不定積分 $\displaystyle\int \frac{1}{x^3+1}\,dx$ の計算

$$\begin{aligned}
\int \frac{1}{x^3+1}\,dx &= \int \frac{1}{(x+1)(x^2-x+1)}\,dx \\
&= \frac{1}{3}\int \left\{ \frac{1}{x+1} - \frac{x-2}{x^2-x+1} \right\} dx \\
&= \frac{1}{3}\ln|x+1| - \frac{1}{6}\int \frac{2x-4}{x^2-x+1}\,dx \qquad (1)
\end{aligned}$$

ここで、

$$\begin{aligned}
\int \frac{2x-4}{x^2-x+1}\,dx &= \int \frac{(x^2-x+1)'}{x^2-x+1}\,dx - 3\int \frac{1}{x^2-x+1}\,dx \\
&= \ln(x^2-x+1) - 3\int \frac{1}{x^2-x+1}\,dx \qquad (2)
\end{aligned}$$

である。さらに、

$$\begin{aligned}
\int \frac{1}{x^2-x+1}\,dx &= \int \frac{1}{\left(x-e^{-\frac{\pi}{3}i}\right)\left(x-e^{\frac{\pi}{3}i}\right)}\,dx \\
&= \frac{i}{\sqrt{3}}\int \left(\frac{1}{x-e^{-\frac{\pi}{3}i}} - \frac{1}{x-e^{\frac{\pi}{3}i}} \right) dx \\
&= \frac{i}{\sqrt{3}}\ln\left(\frac{x-e^{-\frac{\pi}{3}i}}{x-e^{\frac{\pi}{3}i}} \right) \qquad (3)
\end{aligned}$$

ここで、

$$\begin{aligned}
x-e^{-\frac{\pi}{3}i} &= \left(x-\frac{1}{2}\right) + \frac{\sqrt{3}}{2}\,i \\
&= i\sqrt{x^2-x+1}\left\{ \frac{\sqrt{3}}{2\sqrt{x^2-x+1}} - i\frac{2x-1}{2\sqrt{x^2-x+1}} \right\} \\
&= i\sqrt{x^2-x+1}\,\{\cos\varphi - i\sin\varphi\} \\
&= i\sqrt{x^2-x+1}\;e^{-i\varphi}
\end{aligned}$$

とおくと、

$$\cos\varphi = \frac{\sqrt{3}}{2\sqrt{x^2-x+1}}, \quad \sin\varphi = \frac{2x-1}{2\sqrt{x^2-x+1}}$$

すなわち、

$$\tan\varphi = \frac{2x-1}{\sqrt{3}}, \quad \text{または、} \quad \varphi = \arctan\left(\frac{2x-1}{\sqrt{3}}\right) \qquad (4)$$

また、

$$x-e^{\frac{\pi}{3}i} = -i\sqrt{x^2-x+1}\;e^{i\varphi}$$

となるので、

$$\ln\left(\frac{x-e^{-\frac{\pi}{3}i}}{x-e^{\frac{\pi}{3}i}} \right) = \ln\left(-e^{-2i\varphi}\right) = \ln e^{-2i\varphi+\pi i} = -2i\varphi + \pi i \qquad (5)$$

を得る。

(1) ～ (5) 式より、結局、

$$\int \frac{1}{x^3+1}\,dx = \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln(x^2-x+1) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)$$

を得る。