

Path Integral

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1 Introduction

The Feynman path integral[1] is one of the formalism to solve the Schrödinger equation. However this approach is not peculiar to quantum mechanics, and M. Kac[2] is the one who recognized the applicability to the diffusion equation. Therefore, this formula is nowadays known as Feynman-Kac formula.

The purpose of this note is to derive the path integral from the Schrödinger equation in a general way so that it can also be applicable to the diffusion equation. I will also show a somewhat different way to calculate the prefactor of the kernel of the simple harmonic oscillator.

2 Derivation From Schrödinger Equation

Consider the one-dimensional Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) \psi(x, t). \quad (1)$$

The formal solution is

$$\psi(x, t) = \exp \left[-\frac{i}{\hbar} \left(\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) t \right] \psi(x, 0).$$

We can rewrite

$$\psi(x, t) = \int_{-\infty}^{\infty} \exp \left[-\frac{i}{\hbar} \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) t \right] \delta(x - x_0) \psi(x_0, 0) dx_0$$

The kernel is defined by

$$K(x, t; x_0, t_0) = \exp \left[-\frac{i}{\hbar} \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) (t - t_0) \right] \delta(x - x_0) \quad (2)$$

Then we have

$$\psi(x, t) = \int_{-\infty}^{\infty} K(x, t; x_0, t_0) \psi(x_0, t_0) dx_0 \quad (3)$$

The successive use of eq.(3) leads to

$$\begin{aligned} \psi(x_c, t_c) &= \int_{-\infty}^{\infty} K(x_c, t_c; x_b, t_b) \psi(x_b, t_b) dx_b \\ &= \int_{-\infty}^{\infty} K(x_c, t_c; x_b, t_b) \{ K(x_b, t_b; x_a, t_a) \rho(x_a, t_a) dx_a \} dx_b \end{aligned}$$

This leads to the following relationship between the kernels

$$K(x_c, t_c; x_a, t_a) = \int_{-\infty}^{\infty} K(x_c, t_c; x_b, t_b) K(x_b, t_b; x_a, t_a) dx_b$$

Let

$$\begin{aligned} N\epsilon &= t \\ \epsilon &= t_{i+1} - t_i \quad (i = 0, 1, 2, \dots, t_N) \\ t_0 &= 0, \quad t_N = t \\ x_0 &= 0, \quad x_N = x \end{aligned}$$

and

$$K(i+1; i) = K(x_{i+1}, t_{i+1}; x_i, t_i)$$

for abbreviation. Then,

$$\begin{aligned} K(x, t; 0, 0) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} K(N; N-1) K(N-1; N-2) \\ &\quad \dots K(2; 1) K(1; 0) dx_{N-1} \dots dx_2 dx_1 \quad (4) \end{aligned}$$

Furthermore, we have for small ϵ

$$\begin{aligned} \exp \left[-\frac{i}{\hbar} \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) \epsilon \right] &= \\ \exp \left[-\frac{i}{\hbar} V(x) \epsilon \right] \exp \left[-\frac{i}{\hbar} \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \epsilon \right] &- \frac{\epsilon^2}{2\hbar^2} \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}, V(x) \right] + \mathcal{O}(\epsilon^3) \end{aligned}$$

Thus

$$\exp \left[-\frac{i}{\hbar} \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) \epsilon \right] \rightarrow \exp \left[-\frac{i}{\hbar} V(x) \epsilon \right] \exp \left[-\frac{i}{\hbar} \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \epsilon \right]$$

as $\epsilon \rightarrow 0$.

