Question of the Day

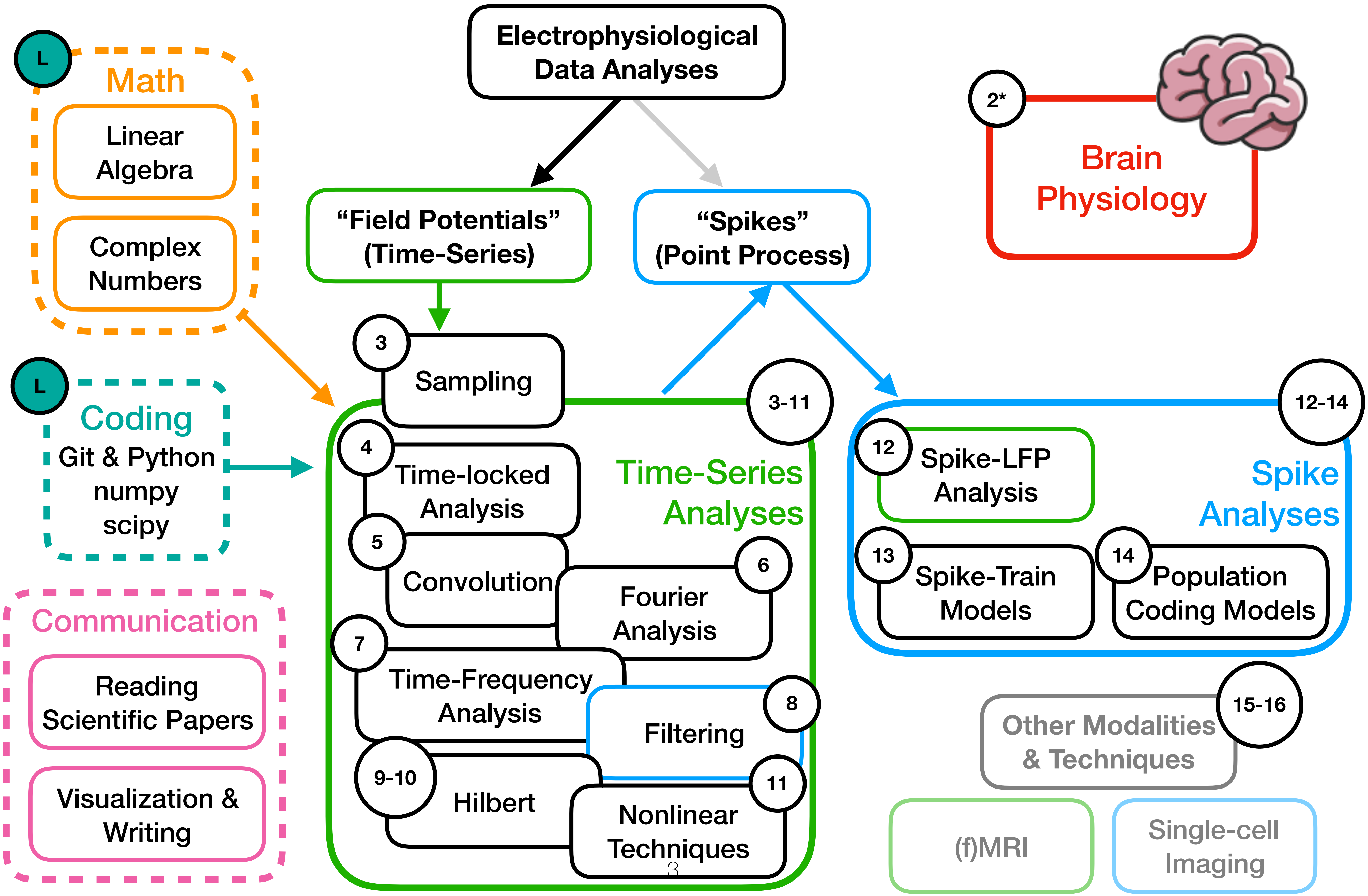
What do these things have in common?

Fourier (Frequency) Analysis

Lecture 6

July 10, 2019

Course Outline: Road Map



1. Introduce (brain) oscillations & Fourier analysis

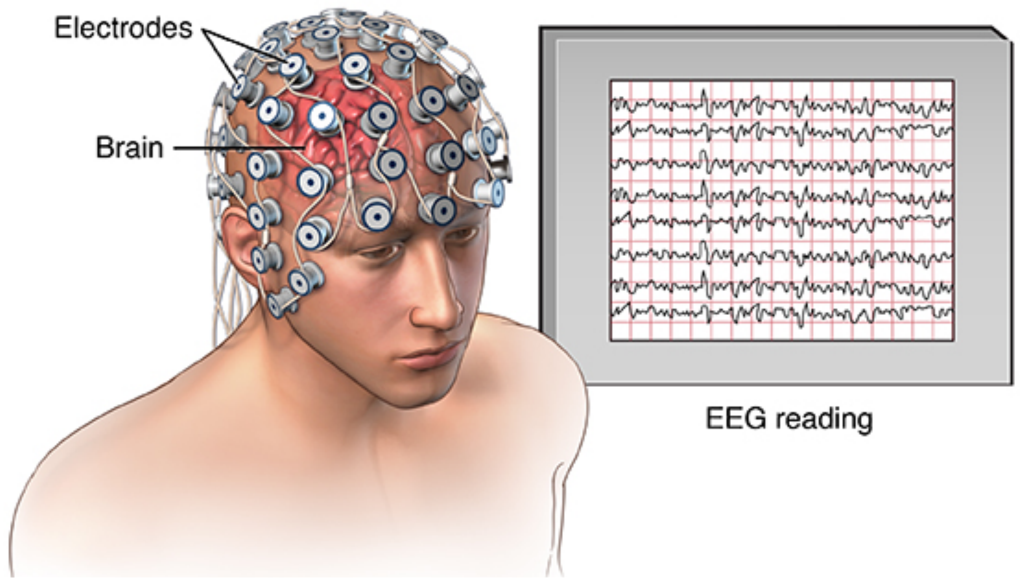
2. **Fourier analysis:** complex exponential & sinusoids

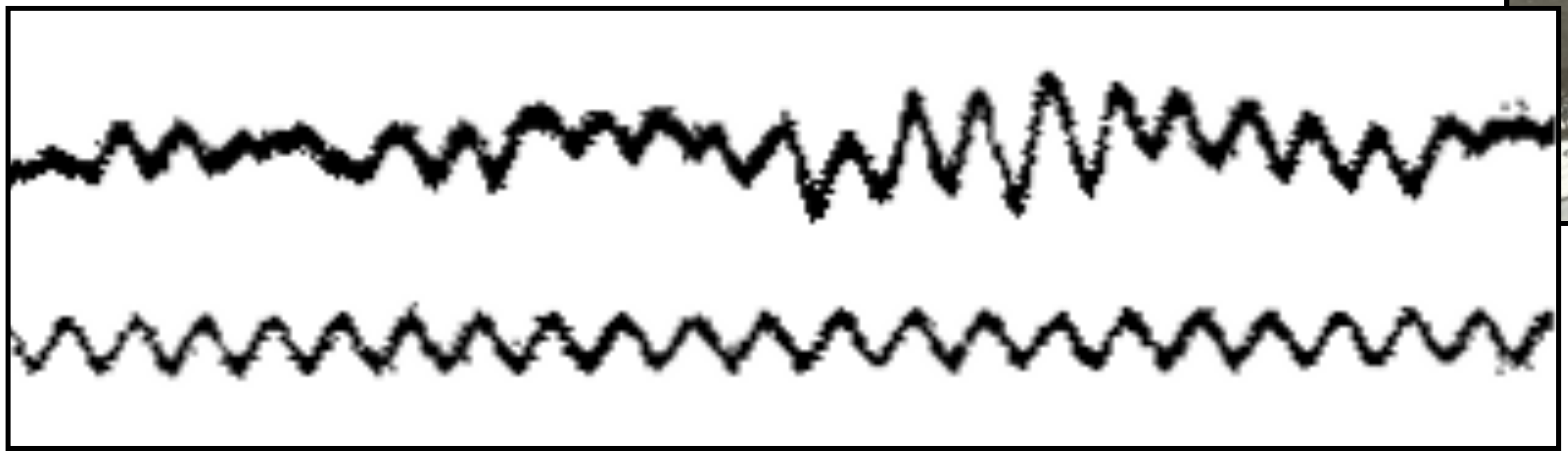
3. **Fourier analysis:** frequency & wavenumber

Brain Oscillations

"Alpha oscillation"

(10 cycles per second)





Berger, *Über das Elektroenkephalogramm des Menschen* 1929

Brain Oscillations & Cognition

Spontaneous Neural Oscillations Bias Perception by Modulating Baseline Excitability

Luca Ien

Oscillatory dynamics coordinating human frontal networks in support of goal maintenance

Bradley Voy
Josef Parviz

Opinion TRENDS in Cognitive Sciences Vol.9 No.10 October 2005
Full text provided by www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER

A mechanism for cognitive dynamics: neuronal communication through neuronal coherence

Pascal Fries[1,2]

Neural oscillations are **ubiquitous** and tightly linked to **cognition**.

Neurological Progress

High-frequency oscillations as a new biomarker in epilepsy

Maeike Zijlman

Frans S.S. Leijte

Abnormal neural oscillations and synchrony in schizophrenia

Peter J. Uhlhaas‡ and Wolf S*

Theta and Alpha Oscillation Impairments in Autistic Spectrum Disorder Reflect Working Memory Deficit

Josefina Larrain-Valenzuela[1], Francisco Zamorano[1,2], Patricia Soto-Icaza[3], Ximena Carrasco[3], Claudia Herrera[4], Francisca Daiber[3], Francisco Aboitiz[3] & Pablo Billeke[1]

Neural oscillations often go awry in **developmental**, **neurological**, and **psychiatric** disorders.

Neural Oscillation

Neural oscillations are a fundamental mechanism that enables the synchronization of neural activity within and across brain regions and promotes the precise temporal coordination of neural processes underlying cognition, memory, perception, and behavior.

