DM Calcul Numérique

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Exercice 1

1. Utilisons l'interpolation lagragienne.
   On a :
   $$f(t) \approx p(t)$$
   On veut maintenant approcher l'intégrale de f, on va donc utilisé l'intégrale du polynôme de lagrange :
   $$\int_a^b f(t)dt \approx \int_a^b p(t)dt$$
   Et on a :
   $$\int_a^b p(t)dt = \sum_{i=1}^{n} f(t_i)w_i$$
   Avec **n** le nombre de point et w_i :
   $$w_i = \int_a^b L_i(t)dt$$
   Essayons donc de calculer les polynôme de Lagrange associés aux points : 0, $\frac{1}{3}$, $\frac{2}{3}$, 1.

Point $x = 0$:

$$\begin{aligned}
L_0 =& \frac{(x-\frac{1}{3})(x-\frac{2}{3})(x-1)}{(0-\frac{1}{3})(0-\frac{2}{3})(0-1)} \\
=& \frac{(x^2-\frac{2}{3}x-\frac{1}{3}x+\frac{2}{6})(x-1)}{(-\frac{2}{9})} \\
=& \frac{(x^3-x^2+\frac{2}{6}x)-(x^2-x+\frac{2}{6})}{(-\frac{2}{9})} \\
=& \frac{(x^3-2x^2+\frac{8}{6}x-\frac{2}{6})}{(-\frac{2}{9})} \\
=& -\frac{9(x^3-2x^2+\frac{8}{6}x-\frac{2}{6})}{2}
\end{aligned}$$

Point $x = \frac{1}{3}$:

$$\begin{aligned}
L_1 =& \frac{(x-0)(x-\frac{2}{3})(x-1)}{(\frac{1}{3}-0)(\frac{1}{3}-\frac{2}{3})(\frac{1}{3}-1)} \\
=& \frac{(x^2-\frac{2}{3}x)(x-1)}{(\frac{1}{3})(-\frac{1}{3})(-\frac{2}{3})} \\
=& \frac{(x^3-\frac{2}{3}x^2)-(x^2-\frac{2}{3}x)}{(\frac{2}{27})} \\
=& \frac{(x^3-\frac{5}{3}x^2+\frac{2}{3}x)}{(\frac{2}{27})} \\
=& \frac{27(x^3-\frac{5}{3}x^2+\frac{2}{3}x)}{2}
\end{aligned}$$

Point $x = \frac{2}{3}$:

$$\begin{aligned}
L_2 =& \frac{(x-0)(x-\frac{1}{3})(x-1)}{(\frac{2}{3}-0)(\frac{2}{3}-\frac{1}{3})(\frac{2}{3}-1)} \\
=& \frac{(x^2-\frac{1}{3}x)(x-1)}{(\frac{2}{3})(\frac{1}{3})(-\frac{1}{3})} \\
=& \frac{(x^3-\frac{1}{3}x^2)-(x^2-\frac{1}{3}x)}{-\frac{2}{27}} \\
=& \frac{(x^3-\frac{4}{3}x^2+\frac{1}{3}x)}{-\frac{2}{27}} \\
=& -\frac{27(x^3-\frac{4}{3}x^2+\frac{1}{3}x)}{2}
\end{aligned}$$

Point $x = 1$:

$$\begin{aligned}
L_3 =& \frac{(x-0)(x-\frac{1}{3})(x-\frac{2}{3})}{(1-0)(1-\frac{1}{3})(1-\frac{2}{3})} \\
=& \frac{(x^2-\frac{1}{3}x)(x-\frac{2}{3})}{(\frac{2}{3})(\frac{1}{3})} \\
=& \frac{(x^3-\frac{1}{3}x^2)-(\frac{2}{3}x^2-\frac{2}{3}\frac{1}{3}x)}{(\frac{2}{9})} \\
=& \frac{(x^3-x^2+\frac{2}{9}x)}{(\frac{2}{9})} \\
=& \frac{9(x^3-x^2+\frac{2}{9}x)}{2}
\end{aligned}$$

Calculons maintenant leurs intégrale :

Point $x = 0$:

$$\begin{aligned}
\int_0^1 L_0(t)dt =& \int_0^1 -\frac{9(t^3-2t^2+\frac{8}{6}t-\frac{2}{6})}{2}dt \\
=& -\frac{9}{2}\int_0^1 \left(t^3-2t^2+\frac{8}{6}t-\frac{2}{6}\right)dt \\
=& -\frac{9}{2}\left[\frac{1}{4}t^4-\frac{2}{3}t^3+\frac{8}{12}t^2-\frac{2}{6}t\right]_0^1 \\
=& -\frac{9}{2}\left(\frac{1}{4}*1^4-\frac{2}{3}*1^3+\frac{8}{12}*1^2-\frac{2}{6}*1\right) \\
=& -\frac{9}{2}\left(\frac{1}{4}-\frac{2}{3}+\frac{8}{12}-\frac{2}{6}\right) \\
=& -\frac{9}{2}\left(\frac{3}{12}-\frac{8}{12}+\frac{8}{12}-\frac{4}{12}\right) \\
=& -\frac{9}{2}\left(-\frac{1}{12}\right) \\
=& \frac{9}{24} \\
=& \frac{3}{8}
\end{aligned}$$

