Syllabus

Unit I : Differential Calculus.
Unit II :- Limit and continuity
Unit III :-
Unit IV :- Integral Unit
Unit V :- Vector Calculus

Exam Pattern

Mid term - I (unit I & II)
mid term - II (unit III, IV & V)

⇒ Mid term Pattern :-

Q-1 — Total 20 marks.
↳ 12 question
↳ any 6 should attempt → 8 marks

Q-2 :
a — 2 marks
b — 2 marks
c — 2 marks
→ 6 marks

Q-3 Same as 2nd question - 6 marks

$20/2 + 20/2 + 10 \rightarrow 30 \rightarrow$ [mid term - I.]

10 = 5 (attendence) + 5 (assignment)

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#

$$\boxed{C = -\frac{\phi_2(m)}{\phi_3'(m)}}$$ when 'm' has different roots.

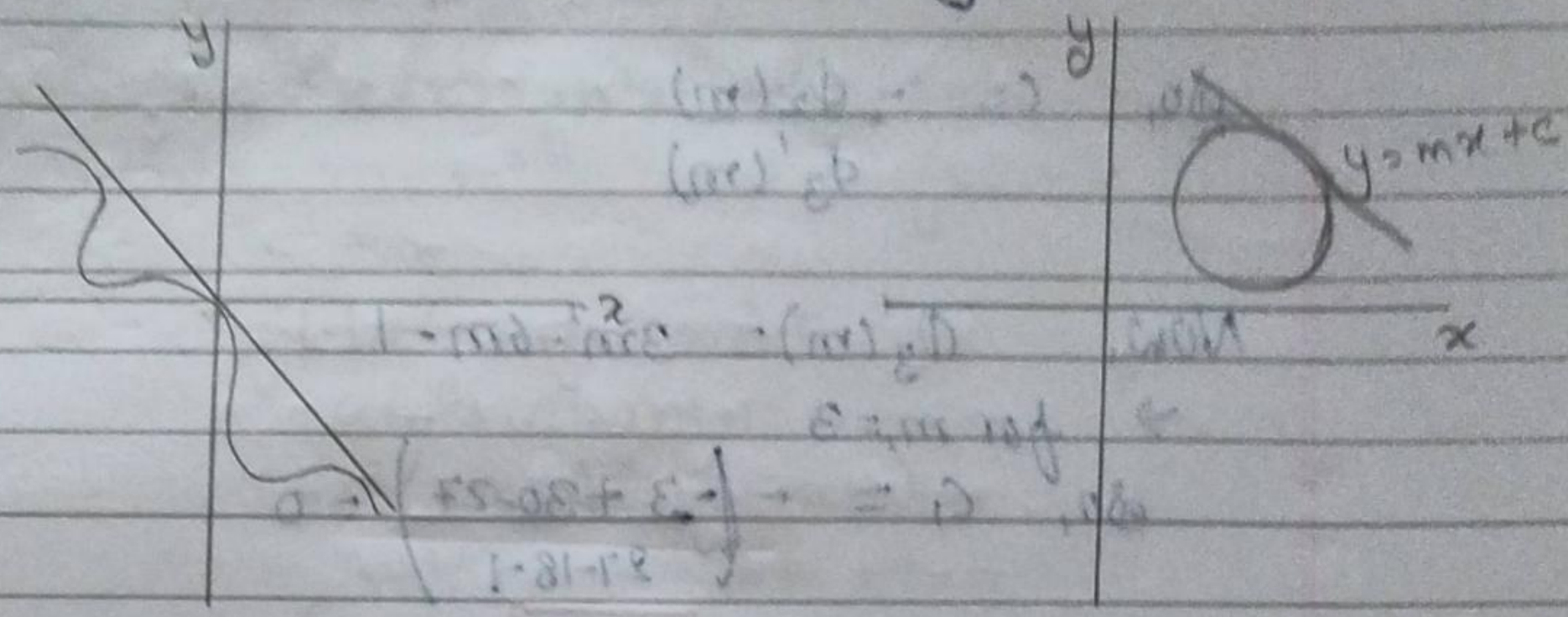
$$\boxed{\frac{C^2}{2!}\phi_3''(m) + \frac{C}{1!}\phi_2'(m) + \phi_1 m = 0}$$ when 'm' has equal roots.

$$\left\{ C = -\frac{\phi_{(n-1)}(m)}{\phi_n'(m)} \right\}$$ general

(BSC 103)

ASYMPTOTE

Asymptotes :- (tangent to infinity)



open curve close curve

** $\boxed{\text{no. of } \overset{\text{values of } m}{\cancel{\text{tangents}}} = \text{highest degree of eq}^n}$

Eg. $y^3 - 3xy^2 - x^2y + 3x^3 - 3x^2 + 10xy - 3y^2 - 10x - 10y + 7 = 0$

Sol^n collect same degree terms.

$\Rightarrow$ ~~put $x = 1$ & $y = m$~~

$\phi_3(x) = y^3 - 3xy^2 - x^2y + 3x^3$

$\phi_2(x) = -3x^2 + 10xy - 3y^2$

$\phi_1(x) = -10x - 10y$

put $x = 1$ & $y = m$

$\phi_3(m) = m^3 - 3m^2 - m + 3 = 0$ —①

$\phi_2(m) = -3 + 10m - 3m^2$

$\phi_1(m) = -10 - 10m$

put eq^n ① is equal to zero

$m^3 - 3m^2 - m + 3 = 0$

$m^2(m-3) - 1(m-3) = 0$

$m - 3 = 0$, $m^2 - 1 = 0$

$m = 3$, $m = \pm 1$

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$\{m = 3, 1, -1\}$ different values

so, $c = \dfrac{-\phi_2(m)}{\phi_3'(m)}$

Now, $\phi_3'(m) = 3m^2 - 6m - 1$

→ for $m_1 = 3$

so, $c_1 = -\left(\dfrac{-3 + 30 - 27}{27 - 18 - 1}\right) = 0$

$\boxed{c_1 = 0}$

→ for $m_2 = 1$

$c_2 = -\left(\dfrac{-3 + 10 - 3}{3 - 6 - 1}\right)$

$\boxed{c_2 = 1}$

→ for $m_3 = -1$

$\boxed{c_3 = 2}$

$y_1 = m_1 x + c_1 = 3x + 0$

$\boxed{y - 3x = 0}$

$y_2 = m_2 x + c_2 = x + 1$

$\boxed{y - x - 1 = 0}$

$y_3 = m_3 x + c_3 = -x + 2$

$\boxed{y + x - 2 = 0}$

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eg $x^3+2x^2y-xy^2-2y^3+xy-y^2-1=0$

Soln put $x=1$ & $y=m$

$\phi_3(m)=1+2m-m^2-2m^3$

$\phi_2(m)=m-m^2$

$1+2m-m^2-2m^3=0$

$2m^3+m^2-2m-1=0$

$m^2(2m+1)-1(2m+1)=0$

$m^2=1 \qquad 2m+1=0$

$\{m=\pm 1 \qquad m=-1/2\}$

$c=-\dfrac{\phi_2(m)}{\phi_3'(m)} \Rightarrow \phi_3'(m)=2-2m-6m^2$

$m=1,\ c_1=-\left(\dfrac{1-1^2}{2-2-6}\right)=0$. $\boxed{c_1=0}$

$m_2=-1,\ c_2=\left(\dfrac{-1-1}{-6+2+2}\right)=-\left(\dfrac{-2}{-2}\right)=-1$, $\boxed{c_2=-1}$

$m_3=-1/2 \qquad c_3=-\left(\dfrac{-1/2-1/4}{-6\times\frac{1}{4}+2\times\frac{1}{2}+2}\right) \Rightarrow \boxed{c_3=\frac{1}{2}}$

So, $y=x+0$, $\boxed{y-x=0}$

$y=-x-1$, $\boxed{y+x+1=0}$

$y=-\frac{1}{2}x+\frac{1}{2}$, $\boxed{y+\frac{x}{2}-\frac{1}{2}=0}$

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eg. $(x^2-y^2)(y^2-4x^2)+6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1=0$

Sol^n $x^2y^2-4x^4-y^4+4x^2y^2+6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1=0$

$\phi_4 = x^2y^2-4x^4-y^4+4x^2y^2$
$\phi_3 = 6x^3-3xy^2+2y^3-5x^2y$
$\phi_2 = -x^2+3xy$

put $x=1$ & $y=m$

$\phi_4(m) = m^2-4-m^4+4m^2$
$\phi_3(m) = 6-3m^2+2m^3-5m$
$\phi_2(m) = -1+3m$

put $\phi_4(m)=0$

Now, $m^2-4-m^4+4m^2=0$
$4m^2-m^4+m^2-4=0$
$m^2(4-m^2)-1(4-m^2)=0$
$\{m^2=1, \quad 4=m^2>0$
$\{m=\pm 1, \quad m=\pm 2\}$

$\left\{c=\frac{-\phi_3(m)}{\phi_4'(m)}\right\}$ $\quad \phi_4'(m)=2m-4m^3+8m$

$m_1=1$, $c_1=-\left[\frac{6-3+2-5}{2-4+8}\right]$, $\boxed{c_1=0}$

$m_2=-1$, $c_2=-\left[\frac{6-3-2+5}{-2+4-8}\right]$, $\boxed{c_2=1}$

$m_3=2$, $c_3=-\left[\frac{6-12+16-10}{20-32}\right]$, $\boxed{c_3=0}$

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$m_4 = -2$, $\boxed{c_4 = 1}$

$y_1 = m_1 x + c$, $\boxed{y = -x + 1}$, $\boxed{y = -2x + 1}$

$\boxed{y = x}$, $\boxed{y = 2x}$

If roots are equal:-

eg. $x^3 - 5x^2y + 8xy^2 - 4y^3 + x^2 - 3xy + 2y^2 - 1 = 0$

Sol^n put $x = 1$ and $y = m$

$\phi_3(m) = 1 - 5m + 8m^2 - 4m^3$

$\phi_2(m) = 1 - 3m + 2m^2$

$\phi_1(m) = 0$

Let, $\phi_3(m) = 0$

$1 - 5m + 8m^2 - 4m^3 = 0$

$4m^3 - 8m^2 + 5m - 1 = 0$

To solve this eq^n, methods are-

① ~~put value~~ first find a value, which satisfies the eq^n.

Here put $m = 1$, $4(1)^3 - 8(1)^2 + 5(1) - 1 = 0$

we get one root $(m-1)$ $0 = 0$

Now, $[4m^2(m-1) - 4m(m-1) + 1(m-1)]$

② by division-

$$
\begin{array}{r|l}
(m-1) & 4m^3 - 8m^2 + 5m - 1 \quad \underbrace{4m^2 - 4m + 1}_{2^{nd}\text{ root}} \\
 & 4m^3 - 4m^2 \\
 & - \quad + \\
\hline
 & -4m^2 + 5m - 1 \\
 & -4m^2 + 4m \\
\hline
 & m - 1 \\
 & m - 1 \\
\hline
 & 0
\end{array}
$$

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③ from eq^n → $4m^3 - 8m^2 + 5m - 1$

Here put 1st root (1) (which we find from ① method)

1	4	-8	5	-1
	last ltp.	4 (add)	-4	1
	4	-4	1	0

उपर digit को '1' से multiply करके आगे '-8' के नीचे लिखेंगे फिर '-8 & 4' को add करेंगे।

फिर इसी तरह '-4' को '1' से multiply करके '5' के नीचे लिखेंगे और ऐसे आगे भी ऐसे ही करना है।

एक-एक power कम करके लिखेंगे

'm' की highest power '3' है।

NOW,

$$4m^2 - 4m + 1$$

$$(m-1)(4m^2 - 4m + 1) = 0$$

$$(m-1)[4m^2 - 2m - 2m + 1] = 0$$

$$(m-1)[2m(2m-1) - 1(2m-1)] = 0$$

$$(m-1)(2m-1)(2m-1) = 0$$

$$\{m = 1, \tfrac{1}{2}, \tfrac{1}{2}\}$$

Here In $m_1 = 1$, ope. $c_1 = -\dfrac{\phi_2(m)}{\phi_3'(m)}$

$$c_1 = -\left[\frac{1 - 3m + 2m^2}{-5 + 16m - 12m^2}\right]$$

$$\boxed{c_1 = 0}$$

$$\boxed{y - x = 0} \quad \text{--- ①}$$

→ For equal roots:

$$\frac{c^2}{2!}\phi_3''(m) + \frac{c}{1}\phi_2'(m) + \phi_1(m) = 0$$

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$\phi_2'(m) = -3+4m$

$\phi_3''(m) = 16-24m$

for $m = 1/2$ ⇒ $\frac{c^2}{2!}(16-24m) + \frac{c}{1!}(-3+4m) + 0 = 0$

Now, for $m = 1/2 = m_2 = m_3$

⇒ $\frac{c^2}{2}\left(16-24\times\frac{1}{2}\right) + \frac{c}{1}\left(-3+4\times\frac{1}{2}\right) = 0$

$\frac{c^2}{\cancel{2}} \times \cancel{4}^{2} - c = 0$

$2c^2 - c = 0$

⇒ $c(2c-1) = 0$

$\{c=0\}$ $2c = 1$

$\{c = 1/2\}$

$y - x = 0$

$y - \frac{1}{2}x = 0$, $y - \frac{1}{2}x - \frac{1}{2} = 0$ Ans

⇒ If highest degree is '3' then:-

$$\left\{\frac{c^3}{3!}\phi_4'''(m) + \frac{c^2}{2!}\phi_3''(m) + \frac{c}{1!}\phi_2'(m) + \phi_1(m) = 0\right\}$$

eg. $(x+y)^2(x+2y+2) = x+9y-2$

Sol: $(x^2+y^2+2xy)(x+2y+2) = x+9y-2$

$x^3+2yx^2+2x^2+xy^2+2y^3+2y^2+2x^2y+4xy^2+4xy$
$= x+9y-2$

$x^3+4x^2y+4xy^2+xy^2+2y^3 = \phi_3(x)$

$2x^2+2y^2+4xy = \phi_2(x)$

$-x-9y = \phi_1(x)$

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put $x=1$ and $y=m$

$1+4m+5m^2+2m^3=0$

find satisfying value of this eqn

put $(m=-1)$

$1+4(-1)+5(-1)^2+2(-1)^3=0$

$[6-6=0]$

$$\begin{array}{l|l} m+1 & 2m^3+5m^2+4m+1 \quad (2m^2+3m+1 \\ & \underline{2m^3+2m^2} \\ & \quad 3m^2+4m+1 \\ & \quad \underline{3m^2+3m} \\ & \qquad m+1 \\ & \qquad \underline{m+1} \\ & \qquad 0 \end{array}$$

$(m+1)(2m^2+3m+1)$

$(m+1)(2m^2+2m+m+1)$

$(m+1)(2m+1)(m+1)$

$\{m=-1,-1 \,\&\, -1/2\}$

$m=-1/2$

$$c=-\frac{\phi_2(m)}{\phi_3(m)}=-\left[\frac{2+2m^2+4m}{6m^2+10m+4}\right]$$

$$c=-\left[\frac{2+2(-1/2)^2+4(-1/2)}{6\times(-1/2)^2+10(-1/2)+4}\right] \quad \boxed{c=-1}$$

$$c=\frac{c^2}{2!}(\phi_3)''m+\frac{c}{1!}\phi_2'(m)+\phi_1(m)=0$$

$$\frac{c^2}{2}(12m+10)+\frac{c}{1}(4m+4)+(-10m)=0$$

put $m = -1$

$$\frac{c^2}{2}(-12+10) + \frac{c}{1}(-4+4) + (-1+9) = 0$$

$$-c^2 + 8 = 0$$

$$\boxed{c = \pm 2\sqrt{2}}$$

$$y = -\frac{1}{2}x - 1, \quad y_2 = -1(x) + 2\sqrt{2}$$

$$y_3 = -x - 2\sqrt{2} \quad \underline{\text{Ans}}$$

<u>Asymptotes:-</u>

- Definition:- A straight line at a finite distance from the origin to which a tangent to a curve tends as the distance from the origin of the point of contact tends to infinity is called an <u>asymptote of the curve.</u>

→ In other words, asymptote is a line that a curve approaches as it move towards infinity.

$$\left\{ \text{no. of degree} = \text{no. of values of 'm'} \right\}$$
not equal to no. of asymptotes.

→ Eq^n of line parallel to y-axis | parallel to x-axis

$x = \pm k$ | $y = \pm k$

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PARALLEL TO COORDINATE AXIS (EITHER x & y):-

① Find asymptote of-

$$x^2y^2 - a^2(x^2+y^2) - a^3(x+y) = 0$$

Ans $x^2y^2 - a^2x^2 - a^2y^2 - a^3(x+y) = 0$

its coefficient is const. so we can't put it equal to 0.

In this, put highest degree term = 0 but it should be variable not const.