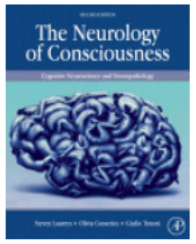
From: The Neurobiology of Schizophrenia, 2016

ACADEMIC PRESS

The Neurology of Conciousness (Second Edition)

Cognitive Neuroscience and Neuropathology

2016, Pages 49-60



Chapter 3 - Neuronal Oscillations, Coherence, and Consciousness

Andreas K. Engel [1], Pascal Fries [2, 3]

Brain Oscillations & Development

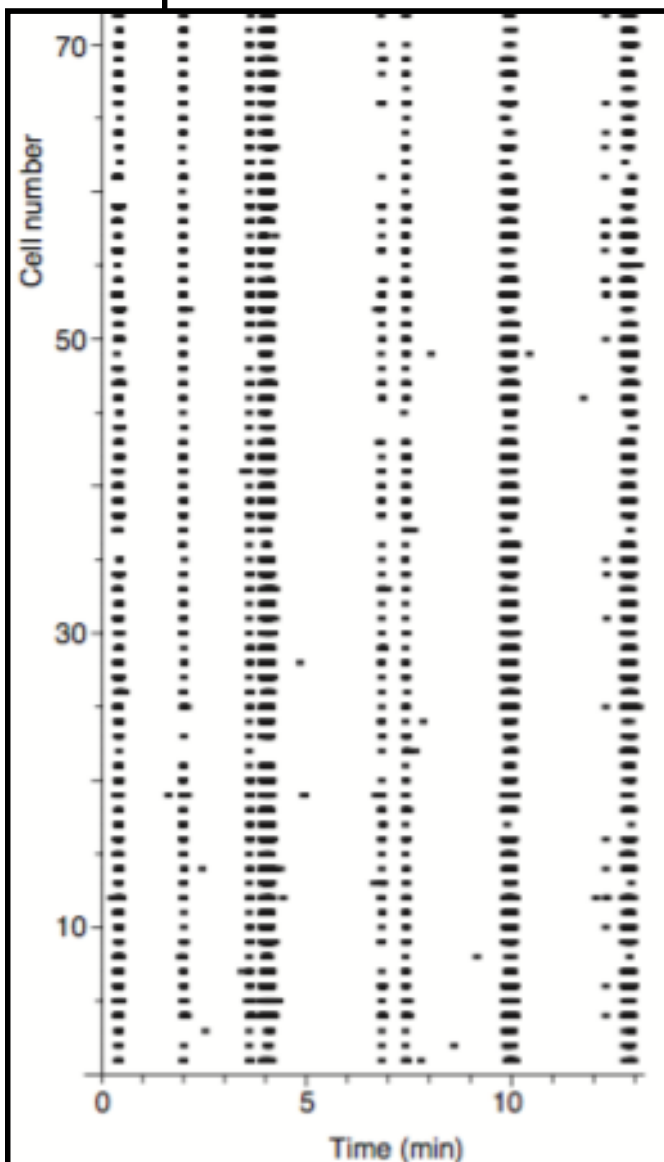
Synchronous bursts of action potentials in ganglion cells of the developing mammalian retina

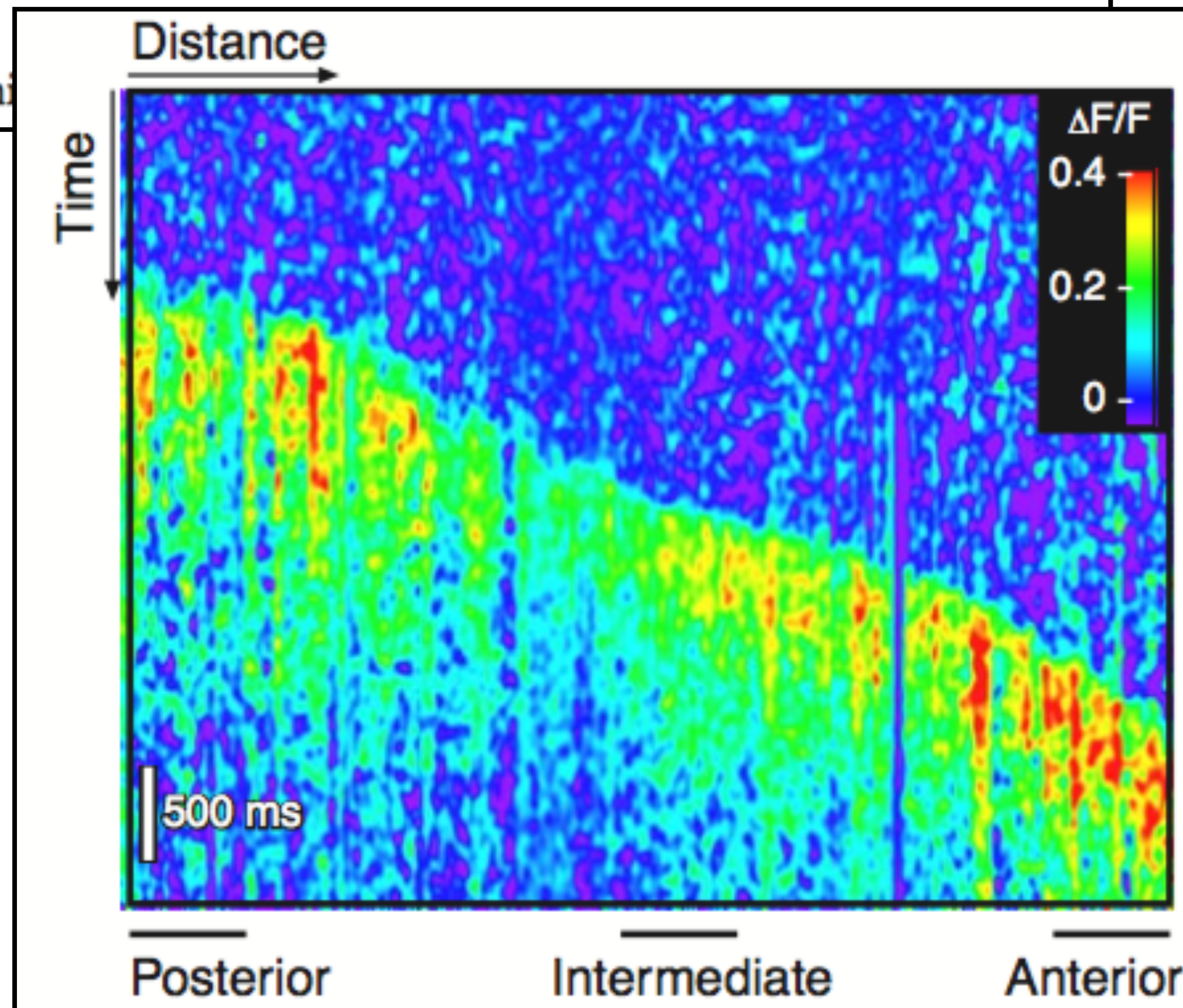
M Meister, RO Wong, D

+ See all authors and af

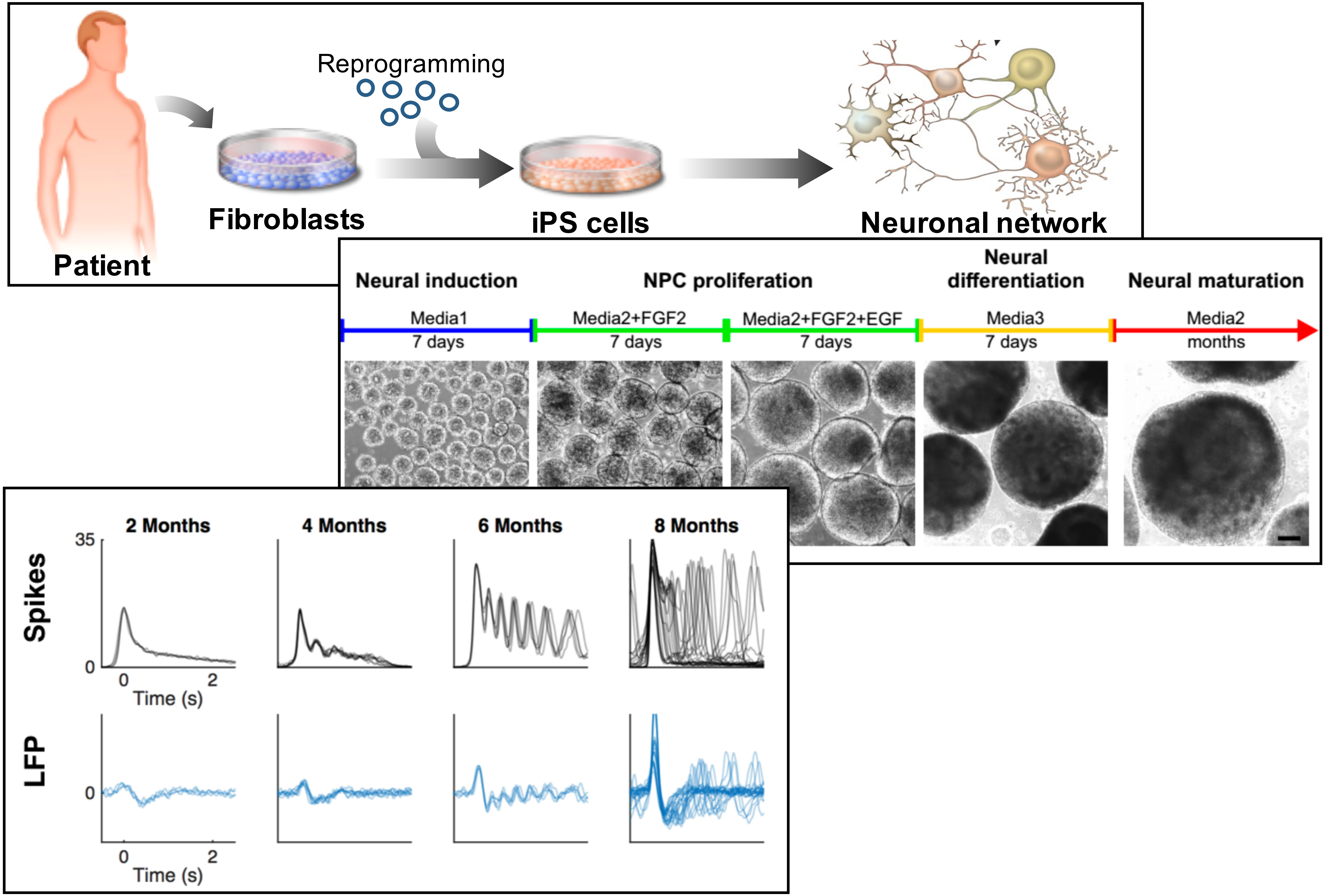
Science 17 May 1991:

Large-scale oscillatory calcium waves in the immature cortex

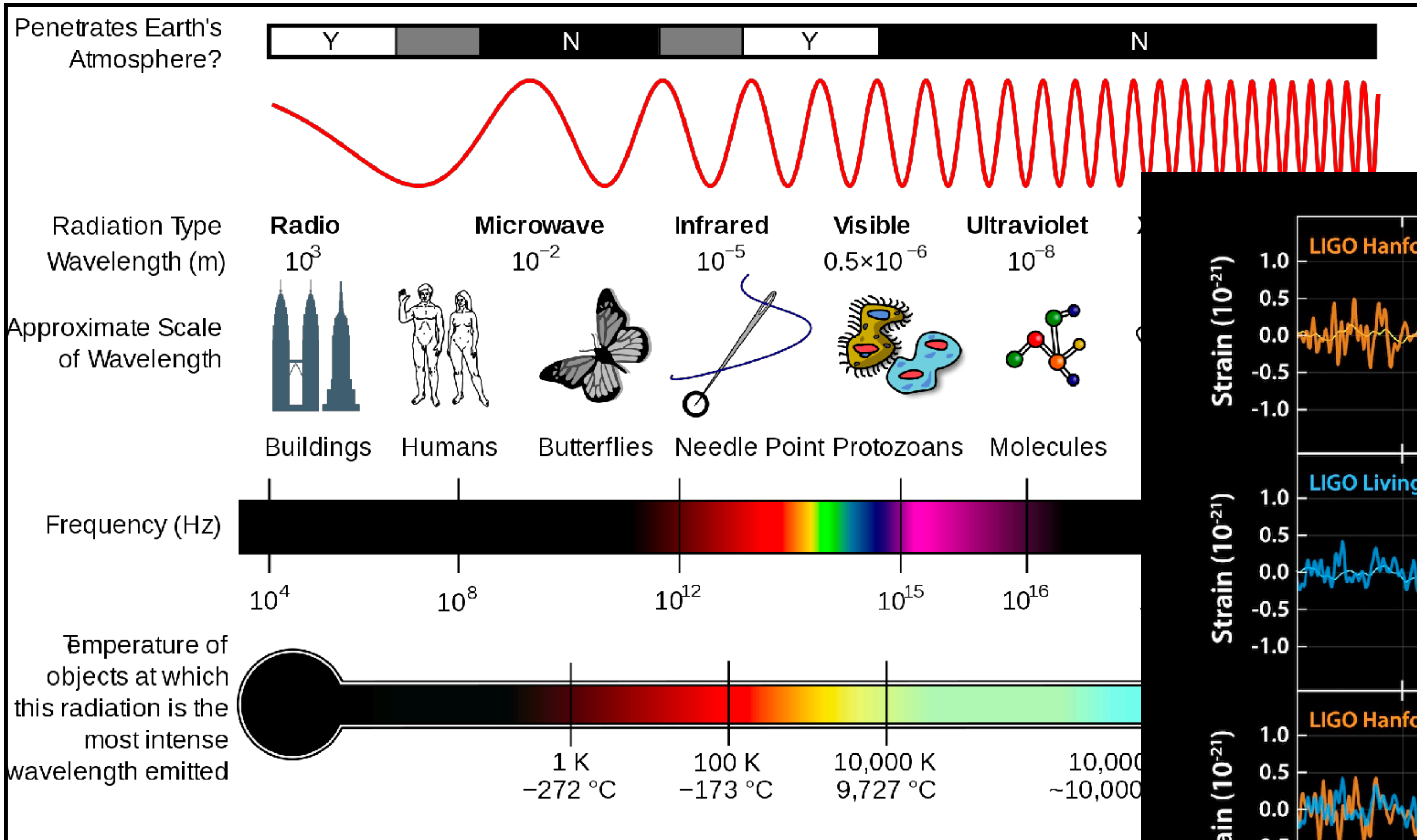


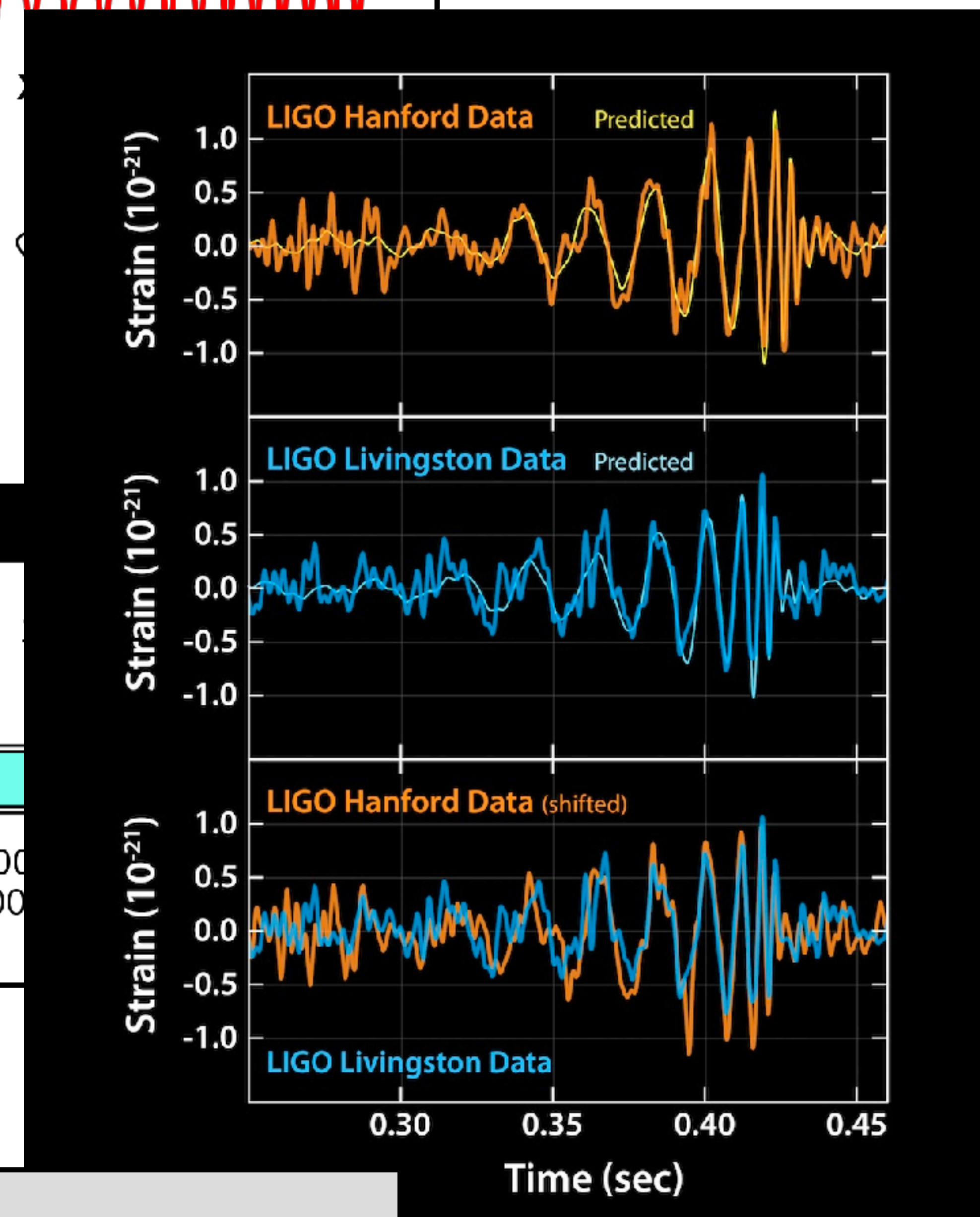


Brain Oscillations & Development



Oscillations in Nature





Gravitational Waves

Gravitational waves are ripples in space-time (the fabled "fabric" of the Universe) caused by massive objects moving with extreme accelerations (in outer space that means objects like neutron stars or black holes orbiting around each other at ever increasing rates, or stars that blow themselves up). Explore the links below to learn more about these ephemeral phenomena.

Oscillations in Nature

Analyzing Oscillations

These oscillations are often combined together and embedded in noises.

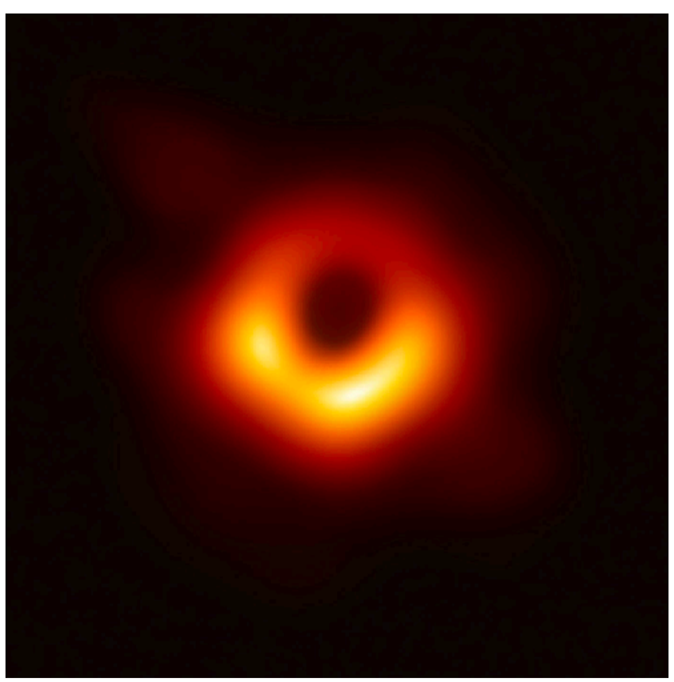
We want to have a method for separating them.

How the Event Horizon Telescope imaged an invisible black hole

Hint: it's much harder than it sounds.

By Erika K. Carlson | Published: Wednesday, April 10, 2019

RELATED TOPICS: UNIVERSE | BLACK HOLES



Black hole photography 101

Radio astronomers can get around this problem by linking together lots of smaller radio dishes into a single array, where they effectively act as one giant telescope. Each dish in the array collects light from a target object, like the glowing disk around a black hole, and converts the radio waves it receives into an electronic signal.

