CS201

Mathematics For Computer Science

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Group Number: 35

Parag Gupta (180500), Samarth Varma (180655), Satvik Jain (180678)

Assignment 3

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Question 1

1. Find the generating function for the following recurrence relation.

$$f(n+1) = \begin{cases} 1 & \text{if } n+1=0 \\ \sum_{i=0}^{n} f(i)f(n-i) & \text{if } n \geq 0 \end{cases}$$

2. Using the generating function and generalised binomial theorem for $\sqrt{1+y}$, find a closed form for $f(n)$.

Solution

We define the ordinary generating function of $f(n)$ as $F(x)$. So,

$$F(x) = \sum_{n \geq 0} f(n)x^n$$

Consider

$$\sum_{n \geq 0} f(n+1)x^{n+1}$$

$$\sum_{n \geq 0} f(n+1)x^{n+1} = \sum_{n \geq 0} \sum_{i=0}^{n} f(i)f(n-i)x^{n+1} = x \sum_{n \geq 0} \sum_{i=0}^{n} f(i)x^i f(n-i)x^{n-i}$$

The LHS is easy to interpret. It is simply given by $F(x) - 1$.
We claim that the RHS is $xF(x)^2$.

Consider the expansion of $F(x)^2$.

$$F(x)^2 = (f(0) + f(1)x + f(2)x^2 + \dots)(f(0) + f(1)x + f(2)x^2 + \dots)$$

Let us find the coefficient of x^n in RHS
x^n can only be obtained by picking $f(i)x^i$ from the first bracket and $f(n-i)x^{n-i}$ from the second bracket and then we sum over i.
Hence, the coefficient of x^n in $F(x)^2$ is $\sum_{i=0}^{n} f(i)f(n-i)$ as desired. So,

$$F(x)^2 = \sum_{n\geq 0}\sum_{i=0}^{n} f(i)f(n-i)x^n$$

Hence our RHS is precisely given by $xF(x)^2$
Equating the LHS and RHS, $F(x) - 1 = xF(x)^2$.
This is a quadratic equation in $F(x)$. Using the quadratic formula, we can obtain,

$$F(x) = \frac{1 \pm (1-4x)^{1/2}}{2x}$$

Since $F(x)$ can be expanded as a power series at $x = 0$, $F(x)$ must converge at $x = 0$.
If we choose the $+$ sign, then numerator $\rightarrow$ 2 while the denominator $\rightarrow$ 0 as x $\rightarrow$ 0.
So, the ratio becomes ∞ at 0.
Hence we must choose the $-$ sign.
Using L'Hopital Rule, we can show that $F(x)$ converges to 1 at $x = 0$. Hence

$$F(x) = \frac{1 - (1-4x)^{1/2}}{2x}$$

We apply the generalized binomial expansion for $(1-4x)^{1/2}$.

$$(1-4x)^{1/2} = \sum_{n\geq 0}\binom{1/2}{n}(-4x)^n = \sum_{n\geq 0}\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})\dots(-\frac{2n-3}{2})}{n!}(-4x)^n$$

Trying to produce $(2n-3)!$ in the numerator

$$= \sum_{n\geq 0} (-1)^{n-1} \frac{(2n-3)!}{2^{2n-2}(n-2)!n!}(-4x)^n$$

$$= \sum_{n\geq 0} \frac{-4(2n-3)!}{(n-2!)n!}x^n = -2\sum_{n\geq 0} \frac{(2n-2)!}{n((n-1)!)^2}x^n = -2\sum_{n\geq 0} \frac{\binom{2n-2}{n-1}}{n}x^n$$

Since the series is not defined for $n=0$, we take the constant term out and start our summation from 1.

$$\implies (1-4x)^{1/2} = 1 - 2\sum_{n\geq 1} \frac{1}{n}\binom{2n-2}{n-1}x^n$$

$$F(x) = \frac{1-(1-4x)^{1/2}}{2x}$$

$$= \frac{1-(1-2\sum_{n\geq 1}\frac{1}{n}\binom{2n-2}{n-1}x^n)}{2x} = \sum_{n\geq 1} \frac{1}{n}\binom{2n-2}{n-1}x^{n-1}$$

Replace n by $n+1$.

