Cryptography and Network Security Chapter 9

Fourth Edition

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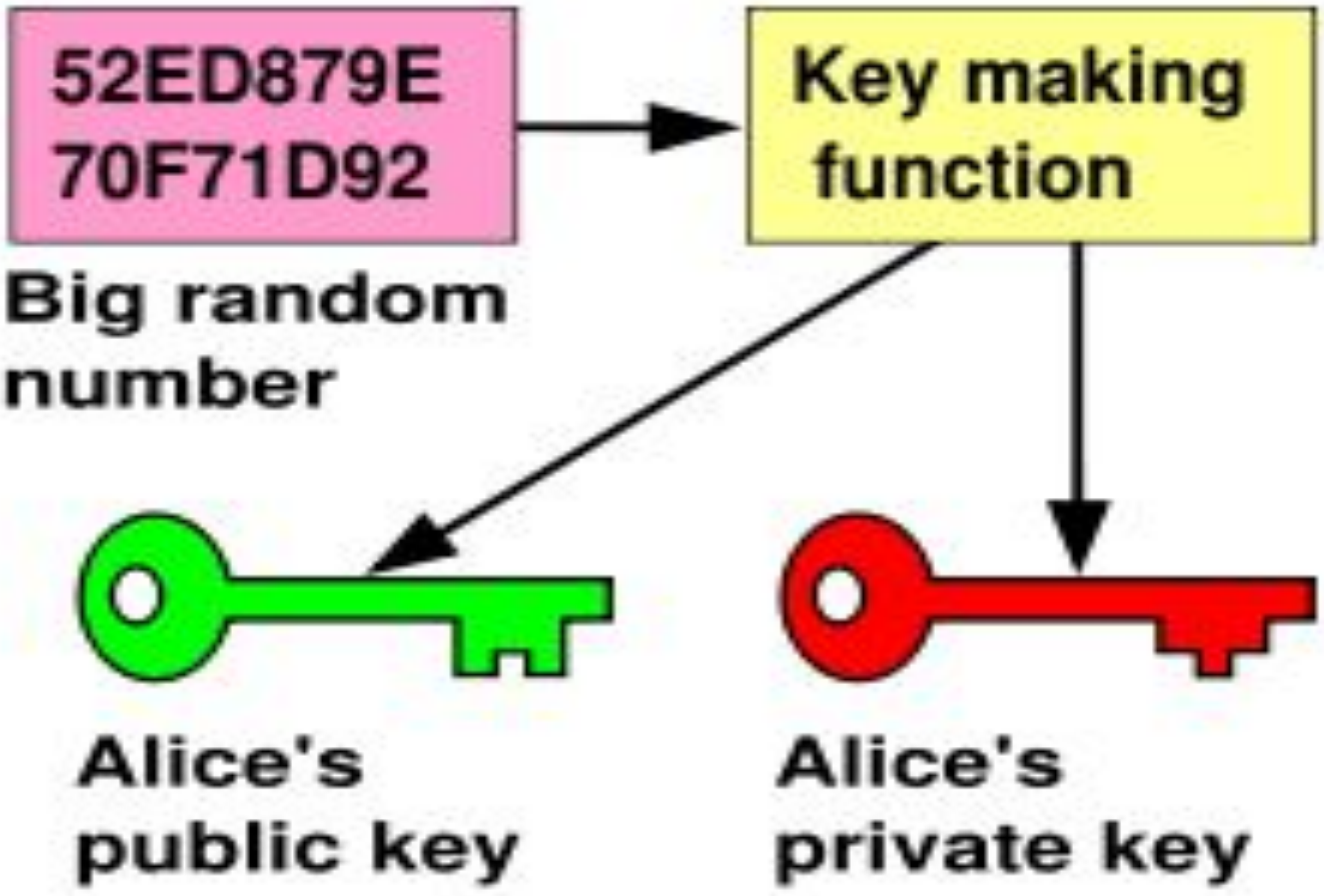
Chapter 9 – Public Key Cryptography and RSA

Every Egyptian received two names, which were known respectively as the true name and the good name, or the great name and the little name; and while the good or little name was made public, the true or great name appears to have been carefully concealed.

—*The Golden Bough*, Sir James George Frazer

Alice



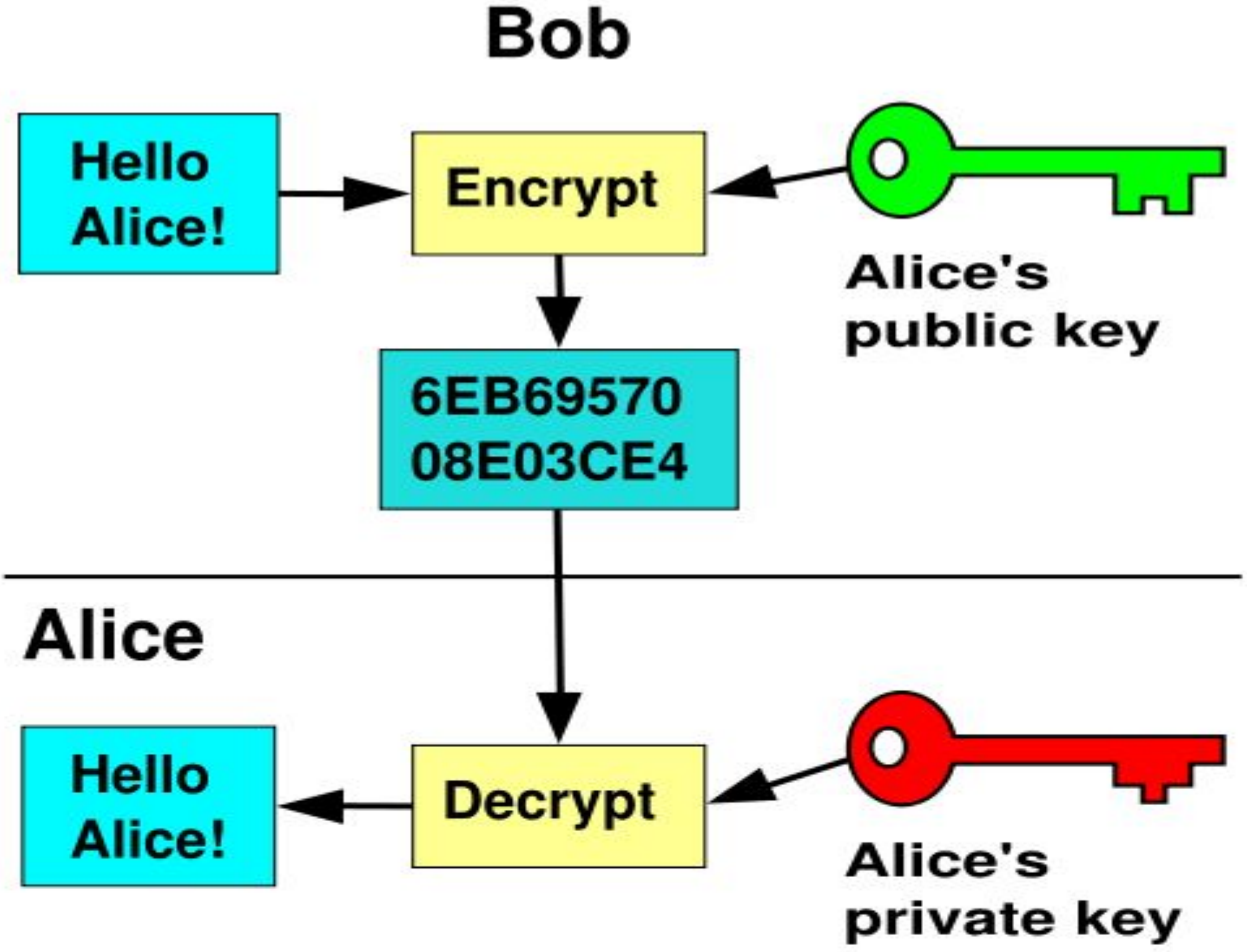
52ED879E
70F71D92

Big random number

Key making function

Alice's public key

Alice's private key



Bob

Hello Alice!