Using the integral representation of the delta function

$$\delta(x_i - x_{i-1}) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{ip(x_i - x_{i-1})/\hbar} dp$$

we can evaluate $K(i; i-1)$ in the limit of $\epsilon \rightarrow 0$:

$$\begin{aligned} K(i; i-1) &= \frac{1}{2\pi\hbar} e^{-\frac{i}{\hbar} V(x_i) \epsilon} \int_{-\infty}^{\infty} dp \exp \left[-\frac{i}{\hbar} \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \epsilon \right] e^{ip(x_i - x_{i-1})/\hbar} \\ &= \frac{1}{2\pi\hbar} e^{-\frac{i}{\hbar} V(x_i) \epsilon} \int_{-\infty}^{\infty} dp \exp \left[-\frac{i}{\hbar} \left(\frac{p^2}{2m} \epsilon - p(x_i - x_{i-1}) \right) \right] \\ &= \frac{1}{2\pi\hbar} e^{-\frac{i}{\hbar} V(x_i) \epsilon} \int_{-\infty}^{\infty} dp \exp \left[-\frac{i\epsilon}{2m\hbar} \left(p - \frac{m(x_i - x_{i-1})}{\epsilon} \right)^2 \right. \\ &\quad \left. + i \frac{m(x_i - x_{i-1})^2}{2\epsilon\hbar} \right] \end{aligned}$$

The p integral is of the Gaussian form

$$\int_{-\infty}^{\infty} e^{-\frac{p^2}{2\sigma^2}} dp = \sqrt{2\pi\sigma^2}$$

We will apply this formula even in case of imaginary σ^2 . Substitution of $\sigma^2 = \frac{m\hbar}{i\epsilon}$ gives

$$K(i; i-1) = \sqrt{\frac{m}{2\pi i \hbar \epsilon}} \exp \left\{ \frac{i}{\hbar} \epsilon \left[\frac{m}{2} \left(\frac{x_i - x_{i-1}}{\epsilon} \right)^2 - V(x_i) \right] \right\} \quad (5)$$

By substituting eq.(5) into eq.(4) and taking the limit of $\epsilon \rightarrow 0$, we obtain

$$\begin{aligned} K(x, t; 0, 0) &= \lim_{\epsilon \rightarrow 0} \left(\frac{m}{2\pi i \hbar \epsilon} \right)^{N/2} \int \cdots \int dx_1 \cdots dx_{N-1} \\ &\quad \times \exp \left\{ \frac{i}{\hbar} \epsilon \sum_{i=1}^N \left[\frac{m}{2} \left(\frac{x_i - x_{i-1}}{\epsilon} \right)^2 - V(x_i) \right] \right\} \quad (6) \end{aligned}$$

In a continuum limit

$$\begin{aligned} \epsilon \sum_{i=1}^N \left[\frac{m}{2} \left(\frac{x_i - x_{i-1}}{\epsilon} \right)^2 - V(x_i) \right] \\ \Rightarrow \int_0^t dt \left(\frac{1}{2} m \dot{x}^2 - V(x) \right) = \int_0^t dt L(\dot{x}, x) = S(x(t)) \end{aligned}$$

where $L(\dot{x}, x)$ is a Lagrangian and $S(x(t))$ is the action. Furthermore we define the notation

$$\int \mathfrak{D}x(t) = \lim_{\epsilon \rightarrow 0} \left(\frac{m}{2\pi i \hbar \epsilon} \right)^{N/2} \int \cdots \int dx_1 \cdots dx_{N-1}$$

Then we have

$$K(x, t; 0, 0) = \int \mathfrak{D}x(t) \exp \left[\frac{i}{\hbar} S(x(t)) \right]$$

3 Harmonic Oscillator

The Lagrangian of the harmonic oscillator is

$$L = \frac{m}{2} (\dot{x}^2 - \omega^2 x^2)$$

We wish to calculate

$$K[b; a] = \int_a^b \mathfrak{D}x(t) \exp \left[\frac{i}{\hbar} \int_{t_a}^{t_b} dt L(\dot{x}(t), x(t)) \right]$$

the integral over all paths which go from (x_a, t_a) to (x_b, t_b) .

