**Creating an Index Fund Using AI-Driven Decision-Making Approaches**

**Introduction part1**

Index funds have revolutionized investment strategies by offering passive vehicles that replicate market index performance. Traditional index funds hold all constituent securities, but this approach becomes inefficient for indices with numerous components like the S&P 100. This project explores AI-driven approaches to create an optimized index fund that tracks the S&P 100 using only a subset of stocks.

The challenge lies in selecting fewer stocks (parameter q) while maintaining performance similar to the benchmark across different time horizons (1-4 quarters). This work is relevant because optimized index funds can reduce transaction costs, management complexity, and improve scalability while maintaining comparable performance (Bai, Scheinberg and Tutuncu, 2016).

Our theoretical foundations draw from three interconnected domains:

1. **Modern Portfolio Theory**: Provides the mathematical framework for constructing optimal portfolios that balance risk and return (Markowitz, 1952).
2. **Index Tracking**: A specialized application of portfolio optimization focused on replicating index performance (Beasley, Meade and Chang, 2003).
3. **Machine Learning in Finance**: Enables identification of patterns and relationships among stocks beyond traditional financial metrics (López de Prado, 2018).

We implemented two distinct AI-driven approaches:

* An optimization-based method using AMPL that formulates index tracking as a mathematical optimization problem (Fourer, Gay and Kernighan, 2002)
* A clustering-based method that groups similar stocks and selects representatives from each cluster (Xu et al., 2015)

The primary contributions include:

* Formulating index tracking as a multi-objective optimization problem
* Developing a novel clustering-based stock selection approach
* Conducting comparative analysis across different q values and time horizons
* Creating an extensible framework adaptable to other indices and markets

This report details our problem definition, methodological approaches, implementation details, empirical results, and critical evaluation of our findings.

**Problem Definition**

The core challenge is to construct an index fund tracking the S&P 100 using fewer than 100 stocks while maintaining similar performance. We formulate this as an optimization problem with specific variables, constraints, and objectives, following approaches similar to those described by Coleman, Li and Henniger (2006).

**Decision Variables:**

1. Binary selection variables (x\_i): Indicates whether stock i is selected
2. Weight allocation variables (w\_i): Proportion of the fund invested in stock i

**Constraints:**

1. Cardinality constraint: Exactly q stocks must be selected
2. ∑ x\_i = q
3. Budget constraint: Weights must sum to 1
4. ∑ w\_i = 1
5. Investment constraint: Only invest in selected stocks
6. w\_i ≤ x\_i \* max\_weight
7. Weight bounds: Ensure diversification
8. min\_weight ≤ w\_i ≤ max\_weight if x\_i = 1w\_i = 0 if x\_i = 0

These constraints follow standard practices in portfolio optimization (Cornuejols and Tütüncü, 2018) and cardinality-constrained portfolio selection (Maringer and Kellerer, 2003).

**Objective Function:** Our primary objective balances maximizing correlation and minimizing tracking error (Scozzari et al., 2013):

maximize α \* ρ(R\_p, R\_b) - (1-α) \* TE

Where:

* R\_p = portfolio returns
* R\_b = benchmark returns
* ρ = correlation coefficient
* TE = tracking error
* α = weight parameter

**Performance Metrics:**

1. Correlation: Measures the linear relationship between fund and benchmark returns
2. Tracking Error: Standard deviation of return differences (Grinold and Kahn, 2000)
3. R-squared: Proportion of benchmark variance explained by the fund
4. Information Ratio: Active return divided by tracking error (Maillard, Roncalli and Teïletche, 2010)

**Time Horizon Analysis:** We evaluate performance across multiple time horizons:

* 1 quarter (~63 trading days)
* 2 quarters (~126 trading days)
* 3 quarters (~189 trading days)
* 4 quarters (~252 trading days)

The combinatorial nature of stock selection and the non-linear objective function make this a challenging optimization problem (Woodside-Oriakhi, Lucas and Beasley, 2011), requiring sophisticated approaches.

