DM Calcul Numérique

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November 12, 2020

Exercice 1

1. Utilisons l'interpolation lagragienne.
   On a :
   $$f(t) \approx p(t)$$
   On veut maintenant approcher l'intégrale de f, on va donc utilisé l'intégrale du polynôme de lagrange :
   $$\int_a^b f(t)dt \approx \int_a^b p(t)dt$$
   Et on a :
   $$\int_a^b p(t)dt = \sum_{i=1}^{n} f(t_i)w_i$$
   Avec **n** le nombre de point et w_i :
   $$w_i = \int_a^b L_i(t)dt$$
   Essayons donc de calculer les polynôme de Lagrange associés aux points : 0, $\frac{1}{3}$, $\frac{2}{3}$, 1.

Point $x = 0$:

$$\begin{aligned}
L_0 =& \frac{(x-\frac{1}{3})(x-\frac{2}{3})(x-1)}{(0-\frac{1}{3})(0-\frac{2}{3})(0-1)} \\
=& \frac{(x^2-\frac{2}{3}x-\frac{1}{3}x+\frac{2}{9})(x-1)}{(-\frac{2}{9})} \\
=& \frac{(x^3-x^2+\frac{2}{9}x)-(x^2-x+\frac{2}{9})}{(-\frac{2}{9})} \\
=& \frac{(x^3-2x^2+\frac{11}{9}x-\frac{2}{9})}{(-\frac{2}{9})} \\
=& -\frac{9(x^3-2x^2+\frac{11}{9}x-\frac{2}{9})}{2} \\
=& -\frac{9}{2}x^3+9x^2-\frac{11}{2}x+1
\end{aligned}$$

Point $x = \frac{1}{3}$:

$$\begin{aligned}
L_1 =& \frac{(x-0)(x-\frac{2}{3})(x-1)}{(\frac{1}{3}-0)(\frac{1}{3}-\frac{2}{3})(\frac{1}{3}-1)} \\
=& \frac{(x^2-\frac{2}{3}x)(x-1)}{(\frac{1}{3})(-\frac{1}{3})(-\frac{2}{3})} \\
=& \frac{(x^3-\frac{2}{3}x^2)-(x^2-\frac{2}{3}x)}{(\frac{2}{27})} \\
=& \frac{(x^3-\frac{5}{3}x^2+\frac{2}{3}x)}{(\frac{2}{27})} \\
=& \frac{27(x^3-\frac{5}{3}x^2+\frac{2}{3}x)}{2}
\end{aligned}$$

Point $x = \frac{2}{3}$:

$$\begin{aligned}
L_2 =& \frac{(x-0)(x-\frac{1}{3})(x-1)}{(\frac{2}{3}-0)(\frac{2}{3}-\frac{1}{3})(\frac{2}{3}-1)} \\
=& \frac{(x^2-\frac{1}{3}x)(x-1)}{(\frac{2}{3})(\frac{1}{3})(-\frac{1}{3})} \\
=& \frac{(x^3-\frac{1}{3}x^2)-(x^2-\frac{1}{3}x)}{-\frac{2}{27}} \\
=& \frac{(x^3-\frac{4}{3}x^2+\frac{1}{3}x)}{-\frac{2}{27}} \\
=& -\frac{27(x^3-\frac{4}{3}x^2+\frac{1}{3}x)}{2}
\end{aligned}$$

Point $x = 1$:

$$\begin{aligned}
L_3 =& \frac{(x-0)(x-\frac{1}{3})(x-\frac{2}{3})}{(1-0)(1-\frac{1}{3})(1-\frac{2}{3})} \\
=& \frac{(x^2-\frac{1}{3}x)(x-\frac{2}{3})}{(\frac{2}{3})(\frac{1}{3})} \\
=& \frac{(x^3-\frac{1}{3}x^2)-(\frac{2}{3}x^2-\frac{2}{3}\frac{1}{3}x)}{(\frac{2}{9})} \\
=& \frac{(x^3-x^2+\frac{2}{9}x)}{(\frac{2}{9})} \\
=& \frac{9(x^3-x^2+\frac{2}{9}x)}{2}
\end{aligned}$$

Calculons maintenant leurs intégrale :