Encrypt

Alice's public key

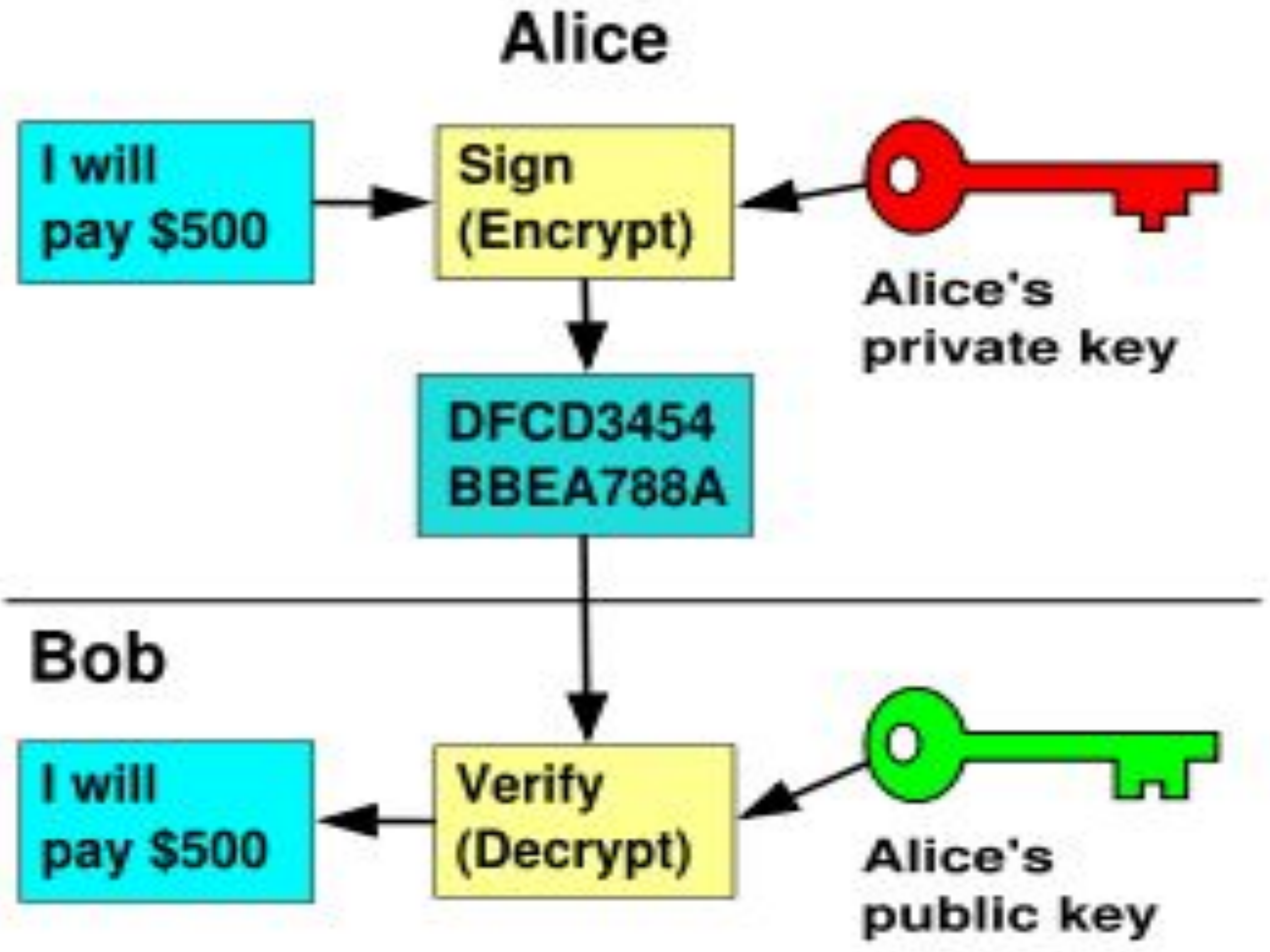
6EB69570 08E03CE4

Alice

Hello Alice!

Decrypt

Alice's private key



Alice

I will pay $500

Sign (Encrypt)

