UNIT – I

MATRICES

Q.1. Find the inverse of the following matrix employing elementary transformation

$$A = \begin{bmatrix} 2 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

Ans. From given matrix, $|A| = 0$, so that A is singular matrix

Let us consider

$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

The elementary row transformations reduce $A = IA$ to the form $I = PA$, then P becomes the inverse of matrix A.

Now we have $\quad A = IA$

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & -1 & 4/3 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A, \text{ by } R_1 \to \frac{1}{3}R_1$$

$$\begin{bmatrix} 1 & -1 & 4/3 \\ 0 & -1 & 4/3 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ -2/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A, \text{ by } R_2 \to R_2 - 2R_1$$

$$\begin{bmatrix} 1 & -1 & 4/3 \\ 0 & 1 & -4/3 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ 2/3 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A, \text{ by } R_2 \to -R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4/3 \\ 0 & 0 & -1/3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2/3 & -1 & 0 \\ 2/3 & -1 & 1 \end{bmatrix} A, \text{ by } R_1 \to R_1 + R_2 \text{ and } R_3 \to R_3 + R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4/3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2/3 & -1 & 0 \\ -2 & 3 & -3 \end{bmatrix} A, \text{ by } R_3 - 3R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A, \; R_2 \to R_2 + \frac{4}{3} R_3$$

$\therefore \quad I = PA$

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

Q.2. Using elementary transformations, find the inverse of the matrix-

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Ans. We have $A = IA$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A'$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A' \text{ by } R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A' \text{ by } R_3 \to R_3 + R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A' \text{ by } R_1 \to R_1 - 2R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ 5 & -3 & 1 \end{bmatrix} A' \text{ by } R_1 \to R_1 + \frac{1}{2} R_3 \text{ and } R_2 \to R_2 - R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ 5/2 & -3/2 & \frac{1}{2} \end{bmatrix} A, \text{ by } R_3 \to \frac{1}{2} R_3$$

$$\therefore \quad A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ 5/2 & -3/2 & \frac{1}{2} \end{bmatrix}.$$

Q.3. Find the rank of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 5 & 5 \end{bmatrix}$$ **by reducing it to normal form.**

Ans.

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 5 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{matrix} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 - 3C_1 \\ C_4 \to C_4 - 2C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -1 & -3 \end{bmatrix} R_2 \to R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix} R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} C_3 \to C_3 - 2C_2 \\ C_4 \to C_4 - 3C_2 \end{matrix}$$

$$\sim [I_3 \; 0]$$

which is the required normal form.

$\therefore$ Rank of A = order of identity matrix in the normal form = 3.

Scanned by CamScanner

Q.4. Find the rank of the following matrix by reducing it to normal form -

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

Ans.

$$A \sim \begin{bmatrix} 1 & 2 & =-1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad \text{by } R_2 \to R_2 - 4R_1,\ R_3 \to R_3 - 3R_1,\ R_4 \to R_4 - R_1$$

$$\Rightarrow \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad \text{by } C_2 \to C_2 - 2C_1,\ C_3 \to C_3 - C_1,\ C_4 \to C_4 - 3C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad \text{by } R_3 \to R_3 - R_2$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad \text{by } C_3 \to C_3 + \frac{6}{7}C_2,\ C_4 \to C_4 + \frac{11}{7}C_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{by } R_2 \to R_2 + -\frac{1}{7}R_2,\ R_4 \to R_4 + \frac{1}{2}R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to -\frac{1}{2}R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_4 \to C_4 + 2C_3$$

$$\sim \begin{bmatrix} I & O \\ O & O \end{bmatrix}$$

which in the required normal form

Rank $\qquad fA = 3.$

Q.5. Find the rank of matrix.

$$\begin{bmatrix} 2 & 3 & -2 & 4 \\ 3 & -2 & 1 & 2 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{bmatrix}$$

Ans.

Then $\qquad A = \begin{bmatrix} 2 & 3 & -2 & 4 \\ 3 & -2 & 1 & 2 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{bmatrix}$

$$\sim \begin{bmatrix} -1 & 5 & -3 & 2 \\ 3 & -2 & 1 & 2 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{bmatrix} \quad R_1 \to R_1 \to R_2$$

$$\sim \begin{bmatrix} 1 & -5 & 3 & -2 \\ 3 & -2 & 1 & 2 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{bmatrix} \quad R_1 \to -R_1$$

$$\sim \begin{bmatrix} 1 & -5 & 3 & -2 \\ 0 & 13 & -8 & 8 \\ 0 & 17 & -6 & 10 \\ 0 & -6 & 6 & -1 \end{bmatrix} \quad R_2 \to R_2 - 3R_1,\ R_3 \to R_3 - 3R_1,\ R_4 \to R_4 + 2R_1$$

$$\sim \begin{bmatrix} 1 & -5 & 3 & -2 \\ 0 & 1 & 4 & 6 \\ 0 & 17 & -6 & 10 \\ 0 & -6 & 6 & -1 \end{bmatrix} \quad R_2 \to R_2 + 2R_4$$

$$\sim \begin{bmatrix} 1 & -5 & 3 & -2 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & -74 & -92 \\ 0 & 0 & 30 & 35 \end{bmatrix} \quad R_3 \to R_3 - 17R_2,\ R_4 \to R_4 + 6R_2$$

$$\sim \begin{bmatrix} 1 & -5 & 3 & -2 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & \frac{92}{74} \\ 0 & 0 & 30 & 35 \end{bmatrix} \quad R_3 \to -\frac{R_3}{74}$$

$$\sim \begin{bmatrix} 1 & -5 & 3 & -2 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & \frac{92}{74} \\ 0 & 0 & 0 & -\frac{95}{27} \end{bmatrix} \quad R_4 \to R_4 - 30 R_3$$

which is in the required form.

$\therefore$ Rank of A = Number of non zero in echelon form
= 4.

Q.6. Find the rank of the matrix :

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix}$$ **by reducing it to normal form.**

[U.T.U., 2006-07]

Ans.

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & -7 & -5 & -2 \\ 0 & -14 & -10 & -4 \\ 0 & -21 & -16 & -6 \end{bmatrix} \quad \begin{aligned} R_2 &\to R_2 - 2R_1 \\ R_3 &\to R_3 - 3R_1 \\ R_4 &\to R_4 - 6R_1 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & -14 & -10 & -4 \\ 0 & -21 & -16 & -6 \end{bmatrix} \quad \begin{aligned} C_2 &\to C_2 - 3C_1 \\ C_3 &\to C_3 - 4C_1 \\ C_4 &\to C_4 - 2C_1 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & -2 \\ 0 & 2 & -10 & -4 \\ 0 & 3 & -16 & -6 \end{bmatrix} \quad C_2 \to \frac{C_2}{-7}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \begin{aligned} R_3 &\to R_3 - 2R_2 \\ R_4 &\to R_4 - 3R_2 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \begin{aligned} C_3 &\to C_3 + 5C_2 \\ C_4 &\to C_4 + 2C_2 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_4 \quad \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

which is the required normal form

$\therefore$ Rank of A = order of identify matrix in the normal form.
= 3.

Q.7. Find the normal form for the matrix.

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 3 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

[U.T.U., 2009-10]

Ans.

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 3 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & -1 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix} \quad \text{by } \begin{aligned} R_2 &\to R_2 - 2R_1 \\ R_3 &\to R_3 - R_1 \\ R_4 &\to R_4 - R_1 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix} \quad \text{by } \begin{aligned} C_2 &\to C_2 - 2C_1 \\ C_3 &\to C_3 + C_1 \\ C_4 &\to C_4 - 4C_1 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -5 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix} \quad \text{by } R_2 \to -R_2$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 4 & 0\\0 & 0 & 5 & -3\end{bmatrix} \text{ by } \begin{matrix}C_3 \to C_3 + 5C_2\\ C_4 \to C_4 - 4C_2\end{matrix}$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 5 & -3\end{bmatrix} R_3 \to \frac{R_3}{4}$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & -3\end{bmatrix} R_4 \to R_4 - 5R_3$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} R_4 \to \frac{R_4}{-3}$$

which is desired normal form.