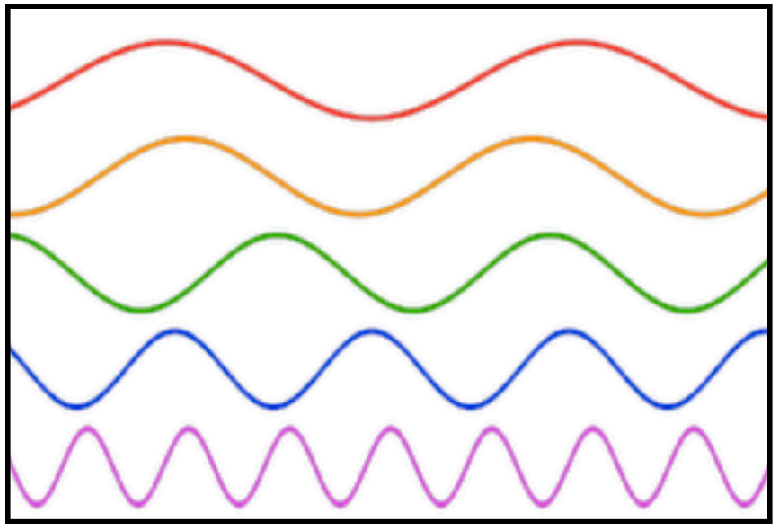
They then used a computer called a correlator to combine all the electronic signals from the various dishes into what's called an interference pattern. Finally, astronomers tapped a special kind of math (Fourier transforms for the curious) to decode that pattern, showing what the target would look like in the sky if our eyes could see in radio wavelengths.

Fourier Decomposition: represent an **infinite-length** signal as the summation of a number of **infinite-length** cosine and sine waves.

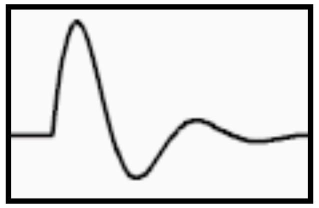
$$cos(\omega t)$$

$$sin(\omega t)$$

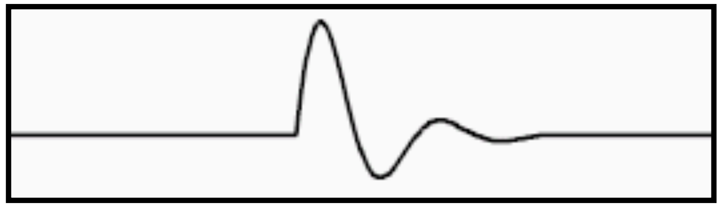
$$for\ t \in (-\infty, \infty)$$



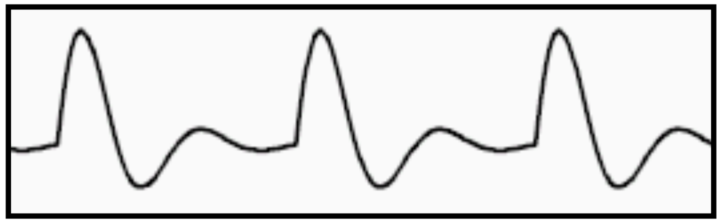
But real life signals are always **finite**.



extend as aperiodic signal (with 0s)



extend as periodic signal



Family of Fourier Transforms **(read this link)**

Continuous Time | **Discrete Time**

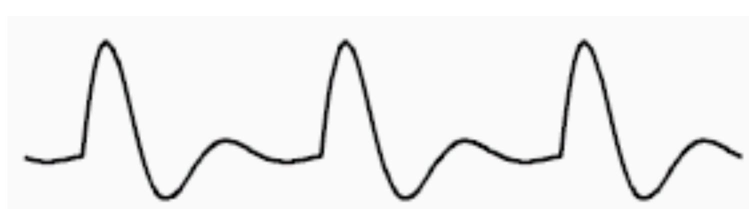
Aperiodic

(Continuous)
Fourier Transform (CFT)

Discrete Time
Fourier Transform (DTFT)

Periodic

Fourier Series (CFS)



Discrete Fourier
Transform (DFT)

Discrete Fourier Transform

Definition [edit]

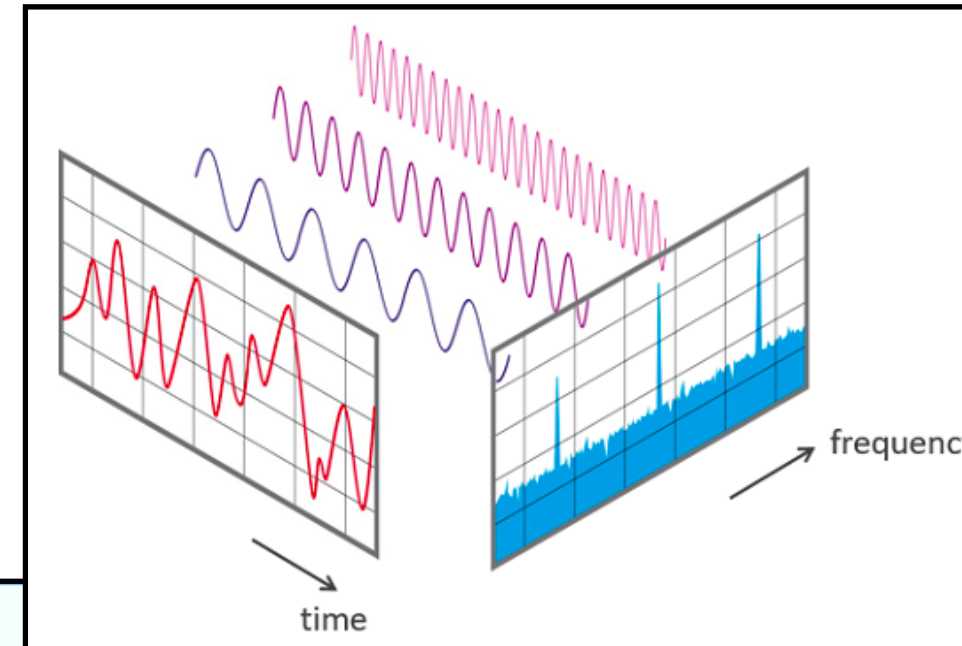
The *discrete Fourier transform* transforms a sequence of N complex numbers $\{\mathbf{x_n}\} := x_0, x_1, \ldots, x_{N-1}$ into another sequence of complex numbers, $\{\mathbf{X_k}\} := X_0, X_1, \ldots, X_{N-1}$, which is defined by

$$
\begin{aligned}
X_k &= \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn} \\
&= \sum_{n=0}^{N-1} x_n \cdot [\cos(2\pi kn/N) - i \cdot \sin(2\pi kn/N)],
\end{aligned}
\qquad \textbf{(Eq.1)}
$$

where the last expression follows from the first one by Euler's formula.

The transform is sometimes denoted by the symbol $\mathcal{F}$, as in $\mathbf{X} = \mathcal{F}\{\mathbf{x}\}$ or $\mathcal{F}(\mathbf{x})$

Components of the DFT



Complex Exponential

x_n's Fourier Transform at frequency *k*

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn}$$

Euler's Formula

$$= \sum_{n=0}^{N-1} x_n \cdot [\cos(2\pi kn/N) - i \cdot \sin(2\pi kn/N)], \qquad \textbf{(Eq.1)}$$

