CS201

Mathematics For Computer Science

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Question 1

We have seen generating functions for $\binom{n}{m}$ for variable m keeping n fixed, and for variable n keeping m fixed. If we wish to make both variable then the generating function needs to be over two variables.

1. Prove that $\frac{1}{1-y-xy} = \sum_{n\geq 0}\sum_{m\geq 0}\binom{n}{m}x^m y^n$.
2. Derive the generating function $\binom{2n}{n}$ from above two-variable generating function by judicious substitution for one of the two variable.

Solution

Let us denote $\binom{n}{m}$ by $f(n,m)$
The binomial coefficient is defined by the following recurrence

$$f(n,0) = 1 \;\; \forall n \geq 0$$

$$f(0,m) = 0 \;\; \forall m \geq 1$$

$$f(n,m) = f(n-1,m) + f(n-1,m-1) \;\; \forall n,m \geq 0$$

We define the ordinary generating function of $f(n,m)$ as $F(x,y)$. So,

$$F(x,y) = \sum_{n\geq 0}\sum_{m\geq 0} f(n,m)x^m y^n$$

Consider

$$\sum_{n\geq 1}\sum_{m\geq 1} f(n,m)x^m y^n$$

Replace $f(n,m)$ by the recurrence relation obtained before to get,

$$\sum_{n\geq 1}\sum_{m\geq 1} f(n,m)x^m y^n = \sum_{n\geq 1}\sum_{m\geq 1}(f(n-1,k)+f(n-1,k-1))x^m y^n$$

First we try to simplify the RHS.
RHS :

$$\sum_{n\geq 1}\sum_{m\geq 1} f(n-1,m)x^m y^n + \sum_{n\geq 1}\sum_{m\geq 1} f(n-1,m-1)x^m y^n$$

$$= y\sum_{n\geq 1}\sum_{m\geq 1} f(n-1,m)x^m y^{n-1} + xy\sum_{n\geq 1}\sum_{m\geq 1} f(n-1,m-1)x^{m-1} y^{n-1}$$

$$= y\sum_{n\geq 0}\sum_{m\geq 1} f(n,m)x^m y^n + xy\sum_{n\geq 0}\sum_{m\geq 0} f(n,m)x^m y^n$$

$$= y(F(x,y) - \sum_{n\geq 0} y^n) + xyF(x,y)$$

$$= y(F(x,y) - \frac{1}{1-y}) + xyF(x,y)$$

Now we look at the LHS:
Since $f(0,m)=0$,

$$\sum_{n\geq 1}\sum_{m\geq 1} f(n,m)x^m y^n = \sum_{n\geq 0}\sum_{m\geq 1} f(n,m)x^m y^n$$

$$= F(x,y) - \sum_{n\geq 0} y^n$$

$$= F(x,y) - \frac{1}{1-y}$$

Since LHS = RHS, we have

$$F(x,y) - \frac{1}{1-y} = y(F(x,y) - \frac{1}{1-y}) + xyF(x,y)$$

$$\implies (1-y-xy)F(x,y) = \frac{1}{1-y} - \frac{y}{1-y} = 1$$

$$\implies F(x,y) = \frac{1}{1-y-xy}$$

Hence the OGF of $f(n,m) = \binom{n}{m}$ is $\frac{1}{1-y-xy}$

We have $\frac{1}{1-y-xy} = \sum_{n\geq 0}\sum_{m\geq 0}\binom{n}{m}x^m y^n$
Substitute $x = 3$ to obtain

$$\frac{1}{1-4y} = \sum_{n\geq 0}\sum_{m\geq 0}\binom{n}{m}3^m y^n$$

The binomial theorem is given by $(1+x)^m = \sum_{i=0}^{m}\binom{n}{m}x^m$. We replace x = 3 to simplify the inner summation as follows

