ECE2150J Introduction to Circuits
Chapter 10. Sinusoidal Steady-State Analysis

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10.1 Introduction

In this chapter, we will see how nodal analysis, mesh analysis, Thevenin's theorem, Norton's theorem, superposition, and source transformations are applied in analyzing ac circuits.

Quick Summary of Chapter 10 → Basically, all methods that we have learned in previous chapters are applicable to ac circuits, too.

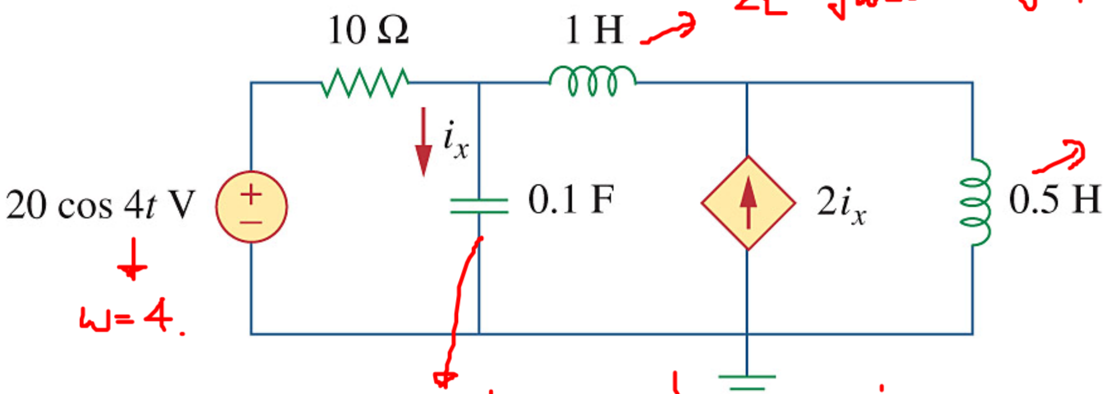
Steps to analyze AC circuits

1. Transform the circuit to the phasor domain.
2. Find the circuit output using nodal analysis, mesh analysis, superposition, etc.
3. Transform the resulting phasor to the time domain.

Step 1 is not necessary if the problem is specified in the frequency domain.
Step 2, the analysis is performed in the same manner as dc circuit analysis except that complex numbers are involved.

Example 10.1 Find i_x in the circuit of Fig. 10.1 using nodal analysis.

$Z_L = j\omega L = j\,4\times1 = j4$

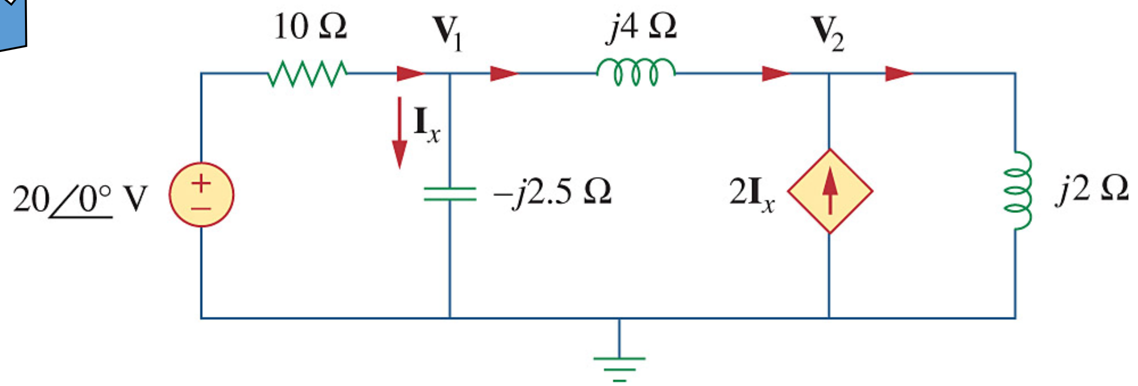


$\rightarrow j4\times0.5 = j2$

$\omega = 4.$

$Z_C = \frac{1}{j\omega C} = \frac{1}{j\times4\times0.1} = -j2.5$

ω = 4

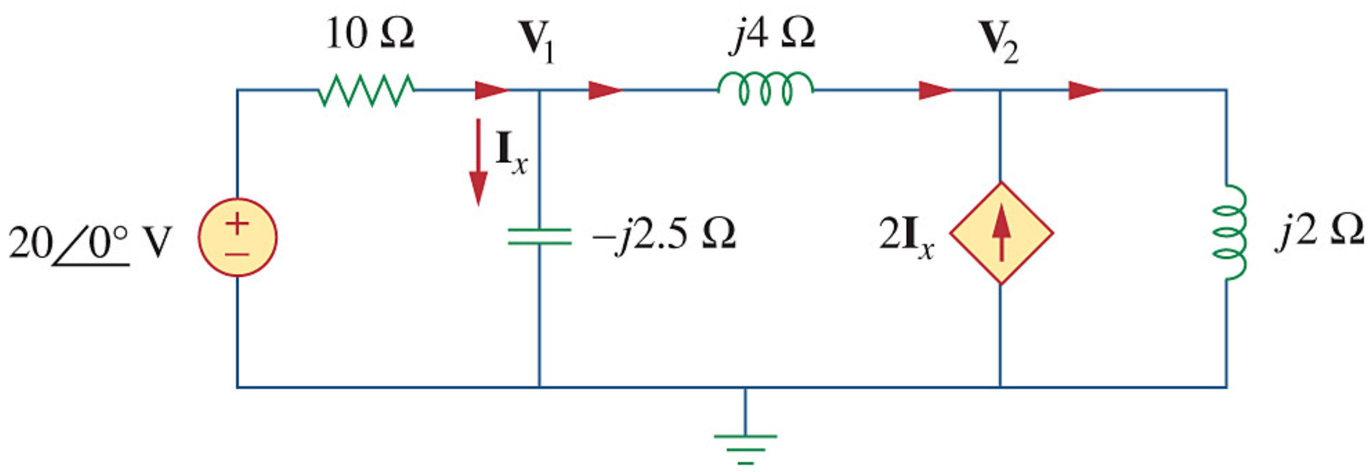


$$20\cos 4t \Rightarrow 20\angle 0^\circ \text{ V}, \ \omega = 4 \text{ rad/s}$$

$$1 \text{ H} \Rightarrow j\omega L = j4 \ \Omega$$

$$0.5 \text{ H} \Rightarrow j\omega L = j2 \ \Omega$$

$$0.1 \text{ F} \Rightarrow \frac{1}{j\omega C} = -j2.5 \ \Omega$$



<u>Review of nodal analysis</u>:

1. Non-reference node and reference node
2. KCL equations for non-reference nodes
3. Solve simultaneous KCL equations

Or by inspection method: **YV=I** (GV = I in DC)
Where G: conductance matrix; V: input vector; I: output vector

Recall: Nodal analysis inspection

$$\begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1N} \\ G_{21} & G_{22} & \cdots & G_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ G_{N1} & G_{N2} & \cdots & G_{NN} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{bmatrix}$$

G_{kk} = Sum of the conductances connected to node k

$G_{kj} = G_{jk}$ = Negative of the sum of the conductances directly connecting nodes k and j, $k \neq j$.

v_k = Unknown voltage at node k

i_k = Sum of all independent current sources directly connected to node k, with currents entering the node treated as positive

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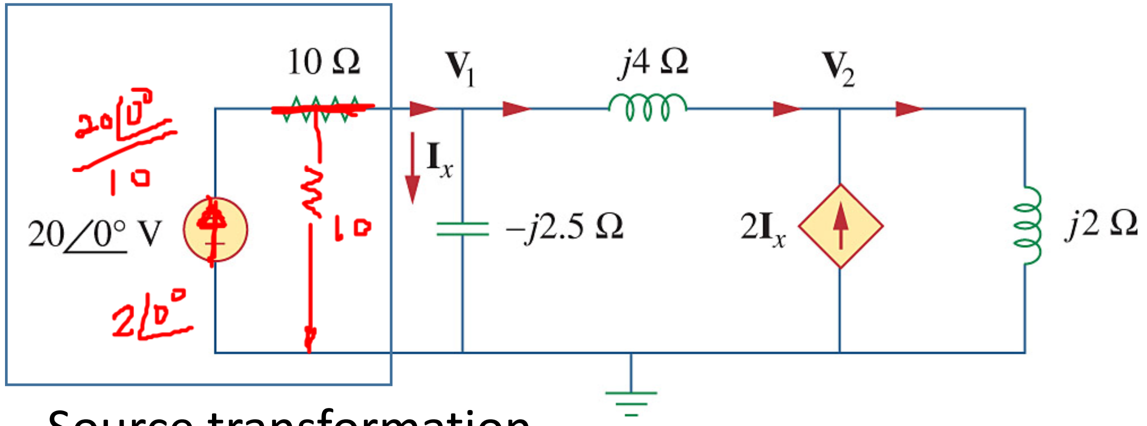
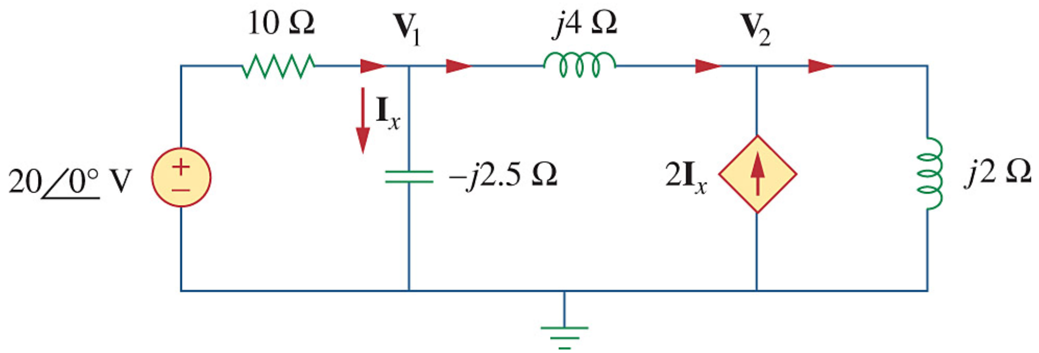


Source transformation

$$
\begin{array}{cc}
\Sigma_i G_i & -G_{ij} \\
\end{array}
$$

$$
\begin{bmatrix} \frac{1}{10} + \frac{1}{j4} + \frac{1}{-j2.5} & -\frac{1}{j4} \\ -\frac{1}{j4} & \frac{1}{j4} + \frac{1}{j2} \end{bmatrix} \begin{bmatrix} \tilde{V}_1 \\ \tilde{V}_2 \end{bmatrix} = \begin{bmatrix} \frac{20\angle 0^\circ}{10} \\ 2\tilde{I}_x \end{bmatrix}
$$

$$
\tilde{I}_x = \frac{\tilde{V}_1}{-j2.5}
$$



$$\begin{bmatrix} \frac{1}{10}+\frac{1}{j4}+\frac{1}{-j2.5} & -\frac{1}{j4} \\ -\frac{1}{j4} & \frac{1}{j4}+\frac{1}{j2} \end{bmatrix}\begin{bmatrix} \tilde{V}_1 \\ \tilde{V}_2 \end{bmatrix}=\begin{bmatrix} \frac{20\angle 0^\circ}{10} \\ 2\tilde{I}_x \end{bmatrix}$$

