問題　不定積分 $\displaystyle\int \frac{\boldsymbol{1}}{\boldsymbol{x^4+1}}\,\boldsymbol{dx}$ を求めよ

解答

$$
\begin{aligned}
\int \frac{1}{x^4+1}\,dx &= \int \frac{1}{(x^2-e^{\frac{\pi}{2}i})(x^2+e^{\frac{\pi}{2}i})}\,dx \\
&= -\frac{i}{2}\int\left\{\frac{1}{x^2-e^{\frac{\pi}{2}i}}-\frac{1}{x^2+e^{\frac{\pi}{2}i}}\right\}dx \\
&= -\frac{i}{2}\int\left\{\frac{1}{(x-e^{\frac{\pi}{4}i})(x+e^{\frac{\pi}{4}i})}-\frac{1}{(x-e^{-\frac{\pi}{4}i})(x+e^{-\frac{\pi}{4}i})}\right\}dx \\
&= -\frac{i}{2}\int\left\{\frac{e^{-\frac{\pi}{4}i}}{2}\left[\frac{1}{x-e^{\frac{\pi}{4}i}}-\frac{1}{x+e^{\frac{\pi}{4}i}}\right]-\frac{e^{\frac{\pi}{4}i}}{2}\left[\frac{1}{x-e^{-\frac{\pi}{4}i}}-\frac{1}{x+e^{-\frac{\pi}{4}i}}\right]\right\}dx \\
&= -\frac{i}{2}\left\{\frac{e^{-\frac{\pi}{4}i}}{2}\ln\left(\frac{x-e^{\frac{\pi}{4}i}}{x+e^{\frac{\pi}{4}i}}\right)-\frac{e^{\frac{\pi}{4}i}}{2}\ln\left(\frac{x-e^{-\frac{\pi}{4}i}}{x+e^{-\frac{\pi}{4}i}}\right)\right\} \\
&= -\frac{i}{2}\left\{\frac{1}{2\sqrt{2}}\ln\left(\frac{(x-e^{\frac{\pi}{4}i})(x+e^{-\frac{\pi}{4}i})}{(x+e^{\frac{\pi}{4}i})(x-e^{-\frac{\pi}{4}i})}\right)-\frac{i}{2\sqrt{2}}\ln\left(\frac{(x-e^{\frac{\pi}{4}i})(x-e^{-\frac{\pi}{4}i})}{(x+e^{\frac{\pi}{4}i})(x+e^{-\frac{\pi}{4}i})}\right)\right\} \\
&= -\frac{i}{4\sqrt{2}}\ln\left(\frac{(x-e^{\frac{\pi}{4}i})(x+e^{-\frac{\pi}{4}i})}{(x+e^{\frac{\pi}{4}i})(x-e^{-\frac{\pi}{4}i})}\right)-\frac{1}{4\sqrt{2}}\ln\left(\frac{(x-e^{\frac{\pi}{4}i})(x-e^{-\frac{\pi}{4}i})}{(x+e^{\frac{\pi}{4}i})(x+e^{-\frac{\pi}{4}i})}\right)
\end{aligned}
$$

ここで、

$$
\begin{aligned}
(x-e^{\frac{\pi}{4}i})(x+e^{-\frac{\pi}{4}i}) &= (x^2-1)-\sqrt{2}\,ix \\
&= -i\sqrt{x^4+1}\,\left(\frac{\sqrt{2}\,x}{\sqrt{x^4+1}}+\frac{x^2-1}{\sqrt{x^4+1}}\,i\right) \\
&= -i\sqrt{x^4+1}\,e^{i\varphi}
\end{aligned}
$$

とおくと、

$$
\cos\varphi = \frac{\sqrt{2}\,x}{\sqrt{x^4+1}},\quad \sin\varphi = \frac{x^2-1}{\sqrt{x^4+1}}
$$

すなわち、

$$
\tan\varphi = \frac{x^2-1}{\sqrt{2}\,x},\quad \text{または、}\quad \varphi = \arctan\left(\frac{x^2-1}{\sqrt{2}\,x}\right)
$$

また、

$$
(x+e^{\frac{\pi}{4}i})(x-e^{-\frac{\pi}{4}i}) = i\sqrt{x^4+1}\,e^{-i\varphi}
$$

なので、

$$
\ln\left(\frac{(x-e^{\frac{\pi}{4}i})(x+e^{-\frac{\pi}{4}i})}{(x+e^{\frac{\pi}{4}i})(x-e^{-\frac{\pi}{4}i})}\right) = \ln\left(-e^{2\varphi}\right) = \ln e^{2\varphi i+\pi i} = 2\varphi i+\pi i
$$

一方、

$$
\begin{aligned}
(x-e^{\frac{\pi}{4}i})(x-e^{-\frac{\pi}{4}i}) &= |x^2-\sqrt{2}\,x+1| \\
(x+e^{\frac{\pi}{4}i})(x+e^{-\frac{\pi}{4}i}) &= |x^2+\sqrt{2}\,x+1|
\end{aligned}
$$

となるので、結局、

$$
\int \frac{1}{x^4+1}\,dx = \frac{1}{2\sqrt{2}}\;\arctan\left(\frac{x^2-1}{\sqrt{2}\,x}\right)-\frac{1}{4\sqrt{2}}\ln\left|\frac{x^2-\sqrt{2}\,x+1}{x^2+\sqrt{2}\,x+1}\right|+C
$$

を得る。