Alice's private key

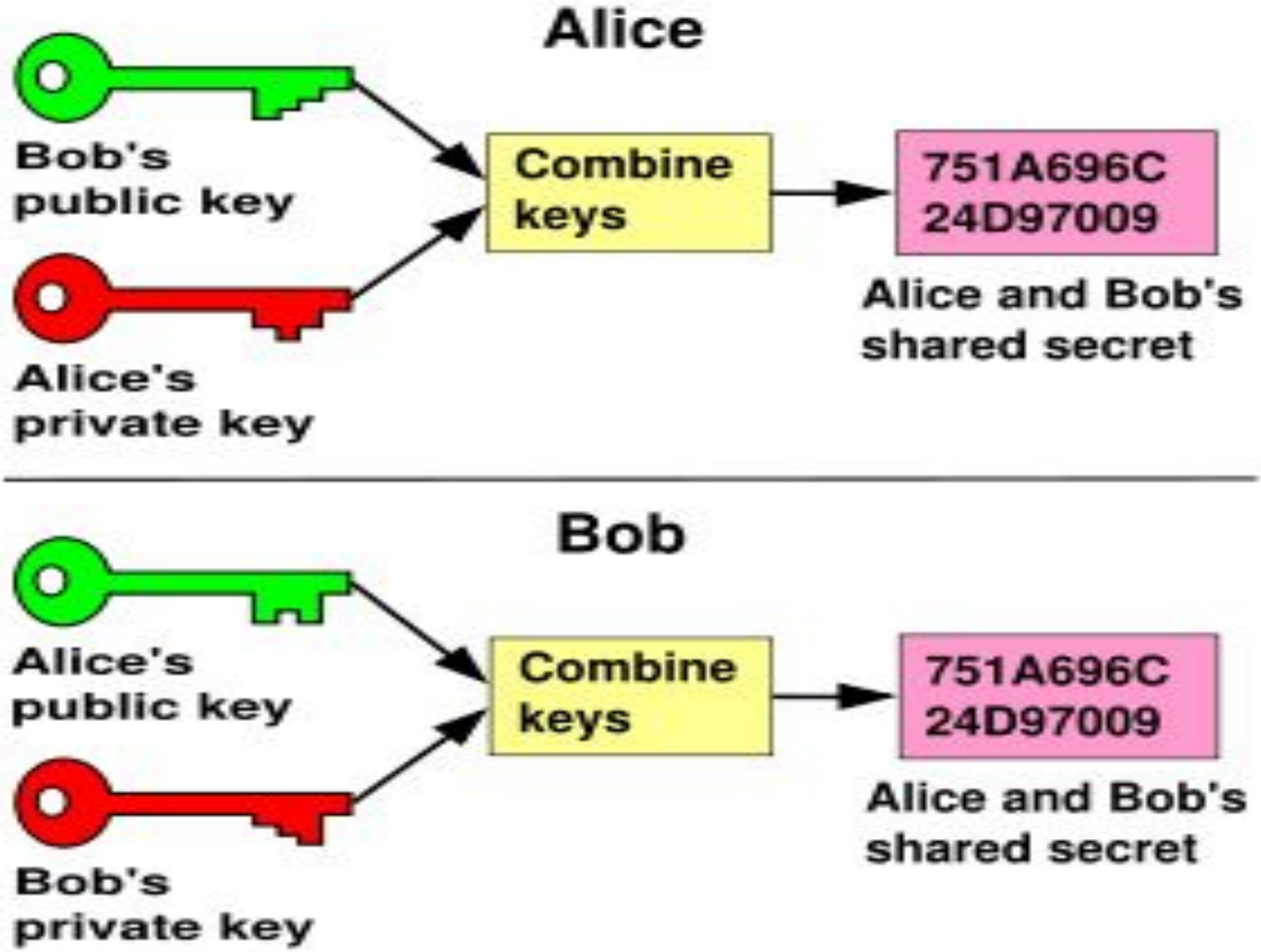
DFCD3454 BBEA788A

Bob

I will pay $500

Verify (Decrypt)

Alice's public key



Alice

Bob's public key

Alice's private key

Combine keys

751A696C 24D97009

Alice and Bob's shared secret

Bob

Alice's public key

Bob's private key

Combine keys

751A696C 24D97009

Alice and Bob's shared secret

Private-Key Cryptography

- traditional **private/secret/single key** cryptography uses **one** key
- shared by both sender and receiver
- if this key is disclosed communications are compromised
- also in **symmetric**, parties are equal
- hence does not protect sender from receiver forging a message & claiming is sent by sender

Public-Key Cryptography

- probably most significant advance in the 3000 year history of cryptography
- uses **two** keys – a public & a private key
- **asymmetric** since parties are **not** equal
- uses clever application of number theoretic concepts to function
- complements **rather than** replaces private key crypto

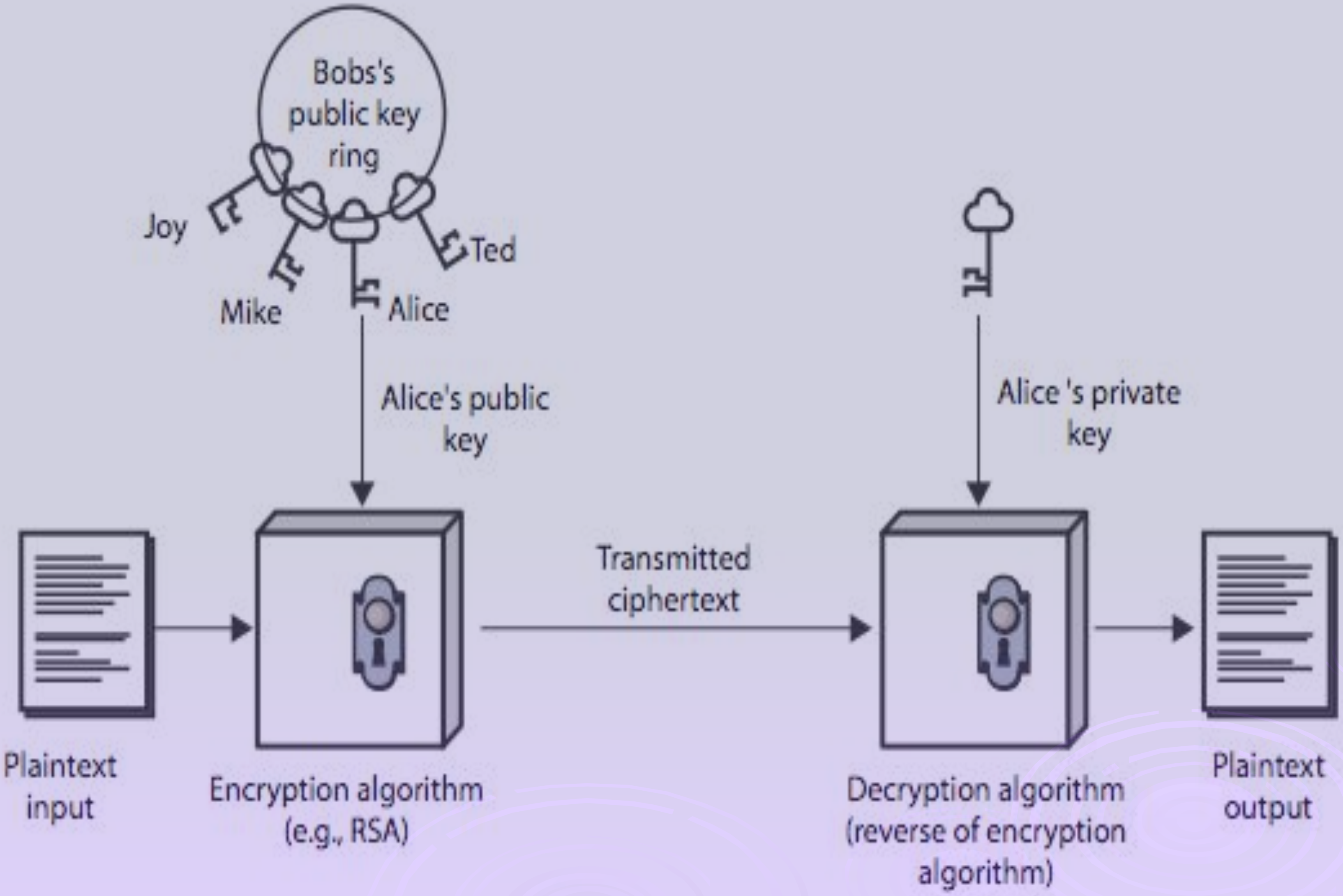
Why Public-Key Cryptography?