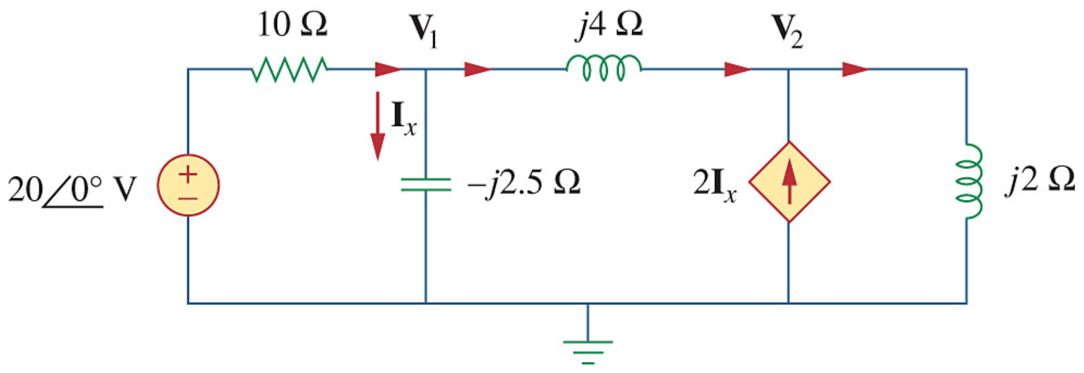
$$\Rightarrow \begin{bmatrix} \frac{1}{10}+\frac{1}{j4}+\frac{1}{-j2.5} & -\frac{1}{j4} \\ -\frac{1}{j4}-\frac{2}{-j2.5} & \frac{1}{j4}+\frac{1}{j2} \end{bmatrix}\begin{bmatrix} \tilde{V}_1 \\ \tilde{V}_2 \end{bmatrix}=\begin{bmatrix} \frac{20\angle 0^\circ}{10} \\ 0 \end{bmatrix}$$

$$\tilde{I}_x=\frac{\tilde{V}_1}{-j2.5}$$

Express I_x in terms of V_1 or V_2

$$\begin{bmatrix} 2+j3 & j5 \\ 11 & 15 \end{bmatrix}\begin{bmatrix} \tilde{V}_1 \\ \tilde{V}_2 \end{bmatrix}=\begin{bmatrix} 40 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2+j3 & j5 \\ 11 & 15 \end{bmatrix}\begin{bmatrix} \tilde{V}_1 \\ \tilde{V}_2 \end{bmatrix} = \begin{bmatrix} 40 \\ 0 \end{bmatrix}$$



$$\Delta = \begin{vmatrix} 2+j3 & j5 \\ 11 & 15 \end{vmatrix} = 30 - j10$$

$$\Delta_1 = \begin{vmatrix} 40 & j5 \\ 0 & 15 \end{vmatrix} = 600$$

$$\tilde{V}_1 = \frac{\Delta_1}{\Delta} = \frac{600}{30-j10} = \frac{60}{3-j} \approx \frac{60}{3.1623\angle -18.43^\circ}$$

$$\approx 18.9735\angle 18.43^\circ \text{ (V)}$$

$$\tilde{I}_x = \frac{\tilde{V}_1}{-j2.5} = \frac{18.9735\angle 18.43^\circ}{2.5\angle -90^\circ}$$

$$\approx 7.59\angle 108.43^\circ \text{ (A)}$$

$$i_x = 7.59\cos(4t + 108.43^\circ) \text{ (A)}$$

10.3 Mesh Analysis

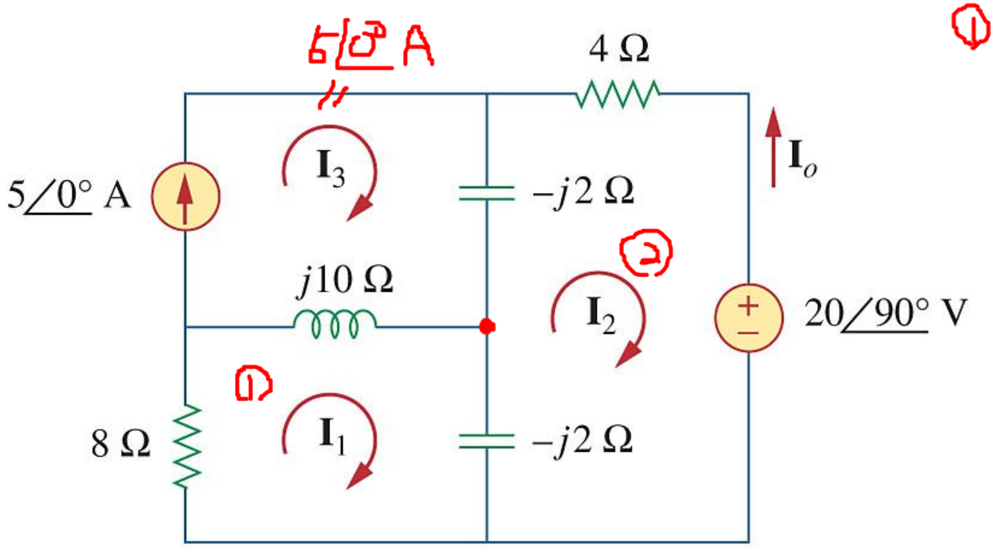
Kirchhoff's voltage law (KVL) forms the basis of mesh analysis. The validity of KVL for ac circuits was shown in Chapter 9.

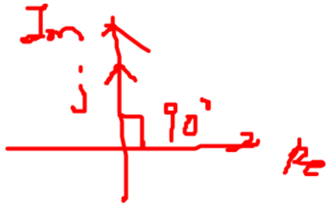
Review of mesh analysis:

1. Mesh currents
    -deal with the case with current source first
2. KVL equations for meshes
3. Solve simultaneous KVL equations

Or by inspection method: **ZI**=**V** (RI = V in DC)
Where R: resistance matrix; I: input vector; V: output vector

Example 10.3 Determine current $\tilde{I}_o$ in the circuit of Fig. 10.7 using mesh analysis.



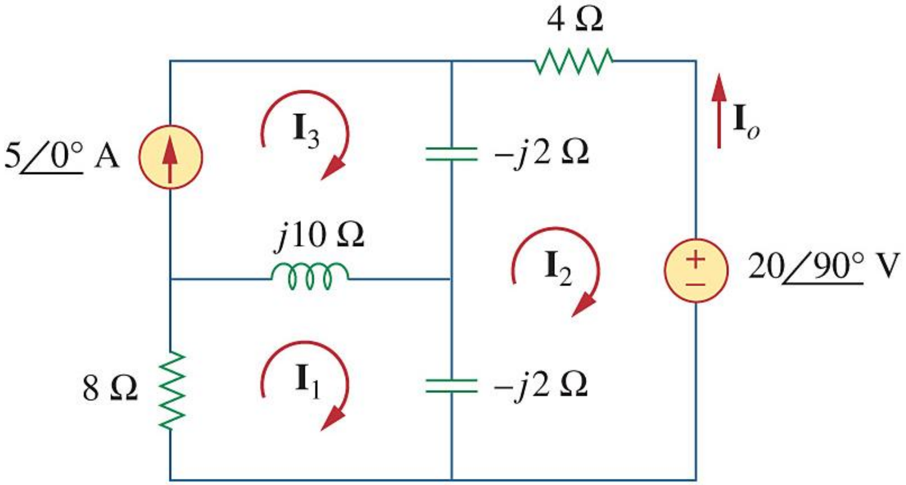


① $8I_1 + j10(I_1 - 5) - j2(I_1 - I_2) = 0$.

$(8+j8)I_1 + j2I_2 = j50$

② $-j2(I_2 - 5) + 4I_2 + j20 - j2(I_2 - I_1) = 0$.

$j2I_1 + (4 - j4)I_2 = -j30$.



Solution :

$$\tilde{I}_3 = 5\angle 0° = 5 \text{ (A)}$$

$$20\angle 90° = j20$$

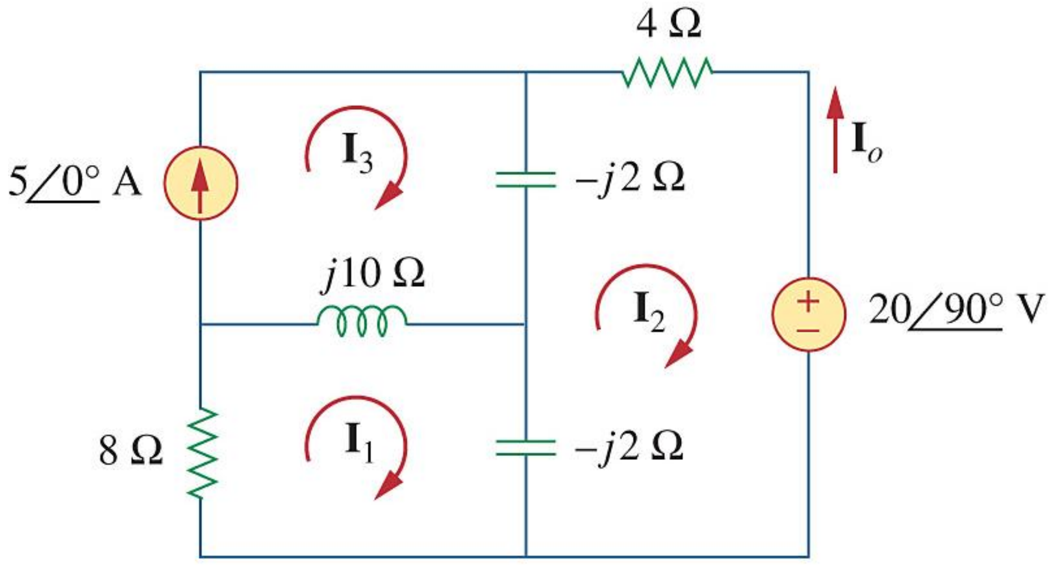
$$\begin{array}{ccc} \Sigma_i \mathbf{R}_i & -\mathbf{R}_{ij} & \end{array}$$
$$\begin{bmatrix} 8 + j10 - j2 & j2 & -j10 \\ j2 & 4 - j2 - j2 & j2 \end{bmatrix} \begin{bmatrix} \tilde{I}_1 \\ \tilde{I}_2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ -j20 \end{bmatrix}$$

I_3 is known, so just neglect 3rd row

$$\begin{bmatrix} 8+j8 & j2 & -j10 \\ j2 & 4-j4 & j2 \end{bmatrix} \begin{bmatrix} \tilde{I}_1 \\ \tilde{I}_2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ -j20 \end{bmatrix}$$

$$\begin{bmatrix} 8+j8 & j2 \\ j2 & 4-j4 \end{bmatrix} \begin{bmatrix} \tilde{I}_1 \\ \tilde{I}_2 \end{bmatrix} = \begin{bmatrix} j50 \\ -j30 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 8+j8 & j2 \\ j2 & 4-j4 \end{vmatrix} = 68$$



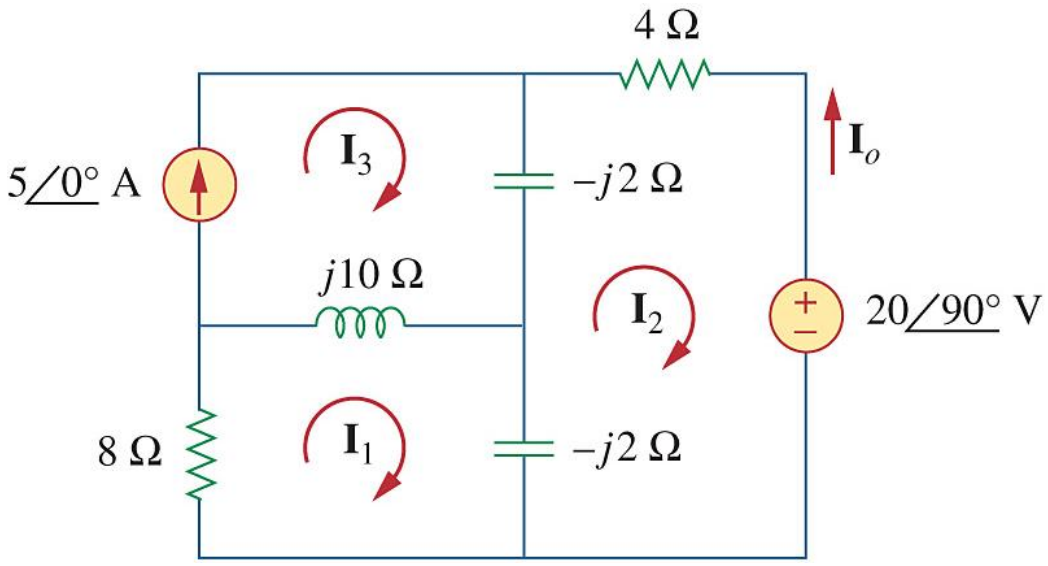
$$\Delta_2 = \begin{vmatrix} 8+j8 & j50 \\ j2 & -j30 \end{vmatrix} = 340 - j240$$

$$\approx 416.1730\angle -35.22°$$

$$\tilde{I}_2 = \frac{\Delta_2}{\Delta} = \frac{416.1730\angle -35.22°}{68}$$

$$\approx 6.12\angle -35.22° \text{ (A)}$$

$$\tilde{I}_o = -\tilde{I}_2 = 6.12\angle 144.78° \text{ (A)}$$



Only step 2 is needed in this problem.