$$\frac{1}{1-4y} = \sum_{n\geq 0}(4y)^n$$

Replace y by x just because I like dealing with the variable x

$$\frac{1}{1-4x} = \sum_{n\geq 0}(4x)^n$$

$$\implies \frac{1}{1-4x} = \sum_{n\geq 0}2^{2n}x^n$$

Now we claim that

$$\sum_{n\geq 0}2^{2n}x^n = (\sum_{n\geq 0}\binom{2n}{n}x^n)^2 \tag{1.1}$$

$$\implies \frac{1}{1-4x} = (\sum_{n\geq 0}\binom{2n}{n}x^n)^2$$

$$\implies \frac{1}{(1-4x)^{1/2}} = \sum_{n\geq 0}\binom{2n}{n}x^n$$

Hence $(1-4x)^{-1/2}$ is the generating function for the central binomial coefficient $\binom{2n}{n}$
Here we provide a proof for equation 1.1
Equation 1.1 is given by -

$$\sum_{n\geq 0}2^{2n}x^n = (\sum_{n\geq 0}\binom{2n}{n}x^n)^2$$

RHS can be simplified further by collecting all the coefficients of x^n.

$$(\sum_{n\geq 0}\binom{2n}{n}x^n)^2 = (\sum_{n\geq 0}\binom{2n}{n}x^n)(\sum_{n\geq 0}\binom{2n}{n}x^n)$$

We take the coefficient of x^k from the first operand and coefficient of x^{n-k} from the second operand, multiply them and sum where k varies from 0 to n.

$$\implies (\sum_{n\geq 0}\binom{2n}{n}x^n)^2 = \sum_{n\geq 0}\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}x^n$$

So we need to prove

$$\sum_{n\geq 0}2^{2n}x^n = \sum_{n\geq 0}\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}x^n$$

Since both are power series, their coefficients should match for every n.
This boils down to proving the following

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k} = 2^{2n} \tag{1.2}$$

This can be shown by considering the generalized binomial theorem for the function $F(x) = (1-x^2)^{-1/2}$ where $x \in (-1, 1)$

$$(1-x^2)^{-1/2} = \sum_{n\geq 0}\binom{n-1/2}{n}x^{2n} \tag{1.3}$$

where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}{n!}$$

for any real α
Substituting $\alpha = n - 1/2$ to obtain

$$\binom{n-1/2}{n} = \frac{(n-1/2)(n-3/2)(n-5/2)\dots 1/2}{n!}$$

Multiplying both the denominator and the numerator by $2^{2n}n!$ and manipulating the terms so as to obtain

$$\binom{n-1/2}{n} = \frac{(n-1/2)(n-3/2)(n-5/2)\ldots 1/2}{n!} = \frac{(2n!)}{(n!)^2 2^{2n}} = \frac{\binom{2n}{n}}{2^{2n}}$$

Substituting $\binom{n-1/2}{n} = \frac{\binom{2n}{n}}{2^{2n}}$ in equation 1.3 to obtain

$$(1-x^2)^{-1/2} = \sum_{n\geq 0} \binom{2n}{n} 2^{-2n} x^{2n} \tag{1.4}$$

Now, we square equation 1.3 to get

$$(1-x^2)^{-1} = (\sum_{n\geq 0} \binom{2n}{n} 2^{-2n} x^{2n})^2$$

LHS can be further written as

$$(1-x^2)^{-1} = \sum_{n\geq 0} x^{2n}$$

by writing the infinite sum G.P. formula

Since LHS = RHS, we obtain

$$\sum_{n\geq 0} x^{2n} = (\sum_{n\geq 0} \binom{2n}{n} 2^{-2n} x^{2n})^2 \tag{1.5}$$

Now, we try to evaluate the coefficient of x^{2n} in RHS

$$(\sum_{n\geq 0} \binom{2n}{n} 2^{-2n} x^{2n})^2 = (\sum_{n\geq 0} \binom{2n}{n} 2^{-2n} x^{2n})(\sum_{n\geq 0} \binom{2n}{n} 2^{-2n} x^{2n})$$

To get the coefficient of x^{2n}, we take the coefficient of x^{2k} from the first operand and coefficient of x^{2n-2k} from the second operand and sum k from 0 to n.