$$S(\omega) = \int_{-\infty}^{\infty} s(t) e^{-i\omega t} dt$$

CFT

Dot Product

(angular) Wave number

Sine Wave

Goals for Today

1. Introduce (brain) oscillations & Fourier analysis

2. **Fourier analysis:** complex exponential & sinusoids

3. **Fourier analysis:** frequency & wavenumber

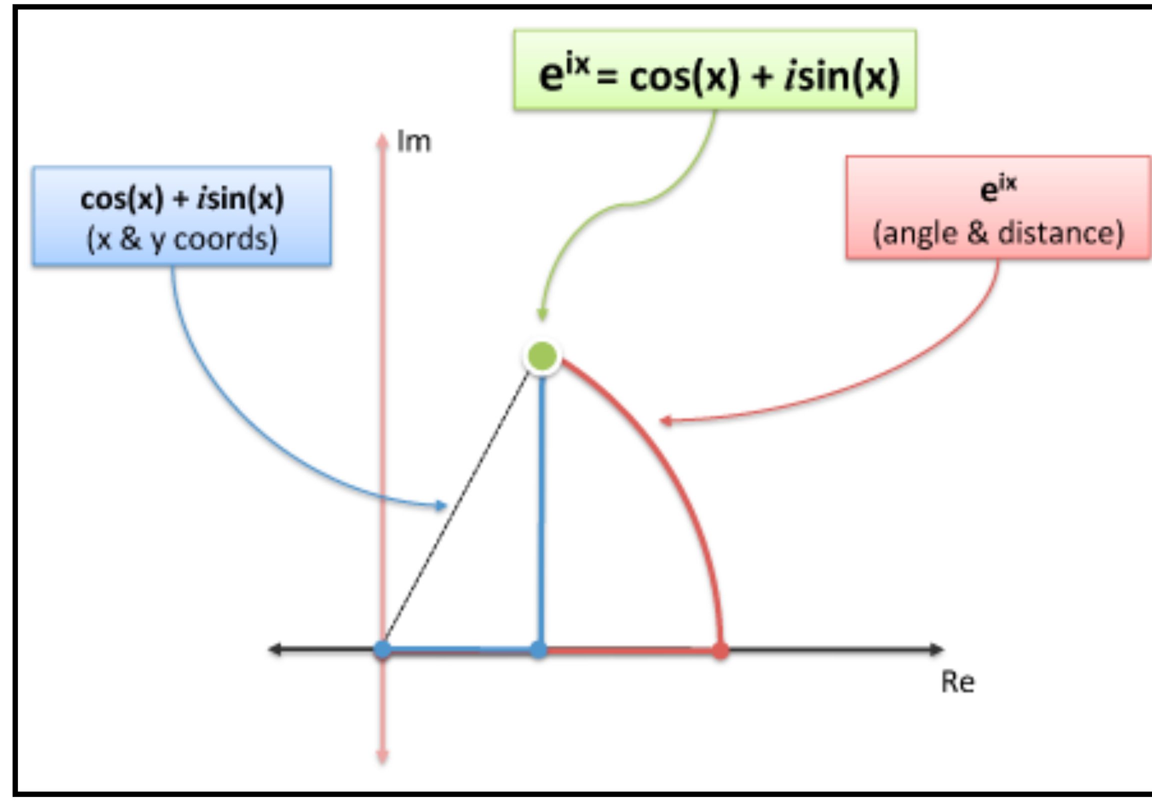
(Co)Sine Waves and Complex Exponentials

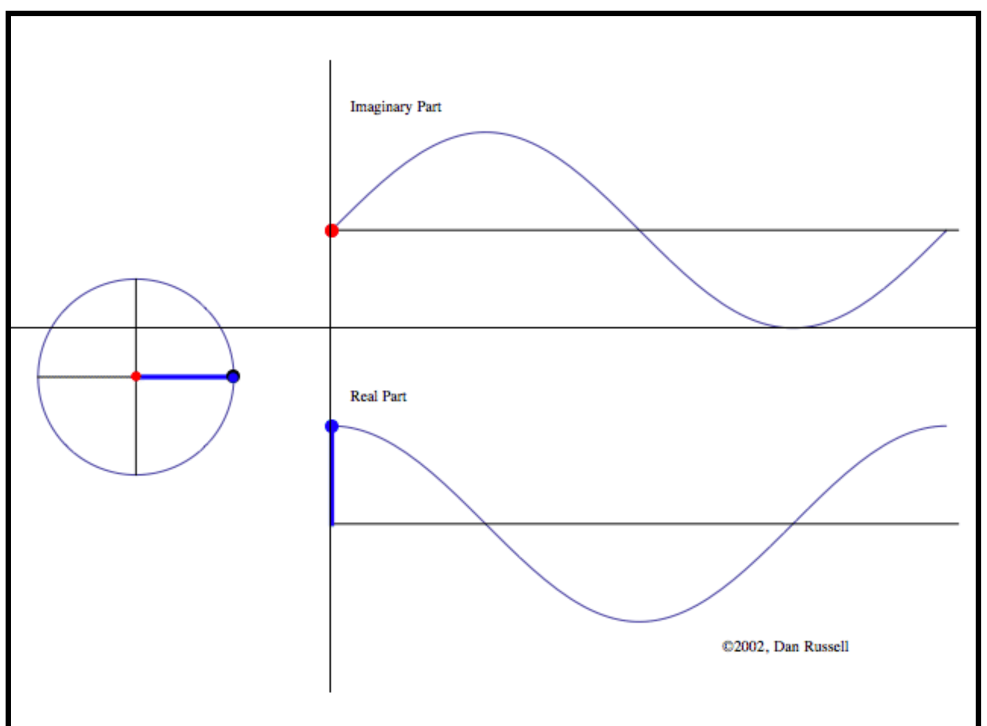
Complex Exponential

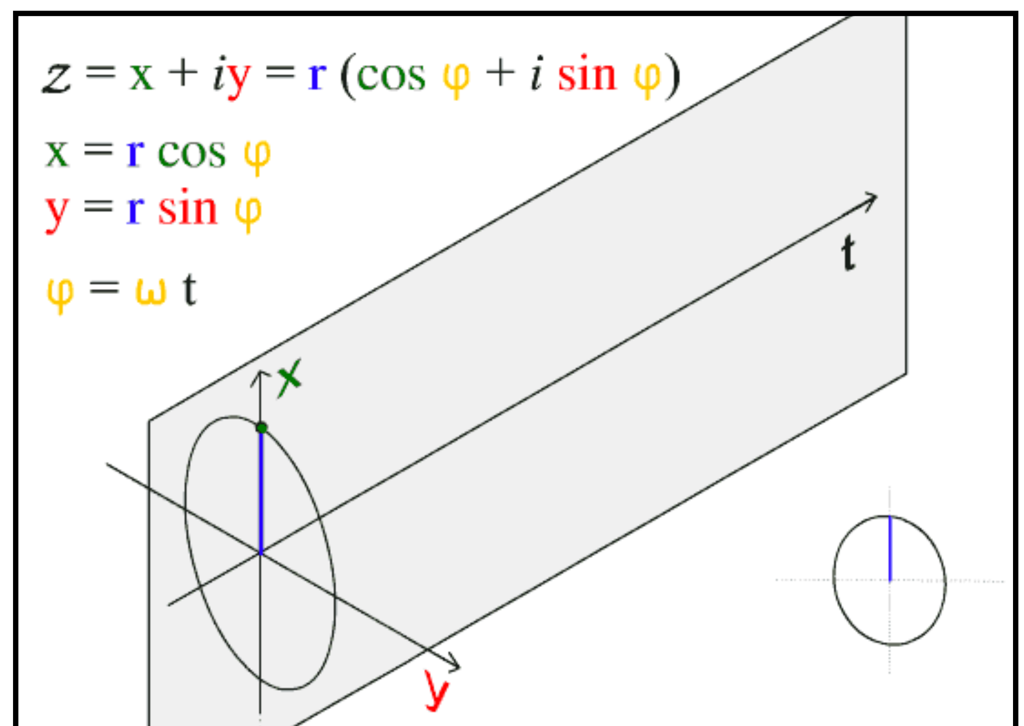
$$\boxed{\sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn}}$$

$$\omega = \frac{2\pi k}{N} \rightarrow e^{-i\omega n}$$

$$\boxed{[\cos(2\pi kn/N) - i \cdot \sin(2\pi kn/N)]}$$





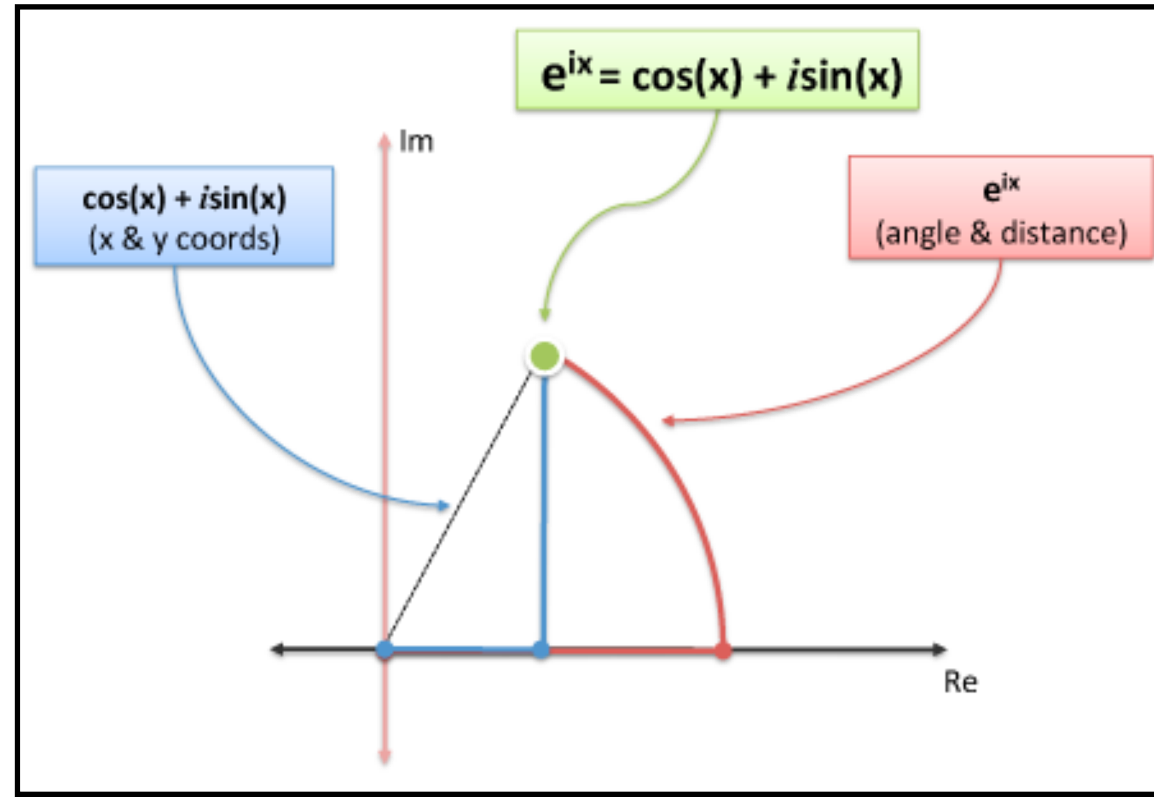


(Co)Sine Waves and Complex Exponentials

Complex Exponential

$$\boxed{X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn}} \quad \omega = \frac{2\pi k}{N} \rightarrow e^{-i\omega n}$$

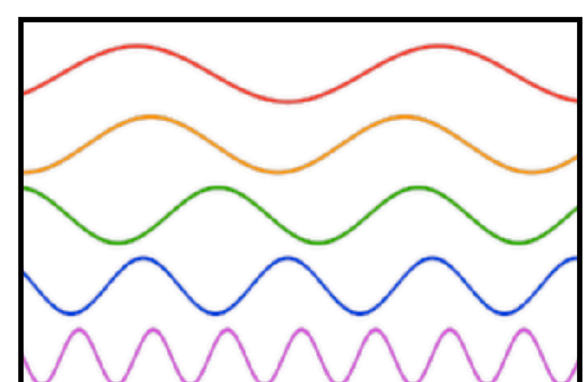
$$\boxed{[\cos(2\pi kn/N) - i \cdot \sin(2\pi kn/N)]}$$



Cosine is the projection of the complex exponential onto the real axis.

Sine is the projection of the complex exponential onto the imaginary axis.

$$\omega = \frac{2\pi k}{N}$$ (discrete angular velocity) is how much the vector rotates per time step, n.