I st possibility $x^2(y^2-a^2) - a^2y^2 - a^3(x+y) = 0$

put $x^2(y^2-a^2) = 0$

$y^2 = a^2$

$\boxed{y = \pm a}$

II nd

$y^2(x^2-a^2) - a^2x^2 - a^3(x+y) = 0$

$x^2 - a^2 = 0$

$\boxed{x = \pm a}$

② $a^2y^2 + b^2y^2 = x^2y^2$

$y^2(a^2+b^2-x^2) = 0$

$a^2+b^2 = x^2$

$\boxed{x = \pm\sqrt{a^2+b^2}}$

Eg. $y^3 - 6xy^2 + 11x^2y - 6x^3 + y + x = 0$

Soln $\phi_3(x) = y^3 - 6xy^2 - 6x^3 + 11x^2y$

$\phi_2(x) = 0$

$\phi_1(x) = y + x$

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put $x=1$ & $y=m$

$\phi_3(m) = m^3 - 6m^2 - 6 + 11m$

$\phi_2(m) = 0$

$\phi_1(m) = m+1$

$\phi_3(m) = 0$

$m^3 - 6m^2 - 6 + 11m = 0$

Satisfying value of this eqn is $m=1$

so one factor of this eqn is $(m-1)$

$$
\begin{array}{r|l}
(m-1) & m^3 - 6m^2 + 11m - 6 \quad m^2 - 5m + 6 \\
 & m^3 - m^2 \\
 & - \quad + \\
\hline
 & -5m^2 + 11m - 6 \\
 & -5m^2 + 5m \\
 & + \quad - \\
\hline
 & 6m - 6 \\
 & 6m - 6 \\
\hline
 & \times
\end{array}
$$

$(m-1)(m^2 - 5m + 6) = 0$

$(m-1)[m(m-3) - 2(m-3)] = 0$

$(m-1)(m-2)(m-3) = 0$

$\boxed{m = 1, 2, 3}$

$$C = -\frac{\phi_2(m)}{\phi_3'(m)} = 0 \quad (\because \phi_2(m) = 0)$$

So, $y = x$, $y = 2x$, $y = 3x$ Ans

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Eg. Find asymptotes of the following -

(i) $\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$

Mth-I

Soln

$$a^2y^2 - b^2x^2 = x^2y^2$$
$$a^2y^2 - x^2y^2 = b^2x^2$$
$$y^2(a^2 - x^2) - b^2x^2 = 0$$
$$a^2 - x^2 = 0$$
$$\boxed{a = \pm x}$$

Mth-II

$$a^2y^2 - b^2x^2 = x^2y^2$$
$$\phi_4(x) = x^2y^2 \quad , \phi_2(x) = a^2y^2 - b^2x^2$$
$$\phi_4(m) = m^2 \quad , \phi_2(m) = a^2m - b^2$$
$$m^2 = 0$$
$$\boxed{m_1 = 0 \ , \ m_2 = 0}$$

so, $c = \frac{c^2}{2}\phi_4''(m) + c\,\phi_3'(m) + \phi_2(m) = 0$

$$= \frac{c^2}{2}(2) + c \times 0 + (a^2m - b^2) = 0$$

put $m = 0$

$$\Rightarrow c^2 - b^2 = 0$$
$$\boxed{c = \pm b} \quad \left(\begin{array}{l} y = b \\ \& \ y = -b \end{array} \right)$$

अगर 'm' की value '3' बार आएगी तो 'c' आएगा और '3' बार diff^n होगा।

(ii) $2x(y-3)^2 = 3y(x-1)^2$

Soln

$$2x[y^2 + 9 - 6y] = 3y[x^2 + 1 - 2x]$$
$$2xy^2 + 18x - 12xy = 3yx^2 + 3y - 6xy$$

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$2xy^2 - 3yx^2 + 18x - 3y - 6xy = 0$

put $x=1$ & $y=m$

$\phi_3(x) = 2xy^2 - 3yx^2$ $\qquad$ $\phi_3(m) = 2m^2 - 3m$

$\phi_2(x) = -6xy$ $\qquad$ $\phi_2(m) = -6m$

$\phi_1(x) = 18x - 3y$ $\qquad$ $\phi_1(m) = 18 - 3m$

put $\phi_3(m) = 0$

$2m^2 - 3m = 0$

$m(2m-3) = 0$

$m = 0 \; ; \; 2m - 3 = 0$

$\{m_1 = 0, m_2 = 3/2\}$

So, $c = \dfrac{-\phi_2(m)}{\phi_3'(m)} = -\left[\dfrac{-6m}{4m-3}\right]$

for $m_1 = 0$, $\boxed{c_1 = 0}$

$m_2 = 3/2 = -\left[\dfrac{-6 \times 3/2}{4 \times 3/2 - 3}\right]$ & $\boxed{c_2 = 3}$

So, $\boxed{y = 0}$

and $y = \frac{3}{2}x + 3$

$\boxed{2y - 3x - 6 = 0}$ Ans

Imp.

Some other Examples:-

eg. Given that $4(x^4 + y^4) - 17x^2y^2 - 4x(4y^2 - x^2) + 2(x^2 - 2) = 0$ show that they passes through the point of intersection of the curve with ellipse which is $\boxed{x^2 + 4y^2 = 4}$

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$2xy^2 - 3yx^2 + 18x - 3y - 6xy = 0$

put $x=1$ & $y=m$

$\phi_3(x) = 2xy^2 - 3yx^2$ $\quad$ $\phi_3(m) = 2m^2 - 3m$

$\phi_2(x) = -6xy$ $\quad$ $\phi_2(m) = -6m$

$\phi_1(x) = 18x - 3y$ $\quad$ $\phi_1(m) = 18 - 3m$

put $\phi_3(m) = 0$

$2m^2 - 3m = 0$

$m(2m-3) = 0$

$m = 0, \; 2m - 3 = 0$

$\{m_1 = 0, \; m_2 = 3/2\}$

So, $c = \dfrac{-\phi_2(m)}{\phi_3'(m)} = -\left[\dfrac{-6m}{4m-3}\right]$

for $m_1 = 0$, $\boxed{c_1 = 0}$

$m_2 = 3/2 = -\left[\dfrac{-6 \times 3/2}{4 \times 3/2 - 3}\right]$ & $\boxed{c_2 = 3}$

So, $\boxed{y = 0}$

and $y = \frac{3}{2}x + 3$

$\boxed{2y - 3x - 6 = 0}$ Ans

Imp.

Some other Examples:-

eg. Given that $4(x^4 + y^4) - 17x^2y^2 - 4x(4y^2 - x^2) + 2(x^2 - 2) = 0$ show that they passes through the point of intersection of the curve with ellipse which is $\boxed{x^2 + 4y^2 = 4}$

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Sol^n Step 1:- Find asymptote of given equation

$$4(x^4+y^4)-17x^2y^2-4x(4y^2-x^2)+2(x^2-2)=0$$

$$4x^4+4y^4-17x^2y^2-16xy^2+4x^3+2x^2-4=0 \quad -(1)$$

$$\phi_4(x)=4x^4+4y^4-17x^2y^2$$

$$\phi_3(x)=4x^3-16xy^2$$

$$\phi_2(m)=2x^2$$

Step 2: put $x=1$ and $y=m$

$$\phi_4(m)=4+4m^4-17m^2$$

$$\phi_3(m)=4-16m^2 \quad , \quad \phi_2(m)=2$$

Step 3: put $\phi_4(m)=0$

$$4+4m^4-17m^2=0$$

Satisfying value of this eq^n is $+2$ & -2

So factor will be $(m\pm 2)$

$$(m+2)(m-2)=m^2-4$$

$$m^2-4 \,\big|\, 4+4m^4-17m^2 \,\big|\, 4m^2-1$$

$$\underline{4m \quad 4m^4 \quad -16m^2}$$

$$\underline{4 \quad -m^2}$$

$$\underline{4 \quad -m^2}$$

$$0$$

$$4m^2-1=0$$

$$\left\{m=\pm\tfrac{1}{2}\right\}$$

$$\left\{\text{Values of } \boxed{m=\pm 2, \pm\tfrac{1}{2}}\right\}$$

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$$C = -\frac{\phi_3(m)}{\phi_4'(m)} = -\left[\frac{4-16m^2}{16m^3-34m}\right]$$

for, $m_1 = 2$, $C_1 = -\left[\frac{4-64}{128-68}\right] = -\left[\frac{-60}{60}\right] \Rightarrow \boxed{C_1 = 1}$

$m_2 = -2$, $C_2 = -\left[\frac{4-64}{-128+68}\right] \Rightarrow \boxed{C_2 = -1}$

$m_3 = +1/2$, $C_3 = -\left[\frac{4-16\times 1/4}{16\times 1/8 - 34\times 1/2}\right]$, $\boxed{C_3 = 0}$

$m_4 = -1/2$ $\boxed{C_4 = 0}$

eq^n of tangents are-

$y - 2x - 1 = 0$ — (1) $\qquad$ $y - \frac{1}{2}x = 0$ — (3)

$y + 2x + 1 = 0$ — (2) $\qquad$ $y + \frac{1}{2}x = 0$ — (4)

Multiply all eq^{n}s of tangent

$$(y-2x-1)(y+2x+1)\left(y-\frac{x}{2}\right)\left(y+\frac{x}{2}\right) = 0$$

$$\left(y^2 + \cancel{2xy} + \cancel{y} - \cancel{2xy} - 4x^2 - 2x - \cancel{y} - 2x - 1\right)\left(y^2 - \frac{x^2}{4}\right) = 0$$

$$\left(y^2 - 4x^2 - 4x - 1\right)\left(y^2 - \frac{x^2}{4}\right) = 0$$

$$4y^4 - x^2y^2 - 16x^2y^2 + 4x^4 - 16xy^2 + 4x^3 - 4y^2 + x^2 = 0 \quad \text{— (2)}$$

Subtract (2) − (1)

$$\cancel{4y^4} - \cancel{17x^2y^2} + \cancel{4x^4} - \cancel{16xy^2} + \cancel{4x^3} - \cancel{4y^2} + x^2 = 0$$

$$\cancel{4y^4} - \cancel{17x^2y^2} + \cancel{4x^4} - \cancel{16xy^2} + \cancel{4x^3} + 2x^2 - 4 = 0$$

$$- \quad + \quad - \quad + \quad - \quad + \quad - \quad +$$

$$-x^2 - 4y^2 + 4 = 0$$

$$-(x^2 + 4y^2) = -4$$

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$$\boxed{x^2+4y^2=4}$$

Hence Proved.

Imp

Eg Show that the 4 asymptotes of the curve $(x^2-y^2)(y^2-4x^2)+6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1=0$. cut the curve again in 8 points which lies on the circle.

$$\boxed{x^2+y^2=1}$$

$$\boxed{\text{no. of points of intersection} = n(n-2)}$$

$\therefore n =$ no. of highest degree

In this question, $n=4$

$$4(4-2)=8$$

Soln $x^2y^2-4x^4-y^4+4x^2y^2+6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1=0$

$$-4x^4-y^4+5x^2y^2+6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1=0 \quad (1)$$

$\phi_4(x) = -4x^4-y^4+5x^2y^2$

$\phi_3(x) = 6x^3-3xy^2+2y^3-5x^2y$

$\phi_2(x) = -x^2+3xy$

$\phi_4(m) = -4-m^4+5m^2$

$\phi_3(m) = 6-3m^2+2m^3-5m$

$\phi_2(m) = -1+3m$

$$\phi_4(m)=0$$

$$-4-m^4+5m^2=0$$

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$m^4 - 5m^2 + 4 = 0$

$m^4 - 4m^2 - m^2 + 4 = 0$

$m^2(m^2-4) - 1(m^2-4)$

$\{m = \pm 1 \; ; \; m = \pm 2\}$

$$c = -\left[\begin{matrix} 6 - 3m^2 + 2m^3 - 5m \\ -4m^3 + 10m \end{matrix}\right]$$

$m = 1$, $\boxed{c_1 = 0}$ $\quad m = 2$, $\boxed{c_3 = 0}$

$m = -1$, $\boxed{c_2 = 1}$ $\quad m = -2$, $\boxed{c_4 = 1}$

$y = x$

$y - x = 0$, $y - x - 1 = 0$, $y - 2x = 0$, $y + 2x - 1 = 0$

$(y - x)(y + x - 1)(y - 2x)(y + 2x - 1) = 0$

$(y^2 + \cancel{xy} - y - \cancel{xy} - x^2 + x)(y^2 + \cancel{2xy} - y - \cancel{2xy} - 4x^2 + 2x) = 0$

$(y^2 - x^2 - y + x)(y^2 - 4x^2 - y + 2x) = 0$

$= 0$

(1)

2) $y^4 - 4x^2y^2 - y^3 + 2xy^2 - x^2y^2 + 4x^4 + x^2y - 2x^3 - y^3 + 4x^2y$
$+ y^2 - 2xy + xy^2 - 4x^3 - xy + 2x^2 = 0$ (2)

(1) + (2)

$\Rightarrow \cancel{y^4} + \cancel{4x^4} - \cancel{5x^2y^2} - \cancel{2y^3} + \cancel{3xy^2} + \cancel{5x^2y} - \cancel{6x^3} - \cancel{3xy} + y^2 + 2x^2 = 0$

$-\cancel{y^4} - \cancel{4x^4} + \cancel{5x^2y^2} + \cancel{2y^3} - \cancel{3xy^2} - \cancel{5x^2y} + \cancel{6x^3} + \cancel{3xy} - x^2 - 1 = 0$

$\Rightarrow y^2 + x^2 - 1 = 0$

$\boxed{y^2 + x^2 = 1}$ Ans

Hence Proved.

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$\boxed{\text{Limit Method}}$:-

eg. $(x-y+2)(2x-3y+4)(4x-5y+6)+5x-6y+7=0$

first find slope of $(x-y+2)=0$

$x+2=y$

acc. to $y=mx+c$

here $\boxed{m=1}$

means $\frac{y}{x}=1$ (slope)

Now we put limit $x\to\infty$ in other form $\frac{1}{x}\to 0$

~~so app~~ Now, we have to find asymptotes parallel to $(x-y+2)$. For this we have to divide whole eq^n by $(2x-3y+4)(4x-5y+6)$ (जिसके लिए निकालना है उसे छोड़कर बाकी बची हुई value से divide करना है)

$$\frac{(x-y+2)\cancel{(2x-3y+4)(4x-5y+6)}}{\cancel{(2x-3y+4)(4x-5y+6)}}+\frac{5x-6y+7}{(2x-3y+4)(4x-5y+6)}=0$$

$$x-y+2+\lim_{\substack{1/x\to 0\\ \frac{y}{x}\to 1}}\frac{\cancel{x}\left[5-6y/x+7/x\right]}{\cancel{x}\left[2-3y/x+\frac{y}{x}\right]x\left[4-5y/x+6/x\right]}$$

now put $\frac{1}{x}=0$ & $\frac{y}{x}=1$

In denominator there will be an 'x' so whole term becomes '0'

$x-y+2+0=0$

$\boxed{x-y+2=0}$

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Similarly, $\left\{ \begin{array}{l} 2x-3y+4=0 \\ 4x-5y+6=0 \end{array} \right\}$

These are the eqn of asymptotes.

Eg $(x+y)^2(x+2y+2) = x+9y-2$

Soln

$$(x+y)^2 = 0$$

$$x+y=0$$

$$y \quad \boxed{m=-1}$$

$$\frac{y}{x} = -1$$

$$(x+y)^2 = \frac{x+9y-2}{x+2y+2}$$

$$(x+y)^2 = \lim_{\substack{1/x \to 0 \\ y/x \to 1}} \frac{\cancel{x}\left[1+9\,y/x - 2/x\right]}{\cancel{x}\left[1+2\,y/x + 2/x\right]}$$

put $1/x = 0$ & $y/x = -1$

$$(x+y)^2 = \frac{1+9(-1)}{1+2(-1)} = \frac{-8}{-1} = 8$$

$$(x+y)^2 = 8$$

$$x+y = \pm 2\sqrt{2}$$

eqn $\Rightarrow \left[x+y-2\sqrt{2}=0 \;(1) \;\&\; x+y+2\sqrt{2}=0 \;(2) \right]$

for $(x+2y+2)=0$

$$\{m = -1/2\}$$

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$$x+2y+2 = \frac{x+9y-2}{(x+y)^2}$$

$$= \lim_{\substack{1/x \to 0 \\ \frac{y}{x} \to -1/2}} \frac{x\left(1+9\,y/x - 2/x\right)}{x^2\left(1+y/x\right)^2}$$

$$\boxed{x+2y+2 = 0} \quad \text{--- (3)}$$

Concavity and Convexity:- particular maxima - local max/min. ~~pra~~ 2) curve ki - global max/min.

- Concave upward or Convex downward:-
  A curve is said to be concave upward or convex downward on (a,b) if all the points of curve lies ~~on~~ above tangent to it on the interval.

- Concave downward or Convex Upward:-
  A curve is said to be concave downward or convex upward on (a,b) if all the points of the curve lies below tangent to it on the interval.

Test for Concavity & Convexity:-

1) A curve is ~~said~~ convex or concave at point P w.r.t x-axis according as $\boxed{y\frac{d^2y}{dx^2}}$ is positive or negative at point P.

2) A curve is said to be convex or convex at pt. 'P' w.r.t y-axis ~~acc as~~ according

as $\boxed{x \cdot \frac{d^2x}{dy^2}}$ is +ive or -ive at pt 'P'.

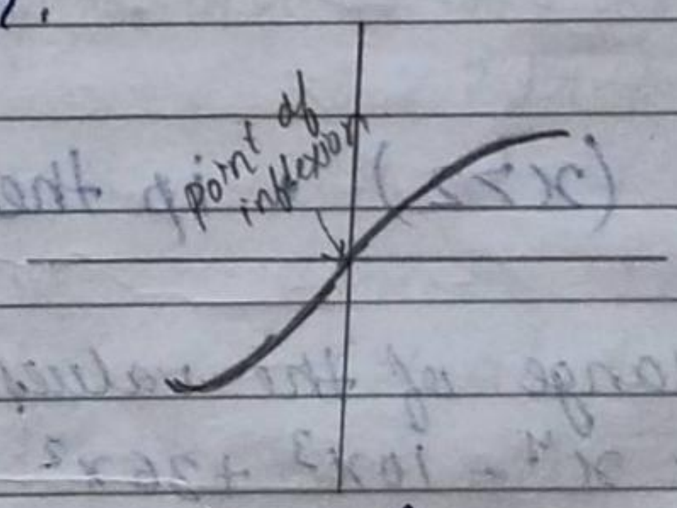
⇒ A curve is convex downward or concave upward $[a, b]$ if $\frac{d^2y}{dx^2}$ is greater than '0'.

$$\boxed{\frac{d^2y}{dx^2} > 0 \; ; \forall \; x \in [a, b]}$$

⇒ Similarly, concave downward or convex upward.

$$\boxed{\frac{d^2y}{dx^2} < 0 \; ; \forall \; x \in [a, b]}$$

<u>Point of Inflexion</u>: A point on a curve at which the curve changes from concavity to convexity or convexity to concavity is called pt. point of inflexion.