10.4 Superposition Theorem

The superposition theorem becomes important if the circuit has sources operating at **different frequencies**. In this case, **since the impedances depend on frequency**, we must have a different frequency domain circuit for each frequency. The total response must be obtained by **adding the individual responses** in the time domain.

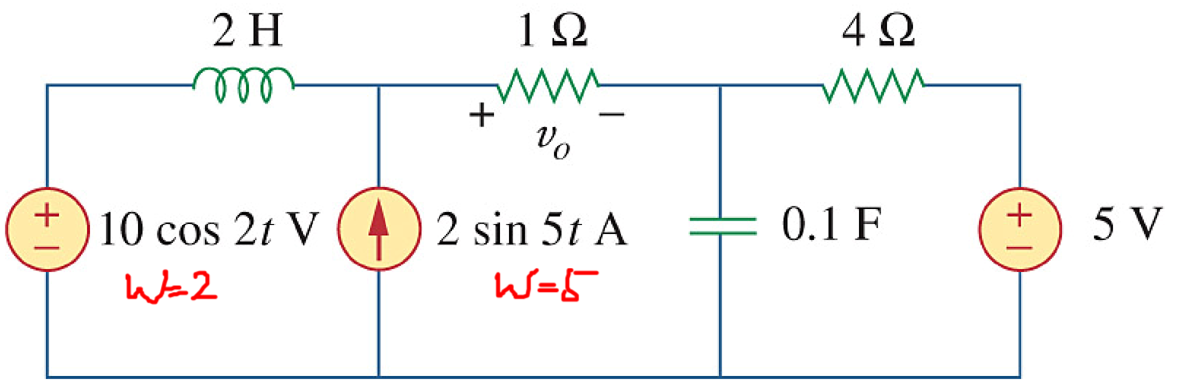
Review of superposition

- Superposition principle is based on additivity.

- It states that whenever a linear system is excited, or driven, by **more than one independent source**, the total response is the sum of the individual responses.

- The principle of superposition helps us to analyze a linear circuit with **more than one independent source** by calculating the contribution of each independent source separately and adding algebraically all the contributions to find the total contribution.

Example 10.6 Find v_o of the circuit of Fig. 10.13 using the superposition theorem.



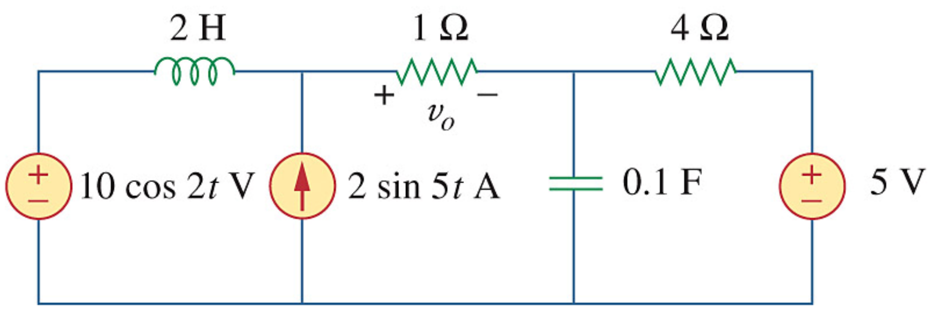
$\left.\begin{array}{l} j\omega L \\ \dfrac{1}{j\omega C} \end{array}\right\} \rightarrow$ depends on freq.

i) $\omega = 2$ $\quad 2H \rightarrow j4$

ii) $\omega = 5$ $\quad 2H \rightarrow j10.$

iii) DC. $\omega = 0$ $\quad j\omega L \rightarrow 0$

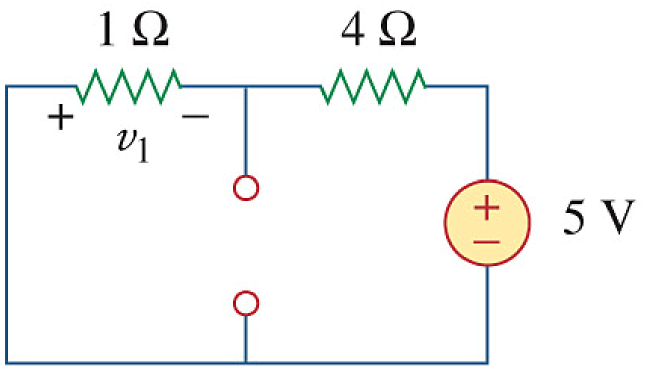
$\dfrac{1}{j\omega C} \rightarrow \infty.$



$$v_o = v_1 + v_2 + v_3$$

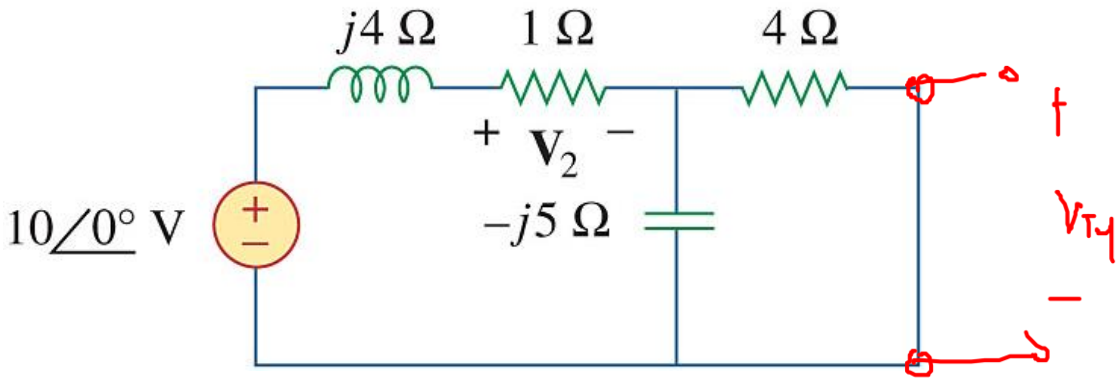
where v_1 is due to the 5-V dc source, v_2 is due to the $10\cos 2t$ V voltage source, and v_3 is due to the $2\sin 5t$ current source.

(1) From 5V source



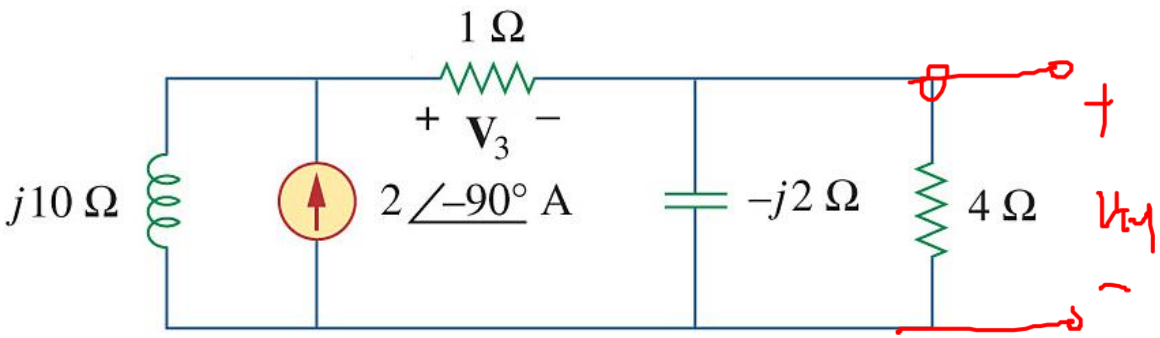
(2) From 10cos(2t) V source

ω=2 rad/s

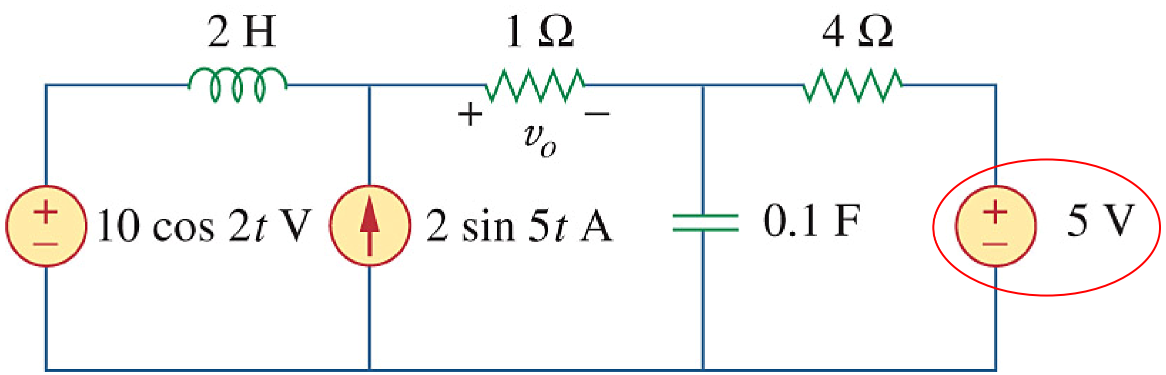


(3) From 2sin(5t) V source

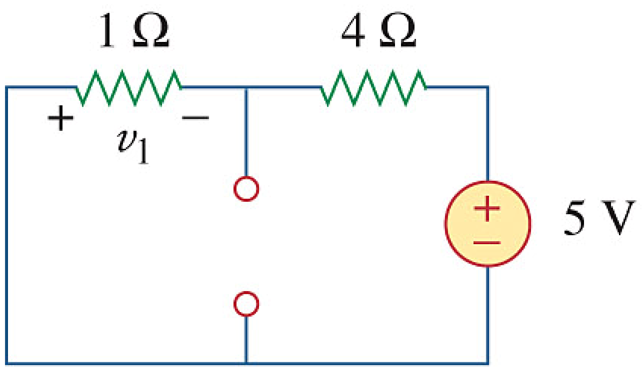
ω=5 rad/s



(1) From 5V source



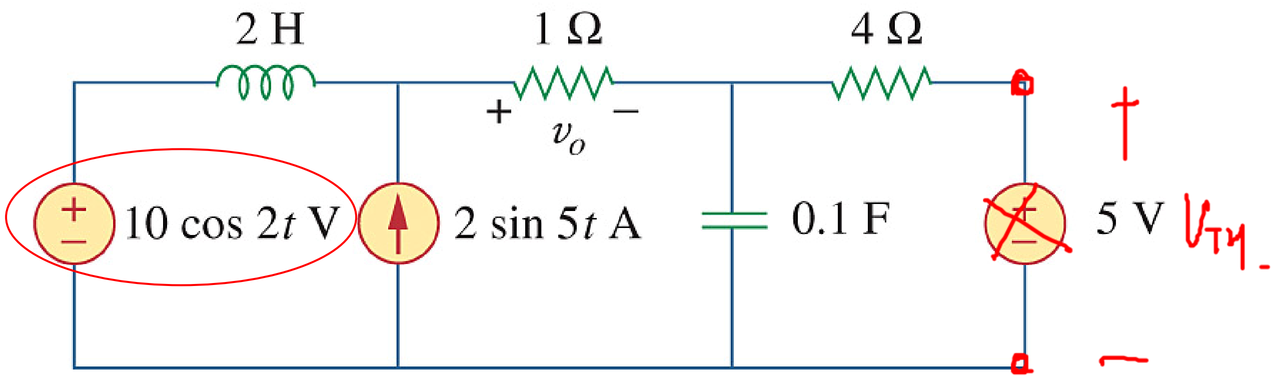
An inductor becomes short in a DC steady state



