r(t) = 4\cos t\, i + \sin t\, j$$

$$\Rightarrow r'(t) = -4\sin t\, i + \cos t\, j$$

$$\Rightarrow r''(t) = -4\cos t\, i - \sin t\, j$$

$$\Rightarrow r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ -4\sin t & \cos t & 0 \\ -4\cos t & -\sin t & 0 \end{vmatrix}$$

$$\Rightarrow i(0-0) - j(0-0) + k(\cancel{-4\cos t\sin t + 4\sin t\cos t}$$
$$+ k(4\sin^2 t + 4\cos^2 t)$$

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$\Rightarrow$ 4k.

$$\|r'(t) \times r''(t)\| = \sqrt{(0)^2 + (0)^2 + (4)^2}$$

$$= \sqrt{16} = 4.$$

$$\|r'(t)\| = \sqrt{(-4\sin t)^2 + (\cos t)^2}$$

$$\Rightarrow \|r'(t)\| = \sqrt{16\sin^2 t + \cos^2 t}$$

$\Rightarrow$ (A) $k(t) = \dfrac{4}{\left(\sqrt{16\sin^2 t + \cos^2 t}\right)^3}$ $\Rightarrow$ Ans

($k(t)$ is not constant b/c the t variable appears in curvature)

---

(12) $r(t) = i + tj + t^2 k$ Find $k(t)$

$\Rightarrow$ $r(t) = i + tj + t^2 k$

$r'(t) = j + 2t\,k$

$\Rightarrow$ $r''(t) = 2$

$$\Rightarrow r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 0 & 1 & 2t \\ 0 & 0 & 2 \end{vmatrix}$$

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$\Rightarrow \; = i(2-0) - j(0-0) + k(0-0)$

$= 2i$

$\|r'(t) \times r''(t)\| = \sqrt{(0)^2 + (0)^2 + (2)^2}$

$= \sqrt{4} = 2.$

$\|r'(t)\| = \sqrt{(0)^2 + (1)^2 + (2t)^2}$

$= \sqrt{1+4t^2}$

$\Rightarrow \; k(t) = \dfrac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$

$\Rightarrow \; k(t) = \dfrac{2}{(\sqrt{1+4t^2})^3}$

$\Rightarrow \; k(t) = \dfrac{2}{(1+4t^2)^{3/2}} \rightarrow$ Ans.

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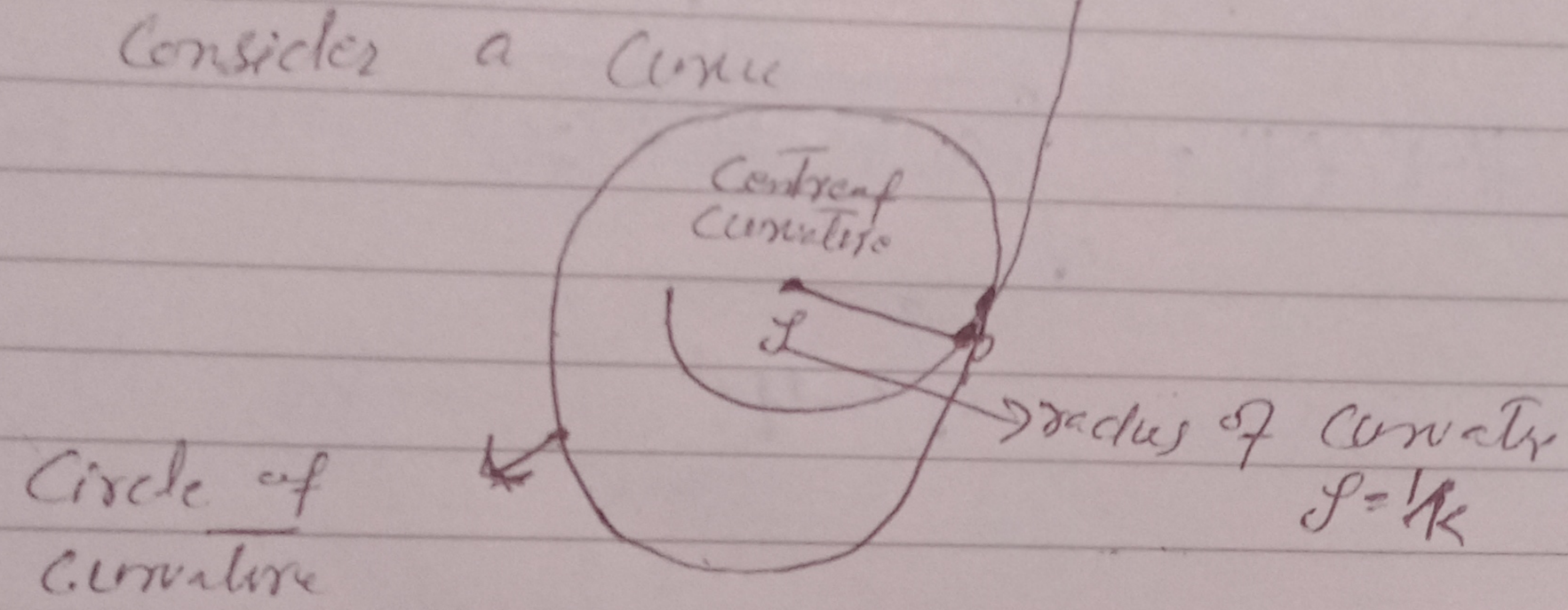
Radius of curvature :-