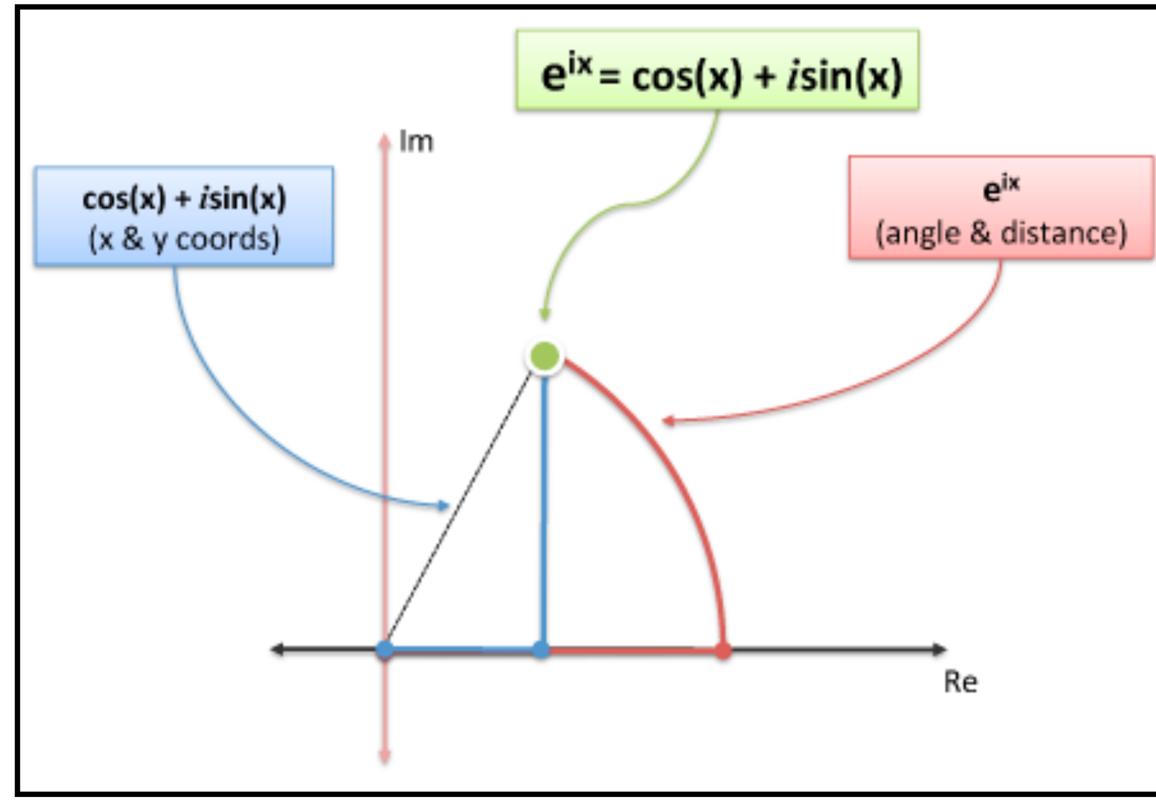
bigger k

(Co)Sine Waves and Complex Exponentials

Complex Exponential

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn}$$

$$[\cos(2\pi kn/N) - i \cdot \sin(2\pi kn/N)]$$



Cosine is the projection of the complex exponential onto the real axis.

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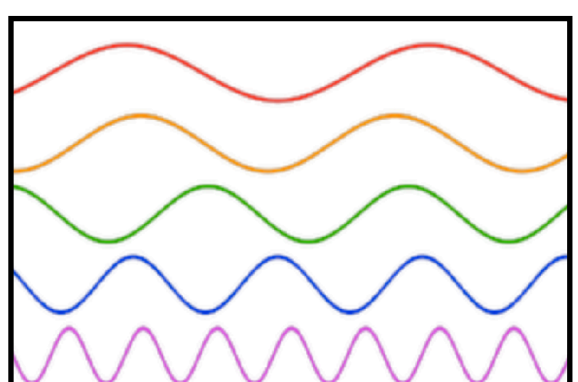
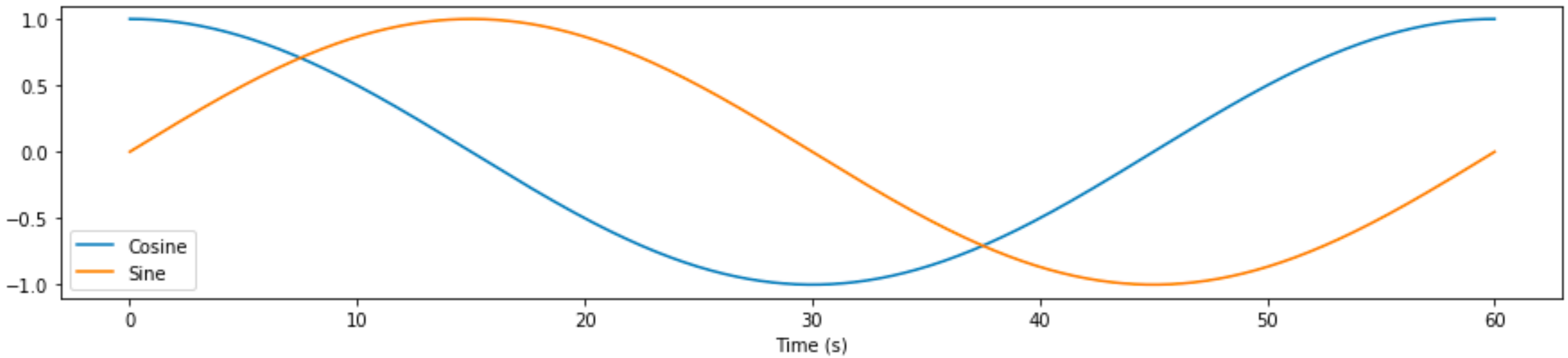


bigger k (wave number)

Cosine & Sine Wave



What is the frequency of this cosine/sine wave? 1/60 Hz

To make a sine wave of frequency *f: (Hz)*

$$s(t) = sin(2\pi f t)$$

in python: np.sin(2 * np.pi * f * t_vector)

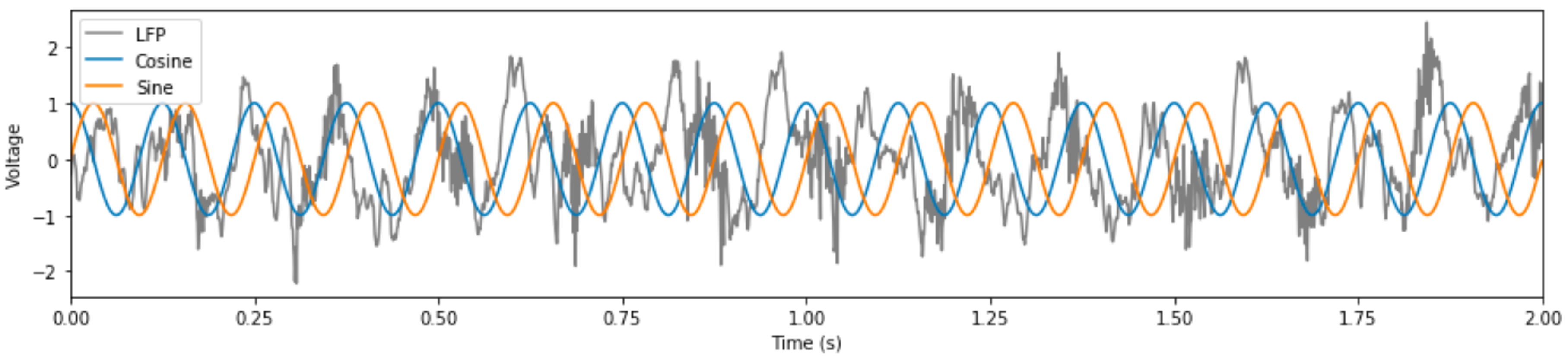
Tip: when t=1 (after 1 second), it has gone through *f* revolutions (cycles), or 2πf radians.

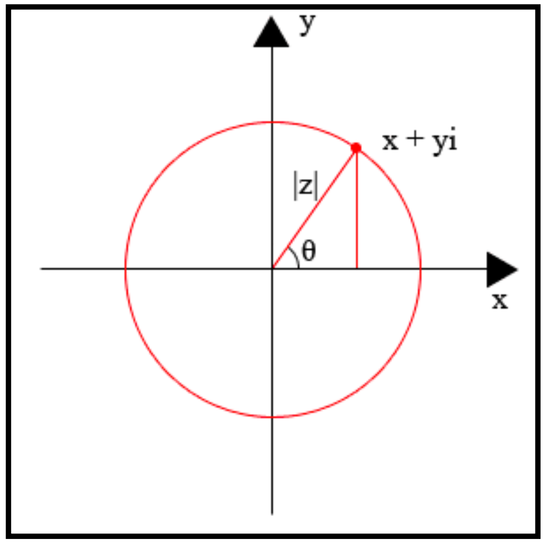
Dot Product & Projection

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn}$$
$$= \sum_{n=0}^{N-1} x_n \cdot [\cos(2\pi kn/N) - i \cdot \sin(2\pi kn/N)]$$

$$dot(x(n), cos_k(n)) - i * dot(x(n), sin_k(n))$$

$$\boxed{\Rightarrow X_k = a_k + ib_k}$$

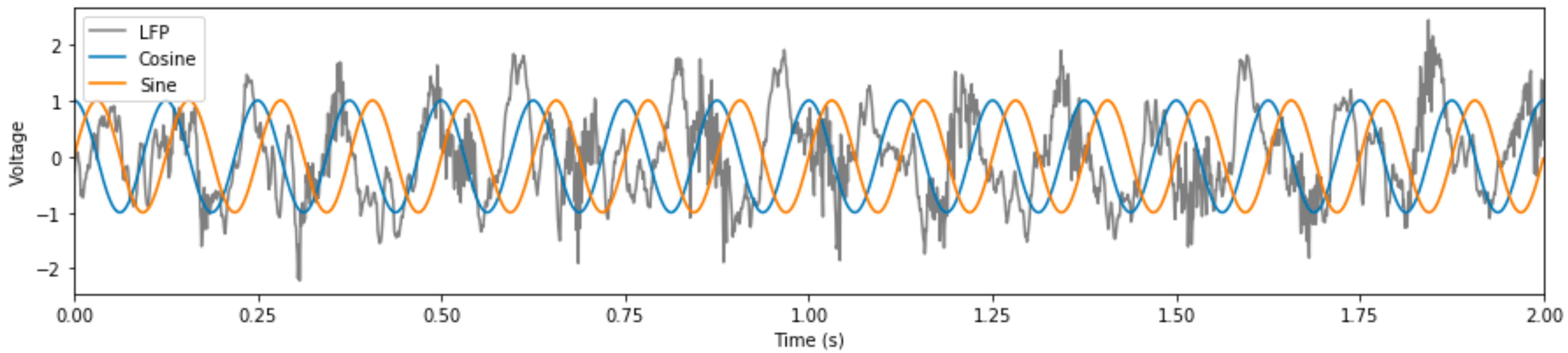




Dot Product & Projection

The dot product acts like sieve for frequency f: it measures the contribution of the cosine & sine waves at f in the signal of interest.

Why not just use cosine OR sine? Why both?