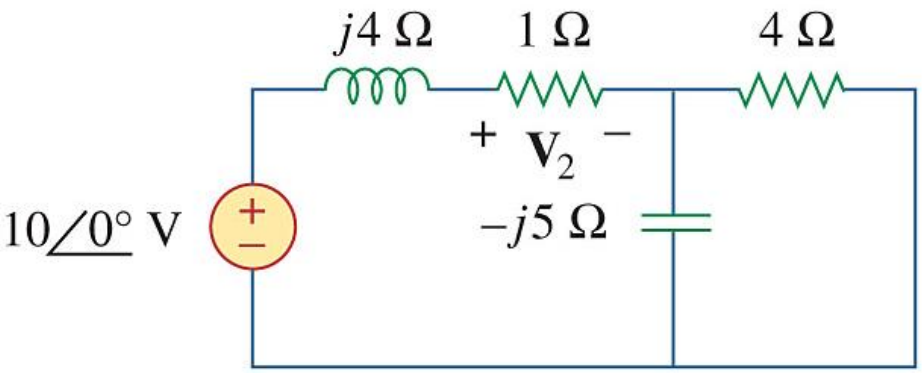
$$(1)\ v_1 = -5 \times \frac{1}{1+4} = -1\ (\mathrm{V})$$

(2) From 10cos(2t) V source

Z depends on w



Impedance is based on ω=2 rad/s

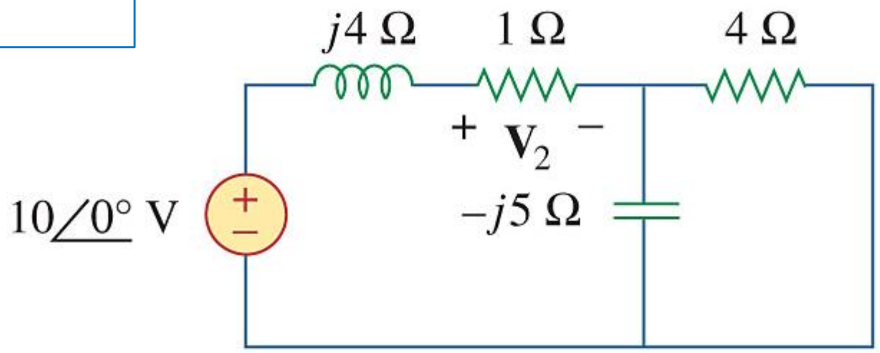


$$10\cos 2t \text{ V} \Rightarrow 10\angle 0° \text{ V}, \omega = 2 \text{ rad/s}$$

$$2 \text{ H} \Rightarrow j\omega L = j4\ (\Omega)$$

$$0.1 \text{ F} \Rightarrow 1/(j\omega C) = -j5\ (\Omega)$$

Complex number calculation:
For +,– calculation: rectangular form suggested
For ×,÷ calculation: polar form suggested



$$\tilde{V}_2 = 10\angle 0° \frac{1}{j4+1+(-j5)\parallel 4}$$

$$(-j5)\parallel 4 = \frac{(-j5)\times 4}{(-j5)+4} \approx \frac{20\angle -90°}{6.4031\angle -51.34°} \approx 3.1235\angle -38.66° \approx 2.4390 - j1.9512$$

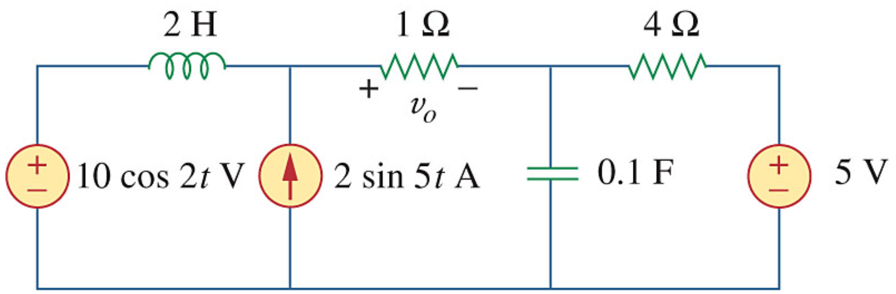
$$\tilde{V}_2 = 10\angle 0° \frac{1}{j4+1+(2.4390-j1.9512)}$$

$$= \frac{10\angle 0°}{3.4390+j2.0488} = \frac{10\angle 0°}{4.0030\angle 30.78°}$$

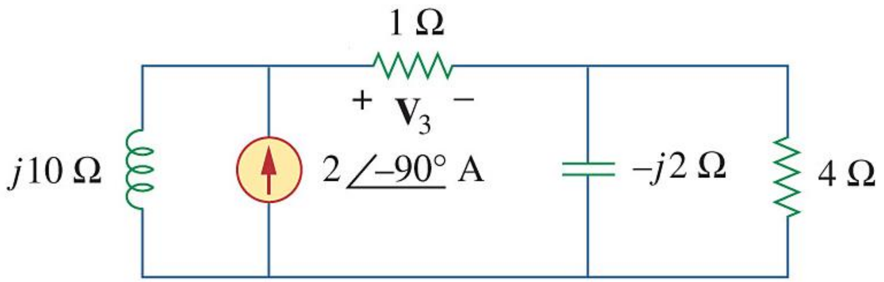
$$\approx 2.50\angle -30.78° \text{ (V)}$$

$$v_2 = 2.50\cos(2t-30.78°) \text{ (V)}$$

(3) From 2sin(5t) V source



Impedance is based on ω=5 rad/s

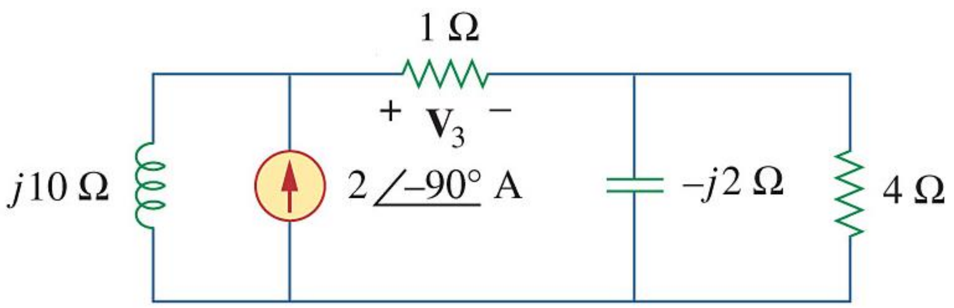


$$2\sin 5t \text{ A} \Rightarrow 2\angle -90^\circ \text{ A}, \ \omega = 5 \text{ rad/s}$$

$$2 \text{ H} \Rightarrow j\omega L = j10 \ (\Omega)$$

$$0.1 \text{ F} \Rightarrow 1/(j\omega C) = -j2 \ (\Omega)$$

sin(5t) = cos(5t–90°)



Source transformation

$$\tilde{V}_3 = \boxed{(2\angle -90°)\times(j10)}\times\frac{1}{j10+1+(-j2)\parallel 4}$$

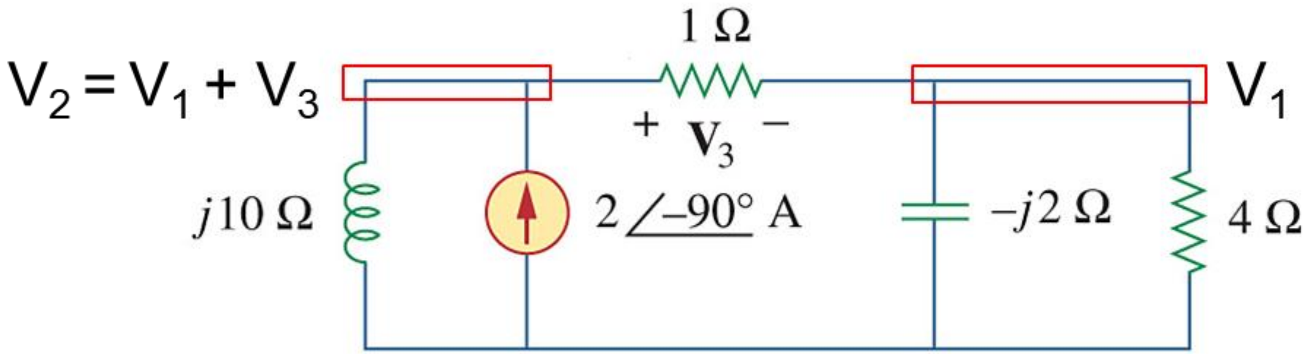
$$(-j2)\parallel 4 = \frac{(-j2)\times 4}{(-j2)+4} = 0.8 - j1.6$$

$$\tilde{V}_3 = (2\angle -90°)\times(j10)\times\frac{1}{j10+1+(0.8-j1.6)}$$

$$= \frac{20}{1.8+j8.4} \approx \frac{20}{8.5907\angle 77.91°}$$

$$\approx 2.33\angle -77.91°\ \text{(V)}$$

$$v_3 = 2.33\cos(5t - 77.91°)\ \text{V}$$



By KCL,

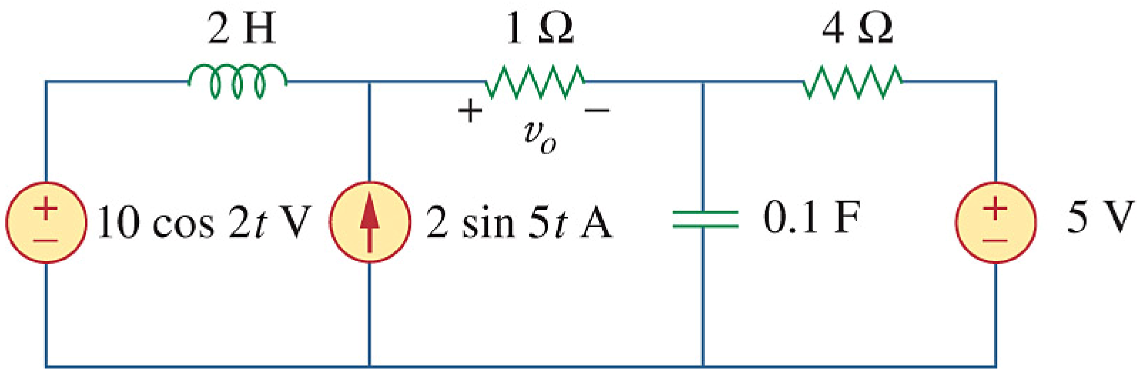
$$(1)\ \frac{V_1+V_3}{j10} - 2\angle-90° + \frac{(V_1+V_3)-V_1}{1} = 0$$

$$\rightarrow\ V_1 + V_3 - j20\angle-90° + j10V_3 = 0$$

$$(2)\frac{V_1-(V_1+V_3)}{1} + \frac{V_1}{-j2} + \frac{V_1}{4} = 0\ \ \rightarrow -4V_3 + j2V_1 + V_1 = 0$$

Therefore, $V_1 = \frac{4V_3}{1+j2}$, and then the eq(1) becomes

$$(9+j42)V_3 = 100 \rightarrow\ V_3 = \frac{100}{9+j42} = \mathbf{2.33\angle-77.9°}$$