- developed to address two key issues:
  - **key distribution** – how to have secure communications in general without having to trust a KDC with your key
  - **digital signatures** – how to verify a message comes intact from the claimed sender
- public invention due to Whitfield Diffie & Martin Hellman at Stanford Uni in 1976
  - known earlier in classified community

Public-Key Cryptography

- **public-key/two-key/asymmetric** cryptography involves the use of **two** keys:
  - a **public-key**, which may be known by anybody, and can be used to **encrypt messages**, and **verify signatures**
  - a **private-key**, known only to the recipient, used to **decrypt messages**, and **sign** (create) **signatures**
- is **asymmetric** because
  - those who encrypt messages or verify signatures **cannot** decrypt messages or create signatures

Public-Key Cryptography



(a) Encryption

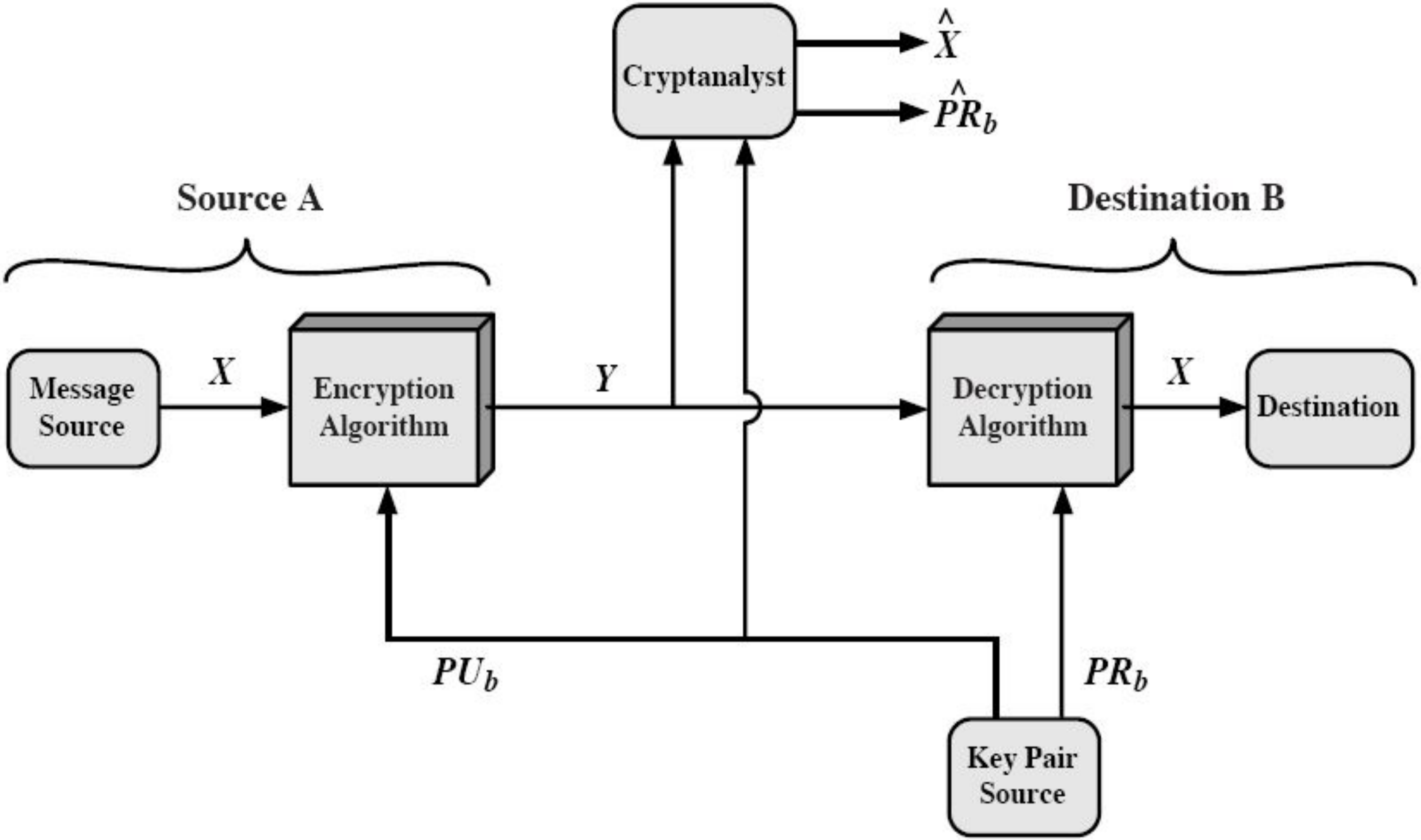
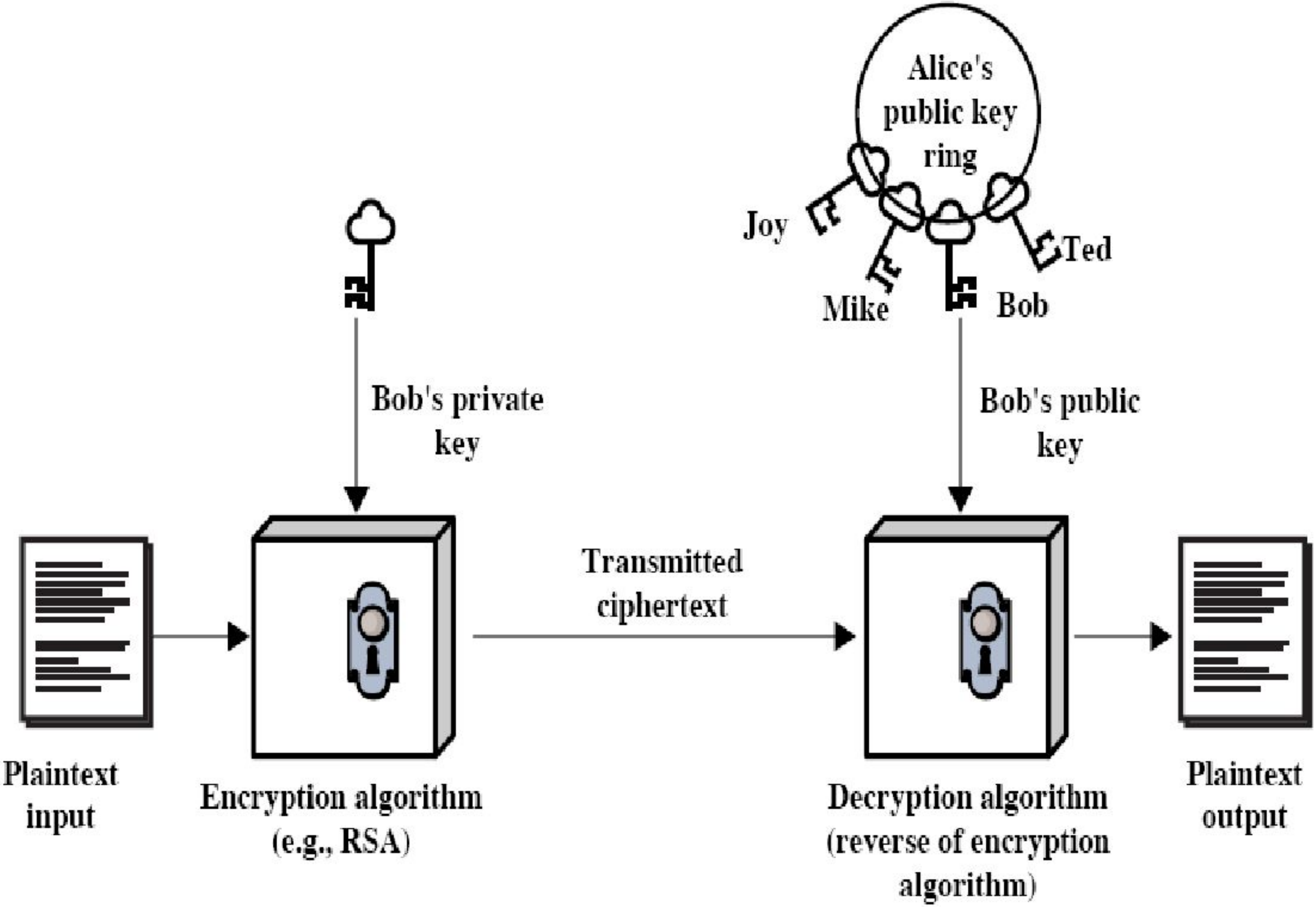