$$= \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n}x^n$$

Hence,

$$f(n) = \frac{1}{n+1}\binom{2n}{n}$$

Question 2

Define n-varaiate polynomials P_d and Q_d as:

$$\begin{aligned} P_d(x_1, x_2, \ldots, x_n) &= \sum_{\substack{J \subseteq [1,n] \\ |J|=d}} \prod_{r \in J} x_r \\ Q_d(x_1, x_2, \ldots, x_n) &= \sum_{\substack{0 \le i_1, i_2, \ldots, i_n \le d \\ i_1+i_2+\cdots+i_n=d}} \prod_{r=1}^{n} x_r^{i_r}, \end{aligned}$$

and $P_0(x_1, x_2, \ldots, x_n) = 1 = Q_0(x_1, x_2, \ldots, x_n)$. Show that for any $d > 0$:

$$\sum_{m=0}^{d} (-1)^m P_m(x_1, x_2, \ldots, x_n) Q_{d-m}(x_1, x_2, \ldots, x_n) = 0.$$

Solution

Let us define a function

$$F(y) = \prod_{i=1}^{N} (1 - x_i y) \tag{2.1}$$

$$\implies \frac{1}{F(y)} = \frac{1}{\prod_{i=1}^{N}(1 - x_i y)} \tag{2.2}$$

We will now prove that the co-efficients of y^d is related to $P_d(x_1, x_2, \ldots, x_n)$ and $Q_d(x_1, x_2, \ldots, x_n)$.

For equation (2.1), $F(y) = \prod_{i=1}^{N}(1 - x_i y) = (1 - x_1 y)(1 - x_2 y)(1 - x_3 y)...(1 - x_n y)$.

Here, the coefficients of y^d are given by collecting x_is d times at once and summing over all the possible combinations.

Therefore, the coefficient is given by:

$$F(y) = \sum_{d=0}^{n} k_d y^d$$

Where

$$k_d = \sum_{\substack{J \subseteq [1,n] \\ |J|=d}} \prod_{r \in J} -x_r$$

$$= (-1)^d \sum_{\substack{J\subseteq[1,n]\\|J|=d}} \prod_{r\in J} x_r$$
$$= (-1)^d P_d(x_1, x_2, \dots, x_n) \tag{2.3}$$

Now, For equation (2.2), $\frac{1}{F(y)}$ can be written as

$$\frac{1}{F(y)} = \frac{1}{(1-x_1y)(1-x_2y)\dots(1-x_ny)}$$

From each basket $(1-x_dy)^{-1}$, take out y^{d_k}, whose coefficient is given by $\binom{-1}{d_k}(-x_k)^{d_k}$ The collection of y^{d_k}'s such that $\sum_{k=1}^n d_k = d$ gives the coefficient of y^d. Here $0 \le d_1, d_2, \dots, d_n \le d$. Therefore the coefficient of y^d becomes

$$\sum_{\substack{0\le i_1,i_2,\dots,i_n\le d\\ i_1+i_2+\dots+i_n=d}} \prod_{r=1}^n \binom{-1}{d_r}(-x_r)^{d_r}$$
$$= \sum_{\substack{0\le i_1,i_2,\dots,i_n\le d\\ i_1+i_2+\dots+i_n=d}} \prod_{r=1}^n x_r^{d_r}$$
$$= \sum_{d\ge 0} Q_d(x_1, x_2, \dots x_n) \tag{2.4}$$

Therefore from equation (2.3) and (2.4), we get that

$$F(y) = \sum_{d=0}^n (-1)^d P_d(x_1, x_2, x_3 \dots, x_n) y^d$$

$$\frac{1}{F(y)} = \sum_{d=0}^n Q_d(x_1, x_2, x_3 \dots, x_n) y^d$$

Multiplying both $F(y)$ and $\frac{1}{F(y)}$, we get

$$1 = \left(\sum_{d\ge 0}(-1)^d P_d(x_1, x_2, x_3, \dots, x_n) y^d\right)\left(\sum_{d\ge 0} Q_d(x_1, x_2, x_3, \dots, x_n) y^d\right)$$
$$= \sum_{d\ge 0}\left(\sum_{m=0}^d (-1)^m P_m(x_1, x_2, x_3, \dots, x_n) Q_{d-m}(x_1, x_2, x_3, \dots, x_n)\right) y^d \tag{2.5}$$