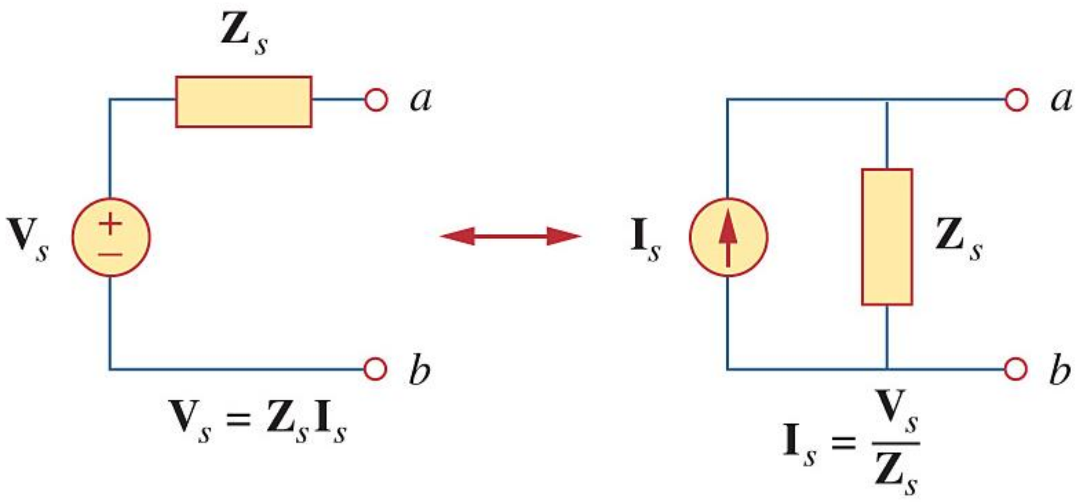


A complete response $v = v_1 + v_2 + v_3$

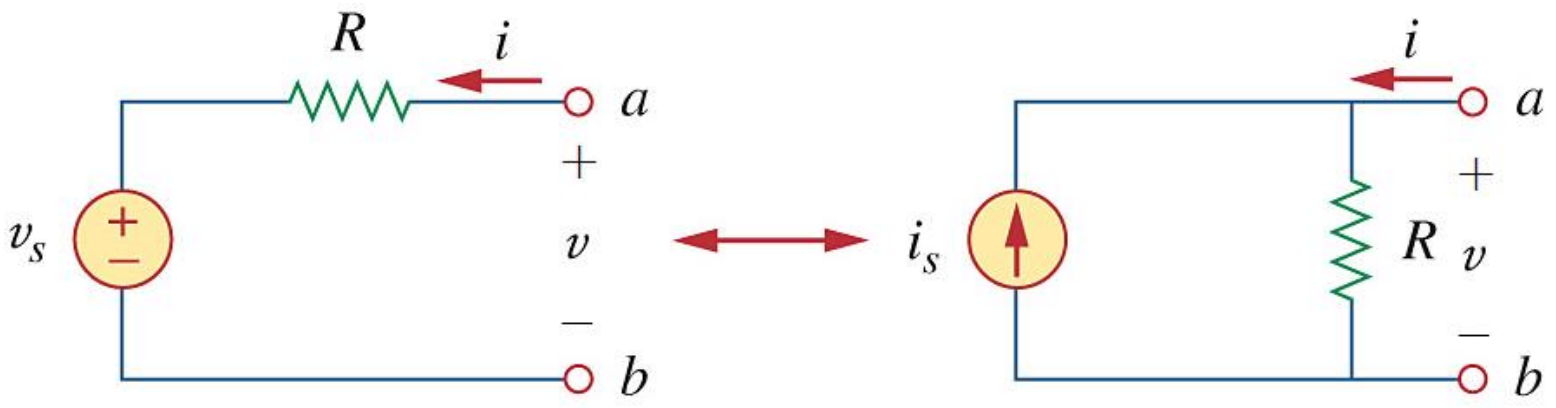
$$v = -1 + 2.50\cos(2t - 30.78°)$$
$$+2.33\cos(5t - 77.91°) \text{ (V)}$$

10.5 Source Transformation

Source transformation in the frequency domain involves transforming a voltage source in series with an impedance to a current source in parallel with an impedance, or vice versa.



Review of source transformation



$$v_s = i_s R \quad \text{or} \quad i_s = \frac{v_s}{R}$$

The source transformation is the process of replacing a **voltage source in series with a resistor** by **a current source in parallel with a resistor**, or vice versa.

Practice Problem 10.7 Find $\dot{I}_o$ in the circuit of Fig. 10.19 using the concept of source transformation.

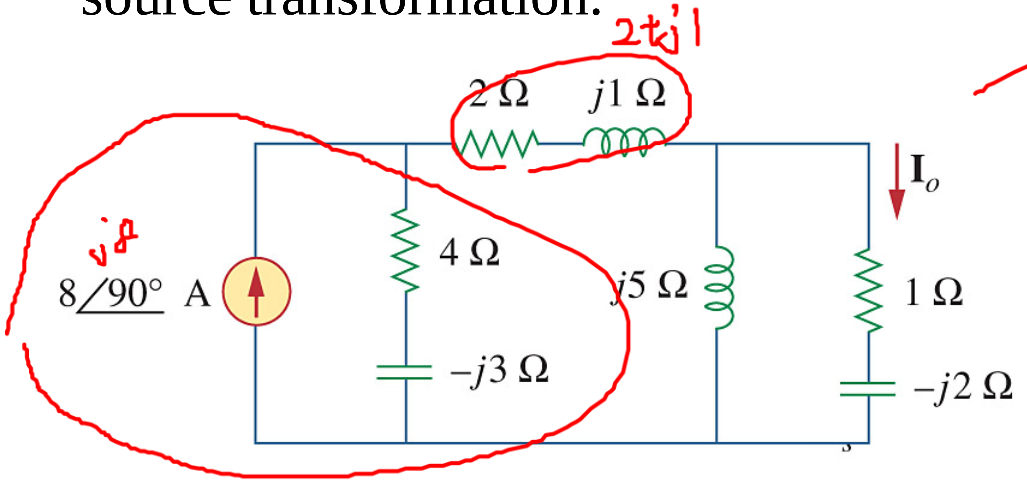
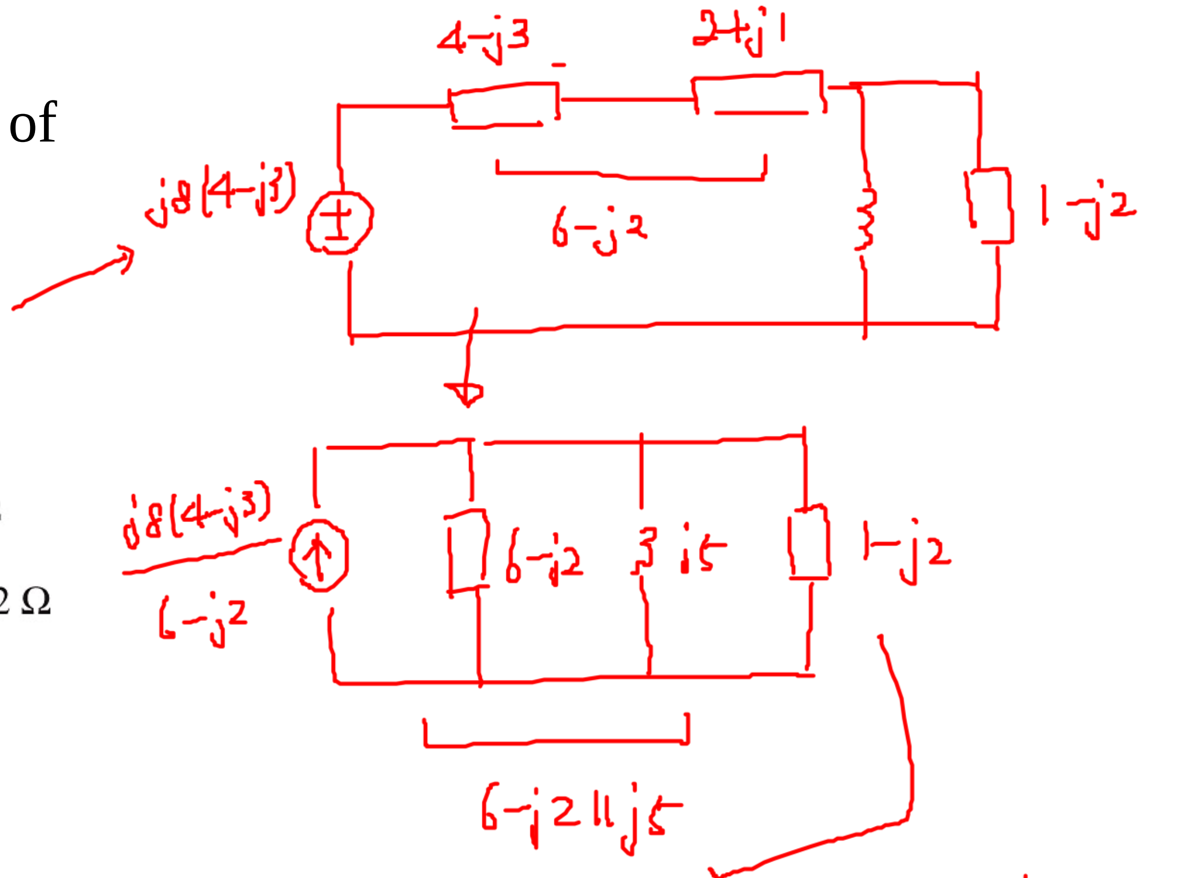


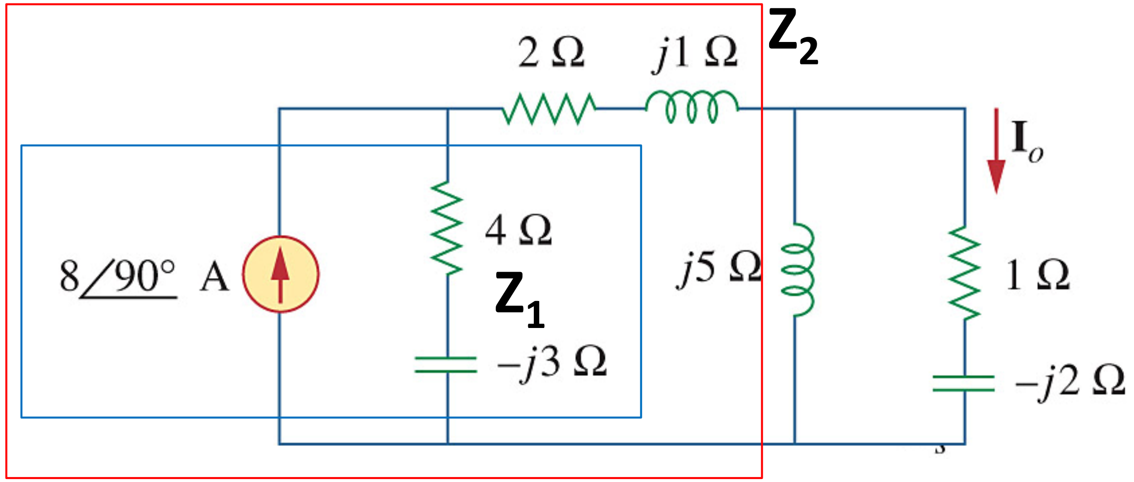
Figure 10.19

$\hookrightarrow V = IZ$

$= (4-j3)\,j8.$



current division.