Figure 9.2 Public-Key Cryptosystem: Secrecy



(b) Authentication

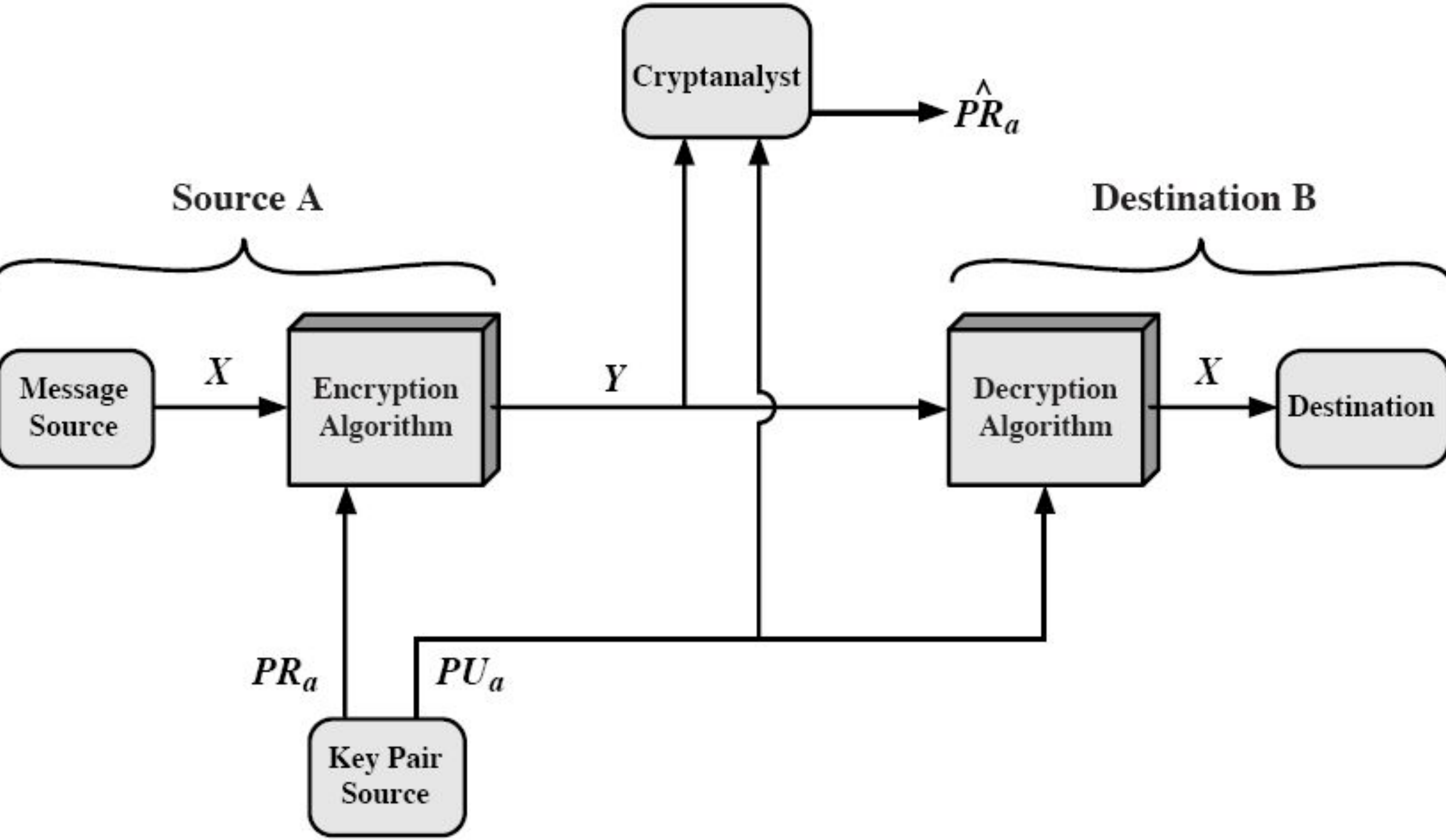
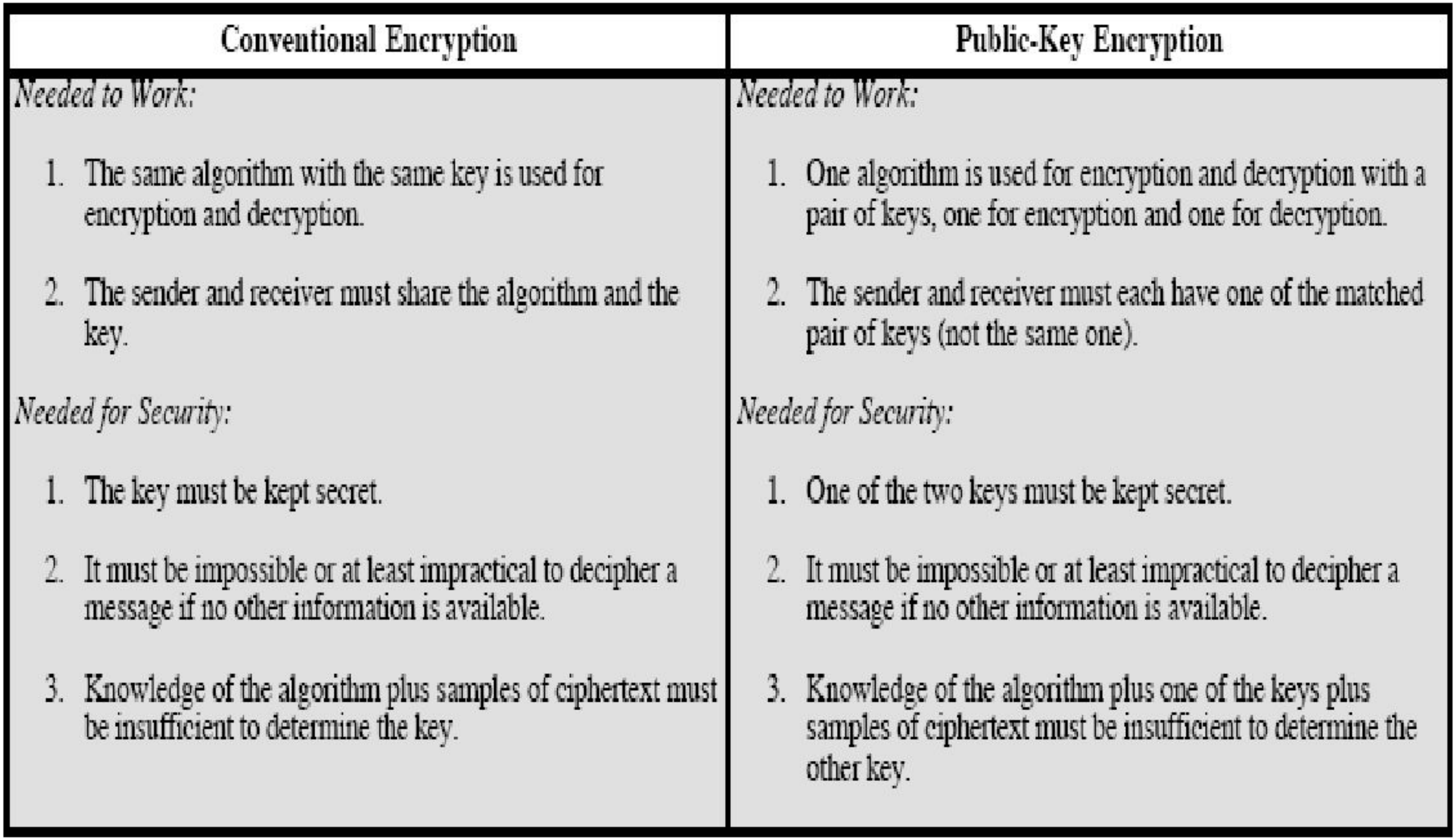