Q.8. Show that the vectors :
$X_1 = (1, 2, 4)$, $X_2 = (2, -1, 3)$, $X_3 = (0, 1, 2)$ [UTU, 2007-08]
and $X_4 = (-3, 7, 2)$ are linearly dependent. Find the relation between them.

Ans. Consider the relation

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 = 0$$

$$\lambda_1(1, 2, 4) + \lambda_2(2, -1, 3) + \lambda_3(0, 1, 2) + \lambda_4(-3, 7, 2) = 0$$

$\Rightarrow$ $(\lambda_1, 2\lambda, 4\lambda_1) + (2\lambda_2, -\lambda_2, 3\lambda_2) + (0, 1\lambda_3, 2\lambda_3) + (-3\lambda_4, 7\lambda_4, 2\lambda_4)$

$\Rightarrow$ $= (0, 0, 0)$

Comparing both sides, we get

$$\lambda_1 + 2\lambda_2 + 0\lambda_3 - 3\lambda_4 = 0$$
$$2\lambda_1 - \lambda_2 + \lambda_3 + 7\lambda_4 = 0$$
$$4\lambda_1 + 3\lambda_2 + 2\lambda_3 + 2\lambda_4 = 0$$

We shall show that above system has non-zero solution to prove that given vectors are linearly dependent. For this we shall reduce coefficient matrix of above system to echelon form.

Now coefficient matrix $A = \begin{bmatrix}1 & 2 & 0 & -3\\2 & -1 & 1 & 7\\4 & 3 & 2 & 2\end{bmatrix}$

$$\sim\begin{bmatrix}1 & 2 & 0 & -3\\0 & -5 & 1 & 13\\0 & -5 & 2 & 14\end{bmatrix} \begin{matrix}R_2 \to R_2 - 2R_1\\ R_3 \to R_3 - 4R_1\end{matrix}$$

$$\sim\begin{bmatrix}1 & 2 & 0 & -3\\0 & -5 & 1 & 13\\0 & 0 & 1 & 1\end{bmatrix} R_3 \to R_3 - R_2$$

which is in echelon form.
Rank of A = 3
Number of unknown = 4
Since Rank of A < Number of unknowns therefore System has non zero solution. Thus given vectors are linearly dependent.

To find the relation between then, we rewrite the system as below:

$$1\lambda_1 + 2\lambda_2 + 0\lambda_3 - 3\lambda_4 = 0 \quad \text{...(1)}$$
$$-5\lambda_2 + \lambda_3 + 13\lambda_4 = 0 \quad \text{...(2)}$$
$$\lambda_3 + 1\lambda_4 = 0 \quad \text{...(3)}$$
$$\lambda_4 = k \text{ (let)}$$

$\Rightarrow$ $\lambda_3 = -k$

Putting λ_3 and λ_4 in (2), we get,

$$-512 - k + 13k = 0$$

$\Rightarrow$ $\lambda_2 = \frac{12}{5}k$

Putting λ_2, λ_3 and λ_4 in (*l*), we get,

$$\lambda_1 + 2\left(\frac{12}{5}k\right) - 3(k) = 0$$

$\Rightarrow$ $\lambda_1 = \frac{-9k}{5}$

$\because$ $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 = 0$

$\Rightarrow$ $\frac{-9k}{5}x_1 + \frac{12k}{5}x_2 - kx_3 + kx_4 = 0$

$\Rightarrow$ $9x_1 - 12x_2 + 5x_3 - 5x_4 = 0$

Which is desired relation.

Q.9. Find the value of λ for which the vectors $(1, -2, \lambda)$, $(2, -1, 5)$ and $(3, -5, 7\lambda)$ are linearly dependent.

Ans. The given vectors are

$$X_1 = (1, -2, \lambda)$$
$$X_2 = (2, -1, 5)$$
$$X_3 = (3, -5, 7\lambda)$$

Consider the relation

Scanned by CamScanner

$$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 = 0$$

$$\Rightarrow \quad \lambda_1(1,-2,\lambda) + \lambda_2(2,-1,5) + \lambda_3(3,-5,7\lambda) = 0$$

$$\Rightarrow \quad (\lambda_1, -2\lambda_1, \lambda\lambda_1) + (2\lambda_2, \lambda_2, 5\lambda_2) + (3\lambda_3, -5\lambda_3, 7\lambda\lambda_3) = 0$$

$$\Rightarrow \quad \lambda_1 + 2\lambda_2 + 3\lambda_3 = 0$$

$$2\lambda_1 - \lambda_2 - 5\lambda_3 = 0$$

$$\lambda\lambda_1 + 5\lambda_2 + 7\lambda\lambda_3 = 0$$

For X_1, X_2 and X_3 to be linearly dependent if above system must have a non zero solution.

$\therefore$ Rank of coefficient matrix < 3

$$\Rightarrow \quad \begin{vmatrix} 1 & 2 & 3 \\ -2 & -1 & -5 \\ \lambda & 5 & 7\lambda \end{vmatrix} = 0$$

$$1(-7\lambda + 25) - 2(-14\lambda + 15\lambda) + 3(-10 + \lambda) = 0$$

$$\Rightarrow \quad 14\lambda - 5 = 0$$

$$\Rightarrow \quad \lambda = \frac{5}{14}$$

Q.10. Show that the matrix

$$\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$$

has less than three linearly independent eigen vectors. Also find them.

Ans. Given that $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

The characteristics equation of matrix A is

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \quad \lambda^3 - 7\lambda^2 + 16\lambda - 12 = 9$$

$$\Rightarrow \quad (\lambda - 2)(\lambda - 2)(\lambda - 3) = 0$$

$$\Rightarrow \quad \lambda = 1, 2, 3.$$

(*i*) Eigen vector for $\lambda = 3$

$$\begin{bmatrix} 3-3 & 10 & 5 \\ -2 & -3-3 & -4 \\ 3 & 5 & 7-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} 0 & 10 & 5 \\ -2 & -6 & -4 \\ 3 & 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 0x + 10y + 5z = 0$$

$$-2x - 6y - 4z = 0$$

$$\Rightarrow \quad \frac{x}{-40+30} = \frac{-y}{0+10} = \frac{z}{0+20}$$

$$\Rightarrow \quad \frac{x}{-10} = \frac{y}{-10} = \frac{z}{20}$$

$$\Rightarrow \quad \frac{x}{1} = \frac{y}{1} = \frac{z}{-2}$$

Therefore, eigen vector is $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

(*ii*) Eigen vector for $\lambda = 2$

$$\begin{bmatrix} 3-2 & 10 & 5 \\ -2 & -3-2 & -4 \\ 3 & 5 & 7-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x + 10y + 5z = 0$$

$$2x + 5y + 4z = 0$$

$$\Rightarrow \quad \frac{x}{40-25} = \frac{y}{10-5} = \frac{z}{5-20}$$

$$\Rightarrow \quad \text{Let } \frac{x}{5} = \frac{y}{2} = \frac{z}{-5} = K$$

Therefore, eigen vector is $\begin{bmatrix} 5K \\ 2K \\ -5K \end{bmatrix}$

There is one eigen vector corresponding to repeated root $\lambda_2 = 2 = \lambda_3$.

Eiven vectors corresponding to $\lambda_2 = 2 = \lambda_3$ are not linearly independent.

Q.11. Test the consistency for the following system of equations and if consistent, solv

$$x_1 + 2x_2 - x_3 = 3$$

Scanned by CamScanner

$$3x_1 - x_2 + 2x_3 = 1$$
$$2x_1 - 2x_2 + 3x_3 = 2$$
$$x_1 - x_2 + x_3 = -1$$

[UTU, 2007-08]

Ans. To test the consistency of given system we shall make augmented matrix and reduce it to echelon form by using row transformation.