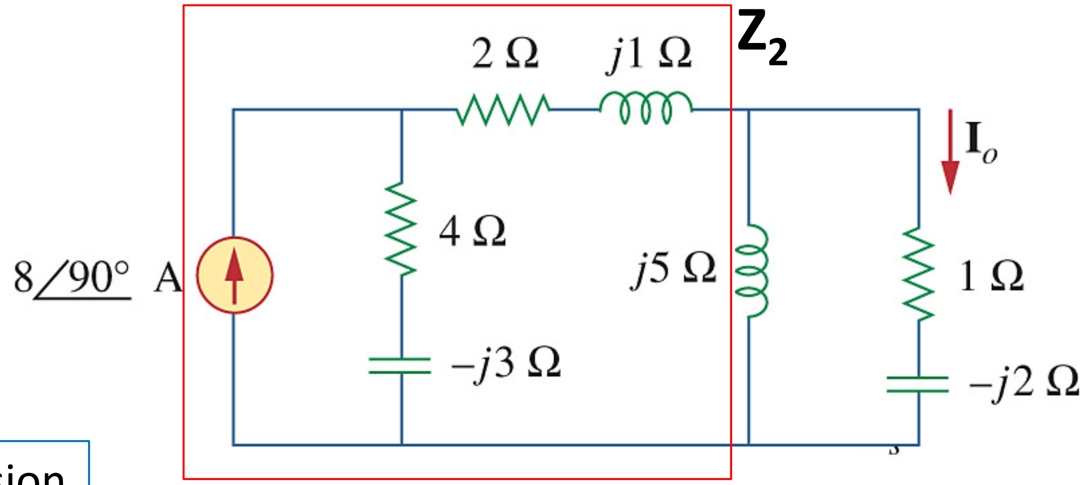
$$Z_2 = Z_1 + (2 + j1) = (4 - j3) + (2 + j1)$$

$$= 6 - j2\ (\Omega)$$

$$\begin{cases} \tilde{V}_1 = 24 + j32\ \text{V} \\ Z_2 = 6 - j2\ \Omega \end{cases} \Rightarrow \begin{cases} \tilde{I}_2 = 6.3245\angle 71.56^\circ\ \text{A} \\ Z_2 = 6 - j2\ \Omega \end{cases}$$

$$\tilde{I}_2 = \frac{\tilde{V}_1}{Z_2} = \frac{24 + j32}{6 - j2} = \frac{40\angle 53.13^\circ}{6.3246\angle -18.43^\circ}$$

$$\approx 6.3245\angle 71.56^\circ\ (\text{A})$$



Current division

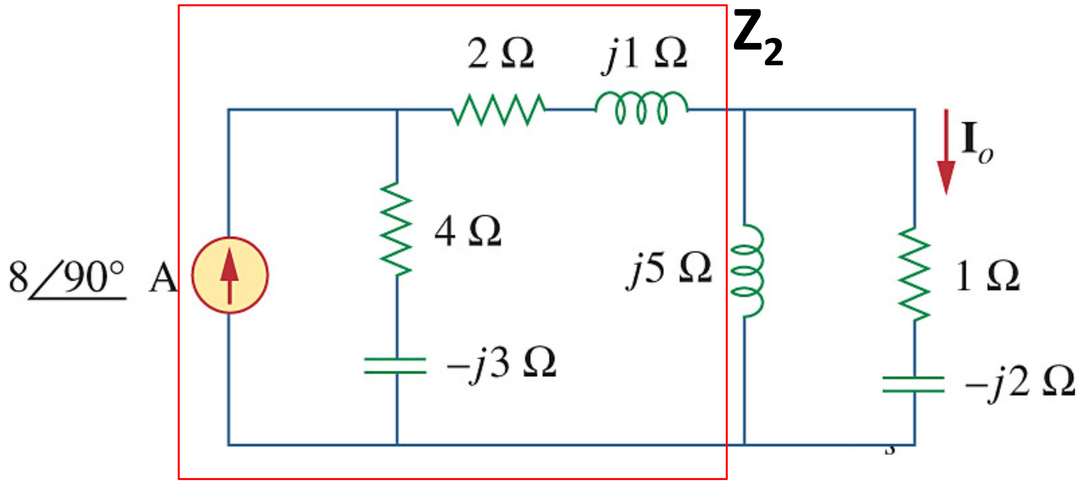
$\mathbf{I_o = I_2 \times (Y_3/(Y_1+Y_2+Y_3))}$

$$\tilde{I}_o = \tilde{I}_2 \frac{1/(1-j2)}{1/(1-j2)+1/(j5)+1/(6-j2)}$$

$$\frac{1/(1-j2)}{1/(1-j2)+1/(j5)+1/(6-j2)}$$

$$= \frac{1}{1+(1-j2)/(j5)+(1-j2)/(6-j2)}$$

$$= \frac{1}{1+(-2-j)/5+(1-j)/4}$$



$$= \frac{20}{17 - j9}$$

$$\approx 1.0398\angle 27.90^\circ$$

$$\tilde{I}_o = \tilde{I}_2 \times 1.0398\angle 27.90^\circ$$

$$= 6.3245\angle 71.56^\circ \times 1.0398\angle 27.90^\circ$$

$$\approx 6.58\angle 99.46^\circ \text{ (A)}$$

10.6 Thevenin and Norton Equivalent Circuits

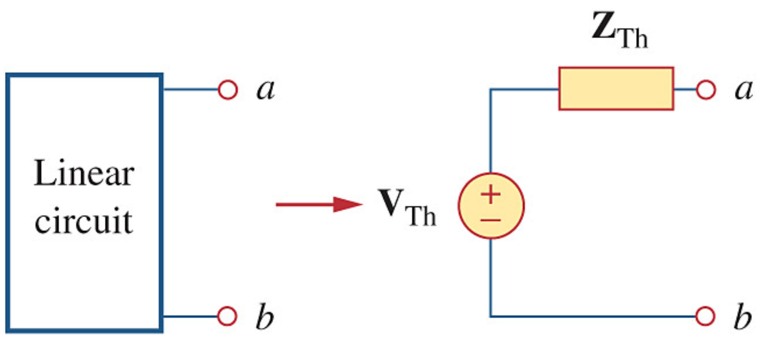


Figure 10.20 Thevenin equivalent.

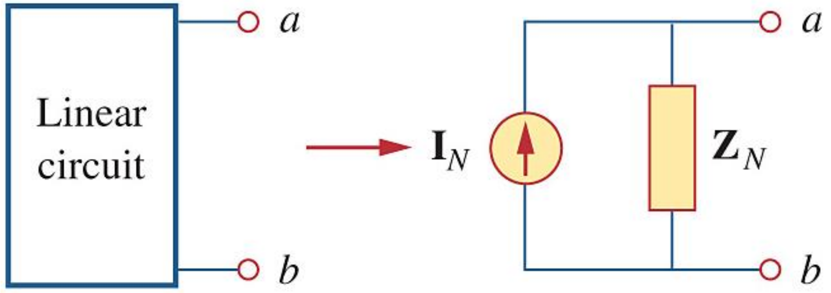
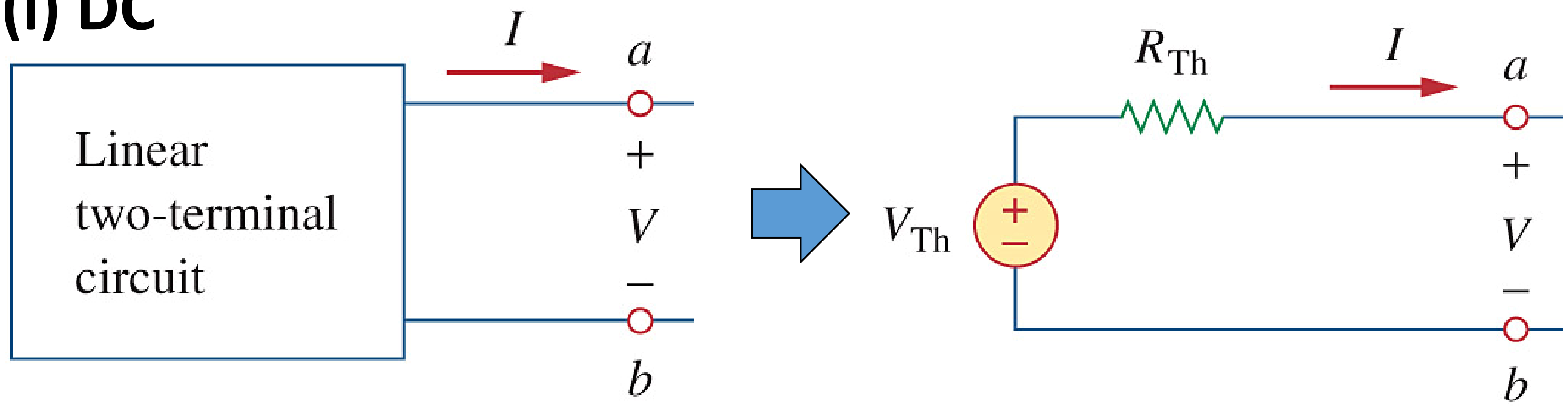


Figure 10.21 Norton equivalent.

Thevenin's and Norton's theorems are applied to ac circuits in the same way as they are to dc circuits.

Thevenin's theorems

(i) DC



(ii) AC

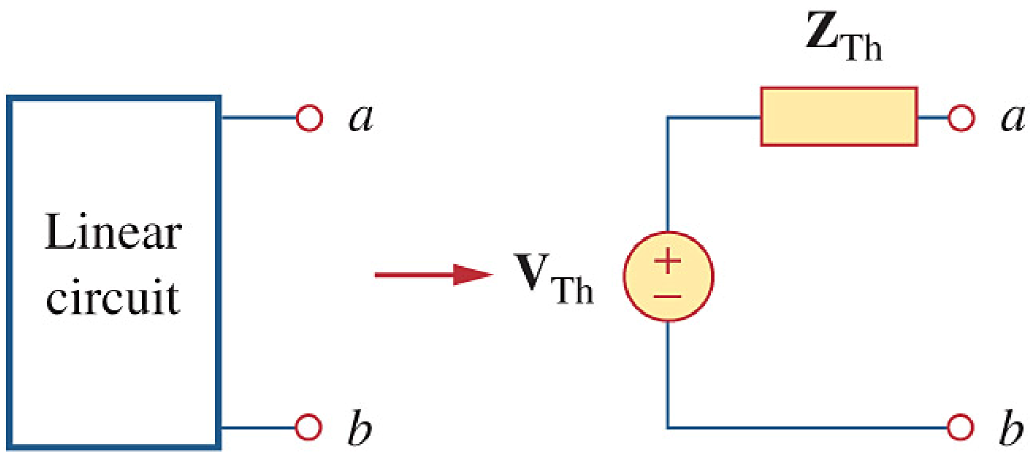
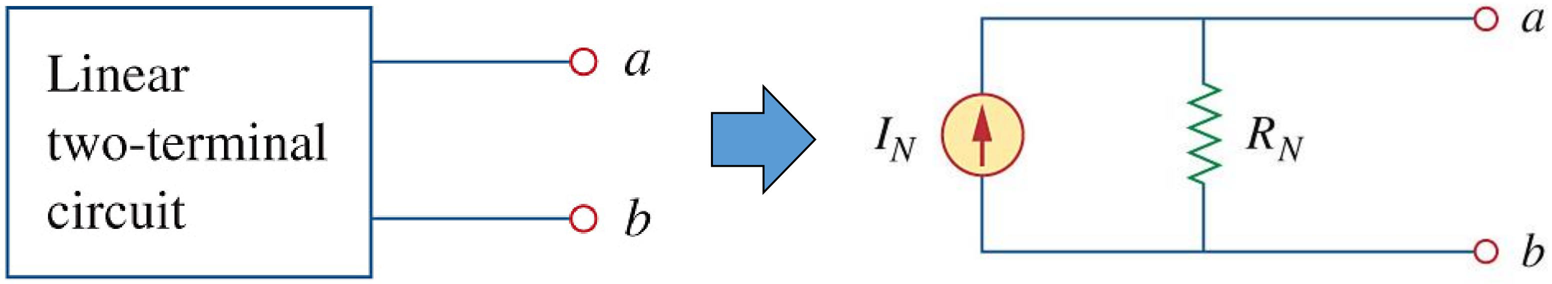


Figure 10.20 Thevenin equivalent.