Point $x = 0$:

$$\begin{aligned}
\int_0^1 L_0(t)dt =& \int_0^1 \left(-\frac{9}{2}t^3+9t^2-\frac{11}{2}t+1\right)dt \\
=& \left[-\frac{9}{8}t^4+\frac{9}{3}t^3-\frac{11}{4}t^2+t\right]_0^1 \\
=& -\frac{9}{8}*1^4+\frac{9}{3}*1^3-\frac{11}{4}*1^2+1 \\
=& \frac{3}{24} \\
=& \frac{1}{8}
\end{aligned}$$

<u>Point $x = \frac{1}{3}$:</u>

$$\begin{aligned}
\int_0^1 L_1(t)dt &= \int_0^1 \frac{27(t^3 - \frac{5}{3}t^2 + \frac{2}{3}t)}{2} dt \\
&= \frac{27}{2} \int_0^1 \left(t^3 - \frac{5}{3}t^2 + \frac{2}{3}t\right) dt \\
&= \frac{27}{2} \left[\left(\frac{1}{4}t^4 - \frac{5}{9}t^3 + \frac{2}{6}t^2\right)\right]_0^1 \\
&= \frac{27}{2} \left(\frac{1}{4} - \frac{5}{9} + \frac{2}{6}\right) \\
&= \frac{27}{2} \left(\frac{9}{36} - \frac{20}{36} + \frac{12}{36}\right) \\
&= \frac{27}{2} \left(\frac{1}{36}\right) \\
&= \frac{27}{72} \\
&= \frac{3}{8}
\end{aligned}$$

<u>Point $x = \frac{2}{3}$:</u>

$$\begin{aligned}
\int_0^1 L_2(t)dt &= \int_0^1 -\frac{27(t^3 - \frac{4}{3}t^2 + \frac{1}{3}t)}{2} dt \\
&= -\frac{27}{2} \int_0^1 \left(t^3 - \frac{4}{3}t^2 + \frac{1}{3}t\right) dt \\
&= -\frac{27}{2} \left[\frac{1}{4}t^4 - \frac{4}{9}t^3 + \frac{1}{6}t^2\right]_0^1 \\
&= -\frac{27}{2} \left(\frac{1}{4} - \frac{4}{9} + \frac{1}{6}\right) \\
&= -\frac{27}{2} \left(\frac{9}{36} - \frac{16}{36} + \frac{6}{36}\right) \\
&= -\frac{27}{2} \left(-\frac{1}{36}\right) \\
&= \frac{27}{72} \\
&= \frac{3}{8}
\end{aligned}$$

Point $x = 1$:

$$\begin{aligned}
\int_0^1 L_3(t)dt &= \int_0^1 \frac{9(t^3 - t^2 + \frac{2}{9}t)}{2} dt \\
&= \frac{9}{2} \int_0^1 \left(t^3 - t^2 + \frac{2}{9}t\right) dt \\
&= \frac{9}{2} \left[\frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{2}{18}t^2\right]_0^1 \\
&= \frac{9}{2} \left(\frac{1}{4} - \frac{1}{3} + \frac{2}{18}\right) \\
&= \frac{9}{2} \left(\frac{9}{36} - \frac{12}{36} + \frac{4}{36}\right) \\
&= \frac{9}{2} \left(\frac{1}{36}\right) \\
&= \frac{9}{72} \\
&= \frac{1}{8}
\end{aligned}$$

Résultat :

$$w_0 = \frac{1}{8}, w_1 = \frac{3}{8}, w_2 = \frac{3}{8}, w_3 = \frac{1}{8}$$

2. Ici la méthode ne fonctionne pas.
   Il nous faudrai un point en plus tel que :

$$\int_0^1 f(t)dt \approx w_0 f(t_0) + w_1 f(t_1) + w_2 f(t_2) + w_3 f(t_3) + w_4 f(t_4)$$