Now,

$$\text{Augmented matrix } [A : B] = \begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 3 & -1 & 2 & : & 1 \\ 2 & -2 & 3 & : & 2 \\ 1 & -1 & 1 & : & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 0 & -7 & 5 & : & -8 \\ 0 & -6 & 5 & : & -4 \\ 0 & -3 & 2 & : & -4 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 0 & -1 & 0 & : & -4 \\ 0 & -6 & 5 & : & -4 \\ 0 & -3 & 2 & : & -4 \end{bmatrix} R_2 \to R_2 - R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 0 & -1 & 0 & : & -4 \\ 0 & 0 & 5 & : & 20 \\ 0 & 0 & 2 & : & 8 \end{bmatrix} \begin{matrix} R_3 \to R_3 - 6R_2 \\ R_4 \to R_4 - 3R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 0 & -1 & 0 & : & -4 \\ 0 & 0 & 1 & : & 4 \\ 0 & 0 & 2 & : & 8 \end{bmatrix} R_3 \to \frac{R_3}{4}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 0 & -1 & 0 & : & -4 \\ 0 & 0 & 1 & : & 4 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} R_4 \to R_4 - 2R_3$$

which is in echelon form.

Now,

Rank of A = 3, Rank of [A : B] = 3, Number of unknowns = 3

Since Rank of A = Rank of [A : B] = No. of unknowns, therefore, system is consistent with unique solution.

To find the solution, we rewrite the system form reduced augmented matrix as below : ...(1)

$$x_1 + 2x_2 - x_3 = 3$$

$$-x_2 = -4$$
$$x_3 = 4$$
$$\Rightarrow \quad x_3 = 4, x_2 = 4$$

Putting x_3 and x_2 in (1), we get

$$x_1 + 2(4) - 4 = 3$$
$$\Rightarrow \quad x_1 = -1$$

The Solution of the system is

$$x_1 = -1, x_2 = 4, x_3 = 4$$

Q.12. Find the value of λ such that the following equations have unique solution:

$$\lambda x + 2y - 2z - 1 = 0$$
$$4x + 2\lambda y - z - 2 = 0$$
$$6x + 6y + \lambda z - 3 = 0$$

and use matrix method to solve these equation, when λ = 2.

Ans. The system will have a unique solution if coefficient matrix A is non-singular i.e.,

$$\begin{bmatrix} \lambda & -2 & -2 \\ 4 & 2\lambda & -2 \\ 6 & 6 & \lambda \end{bmatrix} \neq 0$$

$$\Rightarrow (\lambda - 2)(\lambda^2 + 2\lambda + 15) \neq 0 \quad \Rightarrow \quad \lambda \neq 2$$

The solution can be found by Cramer's rule. In case when λ = 2

The Augmented matrix

$$[A : B] = \begin{bmatrix} 2 & -2 & -2:1 \\ 4 & 4 & -1:2 \\ 6 & 6 & 2:3 \end{bmatrix} \qquad \text{by } R_1 = \frac{1}{2}R_1, \; R_2 = \frac{1}{4}R_2, \; R_3 = \frac{1}{6}R_3$$

$$= \begin{bmatrix} 1 & 1 & -1:\frac{1}{2} \\ 1 & 1 & -\frac{1}{4}:\frac{1}{2} \\ 1 & 1 & \frac{1}{3}:\frac{1}{2} \end{bmatrix} \qquad \text{by } R_2 = R_2 - R_1, \; R_3 = R_3 - R_1$$

$$= \begin{bmatrix} 1 & 1 & -1:\frac{1}{2} \\ 0 & 0 & 3/4:0 \\ 0 & 0 & 4/3:0 \end{bmatrix} \qquad \text{by } R_2 \to \frac{4}{3}R_2, \; R_3 \to \frac{3}{4}R_3$$

$$= \begin{bmatrix} 1 & 1 & -1:\frac{1}{2} \\ 0 & 0 & 1:0 \\ 0 & 0 & 1:0 \end{bmatrix} \qquad \text{by } R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & : & \frac{1}{2} \\ 0 & 0 & 1 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

Since $\rho[A : B] = 2 = \rho(A) <$ Number of unknown, therefore the system have infinite many solutions.

Equivalent system of equation is

$$x + y - z = \frac{1}{2}$$

$$z = 0$$

Let $\qquad y = k$

$$x = \frac{1}{2} - k$$

Hence solutions are $x = \frac{1}{2} - k$, $y = k$, $z = 0$, where k is an arbitrary constant.

Q.13. Solve the system of equations

$$2x_1 + 3x_2 + x_3 = 9$$
$$x_1 + 2x_2 + 3x_3 = 6$$
$$3x_1 + x_2 + 2x_3 = 8$$

Ans. Given system of equation is

$$2x_1 + 3x_2 + x_3 = 9$$
$$x_1 + 2x_2 + 3x_3 = 6$$
$$3x_1 + x_2 + 2x_3 = 8$$

The Augmented matrix

$$[A : B] = \begin{bmatrix} 2 & 3 & 1 & \vdots & 9 \\ 1 & 2 & 3 & \vdots & 6 \\ 3 & 1 & 2 & \vdots & 8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & \vdots & 6 \\ 2 & 3 & 1 & \vdots & 9 \\ 3 & 1 & 2 & \vdots & 8 \end{bmatrix} \text{ by } R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & \vdots & 6 \\ 0 & -1 & -5 & \vdots & -3 \\ 0 & -5 & -7 & \vdots & -10 \end{bmatrix} \text{ by } R_2 \to R_2 - 2R_1, \; R_3 \to R_2 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & \vdots & 6 \\ 0 & -1 & -5 & \vdots & -3 \\ 0 & 0 & 18 & \vdots & 5 \end{bmatrix} \text{ by } R_3 \to R_3 - 5R_2$$

which is in echelon form.

Here, rank of $A = 3 =$ Rank of $[A : B] =$ Number of unknowns

$\Rightarrow$ System has unique solution.

To obtain the solution, we rewrite the system as

$$x_1 + 2x_2 + 3x_3 = 6 \qquad \text{...(1)}$$
$$-x_2 - 5x_3 = -3 \qquad \text{...(2)}$$
$$18x_3 = 5 \qquad \text{...(3)}$$

From (1), we get

$$x_3 = \frac{5}{18}$$

Putting x_3 in (2), we get

$$-x_2 - \frac{25}{18} = -3$$

$$\Rightarrow \qquad x_2 = 3 - \frac{25}{18}$$

$$x_2 = \frac{29}{18}$$

Putting x_2 and x_3 in (1), we get

$$x_1 + \frac{29}{9} + \frac{5}{6} = 6$$

$$\Rightarrow \qquad x_1 = \frac{6}{1} - \frac{29}{9} - \frac{5}{6} = \frac{108 - 58 - 1}{18} = \frac{35}{18}$$

Therefore, the required solution is

$$x_1 = \frac{35}{18}, \; x_2 = \frac{29}{18}, \; x_3 = \frac{5}{18}$$

Q.14. Find for what values of λ and υ, the system of linear equations.

$$x + y + z = 6$$
$$x + 2x + 5z = 10$$
$$2x + 3y + \lambda z = \mu$$

has (*i*) a unique solution (*ii*) no solution (*iii*) infinite solutions. Also find the solution for λ = 2 and μ = 8.
[U.T.U., 2006-07]

Ans. Augmented matrix

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 5 & : & 10 \\ 2 & 3 & \lambda & : & \mu \end{bmatrix}$$

We shall convert $[A : B]$ to Deechlon form by applying row transformation only.