Norton's theorems

(i) DC



(ii) AC

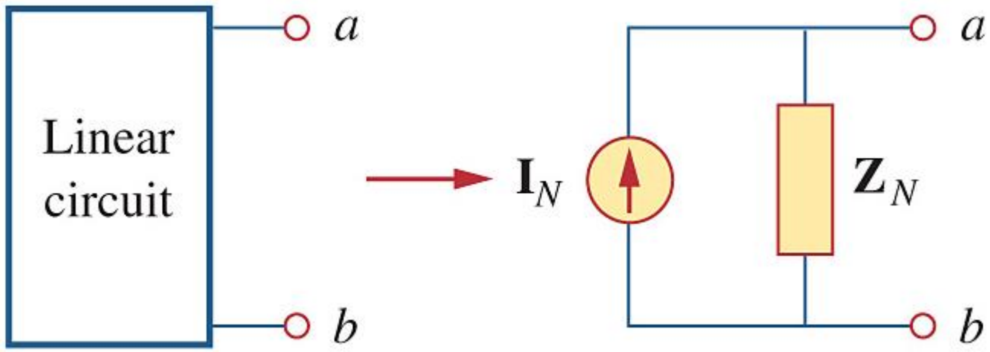
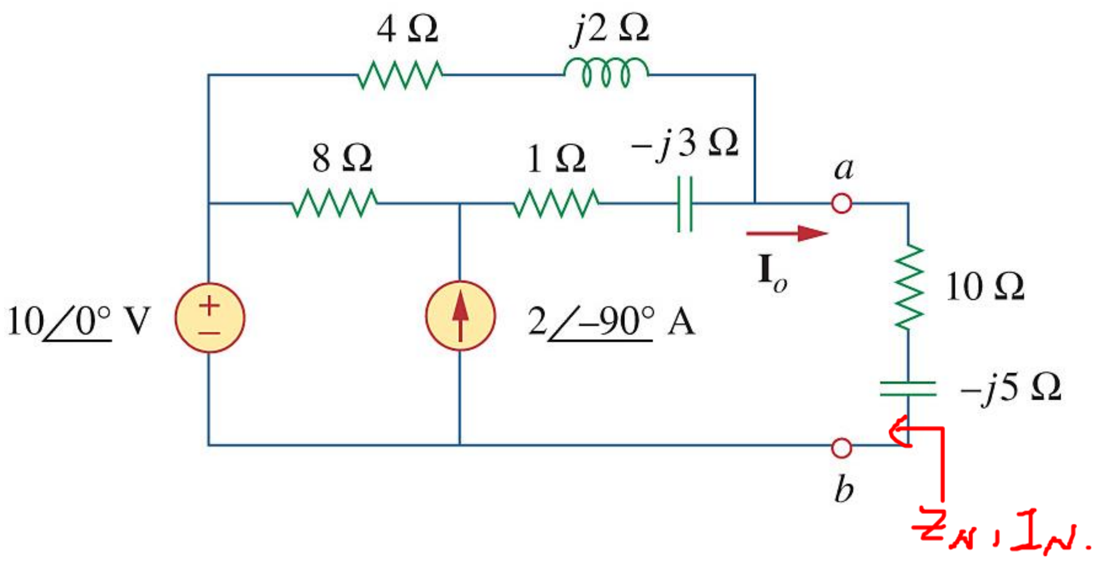


Figure 10.21 Norton equivalent.

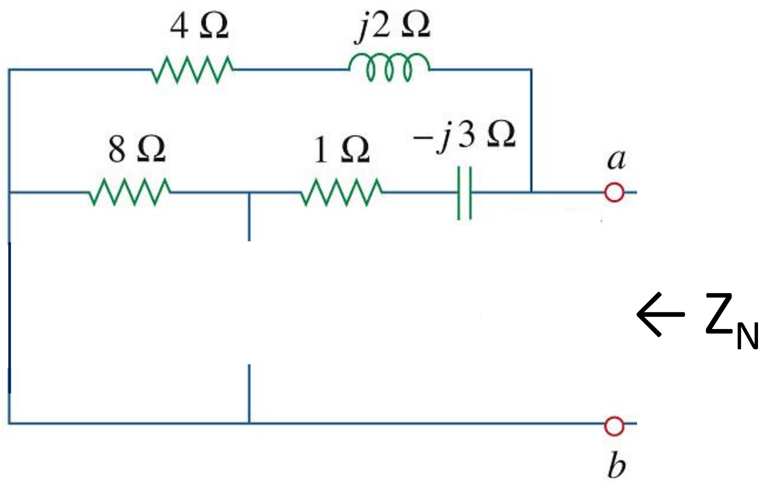
*If the circuit has **sources operating at different frequencies**, the Thevenin or Norton equivalent circuit must be determined **at each frequency**

→ Entirely different equivalent circuits, **one for each frequency**, not one equivalent circuit with equivalent sources and equivalent impedances.

Practice Problem 10.10 Determine the Norton equivalent of the circuit in Fig. 10.30 as seen from terminals *a*-*b*. Use the equivalent to find $\tilde{I}_o$.



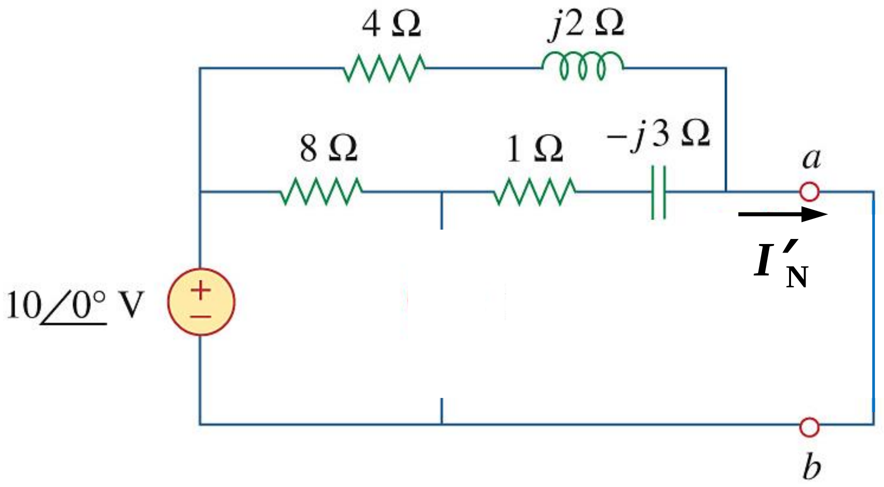
(i) Z_N



$\mathbf{Z_N}$ = (4 + j2) || (9 – j3)

$$= \frac{(4+j2)(9-j3)}{(4+j2)(9-j3)} = \frac{42+j6}{13-j} = 3.25\angle 12.53°$$

(ii) I'_N



Recall: $I_N = I_{SC}$

Use superposition to find $\tilde{I}_N$:

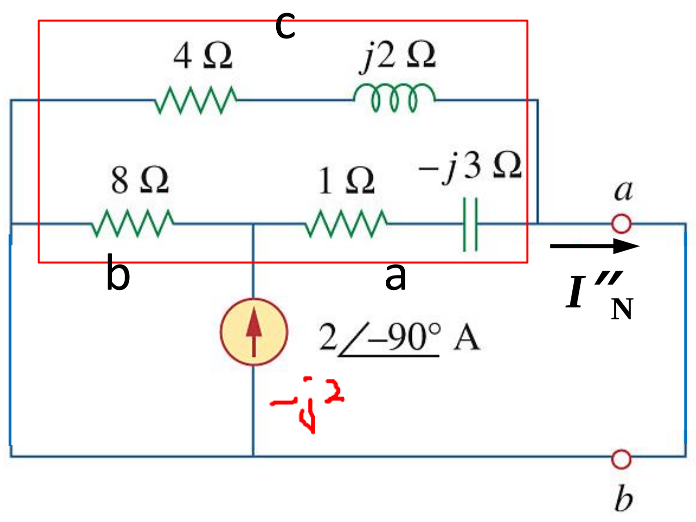
$$\tilde{I}_N = \tilde{I}'_N + \tilde{I}''_N$$

$$\tilde{I}'_N = \frac{10\angle 0^\circ}{(4+j2)\parallel(8+1-j3)}$$

$$= \frac{10}{6(7+j)/(13-j)} = \frac{9-j2}{3}$$

$$\approx 3 - j0.6667 \text{ (A)}$$

(ii) I''_N

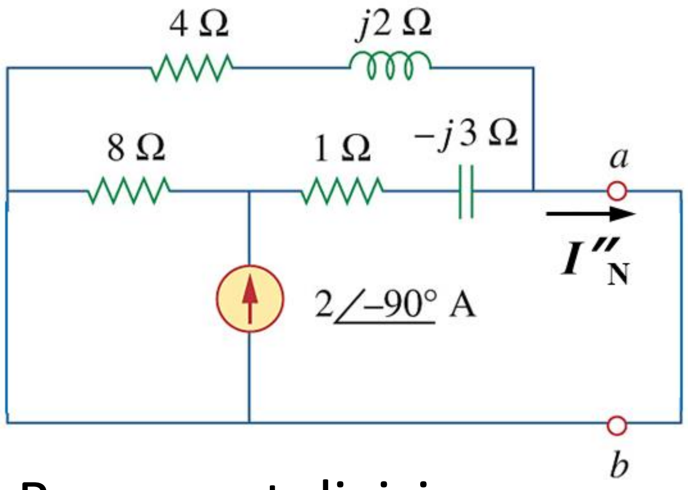


Delta → Y transform

$$Z_1 = \frac{bc}{a+b+c} = \frac{8(4+j2)}{13-j}$$

$$Z_2 = \frac{ac}{a+b+c} = \frac{(4+j2)(1-j3)}{13-j}$$

$$Z_3 = \frac{ba}{a+b+c} = \frac{8(1-j3)}{13-j}$$



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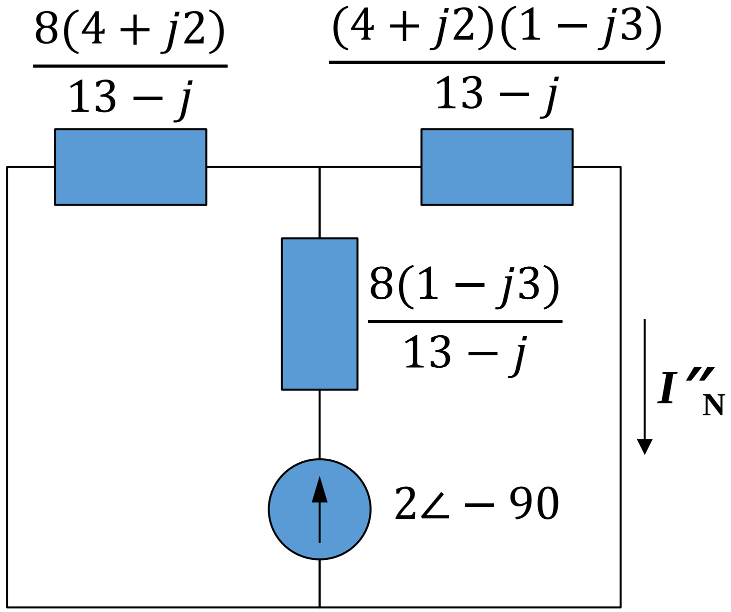
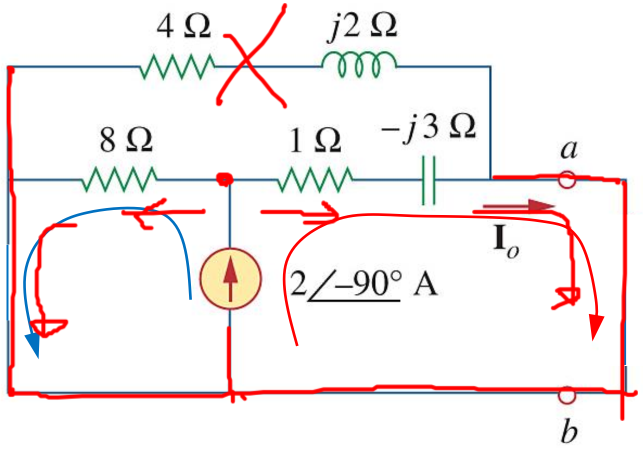