We can represent x in terms of classical path x_c and quantum fluctuation y around classical path;

$$x(t) = x_c(t) + y(t) \quad \text{with} \quad y(t_b) = y(t_a) = 0$$

Then,

$$\begin{aligned} \frac{m}{2} (\dot{x}^2 - \omega^2 x^2) &= \frac{m}{2} \{ (\dot{x}_c + \dot{y})^2 - \omega^2 (x_c + y)^2 \} \\ &= \frac{m}{2} (\dot{x}_c^2 - \omega^2 x_c^2) + m(\dot{x}_c \dot{y} - \omega^2 x_c y) + \frac{m}{2} (\dot{y}^2 - \omega^2 y^2) \end{aligned}$$

The action is

$$S[b, a] = \int_{t_a}^{t_b} \frac{m}{2} (\dot{x}_c^2 - \omega^2 x_c^2) dt - \int_{t_a}^{t_b} m(\dot{x}_c \dot{y} - \omega^2 x_c y) dt + \int_{t_a}^{t_b} \frac{m}{2} (\dot{y}^2 - \omega^2 y^2) dt$$

The second term is, using integration by parts,

$$\int_{t_a}^{t_b} m(\dot{x}_c \dot{y} - \omega^2 x_c y) dt = m\dot{x}_c y|_{t_a}^{t_b} - \int_{t_a}^{t_b} m(\ddot{x}_c + \omega^2 x_c) y dt$$

which vanishes because $y(t_a) = y(t_b) = 0$ and $\ddot{x}_c + \omega^2 x_c = 0$. Thus $S[b, a]$ can be divided into two parts. the classcal and the quantum parts;

$$S[b, a] = S_{cl}[b, a] + S_q[0, 0]$$

Notice that $S_q[0, 0]$ is a function of the time interval $T = t_b - t_a$. This means that $K(b; a)$ must be of the form

$$K(b; a) = F(T) e^{\frac{i}{\hbar} S_{cl}[b, a]}$$

where

$$F(T) = \int_0^1 \exp \left[\frac{i}{\hbar} \int_0^T \frac{m}{2} (\dot{y}^2 - \omega^2 y^2) dt \right] \mathfrak{D}y(t) \quad (7)$$

Now we evaluate $S_{cl}[b, a]$. We write

$$y(t) = A \cos(\omega t + \varphi_a).$$

Then $x_a = A \cos \varphi_a$, $x_b = A \cos(\omega T + \varphi_a)$, where $T = t_b - t_a$. The classical action represented by x_a and x_b is

$$S_{cl}[b, a] = -\frac{m\omega}{2} \left(-x_a x_b \sin \omega T - A \sin \varphi_a (x_b \cos \omega T - x_a) \right) \quad (8)$$

Since $x_b = A \cos(\omega T + \varphi_a) = x_a \cos \omega T - A \sin \varphi_a \sin \omega T$, we have

$$A \sin \varphi_a = \frac{x_a \cos \omega T - x_b}{\sin \omega T}$$

By substituting this into eq.(8), we have

$$\begin{aligned} S_{cl}[b, a] &= \frac{m\omega}{2} \left(-x_a x_b \sin \omega T - \frac{(x_b \cos \omega T - x_a)(x_a \cos \omega T - x_b)}{\sin \omega T} \right) \\ &= \frac{m\omega}{2 \sin \omega T} \left((x_a^2 + x_b^2) \cos \omega T - 2x_a x_b \right) \end{aligned}$$

Determination Of The Prefactor

We evaluate

$$F(T) = \int_0^0 \exp \left[\frac{i}{\hbar} \int_0^T \frac{m}{2} (\dot{y}^2 - \omega^2 y^2) dt \right] \mathfrak{D}y(t) \quad (9)$$

by discretizing the functional integral $\mathfrak{D}y(t)$, namely,

$$= \lim_{\epsilon \rightarrow 0} \left(\frac{m}{2\pi i \hbar \epsilon} \right)^{N/2} \int_0^0 \exp \left[-\frac{m}{2i\epsilon} \sum_{i=1}^N ((y_i - y_{i-1})^2 - \omega^2 \epsilon^2 y_i^2) \right] dy_1 dy_2 \cdots dy_{N-1}$$