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 5 & : & 10 \\ 2 & 3 & \lambda & : & \mu \end{bmatrix}$$

$$\sim\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 4 & : & 4 \\ 0 & 1 & \lambda-2 & : & \mu-12 \end{bmatrix} \text{ by } \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 4 & : & 4 \\ 0 & 0 & \lambda-6 & : & \mu-16 \end{bmatrix} \text{ by } R_3 \to R_3 - R_2$$

which is in echelon form.

Here three cases arises

Case - I : If $\lambda \neq 6$ and μ have any value

Then Rank A = Ran $[A : B]$ = 3 = Number of unknown.

Thus unique solution exist.

Case - II : If $\lambda = 6$ and $\mu = 16$

Then Rank A = Rank K $[A : B] = 2$ < Number of variables.

Thus, infinitely many solution exists.

Case - III : If $\lambda = 6$ and $\mu \neq 16$

Then Rank A $\neq$ Rank $[A : B]$

Thus no solution exists.

2nd Part : For finding the solution for $\lambda = 2$ and $\mu = 8$, we put $\lambda = 2, \mu = 8$ in $[A : B]$ Then $[A : B]$ becomes

$$[A : B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 4 & : & 4 \\ 0 & 0 & 4 & ; & 8 \end{bmatrix}$$

Recording the system, we get

$$x + y + z = 6$$
$$y + 4z = 4$$
$$4z = 8 \Rightarrow z = 2$$

Putting $z = 2$ in $y + 4z = 4$ we get

$$y + 4(2) = 4 \Rightarrow y = -4$$

Putting $z = 2, y = -4$ in $x + y + z = 6$, we get

$$x + (-4) + 2 = 6 \Rightarrow x - 2 = 6 \Rightarrow x = 8$$

Thus, for $\lambda = 2, \mu = 8$

$$x = 8, y = -4, z = 2$$

Q.15. Examine if the following equations are consistent, if consistent, solve them and write the nature of solution [U.T.U., 2009-10]

$$3x + 3y + 2z = 1,\ x + 2y = 4,\ 10y + 3z = -2,\ 2x - 3y - z = 5,$$

Ans. Given system is

$$3x + 3y + 2z = 1$$
$$x + 2y = 4$$

$$10y + 3z = -2$$
$$2x - 3y - z = 5$$

Augmented matrix

$$[A : B] = \begin{bmatrix} 3 & 3 & 2 & : & 1 \\ 1 & 2 & 0 & : & 4 \\ 0 & 10 & 3 & : & -2 \\ 2 & -3 & -1 & : & 5 \end{bmatrix}$$

We shall reduce $[A : B]$ to echelon form by row transformation only. Now,

$$[A : B] = \begin{bmatrix} 3 & 3 & 2 & : & 1 \\ 1 & 2 & 0 & : & 4 \\ 0 & 10 & 3 & : & -2 \\ 2 & -3 & -1 & : & 5 \end{bmatrix} \text{ by } R_1 \to R_2$$

$$\sim\begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 3 & 3 & 2 & : & 1 \\ 0 & 10 & 3 & : & -2 \\ 2 & -3 & -1 & : & 5 \end{bmatrix} \text{ by } \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_4 \to R_4 - 2R_1 \end{array}$$

$$\sim\begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & -3 & 2 & : & -11 \\ 0 & 10 & 3 & : & -2 \\ 0 & -7 & -1 & : & -3 \end{bmatrix} \text{ by } R_2 \to 3R_2$$

$$\sim\begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & -9 & 6 & : & -33 \\ 0 & 10 & 3 & : & -2 \\ 0 & -7 & -1 & : & -3 \end{bmatrix} \text{ by } R_2 \to R_2 + R_3$$

$$\sim\begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & 1 & 9 & : & -35 \\ 0 & 10 & 3 & : & -2 \\ 0 & -7 & -1 & : & -3 \end{bmatrix} \text{ by } \begin{array}{l} R_3 \to R_3 - 10R_2 \\ R_4 \to R_4 - 7R_2 \end{array}$$

$$\sim\begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & 1 & 9 & : & -35 \\ 0 & 0 & -87 & : & 348 \\ 0 & 0 & 62 & : & -248 \end{bmatrix} \text{ by } R_3 \to -\frac{1}{87}R_3$$

$$\sim\begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & 1 & 9 & : & -35 \\ 0 & 0 & 1 & : & -4 \\ 0 & 0 & 82 & : & -248 \end{bmatrix} \text{ by } R_3 \to R_4 - 62R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & 1 & 9 & : & -35 \\ 0 & 0 & 1 & : & -4 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \text{ by } R_4 \to R_4 - 62R_3$$

Since Rank $[A : B]$ = Rank A = Number of unknowns, therefore, system has unique solution.

To find the solution, we rewrite the system as

$$x + 2y + 0z = 4$$
$$y + 9z = -35$$
$$z = -4$$

Putting $z = -4, y = -35 - 9(-4) = 1$

Putting $y = 1, x = 4 - 2(1) = 2$

Thus, $x = 2, y = 1, z = -4$

Q.16. Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

and hence evaluate A^{-1}. **[UTU, 2009-10]**

Ans. The characteristic equation of a matrix A is

$$|A - 2I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

or $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

According to Cayley-Hamilton theorem

$A^3 - 5A^2 + 7A - 3I = 0$ $\qquad A^2 \cdot A \cdot A^{-1} - 5A \cdot A \cdot A^{-1} + 7A \cdot A^{-1} - 3IA^{-1} =$

$A^{-1} = \frac{1}{3}[A^2 - 5A + 7I]$ $\qquad A^2 - 5A + 7I = 3A^{-1}$ ...(1)

Now $$A^2 = A \cdot A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$\therefore A^{-2} - 5A + 5I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

From (1), we get

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

Q.17. Find the characteristic equation of the matrix -

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -2 & 2 \end{bmatrix}$$

Show that the equation is satisfied by A and hence find A^{-1}.

Ans. The charcteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

To verify Caley-Hamilton theorem, we have to show that

$$A^3 - 6A^2 + 9A - 4I = 0$$

Now, $$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & 5 & 6 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -22 & -21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

We have,

$$A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -22 & -21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore A^3 - 6A^2 + 9A - 4I = 0$$

Thus, characteristic equation is statisfied.

Scanned by CamScanner

Now, pre multiplying both the sides of (1) by A^{-1} we have

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

$$4A^{-1} = A^2 - 6A + 9I$$

$$\Rightarrow \quad = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Q.18. Find the characteristic equation of the matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

Also find the eigen values and eigen vectors of this matrix.

Ans. Given matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

Then, characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)[(2-\lambda)^2 - 2] - 2[1] + 2[2-\lambda] = 0$$

$$\Rightarrow \quad (1-\lambda)(4+\lambda^2-4\lambda-2) - 2 + 4 - 2\lambda = 0$$

$$\Rightarrow \quad (1-\lambda)(\lambda^2-4\lambda+2) + 2 - 2\lambda = 0$$

$$\Rightarrow \quad (1-\lambda)(\lambda^2-4\lambda+2) + 2(1-\lambda) = 0$$

$$\Rightarrow \quad (1-\lambda)(\lambda^2+4-4\lambda) = 0$$

$$\Rightarrow \quad (1-\lambda)(\lambda-2)^2 = 0$$

$$\Rightarrow \quad \lambda = 1, 2, 2$$

Let X be an eigen vector corresponding to eigen value λ, then

$$(A - \lambda I)X = 0$$

$$\Rightarrow \quad \begin{bmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{...(1)} \left[\because X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right]$$

For $\lambda = 1$, (1) gives

$$\begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{cases} 0x_1 + 2x_2 + 2x_3 = 0 \\ 0x_1 + 1x_2 + 1x_3 = 0 \\ -x_1 + 2x_2 + x_3 = 0 \end{cases}$$

From last two equations

$$\frac{x_1}{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix}}$$

$$\Rightarrow \quad \frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$\Rightarrow \quad \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$\Rightarrow$ Eigen vector corresponding to eigen value $\lambda = 1$ is $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Now, for $\lambda = 2$, (1) gives,

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{cases} -x_1 + 2x_2 + 2x_3 = 0 \\ 0x_1 + 0x_2 + 1x_3 = 0 \\ -1x_1 + 2x_2 + 0x_3 = 0 \end{cases}$$

From out two equations

$$\frac{x_1}{\begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 0 \\ -1 & 2 \end{vmatrix}}$$

$$\Rightarrow \quad \frac{x_1}{-2} = \frac{x_2}{-1} = \frac{x_3}{0}$$

$$\Rightarrow \quad \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{0}$$

$\therefore$ Eigen vector corresponding to eigen value $\lambda = 2$ is $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Thus eigen values of given matrix are 1, 2, 2.