⇒ for any curve, $f(x, y) = 0$, all points where, $\boxed{\frac{d^2y}{dx^2} = 0}$ but $\boxed{\frac{d^3y}{dx^3} \neq 0}$ is called point of it inflexion.

<u>Eg.</u> Find the values of 'x' for which $y = x^4 - 6x^3 + 12x^2 + 5x + 7$ is Concave upward ?

Ans. $\frac{dy}{dx} = 4x^3 - 18x^2 + 24x$

$$\frac{d^2y}{dx^2} = 12x^2 - 36x + 24.$$

for concave upward $\frac{d^2y}{dx^2} > 0$

$$(12x^2 - 36x + 24) > 0$$
$$12(x^2 - 3x + 2) > 0$$
$$x^2 - 2x - x + 2 > 0$$
$$(x-1)(x-2) > 0$$
$$\boxed{x > 1, x > 2}$$

इस cond^n का मतलब $(x-1)(x-2)$ का product +ve आना चाहिए।

$(x-1)$ और $(x-2)$ +ve तभी होंगे जब $(x-1)$ भी +ve हो और $(x-2)$ भी तो अगर हम $x = 1.5$ लेंगे तो $(x-1)$ वाली term तो +ve आएगी लेकिन $(x-2)$ वाली -ive हो जाएगी इसलिए -

So, $(x > 2)$ is the condition.

Que. Find the range of the values of 'x' in which the curve $y = x^4 - 10x^3 + 36x^2 + 24x + 11$ is convex and concave upward.

Sol^n $\frac{dy}{dx} = 4x^3 - 30x^2 + 72x + 24$

$$\frac{d^2y}{dx^2} = 12x^2 - 60x + 72.$$

for convex upward $\frac{d^2y}{dx^2} < 0$

$$12(x^2 - 5x + 6) < 0$$

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$$x^2 - 5x + 6 < 0$$
$$(x-3)(x-2) < 0$$
$$x < 3 \;,\; x < 2$$

So, cond^n will be $\boxed{2 < x < 3}$ Ans

⇒ for concave upward, $\frac{d^2y}{dx^2} > 0$
$$(x-3)(x-2) > 0$$
So, $(x > 3)$ Ans

Q. Show that the point of inflexion of the curve $y^2 = (x-a)^2(x-b)$ lies on the straight line $3x + a = 4b$.

Sol^n $y^2 = (x-a)^2(x-b)$
$$y = (x-a)\sqrt{x-b}$$

$$\frac{dy}{dx} = (x-a)\frac{1}{2\sqrt{x-b}} + \sqrt{x-b}$$

$$\frac{dy}{dx} = \frac{(x-a)}{2\sqrt{x-b}} + \sqrt{x-b}$$

$$\frac{dy}{dx} = \frac{(x-a) + 2(x-b)}{2\sqrt{x-b}} = \frac{2x - 2b + x - a}{2\sqrt{x-b}}$$

$$\frac{dy}{dx} = \frac{3x - 2b - a}{2\sqrt{x-b}}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}\left[\frac{(\sqrt{x-b})\,3 - (3x - 2b - a)\,\frac{1}{2}(\sqrt{x-b})}{(2\sqrt{x-b})^2}\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}\left[\frac{3(\sqrt{x-b}) - \frac{3x-2b-a}{2\sqrt{x-b}}}{(x-b)}\right]$$

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$$\frac{d^2y}{dx^2} = \frac{3x-4b+a}{4(x-b)^{3/2}}$$

put $\frac{d^2y}{dx^2} = 0$

$$\frac{3x-4b+a}{4(x-b)\sqrt{x-b}} = 0$$

$$\boxed{3x-4b+a=0}$$ Hence proved.

$$x = \frac{4b-a}{3}$$

for pt. of inflexion, $\frac{d^3y}{dx^3} \neq 0$. for value of x.

so, $$\frac{d^3y}{dx^3} = \frac{4(x-b)(\sqrt{x-b})(3) - [(3x-4b+a)(\sqrt{x-b})]}{16(x-b)^3}$$

$$\frac{d^3y}{dx^3} = \frac{\cancel{6}^{3}\left[2(x-b)\sqrt{x-b} - \underbrace{(3x-4b+a)}_{0}\sqrt{x-b}\right]}{\cancel{16}\,8\,(x-b)^3}$$

put $x = \frac{4b-a}{3}$ so, $3x-4b+a=0$

$$\frac{d^3y}{dx^3} = \frac{3\left[2(x-b)^{3/2}\right]}{8\left[(x-b)^3\right]}$$

if we put $x = \frac{4b-a}{3}$

then $\left[\frac{d^3y}{dx^3} \neq 0\right]$ Hence, this is the point of inflexion.

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Eg find the point of inflexion of each of the following

(a) $y=(x^2+4x+5)e^{-x}$

Sol $\frac{dy}{dx} = -e^{-x}(2x+4)$

$= u.v$

$$\frac{dy}{dx} = -(x^2+4x+5)e^{-x} + e^{-x}(2x+4)$$

$$\frac{dy}{dx} = e^{-x}[-x^2-4x-5+2x+4]$$

$$\frac{dy}{dx} = e^{-x}[-x^2-2x-1]$$

$$\frac{d^2y}{dx^2} = e^{-x}[-2x-2] - (-x^2-2x-1)e^{-x}$$

$$\frac{d^2y}{dx^2} = e^{-x}[\cancel{-2x}-2+x^2+\cancel{2x}+1]$$

$$\frac{d^2y}{dx^2} = e^{-x}(x^2-1)$$

for pt. of inflection $\frac{d^2y}{dx^2}=0$

so, $x^2-1=0$

$\boxed{x=\pm 1}$

for pt. of inflection $\frac{d^3y}{dx^3} \neq 0$

$$\frac{d^3y}{dx^3} = e^{-x}(2x) - e^{-x}(x^2-1)$$

$$\frac{d^3y}{dx^3} = e^{-x}[2x-x^2+1]$$

put $x=1$

$$\frac{d^3y}{dx^3} = e^{-1}[2-1+1] = e^{-1}(2)$$

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$x=-1$, $\frac{d^3y}{dx^3} = e^{+1}(-2-1+1)$

$= -2e$ Ans

(ii) $y = 3x - 4\sin x + \sin 2x$

soln $\frac{dy}{dx} = 3 - 4\cos x + \cos 2x \cdot 2$

$\frac{d^2y}{dx^2} = 4\sin x - \sin 2x \cdot 4$

$4\sin x - 4\sin 2x = 0$

$\sin x - \sin(2x) = 0$

$\sin x - 2\sin x \cos x = 0$

$\sin x\,[1 - 2\cos x] = 0$

$\left[\begin{array}{l} x = 0 \\ x = \frac{\pi}{3} \end{array}\right]$, $1 - 2\cos x = 0$ $\Leftarrow$ $x = \frac{1}{2}$

So, $\frac{d^3y}{dx^3} = 4[\sin x - \sin 2x]$

$\frac{d^3y}{dx^3} = 4[\cos x - \cos 2x \cdot 2]$

put $x = 0$, $\frac{d^3y}{dx^3} = 4[1-2] = -4$

$\frac{d^3y}{dx^3} \neq 0$

put $x = \frac{\pi}{3}$, $= 4\left[\frac{1}{2} - 2\left(\frac{-1}{2}\right)\times 2\right]$

$= 4\left[\frac{1}{2} + 1\right]$

$= 6$

$\frac{d^3y}{dx^3} \neq 0$

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Eg. In the curve $y(a^2+x^2)=2x^2$. Prove that for varying values of 'a' the locus of the points of inflection is the straight line $y=\frac{1}{2}$.

Soln.

$$y(a^2+x^2) = 2x^2$$

$$y = \frac{2x^2}{(a^2+x^2)}$$

$$\frac{dy}{dx} = \frac{(a^2+x^2)(4x) - 2x^2(2x)}{(a^2+x^2)^2}$$

$$= \frac{4xa^2 + 4x^3 - 4x^3}{(a^2+x^2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{4xa^2}{(a^2+x^2)^2}$$

$$= \frac{(a^2+x^2)^2(4a^2) - 4xa^2[2(a^2+x^2)(2x)]}{(a^2+x^2)^4}$$

$$\Rightarrow \frac{\cancel{(a^2+x^2)}[4a^2(a^2-x^2) - 4a^2x\cdot 2(2x)]}{(a^2+x^2)^{\cancel{4}}}$$

$$\frac{d^2y}{dx^2} = 0 \quad , \quad \frac{d^2y}{dx^2} = \frac{4a^4 - 12a^2x^2}{(a^2+x^2)^3}$$

$$4a^4 + 4a^2x^2 - 16a^2x^2 = 0$$

$$\cancel{4a^4} \quad 4a^4 - 12a^2x^2 = 0$$

$$4a^2(a^2-3x^2) = 0$$

$$\left\{ x = \pm \frac{a}{\sqrt{3}} \right\}$$

$$\frac{d^3y}{dx^3} = 0$$

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put value of x in given eq^n

$$y(a^2+x^2) = 2x^2$$

$$y\left[a^2+\left(\frac{a}{\sqrt{3}}\right)^2\right] = 2\left(\frac{a}{\sqrt{3}}\right)^2$$

$$y\left[\frac{4a^2}{3}\right] = \frac{2a^2}{3}$$

$$\boxed{y = \frac{1}{2}}$$

Eg. Show that every point in which the curve $y = c\sin\left(\frac{x}{a}\right)$ meets a axis of x is point of inflection. (मतलब y की value zero आनी चाहिए।)

Sol^n

$$y = c\sin\left(\frac{x}{a}\right)$$

$$\frac{dy}{dx} = c\cdot\cos\left(\frac{x}{a}\right)\times\frac{1}{a}$$

$$\frac{d^2y}{dx^2} = -\sin\left(\frac{x}{a}\right)\times\frac{1}{a^2}\cdot c$$

$$-\sin\left(\frac{x}{a}\right)\frac{c}{a^2} = 0$$

$$-\sin\left(\frac{x}{a}\right) = 0$$

$$\frac{x}{a} = 0$$

$$\boxed{x = 0}$$

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General.

Q4 $\sin\left(\frac{x}{a}\right) = 0$

$$\sin\left(\frac{x}{a}\right) = \sin n\pi$$

$$\frac{x}{a} = n\pi \quad , \quad \boxed{x = an\pi}$$

put $x = 0$ in eqn
$y = c\sin\left(\frac{0}{a}\right)$

$\boxed{y = 0}$

put $x = an\pi$
$y = c\sin\left(\frac{an\pi}{a}\right)$

$\boxed{y = 0}$ $[\sin(n\pi) = 0]$

Q4 Prove that the curve $x^3 + y^3 = a^3$ has point of inflection at point where it crosses the coordinate axis

Soln
$$x^3 + y^3 = a^3$$
$$3x^2 + 3y^2 \frac{dy}{dx} = 0$$
$$\cancel{3}y^2 \frac{dy}{dx} = -\cancel{3}x^2$$
$$\frac{dy}{dx} = \frac{-x^2}{y^2} \quad \text{--- (1)}$$

$$\frac{d^2y}{dx^2} = -\left[\frac{y^2(2x) + x^2\, 2y \cdot dy/dx}{y^4}\right]$$

put value of $\frac{dy}{dx}$ from eqn (1)

$$\frac{d^2y}{dx^2} = \frac{-2}{y^4}\left[y^2x + x^2y \cdot \left(\frac{-x^2}{y^2}\right)\right]$$

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$$\frac{d^2y}{dx^2} = -\frac{2}{y^4}\left[xy^2 - \frac{x^4}{y}\right] \Rightarrow -\frac{2}{y^4}\left[xy^3 - x^4\right]$$

$\Rightarrow -\frac{2}{y^4} \times$ put $\frac{d^2y}{dx^2} = 0$

$xy^3 - x^4 = 0$

$x\left[y^3 - x^3\right] = 0$

$x = 0$; $y = x$

for $x = 0$ $\quad x^3 + y^3 = a^3$

$y^3 = a^3 \Rightarrow \boxed{y = a}$ → ll to.

for $x = y$ $\quad y^3 + y^3 = a^3$

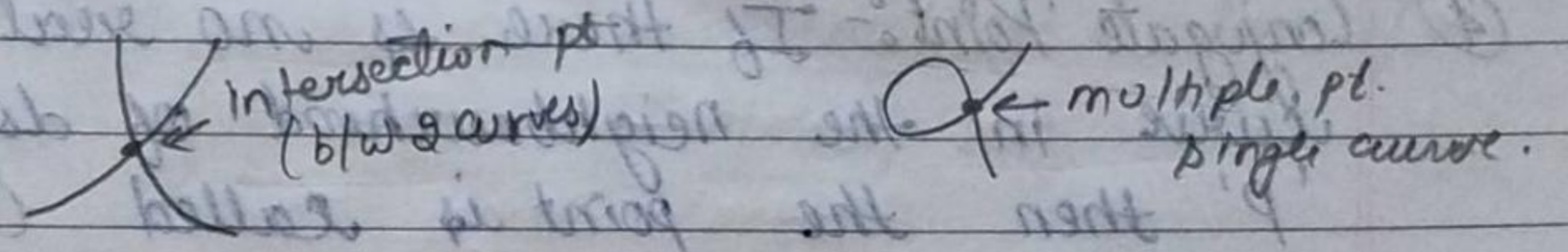
$$\boxed{y = \frac{a}{\sqrt{2}}}$$

$y =$ const means ll to ~~co~~ x-axis.

Curve Tracing

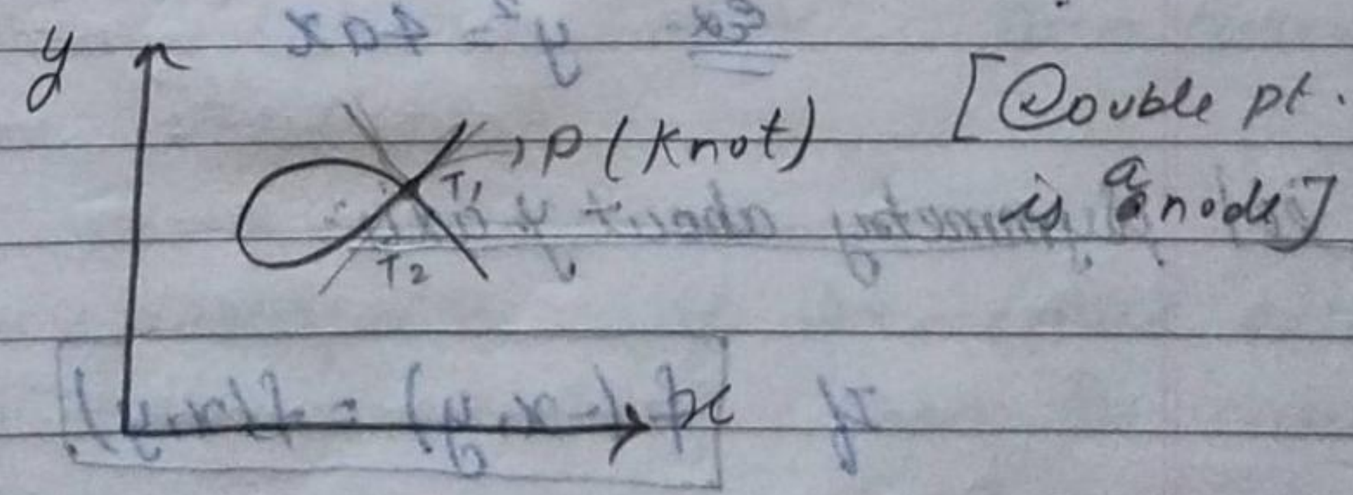
① Multiple Point:-

A point 'p' on a curve is said to be a multiple point of more than one branches of the curve passes through it.



② Double Point:- A point 'P' on a curve is said to be double point if 2 branches of curve passes through it.

③ Node:- If the 2 ~~tangent~~ tangent at a double point 'P' are real and distinct then the point is called Node. Hence, the shape of ~~an~~ curve looks like a knot.



④ Cusp:- If the 2 tangent at a double point 'P' are real & coincident then the point is cusp.

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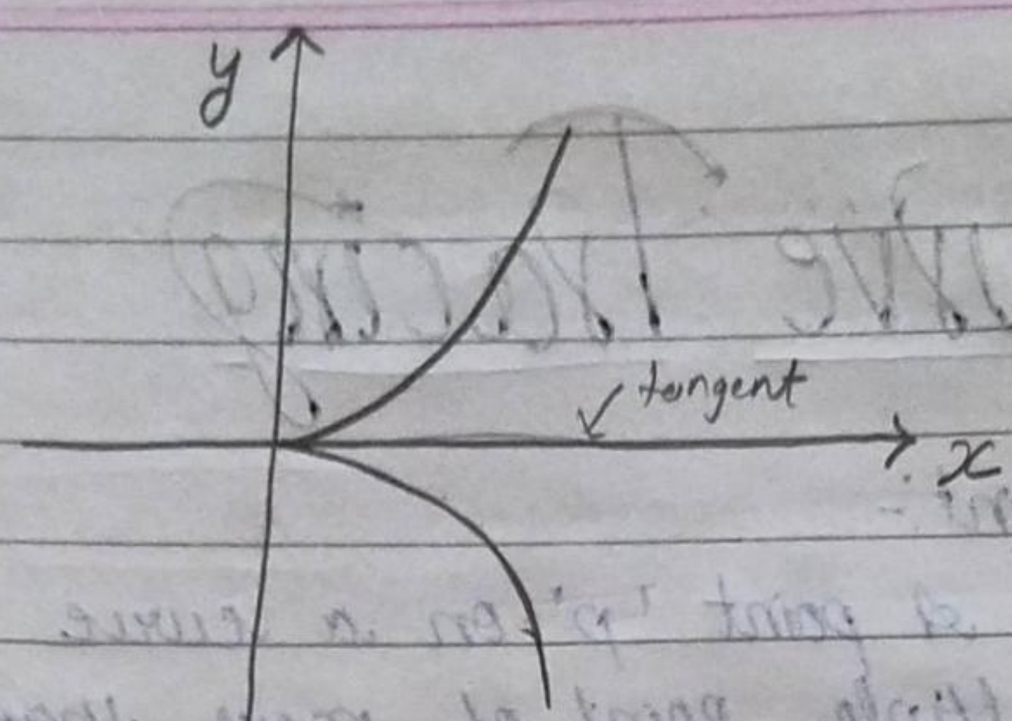

Cusp → represent open curve.