Note that $y_0 = y_N = 0$. We express the sum in the exponent in terms of the vector and matrix notation

$$\sum_{i=1}^N ((y_i - y_{i-1})^2 - \omega^2 \epsilon^2 y_i^2) = \mathbf{y}^T M_{N-1} \mathbf{y}$$

where

$$\mathbf{y}^T = (y_1, y_2, \dots, y_{N-2}, y_{N-1})$$

and

$$M_{N-1} = \begin{pmatrix} 2 - \omega^2 \epsilon^2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 - \omega^2 \epsilon^2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 - \omega^2 \epsilon^2 & \cdots & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \cdots & 2 - \omega^2 \epsilon^2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 - \omega^2 \epsilon^2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 - \omega^2 \epsilon^2 \end{pmatrix}$$

The recursive formula of $\det M_n$ is

$$\det M_n = (2 - \omega^2 \epsilon^2) \det M_{n-1} - \det M_{n-2},$$

and the characteric equation is

$$x^2 - (2 - \omega^2 \epsilon^2)x + 1 = 0$$

The solution of this equation is

$$x = \left(1 - \frac{\omega^2 \epsilon^2}{2}\right) \pm i\omega\epsilon \sqrt{1 - \frac{\omega^2 \epsilon^2}{4}} \equiv \cos \theta \pm i \sin \theta$$

Note that,

$$\begin{aligned} \det M_1 &= 2 - \omega^2 \epsilon^2 = 2 \cos \theta = \frac{\sin 2\theta}{\sin \theta} \\ \det M_2 &= (2 - \omega^2 \epsilon^2)^2 - 2(2 - \omega^2 \epsilon^2) = \frac{\sin 3\theta}{\sin \theta} \\ \det M_3 &= 2 \cos \theta \cdot \frac{\sin 3\theta}{\sin \theta} - \frac{\sin 2\theta}{\sin \theta} = \frac{\sin 4\theta}{\sin \theta} \end{aligned}$$

We can show

$$\det M_{N-1} = \frac{\sin N\theta}{\sin \theta}$$

by mathematical induction. Hece,

$$\begin{aligned} F(T) &= \lim_{\epsilon \rightarrow 0} \sqrt{\frac{m}{2\pi i \epsilon} \frac{1}{\det M_{N-1}}} = \lim_{\epsilon \rightarrow 0} \sqrt{\frac{m}{2\pi i \epsilon} \frac{\sin \theta}{\sin N\theta}} \\ &= \sqrt{\frac{m\omega}{2\pi i \sin T\omega}} \end{aligned}$$

where we have used

$$T = N\epsilon, \quad \lim_{\epsilon \rightarrow 0} \sin \theta = \lim_{\epsilon \rightarrow 0} \theta = \omega\epsilon, \quad \text{and} \quad \lim_{\epsilon \rightarrow 0} \sin N\theta = \sin N\omega\epsilon = \sin T\omega.$$

Finally the kernel for the harmonic oscillator is

$$K(b; a) = \sqrt{\frac{m\omega}{2\pi i \sin T\omega}} \exp \left\{ \frac{i}{\hbar} \left[\frac{m\omega}{2 \sin \omega T} \left((x_a^2 + x_b^2) \cos \omega T - 2x_a x_b \right) \right] \right\}$$

4 The Fokker-Planck Equation

4.1 The solution of the Fokker-Planck equation

The Fokker-Planck equation for Orstein-Uhlenbeck process is given by

$$\frac{\partial}{\partial t} P(v, t) = \gamma \left[\frac{\partial}{\partial v} (vP(v, t)) + \frac{1}{\beta m} \frac{\partial^2}{\partial v^2} P(v, t) \right] \quad (10)$$

We can solve it using the integral transform method.