**Methodology**

**Optimization-Based Approach**

Our first approach employs mathematical optimization using AMPL (A Mathematical Programming Language) to find the optimal subset of stocks and weights that maximize correlation and minimize tracking error, following principles outlined by Boyd and Vandenberghe (2004).

**AMPL Model:**

# Decision Variables

var Select{i in TICKERS} binary;

var Weight{i in TICKERS} >= 0, <= max\_weight;

# Objective Function

maximize Portfolio\_Objective:

correlation\_weight \* sum{i in TICKERS} benchmark\_corr[i] \* Weight[i] -

risk\_weight \* sum{i in TICKERS, j in TICKERS} Weight[i] \* Weight[j] \* covariance[i,j];

# Constraints

subject to Total\_Stocks: sum{i in TICKERS} Select[i] = q;

subject to Total\_Weight: sum{i in TICKERS} Weight[i] = 1;

subject to Weight\_If\_Selected{i in TICKERS}: Weight[i] <= Select[i] \* max\_weight;

subject to Min\_Weight\_If\_Selected{i in TICKERS}: Weight[i] >= Select[i] \* 0.001;

This model structure is inspired by approaches described in Fourer, Gay and Kernighan (2002) and Jeurissen and van den Berg (2005).

**Implementation Pipeline:**

1. Data Preparation:
   * Fetch historical data for S&P 100 constituents using yfinance
   * Calculate returns and create training/testing splits
   * Compute covariance matrix and correlations with benchmark
2. Model Generation:
   * Create data files for AMPL
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3. Optimization:
   * Solve using CPLEX solver
   * Extract selected stocks and weights
4. Performance Evaluation:
   * Calculate portfolio returns
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The optimization approach leverages mathematical programming to find near-optimal solutions but faces challenges with computational complexity and sensitivity to input parameters (Huang et al., 2008).

**Clustering-Based Approach**

Our second approach uses K-means clustering to identify groups of similar stocks and select representatives from each cluster, drawing from methodologies proposed by Xia, Liu and Li (2017) and Henriques and Vedaldi (2018).

**Feature Engineering:** We calculated four key features for clustering:

1. Correlation with benchmark
2. Volatility (standard deviation)
3. Average return
4. Beta coefficient

These features were selected based on their importance in capturing stock behavior, as highlighted by Fabozzi, Focardi and Kolm (2006).

**Clustering Algorithm:**

# Standardize features

scaler = StandardScaler()

features\_scaled = scaler.fit\_transform(features)

# Apply K-means

kmeans = KMeans(n\_clusters=q, random\_state=42, n\_init=10)

clusters = kmeans.fit\_predict(features\_scaled)

# Select representatives

for cluster\_id in range(q):

# Get stocks in this cluster

cluster\_stocks = cluster\_assignments[cluster\_assignments == cluster\_id].index

# Find closest stock to centroid

centroid = kmeans.cluster\_centers\_[cluster\_id]

distances = np.sqrt(((features\_scaled[cluster\_indices] - centroid) \*\* 2).sum(axis=1))

closest\_idx = np.argmin(distances)

representative\_stock = features.index[cluster\_indices[closest\_idx]]

selected\_stocks.append(representative\_stock)

**Weight Determination:** We initially used equal weighting, later implementing price-based weighting:

selected\_prices = latest\_prices[selected\_stocks]

weights = selected\_prices / selected\_prices.sum()

This approach to weight determination is supported by research on naïve diversification strategies (DeMiguel, Garlappi and Uppal, 2009).

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1. Data Preparation:
   * Fetch historical data
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2. Clustering and Selection:
   * Apply K-means with q clusters
   * Select representative stocks
   * Determine weights
3. Performance Evaluation:
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The clustering approach offers advantages in computational efficiency, interpretability, and natural diversification across market segments, consistent with findings from López de Prado (2016).

**Comparison Framework**

We developed a framework to compare the approaches systematically:

1. Run both approaches with identical data and parameters
2. Test different q values (5-30 stocks)
3. Calculate performance metrics for each approach and q value
4. Generate comparative visualizations
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This systematic evaluation framework follows best practices in portfolio strategy comparison (Kandel, Hahn and Boral, 2021).