Node → represent close curve

④ Conjugate Point:- If there is no real point on the curve in the neighbourhood of double point 'P' then the point is called Conjugate Point.

Symmetry:-

i] Symmetry about x-axis:-

If $\boxed{f(x,-y) = f(x,y)}$ Then, the curve contain symmetry about x-axis.

(अर्थात किसी भी curve में 'y' की जगह '-y' रखेंगे तो curve में बदलाव नहीं आना चाहिए)

Ex. $y^2 = 4ax$ (trick y की power even होगी वहाँ symmetry about x-axis होगी।)

ii] Symmetry about y-axis:-

If $\boxed{f(-x,y) = f(x,y)}$

Eg. $x^2 = 4ay$

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iii) Symmetry about origin:-

$$\boxed{f(-x,-y) = f(x,y)}$$

Eg. $\boxed{x^2+y^2=a^2} \rightarrow$ eq^n of circle

(iv) Symmetry about the line $y=x$:-
A curve is symmetric about $y=x$ line if -

$$\boxed{f(x,y) = f(y,x)}$$

eg. $x^3+y^3=3axy$

v) Position about origin:-
If the curve passes through origin then $(0,0)$ will satisfies the equation.

Intersection at axes:-

i) At x-axis:- On putting $(y=0)$ if the eq^n exist and give a real value of 'x'. Then curve will intersect x-axis.

(ii) At y-axis:- On putting $(x=0)$ if the curve exist and give a real value of 'y'. Then the curve exists.

Asymptotes:- We have to find.

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Region of existence :- In the curve we identify those region in which the values of x & y should be real.

Eg. $y^2 = (x-a)$

$y = \sqrt{x-a}$

$x < a$,, imaginary curve नहीं बनेगा।

so, for real $x > a$

Eg. Trace the curve -

$$x^3 + y^3 = 3axy$$

Sol^n First we have to check symmetry in this curve

In this curve symmetry is $(y=x)$

put $(x=y)$ & $(y=x)$

$$y^3 + x^3 = 3axy$$

eq^n remains same.

The given curve is symmetric about the line $y=x$.

position at origin:-

put $(0,0)$ in given curve

$$(0)^3 + (0)^3 = 3a(0)(0)$$

$$0 = 0$$

L.H.S = R.H.S

Hence, it ~~shows~~ curve is passing through the origin

To find tangent -

putting lowest degree term of the curve equal to zero.

$xy=0$

$x=0$ & $y=0$

two different tangents means <u>Node.</u>

Hence x-axis & y-axis are two different tangent of the curve. Hence, the double point of the curve is node.

Find asymptotes:-

$$x^3+y^3=3axy$$

$f_3(x,y)=x^3+y^3$

$f_2(x,y)=-3axy$

$f_3(m)=1+m^3$

$f_2(m)=3am$

put $f_3(m)=0$

$1^3+m^3=0$

$[a^3+b^3=(a+b)(a^2-ab+b^2)]$

$(1+m)(m^2-m+1)=0$ $\left(\frac{1+\sqrt{3}i}{2}\right)$ & $\left(\frac{1-\sqrt{3}i}{2}\right)$

$\boxed{m=-1}$ imaginary roots.

$$c=-\left[\frac{f_2(m)}{f_3'(m)}\right]=-\left[\frac{-3am}{3m^2}\right]=-\left[\frac{a(-1)}{(-1)^2}\right]=-a$$

$\boxed{y=-x-a}$ → eq^n of asymptotes.

$\{x+y+a=0\}$

<u>Intersection point:-</u>

on putting $y=x$

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$$x^3+y^3 = 3axy$$
$$y^3+y^3 = 3ay^2$$
$$2y^3-3ay^2=0$$
$$y^2(2y-3a)=0$$
$$y=0,0 \;/\; y=\frac{3a}{2}$$

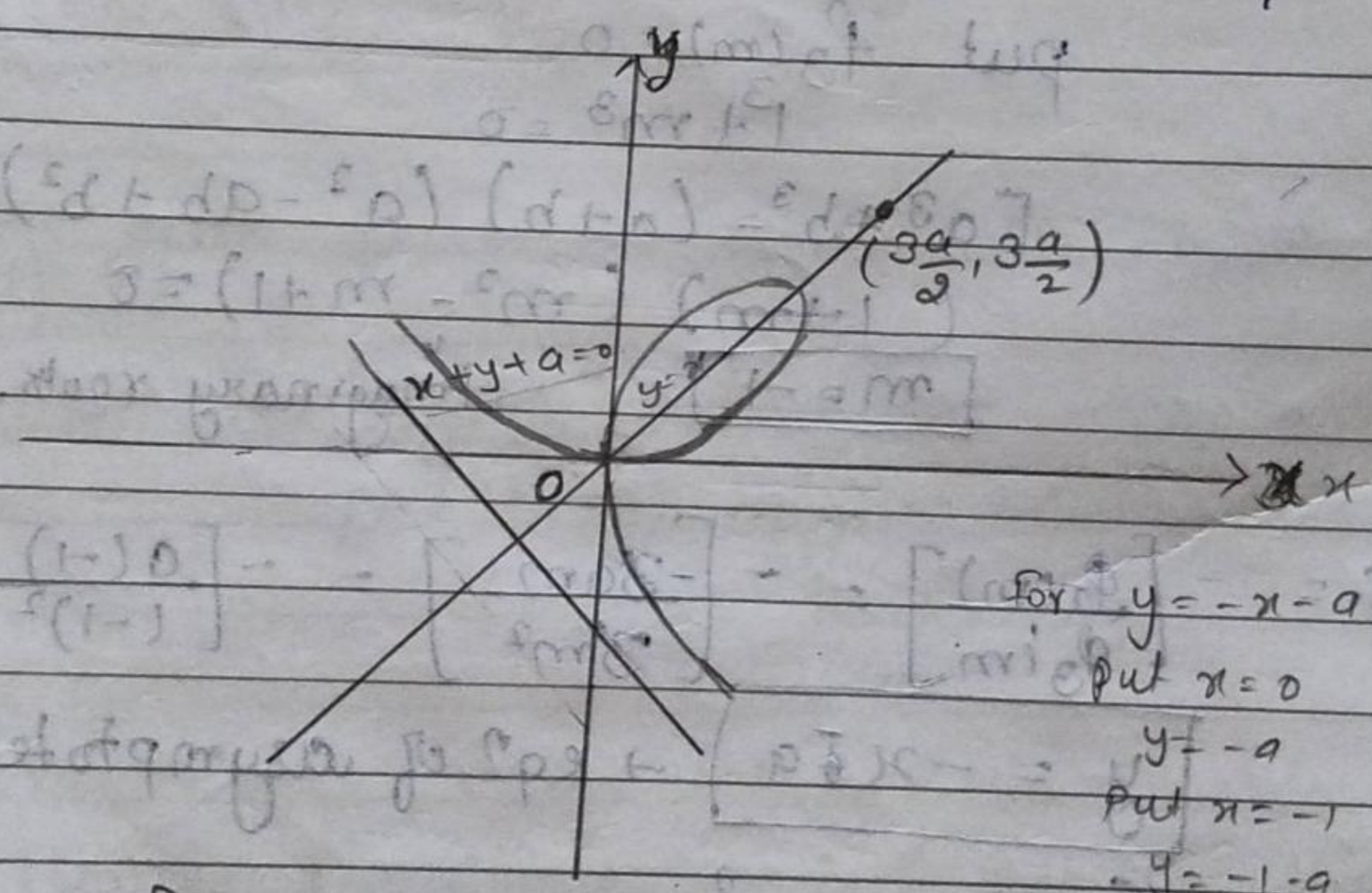
If $y=0 \Rightarrow x=0$ $\quad y=x$

(i) $(x,y)=(0,0)$

(ii) $y=\frac{3a}{2}$, $x=\frac{3a}{2}$

$$\Rightarrow \left(\frac{3a}{2},\frac{3a}{2}\right)$$

all points are +ive so lies in 1st quadrant.



For $y=-x-a$
put $x=0$
$y=-a$
put $x=-1$
$y=-1-a$

end में curve asymptote के Parallel हो जाता है curve
और ऐसा curve $y=x$ से दोनों side curve symmetric होता।

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eg. Trace the curve $\Rightarrow y^2(a+x) = x^2(a-x)$

Sol $y^2a + xy^2 - ax^2 + x^3 = 0$

Put $f(y) = -y$

$(-y)^2(a+x) = x^2(a-x)$

$y^2(a+x) = x^2(a-x)$

$f(x,-y) = f(x,y)$

The given curve is symmetric about x-axis.

position at origin:-

put $x=0$ & $y=0$

$0(a+0) = 0(a-0)$

$0 = 0$

L.H.S = R.H.S

Hence, curve is passing through the origin.

putting lowest degree term of the curve equal to zero.

ay^2

$y^2 - ax^2 = 0$

$y^2 - x^2 = 0$

$\boxed{y = \pm x}$, $y = x$ & $y = -x$

Different points → Nodes

Asymptote:-

$\phi_3(x) = x^3 + xy^2 = 1 + m^2$

$\phi_2(x) = ay^2 - ax^2$

$1 + m^2 = 0$

$m^2 = -1$

$m = \pm\sqrt{-1}$ $\boxed{m = \pm i}$

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from Parallel asymptotes:- highest degree वाले coefficient को zero से equal put करते हैं।

$$y^2(a+x) = x^2(a-x)$$

Tangent will be- $\boxed{a+x=0}$

Pt. of Intersection:- put $x=0$ & $y=0$

when $y=0$, $y^2(a+x)=x^2(a-x)$

$$x^2(a-x)=0$$

$$\boxed{x=0}, \boxed{x=a}$$

when $x=0$, $y^2(a+x)=x^2(a-x)$

$$\cancel{y^2(a+0)=0} \quad y^2(a+0)=0$$

$$\boxed{y=0}$$

Region of existence:-

* इसके लिए 'x' वाले term को एक side करो।

$$y^2 = \frac{x^2(a-x)}{a+x}$$

$$y = \pm x\sqrt{\frac{a-x}{a+x}}$$

$a-x$ -ive नहीं होना चाहिए अर्थात 'a' की value 'x' से बड़ी नहीं होनी चाहिए।

So, condⁿ will be $a-x>0$ (not possible)

$$\boxed{x<a}$$

& In denominator.

$x+a<0$ (not possible)

$$\boxed{x<-a}$$

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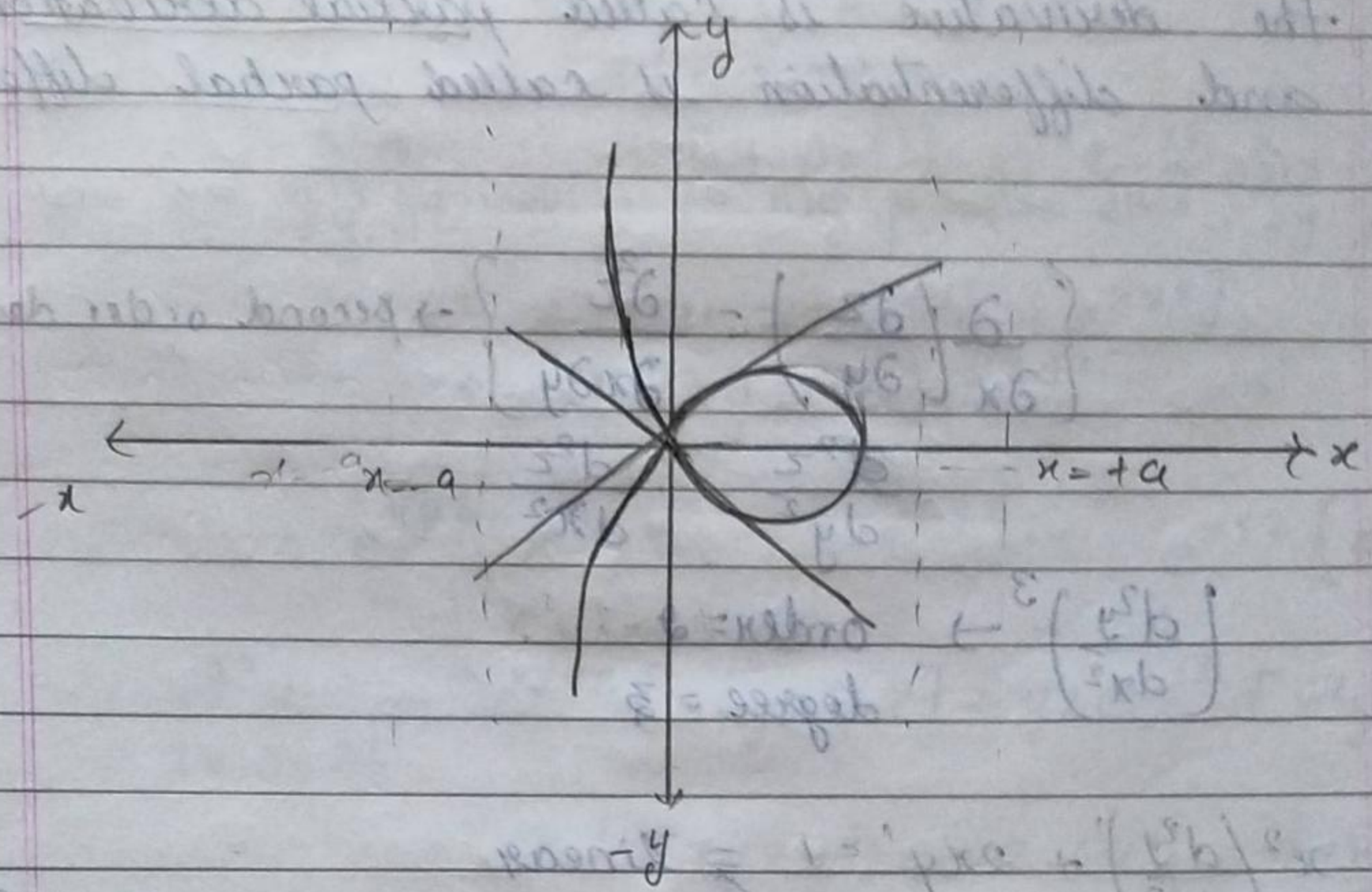
So, condition will be, ~~$-a < x$~~

$$\boxed{a < x < -a}$$



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Unit-2

PARTIAL DIFFERENTIATION

When number of independent variable in the given function are more than 1 then the derivative is called partial derivation and differentiation is called partial differentiation.

जिसके respect में करेंगे वो ही variable होगा बाकी const. होंगे।

$$\left\{ \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \right\} \rightarrow \text{second order derivative.}$$

$$\frac{d^2 z}{dy^2}, \frac{d^2 z}{dx^2}$$

$$\left(\frac{d^2 y}{dx^2}\right)^3 \rightarrow \text{order} = 2$$
$$\text{degree} = 3$$

Ex. $x^2\left(\frac{d^2y}{dx^2}\right)' + 2xy' = 1$ = linear

→ dependendant variable 'y' की degree '1' होनी चाहिए।

$x^2\left(\frac{d^2y}{dx^2}\right) + 2xy^2 = 1$ (non linear)

$\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + 2y\left(\frac{dy}{dx}\right) = 1$ = (non linear)

Eg. If $u = e^{xyz}$ then show that —

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)\, e^{xyz}$$

Sol.

$$\frac{\partial^3 u}{\partial x \partial y \partial z}$$

solve from this side.

$u = e^{xyz}$

$\frac{\partial u}{\partial z} = xy\, e^{xyz}$

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Now, $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial y}\left(xye^{xyz}\right)$. (x & z const.)

$$= x \cdot \frac{\partial}{\partial y}\left(\underset{u}{y} \cdot \underset{v}{e^{xyz}}\right)$$

$$\frac{\partial^2 u}{\partial y \cdot \partial z} = x\left[y \cdot \frac{\partial e^{xyz}}{\partial y} + e^{xyz} \cdot \frac{\partial y}{\partial y}\right]$$

$$= x \cdot \left[y \cdot xz\, e^{xyz} + e^{xyz}\right]$$

$$= \left[x^2yz + x\right] e^{xyz}$$

$$= e^{xyz} \cdot x^2yz + x \cdot e^{xyz} = e^{xyz}\left[x^2yz + x\right]$$

$$\frac{\partial^3 u}{\partial x \cdot \partial y \cdot \partial z} = e^{xyz}\left[2xyz + 1\right] + yz\, e^{xyz}\left[x^2yz + x\right]$$

$$= e^{xyz}\left[2xyz + 1 + yz\left(x^2yz + x\right)\right]$$

$$= e^{xyz}\left[2xyz + 1 + x^2y^2z^2 + xyz\right]$$

$$= e^{xyz}\left[2xyz + 1 + x^2y^2z^2 + xyz\right]$$

$$= e^{xyz}\left[3xyz + x^2y^2z^2 + 1\right]$$

Hence Proved.