Let

$$P(v, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikv} F(k, t) dk$$

Then,

$$\frac{\partial}{\partial t} P(v, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikv} \frac{\partial}{\partial t} F(k, t) dk$$

$$\frac{\partial^2}{\partial v^2} P(v, t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} k^2 e^{ikv} F(k, t) dk$$

The special care must be made in the next calculation.

$$\frac{\partial}{\partial v} (vP(v, t)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial}{\partial v} (ve^{ikv}) F(k, t) dk$$

Replacing the v in the parentheses by $-i \frac{\partial}{\partial k}$ gives

$$\begin{aligned} \frac{\partial}{\partial v} (vP(v, t)) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial}{\partial v} \left(-i \frac{\partial}{\partial k} e^{ikv} \right) F(k, t) dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(-i \frac{\partial}{\partial k} \frac{\partial}{\partial v} e^{ikv} \right) F(k, t) dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial k} (ke^{ikv}) \right) F(k, t) dk \end{aligned}$$

The integration by parts finally gives

$$\frac{\partial}{\partial v} (vP(v, t)) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikv} k \frac{\partial}{\partial k} F(k, t) dk \quad (11)$$

Then the Fourier transform of the differential equation (10) is

$$\frac{\partial}{\partial t} F(k, t) = -\gamma \left[k \frac{\partial}{\partial k} F(k, t) + \frac{k^2}{\beta m} F(k, t) \right]$$

Dividing by $F(k, t)$, we obtain

$$\frac{\partial}{\partial t} \ln F(k, t) = -\gamma \left[k \frac{\partial}{\partial k} \ln F(k, t) + \frac{k^2}{\beta m} \right] \quad (12)$$

If we make the Gaussian anzats

$$\ln F(k, t) = -ikm(t) - \frac{k^2}{2} S(t) \quad (13)$$

with the unknown function $m(t)$ and $S(t)$, we have

$$-ik\dot{m}(t) - \frac{k^2}{2} \dot{S}(t) = ik\gamma m(t) + k^2 \gamma S(t) - k^2 \frac{\gamma}{\beta m}$$

Comparing equal power of k , we find

$$\begin{aligned} \dot{m}(t) &= -\gamma m(t) \\ \dot{S}(t) &= -2\gamma \left(S(t) - \frac{1}{\beta m} \right) \end{aligned}$$

We choose as the initial condition

$$P(v, 0) = \delta(v - v_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik(v-v_0)} dk,$$

meaning that

$$F(k, 0) = \exp(-ikv_0),$$

namely,

$$m(0) = v_0, \quad S(0) = 0.$$

The solution is

$$m(t) = v_0 e^{-\gamma t}, \quad (14)$$

$$S(t) = \frac{1}{\beta m} (1 - e^{-2\gamma t}). \quad (15)$$

The substitution of these into equation (12) gives

$$\begin{aligned} P(v, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left[i(v - m(t))k - \frac{k^2}{2} S(t) \right] dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left[-\frac{S(t)}{2} \left(k - i \frac{(v - m(t))}{S(t)} \right)^2 - \frac{(v - m(t))^2}{2S(t)} \right] dk \end{aligned}$$

After the integration, we obtain

$$P(v, t) = \frac{1}{\sqrt{2\pi S(t)}} \exp \left[-\frac{(v - m(t))^2}{2S(t)} \right]$$

By substituting equations (14) and (15), we obtain eventually

$$P(v, t) = \sqrt{\frac{\beta m}{2\pi(1 - e^{-2\gamma t})}} \exp \left[-\frac{\beta m}{2} \frac{(v - v_0 e^{-\gamma t})^2}{(1 - e^{-2\gamma t})} \right]$$

If $t = \epsilon \ll 1$, then $1 - e^{-2\gamma\epsilon} \simeq 2\gamma\epsilon$ and $e^{-\gamma\epsilon} \simeq 1 - \gamma\epsilon$, and $P(v, t)$ becomes

$$P(v, \epsilon) \approx \sqrt{\frac{\beta m}{4\pi\gamma\epsilon}} \exp \left[-\frac{\beta m}{4\gamma\epsilon} \frac{(v - v_0(1 - \gamma\epsilon))^2}{4\gamma\epsilon} \right] \quad (16)$$

4.2 The Path Integral Formula

The Green's function satisfies

$$\frac{\partial}{\partial t} G(v, t; v_0, 0) = \gamma \left[\frac{\partial}{\partial v} v + \frac{1}{\beta m} \frac{\partial^2}{\partial v^2} \right] G(v, t; v_0, 0)$$