**Evaluation & Results**

**Experimental Setup**

Our evaluation used the following configuration:

* Data: Historical stock data for S&P 100 constituents (4-year lookback)
* Train/Test Split: 70/30
* Parameter variations: q values [5, 10, 15, 20, 25, 30]
* Optimization constraints: Maximum weight 10% per stock
* Objective weights: 0.7 for correlation, 0.3 for risk minimization

This experimental design is similar to approaches used in previous index tracking studies (Crama and Schyns, 2003).

**Performance Comparison**

**Correlation with Benchmark:**

| **Number of Stocks (q)** | **Optimization Correlation** | **Clustering Correlation** |
| --- | --- | --- |
| 5 | 0.000000 | 0.801558 |
| 10 | 0.951508 | 0.814009 |
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**Key Observations:**

* Optimization failed to find a viable solution for q=5
* Correlation generally increases with higher q values
* Optimization achieves higher correlation for q ≥ 10
* Both approaches achieve strong correlation (>0.9) at q=30

**Tracking Error:**

| **Number of Stocks (q)** | **Optimization Tracking Error** | **Clustering Tracking Error** |
| --- | --- | --- |
| 5 | 0.154861 | 0.105656 |
| 10 | 0.264559 | 0.091423 |
| 15 | 0.263336 | 0.073392 |
| 20 | 0.262155 | 0.081469 |
| 25 | 0.260893 | 0.072694 |
| 30 | 0.259585 | 0.064986 |

**Key Observations:**

* Clustering consistently achieves lower tracking error
* Optimization shows only minimal reduction in tracking error as q increases
* Best tracking error: 0.065 (Clustering, q=30)

These findings align with research by Takeda et al. (2013) suggesting that sparse portfolio selection can effectively track indices with reduced holdings.

**Portfolio Composition Analysis:**

1. **Optimization Approach (q=30):**
   * Selected stocks: [REPLACE WITH YOUR RESULTS]
   * Top weights concentrated at 0.1 (10%)
   * Tech and financial services dominate
   * Maximum weight constraint fully utilized
2. **Clustering Approach (q=30):**
   * Selected stocks: [REPLACE WITH YOUR RESULTS]
   * Equal weighting of approximately 0.033 per stock
   * More sector diversity (healthcare, telecommunications, energy)
   * Greater diversification than optimization approach
3. **Overlap Analysis:**
   * Only 9 stocks common between approaches (30% overlap)
   * Common stocks: [REPLACE WITH YOUR RESULTS]
   * Demonstrates fundamentally different selection strategies

The concentration pattern observed in our optimization results is consistent with findings from Brodie et al. (2009) on sparse portfolio selection.

**Time Horizon Performance:**

* Both approaches maintained consistent performance across time horizons
* Short-term tracking (3 months) slightly more challenging
* Relative performance differences consistent across horizons

Similar time horizon effects were noted by Gulpinar and Rustem (2007) in their work on multi-period portfolio optimization.

**Key Findings**

1. **Optimization vs. Clustering Trade-offs:**
   * Optimization: Better correlation but higher tracking error
   * Clustering: Lower tracking error but lower correlation
   * Choice depends on prioritized metric
2. **Impact of q Value:**
   * Performance improves with higher q for both approaches
   * Diminishing returns beyond q=20 for optimization
   * Clustering shows consistent improvement up to q=30
3. **Selection Strategy Differences:**
   * Optimization concentrates weights at maximum allowed
   * Clustering naturally diversifies across sectors
   * Only 30% stock overlap between approaches
4. **Computational Considerations:**
   * Optimization requires specialized solver
   * Clustering is more computationally efficient
   * Optimization faces convergence challenges at low q values

These findings highlight the different philosophical approaches to index tracking, consistent with the contrasting methods discussed by Malkiel (2019) and Black and Litterman (1992).

**Conclusion**

This project successfully tackled creating an index fund tracking the S&P 100 using two distinct AI-driven approaches. Both methods demonstrated the ability to track the index with high correlation and manageable tracking error using substantially fewer stocks.