Eg. If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$ then show that

$$\frac{\partial^2 u}{\partial x \cdot \partial y} = \frac{x^2 - y^2}{x^2 + y^2}.$$

Soln $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)\right]$

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$\frac{y}{x}$ का partial diff. w.r.t. $y = \frac{1}{x}$

$\frac{x}{y}$ " " " " " " $= \frac{y(0)-x}{y^2} = \boxed{\left(\frac{-x}{y^2}\right)}$

$$x^2 \cdot \left(\frac{1}{1+\frac{y^2}{x^2}}\right)\left(\frac{1}{x}\right) - \left[y^2\left(\frac{1}{1+\frac{x^2}{y^2}}\right)\left(-\frac{x}{y^2}\right) + \tan^{-1}\left(\frac{x}{y}\right) 2y\right]$$

$$\frac{x^3}{x^2+y^2} - \left[-\frac{xy^2}{y^2+x^2} + \tan^{-1}\left(\frac{x}{y}\right) 2y\right]$$

$$\frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y \cdot \tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{x(x^2+y^2)}{(x^2+y^2)} - 2y \cdot \tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial v}{\partial y} = x - 2y \tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial x}\left[x - 2y \tan^{-1}\left(\frac{x}{y}\right)\right]$$

$\frac{x}{y}$ का par. diff. w.r.t. x $\left(x = \frac{1}{y}\right)$

$$= 1 - 2y \cdot \left[\frac{1}{1+\frac{x^2}{y^2}}\right]\left(\frac{1}{y}\right)$$

$$= 1 - 2y \cdot \frac{y^2}{y+x^2} \times \frac{1}{y}$$

$$= 1 - \frac{2y^2}{y^2+x^2} = \frac{y^2+x^2-2y^2}{y^2+x^2}$$

$$\left\{\frac{\partial^2 v}{\partial x \cdot \partial y} = \frac{x^2-y^2}{x^2+y^2}\right\}$$

Hence Proved

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Homogeneous Function :-

$$f(x,y) = x^n \phi\left(\frac{y}{x}\right)$$

Let, $f(x,y)$ be a given function. If it can be represent as- $\boxed{f(x,y) = x^n \phi\left(\frac{y}{x}\right)}$ then the function 'f' is called homogeneous function of degree 'n'.

eg. $f(x,y) = a_0 y^n + a_1 y^{n-1} + a_2 y^{n-2} x^2 + \cdots + a_n x^n$

$$= x^n \left[a_0 \left(\frac{y}{x}\right)^n + a_1 \left(\frac{y}{x}\right)^{n-1} + a_2 \left(\frac{y}{x}\right)^{n-2} + \cdots + a_n\right]$$

$$\left\{f(x,y) = x^n \phi\left(\frac{y}{x}\right)\right\}$$

Eg. Check whether the following function is homogeneous or not-

$$f(x,y) = \frac{\sqrt{x} + \sqrt{y}}{x+y}$$

Sol^n :- $= \dfrac{\sqrt{x}\left(1+\sqrt{\frac{y}{x}}\right)}{\sqrt{x}\,x\left(1+y/x\right)} = \dfrac{\left(1+\sqrt{y/x}\right)}{\sqrt{x}\left(1+y/x\right)}$

$$= x^{-1/2}\left(\frac{1+\sqrt{y/x}}{1+y/x}\right) \Rightarrow x^n \phi\left(\frac{y}{x}\right)$$

Hence it is a homogeneous function of degree $-1/2$.

*** V.V. Imp

Euler's Theorem :-

statement → If $f(x,y)$ is a homogeneous function in x & y variables then –

$$\boxed{x\frac{\partial f}{\partial x} + y\cdot\frac{\partial f}{\partial y} = nf}$$

Proof :→

Since it is given that function $f(x,y)$ is a homogeneous function. Hence, –

$$f(x,y) = x^n \phi\left(\frac{y}{x}\right) \quad \text{---} (1)$$

$$\frac{\partial f}{\partial x} = x^n \phi'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + nx^{n-1}\phi\left(\frac{y}{x}\right) \quad \text{---} (2)$$

$$\frac{\partial f}{\partial y} = x^n \cdot \phi'\left(\frac{y}{x}\right)\cdot\left(\frac{1}{x}\right) \quad \text{----} (3)$$

multiply eq^n (2) by 'x' & (3) by 'y'

$$x\cdot\frac{\partial f}{\partial x} = x\left[x^n \phi'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + nx^{n-1}\phi\left(\frac{y}{x}\right)\right]$$

$$\cancel{y\cdot\frac{\partial f}{\partial y}} = -x^{n+1}\phi'\left(\frac{y}{x}\right)y\cdot x^{-2} + nx^{n}\phi\left(\frac{y}{x}\right)$$

$$= -x^{n-1}\phi'\left(\frac{y}{x}\right)(y) + nx^{n}\phi\left(\frac{y}{x}\right) \quad \text{---} (4)$$

$$y\cdot\frac{\partial f}{\partial y} = y\cdot x^n \phi'\left(\frac{y}{x}\right)x^{-1}$$

$$= y\,x^{n-1}\phi'\left(\frac{y}{x}\right) \quad \text{---} (5)$$

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Add eq^n (4) & (5)

$$x\cdot\frac{\partial f}{\partial x}+y\cdot\frac{\partial f}{\partial y}=-x^{n-1}\cancel{\phi'\left(\frac{y}{x}\right)\cdot y}+n\,x^{n-1}\phi\left(\frac{y}{x}\right)+x^{n-1}\cancel{\phi'\left(\frac{y}{x}\right)\cdot y}$$

$$x\cdot\frac{\partial f}{\partial x}+y\cdot\frac{\partial f}{\partial y}=n\cdot x^{n}\phi\left(\frac{y}{x}\right)$$

$\left\{\because x^n\phi\left(\frac{y}{x}\right)=f\right\}$

$$\boxed{x\cdot\frac{\partial f}{\partial x}+y\cdot\frac{\partial f}{\partial y}=n\cdot f}$$

Hence Proved.

eg. If $u=\sec^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ then prove that –

$$x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=2\cot u.$$

Sol^n First check given function is homogeneous or not.

$$u=\sec^{-1}\left[\frac{x^3(1+y^3/x^3)}{x(1+y/x)}\right]$$

$$u=\sec^{-1}\left[\frac{x^2(1+y^3/x^3)}{(1+y/x)}\right]$$

$$\sec u=x^2\left[\frac{1+y^3/x^3}{(1+y/x)}\right]$$

Hence, ~~this~~ 'sec u' function is homogeneous with degree '2'. $(n=2)$

By Euler's Theorem –

$$x\cdot\frac{\partial f}{\partial x}+y\cdot\frac{\partial f}{\partial y}=nf \qquad \{\because f=f(x,y)\}$$

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$$x\frac{\partial}{\partial x}(\sec u) + y\cdot\frac{\partial}{\partial y}(\sec u) = 2\sec u$$

$\left\{\because \frac{d}{dx}(\sec x) = \sec x\cdot\tan x\right\}$

$$x\sec u\cdot\tan u\cdot\frac{\partial u}{\partial x} + y\cdot\sec u\cdot\tan u\cdot\frac{\partial u}{\partial y} = 2\sec u$$

Divide by '$\sec u\cdot\tan u$'

$$x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = \frac{2\cancel{\sec u}}{\cancel{\sec u}\cdot\tan u}$$

$$\boxed{x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = 2\cot u}$$

Hence Proved.

Eg If $u = \sin^{-1}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\right)$ then prove that –

$$x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = \frac{1}{20}\tan u$$

Sol^n First check whether the function is homogenous or not.

$$u = \sin^{-1}\left[\frac{x^{1/4}\left(1+y^{1/4}/x^{1/4}\right)}{x^{1/5}\left(1+y^{1/5}/x^{1/5}\right)}\right]$$

$$= \sin^{-1}\left[x^{1/20}\left(\frac{1+(y/x)^{\frac{1}{4}}}{1+\left(\frac{y}{x}\right)^{1/5}}\right)\right]$$

$$\sin u = x^{1/20}\left[\frac{1+(y/x)^{1/4}}{1+\left(\frac{y}{x}\right)^{1/5}}\right]$$

given function is homogeneous function of ~~with~~ degree $\frac{1}{20}$. $\left(n = \frac{1}{20}\right)$

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So, $x \cdot \frac{\partial}{\partial x}(\sin u) + y \cdot \frac{\partial}{\partial y}(\sin u) = \frac{1}{2\theta}(\sin u)$

$$x \cdot \cos u \cdot \frac{\partial u}{\partial x} + y \cdot \cos u \cdot \frac{\partial u}{\partial y} = \frac{1}{2\theta} \sin u$$

Divide whole equation by '$\cos u$'

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \frac{1}{2\theta} \frac{\sin u}{\cos u}$$

$$\left\{ x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \frac{1}{2\theta} \tan u \right\}$$

$\left\{ \because \frac{\sin u}{\cos u} = \tan u \right\}$

Hence Proved.

eg If $x^x y^y z^z = c$ then prove that $\frac{\partial^2 z}{\partial x \cdot \partial y} = [x(1+\log x)]^{-1}$ at $x = y = z$.

Sol^n

$$x^x y^y z^z = c$$

$\left(\begin{array}{l} x \to \text{independent} \\ z \to \text{dependent} \end{array} \right)$

Both side multiply by 'log'

$$(x \log x + y \log y + z \log z) = \log c \quad \text{--- (1)}$$

differentiating eq^n no. 1 w.r.t. 'y'

$$\frac{\partial z}{\partial y} = 0 + \cancel{y} \cdot \frac{1}{\cancel{y}} + \log y + \cancel{z} \cdot \frac{1}{\cancel{z}} \cdot \frac{\partial z}{\partial y} + \log z \cdot \frac{\partial z}{\partial y} = 0$$

$$1 + \log y + \frac{\partial z}{\partial y} + \log z \cdot \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y}(1 + \log z) = -(1 + \log y)$$

$$\frac{\partial z}{\partial y} = -\frac{(1+\log y)}{(1+\log z)} \quad \text{--- (2)}$$

$$\frac{\partial^2 z}{\partial x \cdot \partial y} = \frac{\partial}{\partial x}\left[-\frac{(1+\log y)}{(1+\log z)} \right]$$

$$= -(1+\log y) \frac{\partial}{\partial x}\left(\frac{1}{1+\log z} \right)$$

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$$u = -(1+\log y) \cdot \frac{1}{(1+\log z)^2}\left(\frac{1}{z}\right)\cdot\frac{\partial z}{\partial x}$$

$$u = \frac{(1+\log y)}{(1+\log z)^2}\cdot\left(\frac{1}{z}\right)\cdot\frac{\partial z}{\partial x} \quad \text{—} (3)$$

diff. eqn (1) w.r.t. 'x'

$$\frac{\partial z}{\partial x} = x\cdot\frac{1}{x} + \log x + 0 + z\times\frac{1}{z}\cdot\frac{\partial z}{\partial x} + \log z\cdot\frac{\partial z}{\partial x} = 0$$

$$1 + \log x + \frac{\partial z}{\partial x} + \log z\cdot\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}(1+\log z) = -(1+\log x)$$

$$\frac{\partial z}{\partial x} = -\frac{(1+\log x)}{1+\log z}$$

put the value of $\frac{\partial z}{\partial x}$ in eqn (3)

$$\frac{\partial^2 u}{\partial x\cdot\partial y} = \frac{(1+\log y)}{(1+\log z)^2}\times\frac{-(1+\log x)}{(1+\log z)}\times\frac{1}{z}$$

<u>at $x = y = z$</u> $\Rightarrow \frac{\cancel{(1+\log x)}^{-}(-)\cancel{(1+\log x)}}{(1+\log x)^{\cancel{3}}}\times\frac{1}{x}$

$$= -\frac{1}{x(1+\log x)} \Rightarrow -x\,[(1+\log x)]^{-1}$$

<u>Hence Proved.</u>

Eg (i) If $u = \log(x^3+y^3+z^3-3xyz)$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$

(ii) $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = \frac{1}{3(x+y+z)^2}$$

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(ii) $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$

<u>sol^n</u> $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$ ---- (1)

first we solve this part

$$\frac{\partial(u)}{\partial x}=\frac{\partial}{\partial x}\left[\log x\,(x^3+y^3+z^3-3xyz)\right]$$

$$=\frac{1\,(3x^2+0+0-3yz)}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial(u)}{\partial y}=\frac{(3y^2-3xz)}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial(u)}{\partial z}=\frac{(3z^2-3xy)}{x^3+y^3+z^3-3xyz}$$

Now, $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3x^2-3yz+3y^2-3xz+3z^2-3xy}{x^3+y^3+z^3-3xyz}$

$$=\frac{3\,[x^2+y^2+z^2-xy-yz-xz]}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3\,(x^2+y^2+z^2-xy-yz-xz)}{x^3+y^3+z^3-3xyz} \quad \text{---- (2)}$$

$\{\because x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$

$$=\frac{3\,\cancel{(x^2+y^2+z^2-xy-yz-xz)}}{(x+y+z)\cancel{(x^2+y^2+z^2-xy-yz-zx)}}$$

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$$= \frac{3}{x+y+z}$$

from eq^n ①

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$\Rightarrow 3\left[\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right)\right]$$

$$\Rightarrow 3\left[\frac{-1}{(x+y+z)^2}\frac{-1}{(x+y+z)^2}\frac{-1}{(x+y+z)^2}\right]$$

$$= \frac{-9}{(x+y+z)^2}$$

<u>Hence Proved.</u>

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Que: If $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ then prove that –

(i) $$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \sin 2u.$$

Sol^n

$$u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$$

$$\tan u = \frac{x^3+y^3}{x+y}$$

$$\tan u = \frac{x^{\cancel{3}\,2}\left[1+(y/x)^3\right]}{\cancel{x}\left[1+y/x\right]}$$

$$= x^2 \frac{\left[1+(y/x)^3\right]}{\left[1+y/x\right]}$$

So, $\boxed{n=2}$

$$x \cdot \frac{\partial}{\partial x}(\tan u) + \frac{\partial}{\partial y}(\tan u) = 2\tan u$$

$$x \cdot \sec^2 u \cdot \frac{\partial u}{\partial x} + y \cdot \sec^2 u \cdot \frac{\partial u}{\partial y} = 2\tan u$$

Divide whole eq^n by '$\sec^2 u$'

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 2\frac{\tan u}{\sec^2 u}$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 2\frac{\sin u/\cos u}{1/\cos^2 u}$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 2\sin u \cos u$$

$[\because 2\sin x \cos x = \sin 2x]$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \sin 2u.$$

Hence Proved.

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Euler's Theorem formula for higher order derivative:-

$$\left\{ x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \cdot \frac{\partial^2 y}{\partial x \cdot \partial y} + y^2 \cdot \frac{\partial^2 f}{\partial y^2} = n(n-1) f \right\}$$

(ii) $x^2 \cdot \frac{\partial^2 u}{\partial x^2} + 2xy \cdot \frac{\partial^2 u}{\partial x \cdot \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \sin 2u \,(1 - 4\sin^2 u)$

Sol^n from (1) $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \sin 2u$ — ①

diff. eq^n ① w.r.t 'x'

$$x \cdot \frac{\partial^2 u}{\partial x^2} + 1 \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial^2 u}{\partial x \cdot \partial y} = 2\cos 2u \cdot \frac{\partial u}{\partial x} \quad \text{— ②}$$

diff. eq^n ① w.r.t 'y'

$$x \cdot \frac{\partial^2 u}{\partial x \cdot \partial y} + 1 \cdot \frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2} \cdot y = 2\cos 2u \cdot \frac{\partial u}{\partial y} \quad \text{— ③}$$

Multiply eq^n ② by 'x' & ~~by~~ eq^n ③ by 'y'.

$$\cancel{y \cdot \frac{\partial^2 y}{\partial x^2}} \cdot x \cdot \frac{\partial u}{\partial x} + x^2 \frac{\partial^2 u}{\partial x^2} + xy \cdot \frac{\partial^2 u}{\partial x \cdot \partial y} = 2x\cos 2u \cdot \frac{\partial u}{\partial x} \quad \text{— ④}$$

$$y^2 \cdot \frac{\partial^2 u}{\partial x^2} + y \cdot \frac{\partial u}{\partial y} + xy \cdot \frac{\partial^2 u}{\partial x \cdot \partial y} = 2y\cos 2u \cdot \frac{\partial u}{\partial x} \quad \text{— ⑤}$$

add eq^n ④ & ⑤

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \cdot \frac{\partial^2 u}{\partial x^2} + 2xy \cdot \frac{\partial^2 u}{\partial x \cdot \partial y} + x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y}$$

$$= 2\cos 2u \left(x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} \right).$$

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pvt $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x\cdot\partial y} + \sin 2u = 2\cos 2u \cdot \sin 2u$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x\cdot\partial y} = 2\cos 2u \sin 2u - \sin 2u$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x\cdot\partial y} = \sin 2u\,(2\cos 2u - 1)$$

$[\because \cos 2u = 1 - 2\sin^2 u]$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x\cdot\partial y} = \sin 2u\,[2(1-2\sin^2 u) - 1]$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x\cdot\partial y} = \sin 2u\,[2 - 4\sin^2 u - 1]$$

$$\cancel{= \sin 2u}$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x\cdot\partial y} = \sin 2u\,[1 - 4\sin^2 u]$$

Hence Proved

Maxima & Minima

$y = f(x)$

If $\frac{dy}{dx} = 0 \Rightarrow x = x^*$

$\left.\frac{d^2y}{dx^2}\right|_{x=x^*}$
- $> 0 \Rightarrow$ minima
- $< 0 \Rightarrow$ maxima
- $= 0 \Rightarrow$ pt. of inflexion.