Therefore from the above equation, the coefficient of y^d should be zero for every $d \geq 1$. Hence, for any $d > 0$

$$\sum_{m=0}^{d} (-1)^m P_m(x_1, x_2, \ldots, x_n) Q_{d-m}(x_1, x_2, \ldots, x_n) = 0.$$

Question 3

1. Let $\alpha \in \mathbb{R}$ and N be a natural number. Using pigeon-hole principle, show that there exists integers p and q such that $1 \leq q \leq N$ and
$$|q\alpha - p| \leq \frac{1}{N}$$
2. Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in R$ and N be a natural number. Using pigeon-hole principle, show that there exists integers $p_1, p_2, \ldots, p_n, q$ such that $1 \leq q \leq N^n$ and for all $i \in \{1, \ldots, n\}$
$$|\alpha_i - \frac{p_i}{q}| \leq \frac{1}{q^{1+1/n}}$$

Solution

1. - We have q as an integer such that $1 \leq q \leq N$ and $\alpha \in \mathbb{R}$.
   - Consider the fractional parts $\{0.\alpha\}$, $\{1.\alpha\}$, $\{2.\alpha\}$ $\ldots\{N.\alpha\}$. They all belongs to the interval $[0, 1)$.
   - We can represent the interval $[0, 1)$ as union of N subintervals
   $$[0,1) = \left[0, \frac{1}{N}\right) \cup \left[\frac{1}{N}, \frac{2}{N}\right) \cdots \cup \left[\frac{N-1}{N}, 1\right)$$
   - Now, We have $N+1$ fractional parts and only N sub-intervals ,by pigeonhole principle there are atleast 2 integers k and l such that $0 \leq k < l \leq N$ and $\{k\alpha\}$ and $\{l\alpha\}$ belongs to same Interval.
   - Now consider the difference $|\{l\alpha\} - \{k\alpha\}| \leq \frac{1}{N}$.
   - Now $\{l\alpha\} = l\alpha - \lfloor l\alpha \rfloor$ and $\{k\alpha\} = k\alpha - \lfloor k\alpha \rfloor$.
   - Substituting we get $|l\alpha - \lfloor l\alpha \rfloor - k\alpha + \lfloor k\alpha \rfloor| \leq \frac{1}{N}$
   - which gives$|(l-k)\alpha - (\lfloor l\alpha \rfloor - \lfloor k\alpha \rfloor)| \leq \frac{1}{N}$. Let $l - k = q$ and $\lfloor l\alpha \rfloor - \lfloor k\alpha \rfloor = p$.
   - Hence we have $|q\alpha - p| \leq \frac{1}{N}$ where $1 \leq q \leq N$ and p is integer.
2. - From the first part we know that given any $\alpha \in \mathbb{R}$ and $N \in \mathbb{N}$ then there exist integers p,q such that $1 \leq q \leq N$ and $|q\alpha - p| \leq \frac{1}{N}$.

- So for any $\alpha_i \in \mathbb{R}$ and $K = N^n \in \mathbb{N}$ there exist integers p_i, q such that $1 \leq q \leq K$ and $|q\alpha_i - p_i| \leq \frac{1}{K}$.
- Now $1 \leq q \leq K \iff 1 \geq \frac{1}{q} \geq \frac{1}{K}$.
- Putting $k = N^n$ and rearranging we get $\frac{1}{N} \leq \frac{1}{q^{1/n}}$. Since $\frac{1}{N^n} \leq \frac{1}{N}$. We have $\frac{1}{N^n} \leq \frac{1}{q^{1/n}}$.
- Now $|q\alpha_i - p_i| \leq \frac{1}{N^n} \leq \dfrac{1}{q^{\frac{1}{n}}}$ which gives $|\alpha_i - \dfrac{p_i}{q}| \leq \dfrac{1}{q^{1+1/n}}$
- Hence for all $i \in \{1, \ldots, n\}$ there exist integers p_i , q such that $1 \leq q \leq N^n$ and
$$|\alpha_i - \frac{p_i}{q}| \leq \frac{1}{q^{1+1/n}}$$
.

Question 4

Give a proof for Ramsey's theorem for general case.