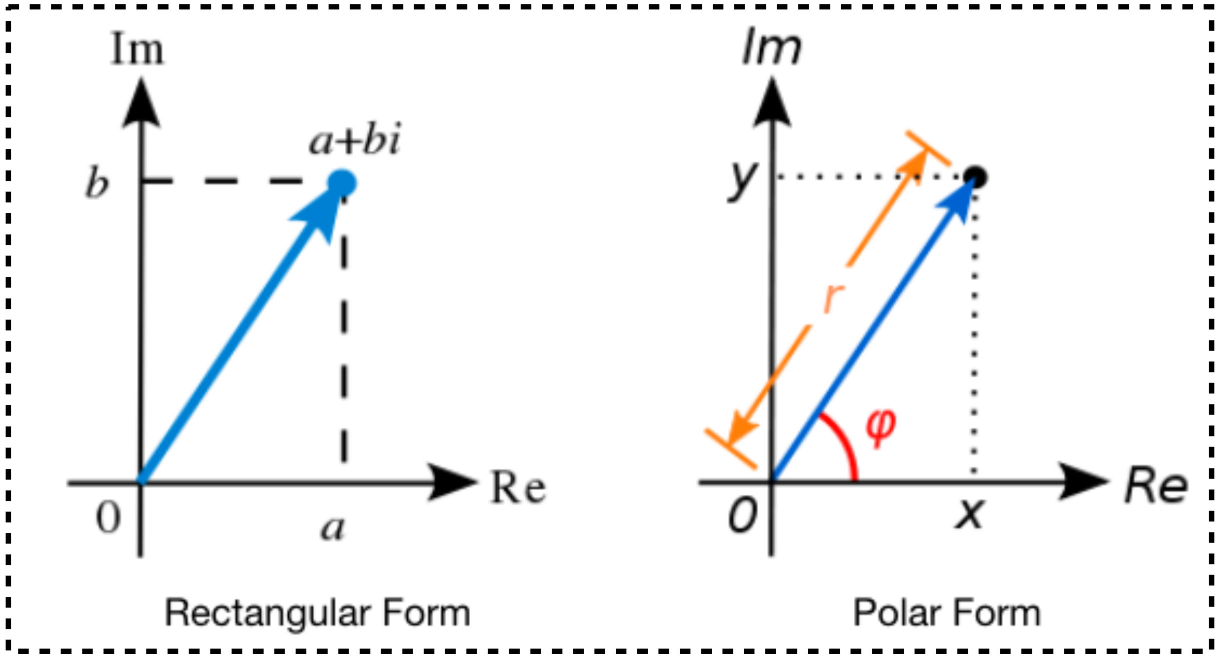
Answer: **phase shift**
Need both cosine and sine to completely capture all possible contributions.

Analogy: the 2 dot products are like measuring with 2 rulers.

Fourier Coefficient: Power & Phase

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn}$$
$$= \sum_{n=0}^{N-1} x_n \cdot [\cos(2\pi kn/N) - i \cdot \sin(2\pi kn/N)]$$

$$\Rightarrow X_k = a_k + ib_k$$



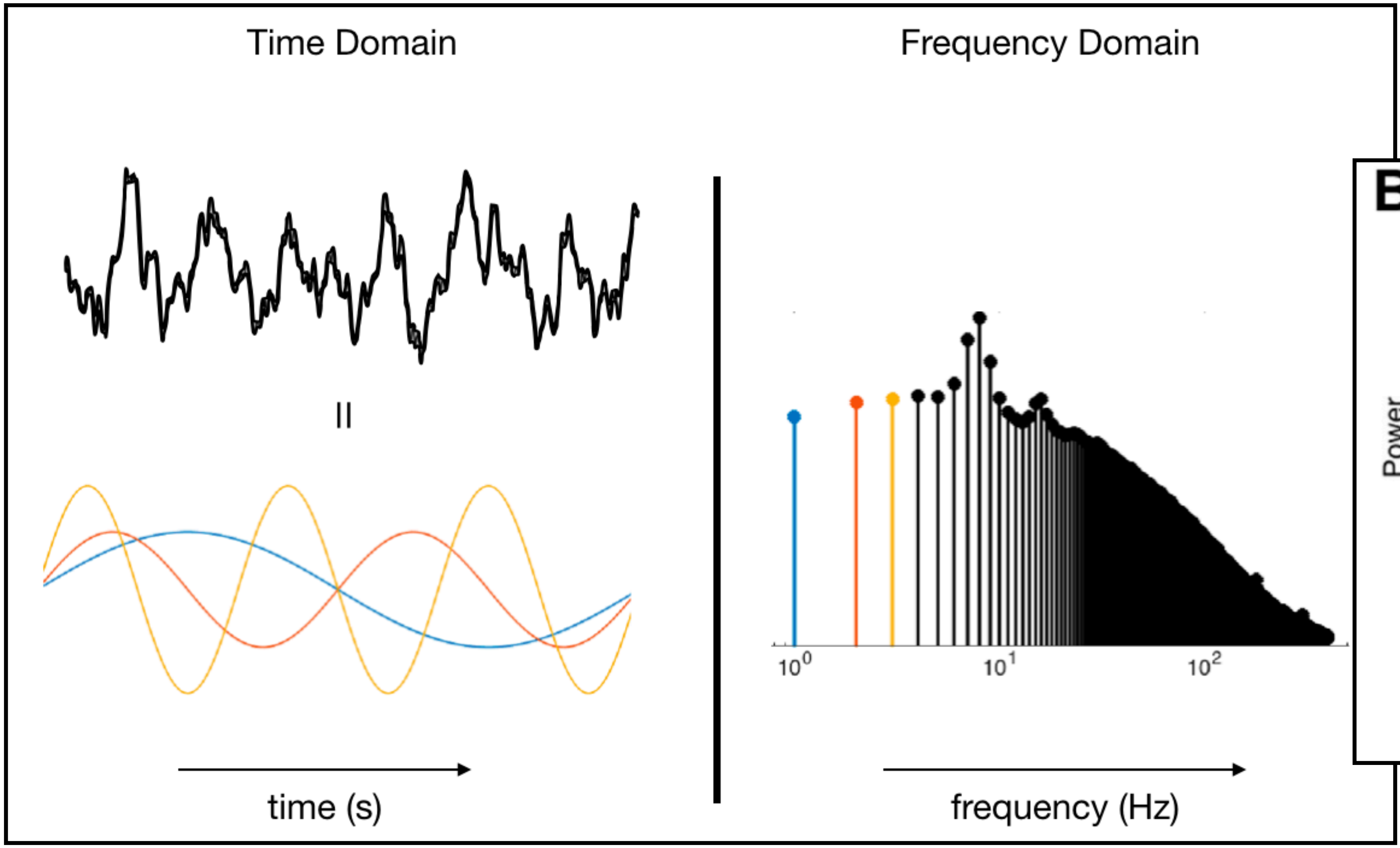
$$|X_k| = \sqrt{a_k^2 + b_k^2}$$

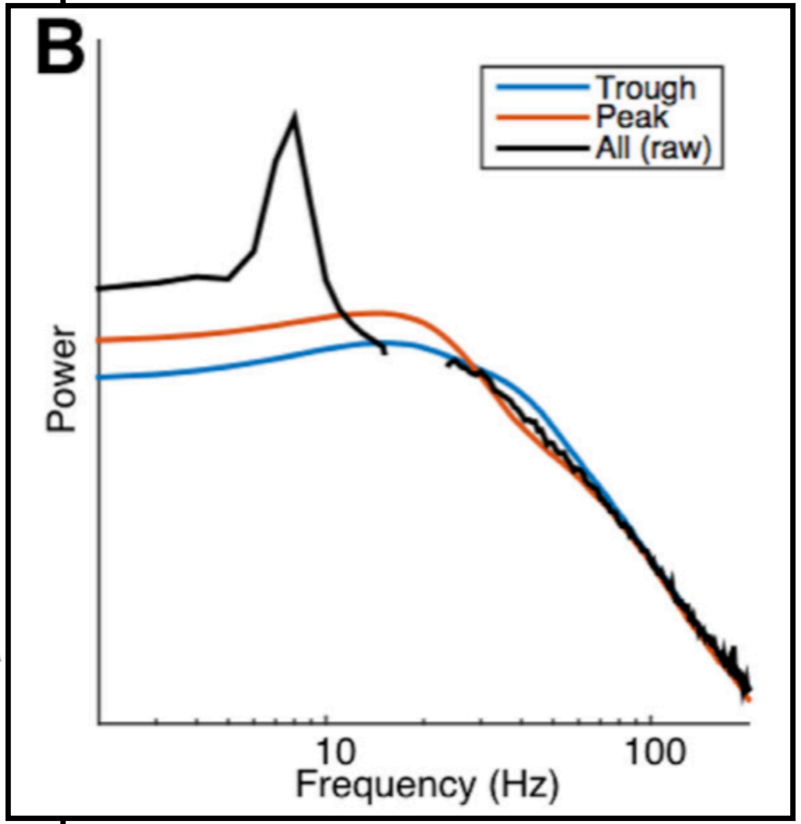
$$power(k) = |X_k|^2 = a_k^2 + b_k^2$$

$$phase(k) = tan^{-1}\frac{b_k}{a_k}$$

Power measures the energy (variance) contribution to the signal
Phase measures the angular delay with respect to the cosine

Power Spectrum





We've thrown out the phase information.

To reconstruct the signal from its Fourier Transform, you need **both power and phase**, or both coefficients a and b.

1. Introduce (brain) oscillations & Fourier analysis

2. **Fourier analysis:** complex exponential & sinusoids

3. **Fourier analysis:** frequency & wavenumber

How Many Frequencies?

$\{\mathbf{x_n}\} := x_0, x_1, \ldots, x_{N-1}$ into another sequence of complex numbers, $\{\mathbf{X_k}\} := X_0, X_1, \ldots, X_{N-1}$, which is defined by

$$\begin{aligned} X_k &= \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn} \\ &= \sum_{n=0}^{N-1} x_n \cdot [\cos(2\pi kn/N) - i \cdot \sin(2\pi kn/N)], \end{aligned} \qquad \textbf{(\textit{Eq.1})}$$

We can measure the contribution of a single frequency, f, but which frequencies to include so we can fully decompose the signal?

The answer is in the equation: **k goes from 0 to N-1**

You need as many frequencies as you have time points!