Eigen vector for $\lambda = 1$ is $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and eigen vector for $\lambda = 2$ is $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Q.19. Show that for any square matrix A, the product of all eigen values of A is equal to det (A).

Ans. Let $A = [aij]_{n \times n}$ be a given square matrix and $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ be its eigen values. If $\phi(\lambda)$ be the characteristics polynomial, then

$$\phi(\lambda) = |A - \lambda I|$$

$$= \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix}$$

$$= (-1)^n\{\lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \ldots + p_n\}$$
$$= (-1)^n\{(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \ldots (\lambda - \lambda_n)\}$$

Putting $\lambda = 0$, we get

$$\phi(0) = (-1)^n(-1)^n\lambda_1\lambda_2\lambda_3 \ldots \lambda_n$$
$$|A| = \lambda_1\lambda_2\lambda_3 \ldots \lambda_n$$

Hence the product of all eigen values of A is equal to det (A).

Q.20. State Cayley-Hamilton theorem and verify it for the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and hence, find A^{-1}.

[UTU, 2006-07]

Ans. Cayley-Hamilton Theorem : Every square matrix satisfies its own characteristics equation.

2nd Part :

Given matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

The characteristics equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & -2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)[(-2-\lambda)(3-\lambda)-0] = 0$$
$$\Rightarrow \quad (1-\lambda)(\lambda-3)(\lambda+2) = 0$$
$$\Rightarrow \quad (\lambda-1)[\lambda^2+2\lambda-3\lambda-3] = 0$$

$$\Rightarrow \quad (\lambda-1)[\lambda^2-\lambda-6] = 0$$
$$\Rightarrow \quad \lambda^3-\lambda^2-6\lambda-\lambda^2+\lambda+6 = 0$$
$$\Rightarrow \quad \lambda^3-2\lambda^2-5\lambda+6 = 0$$

Which is characteristics equation of A. To verify cayley Hamilton theorem, we shall show that

$$A^3-2A^2-5A+6I = 0$$

Now, $A^2 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 8 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

and $A^3 = \begin{bmatrix} 1 & -1 & 8 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 26 \\ 0 & -8 & 0 \\ 0 & 0 & 27 \end{bmatrix}$

Now, we have $A^3-2A^2-5A+6I$

$$= \begin{bmatrix} 1 & 3 & 26 \\ 0 & -8 & 0 \\ 0 & 0 & 27 \end{bmatrix} - 2\begin{bmatrix} 1 & -1 & 8 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} - 5\begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 26 \\ 0 & -8 & 0 \\ 0 & 0 & 27 \end{bmatrix} - \begin{bmatrix} 2 & -2 & 16 \\ 0 & 8 & 0 \\ 0 & 0 & 18 \end{bmatrix} - \begin{bmatrix} 5 & 5 & 10 \\ 0 & -10 & 0 \\ 0 & 0 & 15 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \ldots(1)$$

Thus, $A^3-2A^2-5A+6I = 0$

i.e., Caylay Hamilton theorem is verified.

To calculate A^{-1}, we multiply (1) by A^{-1}. Then, we get

$$A^{-1}(A^3-2A^2-5A+6I) = 0$$
$$\Rightarrow \quad A^2-2A-5A^{-1}A+6A^{-1} = 0$$
$$\Rightarrow \quad 6A^{-1} = -A^2+2A+5I \quad [\because AA^{-1} = I]$$

$$\Rightarrow \quad 6A^{-1} = \begin{bmatrix} 1 & -1 & 8 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} + 2\begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 8 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 4 \\ 0 & -4 & 0 \\ 0 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\Rightarrow \quad 6A^{-1} = \begin{bmatrix} 6 & 3 & -4 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow \qquad A^{-1} = \frac{1}{6}\begin{bmatrix} 6 & 3 & -4 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Q.21. Verify the Cayley Hamilton theorem for the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$. Also find its inverse using this theorem.

Ans. Given matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

Characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \qquad \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 5 \\ 3 & 5 & 6-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \qquad (1-\lambda)[(4-\lambda)(6-\lambda)-25]-2[2(6-\lambda)-15]+3[10-3(4-\lambda)]=0$$

$$\Rightarrow \qquad (1-\lambda)(\lambda^2-10\lambda+24-25)-2[12-2\lambda-15]+3[10-12+3]=0$$

$$\Rightarrow \qquad (1-\lambda)(\lambda^2-10\lambda-1)-2(-2\lambda-3)+3(3\lambda-2)=0$$

$$\Rightarrow \qquad \lambda^3-10\lambda-1-\lambda 3+10\lambda^2+\lambda+4\lambda+6+9\lambda-6=0$$

$$\Rightarrow \qquad -\lambda^3+11\lambda^2+4-4\lambda-1=0$$

$$\Rightarrow \qquad \lambda^3-11\lambda^2-4\lambda+1=0$$

To verify Cayley Hamilton theorem we have to show

$$A^3 - 11A^2 - 4A + I = 0$$

Now

$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$$

$$A^3 = A \cdot A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} = \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 383 & 636 & 793 \end{bmatrix}$$

Now, we have

$$A^3 - 11A^2 - 4A + I = \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix} - 11\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 383 & 636 & 793 \end{bmatrix} - \begin{bmatrix} 154 & 275 & 341 \\ 275 & 495 & 610 \\ 341 & 616 & 770 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 12 \\ 8 & 16 & 20 \\ 12 & 20 & 24 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \qquad A^3 - 11A^2 - 4A + I = 0 \qquad \ldots(1)$$

i.e., Cayley Hamilton theorem is verified.

Computation of Inverse:

Multiplying both sides of (1) by A^{-1}, we get

$$A^2 - 11A - 4AA^{-1} + IA^{-1} = 0$$

$$\Rightarrow \qquad A^{-1} = -A^2 + 11A + 4I$$

$$= -\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} + 11\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -14 & -25 & -31 \\ -25 & -45 & -56 \\ -31 & -56 & -70 \end{bmatrix} + \begin{bmatrix} 11 & 22 & 33 \\ 22 & 44 & 55 \\ 33 & 55 & 66 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

$$\Rightarrow \qquad A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

Q.22. Find the eigen values and eigen vectors of matrix $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$. **[UTU, 2009-10]**

Ans. The characteristics equation of given matrix A is given by

$$|A - \lambda I| = 0$$

$$\Rightarrow \qquad \begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \qquad (3-\lambda)(2-\lambda)(5-\lambda) = 0$$

$$\therefore \qquad \lambda = 2, 3, 5.$$

The eigen values are $\lambda = 2, 3, 5$

Let X be an eigen vector corresponding to the eigen value λ, then

$$(A - \lambda I)X = 0$$

$$\Rightarrow \quad \begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \text{...(1)}$$

If $\lambda = 2$, the corresponding eigen vector is given by

$$\begin{bmatrix} 3-2 & 1 & 4 \\ 0 & 2-2 & 6 \\ 0 & 0 & 5-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_3 \to R_3 - \frac{1}{2}R_2 .$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} x_1 + x_2 + 4x_3 \\ 6x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_1 + x_2 + 4x_3 = 0,\ 3x_3 = 0$

$\Rightarrow \quad x_3 = 0.$

$\therefore \quad x_2 = -x_1$

Thus $X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ be an eigen vector of A corresponding to the eigen value $\lambda = 2$.