Solution

Ramsey Theorem General Form is given as for any $c, n_1, n_2, n_3..., n_c, k \geq 1$, there exists a number $N(n_1, n_2, n_3...n_c, k) > 0$ such that for any set X with $|X| \geq N_{n_1,n_2...n_c,k}$ and any mapping $f : X^k \mapsto 1, 2, ...c$, there exists a $\tilde{c}, 1 \leq \tilde{c} \leq c$ and a subset $Y \subseteq X, |Y| = n_{\tilde{c}}$,with $f(Y^k) = \tilde{c}$

We will prove this by induction

- Let us view f as assigning one of c colours to elements of set X^k
- Proof is by induction on c
- For $c = 1, Y = X$ satisfies the condition trivially.
- to prove for $c = 2$ case, we will do induction on $n_1 + n_2 = 2$
- Assume it is true for $n_1 + n_2 - 1$ and consider for $n_1 + n_2$
- Consider X of size $N(n_1, n_2 - 1) + N(n_1 - 1, n_2)$
- Take $a \in X$ and let

$$Y_{c_1} = \{b_i | b_i \neq a, b_i \in X, 0 < i \leq k-1, s.t. f(a, b_1, b_2, b_3 \dots b_{k-1}) = c_1\}$$

$$Y_{c_2} = \{b_i | b_i \neq a, b_i \in X, 0 < i \leq k-1, s.t. f(a, b_1, b_2, b_3 \dots b_{k-1}) = c_2\}$$

- Since every set of a single subset a and $k - 1$ subset b will be assigned a color either c_1 or c_2 because of induction hypothesis, the set Y_{c_1} and Y_{c_1} spans the entire set X except a. Therefore $|Y_{c_1}| + |Y_{c_2}| = |X| - 1 = N(n_1, n_2 - 1) + N(n_1 - 1, n_2) - 1$
- Therefore we have $|Y_{c_1}| \geq N(n_1 - 1, n_2)$ or $|Y_{c_2}| \geq N(n_1, n_2 - 1)$ by pigeonhole theory.
- If former, then by induction, Y_{c_1} either has a subset Z_1 of size $n_1 - 1$ with $f(Z_1^k) = c_1$ or a subset Z_2 of size n_2 with $f(Z_2^k) = c_2$.

- In the First Case, $Z_1' = \{a\} \cup Z_1$ is a set of size n_1 with $f(Z_1'^k) = c_1$. This is because by the induction hypothesis every subset of size k in Z_1 is assigned the color c_1 and the only case remaining is a single a in the subset, which by the definition of Y_{c_1} is also assigned color c_1.
- Similar Argument can be made for the other case, $|Y_{c_2}| \geq N(n_1, n_2 - 1)$
- In this case, by induction, Y_{c_2} either has a subset Z_2 of size $n_2 - 1$ with $f(Z_2^k) = c_2$ or a subset Z_2 of size n_2 with $f(Z_2^k) = c_2$.
- In the First Case, $Z_2' = \{a\} \cup Z_2$ is a set of size n_2 with $f(Z_2'^k) = c_2$. This is because by the induction hypothesis every subset of size k in Z_2 is assigned the color c_2 and the only case remaining is a single a in the subset, which by the definition of Y_{c_2} is also assigned color c_2.
- Now assume true for $c - 1$ colors
- Consider any X of size $N(n_1, n_2, n_3....N(n_{c-1}, n_c), k)$
- Now define the Mapping

$$g(a, b^{k-1}) = \begin{cases} f(a, b^{k-1}) & f(a, b^{k-1}) \leq c - 2 \\ c - 1 & f(a, b) = c - 1, c \end{cases}$$

  where b^{k-1} is a $k - 1$ dimension hypergraph.
- By induction hypothesis, there exists a $\hat{c}$, $1 \leq \hat{c} \leq c - 1$ and subset $Y \subseteq X$ such that

$$|Y| = \begin{cases} n_{\hat{c}} & \hat{c} \leq c - 2 \\ N(n_{c-1}, n_c) & \hat{c} = c - 1 \end{cases}$$

  with $g(Y^k) = \hat{c}$
- If $\hat{c} \leq c - 2$, the above condition satisfies the Ramsey Theorem
- For $\hat{c} = c - 1$, we have $|Y| = N(n_{c-1}, n_c)$ and $g(Y^k) = c - 1$.
- Therefore, $f(Y^k) \in \{c - 1, c\}$ and $|Y| = N(n_{c-1}, n_c)$

- By induction hypothesis for $c = 2$, we get the either there exists a set $Z \subseteq Y, |Z| = n_{c-1}$ and $f(Z^k) = c - 1$ or there exists a set $Z \subseteq Y, |Z| = n_c$ and $f(Z^k) = c$ which is the Ramsey theorem for general case.