Figure 9.3 Public-Key Cryptosystem: Authentication

Table 9.1 CONVENTIONAL AND PUBLIC-KEY ENCRYPTION

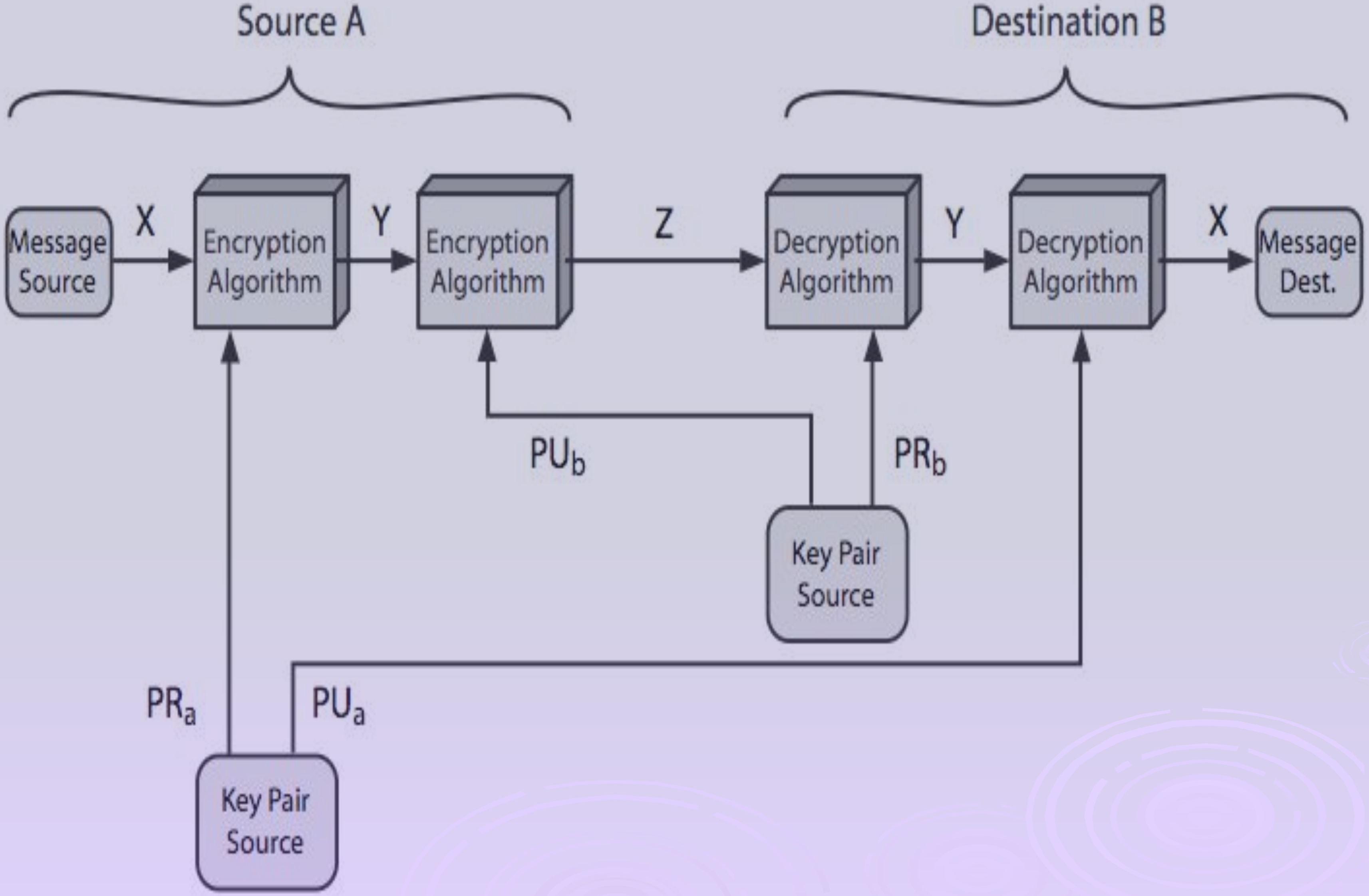


Conventional Encryption	Public-Key Encryption
Needed to Work: 1. The same algorithm with the same key is used for encryption and decryption. 2. The sender and receiver must share the algorithm and the key.	*Needed to Work:* 1. One algorithm is used for encryption and decryption with a pair of keys, one for encryption and one for decryption. 2. The sender and receiver must each have one of the matched pair of keys (not the same one).
Needed for Security: 1. The key must be kept secret. 2. It must be impossible or at least impractical to decipher a message if no other information is available. 3. Knowledge of the algorithm plus samples of ciphertext must be insufficient to determine the key.	*Needed for Security:* 1. One of the two keys must be kept secret. 2. It must be impossible or at least impractical to decipher a message if no other information is available. 3. Knowledge of the algorithm plus one of the keys plus samples of ciphertext must be insufficient to determine the other key.

Public-Key Characteristics

- Public-Key algorithms rely on two keys where:
  - it is computationally infeasible to find decryption key knowing only algorithm & encryption key
  - it is computationally easy to en/decrypt messages when the relevant (en/decrypt) key is known
  - either of the two related keys can be used for encryption, with the other used for decryption (for some algorithms)

Public-Key Cryptosystems



Public-Key Applications

- can classify uses into 3 categories:
  - **encryption/decryption** (provide secrecy)
  - **digital signatures** (provide authentication)
  - **key exchange** (of session keys)
- some algorithms are suitable for all uses, others are specific to one

Security of Public Key Schemes

- like private key schemes brute force **exhaustive search** attack is always theoretically possible
- but keys used are too large (>512bits)
- security relies on a **large enough** difference in difficulty between **easy** (en/decrypt) and **hard** (cryptanalyse) problems
- more generally the **hard** problem is known, but is made hard enough to be impractical to break
- requires the use of **very large numbers**
- hence is **slow** compared to private key schemes

RSA

- by Rivest, Shamir & Adleman of MIT in 1977
- best known & widely used public-key scheme
- based on exponentiation in a finite (Galois) field over integers modulo a prime
  - nb. exponentiation takes $O((\log n)^3)$ operations (easy)
- uses large integers (eg. 1024 bits)
- security due to cost of factoring large numbers
  - nb. factorization takes $O(e^{\log n \log \log n})$ operations (hard)

RSA Key Setup

- each user generates a public/private key pair by:
- selecting two large primes at random - p, q
- computing their system modulus $n=p.q$
  - note $ø(n)=(p-1)(q-1)$
- selecting at random the encryption key e
  - where $1<e<ø(n)$, $\gcd(e,ø(n))=1$
- solve following equation to find decryption key d
  - $e.d=1 \bmod ø(n)$ and $0≤d≤n$
- publish their public encryption key: PU={e,n}
- keep secret private decryption key: PR={d,n}