The radius of curvature of a curve at a point P is the inverse of the curvature k of the curve at this point

Consider a curve



$\Rightarrow$ radius of curvature $\rho = 1/k$

Find the curvature and radius of the curvature at the stated point.

(14) $\underline{r}(t) = e^{t} i + e^{-t} j + t k \quad , \; t = 0$

First we find the curvature of $\underline{r}(t)$

$$\Rightarrow \quad k(t) = \frac{\| \underline{r}'(t) \times \underline{r}''(t) \|}{\| \underline{r}'(t) \|^{3}} \longrightarrow (A)$$

$\underline{r}'(t) = e^{t} i - e^{-t} j + k$

$\underline{r}''(t) = e^{t} i + e^{-t} j + 0$

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$\Rightarrow \quad \vec{r}'(t) \times \vec{r}''(t)$

$$\begin{vmatrix} i & j & k \\ e^{t} & -e^{-t} & 1 \\ e^{t} & e^{-t} & 0 \end{vmatrix}$$

$\Rightarrow \quad i(-e^{-t}) - j(e^{t}) + k(e^{-t}.e^{t} + e^{t}.e^{-t})$

$\Rightarrow \quad -e^{-t}i - e^{t}j + k(\overset{0}{e^{-t+t}} + \overset{0}{e^{t-t}})$

$\Rightarrow \quad -e^{-t}i - e^{t}j + 2k$

$\vec{r}'(t) \times \vec{r}''(t) = -e^{-t}i - e^{t}j + 2k$

$\Rightarrow$

$$\|\vec{r}'(t) \times \vec{r}''(t)\| = \sqrt{(-e^{-t})^2 + (-e^{t})^2 + (2)^2}$$

$$\Rightarrow \|\vec{r}'(t) \times \vec{r}''(t)\| = \sqrt{e^{2t} + e^{2t} + 4}$$

$$= \sqrt{2e^{2t} + 4}$$

$$\|\vec{r}'(t)\| = \sqrt{(e^{t})^2 + (-e^{-t})^2 + (1)^2}$$

$$= \sqrt{e^{2t} + e^{2t} + 1} = \sqrt{2e^{2t} + 1}$$

$\Rightarrow$ A

$$k(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|}$$

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$$k(t) = \frac{\sqrt{2e^{2t}+4}}{\left(\sqrt{2e^{t}+1}\right)^{3}}$$

at point $t=0$

$$k(0) = \frac{\sqrt{2e^{2(0)}+4}}{(2e^{0}+1)^{3/2}} \Rightarrow \frac{\sqrt{2+4}}{(3)^{3/2}}$$

$$\Rightarrow \frac{\sqrt{6}}{(3)^{3/2}} = \frac{(3\times 2)^{1/2}}{(3)^{3/2}}$$

$$\Rightarrow 3^{\frac{1}{2}-\frac{3}{2}} \times \sqrt{2} \Rightarrow \cancel{\frac{\sqrt{2}\times\sqrt{3}}{\sqrt{3}\ \sqrt{3}}}$$

$$\cancel{k(0) = \frac{\sqrt{2}\sqrt{3}}{3}} \qquad 3^{-1} \times \sqrt{2}$$

$$k(0) = \frac{\sqrt{2}}{3}$$

So radius of curvature $\rho = 1/k$.

$$\Rightarrow \rho = \frac{1}{\frac{\sqrt{2}}{3}}$$

$$\Rightarrow \rho = 3/\sqrt{2} \Rightarrow \text{Ans}$$

at point $t=0$

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Practise question

Ex 12.5 Q 5 to 16.

Chapter 12 Complete