Question 5

Consider the set $S_n = \{f \mid f : [n] \to [n] \text{ and } f \text{ is a bijection}\}$ which contains all bijective mapping from $[n]$ to $[n]$ where $[n] = \{1, 2, 3, \dots, n\}$. In other words, any $f \in S_n$ simply permutes the elements in $[n]$.

1. A mapping $f \in S_n$ is called a **transposition** if there exists (i, j) such that $0 \le i \ne j \le n$ and

$$f(k) = \begin{cases} j & \text{if } k = i \\ i & \text{if } k = j \\ k & \text{otherwise} \end{cases}$$

   Show that any $g \in S_n$ can be written as a finite product $f_1 \circ f_2 \circ \dots \circ f_m$ where each f_i is a transposition in S_n.

2. The **parity** of a function f in S_n denoted by $N(f)$ is defined as the number of pairs (i, j) such that $1 \le i < j \le n$ and $f(i) > f(j)$. Show that

$$N(f) \equiv m \ (\ mod\ 2)$$

   where $f = g_1 \circ g_2 \circ \dots \circ g_m$ and each g_i is a transposition in S_n.

Solution

1. We will prove this by induction on n

   - Consider $n = 2$. S_2 has only two functions of which one is identity function (i.e $f(t) = t$) and other transposition given by

$$f_1(t) = \begin{cases} 2 & \text{if } t = 1 \\ 1 & \text{if } t = 2 \end{cases}$$

   - Clearly the identity function $f = f_1 \circ f_1$.Hence the Statement is true for $n = 2$.
   - Assume it to be true for $n = k - 1$. Any function $g \in S_{k-1}$ can be expressed as product of finite transpositions of S_{k-1}.

- Now for every function $g \in S_{k-1}$, Define a function $g' : [k] \to [k]$ such that

$$g'(t) = \begin{cases} g(t) & \text{if } t \in [k-1] \\ k & \text{if } t = k \end{cases}$$

- Now $g(t) = f_1 \circ f_2 \circ \cdots \circ f_m$ And $g' = f_1' \circ f_2' \circ \cdots \circ f_m'$ where f_i' is defined similar to g' for f_i. Clearly all f_i' are transposition of S_k.
- Hence, any function belonging to S_k having k mapped to itself can be written as product of finite transpositions.
- Now take any function $p \in S_k$ in which k is not mapped to itself. Let $p(k) = j$ and $p(i) = k$ for some $j, i \in [k-1]$. Consider this particular transposition of S_k

$$f(t) = \begin{cases} k & \text{if } t = j \\ j & \text{if } t = k \\ t & \text{otherwise} \end{cases}$$

- Now $f \circ p$ is given by

$$f \circ p(t) = \begin{cases} f \circ p(i) = j & \text{if } t = i \\ f \circ p(k) = k & \text{if } t = k \\ f \circ p(t) = p(t) & \text{otherwise} \end{cases}$$

- $f \circ p$ is function having k mapped to itself, hence it can be written as product of finite transposition.
- Also $f \circ f$ is identity function($f(t) = t$). Hence $f \circ f \circ p = p$
- Now f is a transposition $f \circ p$ is product of finite transposition, hence $p \in S_k$ is also a product of finite transpositions.
- Hence any function $f \in S_k$ can be written as product of transpositions given any function from S_{k-1} can be written as product of transpositions.
- Hence by induction any $f \in S_n$ can be written as product of transpositions.

2. Let $f_{i,j}$ denotes the transposition of i and j.

$$f_{i,j}(k) = \begin{cases} j & \text{if } k = i \\ i & \text{if } k = j \\ k & \text{otherwise} \end{cases}$$

Also we will use a special type of transposition called as Adjacent Transposition which is of type $f_{k,k+1}$.
We need to show

$$N(f) \equiv m \ (\ mod\ 2)$$

where $f = g_1 \circ g_2 \circ \cdots \circ g_m$ and each g_i is a transposition in S_n. which basically means given any two decomposition of $f \in S_n$, parity of both will be same and will be equivalent to $N(f)$.
Also if $i < j$ and $f(i) > f(j)$ then we call it **inversion**. $N(f)$ give total number of inversion. Let the two different decomposition in transposition of f be $g_1 \circ g_2 \circ \cdots \circ g_k$ and $g_1 \circ g_2 \circ \cdots \circ g_k$.