If $\lambda = 3$, then corresponding eigen vector is given by

$$\begin{bmatrix} 3-3 & 1 & 4 \\ 0 & 2-3 & 6 \\ 0 & 0 & 5-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \quad \begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} x_2 + 4x_3 \\ -x_2 + x_3 \\ 2x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_2 + 4x_3 = 0,\ -x_2 + 6x_3 = 0,\ 2x_3 = 0$$

Clearly $x_2 = 0$, $x_3 = 0$ and x_1 is arbitrary.

Thus $X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be an eigen vector of A corresponding to the eigen value $\lambda = 3$.

If $\lambda = 5$, then corresponding eigen vector is given by

$$\begin{bmatrix} 3-5 & 1 & 4 \\ 0 & 2-5 & 6 \\ 0 & 0 & 5-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \quad \begin{bmatrix} -2x_1 + x_2 + 4x_3 \\ -3x_2 + 6x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$\Rightarrow \quad -2x_1 + x_2 + 4x_3 = 0,\ -3x_2 + 6x_3 = 0$$

$$\Rightarrow \quad x_2 = 2x_3 \Rightarrow \frac{x_2}{2} = \frac{x_3}{1} = k \text{ (say)}$$

$$\therefore \quad x_2 = 2k,\ x_3 = k$$

Then $\quad 2x_1 = x_2 + 4x_3 \quad \Rightarrow 2k + 4k = 6k \quad \Rightarrow x_1 = 3k$

$$\Rightarrow \quad \frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1}$$

Thus $X_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ be an eigen vector of A corresponding to the eigen value $\lambda = 5$.

Q.23. Reduce the matrix $P = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ to the diagonal form.

Ans. The characteristic equation of the matrix P is

$$|P - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & 0-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(2-\lambda)(-\lambda) + 1] - 2(-\lambda + 1) - 2(-1 + 2 - \lambda) = 0$$

$$(1-\lambda)(1-\lambda)^2 - 4(1-\lambda) = 0$$

$$(1-1)[(1-\lambda)^2 - 4] = 0$$

$\Rightarrow \quad \lambda = 1, -1, 3$

Therefore, $\lambda = 1, -1, 3$, and the eigen values of the matrix P.

Let X be the eigen vectors corresponding to eigen value λ. Then

$$(P - \lambda 1)X = 0$$

For $\lambda = 1$, we have

$$\begin{bmatrix} 0 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 2x_2 - 2x_3 = 0,\ x_1 + x_2 + x_3 = 0,\ x_1 - x_2 - x_3 = 0$$

$$x_2 = 1,\quad x_3 = 1 \quad \text{and} \quad x_1 = -2$$

Eigen vector $\quad X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda = -1$, we have

$$\begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 2x_1 + 2x_2 - 2x_3 = 0,\ x_1 + 3x_2 + x_3 = 0,\ -x_1 - x_2 + x_3 = 0$$

$$\Rightarrow x_1 = 2,\quad x_2 = 1,\quad x_3 = -1$$

Eigen vecotr $\quad X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$

For $\lambda = 3$, we have

$$\begin{bmatrix} -2 & 2 & -2 \\ 1 & -1 & 1 \\ -1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad -2x_1 + 2x_2 - 2x_3 = 0,\ x_1 + x_2 + x_3 = 0,\ -x_1 - x_2 - x_3 = 0$$

$$\Rightarrow x_1 = 2,\quad x_2 = 1,\quad x_3 = -1$$

Eigen vector $\quad X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

Now the matrix $\quad B = \begin{bmatrix} -2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$ (matrix eigen vectors)

The Cofactor matrix of $B = \begin{bmatrix} 0 & 2 & -2 \\ -4 & 0 & -4 \\ -4 & 4 & 0 \end{bmatrix}$

$$\text{Adj } B = \begin{bmatrix} 0 & -4 & -4 \\ 2 & 0 & 4 \\ -2 & -4 & 0 \end{bmatrix}$$

$$B^{-1} = \frac{Adj}{|B|} = -\frac{1}{8}\begin{bmatrix} 0 & -4 & -4 \\ 2 & 0 & 4 \\ -2 & -4 & 0 \end{bmatrix}$$

The diagonal for of matrix P is

$$B^{-1}PB = D$$

Now $\quad B^{-1}PB = \frac{1}{8}\begin{bmatrix} 0 & -4 & -4 \\ 2 & 0 & 4 \\ -2 & -4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} -2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$

$$= -\frac{1}{8}\begin{bmatrix} -8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -24 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$\therefore \quad D = \text{diag}(1, -1, 3).$

Q.24. Reduce the matrix A to diagonal form

$$A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Ans. The charactristic equation of the matrix A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (-1-\lambda)[-\lambda(2-\lambda)+1] - 2[-\lambda+1] - 2[-1+2-\lambda] = 0$$

$$\Rightarrow \quad (-1-\lambda)[-2\lambda+\lambda^2+1] + 2(\lambda-1) - 2(1-\lambda) = 0$$

$$\Rightarrow \quad -(\lambda+1)(\lambda-1)^2 + 2(\lambda-1) + 2(\lambda-1) = 0$$

$$\Rightarrow \quad (\lambda-1)[-(\lambda^2-1)+4] = 0$$

$$\Rightarrow \quad (\lambda-1)(5-\lambda^2) = 0$$

$$\lambda = 1,\ \sqrt{5}, -\sqrt{5}$$

Thus, eigen values of A are $1, \sqrt{5}, -\sqrt{5}$.

Let X be the eigen vector corresponding to the eigen value λ.

$$(A - \lambda I)X = 0$$

$$\Rightarrow \quad \begin{bmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{...(1)}$$

For $\lambda = 1$, we have

$$\begin{bmatrix} -2 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Scanned by CamScanner

$$\Rightarrow \quad \begin{cases} -2x_1 + 2x_2 - 2x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \\ -x_1 - x_2 - x_3 = 0 \end{cases}$$

Taking first two equation and using cross multiplication method,

$$\frac{x_1}{\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} -2 & -2 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 2 \\ 1 & 1 \end{vmatrix}}$$

$$\Rightarrow \quad \frac{x_1}{4} = \frac{x_2}{0} = \frac{x_3}{-4}$$

$$\Rightarrow \quad \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$\Rightarrow$ Eigen vector for the eigen value $\lambda = 1$ is $X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

For $\lambda = \sqrt{5}$, we have,

$$\begin{bmatrix} -1-\sqrt{5} & 2 & -2 \\ 1 & 2-\sqrt{5} & 1 \\ -1 & -1 & -\sqrt{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} &(-1-\sqrt{5})x_1 + 2x_2 - 2x_3 = 0 \\ &x_1 + (2-\sqrt{5})x_2 + x_3 = 0 \\ &-x_1 - x_2 - \sqrt{5}x_3 = 0. \end{aligned}$$