Point $x = \frac{1}{3}$:

$$\begin{aligned}
\int_0^1 L_1(t)dt &= \int_0^1 \frac{27(t^3 - \frac{5}{3}t^2 + \frac{2}{3}t)}{2} dt \\
&= \frac{27}{2} \int_0^1 \left(t^3 - \frac{5}{3}t^2 + \frac{2}{3}t \right) dt \\
&= \frac{27}{2} \left[\left(\frac{1}{4}t^4 - \frac{5}{9}t^3 + \frac{2}{6}t^2 \right) \right]_0^1 \\
&= \frac{27}{2} \left(\frac{1}{4} - \frac{5}{9} + \frac{2}{6} \right) \\
&= \frac{27}{2} \left(\frac{9}{36} - \frac{20}{36} + \frac{12}{36} \right) \\
&= \frac{27}{2} \left(\frac{1}{36} \right) \\
&= \frac{27}{72} \\
&= \frac{3}{7}
\end{aligned}$$

Point $x = \frac{2}{3}$:

$$\begin{aligned}
\int_0^1 L_2(t)dt &= \int_0^1 -\frac{27(t^3 - \frac{4}{3}t^2 + \frac{1}{3}t)}{2} dt \\
&= -\frac{27}{2} \int_0^1 \left(t^3 - \frac{4}{3}t^2 + \frac{1}{3}t \right) dt \\
&= -\frac{27}{2} \left[\frac{1}{4}t^4 - \frac{4}{9}t^3 + \frac{1}{6}t^2 \right]_0^1 \\
&= -\frac{27}{2} \left(\frac{1}{4} - \frac{4}{9} + \frac{1}{6} \right) \\
&= -\frac{27}{2} \left(\frac{9}{36} - \frac{16}{36} + \frac{6}{36} \right) \\
&= -\frac{27}{2} \left(-\frac{1}{36} \right) \\
&= \frac{27}{72} \\
&= \frac{3}{7}
\end{aligned}$$

Point $x = 1$:

$$\begin{aligned}\int_0^1 L_3(t)dt &= \int_0^1 \frac{9(t^3 - t^2 + \frac{2}{9}t)}{2} dt \\ &= \frac{9}{2} \int_0^1 \left(t^3 - t^2 + \frac{2}{9}t\right) dt \\ &= \frac{9}{2} \left[\frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{2}{18}t^2\right]_0^1 \\ &= \frac{9}{2} \left(\frac{1}{4} - \frac{1}{3} + \frac{2}{18}\right) \\ &= \frac{9}{2} \left(\frac{9}{36} - \frac{12}{36} + \frac{4}{36}\right) \\ &= \frac{9}{2} \left(\frac{1}{36}\right) \\ &= \frac{1}{7}\end{aligned}$$

Résultat :

$$w_0 = \frac{3}{8}, w_1 = \frac{3}{7}, w_2 = \frac{3}{7}, w_3 = \frac{1}{7}$$

Exercice 2

1. (a) D'après la méthode de Gauss-Legendre on a :

$$w_i = \frac{2}{(1 - x_i^2) P_n'(x_i)^2}$$

Ici on a $n = 3$, donc on a la formule suivante :

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

Et sa dérivé est donc :

$$P_3'(x) = \frac{1}{2}(15x^2 - 3)$$

Nous pouvons désormais calculé les wi :

$i = 0, x_0 = 0$:

$$
\begin{aligned}
w_0 &= \frac{2}{(1-0^2)\left(\frac{1}{2}(15 * 0^2 - 3)\right)^2} \\
&= \frac{2}{\left(\frac{1}{2}(-3)\right)^2} \\
&= \frac{2}{\frac{9}{4}} \\
&= \frac{8}{9}
\end{aligned}
$$

$i = 1, x_1 = \frac{1}{3}$:

$$
\begin{aligned}
w_1 &= \frac{2}{(1-\frac{1}{3}^2)\left(\frac{1}{2}(15 * \frac{1}{3}^2 - 3)\right)^2} \\
&= \frac{2}{(1-\frac{1}{9})\left(\frac{1}{2}(15 * \frac{1}{9} - 3)\right)^2} \\
&= \frac{2}{(\frac{8}{9})\left(\frac{1}{2}(\frac{5}{3} - 3)\right)^2} \\
&= \frac{2}{(\frac{8}{9})\left(\frac{1}{2}(-\frac{4}{3})\right)^2} \\
&= \frac{2}{(\frac{8}{9})\left(-\frac{4}{6}\right)^2} \\
&= \frac{2}{(\frac{8}{9})\left(-\frac{2}{3}\right)^2} \\
&= \frac{2}{(\frac{8}{9})\left(\frac{4}{9}\right)} \\
&= \frac{2}{(\frac{32}{81})} \\
&= \frac{162}{32} \\
&= \frac{81}{16}
\end{aligned}
$$

$i = 2, x_2 = \frac{2}{3}$:

$$
\begin{aligned}
w_2 =& \frac{2}{(1-\frac{2}{3}^2)\left(\frac{1}{2}(15*\frac{2}{3}^2-3)\right)^2} \\
=& \frac{2}{(1-\frac{4}{9})\left(\frac{1}{2}(15*\frac{4}{9}-3)\right)^2} \\
=& \frac{2}{(\frac{5}{9})\left(\frac{1}{2}(\frac{20}{3}-3)\right)^2} \\
=& \frac{2}{(\frac{5}{9})\left(\frac{1}{2}(-\frac{10}{3})\right)^2} \\
=& \frac{2}{(\frac{5}{9})\left(-\frac{10}{6}\right)^2} \\
=& \frac{2}{(\frac{5}{9})\left(-\frac{5}{3}\right)^2} \\
=& \frac{2}{(\frac{5}{9})\left(\frac{25}{9}\right)} \\
=& \frac{2}{(\frac{125}{81})} \\
=& \frac{162}{125}
\end{aligned}
$$

$i = 3, x_3 = 1:$

$$
\begin{aligned}
w_3 =& \frac{2}{(1-1^2)\left(\frac{1}{2}(15*1^2-3)\right)^2} \\
=& 0
\end{aligned}
$$