* $z = f(x, y)$

① $\frac{\partial z}{\partial x} = 0$, $\frac{\partial z}{\partial y} = 0$

② Let, $\left.\frac{\partial^2 z}{\partial x^2}\right|_{\substack{x=x^* \\ y=y^*}} = A$, $\frac{\partial^2 z}{\partial x\cdot\partial y} = B$, $\frac{\partial^2 z}{\partial y^2} = C$

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Case I:- $AC - B^2 > 0$ — $A < 0$ (maxima); $A > 0$ (minima)

Case II. $AC - B^2 = 0$ (saddle point / point of inflexion)

Case III $AC - B^2 < 0$

If $x = x_1$ & x_2, $y = y_1$ & y_2

so points are ⇒ (x_1, y_1) (x_1, y_2), (x_2, y_1), (x_2, y_2)

Ques:- Obtain the extreme point (if any) of

(i) $f = 20x_1 + 26x_2 + 4x_1 \cdot x_2 - 4x_1^2 - 3x_2^2$

(ii) $f = x_1^3 + x_2^3 + 2x_1^2 + 4x_2^2 + 6$

Sol^n (i) Step 1:- $\frac{\partial f}{\partial x_1} = 0$, $\frac{\partial f}{\partial x_2} = 0$

$$\frac{\partial f}{\partial x_1} = 20 + 4x_2 - 8x_1 = 0 \quad \text{— (1)}$$

$$\frac{\partial f}{\partial x_2} = \cancel{20x_1} + 26 + 4x_1 - 6x_2 = 0 \quad \text{— (2)}$$

By solving eq^n (1) & (2)

$$20 + 4x_2 - 8x_1 = 0$$

$$2(26 - 6x_2 + 4x_1 = 0)$$

$$20 + 4x_2 - 8x_1 = 0$$

$$52 - 12x_2 + 8x_1 = 0$$

$\boxed{x_2 = 9}$ & $\boxed{x_1 = 7}$

so, $\frac{\partial^2 f}{\partial x_1^2} = -8 = A$, $\frac{\partial^2 f}{\partial x_2^2} = -6 = C$

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$$B = \frac{\partial f}{\partial x_1 \cdot \partial x_2} = \frac{\partial f}{\partial x_1} = \frac{\partial(6 + 4x_1 - 6x_2)}{\partial x_1} = 4 = B$$

So, $AC - B^2 = ?$

$$(-8)(-6) - (4)^2 = \boxed{32 > 0}$$

So, $AC - B^2 > 0$,

and $\boxed{A < 0}$ → so this is <u>maxima</u>.

$-8 < 0$

(ii) $f = x_1^3 + x_2^3 + 2x_1^2 + 4x_2^2 + 6$

<u>Sol^n</u> $\frac{\partial f}{\partial x_1} = 3x_1^2 + 4x_1 = 0$

$$x_1(3x_1 + 4) = 0$$

$$x_1 = 0, \quad x_1 = -4/3$$

$$\frac{\partial f}{\partial x_2} = 3x_2^2 + 8x_2 = 0$$

$$x_2(3x_2 + 8) = 0$$

$$x_2 = 0, \quad x_2 = -8/3$$

So points are $(0,0)$, $(0, -8/3)$, $(-\frac{4}{3}, 0)$, $(-\frac{4}{3}, -\frac{8}{3})$

⇒ ~~for (0,0)~~ $A = \frac{\partial^2 f}{\partial x_1^2} = 6x_1 + 4$

$$B = 0$$

$$C = \frac{\partial^2 f}{\partial x_2^2} = 6x_2 + 8$$

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for $(0,0)$:-

$A = 6(0) + 4 \Rightarrow \boxed{A = 4}$ so $A > 0$

$C = 6 \times 0 + 8 \Rightarrow \boxed{A = 8}$

Now, $AC - B^2$

$(4)(8) - 0 = \boxed{32 > 0}$

and $A > 0$, so this is minima.

for $\left(-\frac{4}{3}, 0\right)$:-

$A = 6 \times \left(-\frac{4}{3}\right) + 4 = -4$

$B = 6(0) + 8 = 8$

so, $AC - B^2$

$(-4)(8) - 0 \Rightarrow \boxed{-32 < 0}$ → III case

for $\left(0, -8/3\right)$:-

$A = 6 \times 0 + 4 = 4$

$B = 6 \times -\frac{8}{3} + 8 = -8$

$AC - B^2$

$(4)(-8) = 0$ → III case

$\boxed{-32 < 0}$

for $\left(-\frac{4}{3}, -\frac{8}{3}\right)$:-

$A = 6 \times \left(-\frac{4}{3}\right) + 4 = -4$

$C = 6 \times -\frac{8}{3} + 8 = -8$

$AC - B^2$

$(-4)(-8)$.

$\boxed{32 > 0}$

& $A < 0$

so this is maxima.

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Ques. $f = x^3 + 2y^3 + 3x^2 + 12y^2 + 24$

Sol^n $\frac{\partial f}{\partial x} = 3x^2 + 6x = 0$

$x(3x+6) = 0$

$x = 0, \quad x = \frac{-6}{3} = -2$

$\frac{\partial f}{\partial y} = 6y^2 + 24y = 0$

$y(6y + 24) = 0$

$y = 0, \quad y = -\frac{24}{6} = -4$

$(0,0), (0,-4), (-2,0), (-2,-4)$

$A = \frac{\partial^2 f}{\partial x^2} = 6x + 6,$

$|A - \lambda I| = 0$, igen value.

Langrange Multiplier Method:-

Max / Min. $f(x_1, x_2, x_3)$

Subjected to- $g(x_1, x_2, x_3) = 0$

Step I $\rightarrow$ $L(x_1, x_2, x_3, \lambda) = f + \lambda g$. λ → Lagrange multiplier

Necessary Condition -

$$\frac{\partial L}{\partial x_1} = 0 , \frac{\partial L}{\partial x_2} = 0 , \frac{\partial L}{\partial x_3} = 0 , \frac{\partial L}{\partial \lambda} = 0$$

$$H = \begin{vmatrix} \frac{\partial^2 L}{\partial x_1^2} - Z & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial L}{\partial x_3 \partial x_1} & \frac{\partial g}{\partial x_1} \\ \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_2^2} - Z & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial g}{\partial x_2} \\ \frac{\partial^2 L}{\partial x_1 \partial x_3} & \frac{\partial^2 L}{\partial x_2 \partial x_3} & \frac{\partial^2 L}{\partial x_3^2} - Z & \frac{\partial g}{\partial x_3} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_3} & 0 \end{vmatrix}$$

Note :-

If all igen values are +ive then matrix is +ive definite implies minima. and values are -ive then matrix is -ive definite implies maxima. , and values are some +ve saddle point.

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Q: Find the extreme points and extreme value of $u = \sin A \cdot \sin B \cdot \sin C$.
subject to,

$$A + B + C = \pi$$

Sol^n $f = \sin A \cdot \sin B \cdot \sin C$

$g = A + B + C - \pi = 0$

$L(A, B, C, \lambda) = \sin A \cdot \sin B \cdot \sin C + \lambda(A + B + C - \pi)$ — (1)

~~Sin A~~

$\frac{\partial L}{\partial A} = 0$, $\frac{\partial L}{\partial B} = 0$, $\frac{\partial L}{\partial C} = 0$, $\frac{\partial L}{\partial \lambda} = 0$

$\frac{\partial L}{\partial A} = \sin B \sin C \cdot \cos A + \lambda = 0 \Rightarrow \cos A \cdot \sin B \cdot \sin C = -\lambda$ — (2)

$\frac{\partial L}{\partial B} = \sin A \cdot \cos B \cdot \sin C + \lambda = 0 \Rightarrow \sin A \cdot \cos B \cdot \sin C = -\lambda$. — (3)

$\frac{\partial L}{\partial C} = \sin A \sin B \cdot \cos C + \lambda = 0 \Rightarrow \sin A \sin B \cdot \cos C = -\lambda$ — (4)

$\frac{\partial L}{\partial \lambda} = A + B + C = \pi$ — (5)

from eq^n (2), (3) & (4)

$\cos A \cdot \sin B \cdot \sin C = \sin A \cdot \cos B \cdot \sin C = \sin A \sin B \cdot \cos C$

Divide whole eq^n by $\sin A \cdot \sin B \cdot \sin C$.

then $\frac{\cos A}{\sin A} = \frac{\cos B}{\sin B} = \frac{\cos C}{\sin C}$

$\cot A = \cot B = \cot C$

$A = B = C$

put $A = B = C$ in eq^n (5)

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$\Rightarrow A+B+C=\pi$

$A+A+A=\pi$

$$\boxed{A=\frac{\pi}{3}=60^\circ}$$

similarly, $B=60^\circ$ & $C=60^\circ$

$$\frac{\partial^2 L}{\partial A^2}=-\sin A\cdot\sin B\cdot\sin C=-\sin 60\times\sin 60\times\sin 60=-\frac{3\sqrt{3}}{8}$$

$$\frac{\partial^2 L}{\partial B^2}=-\sin A\cdot\sin B\cdot\sin C=-\frac{3\sqrt{3}}{8}$$

$$\frac{\partial^2 L}{\partial C^2}=-\sin A\sin B\cdot\sin C=-\frac{3\sqrt{3}}{8}$$

$$\frac{\partial^2 L}{\partial A\cdot\partial B}=\frac{\partial}{\partial A}(\sin A\cos B\cdot\sin C)=\cos A\cos B\cdot\sin C$$

$$=\frac{1}{2}\times\frac{1}{2}\times\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{8}$$

$$\frac{\partial^2 L}{\partial B\cdot\partial C}=\frac{\partial}{\partial B}(\sin A\sin B\cos C)=\sin A\cos C\cdot\cos B=\frac{\sqrt{3}}{8}$$

$$\frac{\partial^2 L}{\partial A\cdot\partial C}=\frac{\partial L}{\partial A}(\sin A\sin B\cdot\cos C)=\cos A\cdot\sin B\cdot\cos C=\frac{\sqrt{3}}{8}$$

$$H=\begin{bmatrix}-\frac{3\sqrt{3}}{8}-z & \frac{\sqrt{3}}{8} & \frac{\sqrt{3}}{8} & 1\\ \frac{\sqrt{3}}{8} & -\frac{3\sqrt{3}}{8}-z & \frac{\sqrt{3}}{8} & 1\\ \frac{\sqrt{3}}{8} & \frac{\sqrt{3}}{8} & -\frac{3\sqrt{3}}{8}-z & 1\\ 1 & 1 & 1 & 0\end{bmatrix}$$

operation: $C_2\to C_2-C_1$ & $C_3\to C_3-C_1$

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$$\begin{array}{cccc}
-\frac{3\sqrt{3}}{8}-z & \frac{\sqrt{3}}{8}+\frac{3\sqrt{3}}{8}+z & \frac{\sqrt{3}}{8}+\frac{3\sqrt{3}}{8}+z & 1 \\
\frac{\sqrt{3}}{8} & -\frac{3\sqrt{3}}{8}-z-\frac{8\sqrt{3}}{8} & 0 & 1 \\
\frac{\sqrt{3}}{8} & 0 & -\frac{3\sqrt{3}}{8}z-\frac{\sqrt{3}}{8} & 1 \\
1 & 0 & 0 & 0
\end{array}$$

Now solve the determinant -

$$\begin{vmatrix}
\frac{4\sqrt{3}}{8}+z & \frac{4\sqrt{3}}{8}+z & 1 \\
-\frac{4\sqrt{3}}{8}-z & 0 & 1 \\
0 & -\frac{4\sqrt{3}}{8}-z & 1
\end{vmatrix} = 0$$

$$\Rightarrow \frac{4\sqrt{3}}{8}+z\left(0+\frac{4\sqrt{3}}{8}-z\right)-\left[\left(\frac{4\sqrt{3}}{8}+z\right)\left(-\frac{4\sqrt{3}}{8}-z\right)\right]$$
$$+1\left[\left(-\frac{4\sqrt{3}}{8}-z\right)^2\right]=0$$

$$\Rightarrow \frac{\cancel{48}^{3}}{\cancel{64}_{4}}-z^2-\left[-\frac{\cancel{48}^{3}}{\cancel{64}_{4}}-\frac{4\sqrt{3}}{8}z-\frac{4\sqrt{3}}{8}z-\cancel{4}z^2\right]$$
$$+\frac{\cancel{48}^{3}}{\cancel{64}_{4}}+z^2-\frac{\cancel{8}\sqrt{3}}{\cancel{8}}\cdot z$$

$$\Rightarrow \frac{9}{12}-\cancel{z^2}+\frac{4\sqrt{3}}{8}z+\cancel{\frac{4\sqrt{3}}{8}\cdot z}+z^2+\cancel{z^2}-\cancel{\frac{4\sqrt{3}}{8}\cdot z}=0$$

$$z^2+\frac{\cancel{4}\sqrt{3}}{\cancel{8}_{2}}\cdot z+\frac{3}{4}=0$$

$$z^2+\frac{\sqrt{3}}{2}\cdot z+\left(\frac{\sqrt{3}}{2}\right)^2=0$$

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Now $\{(a+b)^2 = a^2+b^2+2ab\}$

$$\left(z+\frac{\sqrt{3}}{2}\right)^2 = 0$$

$$z = -\frac{3}{2}$$

Next

$$\begin{vmatrix} \frac{\sqrt{3}}{2}+z & \frac{\sqrt{3}}{2}+z & 1 \\ -\frac{\sqrt{3}}{2}-z & 0 & 1 \\ 0 & -\frac{\sqrt{3}}{2}-z & 1 \end{vmatrix} = 0$$

$$\Rightarrow \left(\frac{\sqrt{3}}{2}+z\right)\left(\frac{\sqrt{3}}{2}+z\right) - \left[\left(\frac{\sqrt{3}}{2}+z\right)\left(-\frac{\sqrt{3}}{2}-z\right)\right] + 1\left[\left(-\frac{\sqrt{3}}{2}-z\right)^2\right] = 0$$

$$\Rightarrow \left(\frac{\sqrt{3}}{2}+z\right)^2 + \cancel{\left(\frac{\sqrt{3}}{2}+z\right)^2} - \cancel{\left(\frac{\sqrt{3}}{2}+z\right)^2} = 0$$

$$\left(\frac{\sqrt{3}}{2}+z\right)^2 = 0$$

$$z = -\sqrt{3}/2,\ -\sqrt{3}/2$$

Both values are -ive so it is maxima.

11/12

Ques. Find the dimension of a rectangular box, open at the top of maximum volume whose surface is 108 sq. cm

Sol^n Let, $l = x$, $b = y$, $h = z$

so, volume (V) $= xyz$

~~Sub to~~.

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Surface area of cubord $= 2lb + 2bh + 2hl$
but in this it is open so formula will be-
$= \cancel{xy +} \; = lb + 2bh + 2hl$
sub. to,

$$xy + 2yz + 2zx = 108$$

So $L = (x, y, z, \lambda) = x \cdot y \cdot z + \lambda(xy + 2yz + 2xz - 108)$

$$\frac{\partial L}{\partial x} = y \cdot z + \lambda(y + 2z) = 0$$

$$\frac{\partial L}{\partial y} = x \cdot z + \lambda(x + 2z) = 0$$

$$\frac{\partial L}{\partial z} = x \cdot y + \lambda(2y + 2x) = 0$$

$$\frac{\partial L}{\partial \lambda} = (xy + 2yz + 2xz - 108) = 0 \quad \text{--- (1)}$$

$$yz + \lambda y + 2x\lambda = 0$$
$$x \cdot z + x\lambda + 2z\lambda = 0$$
$$xy + 2y\lambda + 2x\lambda = 0$$

$$y \cdot z = -\lambda(y + 2z)$$
$$-\lambda = \frac{yz}{(y + 2z)} \quad \text{--- (2)}$$

Similarly, $-\lambda = \dfrac{x \cdot z}{(x + 2z)}$ --- (3)

$$-\lambda = \frac{xy}{\lambda(2y + 2x)} \quad \text{--- (4)}$$

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equate all the above equations

$$\frac{y \cdot z}{y+2z} = \frac{x \cdot z}{x+2z} = \frac{y \cdot x}{2y+2x}$$

$$\frac{y \cdot \cancel{z}}{y+2z} = \frac{x \cdot \cancel{z}}{x+2z} \qquad \frac{\cancel{y}z}{y+2z} = \frac{\cancel{y} \cdot x}{2y+2x}$$

$$\cancel{xy} + 2yz = \cancel{xy} + 2xz \qquad 2yz + \cancel{2xz} = xy + \cancel{2xz}$$

$$2yz - 2xz = 0 \qquad 2yz - xy = 0$$

$$y - x = 0 \qquad y(2z - x) = 0$$

$$\boxed{x = y} \qquad \boxed{x = 2z}$$

from this we get $\boxed{x = y = 2z}$

put values of $x = y = 2z$ in eq^n ①

$$xy + 2yz + 2zx = 108$$

$$(2z)(2z) + 2(2z)(z) + 2(z)(2z) = 0$$

$$12z^2 = 108$$

$$\boxed{z = \pm 3}$$

But -ive value is not possible

so $\boxed{z = 3}$

& $\boxed{x = 2z = 6}$, $\boxed{y = 6}$

put value of x, y, z in eq^n ②

$$\text{so, } -\lambda = \frac{6 \times 3}{6+6} \Rightarrow \frac{\cancel{18}}{\cancel{12}} = \frac{3}{2}$$

$$\boxed{\lambda = -\frac{3}{2}}$$

so, $\frac{\partial^2 L}{\partial x^2} = 0$, $\frac{\partial^2 L}{\partial y^2} = 0$, $\frac{\partial^2 L}{\partial z^2} = 0$

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$$\frac{\partial L}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}[x_1 + x_2 + \lambda x_1 + x_1 x_2]$$

$$= 2 + \lambda = 3 - \frac{2}{3} = \frac{7}{3}$$

$$\frac{\partial L}{\partial x_1 \partial \lambda} = x_1 + x_2$$

$$= 6 + 3\left(-\frac{2}{3}\right) = 3$$

$$\frac{\partial L}{\partial x_2 \partial \lambda} = x_1 + x_2$$

$$= 6 + 3\left(-\frac{2}{3}\right) = 3$$

let the eigen value is k in the matrix we

$$H = \begin{bmatrix} 0-k & 7+\lambda-k & y_1+\lambda_1-k & 12 \\ 7+\lambda-k & 0-k & x_1+\lambda_1-k & 12 \\ y_1+\lambda_1-k & x_1+\lambda_1-k & 0-k & 24 \\ 12 & 12 & 24 & 0 \end{bmatrix}$$

$$H = \begin{bmatrix} 0-k & 2/3 & 3 & 12 \\ 2/3 & 0-k & 3 & 12 \\ 3 & 3 & 0-k & 24 \\ 12 & 12 & 24 & 0 \end{bmatrix}$$

$$0-k = 2/3 \quad , \quad 0-k = 2/3$$

$$\left[k = -2/3\right] \quad \left[k = -2/3\right]$$

So, this is maximum

13/12

Taylor's series for function of two variables :-

for one variable. $y = f(x)$, $x = a$

$$f(x) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \cdots + \frac{h^n f^n(x)}{n!}$$

(where, $h = x - a$)

(or)

$$f(x) = f(x) + (x-a) f'(x) + \frac{(x-a)^2}{2!} f''(x) + \cdots$$

eg. $f(x) = x^2 + 2x + 1$, $x = 2$

So $f(x) = f(2) + (x-2) f'(2) + \frac{(x-2)^2}{2!} f''(2)$

For two variable :-

eg. $z = f(x, y)$, $x = a$ & $y = b$ let $x - a = h$, $y - b = k$

$$\left\{ f(x,y) = f(a,b) + h f_x(a,b) + k f_y(a,b) + \frac{h^2}{2!} f_{xx}(a,b) + \frac{k^2}{2!} f_{yy}(a,b) + h.k f_{x.y}(a,b) + \cdots \right\}$$

diff. w.r.t. x diff. w.r.t y

Ques:- Determine the first & second degree Taylor's polynomial approximation for the following functions of x & y variable near the given points.