So coefficient of x^{2n} is $\sum_{k=0}^{n} \binom{2k}{k} 2^{-2k} \binom{2n-2k}{n-k} 2^{2k-2n} = \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} 2^{-2n}$

In equation 1.5, the coefficient of x^{2n} in the LHS is simply 1 and that for the RHS is given above. Now we just have to equate both of them to obtain

$$\sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} 2^{-2n} = 1$$

$$\implies \sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k} = 2^{2n}$$

This proves equation 1.2 thereby proving equation 1.1

Question 2

For a fixed number $k > 0$, find the recurrence relation and generating function for the sequence $a_n^k = \lfloor \frac{n}{k} \rfloor$. Use these two to derive the generating function for the sequence $b_n^k = \left(\lfloor \frac{n}{k} \rfloor\right)^2$.

Solution

We have $a_n^k = \lfloor \frac{n}{k} \rfloor \implies a_{n-k}^k = \lfloor \frac{n-k}{k} \rfloor = \lfloor \frac{n}{k} - 1 \rfloor = \lfloor \frac{n}{k} \rfloor - 1 = a_n^k - 1 \quad \forall n \geq k$.
For $n < k,\ a_n^k = 0$
Hence, the recurrence relation is,

$$a_n^k = a_{n-k}^k + 1 \ \ \forall n \geq k$$

$$a_n^k = 0 \ \ \forall n \in [0, k-1]$$

Let us define an ordinary generating function $A(x)$ as follows,

$$A(x) = \sum_{n \geq 0} a_n^k x^n$$

Now, Since $a_n^k = 0$ for $n = 0$ to $n = k - 1$. We have,

$$\sum_{n \geq 0} a_n^k x^n = \sum_{n \geq k} a_n^k x^n \implies \sum_{n \geq 0} a_n^k x^n = \sum_{n \geq k} (a_{n-k}^k + 1) x^n$$

LHS :

$$\sum_{n \geq 0} a_n^k x^n = A(x)$$

RHS :

$$\sum_{n \geq k} (a_{n-k}^k + 1) x^n = \sum_{n \geq k} a_{n-k}^k x^n + \sum_{n \geq k} x^k$$

$$= \sum_{n \geq 0} a_n^k x^{k+n} + \frac{x^k}{1-x}$$

$$= x^k \sum_{n \geq 0} a_n^k x^n + \frac{x^k}{1-x}$$

$$= x^k A(x) + \frac{x^k}{1-x}$$

Since LHS = RHS,

$$A(x) = x^k A(x) + \frac{x^k}{1-x}$$

$$\implies A(x) = \frac{x^k}{(1-x)(1-x^k)}$$

Hence the generating function for the sequence $a_n^k = \lfloor \frac{n}{k} \rfloor$ is given by $A(x) = \frac{x^k}{(1-x)(1-x^k)}$

For the second part, we have $b_n^k = (\lfloor \frac{n}{k} \rfloor)^2 = (a_n^k)^2$
Since $a_n^k = 0 \ \ \forall n < k; \ \ b_n^k = 0 \ \ \forall n < k$
For $n \geq k$,

$$b_{n-k}^k = (a_{n-k}^k)^2 = (a_n^k - 1)^2 = (a_n^k)^2 - 2a_n^k + 1 = b_n^k - 2a_n^k + 1$$

$$\implies b_n^k = b_{n-k}^k + 2a_n^k - 1$$

The equation above is our recurrence relation for b_n^k
Let us denote $B(x)$ as the generating function for the sequence b_n^k.

$$B(x) = \sum_{n \geq 0} b_n^k x^n$$

As usual, we try to relate B with itself using recurrence. Since $b_n^k = 0 \ \ \forall n < k$,

$$B(x) = \sum_{n \geq 0} b_n^k x^n = \sum_{n \geq k} b_n^k x^n$$

Substituting for b_n^k using the recurrence relation derived above,

$$\sum_{n \geq k} b_n^k x^n = \sum_{n \geq k} (b_{n-k}^k + 2a_n^k - 1) x^n$$

We know LHS = $B(x)$, we try to simplify the RHS.
RHS :

$$\sum_{n \geq k} (b_{n-k}^k + 2a_n^k - 1) x^n = \sum_{n \geq k} b_{n-k}^k x^n + 2 \sum_{n \geq k} a_n^k x^n - \sum_{n \geq k} x^n$$

$$= \sum_{n \geq 0} b_n^k x^{n+k} + 2 \sum_{n \geq 0} a_n^k x^n - \sum_{n \geq k} x^n$$