RSA Use

- to encrypt a message M the sender:
  - obtains **public key** of recipient $PU=\{e,n\}$
  - computes: $C = M^e \bmod n$, where $0 \le M < n$
- to decrypt the ciphertext C the owner:
  - uses their private key $PR=\{d,n\}$
  - computes: $M = C^d \bmod n$
- note that the message M must be smaller than the modulus n (block if needed)

Why RSA Works

- because of Euler's Theorem:
  - $a^{\varnothing(n)} \bmod n = 1$ where $\gcd(a,n)=1$
- in RSA have:
  - $n=p.q$
  - $\varnothing(n)=(p-1)(q-1)$
  - carefully chose e & d to be inverses $\bmod \varnothing(n)$
  - hence $e.d=1+k.\varnothing(n)$ for some k
- hence :

$$C^{d} = M^{e.d} = M^{1+k.\varnothing(n)} = M^{1}.(M^{\varnothing(n)})^{k}$$

$$= M^{1}.(1)^{k} = M^{1} = M \bmod n$$

Key Generation

Select p, q	p and q both prime, $p \neq q$
Calculate $n = p \times q$	
Calculate $\phi(n) = (p-1)(q-1)$	
Select integer e	$\gcd(\phi(n), e) = 1;\ 1 < e < \phi(n)$
Calculate d	$d = e^{-1} \pmod{\phi(n)}$
Public key	$PU = \{e, n\}$
Private key	$PR = \{d, n\}$

Encryption

Plaintext:	$M < n$
Ciphertext:	$C = M^{e} \bmod n$

Decryption

Ciphertext:	C
Plaintext:	$M = C^{d} \bmod n$

Figure 9.5 The RSA Algorithm

RSA Example - Key Setup

1. Select primes: $p=17$ & $q=11$
2. Compute $n = pq = 17 \text{ x } 11=187$
3. Compute ø$(n)=(p-1)(q-1)=16 \text{ x } 10=160$
4. Select e: gcd(e,160)=1; choose $e=7$
5. Determine d: $de=1 \bmod 160$ and $d < 160$
   Value is d=23 since 23x7=161= 10x16+1
6. Publish public key PU={7,187}
7. Keep secret private key PR={23,187}

RSA Example - En/Decryption

- sample RSA encryption/decryption is:
- given message `M = 88` (nb. `88<187`)
- encryption:

  $C = 88^{7} \bmod 187 = 11$
- decryption:

  $M = 11^{23} \bmod 187 = 88$

Exponentiation

- can use the Square and Multiply Algorithm
- a fast, efficient algorithm for exponentiation
- concept is based on repeatedly squaring base
- and multiplying in the ones that are needed to compute the result
- look at binary representation of exponent
- only takes $O(\log_2 n)$ multiples for number n
  - eg. $7^5 = 7^4.7^1 = 3.7 = 10 \bmod 11$
  - eg. $3^{129} = 3^{128}.3^1 = 5.3 = 4 \bmod 11$

Exponentiation

```
c = 0; f = 1
for i = k downto 0
    do c = 2 x c
       f = (f x f) mod n
    if $b_i$ == 1 then
       c = c + 1
      f = (f x a) mod n
return f
```

i	9	8	7	6	5	4	3	2	1	0
b_i	1	0	0	0	1	1	0	0	0	0
c	1	2	4	8	17	35	70	140	280	560
f	7	49	157	526	160	241	298	166	67	1

Table 9.3 Result of the Fast Modular Exponentiation Algorithm for a^b mod n, where $a = 7$, $b = 560 = 1000110000$, $n = 561$

Efficient Encryption

- encryption uses exponentiation to power e
- hence if e small, this will be faster
  - often choose e=65537 (2^{16}-1)
  - also see choices of e=3 or e=17
- but if e too small (eg e=3) can attack
  - using Chinese remainder theorem & 3 messages with different modulii
- if e fixed must ensure gcd(e,ø(n))=1
  - ie reject any p or q not relatively prime to e

Efficient Decryption

- decryption uses exponentiation to power d
  - this is likely large, insecure if not
- can use the Chinese Remainder Theorem (CRT) to compute mod p & q separately. then combine to get desired answer
  - approx 4 times faster than doing directly
- only owner of private key who knows values of p & q can use this technique

RSA Key Generation

- users of RSA must:
  - determine two primes at random - `p, q`
  - select either `e` or `d` and compute the other
- primes `p,q` must not be easily derived from modulus `n=p.q`
  - means must be sufficiently large
  - typically guess and use probabilistic test
- exponents `e, d` are inverses, so use Inverse algorithm to compute the other

RSA Security

- possible approaches to attacking RSA are:
  - brute force key search (infeasible given size of numbers)
  - mathematical attacks (based on difficulty of computing ø(n), by factoring modulus n)
  - timing attacks (on running of decryption)
  - chosen ciphertext attacks (given properties of RSA)

Factoring Problem

- mathematical approach takes 3 forms:
  - factor `n=p.q`, hence compute `ø(n)` and then d
  - determine `ø(n)` directly and compute d
  - find d directly
- currently believe all equivalent to factoring
  - have seen slow improvements over the years
    - as of May-05 best is 200 decimal digits (663) bit with LS
  - biggest improvement comes from improved algorithm
    - cf QS to GHFS to LS
  - currently assume 1024-2048 bit RSA is secure
    - ensure p, q of similar size and matching other constraints

RSA-640 is factored!

The factoring research team of F. Bahr, M. Boehm, J. Franke, T. Kleinjung continued its productivity with a successful factorization of the challenge number RSA-640, reported on November 2, 2005. The factors [verified by RSA Laboratories] are:

16347336458092538484431338838650908598417836700330
92312181110852389333100104508151212118167511579

and

1900871281664822113126851573935413975471896789968
515493666638539088027103802104498957191261465571

The effort took approximately 30 2.2GHz-Opteron-CPU years according to the submitters, over five months of calendar time. (This is about half the effort for RSA-200, the 663-bit number that the team factored in 2004.)

Table 9.4 Progress in Factorization

Number of Decimal Digits	Approximate Number of Bits	Date Achieved	MIPS-years	Algorithm
100	332	April 1991	7	quadratic sieve
110	365	April 1992	75	quadratic sieve
120	398	June 1993	830	quadratic sieve
129	428	April 1994	5000	quadratic sieve
130	431	April 1996	1000	generalized number field sieve
140	465	February 1999	2000	generalized number field sieve
155	512	August 1999	8000	generalized number field sieve
160	530	April 2003	–	Lattice sieve
174	576	December 2003	–	Lattice sieve
200	663	May 2005	–	Lattice sieve

