

問題  $\int \frac{1}{x^4+1} dx$  を求めよ

解答

$$\begin{aligned}\int \frac{1}{x^4+1} dx &= \int \frac{1}{(x^2 - e^{\frac{\pi}{2}i})(x^2 + e^{\frac{\pi}{2}i})} dx \\&= -\frac{i}{2} \int \left\{ \frac{1}{x^2 - e^{\frac{\pi}{2}i}} - \frac{1}{x^2 + e^{\frac{\pi}{2}i}} \right\} dx \\&= -\frac{i}{2} \int \left\{ \frac{1}{(x - e^{\frac{\pi}{4}i})(x + e^{\frac{\pi}{4}i})} - \frac{1}{(x - e^{-\frac{\pi}{4}i})(x + e^{-\frac{\pi}{4}i})} \right\} dx \\&= -\frac{i}{2} \int \left\{ \frac{e^{-\frac{\pi}{4}i}}{2} \left[ \frac{1}{x - e^{\frac{\pi}{4}i}} - \frac{1}{x + e^{\frac{\pi}{4}i}} \right] - \frac{e^{\frac{\pi}{4}i}}{2} \left[ \frac{1}{x - e^{-\frac{\pi}{4}i}} - \frac{1}{x + e^{-\frac{\pi}{4}i}} \right] \right\} dx \\&= -\frac{i}{2} \left\{ \frac{e^{-\frac{\pi}{4}i}}{2} \ln \left( \frac{x - e^{\frac{\pi}{4}i}}{x + e^{\frac{\pi}{4}i}} \right) - \frac{e^{\frac{\pi}{4}i}}{2} \ln \left( \frac{x - e^{-\frac{\pi}{4}i}}{x + e^{-\frac{\pi}{4}i}} \right) \right\} \\&= -\frac{i}{2} \left\{ \frac{1}{2\sqrt{2}} \ln \left( \frac{(x - e^{\frac{\pi}{4}i})(x + e^{-\frac{\pi}{4}i})}{(x + e^{\frac{\pi}{4}i})(x - e^{-\frac{\pi}{4}i})} \right) - \frac{i}{2\sqrt{2}} \ln \left( \frac{(x - e^{\frac{\pi}{4}i})(x - e^{-\frac{\pi}{4}i})}{(x + e^{\frac{\pi}{4}i})(x + e^{-\frac{\pi}{4}i})} \right) \right\} \\&= -\frac{i}{4\sqrt{2}} \ln \left( \frac{(x - e^{\frac{\pi}{4}i})(x + e^{-\frac{\pi}{4}i})}{(x + e^{\frac{\pi}{4}i})(x - e^{-\frac{\pi}{4}i})} \right) - \frac{1}{4\sqrt{2}} \ln \left( \frac{(x - e^{\frac{\pi}{4}i})(x - e^{-\frac{\pi}{4}i})}{(x + e^{\frac{\pi}{4}i})(x + e^{-\frac{\pi}{4}i})} \right)\end{aligned}$$

ここで、

$$\begin{aligned}(x - e^{\frac{\pi}{4}i})(x + e^{-\frac{\pi}{4}i}) &= (x^2 - 1) - \sqrt{2} ix \\&= -i\sqrt{x^4+1} \left( \frac{\sqrt{2} x}{\sqrt{x^4+1}} + \frac{x^2-1}{\sqrt{x^4+1}} i \right) \\&= -i\sqrt{x^4+1} e^{i\varphi}\end{aligned}$$

とおくと、

$$\cos \varphi = \frac{\sqrt{2} x}{\sqrt{x^4+1}}, \quad \sin \varphi = \frac{x^2-1}{\sqrt{x^4+1}}$$

すなわち、

$$\tan \varphi = \frac{x^2-1}{\sqrt{2} x}, \quad \text{または、} \quad \varphi = \arctan \left( \frac{x^2-1}{\sqrt{2} x} \right)$$

また、

$$(x + e^{\frac{\pi}{4}i})(x - e^{-\frac{\pi}{4}i}) = i\sqrt{x^4+1} e^{-i\varphi}$$

なので、

$$\ln \left( \frac{(x - e^{\frac{\pi}{4}i})(x + e^{-\frac{\pi}{4}i})}{(x + e^{\frac{\pi}{4}i})(x - e^{-\frac{\pi}{4}i})} \right) = \ln (-e^{2\varphi}) = \ln e^{2\varphi i + \pi i} = 2\varphi i + \pi i$$

一方、

$$\begin{aligned}(x - e^{\frac{\pi}{4}i})(x - e^{-\frac{\pi}{4}i}) &= |x^2 - \sqrt{2} x + 1| \\(x + e^{\frac{\pi}{4}i})(x + e^{-\frac{\pi}{4}i}) &= |x^2 + \sqrt{2} x + 1|\end{aligned}$$

となるので、結局、

$$\int \frac{1}{x^4+1} dx = \frac{1}{2\sqrt{2}} \arctan \left( \frac{x^2-1}{\sqrt{2} x} \right) - \frac{1}{4\sqrt{2}} \ln \left| \frac{x^2 - \sqrt{2} x + 1}{x^2 + \sqrt{2} x + 1} \right| + C$$

を得る。