3. Formule de quadrature : Faisons un changement de variable.
   On a $x = \frac{t - x_k}{x_{k+1} - x_k}$ donc $t = (x(x_{k+1} - x_k) + x_k)$

$$
\begin{aligned}
\int_{x_k}^{x_{k+1}} f(x)dx &\approx (x_{k+1}-x_k)\int_0^1 f(x(x_{k+1}-x_k)+x_k)dx \\
&\approx (x_{k+1}-x_k) \\
&* [w_0 f((x_{k+1}-x_k)0+x_k) \\
&+ w_1 f\left((x_{k+1}-x_k)\frac{1}{3}+x_k\right) \\
&+ w_2 f\left((x_{k+1}-x_k)\frac{2}{3}+x_k\right) \\
&+ w_3 f\left((x_{k+1}-x_k)1+x_k\right)] \\
&\approx (x_{k+1}-x_k) \\
&* \left[\frac{1}{8} f((x_{k+1}-x_k)0+x_k)\right. \\
&+ \frac{3}{8} f\left((x_{k+1}-x_k)\frac{1}{3}+x_k\right) \\
&+ \frac{3}{8} f\left((x_{k+1}-x_k)\frac{2}{3}+x_k\right) \\
&+ \left.\frac{1}{8} f\left((x_{k+1}-x_k)1+x_k\right)\right] \\
&\approx \frac{(x_{k+1}-x_k)}{8}\left[f(x_k)+3f\left(\frac{1}{3}x_{k+1}+\frac{2}{3}x_k\right)\right. \\
&+ \left.3f\left(\frac{2}{3}x_{k+1}+\frac{1}{3}x_k\right)+f(x_{k+1})\right]
\end{aligned}
$$

4. Formule composite :

   On a $x_k = a + kh$ et $h = \frac{b-a}{N}$

$$
\begin{aligned}
\int_a^b f(x)dx &\approx \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} f(x)dx \\
&\approx \sum_{k=0}^{N-1} h\frac{(x_{k+1}-x_k)}{8}\left[f(x_k)+3f\left(\frac{1}{3}x_{k+1}+\frac{2}{3}x_k\right)\right. \\
&\left.+3f\left(\frac{2}{3}x_{k+1}+\frac{1}{3}x_k\right)+f(x_{k+1})\right] \\
&\approx \sum_{k=0}^{N-1} h\frac{(a*(k+1)h-a-kh)}{8}\left[f(x_k)+3f\left(\frac{1}{3}x_{k+1}+\frac{2}{3}x_k\right)\right. \\
&\left.+3f\left(\frac{2}{3}x_{k+1}+\frac{1}{3}x_k\right)+f(x_{k+1})\right] \\
&\approx \sum_{k=0}^{N-1} \frac{h}{8}\left[f(x_k)+3f\left(\frac{1}{3}x_{k+1}+\frac{2}{3}x_k\right)\right. \\
&\left.+3f\left(\frac{2}{3}x_{k+1}+\frac{1}{3}x_k\right)+f(x_{k+1})\right]
\end{aligned}
$$

5. Nom de la fontion : **fourpoints**

   Paramètre d'entrées :

   - **a** : borne inférieur de l'intervale
   - **b** : borne supérieur de l'intervale
   - **N** : nombre de pas
   - **f** : fonction f

   Paramètre de sortie :

   - **res** : résultat de l'intégrale $\int_a^b f(x)dx$

   Explication du code :

   - Pour moi $x_k = k * \frac{b-a}{N}$
   - Le reste est triviale

   Résultat numérique :

   1er test : **fourpoints(0, 1, 1, mysquare)**

   - $a = 0$
   - $b = 1$

- $N = 1$
- f : mysquare fonction $f(x) = x^2$
- **résultat :** 0.3333333

2ème test : **fourpoints(0, 10, 1, mysquare)**

- $a = 0$
- $b = 10$
- $N = 1$
- f : mysquare fonction $f(x) = x^2$
- **résultat :** 0.3333333

3ème test : **fourpoints(0, 100, 1, mypolynome)**

- $a = 0$
- $b = 100$
- $N = 1$
- f : mypolynome fonction $f(x) = -\frac{1}{3}x^3 + \frac{67}{4}x^2 + 7$
- **résultat :** -2749300.0

Exercice 2

1. (a) S_f est un polynôme de degré inférieur ou égale à 3.
   Donc S''_f est un polynôme de degré inférieur ou égale à 1.
   On connait deux point :
   - au point x_k : $S''_f(x_k) = M_k$
   - au point x_{k+1} : $S''_f(x_{k+1}) = M_{k+1}$