We can see that,For $k < l$,

$$f_{k,l} = f_{l-1,l} \circ f_{l-2,l-1} \ldots f_{k+1,k+2} \circ f_{k,k+1} \circ f_{k+1,k+2} \ldots f_{l-2,l-1} \circ f_{l-1,l} \tag{5.1}$$

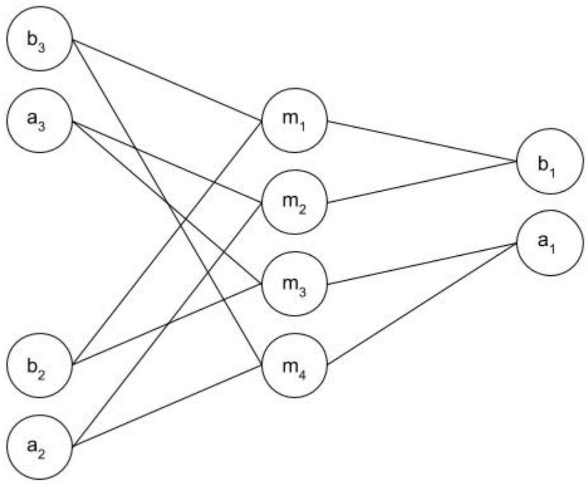
Hence any $f_k l$ can be broken into product of $2(l-k)+1$ transposition i.e odd number of transposition.

- By above statement we can write any transposition as product of adjacent transposition without changing its parity.
- $N(p)$ denotes the number of inversions for any $p \in S_n$.
- We show that if $g = f_{j,j+1}$is an adjacent transposition then $N(g \circ p)$ and $N(p)$ have opposite parities (one odd, the other even).Then
- That means that if p is written as any product of adjacent transpositions, then N(p) has exactly the parity of that of the number of transpositions.
- But with $q = g \circ p$, the number of inversions $N(q)$ for q differs from $N(p)$ by 1.

- Now for a pair k, l with $k < l$,$p(l) > p(k)$ and $q(l) > q(k)$ can be different statements only if one of $p(l), p(k)$ is j and the other is $j+1$.
  - if neither is j or$j+1$ then the p is the same as the q.
  - if one is but the other is not, say $p(l) = j$ or$j+1$ then $q(k) = p(k)$, and whether $p(l) = j$ or $j+1$, we we will have $p(l) = j+1$ or j , the inequalities $p(k) < p(l)$ and $q(k) < q(l)$ are equivalent as $k \neq j$ and $k \neq j+1$.
- When one of $p(l), p(k)$ is j and the other is $j+1$, $N(p)$ and $N(q)$ differ by 1. Thus,$N(p)$ and $N(q)$ have the opposite parity which proves the statement.

Question 6

Let $G = (V, E)$ be a graph where V is the vertex set and E is the edge set. A bijective mapping $f : V \to V$ is an **automorphism** if it has the property that $(u, v) \in E \iff (f(u), f(v)) \in E$. Consider the following graph.



Let $A = \{a_1, a_2, a_3\}, B = \{b_1, b_2, b_3\}, M = \{m_1, m_2, m_3, m_4\}$. Then, the vertex set of the above graph is $V = A \cup B \cup M$. Consider a bijective mapping $g : A \cup B \to A \cup B$ such that $g(a_i) \in \{a_i, b_i\}$ and $g(b_i) \in \{a_i, b_i\}$ for all $i \in \{1, 2, 3\}$, i.e., g maps the ordered pair $[a_i, b_i]$ to either $[a_i, b_i]$ (no swap) or $[b_i, a_i]$ (swap).

Show that g can be extended to an automorphism f for the above graph if and only if the number of swaps performed by g is even.

Solution

Foremost, we note that there is no edge of the form (a_i, b_j) or (a_i, a_j) or (b_i, b_j) for any $i, j = \{1, 2, 3\}$. Hence all a_i and all b_i are connected to m_j through an edge. We can perform at most 3 swaps. So we have the following four cases -

- exactly 0 swaps
- exactly 1 swap
- exactly 2 swaps

- exactly 3 swaps

We collect all the vertices in the neighbourhood of m_i in the set N_i and call it the neighbour set of m_i $\forall i \in \{1, 2, 3, 4\}$

- $N_1 = \{b_1, b_2, b_3\}$
- $N_2 = \{b_1, a_2, a_3\}$
- $N_3 = \{a_1, b_2, a_3\}$
- $N_4 = \{a_1, a_2, b_3\}$

We assume that g is an automorphism and try to find the number of swaps that are valid. We now consider all the cases one by one.