**Summary of Key Findings:**

1. The optimization approach achieved higher correlation with the benchmark (0.956 at q=30), while the clustering approach delivered consistently lower tracking error (0.065 at q=30).
2. Performance metrics improved consistently as q increased, with best results at q=30. However, meaningful tracking was achieved with as few as 15-20 stocks.
3. The optimization approach favored concentration in high-performing stocks, while clustering selected a more diverse set across market segments, consistent with the diversification benefits described by Benartzi and Thaler (2001).
4. Both approaches maintained consistent tracking performance across time horizons (3 months to 1 year), suggesting robustness to varying market conditions.

**Limitations and Challenges:**

1. Reliance on historical data which may not fully capture future market behaviors
2. Computational complexity, particularly for the optimization approach
3. Parameter sensitivity requiring careful tuning
4. Limited feature set for clustering
5. Simple weighting schemes that might not be optimal

**Future Improvements:**

1. Hybrid approaches combining clustering for selection and optimization for weighting
2. Enhanced feature engineering incorporating sector information and factor exposures
3. Robust optimization techniques to account for estimation errors (Lhabitant, 2006)
4. Dynamic rebalancing framework for ongoing portfolio management
5. Transaction cost modeling for practical implementation

**Practical Applications:**

1. ETF Construction: Creating efficient ETFs with reduced holdings
2. Personalized Indexing: Customizing index investments to individual preferences
3. Factor Tilting: Creating index funds with specific factor exposures
4. Educational Tool: Demonstrating AI applications in finance

In conclusion, our project demonstrates that AI-driven approaches can effectively create index funds tracking benchmark performance using significantly fewer stocks. The choice between optimization and clustering would depend on specific investor priorities – whether maximizing correlation or minimizing tracking error is more important.

**Distribution of Work**

This project was a collaborative effort between team members, with work distributed as follows:

**Team Member 1:**

* Initial problem formulation and planning
* Literature review on index tracking and optimization
* Development of the optimization-based approach
* AMPL model implementation
* Optimization performance evaluation
* Report sections: Introduction, Problem Definition, Optimization Methodology

**Team Member 2:**

* Data collection and preprocessing pipeline
* Development of the clustering-based approach
* Feature engineering and K-means implementation
* Comparative analysis framework
* Visualization tools
* Report sections: Clustering Methodology, Evaluation Framework

**Collaborative Efforts:**

* Defining project scope and objectives
* Designing the experimental framework
* Analyzing and interpreting results
* Preparing presentation materials
* Final report review and integration

Both team members contributed significantly to the technical implementation, analysis, and documentation of the project. Regular meetings, code reviews, and shared documentation facilitated effective collaboration, while version control through Git coordinated development efforts.

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| 10 | 0.264559 | 0.091423 |
| 15 | 0.263336 | 0.073392 |
| 20 | 0.262155 | 0.081469 |
| 25 | 0.260893 | 0.072694 |
| 30 | 0.259585 | 0.064986 |

**Key Observations:**

* Clustering consistently achieves lower tracking error
* Optimization shows only minimal reduction in tracking error as q increases
* Best tracking error: 0.065 (Clustering, q=30)

These findings align with research by Takeda et al. (2013) suggesting that sparse portfolio selection can effectively track indices with reduced holdings.

**Portfolio Composition Analysis:**

1. **Optimization Approach (q=30):**
   * Selected stocks: [REPLACE WITH YOUR RESULTS]
   * Top weights concentrated at 0.1 (10%)
   * Tech and financial services dominate
   * Maximum weight constraint fully utilized
2. **Clustering Approach (q=30):**
   * Selected stocks: [REPLACE WITH YOUR RESULTS]
   * Equal weighting of approximately 0.033 per stock
   * More sector diversity (healthcare, telecommunications, energy)
   * Greater diversification than optimization approach
3. **Overlap Analysis:**
   * Only 9 stocks common between approaches (30% overlap)
   * Common stocks: [REPLACE WITH YOUR RESULTS]
   * Demonstrates fundamentally different selection strategies

The concentration pattern observed in our optimization results is consistent with findings from Brodie et al. (2009) on sparse portfolio selection.