By current division

$$I''_{\mathrm{N}} = \frac{\dfrac{8(4+\mathrm{j}2)}{13-\mathrm{j}}}{\dfrac{8(4+\mathrm{j}2)}{13-\mathrm{j}} + \dfrac{(4+\mathrm{j}2)(1-\mathrm{j}3)}{13-\mathrm{j}}} \times 2\angle(-90^\circ)$$

$$= \frac{32+j16}{42+j6} \times 2\angle(-90^\circ)$$

(iii) I''_N – another approach



By current division

$$\tilde{I}''_N = 2\angle -90° \times \frac{8}{8+1-j3}$$

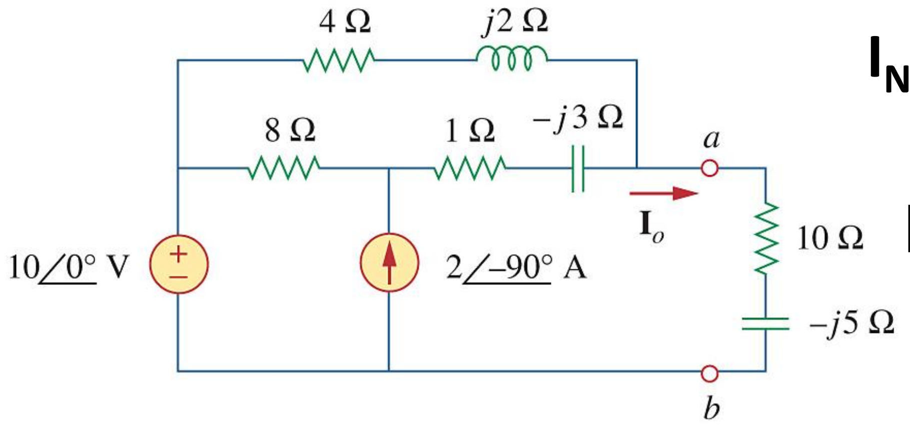
$$\approx 0.5333 - j1.6 \text{ (A)}$$

$$\boldsymbol{I_N = I'_N + I''_N}$$

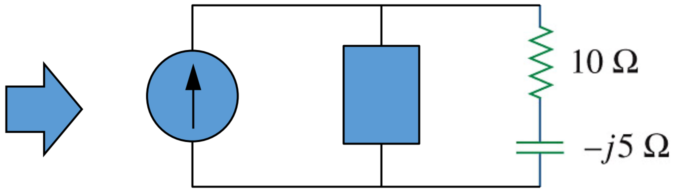
$$\tilde{I}_N = 3.5333 - j2.2667$$

$$\approx 4.1979\angle -32.68° \text{ (A)}$$

(iv) I_o



I_N = 3.53 – j2.27



Z_N = 3.25∠12.53

$$\tilde{I}_o = \tilde{I}_N \frac{Z_N}{Z_N + (10 - j5)} = 4.1979\angle -32.68^\circ \times \frac{3.1765 + j0.7059}{(3.1765 + j0.7059) + (10 - j5)}$$

$$\approx 4.1979\angle -32.68^\circ \times \frac{3.1765 + j0.7059}{13.1765 - j4.2941}$$

$$\approx 4.1979\angle -32.68^\circ \times \frac{3.2540\angle 12.53^\circ}{13.8586\angle -18.05^\circ}$$

$$\approx 0.99\angle -2.10^\circ \text{ (A)}$$

10.7 Op Amp AC Circuits

The three steps also apply to op amp circuits, as long as the op amp is in the linear region.

Steps to analyze AC circuits

1. Transform the circuit to the phasor domain.
2. Find the circuit output using nodal analysis, mesh analysis, superposition, etc.
3. Transform the resulting phasor to the time domain.

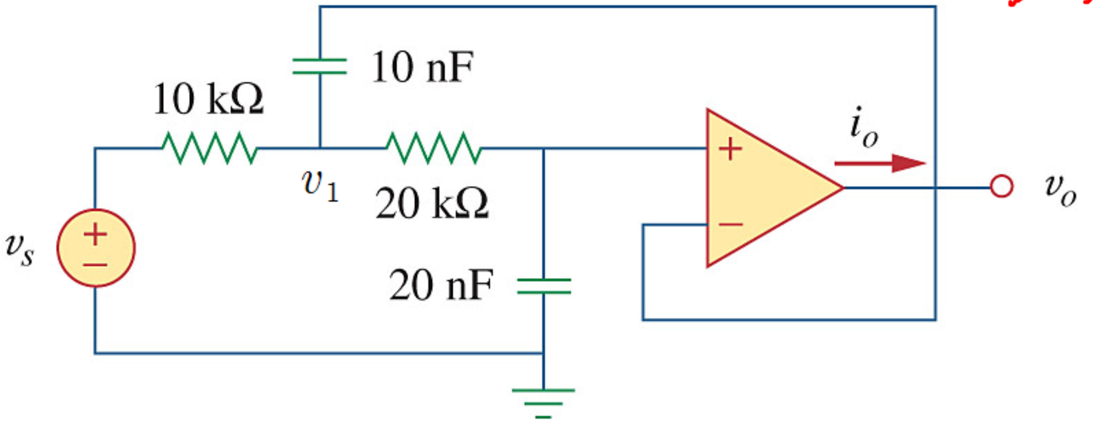
Recall Op Amp (Chapter 5): The key to analyzing op amp circuits is to keep two important properties of an ideal op amp in mind.

(1) **No current** enters either of its input terminals.
(2) **The voltage** across its input terminals is **zero**.

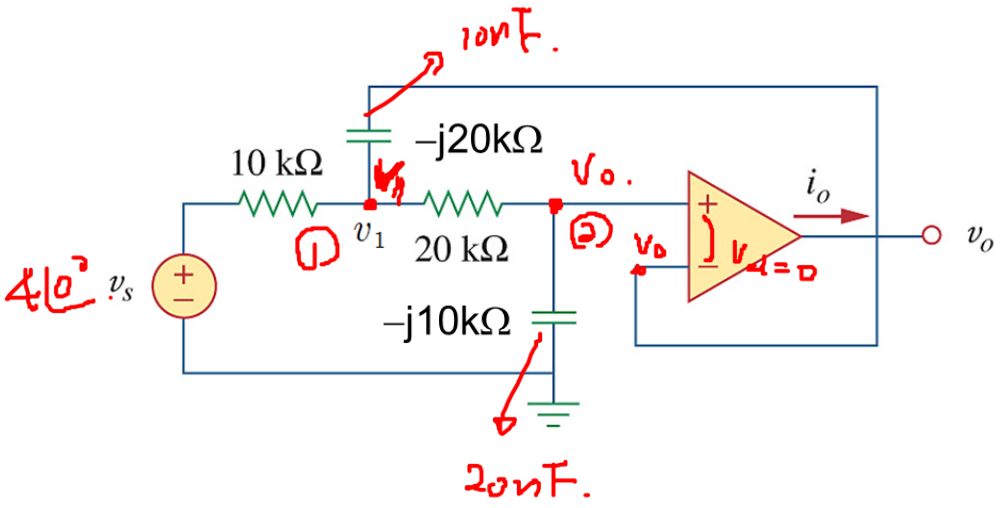
Practice Problem 10.11 Find v_o and i_o in the op amp circuit of Fig. 10.32. Let $v_s = 4\cos 5000t$ V.

w=5000 .

→ freq.



(i) Phasor diagram



$C \to \frac{1}{j\omega C}.$ $\quad \omega = 5000.$

Solution :

$$4\cos 5000t \text{ V} \Rightarrow 4\angle 0° \text{ V}, \ \omega = 5000 \text{ rad/s}$$

$$10 \text{ nF} \Rightarrow \frac{1}{j\omega C} = \frac{1}{j5000 \times 10 \times 10^{-9}}$$

$$= -j2\times 10^4 \ (\Omega) = -j20 \ (\text{k}\Omega)$$

$$20 \text{ nF} \Rightarrow \frac{1}{jWC} = \frac{1}{j5000 \times 20 \times 10^{-9}}$$

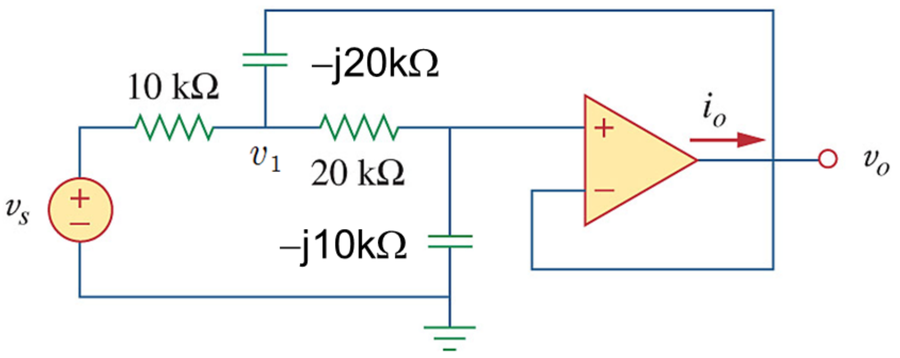
$$= -j1\times 10^4 \ (W) = -j10 \ (\text{k}W)$$

$$① \ \frac{V_1 - 4\angle 0°}{10k} + \frac{V_1 - V_o}{-j20k} + \frac{V_1 - V_o}{20k} = 0$$

$$② \ \frac{V_o - V_1}{20k} + \frac{V_o - 0}{-j10k} = 0$$

(ii) V_o

$$\begin{cases} \dfrac{\tilde{V}_s - \tilde{V}_1}{10} = \dfrac{\tilde{V}_1 - \tilde{V}_o}{-j20} + \cancel{\dfrac{\tilde{V}_o}{-j10}} \\ \tilde{V}_1 = \tilde{V}_o \dfrac{20 - j10}{-j10} = \tilde{V}_o (1 + j2) \end{cases}$$



$$-j2(\tilde{V}_s - \tilde{V}) = (\tilde{V}_1 - \tilde{V}_o) + 2\tilde{V}_o$$

$$-j2\tilde{V}_s = \tilde{V}_1(1 - j2) + \tilde{V}_o$$

$$-j2\tilde{V}_s = \tilde{V}_o(1 + j2)(1 - j2) + \tilde{V}_o = 6\tilde{V}_o$$

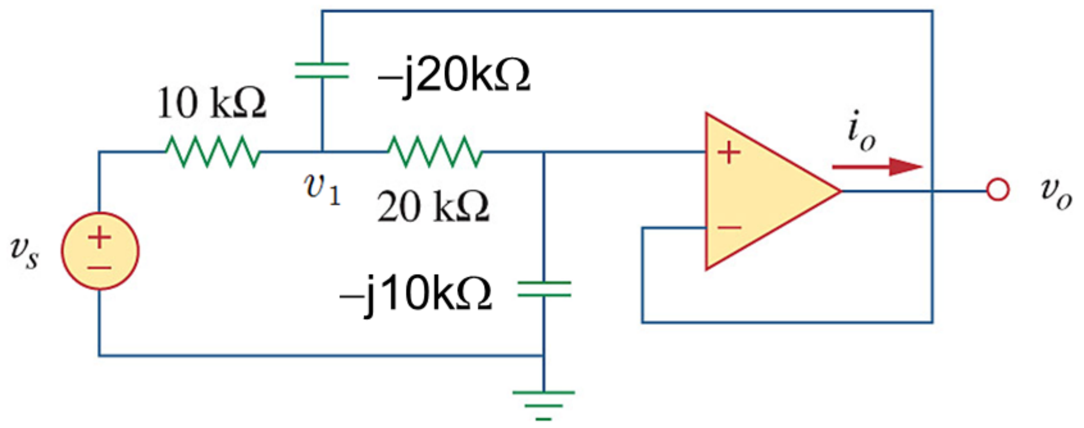
$$\tilde{V}_o = \frac{-j2\tilde{V}_s}{6} = \frac{-j2 \times 4\angle 0^\circ}{6}$$

$$\approx 1.3333\angle -90^\circ \text{ (V)}$$

$$v_o = 1.33\cos(5000t - 90^\circ)$$

$$= 1.33\sin 5000t \text{ (V)}$$

(iii) i_o



$$\tilde{I}_o = \frac{\tilde{V}_o - \tilde{V}_1}{-j20} = \frac{\tilde{V}_o - \tilde{V}_o(1+j2)}{-j20} = 0.1\tilde{V}_o$$

$$= 0.1 \times 1.3333\angle -90° \approx 0.13\angle -90° \text{ (mA)}$$

$$i_o = 0.13\cos(5000t - 90°)$$

$$= 0.13\sin 5000t \text{ (mA)}$$