- **Case 1 - No swaps**
  - Since we have no swaps, g maps the ordered pair $[a_i, b_i]$ to itself for all $i \in \{1, 2, 3\}$.
  - Each of the ordered pair $m_i \times N_i \in E$.
  - Since g is an automorphism, $g(m_i) \times g(N_i) \in E$.
  - Since we have no swaps, $g(N_i) = N_i \implies g(m_i) \times N_i \in E$.
  - Now, none of the N_i and N_j are equal.
  - This implies that $g(m_i) = m_i$ which is a valid solution for g to be an automorphism.
  - Hence g is an automorphism when no swaps are performed.
- **Case 2 - exactly 1 swap**
  - We observe that none of the $N_i s$ have 2 or more elements in common.
  - Suppose we swap b_i and a_i for some arbitrary $i \in \{1, 2, 3\}$.
  - We make the necessary swaps in $N_j s$ that is, if b_i is present in the set, we make it a_i and vice versa.
  - Let us call the sets induced after swapping $F_j s$.

- For g to be an automorphism, we need to map the sets F to N since each N_j corresponds to a particular m_j and the set of edges $m_j \times N_j \forall j \in \{1,2,3,4\}$ is exhaustive.
- Since we swapped exactly one element in each of N_j to form F_j, $F_j \neq N_j \forall j \in \{1,2,3,4\}$.
- We can see by inspection that in fact none of the $F_j s$ are equal to any of the $N_k s$.
- We can also realize the above fact in the following way -
  * Suppose we have $F_j = N_k$ for some $j,k \in \{1,2,3,4\}$ and $j \neq k$
  * It follows that there exists an N_j such that N_j and N_k have exactly two elements in common because swapping the i^{th} element leads to an equality.
  * This is a clear contradiction to the fact established in the beginning that none of the $N_i s$ have 2 or more elements in common.
  * Hence we cannot have $F_j = N_k$.
- Since j, k were arbitrary, no bijection g can be established with exactly one swap.

- **Case 3 - exactly 2 swaps**
  - We have the following observation about $N_i s$ -
    * For each $i \in \{1,2,3\}$, there are exactly two sets that contain b_i and exactly two that contain a_i.
  - By the above observation and the fact that none of the $N_j s$ have two or more elements in common, it immediately follows that for each N_j there must exist an N_k, $j \neq k$ and $j,k \in \{1,2,3,4\}$ such that both N_j and N_k have exactly one element in common.
  - We perform two swaps on each N_j where the i^{th} element is kept fixed. Let's call this the induced set $F_j \forall j \in \{1,2,3,4\}$.
  - By the previous conjecture, there exists a set N_k, $k \neq j$ and $k \in \{1,2,3,4\}$ such that N_j and N_k have exactly the i^{th} entry in common.
  - By definition, the above set N_k is F_j. Hence $F_j = N_k$. Since i and j were arbitrary, this must be true for all i and j.

- Hence by performing 2 arbitrary swaps, g is still an automorphism.

- **Case 4 - exactly 3 swaps**
  - We make yet another basic observation that none of the $N_i s$ are mutually disjoint.
  - Hence swapping all three elements would not produce an F_j that is equal to some other N_k.
  - So, g is not an automorphism if we perform all three swaps.

We have shown that either 0 swaps or exactly 2 swaps that is, an even number of swaps is required.
Now, we try to extend our definition for g.
Since g was already defined for $V \times V$, we redefine it for $A \cup B \cup M \times A \cup B \cup M$ as follows -

> $g(a_i) \in \{a_i, b_i\} \forall i \in \{1, 2, 3\}$ and $g(m_i) = m_j$ such that the induced set F_i of N_i is same as the neighbour set N_j where j is uniquely defined for each $i \in \{1, 2, 3, 4\}$.

References

https://people.math.sc.edu/laszlo/Ramsey.pdf
https://web.stanford.edu/class/math61cm/permutations.pdf