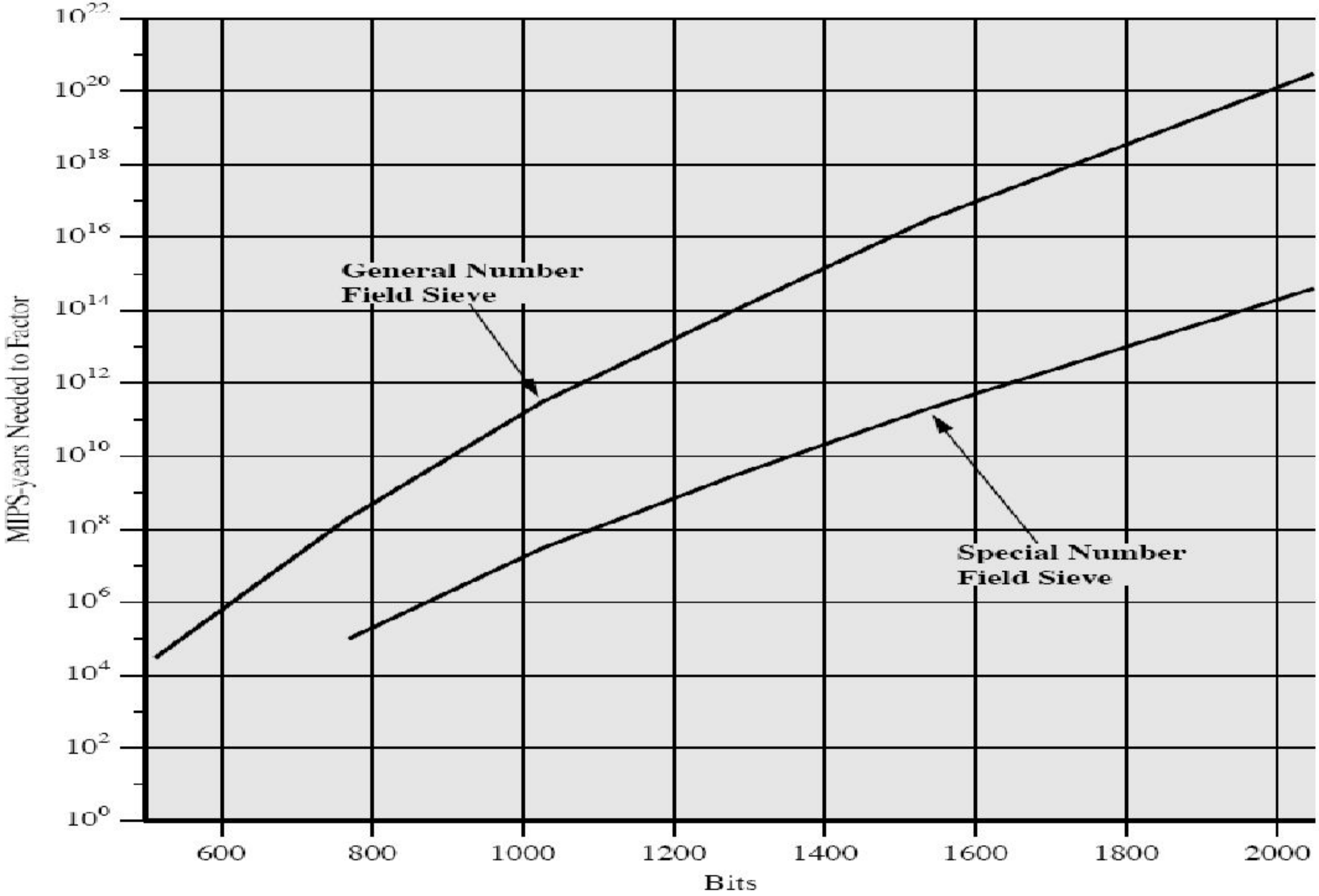


Figure 9.8 MIPS-years Needed to Factor

Timing Attacks

- developed by Paul Kocher in mid-1990’s
- exploit timing variations in operations
  - eg. multiplying by small vs large number
  - or IF's varying which instructions executed
- infer operand size based on time taken
- RSA exploits time taken in exponentiation
- countermeasures
  - use constant exponentiation time
  - add random delays
  - blind values used in calculations

Chosen Ciphertext Attacks

RSA is vulnerable to a Chosen Ciphertext Attack (CCA)

attackers chooses ciphertexts & gets decrypted plaintext back

choose ciphertext to exploit properties of RSA to provide info to help cryptanalysis

- can counter with random pad of plaintext
- or use Optimal Asymmetric Encryption Padding (OASP)

An Example of the RSA Algorithm

1. P = 61 <- first prime number (destroy this after computing E and D)
2. Q = 53 <- second prime number (destroy this after computing E and D)
3. PQ = 3233 <- modulus (give this to others)
4. E = 17 <- public exponent (give this to others)
5. D = 2753 <- private exponent (keep this secret!)
6. Your public key is (E,PQ).
7. Your private key is D.

An Example of the RSA Algorithm

8. The encryption function is:

encrypt(T) = (T^E) mod PQ
= (T^{17}) mod 3233

9. The decryption function is:

decrypt(C) = (C^D) mod PQ
= (C^{2753}) mod 3233

10. To encrypt the plaintext value 123, do this:

encrypt(123) = (123^{17}) mod 3233
= 337587917446653715596592958817679803 mod 3233
= 855

11. To decrypt the ciphertext value 855, do this:

decrypt(855) = (855^{2753}) mod 3233
= 123

An Example of the RSA Algorithm

One way to compute the value of 855^2753 mod 3233 is like this:
2753 = 101011000001 base 2, therefore
2753 = 1 + 2^6 + 2^7 + 2^9 + 2^11
= 1 + 64 + 128 + 512 + 2048
Consider this table of powers of 855:

$855^{1} = 855 \pmod{3233}$
$855^{2} = 367 \pmod{3233}$
$855^{4} = 367^{2} \pmod{3233} = 2136 \pmod{3233}$
$855^{8} = 2136^{2} \pmod{3233} = 733 \pmod{3233}$
$855^{16} = 733^{2} \pmod{3233} = 611 \pmod{3233}$
$855^{32} = 611^{2} \pmod{3233} = 1526 \pmod{3233}$
$855^{64} = 1526^{2} \pmod{3233} = 916 \pmod{3233}$
$855^{128} = 916^{2} \pmod{3233} = 1709 \pmod{3233}$
$855^{256} = 1709^{2} \pmod{3233} = 1282 \pmod{3233}$
$855^{512} = 1282^{2} \pmod{3233} = 1160 \pmod{3233}$
$855^{1024} = 1160^{2} \pmod{3233} = 672 \pmod{3233}$
$855^{2048} = 672^{2} \pmod{3233} = 2197 \pmod{3233}$

An Example of the RSA Algorithm

Given the above, we know this:

$855^{2753} \pmod{3233}$

$= 855^{(1 + 64 + 128 + 512 + 2048)} \pmod{3233}$

$= 855^{1} * 855^{64} * 855^{128} * 855^{512} * 855^{2048} \pmod{3233}$

$= 855 * 916 * 1709 * 1160 * 2197 \pmod{3233}$

$= 794 * 1709 * 1160 * 2197 \pmod{3233}$

$= 2319 * 1160 * 2197 \pmod{3233}$

$= 184 * 2197 \pmod{3233}$

$= 123 \pmod{3233}$

$= 123$

$855^{2753} \bmod 3233 =$

50432888958416068734422899127394466631453878360035509315554967564501055628612082559978744245428110054383498654289336384930246451441507851720917966547826353070
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45697021484375 mod 3233 = 123

Summary

- have considered:
  - principles of public-key cryptography
  - RSA algorithm, implementation, security