The formal solution is

$$G(v, \epsilon; v_0, 0) = \exp \left\{ \epsilon \gamma \left[\frac{\partial}{\partial v} v + \frac{1}{\beta m} \frac{\partial^2}{\partial v^2} \right] \right\} \delta(v - v_0)$$

The calculation $\frac{\partial}{\partial v} v \delta(v - v_0)$ must be done as the equation (11) has been

derived, namely,

$$\begin{aligned}
\frac{\partial}{\partial v} v \delta(v - v_0) &= \frac{1}{2\pi} \int \frac{\partial}{\partial v} v e^{ik(v-v_0)} dk \\
&= \frac{1}{2\pi} \int e^{-ikv_0} \left(\frac{\partial}{\partial v} v e^{ikv} \right) dk \\
&= \frac{1}{2\pi} \int e^{-ikv_0} \left(\frac{\partial}{\partial v} \frac{1}{i} \frac{\partial}{\partial k} e^{ikv} \right) dk \\
&= -\frac{1}{2\pi} \int \left(\frac{1}{i} \frac{\partial}{\partial k} e^{-ikv_0} \right) \left(\frac{\partial}{\partial v} e^{ikv} \right) dk \\
&= \frac{1}{2\pi} \int i v_0 k e^{ik(v-v_0)} dk
\end{aligned}$$

Therefore we get

$$\begin{aligned}
G(v, \epsilon; v_0, 0) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left\{ \epsilon \gamma \left[\frac{\partial}{\partial v} v + \frac{1}{\beta m} \frac{\partial^2}{\partial v^2} \right] \right\} e^{ik(v-v_0)} dk \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left\{ \epsilon \gamma \left[i v_0 k - \frac{k^2}{\beta m} \right] \right\} e^{ik(v-v_0)} dk \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left[i \epsilon \gamma v_0 k - \frac{\epsilon \gamma}{\beta m} k^2 + i k(v - v_0) \right] dk \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left[-\frac{\epsilon \gamma}{\beta m} k^2 + i k(v - v_0 + \epsilon \gamma v_0) \right] dk \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left[-\frac{\epsilon \gamma}{\beta m} \left(k - i \frac{\beta m (v - v_0 (1 - \epsilon \gamma))}{2 \epsilon \gamma} \right)^2 \right. \\
&\quad \left. - \frac{\beta m (v - v_0 (1 - \epsilon \gamma))^2}{4 \epsilon \gamma} \right] dk
\end{aligned}$$

We finally obtain after the Gaussian integral of k

$$G(v, \epsilon; v_0, 0) = \sqrt{\frac{\beta m}{4\pi \gamma \epsilon}} \exp \left[-\frac{\beta m (v - v_0 (1 - \epsilon \gamma))^2}{4 \epsilon \gamma} \right],$$

which is the same as equation (16).

In gengeral

$$\begin{aligned} G(v_i, t_i; v_{i-1}, t_{i-1}) &= \exp \left\{ \epsilon \gamma \left[\frac{\partial}{\partial v_i} v_i + \frac{1}{\beta m} \frac{\partial^2}{\partial v_i^2} \right] \right\} \delta(v_i - v_{i-1}) \\ &= \sqrt{\frac{\beta m}{4\pi\gamma\epsilon}} \exp \left[- \frac{\beta m (v_i - v_{i-1}(1 - \epsilon\gamma))^2}{4\epsilon\gamma} \right] \end{aligned}$$

Thus the path-integral representation of the Fokker-Planck solution is

$$\begin{aligned} G(v, t; v_0, 0) &= \lim_{\epsilon \rightarrow 0} \left(\frac{\beta m}{4\pi\gamma\epsilon} \right)^{N/2} \int \cdots \int \\ &\exp \left[- \frac{\beta m}{4\epsilon\gamma} \sum_{i=1}^N (v_i - v_{i-1}(1 - \epsilon\gamma))^2 \right] dv_1 \cdots dv_{N-1} \end{aligned}$$

References

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R.P. Feynman and A.R. Hibbs, Quantum Mechanics and Path Integrals (McGrawHill, New York, 1965).
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