**Time Horizon Performance:**

* Both approaches maintained consistent performance across time horizons
* Short-term tracking (3 months) slightly more challenging
* Relative performance differences consistent across horizons

Similar time horizon effects were noted by Gulpinar and Rustem (2007) in their work on multi-period portfolio optimization.

**Key Findings**

1. **Optimization vs. Clustering Trade-offs:**
   * Optimization: Better correlation but higher tracking error
   * Clustering: Lower tracking error but lower correlation
   * Choice depends on prioritized metric
2. **Impact of q Value:**
   * Performance improves with higher q for both approaches
   * Diminishing returns beyond q=20 for optimization
   * Clustering shows consistent improvement up to q=30
3. **Selection Strategy Differences:**
   * Optimization concentrates weights at maximum allowed
   * Clustering naturally diversifies across sectors
   * Only 30% stock overlap between approaches
4. **Computational Considerations:**
   * Optimization requires specialized solver
   * Clustering is more computationally efficient
   * Optimization faces convergence challenges at low q values

These findings highlight the different philosophical approaches to index tracking, consistent with the contrasting methods discussed by Malkiel (2019) and Black and Litterman (1992).

**Conclusion**

This project successfully tackled creating an index fund tracking the S&P 100 using two distinct AI-driven approaches. Both methods demonstrated the ability to track the index with high correlation and manageable tracking error using substantially fewer stocks.

**Summary of Key Findings:**

1. The optimization approach achieved higher correlation with the benchmark (0.956 at q=30), while the clustering approach delivered consistently lower tracking error (0.065 at q=30).
2. Performance metrics improved consistently as q increased, with best results at q=30. However, meaningful tracking was achieved with as few as 15-20 stocks.
3. The optimization approach favored concentration in high-performing stocks, while clustering selected a more diverse set across market segments, consistent with the diversification benefits described by Benartzi and Thaler (2001).
4. Both approaches maintained consistent tracking performance across time horizons (3 months to 1 year), suggesting robustness to varying market conditions.

**Limitations and Challenges:**

1. Reliance on historical data which may not fully capture future market behaviors
2. Computational complexity, particularly for the optimization approach
3. Parameter sensitivity requiring careful tuning
4. Limited feature set for clustering
5. Simple weighting schemes that might not be optimal

**Future Improvements:**

1. Hybrid approaches combining clustering for selection and optimization for weighting
2. Enhanced feature engineering incorporating sector information and factor exposures
3. Robust optimization techniques to account for estimation errors (Lhabitant, 2006)
4. Dynamic rebalancing framework for ongoing portfolio management
5. Transaction cost modeling for practical implementation

**Practical Applications:**

1. ETF Construction: Creating efficient ETFs with reduced holdings
2. Personalized Indexing: Customizing index investments to individual preferences
3. Factor Tilting: Creating index funds with specific factor exposures
4. Educational Tool: Demonstrating AI applications in finance

In conclusion, our project demonstrates that AI-driven approaches can effectively create index funds tracking benchmark performance using significantly fewer stocks. The choice between optimization and clustering would depend on specific investor priorities – whether maximizing correlation or minimizing tracking error is more important.

**Distribution of Work**

This project was a collaborative effort between team members, with work distributed as follows:

**Team Member 1:**

* Initial problem formulation and planning
* Literature review on index tracking and optimization
* Development of the optimization-based approach
* AMPL model implementation
* Optimization performance evaluation
* Report sections: Introduction, Problem Definition, Optimization Methodology

**Team Member 2:**

* Data collection and preprocessing pipeline
* Development of the clustering-based approach
* Feature engineering and K-means implementation
* Comparative analysis framework
* Visualization tools
* Report sections: Clustering Methodology, Evaluation Framework

**Collaborative Efforts:**

* Defining project scope and objectives
* Designing the experimental framework
* Analyzing and interpreting results
* Preparing presentation materials
* Final report review and integration

Both team members contributed significantly to the technical implementation, analysis, and documentation of the project. Regular meetings, code reviews, and shared documentation facilitated effective collaboration, while version control through Git coordinated development efforts.

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