Problem what conditions must b_1, b_2 and b_3 satisfy in order for the system of equations

$$x_1 + x_2 + 2x_3 = b_1$$
$$x_1 \qquad + x_3 = b_2$$
$$2x_1 + x_2 + 3x_3 = b_3$$

to be consistent?

Solution

The augmented matrix is

$$\left[\begin{array}{ccc:c} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{array}\right]$$

$R_2 - R_1,\ R_3 - 2R_1$

$$\left[\begin{array}{ccc:c} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2-b_1 \\ 0 & -1 & -1 & b_3-2b \end{array}\right]$$

$-R_2$

$$\left[\begin{array}{ccc:c} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1-b_2 \\ 0 & -1 & -1 & b_3-2b_1 \end{array}\right]$$

$R_3 + R_2$

$$\left[\begin{array}{ccc:c} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1-b_2 \\ 0 & 0 & 0 & b_3-b_1-b_2 \end{array}\right]$$

$$b_3 - b_1 - b_2 = 0$$

$$\boxed{b_3 = b_1 + b_2}$$

To express the condition another way, $AX = b$ is consistent if and only if b is a matrix of the form

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_1 + b_2 \end{bmatrix}$$

where b_1 and b_2 are arbitary constant.