   Et on sait que $h = \frac{b-a}{N}$

   Donc on peut appliquer la méthode de Lagrange :

   $$L_0 = \frac{(x - x_{k+1})}{(x_k - x_{k+1})}$$

   $$L_1 = \frac{(x - x_k)}{(x_{k+1} - x_k)}$$

   $$\begin{aligned} S''_f(x) &= S''_f(x_k)L_0 + S''_f(x_{k+1})L_1 \\ &= M_k \frac{(x - x_{k+1})}{(x_k - x_{k+1})} + M_{k+1} \frac{(x - x_k)}{(x_{k+1} - x_k)} \\ &= -\frac{M_k}{h}(x - x_{k+1}) + \frac{M_{k+1}}{h}(x - x_k) \end{aligned}$$

   car on a : $x_{k+1} - x_k = a + (k+1)h - (a + kh) = h$

   (b) On a :

   $$S''_f(x) = -\frac{M_k}{h}(x - x_{k+1}) + \frac{M_{k+1}}{h}(x - x_k)$$

   En intégrant une fois on obtient :

   $$S'_f(x) = -\frac{M_k}{2h}(x - x_{k+1})^2 + \frac{M_{k+1}}{2h}(x - x_k)^2 + C$$

   avec C une constante.

   En intégrant une nouvelle fois on obtient ;

   $$S_f(x) = -\frac{M_k}{6h}(x - x_{k+1})^3 + \frac{M_{k+1}}{6h}(x - x_k)^3 + Cx + D$$

   avec D une constante.

Calculons $S_f(x_k)$:

$$\begin{aligned} S_f(x_k) &= -\frac{M_k}{6h}(x_k - x_{k+1})^3 + \frac{M_{k+1}}{6h}(x_k - x_k)^3 + Cx_k + D \\ &= -\frac{M_k}{6h}(x_k - x_{k+1})^3 + Cx_k + D \end{aligned}$$

Calculons $S_f(x_{k+1})$:

$$\begin{aligned} S_f(x_{k+1}) &= -\frac{M_k}{6h}(x_{k+1} - x_{k+1})^3 + \frac{M_{k+1}}{6h}(x_{k+1} - x_k)^3 + Cx_{k+1} + D \\ &= \frac{M_{k+1}}{6h}(x_{k+1} - x_k)^3 + Cx_{k+1} + D \end{aligned}$$

On a donc le système suivant :

$$\left\{ \begin{array}{l} -\frac{M_k}{6h}(x_k - x_{k+1})^3 + Cx_k + D = y_k \\ \frac{M_{k+1}}{6h}(x_{k+1} - x_k)^3 + Cx_{k+1} + D = y_{k+1} \end{array} \right.$$

$$\left\{ \begin{array}{l} -\frac{M_k}{6h}(x_k - x_{k+1})^3 + Cx_k + D = y_k \\ \frac{M_{k+1}}{6h}(x_{k+1} - x_k)^3 + Cx_{k+1} + D - \big(-\frac{M_k}{6h}(x_k - x_{k+1})^3 + Cx_k + D\big) = y_{k+1} - y_k \end{array} \right.$$

$$\left\{ \begin{array}{l} -\frac{M_k}{6h}(x_k - x_{k+1})^3 + Cx_k + D = y_k \\ \frac{1}{6h}(M_{k+1}(x_{k+1} - x_k)^3 + M_k(x_k - x_{k+1})^3) + C(x_{k+1} - x_k) = y_{k+1} - y_k \end{array} \right.$$

$$\left\{ \begin{array}{l} -\frac{M_k}{6h}(x_k - x_{k+1})^3 + Cx_k + D = y_k \\ \frac{1}{6h}(M_{k+1}h^3 + M_k(-h^3)) + Ch = y_{k+1} - y_k \end{array} \right.$$

$$\left\{ \begin{array}{l} -\frac{M_k}{6h}(x_k - x_{k+1})^3 + Cx_k + D = y_k \\ C = \frac{y_{k+1}}{h} - \frac{y_k}{h} - \frac{M_{k+1}h^3}{6h^2} + \frac{M_k h^3}{6h^2} \end{array} \right.$$

