The Roger and Coxeter bounds

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This note gives an illustration of the Schläfli function computation algorithm on the example of the conjectured Roger and Coxeter sphere packing bounds.

The Schläfli function $f_n(x)$ is used to define the average density of unit spheres packing [1, 2, 3]. This average density cannot exceed the Rogers bound defined as the proportion of the interior of a regular simplex of side 2 which is interior to unit spheres centred at its vertices,

$$2^{-3n/2}(n+1)^{1/2}(n!)^2 f_n(n)\,. \tag{1}$$

The Schläfli function also appears in a closely related to the sphere packing problem, the *kissing number problem*. This problem asks to find how many equal non-overlapping spheres located in n-dimensional Euclidean space can be arranged so that they all touch one central sphere of the same size. This number cannot exceed the Coxeter bound [1, 4],

$$\left\lfloor \frac{2f_{n-1}(n)}{f_n(n)} \right\rfloor\,, \tag{2}$$

where $\lfloor ... \rfloor$ is the floor function. Computed for some values of n the Roger and Coxeter bounds are presented in the Table 1. The bounds in columns 3 and 5 are computed by *bounds.py* program, which calls the function $q_n(x)$ computed by *qn_cfs.py* program, and then rounded up to twelve digits after the decimal point.

Table 1: Sphere packing bounds.

n	Roger's from [5]	Roger's computed	Coxeter's from [5]	Coxeter's computed
1	0.5	0.5	2	2
2	0.28867	0.288675134595	6	6
3	0.18612	0.186124282174	13	13
4	0.13127	0.131275370036	26	26
5	0.09987	0.099872322205	48	48
6	0.08112	0.081125515126	85	85
7	0.06981	0.069815785660	146	146
8	0.06326	0.063268438691	244	244
9	0.06007	0.060078842708	401	401
10	0.05953	0.059538163269	648	648
11	0.06136	0.061366433551	1,035	1,035
12	0.06559	0.065594886039	1,637	1,637
13	0.07253	0.072531858752	2,569	2,569
14	0.08278	0.082787262972	4,003	4,003
15	0.09735	0.097351974220	6,198	6,198
16	0.11774	0.117743760204	9,544	9,544
17	0.14624	0.146247229214	14,628	14,628
18	0.18629	0.186296973633	22,324	22,324
19	0.24308	0.243086821589	33,940	33,940
20	0.32454	0.324543415782	51,421	51,421
21	0.44289	0.442895182913	77,664	77,664
22	0.61722	0.617225848719	116,965	116,965
23	0.87767	0.877673839461	175,696	175,696
24	1.27241	1.272412247221	263,285	263,285
25	******	1.879375148626	******	393,669
26	******	2.826169287306	******	587,419
27	******	4.324246207878	******	874,863
28	******	6.728170984072	******	1,300,666
29	******	10.639502829693	******	1,930,527
30	******	17.090770970218	******	2,860,992
31	******	27.874677669244	******	4,233,797
32	******	46.139212015066	******	6,256,830
33	******	77.474386119455	******	9,234,729
34	******	131.916657582195	******	13,613,540
35	******	227.682696171162	******	20,045,836
36	******	398.192803837892	******	29,485,604

A table given in [5] contains bounds for $n \leq 24$. As it is noted in [5], the last digit given in the decimal quantities presented in [5] has not been increased where the amount truncated exceeds half a unit of this digit.

References

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