Taking last two and applying cross multiplication

$$\frac{x_1}{\begin{vmatrix} 2-\sqrt{5} & 1 \\ -1 & -\sqrt{5} \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & 1 \\ -\sqrt{5} & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 2-\sqrt{5} \\ -1 & -1 \end{vmatrix}}$$

$$\Rightarrow \quad \frac{x_1}{-2\sqrt{5}+5+1} = \frac{x_2}{-1+\sqrt{5}} = \frac{x_3}{-1+2-\sqrt{5}}$$

$$\Rightarrow \quad \frac{x_1}{6-2\sqrt{5}} = \frac{x_2}{\sqrt{5}-1} = \frac{x_3}{1-\sqrt{5}}$$

$$\Rightarrow \quad \frac{x_1}{(\sqrt{5})^2 + (1^2 - 2(\sqrt{5})(1))} = \frac{x_2}{\sqrt{5}-1} = \frac{x_3}{-(\sqrt{5}-1)}$$

$$\Rightarrow \quad \frac{x_1}{(\sqrt{5}-1)^2} = \frac{x_2}{\sqrt{5}-1} = \frac{x_3}{-(\sqrt{5}-1)}$$

$$\Rightarrow \quad \frac{x_1}{\sqrt{5}-1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

Thus, eigen vector for the eigen value $\lambda = \sqrt{5}$ is $x_2 = \begin{pmatrix} \sqrt{5}-1 \\ 1 \\ -1 \end{pmatrix}$

For $\lambda = -\sqrt{5}$, we have

$$\begin{bmatrix} 1+\sqrt{5} & 2 & -2 \\ 1 & 2+\sqrt{5} & 1 \\ -1 & -1 & +\sqrt{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{cases} (1+\sqrt{5})x_1 + 2x_2 - 2x_3 = 0 \\ 1x_1 + (2+\sqrt{5})x_2 + x_3 = 0 \\ -x_1 - x_2 + \sqrt{5}x_3 = 0 \end{cases}$$

Taking last two and applying cross multiplication method, we get

$$\frac{x_1}{\begin{vmatrix} 2-\sqrt{5} & 1 \\ -1 & \sqrt{5} \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & 1 \\ \sqrt{5} & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 2-\sqrt{5} \\ -1 & -1 \end{vmatrix}}$$

$$\Rightarrow \quad \frac{x_1}{-2\sqrt{5}+5+1} = \frac{x_2}{-1+\sqrt{5}} = \frac{x_3}{-1+2-\sqrt{5}}$$

$$\Rightarrow \quad \frac{x_1}{6-2\sqrt{5}} = \frac{x_2}{\sqrt{5}-1} = \frac{x_3}{1-\sqrt{5}}$$

$$\Rightarrow \quad \frac{x_1}{(\sqrt{5})^2 + (1)^2 - 2(\sqrt{5})(1)} = \frac{x_2}{\sqrt{5}-1} = \frac{x_3}{-(\sqrt{5}-1)}$$

$$\Rightarrow \quad \frac{x_1}{(\sqrt{5}-1)^2} = \frac{x_2}{\sqrt{5}-1} = \frac{x_3}{-(\sqrt{5}-1)}$$

$$\Rightarrow \quad \frac{x_1}{\sqrt{5}+1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Thus, eigen vector for the eigen value $\lambda = -\sqrt{5}$ is $x_3 = \begin{bmatrix} \sqrt{5}+1 \\ -1 \\ +1 \end{bmatrix}$

$\therefore$ Modal matrix P = Matrix formed by eigen vecotrs

$$= \begin{bmatrix} 1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

Required diagonal matrix $D = P^{-1}AP$

We shall now calculate P^{-1}

The cofactor matrix of P

$$= \begin{bmatrix} 0 & +1 & 1 \\ -2\sqrt{5} & 2+\sqrt{5} & 2-\sqrt{5} \\ -2\sqrt{5} & 1 & 1 \end{bmatrix}$$

∴ Adjoined of P = Transpose of above matrix

$$\text{Adj } P = \begin{bmatrix} 0 & -2\sqrt{5} & -2\sqrt{5} \\ 1 & 2+\sqrt{5} & 1 \\ 1 & 2-\sqrt{5} & 1 \end{bmatrix}$$

and

$$|P| = \begin{vmatrix} 1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}$$

$$= 1(1-1) - (\sqrt{5}-1)(-1) + (\sqrt{5}+1)(1)$$

$$= \sqrt{5} - 1 + \sqrt{5} + 1 = 2\sqrt{5}$$

∴

$$P^{-1} = \frac{\text{Adj } P}{|P|} = \frac{1}{2\sqrt{5}} \begin{bmatrix} 0 & -2\sqrt{5} & -2\sqrt{5} \\ 1 & 2+\sqrt{5} & 1 \\ 1 & 2-\sqrt{5} & 1 \end{bmatrix}$$

∴

$$D = P^{-1}AP$$

$$= \frac{1}{2\sqrt{5}} \begin{bmatrix} 0 & -2\sqrt{5} & -2\sqrt{5} \\ 1 & 2+\sqrt{5} & 1 \\ 1 & 2-\sqrt{5} & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2\sqrt{5}} \begin{bmatrix} 0 & -2\sqrt{5} & -2\sqrt{5} \\ 1 & 2+\sqrt{5} & 1 \\ 1 & 2-\sqrt{5} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\sqrt{5}+5 & -\sqrt{5}-5 \\ 0 & \sqrt{5} & \sqrt{5} \\ -1 & -\sqrt{5} & -\sqrt{5} \end{bmatrix}$$

$$= \frac{1}{2\sqrt{5}} \begin{bmatrix} 2\sqrt{5} & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{10}{2\sqrt{5}} & 0 \\ 0 & 0 & -\frac{10}{2\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$$

∴

$$D = \text{diag } (1, \sqrt{5}, -\sqrt{5})$$

Q.25. Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & 6 \\ -1 & -2 & 0 \end{bmatrix}$$

Also find the matrix B such that $B^{-1}AB$ is a diagonal matrix. **[U.T.U., 2006-07]**

Ans. Given matrix is

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

The characteristics equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0; \quad \text{by } C_1 \to C_1 + C_2 + C_3$$

$$\Rightarrow \begin{vmatrix} -3-\lambda & 2 & -3 \\ -3-\lambda & 1-\lambda & -6 \\ -3-\lambda & -2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-3-\lambda) \begin{vmatrix} 1 & 2 & -3 \\ 1 & 1-\lambda & -6 \\ 1 & -2 & -\lambda \end{vmatrix} = 0 \qquad \text{[Taking } (-3-\lambda) \text{ common from } C_1]$$

$$\Rightarrow (3+\lambda) \begin{vmatrix} 1 & 2 & -3 \\ 0 & -1-\lambda & -3 \\ 0 & -4 & -\lambda+3 \end{vmatrix} = 0 \quad \text{by } R_2 \to R_2 - R_1, \; R_3 \to R_3 - R_1$$

$$\Rightarrow (3+\lambda)[1(-1-\lambda)(-\lambda+3) - (-3)(-4)] = 0$$

$$\Rightarrow (3+\lambda)[(\lambda+1)(\lambda-3) - 12] = 0$$

$$\Rightarrow (3+\lambda)(\lambda^2 - 2\lambda - 15) = 0$$

$$\Rightarrow (3+\lambda)[(\lambda-5)(\lambda+3)] = 0$$

$$\Rightarrow (3+\lambda)(\lambda-5)(\lambda+3) = 0$$

$$\Rightarrow \lambda = 5, -3, -3.$$

Thus the eigen values of A are 5, – 3 and – 3.

Let $X = \begin{bmatrix} x_2 \\ x_2 \\ x_3 \end{bmatrix}$ be eigen vector of A correponding to eigen value λ. Then we have

$$(A - \lambda I)X = 0.$$

$$\Rightarrow \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad ...(1)$$

For $\lambda = 5$, we have

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{cases} -7x_1 + 2x_2 - 3x_3 = 0 \\ 2x_1 - 4x_2 - 6x_3 = 0 \\ -1x_1 - 2x_2 - 5x_3 = 0 \end{cases}$$

Applying cross multiplication method in first two equations, we get

$$\frac{x_1}{\begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} -7 & -3 \\ 2 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & 2 \\ +2 & -4 \end{vmatrix}}$$

$$\Rightarrow \quad \frac{x_1}{-12-12} = \frac{x_2}{-(42+6)} = \frac{x_3}{28-4}$$

$$\Rightarrow \quad \frac{x_1}{-24} = \frac{x_2}{-(42+6)} = \frac{x_3}{28-4}$$

[cancelling −24 from ...

$$\Rightarrow \quad \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}$$

Thus, eigen vector for eigen value $\lambda = 5$ is $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

For $\lambda = -3$ we have

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{cases} x_1 + 2x_2 - 3x_3 = 0 \\ 2x_1 + 4x_2 - 6x_3 = 0 \\ -x_1 - 2x_2 - 3x_3 = 0 \end{cases}$$

$$\Rightarrow \quad x_1 + 2x_2 - 3x_3 = 0$$

[Since all euqations ...