$$\left\{ \begin{array}{l} -\frac{M_k}{6h}(x_k - x_{k+1})^3 + Cx_k + D = y_k \\ C = \frac{y_{k+1}}{h} - \frac{y_k}{h} - \frac{M_{k+1}h}{6} + \frac{M_k h}{6} \end{array} \right.$$

$$\left\{ \begin{array}{l} Cx_k + D = y_k + \frac{M_k}{6h}(x_k - x_{k+1})^3 \\ C = \frac{y_{k+1}}{h} - \frac{y_k}{h} - \frac{M_{k+1}h}{6} + \frac{M_k h}{6} \end{array} \right.$$

$$\left\{ \begin{array}{l} Cx_k + D = y_k - \frac{M_k h^2}{6} \\ C = \frac{y_{k+1}}{h} - \frac{y_k}{h} - \frac{M_{k+1}h}{6} + \frac{M_k h}{6} \end{array} \right.$$

$$\left\{ \begin{array}{l} D = y_k - \frac{M_k h^2}{6} - (\frac{y_{k+1}}{h} - \frac{y_k}{h} - \frac{M_{k+1}h}{6} + \frac{M_k h}{6})x_k \\ C = \frac{y_{k+1}}{h} - \frac{y_k}{h} - \frac{M_{k+1}h}{6} + \frac{M_k h}{6} \end{array} \right.$$

Maintenant remplaçons C et D par leurs valeurs :

$$S_f(x) = -\frac{M_k}{6h}(x-x_{k+1})^3 + \frac{M_{k+1}}{6h}(x-x_k)^3 + Cx + y_k - \frac{M_k h^2}{6} - Cx_k$$

$$\begin{aligned} S_f(x) = &-\frac{M_k}{6h}(x-x_{k+1})^3 + \frac{M_{k+1}}{6h}(x-x_k)^3 \\ &+ \left(\frac{y_{k+1}}{h} - \frac{y_k}{h} - \frac{M_{k+1}h}{6} + \frac{M_k h}{6}\right)x \\ &+ y_k - \frac{M_k h^2}{6} \\ &- \left(\frac{y_{k+1}}{h} - \frac{y_k}{h} - \frac{M_{k+1}h}{6} + \frac{M_k h}{6}\right)x_k \end{aligned}$$

Donc on a :

$$\begin{aligned} S_f(x) = &-\frac{M_k}{6h}(x-x_{k+1})^3 + \frac{M_{k+1}}{6h}(x-x_k)^3 \\ &+ \left(\frac{y_{k+1}}{h} - \frac{y_k}{h} - \frac{M_{k+1}h}{6} + \frac{M_k h}{6}\right)(x-x_k) \\ &+ y_k - \frac{M_k h^2}{6} \end{aligned}$$

Et on a :

$$S_f(x) = A_k(x-x_{k+1})^3 + B_k(x-x_k)^3 + C_k(x-x_k) + D_k$$

Finalement on trouve :

$$\begin{cases} A_k = -\frac{M_k}{6h} \\ B_k = \frac{M_{k+1}}{6h} \\ C_k = \frac{y_{k+1}}{h} - \frac{y_k}{h} - \frac{M_{k+1}h}{6} + \frac{M_k h}{6} \\ D_k = y_k - \frac{M_k h^2}{6} \end{cases}$$

(c) Commencons par calculer $S'_f(x)$ les calculs seront plus faciles avec :

$$\begin{aligned} S'_f(x) &= 3A_k(x-x_{k+1})^2 + 3B_k(x-x_k)^2 + C_k \\ &= -\frac{M_k}{2h}(x-x_{k+1})^2 + \frac{M_{k+1}}{2h}(x-x_k)^2 \\ &+ \left(\frac{y_{k+1}}{h} - \frac{y_k}{h} - \frac{M_{k+1}h}{6} + \frac{M_k h}{6}\right) \end{aligned}$$