$$= x^k \sum_{n\geq 0} b_n^k x^n + 2A(x) - \frac{x^k}{1-x}$$

$$= x^k B(x) + 2A(x) - \frac{x^k}{1-x}$$

Since LHS = RHS,

$$B(x) = x^k B(x) + 2A(x) - \frac{x^k}{1-x}$$

$$\implies B(x) = \frac{2A(x) - \frac{x^k}{1-x}}{1-x^k}$$

From the first part, we obtained $A(x) = \frac{x^k}{(1-x)(1-x^k)}$. Substituting this and simplifying, we get

$$B(x) = \frac{x^k(1+x^k)}{(1-x)(1-x^k)^2}$$

Hence the generating function for the sequence $b_n^k = (\lfloor \frac{n}{k} \rfloor)^2$ is given by $B(x) = \frac{x^k(1+x^k)}{(1-x)(1-x^k)^2}$

Question 3

Given numbers from 0 to $2n-1$ in a sequence, what is the number of permutations of this sequence such that no even number is in its original position (express the number of permutations in terms of derangement number d_n)?

Solution

- Here we have a total of $2n$ numbers. Number of permutations such that no number is in its correct position will be equal to the derangement of $2n$ numbers which is d_{2n}.
- Let $P(i)$=Number of permutation when exactly i odd numbers are on their correct position and no other number is on the correct position.Then $P(i) = \binom{n}{i} d_{2n-i}$.
- In above statement $\binom{n}{i}$ is the number of ways of choosing i odd numbers which we keep in correct position (i.e. only one way of arranging).
- Remaining $2n-i$ are arranged such that no one is on the correct position. Number of ways of doing that is d_{2n-i}.
- Above two events are independent. Hence $P(i) = \binom{n}{i} d_{2n-i}$
- For every $P(i)$; $0 \leq i \leq n$; No even number is in their correct position. Hence, the total number of permutations where even is not in it's correct position will be given by the summation over $P(i)$.
- Now, the number of permutations such that no even number is in its original position is simply given by $\sum_{i=0}^{n} \binom{n}{i} d_{2n-i}$.

Question 4

Let A be a set containing non-empty sets and define $A_\times = \prod_{B \in A} B$. Prove that Axiom of Choice is equivalent to the statement that for every set A as above, $A_\times \neq \emptyset$.

Solution

Axiom of Choice says that Let A be a set whose elements are non-empty subsets of set U. Then there exists a mapping f, $f : \mathbb{A} \to \mathbb{U}$ such that $f(X) \in X$ for all $X \in A$.

So we need to prove that given A be a set containing non-empty sets and define $A_\times = \prod_{B \in A} B$. For every set A, $A_\times \neq \emptyset$ implies the above statement and vice versa.

- $A_\times \neq \emptyset \implies$ Axiom of choice
  - $\prod_{B \in A} B$ contains all tuple (having number of elements equal to cardinality of A) which contain one element each from every set $B \in A$.
  - We know that $A_\times$ contains atleast one element.
  - Let U be the Union of all the sets $B \in A$.
  - Let p be any element of $A_\times$. It can be treated as an image of mapping f, $f: A \to U$.
  - p will be tuple having one element from each one of $B \in A$.
  - The value of mapping $f(B)$ for each element $B \in A$ will be the element of p which have come from B.
  - Thus $f(B) \in B$,for all $B \in A$.
  - Hence, there exists a mapping f, $f : \mathbb{A} \to \mathbb{U}$ such that $f(B) \in B$ for all $B \in A$
- Axiom of choice $\implies A_\times \neq \emptyset$
  - Let U be the union of all sets in A.
  - According to Axiom of choice, given a set of non-empty sets, we have a mapping f, $f : A \to U$ such that $f(X) \in X$ for all $X \in A$.
  - Create a tuple P by taking the value of $f(X)$ for each $X \in A$.
  - P will contain one element each from every set $X \in A$ as $f(X) \in X$.

- Now $A_\times$ contains all tuples (having number of elements equal to cardinality of A) which contain one element each from each set X of A.
- Thus $A_\times$ contains P.
- Hence, $A_\times \neq \emptyset$

References

http://www.math.ucsd.edu/ ebender/CombText/ch-10.pdf
https://plato.stanford.edu/entries/axiom-choice/