(i) $f(x,y) = \sin 2x + \cos y$; $(0,0)$

(ii) $f(x,y) = x e^y + 1$; $(1,0)$

sol^n (i) $a = 0$, $b = 0$

$x - a = h$, $y - b = k$

$x - 0 = h$ $y - 0 = k$

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$f(a,b) + h f_x(a,b) + k f_y(a,b)$ — ①

$f(a,b) = f(0,0) = \sin 0 + \cos 0 = 1$

diff. $f(x,y)$ w.r.t. x

$f_x(x,y) = 2.\cos 2x + 0$

$f_x(a,b) = 2\cos 2(0) = 2$

$f_y(a,b) = -\sin y$

$= -\sin(0) = 0$

Put all values in eqn ①

$f(x,y)\ 1 + (x-0)\,2 + (y-0)\,0$

$f(x,y) = 1 + 2x$ Ans

(ii) $f(x,y) = xe^y + 1$ $(1,0)$

Soln so, $a = 1$ & $b = 0$

$x - 1 = h$, $y - 0 = k$

$f(x,y) = f(a,b) + h f_x(a,b) + k f_y(a,b) + \frac{h^2}{2!} f_{xx}(a,b) + \frac{k^2}{2!} f_{yy}(a,b) + h.k\, f_{xy}(a,b) + \dots$ — ①

For I^{st} degree we solve upto 3 terms

so, $f(a,b) = (1)e^0 + 1 \rightarrow 1 + 1 = 2$

$f_x(a,b) = x.e^y + 1$

$f_x(a,b) = e^y$

$f_x(a,b) = e^{(0)} = 1$

$f_y(x,y) =$ diff. (x,y) w.r.t 'y'

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$f_y(x,y) = x \cdot e^y$
$f_y(a,b) = (1) \cdot e^{(0)} = 1$

$f_{xx}(a,b)$ = double diff. ~~$f_x(x,y)$~~ $f_x(x,y)$ w.r.t 'x'
$= 0$

$f_{yy}(a,b) = x \cdot e^y = 1$
$f_{x \cdot y}(a,b) = e^y = 1$

so, $f(x,y) = 2 + (x-1)1 + (y-0)1$
$= 2 + x - 1 + y$
$= 1 + x + y$ Ans

<u>IInd degree.</u> $= 2 + (x-1)1 + (y-0)1 + \frac{(x-1)^2}{2} \cdot 0 +$

$\frac{(y-0)^2}{2}(1) + (x-1)y(1)$

$= 2 + x - 1 + y + 0 + \frac{y^2}{2} + xy - y$

$= 1 + x + xy + \frac{y^2}{2}$ Ans

Ques 1- Determine the I & II degree polynomial of -
(i) $f(x,y) = e^x \cos y$, $(0,0)$
Sol^n $x - 0 = h$, $y - 0 = k$
So, $f(a,b) = e^0 \cos(0) = 1$ — ①
Now, ~~$f(x)$~~ $f_x(x,y) = e^x \cos y$
so $f_x(a,b) = e^0 = 1$ — ②
$f_y(x,y) = -\sin y \cdot e^x$
$f_y(a,b) = 0$ — ③

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For I degree we know that

$f(x,y) = f(a,b) + h f_x(a,b) + k f_y(a,b)$

put value of eq ①, ② & ③

$= 1 + x(1) + y(0)$

$= 1 + x$ Ans

Now $f_{xx}(a,b) = e^x \cos y = 1$

$f_{yy}(a,b) = -e^x \cos y = -1$

$f_{xy}(a,b) = -e^x \sin y = 0$ (diff. of $f_y(a,b)$ w.r.t. x)

so, $f(x,y) = f(a,b) + h f_x(a,b) + k f_y(a,b) + \frac{h^2}{2!} f_{xx}(a,b)$

$+ \frac{k^2}{2!} f_{yy}(a,b) + h.k\, f_{xy}(a,b)$

$= 1 + x(1) + y(0) + \frac{(x-0)^2}{2}(1) + \frac{(y-0)^2}{2}(-1) + 0$

$= 1 + x + \frac{x^2}{2} - \frac{y^2}{2}$ Ans

(ii) $f(x,y) = x^2y + y^2$ $(1,3)$

Sol. $f(a,b) = 1 \times 3 + 9 = 12$

$f_x(x,y) = 2xy$

$f_x(a,b) = 2 \times 1 \times 3 = 6$

$f_y(a,b) = x^2 + 2y$

$= 1 + 6 = 7$

So, I order will be –

$f(x,y) = 12 + (x-1)6 + (y-3)7$

$= 12 + 6x - 6 + 7y - 21$

$= 6x + 7y - 15$ Ans

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IInd order derivative will be-

$f_{xx}(a,b) = 2y = 2\times 3 = 6$

$f_{yy}(a,b) = 2$

$f_{x\cdot y}(a,b) = 2x = 2\times 1 = 2$

So, $f(x,y) = 12 + (x-1)6 + (y-3)7 + \frac{(x-1)^2}{2}\times 6 + \frac{(y-3)^2}{\cancel{2}}(\cancel{2})$

$+ (x-1)(y-3)2$

$= 12 + \cancel{6x} - \cancel{6} + 7y - 21 + 3x^2 + 3 - 6x + y^2 + 9 - 6y + 2xy - \cancel{6x} - 2y + \cancel{6}$

$= 3x^2 + y^2 - 6x - y + 2xy + 3$ Ans

14/12/21

Limit & Continuity UNIT-3

(i) $(a,b) = \{x \in R ; a < x < b\}$

(ii) $[a,b] = \{x \in R ; a \le x \le b\}$

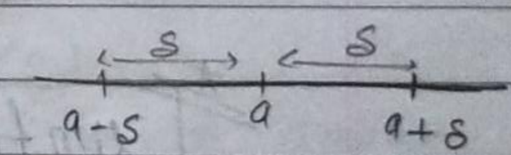
(iii) $[a,b) = \{x \in R ; a \le x < b\}$

(iv) $(a,b] = \{x \in R ; a < x \le b\}$

(v) $|x| = \begin{cases} x & ; x > 0 \\ -x & ; x < 0 \\ 0 & ; x = 0 \end{cases}$

(vi) $|x| \le a \Rightarrow -a \le x \le a$

$|x| < a \Rightarrow -a < x < a$



(7) $(a-\delta, a+\delta)$ = It is neighbour of 'a'

(or) $(a-\delta, a) \cup (a, a+\delta)$ = deleted neighbourhood of a.

Limit of a function:

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Let $f(x)$ be a function defined in neighbourhood of 'a' and 'l' be any number ~~e~~ belongs to R ($l \in R$) such that for any $\varepsilon > 0$, $\exists$ (there exist) $\delta > 0$ such that –

Epsilon

$$\left[\begin{array}{l} 0 < |x-a| < \delta \\ \Rightarrow |f(x) - l| < \varepsilon \end{array}\right]$$

(or)

$$\lim_{x \to a} f(x) = l$$

The 'l' is called limit of function $f(x)$.

Left Hand Limit & Right hand limit:-

RHL:-

$$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$$

LHL:-

$$\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$$

$$\boxed{\lim_{h \to 0} f(a+h) = \lim_{h \to 0} f(a-h) = l}$$

Ques:- Find the limit of (i) $\lim_{x \to 3} \left[\frac{x^2-9}{x-3}\right]$

Solⁿ

$$= \lim_{x \to 3} \left[\frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}\right]$$

$$= 3+3 = 6 \quad \text{Ans}$$

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$$\left[\lim_{x\to 0} \frac{\sin ax}{ax} = 1\right]$$

(ii) $\lim_{x\to 0} \frac{\sin 5x}{3x}$

Sol^n $= \lim_{x\to 0} \frac{\sin 5x}{x} \cdot \frac{1}{3}$

Multiply & divide by 5

$= \lim_{x\to 0} \frac{\sin 5x}{5x} \cdot \frac{5}{3} = 1 \cdot 5/3$

$= \frac{5}{3}$ Ans

(iii) $\lim_{x\to 0} \frac{\sin^2 4x}{x^2}$

Sol^n $= \lim_{x\to 0} \frac{(\sin 4x)^2}{x^2}$

~~Divide wt no. & d.~~

multiply & divide by $(4)^2$

$= \lim_{x\to 0} \frac{(\sin 4x)^2}{(4x)^2} \times 4^2 = \left(\frac{\sin 4x}{4x}\right)^2 \times 4^2$

$= 1 \times 4^2 = 16$ Ans

(iv) $\lim_{x\to a} \frac{(x^n - a^n)}{(x-a)}$

Sol^n This is indeterminant form $\left(\frac{0}{0}\right)$ so we apply L-Hospital rule — $\frac{f'(x)}{g'(x)}$

$= \lim_{x\to a} \left(\frac{nx^{n-1}}{1}\right)$

$= \cancel{a}x^{a-1}$ na^{n-1} Ans

(v) $\lim_{x\to 1} \frac{x^{15}-1}{x^{10}-1}$

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Sol^n L Hoppital Rule $= \lim_{x \to 1} \left(\frac{15x^{14-1}}{10x^{9-1}} \right)$

$$= \frac{15(1)^{13}}{10(1)^8} = \frac{15}{10} = \frac{3}{2} \text{ Ans}$$

Ques. $\lim_{x \to \infty} [\operatorname{cosec}^{-1} x - \cot^{-1} x]$

Sol^n $= \lim_{x \to \infty} \left[\sin^{-1}\left(\frac{1}{x}\right) - \tan^{-1}\left(\frac{1}{x}\right) \right]$

$$= \left[\sin^{-1}\left(\frac{1}{\infty}\right) - \tan^{-1}\left(\frac{1}{\infty}\right) \right]$$

$$= \sin^{-1}(0) - \tan^{-1}(0)$$

$$= 0 - 0 = 0 \text{ Ans}$$

Ques. Prove that –

(i) $\lim_{x \to a} \frac{x^n - a^n}{x^m - a^m} = \frac{n}{m} a^{n-m}$

(ii) $\lim_{x \to 0} \frac{e^x \pm 1}{x} = 1$

(iii) $\lim_{x \to 0} \frac{a^x \pm 1}{x} = ?$

(iv) $\lim_{x \to 1} \frac{x + x^2 + x^3 + \cdots + x^n - n}{x - 1}$

Sol(i) Acc. to L Hospital Rule – $\lim_{x \to a} \left(\frac{nx^{n-1}}{mx^{m-1}} \right)$

$$= \frac{na^{n-1}}{ma^{m-1}} = \frac{n}{m} a^{n-1-m+1}$$

$$= \frac{n}{m} a^{n-m}$$

Hence Proved.

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(ii) $\lim_{x \to 0} \frac{e^x \pm 1}{x}$

L Hospital rule $= \lim_{x \to 0} \frac{e^x + 0}{1}$

$= e^0 = 1$

Hence Proved.

(iii) $\lim_{x \to a} \frac{a^x \pm 1}{x}$

Sol^n L Hoppital Rule $= \lim_{x \to a} \frac{a^x \log_e a + 0}{1}$

$= \lim_{x \to a} a^x \log_e a$

$= a^a \log a$ Ans

(iv) $\lim_{x \to 1} \frac{x + x^2 + x^3 + \cdots + x^n - n}{x - 1}$

Sol^n L Hoppital Rule $= \lim_{x \to 1} \frac{1 + 2x + 3x^2 + \cdots + nx^{n-1} - 0}{1}$

put limit $x \to 1$ then

$= 1 + 2 + 3 + \cdots + n$

this series is equal to $n\left(\frac{n+1}{2}\right)$

$= \frac{n(n+1)}{2}$ Ans

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Continuity :→

A function $f(x)$ defined in the neighbourhood of 'a' & ~~lim~~

$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x) = f(a)$, the function is called continuous function.

If the ~~above~~ above not satisfy at $x=a$ / in the neighbourhood of 'a' the function is called discontinuous function.

* Types of discontinuity –

(i) $\lim_{x \to a} f(x)$ exist $\neq f(a)$ then this is called removable discontinuity.

(ii) If $\lim_{x \to a} f(x)$ does not exist then it is called irremovable discontinuity.

* Properties of Continuity :-

(i) If $f(x)$ & $g(x)$ are continuous at $x=a$ then $f+g$, $f-g$, f/g are all continuous at $x=a$.

(ii) If f is a continuous function at $x=a$ & g is a continuous function at $x=f(a)$ then ~~gof~~ gof will also be a continuous function at $x=a$.

(iii) Every polynomial function, identity function are continuous on R.

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Q. Show that the function -

$$f(x) = \begin{cases} x \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$, at $x=0$ continuous.

Sol^n at $x=0$,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} = \lim_{x \to 0} f(x)$$

$$\lim_{h \to 0} (a-h) = \lim_{h \to 0} f(a+h) = \lim_{x \to 0} f(x)$$

$$0 = 0 = 0$$

$$LHL = RHL = f(a)$$

Hence, continuous.

Q. Find the value of k so that the function

$$f(x) = \begin{cases} \dfrac{1-\cos 4x}{8x^2}; & \text{if } x \neq 0 \\ k; & \text{if } x = 0 \end{cases}$$ so that the $f(x)$ at $x=0$.

Sol^n at $x \neq 0$

$$\lim_{h \to 0} \left(\frac{1-\cos 4h}{8h^2} \right) = \lim_{h \to 0} \left(\frac{1-\cos 4h}{8h^2} \right) = k$$

$$\lim_{h \to 0} \frac{1-1+2\sin^2 2h}{8h^2} = k \qquad [\because \cos 2\theta = 1 - 2\sin^2\theta]$$

$$k = \lim_{h \to 0} \left(\frac{\sin^2 2h}{4h^2} \right)$$

$$\boxed{k = 1}$$ Ans

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Q. $f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2}, & x<0 \\ K, & x=0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & x>0 \end{cases}$

Sol^n $\lim_{h\to 0^-} f(-h) = \lim_{h\to 0^+} f(h) = f(0)$

$\lim_{h\to 0}\left(\dfrac{1-\cos 4h}{h^2}\right) = \lim_{h\to 0}\left(\dfrac{\sqrt{h}}{\sqrt{16+\sqrt{h}}-4}\right) = K$

$\lim_{h\to 0} \dfrac{1-1+2\sin^2 2h}{(2h)^2} \times 4$

Q. find k $f(x) = \begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & -1\le x<0 \\ \dfrac{2x+1}{x-1}, & 0\le x<1 \end{cases}$

is continuous $x=0$.

Sol^n $\lim_{x\to 0^+} f\left(\dfrac{2x+1}{x-1}\right) = \lim_{x\to 0^-} f\left(\dfrac{\sqrt{1-kh}-\sqrt{1+kh}}{x} \times \dfrac{\sqrt{1-kh}+\sqrt{1+kh}}{\sqrt{1-kh}+\sqrt{1+kh}}\right)$

$\dfrac{0+1}{-1} = \lim_{h\to 0} f\left(\dfrac{1-kh-1-kh}{h\sqrt{1-kh}+h\sqrt{1+kh}}\right) = \dfrac{-2k}{2k} = -1$

$-1 = -1$

L.H.S = R.H.S.

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Q. $f(x) = \begin{cases} a\sin\frac{\pi}{2}(x+1) \; ; \; x \le 0 \\ \dfrac{\tan x - \sin x}{x^2} \; ; \; x > 0 \end{cases}$ = continuous at $x=0$.