Calculons maintenant $S'_f(x_k)$:

$$\begin{aligned}
S'_f(x_k) &= -\frac{M_k}{2h}(x_k - x_{k+1})^2 + \frac{M_{k+1}}{2h}(x_k - x_k)^2 \\
&\quad + \left(\frac{y_{k+1}}{h} - \frac{y_k}{h} - \frac{M_{k+1}h}{6} + \frac{M_k h}{6}\right) \\
&= -\frac{M_k}{2h}h^2 + \left(\frac{y_{k+1}}{h} - \frac{y_k}{h} - \frac{M_{k+1}h}{6} + \frac{M_k h}{6}\right) \\
&= \frac{y_{k+1}}{h} - \frac{y_k}{h} - \frac{M_{k+1}h^2}{6h} + \frac{M_k h^2}{6h} - \frac{3M_k h^2}{6h} \\
&= \frac{y_{k+1} - y_k}{h} - \frac{h}{6}M_{k+1} - \frac{h}{3}M_k
\end{aligned}$$

Calculons maintenant $S'_f(x_{k-1})$:

Si on pose $k = k-1$ on a l'équation suivante :

$$S'_f(x_{k-1}) = \frac{y_k - y_{k-1}}{h} - \frac{h}{6}M_k - \frac{h}{3}M_{k-1}$$

Comme on a poser $k = k-1$ ici, on doit s'assurer que $1 \leq k$ car sinon on aurait un k négatif. Et évidemment $k \leq N-1$ car sinon $k+1 > N$.

Supposons maintenant que h est très petit, c'est-à-dire que la différence des dérivé de $S'_f(x_k)$ et $S'_f(x_{k-1})$ est quasiment nulle. On peut donc dire que $S'_f(x_k) = S'_f(x_{k-1})$. Et on a :

$$\begin{aligned}
S'_f(x_k) &= S'_f(x_{k-1}) \\
\frac{y_{k+1} - y_k}{h} - \frac{h}{6}M_{k+1} - \frac{h}{3}M_k &= \frac{y_k - y_{k-1}}{h} - \frac{h}{6}M_k - \frac{h}{3}M_{k-1} \\
\frac{y_{k+1} - y_k}{h} - \frac{y_k - y_{k-1}}{h} &= -\frac{h}{6}M_k - \frac{h}{3}M_{k-1} + \frac{h}{6}M_{k+1} + \frac{h}{3}M_k \\
\frac{y_{k+1} - 2y_k + y_{k-1}}{h} &= -\frac{h}{6}M_k - \frac{2h}{6}M_{k-1} + \frac{h}{6}M_{k+1} + \frac{2h}{6}M_k \\
6\frac{y_{k+1} - 2y_k + y_{k-1}}{h} &= -hM_k - 2hM_{k-1} + hM_{k+1} + 2hM_k
\end{aligned}$$

Il faut calculer $S_f(x_{k-1})$ et $S''_f(x_{k-1})$.

$$
\begin{aligned}
S_f''(x_{k-1}) &= -\frac{M_k}{h}(x_{k-1} - x_{k+1}) + \frac{M_{k+1}}{h}(x_{k-1} - x_k) \\
M_{k-1} &= -\frac{M_k}{h}(-2h) + \frac{M_{k+1}}{h}(-h) \\
M_{k-1} &= 2M_k - M_{k+1}
\end{aligned}
$$

$$
\begin{aligned}
S_f(x_{k-1}) = &-\frac{M_k}{6h}(x_{k-1} - x_{k+1})^3 + \frac{M_{k+1}}{6h}(x_{k-1} - x_k)^3 \\
&+ \left(\frac{y_{k+1}}{h} - \frac{y_k}{h} - \frac{M_{k+1}h}{6} + \frac{M_k h}{6}\right)(x_{k-1} - x_k) \\
&+ y_k - \frac{M_k h^2}{6} \\
S_f(x_{k-1}) = &-\frac{M_k}{6h}(-2h)^3 + \frac{M_{k+1}}{6h}(-h)^3 \\
&+ \left(\frac{y_{k+1}}{h} - \frac{y_k}{h} - \frac{M_{k+1}h}{6} + \frac{M_k h}{6}\right)(-h) \\
&+ y_k - \frac{M_k h^2}{6}
\end{aligned}
$$

2.

Ecercice 3

Annexe