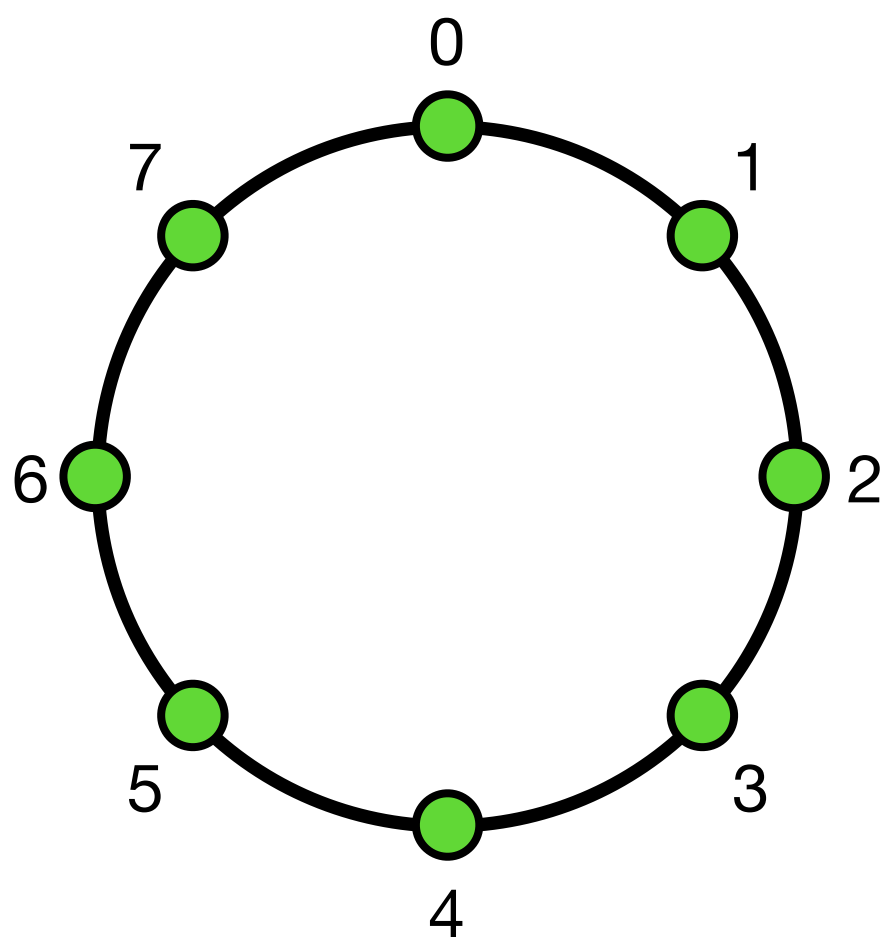
Wave Number to Frequency

$$= \sum_{n=0}^{N-1} x_n \cdot [\cos(2\pi kn/N)$$

Suppose N = 8

-> k = 0, 1, … 7

for a given k, n = 0, 1, … 7



The Real Question: what does the wave number correspond to in frequency (Hz)?

This depends on your signal’s sampling rate.

Wave Number to Frequency

$$= \sum_{n=0}^{N-1} x_n \cdot [\cos(2\pi k n/N)$$

Suppose N = 8

-> k = 0, 1, … 7

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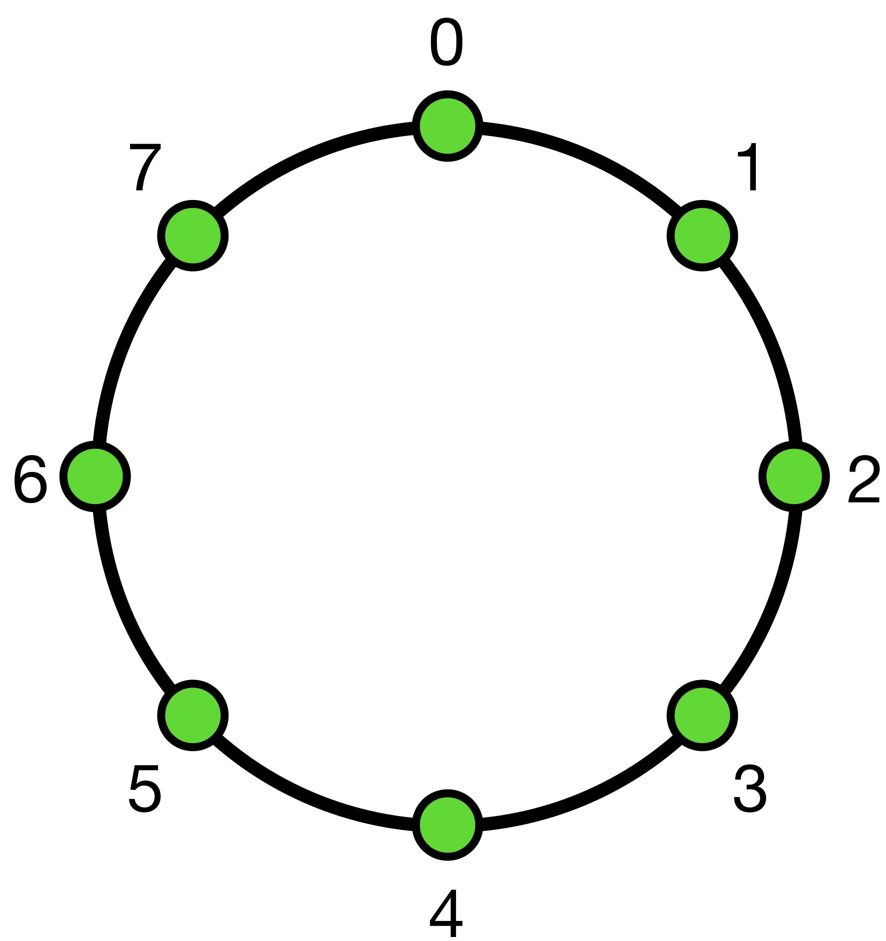


How many times around the circle does k=1 wave go around in N steps?

How much time (in seconds) does each time step represent?

How much time (in seconds) does each time step represent?

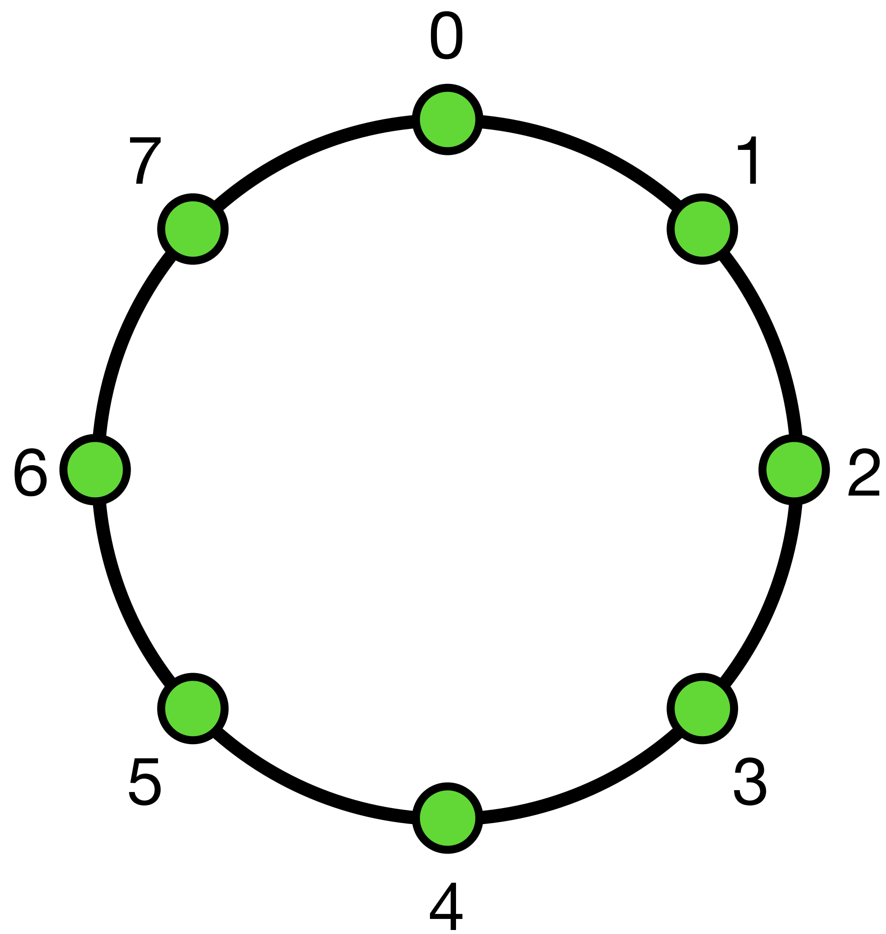
DFT Frequency Axis

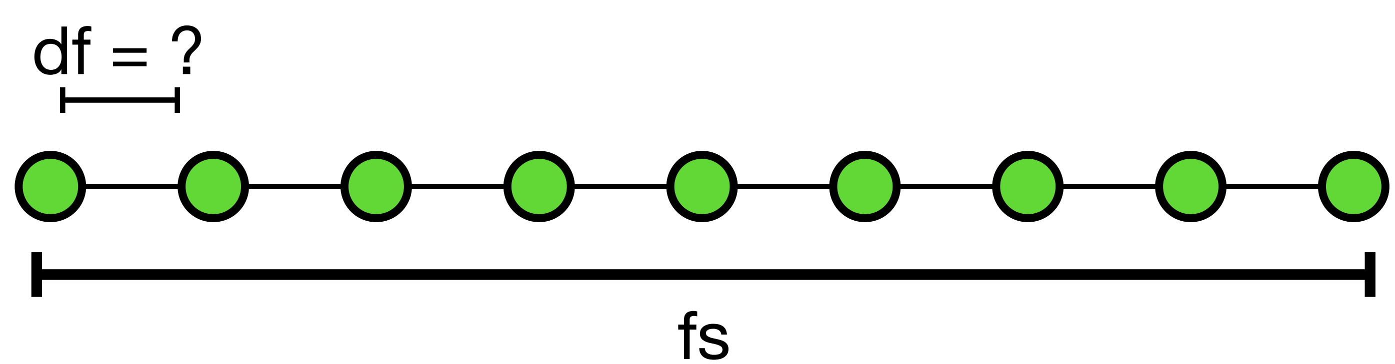
$$= \sum_{n=0}^{N-1} x_n \cdot [\cos(2\pi kn/N)$$

Suppose N = 8

-> k = 0, 1, … 7

for a given k, n = 0, 1, … 7





Summary

1. Introduce (brain) oscillations & Fourier analysis

2. **Fourier analysis:** complex exponential & sinusoids

3. **Fourier analysis:** frequency & wavenumber

https://tinyurl.com/cogs118c-att