Sol^n $\lim_{x\to 0^-} f\left[a\sin\frac{\pi}{2}(x+1)\right] = \lim_{x\to 0^+}\left[\dfrac{\tan x - \sin x}{x^3}\right]$

$\lim_{x\to 0^-} f\left[a\sin\left(\frac{\pi}{2}+\frac{\pi}{2}x\right)\right] = \lim_{x\to 0^+}\left[\dfrac{\sin x - \sin x\cdot\cos x}{\cos x\cdot x^3}\right]$

$\lim_{x\to 0^-} f\left[a\cos\frac{\pi}{2}x\right] = \lim_{x\to 0^+}\left[\dfrac{\sin x(1-\cos x)}{\cos x\cdot x^2}\right]$

$a\times 1 = \lim_{x\to 0^+}\left[\dfrac{1}{\cos x}\times\dfrac{\cancel{1}-\cancel{1}+\cancel{2}\sin^2 x/2}{x^2/4\times\cancel{4}_2}\right]$

$a = \lim_{x\to 0^+}\left[\dfrac{1}{\cos x}\times\dfrac{1}{2}\right]$

$\left\{a = \dfrac{1}{2}\right\}$ Ans

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UNIT-4

DOUBLE INTEGRATION

$\left[\because \int e^{ax}.dx = \frac{e^{ax}}{a}\right]$

An integration of the form:-

$$I = \iint_R f(x,y)\, dx.dy$$ is called

double integration.

w.r.t 'x' and 'y'.

(i) When all the limits of integration are constant.

Q. $I = \int_1^a\int_1^b \frac{1}{x.y}\, dx.dy \Rightarrow \int_1^a \left[\int_1^b \frac{1}{x}.dx\right]\frac{1}{y}.dy$

$\Rightarrow \int_1^a [\log x]_1^b \frac{1}{y}.dy \Rightarrow \int_1^a [\log b - \log 1]\frac{1}{y}.dy$

$\Rightarrow \log b \int_1^a \frac{1}{y}.dy \Rightarrow \log b\,[\log y]_1^a$

$\Rightarrow \log b.\log a + c$ Ans

(ii) When x_1 & x_2 are constant but y_1 & y_2 are funct of x:-

Q $I = \int_0^1\int_0^{x^2} e^{y/x}.dy.dx = \int_0^1 \left[\int_0^{x^2} e^{y/x}.dy\right].dx$

$= \int_0^1 [x.e^{y/x}]_0^{x^2}.dx = \int_0^1 [xe^x - x].dx$

$= \int_0^1 [x.e^x - x].dx = \int_0^1 xe^x dx - \int_0^1 x.dx$

$= x\int e^x.dx - \int\left[\frac{d}{dx}(x).\int e^x.dx\right].dx - \frac{x^2}{2}$

$= [x.e^x - e^x]_0^1 - [x^2/2]_0^1 = (e-e) - (-1) - 1/2$

$= 1 - 1/2 = \frac{1}{2}$ Ans

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Ques. $I = \int_0^1 \int_0^{\sqrt{1-y^2}} y \cdot dy \cdot dx$

Sol^n $I = \int_0^1 \left[\int_0^{\sqrt{1-y^2}} y \cdot dy \cdot dx \right.$

$I = \int_0^1 [x]_0^{\sqrt{1-y^2}} \cdot y \cdot dy$ if $y=0$, $y=1$; $t=1$, $t=0$

$= \int_0^1 y\sqrt{1-y^2} \cdot dy \Rightarrow$ Let, $1-y^2 = t$

$-2y \cdot dy = dt$

$= \int_1^0 \sqrt{t} \cdot \frac{dt}{-2}$ $\quad$ $y \cdot dy = \frac{dt}{-2}$

$= \frac{1}{2}\left[\frac{2}{3} t^{3/2} \right]_1^0$

$= \frac{1}{3}(1-0) = \frac{1}{3}$ **Ans**

Ques Find: $I = \int_0^{\pi} \int_0^{a(1+\cos\theta)} r \cdot d\theta \cdot dr$

Sol^n $I = \int_0^{\pi} \left[\frac{r^2}{2} \right]_0^{a(1+\cos\theta)} \cdot d\theta$ $\quad$ $[\because \cos 2\theta = 2\cos^2\theta - 1]$

$I = \int_0^{\pi} \left[\frac{a(1+\cos\theta)}{2} \right]^2 \cdot d\theta$ $\quad$ $\left\{ \cos^2\theta = \frac{\cos 2\theta + 1}{2} \right\}$

$I = \frac{a^2}{2} \int_0^{\pi} (1 + \cos^2\theta + 2\cos\theta) \cdot d\theta$

$I = \frac{a^2}{2} \int_0^{\pi} \left(1 + \frac{\cos 2\theta + 1}{2} + 2\cos\theta \right) \cdot d\theta$

$I = \frac{a^2}{2} \times \left[\theta + \frac{\sin 2\theta}{2} + \frac{\theta}{2} + 2\sin\theta \right]_0^{\pi}$

$I = \frac{a^2}{2} \times \left[\pi + 0 + \frac{\pi}{2} + 0 \right]$

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$$I = \frac{a^2}{2} \times \frac{3\pi}{2} = \frac{a^2 \cdot 3\pi}{4} \quad \underline{Ans}$$

<u>Change of Variables:-</u>

$$\boxed{\iint_R f(x,y)\cdot dx\cdot dy = \iint_R f(r\cos\theta, r\sin\theta)\, r\cdot d\theta\cdot dr}$$

Ques:- find the value of $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\cdot dx\cdot dy$ by changing it to polar coordinates.

<u>Ans</u> $I = \int_0^\infty\int_0^\infty e^{-r^2}\cdot r\cdot d\theta\cdot dr$

$\therefore \quad x = r\cos\theta$ — ① , $y = r\sin\theta$ — ②

square and add both the equations

$x^2+y^2 = r^2$ — ③

Divide ② ÷ ①

$\left\{\tan\theta = \frac{y}{x}\right\}$. $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

from eqn ③

put $x=0$, $y=0$ then

$\boxed{r=0}$

if x & $y = \infty$ then,

$\boxed{r=\infty}$

$\tan^{-1}\left(\frac{0}{x}\right) = 0$. $\tan^{-1}\left(\frac{\infty}{x}\right) = \frac{\pi}{2}$

$$I = \int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}\cdot r\cdot d\theta\cdot dr$$

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$t=0,\ r=0$ & $t=\infty,\ r=\infty.$

$\therefore r^2 = t$

$2r.dr = dt$

$r.dr = \frac{1}{2}.dt$

$$I = \int_0^{\frac{\pi}{2}}\int_0^{\infty} \frac{dt}{2}.e^{-t}.d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\left[-e^{-t}\right]_0^{\infty}.d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\left[-e^{-t}\right]_0^{\infty}.d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[(-e^{-\infty}+1)\right]_0^{\pi/2} \Rightarrow \frac{1}{2}\left[0+\theta\right]_0^{\pi/2}$$

$$= \frac{1}{2}\times\frac{\pi}{2} = \frac{\pi}{4}$$ Ans

Q:- Find the value of -

(i) $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dx.dy}{1+x^2+y^2}$ (ii) $\int_0^1\left[\int_0^1 \frac{x-y}{(x+y)^3}.dy\right].dx.$

Ans (i) $\int_0^1\int_0^{a\sqrt{1+x^2}} \frac{dx.dy}{1+x^2+y^2}$ let, $1+x^2=a^2$

$a=\sqrt{1+x^2}$

$\left\{\therefore \frac{1}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right\}$

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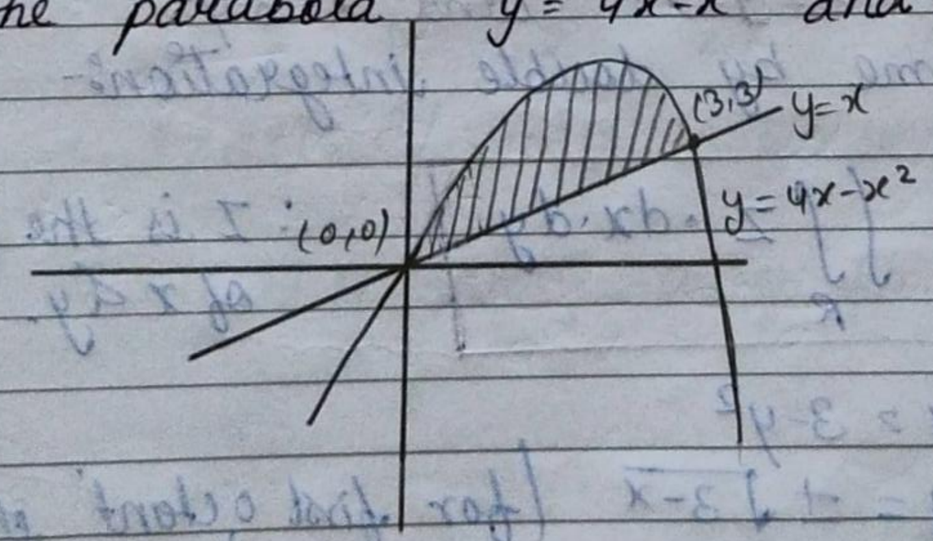
Area & Volume by Double Integration:-

eg. By double Integration find the area lying b/w the parabola $y = 4x - x^2$ and the line $y = x$.



Soln $y = 4x - x^2$, $y = x$

so, $x = 4x - x^2$

$x - 4x + x^2 = 0$

$x^2 = 3x$

$x - 3x = 0$, $x(x-3) = 0$, $\boxed{x = 0 \text{ \& } x = 3}$

$x=0,3$ & $y=0$

$x=3$, $y=3$ do, $x=0$ & $y=0$.

$$A = \int_0^3 \int_x^{4x-x^2} 1 \cdot dx \cdot dy$$

$$A = \int_0^3 [y]_x^{4x-x^2} \cdot dx$$

$$A = \int_0^3 (4x - x^2 - x) \cdot dx$$

$$A = \left[\frac{4x^2}{2} - \frac{x^3}{3} - \frac{x^2}{2}\right]_0^3$$

$$= \frac{\cancel{4}^{2} \times 9}{\cancel{2}} - \frac{27}{3} - \frac{9}{2}$$

$$= 18 - 9 - \frac{9}{2} = \frac{9}{2} \text{ sq. unit Ans}$$

Q. Find the volume in the first octant (quadrant in 3-D) bounded by the parabolic cylinder $z = 9 - x^2$ and $x = 3 - y^2$.

Sol^n Volume by double integration:-

$$\boxed{V = \iint_R z \cdot dx \cdot dy}$$

$\therefore z$ is the function of x & y.

$x = 3 - y^2$

$y = \pm\sqrt{3-x}$ (for first octant it should be +ive)

$\{y = +\sqrt{3-x}\}$

$z = 9 - x^2$ at origin

$z = 0$

$\boxed{x = \pm 3}$

$$V = \int_0^3 \int_0^{\sqrt{3-x}} (9-x^2)\,dx \cdot dy. \rightarrow \int_0^3 9-x^2\left(\int_0^{\sqrt{3-x}} 1\,dy\right)\cdot dx$$

$$= \int_0^3 (9-x^2)\left(\sqrt{3-x}\right) \cdot dx$$

if $x=0$,
$x = 3\cos\theta$
$0 = 3\cos\theta$
$\boxed{\theta = \frac{\pi}{2}}$

if $x = 3$
$\boxed{\theta = 0^\circ}$

On putting $x = 3\cos\theta$
$\{dx = -3\sin\theta \cdot d\theta\}$

$$V = -\int_{\pi/2}^{0} (9 - 9\cos^2\theta)\sqrt{3-3\cos\theta} \cdot 3\sin\theta \cdot d\theta$$

$(1-\cos^2\theta) = \sin^2\theta$

$$V = -9\sqrt{3}\int_{\pi/2}^{0} \sin^2\theta \cdot \sqrt{2}\sin\frac{\theta}{2} \cdot 3\sin\theta \cdot d\theta$$

$\{\because \sqrt{1-\cos\theta}\}$ $\quad 1-\cos\theta = 2\sin^2\frac{\theta}{2}$

$\sqrt{1-1+2\frac{\sin^2\theta}{2}}$

$\sqrt{2}\sin\theta/2$

$$V = -27\sqrt{6}\int_{\pi/2}^{0} \sin^3\theta \cdot \sin\frac{\theta}{2} \cdot d\theta$$

$$V = +27\sqrt{6}\int_0^{\pi/2} (\sin\theta)^3 \cdot \sin\frac{\theta}{2} \cdot d\theta \qquad \left\{\because \int_a^b \sin\theta\, d\theta = -\int_b^a \sin\theta\, d\theta\right\}$$

$\sin 2\theta = 2\sin\theta \cdot \cos\theta$

$$= 27\sqrt{6}\int_0^{\pi/2} \left(2\sin\frac{\theta}{2} \cdot \cos\frac{\theta}{2}\right)^3 \cdot \sin\frac{\theta}{2} \cdot d\theta$$

$$= 8 \times 27\sqrt{6}\int_0^{\frac{\pi}{2}} \sin^3\frac{\theta}{2} \cdot \cos^3\frac{\theta}{2} \times \sin\frac{\theta}{2} \cdot d\theta$$

$$= 216\sqrt{6}\int_0^{\frac{\pi}{2}} \sin^4\frac{\theta}{2} \cdot \cos^3\frac{\theta}{2} \cdot d\theta.$$

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Beta & Gamma functions :-

- **Gamma functions :-** The Gamma functions is denoted by 'Γn' is an improper integral defined as-

$$\boxed{\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1} \cdot dx}$$

where, 'n' is the parameter of integral

- **Beta function :-** It is denoted by '$\beta(m,n)$ or $B(m,n)$' is defined as-

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \cdot dx}$$

Eg. Prove that $\Gamma n+1 = n\Gamma n$

Sol^n Acc. to definition we know that-

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} \cdot dx \quad \text{—(1)}$$

Eg. $\Gamma 5 = 4\Gamma 4$
$= 4\times 3\Gamma 3$
$= 4\times 3\times 2\Gamma 2$

$\boxed{\Gamma 1 = 1}$

It also works on fractional eqn function

$\therefore$ replace $n \to n+1$ in eqn (1)

$$\Gamma n+1 = \int_0^\infty \underset{II}{e^{-x}} \cdot \underset{I}{x^n} \cdot dx$$

$$\Gamma n+1 = -x^n e^{-x}\Big|_0^\infty - \int_0^\infty n x^{n-1} (-e^{-x}) \cdot dx$$

$$= 0 + n \underbrace{\int_0^\infty e^{-x} x^{n-1} \cdot dx}_{\Gamma n \text{ from eqn (1)}}$$

$$\boxed{\Gamma n+1 = n\Gamma n}$$

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हों जिसमें ans. लाना है तो Γ^n use करेंगे जैसे आगे
β में लाना है तो β function use करेंगे।

Imp ***

eg Prove that:- $\Gamma\frac{1}{2} = \sqrt{\pi}$

soln Proof:-

Acc. to definition we know that

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} . dx \quad -(1)$$

On putting $n = \frac{1}{2}$

$$\Gamma\frac{1}{2} = \int_0^\infty e^{-x} x^{-1/2} . dx \quad -(2)$$

Let, $x = u^2$ limits for $x=0, u=0$
$x=\infty, u=\infty$

$dx = 2u . du$

from eq^n (2)

$$\Gamma\frac{1}{2} = \int_0^\infty e^{-u^2} . (u^2)^{-1/2} . 2u . du$$

$$\Gamma\frac{1}{2} = \int_0^\infty e^{-u^2} u^{-1} . 2u . du$$

$$\Gamma\frac{1}{2} = \int_0^\infty e^{-u^2} . \frac{1}{\cancel{u}} \times 2\cancel{u} . du$$

$$\Gamma\frac{1}{2} = 2\int_0^\infty e^{-u^2} . du \quad -(3)$$

Assume, $u = v$

similarly, $\Gamma\frac{1}{2} = 2\int_0^\infty e^{-v^2} . dv \quad -(4)$

Multiply eq^n (3) & (4)

$$\left(\Gamma\frac{1}{2}\right)^2 = 4\int_0^\infty\int_0^\infty e^{-u^2} . e^{-v^2} . du . dv$$

$$\left(\Gamma\frac{1}{2}\right)^2 = 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)} du, dv \quad -(5)$$

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In eq^n (5), putting $u = r\cos\theta$ & $v = r\sin\theta$

so, $u^2 + v^2 = r^2$ limit $u=0$ & $v=0 \Rightarrow r=0$

$\tan\theta = \frac{v}{u}$ $u=\infty, v=\infty \Rightarrow r=\infty$

$\theta = 0$ (lower limit)

$\theta = \tan^{-1}\left(\frac{v}{u}\right)$ $\theta = \pi/2$ (upper limit)

so, $\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\cdot r\cdot dr\cdot d\theta$

& Let $r^2 = t$, $2r\cdot dr = dt$, $r\cdot dr = \frac{dt}{2}$

$= 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} \frac{dt}{2}\cdot e^{-t}\cdot d\theta \Rightarrow \frac{1}{2}\times 4\int_0^{\frac{\pi}{2}}\left[-e^{-t}\right]_0^{\infty}$

$= \frac{\pi}{2}\quad 2\int_0^{\frac{\pi}{2}}\left[-e^{-\infty}+1\right]\cdot d\theta = 2\left[\theta\right]_0^{\pi/2}$

$= 2\times\frac{\pi}{2} = \pi$

so, $\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \pi$

Hence $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Hence Proved.

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Q. Prove that:

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

Sol^n Acc. to definition, we know that -

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx \quad -(1)$$

Let $x = \sin^2\theta$

$dx = 2\sin\theta.\cos\theta.d\theta$

limit, $x=0$, $\sin\theta=0$

$x=1$, $\sin\theta=\pi/2$

$\{\because 1-\sin^2\theta = \cos^2\}$

$$\beta(m,n) = 2\int_0^{\pi/2} (\sin^2\theta)^{m-1} (1-\sin^2\theta)^{n-1} \sin\theta.\cos\theta.d\theta$$

$$\beta(m,n) = 2\int_0^{\pi/2} (\sin^2\theta)^{m-1} (\cos^2\theta)^{n-1} \sin\theta.\cos\theta.d\theta$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-2+1}.\cos^{2n-2+1}.d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\cos^{2n-1} d\theta.$$

Hence Proved.

Formulas:-

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