Let $x_2 = K_1$ and $x_3 = K_2$

Then $x_1 = -2K_1 + 3K_2$

Thus, eigen vector for the eigen value $\lambda = -3$ is $\begin{bmatrix} 3k_2 - 2k_1 \\ k_1 \\ k_2 \end{bmatrix}$

Taking $k_1 = 0$, $k_2 = 1$, eigen vector $= \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Taking $k_1 = 1$, $k_2 = 0$, eigen vector $= \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Thus, eigen vectors for $\lambda = -3$ are $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Thus, eigen vectors of A are $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

To find the matrix B, which diagenalize A, we just write eigen vectors of A as columns. Thus, diagonalizing matrix

$$B = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

Q.26. If A $= \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ resulting diagonal matrix D of A. **[U.T.U., 2009-10]**

Ans. The characterisic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0. \quad \text{by } R_2 + R_2 + R_3$$

$$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 2 & -2 \\ 0 & 1-\lambda & 1-\lambda \\ -1 & -1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)\begin{vmatrix} 1-\lambda & 2 & -2 \\ 0 & 1 & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0, \quad \text{by } C_2 \to C_2 - C_3$$

$$\Rightarrow \quad (1-\lambda)\begin{vmatrix} 1-\lambda & 4 & -2 \\ 0 & 0 & 1 \\ -1 & \lambda-1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad -(1-\lambda)[-\{(\lambda-1)(1-\lambda)+4\}]$$

$$\Rightarrow \quad -(1-\lambda)\{-(\lambda-1)^2+4\}$$

$$\Rightarrow \quad (1-\lambda)[(\lambda-1)^2-4]$$

$$\Rightarrow \quad \lambda = 1, -1, 3$$

Thus, eigen values of given matrix are 1, − 1 and 3.

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector of A corresponding to eigen value λ. Then

$(A - \lambda I)X = 0$

$$\Rightarrow \quad \begin{bmatrix} 1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad ...(1)$$

For λ = 1, we have

$$\begin{bmatrix} 0 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{cases} 0x_1 + 2x_2 - 2x_3 = 0 \\ 1x_1 + 1x_2 + 1x_3 = 0 \\ -1x_1 - 1x_2 - 1x_3 = 0 \end{cases}$$

Apply cross multiplication method in first two equations, we get

$$\frac{x_1}{\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix}}$$

$$\Rightarrow \quad \frac{x_1}{4} = \frac{x_2}{-2} = \frac{x_3}{-2}$$

$$\Rightarrow \quad \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{-1} \qquad \text{[cancelling 2 from each term]}$$

Thus, eigen vector for the eigen value λ = 1 is $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$

For λ = 1, we have

$$\begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{cases} 2x_1 + 2x_2 - 2x_3 = 0 \\ 1x_1 + 3x_2 + 1x_3 = 0 \\ -1x_1 - 1x_2 + 1x_3 = 0 \end{cases}$$

Applying cross multiplying method in first two equations, we get

$$\frac{x_1}{\begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}}$$

$$\Rightarrow \quad \frac{x_1}{8} = \frac{x_2}{-4} = \frac{x_3}{4}$$

$$\Rightarrow \quad \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1} \qquad \text{[cancelling 4 from each term]}$$

Thus, eigen vector for the eigen value λ = − 1 is $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

For λ = 3, we have

$$\begin{bmatrix} -2 & 2 & -2 \\ 1 & -1 & 1 \\ -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{cases} -2x_1 + 2x_2 - 2x_3 = 0 \\ 1x_1 - 1x_2 + 1x_3 = 0 \\ -1x_1 - 1x_2 - 3x_3 = 0 \end{cases}$$

Applying cross multiplication method in last two equations, we get

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ -1 & -3 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & 1 \\ -1 & -3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix}}$$

$$\Rightarrow \quad \frac{x_1}{4} = \frac{x_2}{2} = \frac{x_3}{-2}$$

$$\Rightarrow \quad \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-1} \qquad \text{[cancelling from each term]}$$

Thus, eigen vector for eigen value λ = 3 is $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

Thus, eigen values of A are 1 − 1 and 3 and corresponding eigen vectors are $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ respectively.

Modal matrix P = Matrix formed by eigen vectors

$$= \begin{bmatrix} 2 & 2 & 2 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

Resulting diagonal matrix $D = P^{-1}AP$

The diagonal matrix formed by eigen values of A is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Q.27. If $\begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$ **Show that A*A is a Hermitian matrix, where A^* is the conjugate transpose of A.**

Ans. Let A be the given matrix

$$A' = \begin{bmatrix} 2+i & -5 \\ 3 & i \\ -1+3i & 4-2i \end{bmatrix}$$

$$A^* = (\overline{A'}) = \begin{bmatrix} 2+i & -5 \\ 3 & i \\ -1-3i & 4-2i \end{bmatrix}$$

Now
$$A^*A = \begin{bmatrix} 2+i & -5 \\ 3 & i \\ -1-3i & 4-2i \end{bmatrix} = B \text{ (Consider)}$$

or
$$B = \begin{bmatrix} 30 & 6-8i & -19+17i \\ 6+8i & 10 & -5+5i \\ -19-17i & -5+5i & 30 \end{bmatrix}$$

Now
$$B' = \begin{bmatrix} 30 & 6-8i & -19-17i \\ 6+8i & 10 & -5+5i \\ -19-17i & -5+5i & 30 \end{bmatrix}$$

$$B^* = (\overline{B'}) \begin{bmatrix} 30 & 6-8i & -19-17i \\ 6+8i & 10 & -5+5i \\ -19-17i & -5+5i & 30 \end{bmatrix} = B$$

$\Rightarrow$ $B = A^* A$ is Hermitian matrix.

Q.28. Prove that characteristic roots of unitary matrix are of unit modulus. **[UTU, 2009-10]**

Ans. Let A be a unitary matrix

$\Rightarrow$ $AA^\theta = A^\theta A = 1$, where A^θ is transposed conjugate of A.

Let λ be eigen value of A and X be corresponding eigen vector.

$\Rightarrow$ $AX = \lambda X$

$\Rightarrow$ $(AX)^\theta = (\lambda X)^\theta$

$\Rightarrow$ $X^\theta A^\theta = \bar{\lambda} X^\theta$ $\quad [\because (AB)^\theta = B^\theta A^\theta \text{ and } (kA^\theta) = \bar{k}A^\theta]$

$\Rightarrow$ $X^\theta A^\theta (AX) = \bar{\lambda} X^\theta (AX)$

$\Rightarrow$ $X^\theta (A^\theta A) X = \bar{\lambda} X^\theta (AX)$ $\quad [\because AX = \lambda X]$

$\Rightarrow$ $X^\theta (I) X = \bar{\lambda} \lambda X^\theta X$ $\quad [\because A^\theta A = I]$

$\Rightarrow$ $X^\theta X = |\lambda|^2 X^\theta X$ $\quad [\because a\,z\,\bar{z} = |z|^2]$

$\Rightarrow$ $|\lambda|^2 = 1$

Thus eigen values (characterisitc roots) of a unitary matrix are of unit modulus.