MA 341: HW

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Problem 1

Let X be a set with $A, B, C \subseteq X$

Questions

a Prove that A ∩ B ⊆ A, A ∩ B ⊆ B. Determine when $A \cap\ B = A$

b Prove that $A \subseteq A \cup B$, and $B \subseteq A \cup B$. Determine when $A \cup B = A$

c Prove that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

d Prove that if $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cap B$

e Prove that if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$

Answers

a For all $a \in A \cap B, a \in A$ and $a \in B$ by the definition of intersection
By definition of proper subset it immediately follows that $A \cap B \subseteq A$ and $A \cap B \subseteq B$
$A \cap B = A$ when $A \subseteq B$ as all elements $a \in A$ would also exist in B, which allows the definition of intersection to create A.

b By the definition of union all $a \in A$ and $b \in B$, $a, b \in A \cup B$
By definition of subset $A, B \subseteq A \cup B$ as elements of both sets are included in the union.
$A \cup B = A$ when $B \subseteq A$ as no new elements in B would be added to A in the union.

c $A \subseteq B$ implies for all $a \in A$ $a \in B$.
$B \subseteq C$ implies for all $b \in B$ $b \in C$.
Finally this implies $a \in A, B, C$ and $A \subseteq B \subseteq C \rightarrow A \subseteq C$

d For all $c \in C, C \subseteq A$ implies $c \in A$
For all $c \in C, C \subseteq B$ implies $c \in B$
All $c \in C, c \in A, B$ which is the definition of $A \cap B$.
Therefore $C \subseteq A \cap B$

e $A \subseteq C$ implies for all $a \in A, a \in C$
$B \subseteq C$ implies for all $b \in B, b \in C$
$A \cup B = \{a : a \in A \text{ and } B\}$ which immediatley implies $A \cup B \subseteq C$

Problem 2

Let X be a set and $A \subseteq X$

Questions

a Prove that $A \cap (X - A) = \emptyset$

b Prove that $A \cup (X - A) = X$

Answer

a $X - A = \{x \in X : x \ni A\}$
X and $X - A$ are by definition disjoint and the intersection of disjoint sets is the empty set.
Therefore $A \cap X - A = \emptyset$

b Using $X - A$ as defined above
$A \cup X - A$ states the elements $a \in A$ as well as the elements $x \in X and x \ni A$. Combining the elements of A and the elements of X excluding only A trivially yields X.

Problem 3

Questions

a Prove that for any subset $I \subseteq \mathcal{P}(X)$, we have
$$f(\bigcup_{A \in I} A) = \bigcup_{A \in I} f(A).$$

b Prove that for any subset $I \subseteq \mathcal{P}(X)$, we have
$$f(\bigcap_{A \in I} A) \subseteq \bigcap_{A \in I} f(A).$$

c Prove that for any subset $J \subseteq \mathcal{P}(Y)$, we have
$$f^{-1}(\bigcup_{B \in J} B) = \bigcup_{B \in J} f^{-1}(B).$$

d Prove that for any subset $J \subseteq \mathcal{P}(Y)$, we have
$$f^{-1}(\bigcap_{B \in J} B) = \bigcap_{B \in J} f^{-1}(B)$$

Answers

a Must prove $f(\bigcup_{A \in I} A) \subseteq \bigcup_{A \in I} f(A)$ and $\bigcup_{A \in I} f(A) \subseteq f(\bigcup_{A \in I} A)$
$\exists x \in \bigcup_{A \in I} A$ such that $f(x) = y$
$\exists x \in A_i$ for some i such that $f(x) = y$
$y \in f(A_i) \to f(A_i) \subseteq \bigcup_{A \in I} A$
Therefore $f(\bigcup_{A \in I} A) \subseteq \bigcup_{A \in I} f(A)$
Let $y \in \bigcup_{A \in I} f(A) \to y \in f(A_i)$ for some A_i
$\exists \in A_i$ such that $f(x) = y$
$x \in \bigcup_{A \in I} A$
$y \in f(\bigcup_{A \in I} A)$ therefore $\bigcup_{A \in I} f(A) \subseteq f(\bigcup_{A \in I} A)$
$f(\bigcup_{A \in I} A) \subseteq \bigcup_{A \in I} f(A)$ and $\bigcup_{A \in I} f(A) \subseteq f(\bigcup_{A \in I} A) \to f(\bigcup_{A \in I} A) = \bigcup_{A \in I} f(A)$

b Let $y \in \bigcap f(A_i)$. $y \in f(A_i)$ for all $A_i \in I$.
$\exists x \in A_1$ for all $A \in I$ such that $y = f(x)$
Therefore $y \in f(\bigcap A_i) \to \bigcap F(A) \subseteq F(\bigcap A)$

c By definition of preimage, this proof follows exactly as A.
Must prove $f^{-1}(\bigcup_{B\in J} B) \subseteq \bigcup_{B\in J} f^{-1}(B)$ and $\bigcup_{B\in J} f^{-1}(B) \subseteq f^{-1}(\bigcup_{B\in J} B)$
$\exists x \in \bigcup_{B\in J} B$ such that $f^{-1}(x) = y$
$\exists x \in B_i$ for some i such that $f^{-1}(x) = y$
$y \in f^{-1}(B_i) \rightarrow f^{-1}(B_i) \subseteq \bigcup_{B\in J} B$
Therefore $f^{-1}(\bigcup_{B\in J} B) \subseteq \bigcup_{B\in J} f^{-1}(B)$
Let $y \in \bigcup_{B\in J} f^{-1}(B) \rightarrow y \in f^{-1}(B_i)$ for some B_i
$\exists \in B_i$ such that $f^{-1}(x) = y$
$x \in \bigcup_{B\in J} B$
$y \in f^{-1}(\bigcup_{B\in J} B)$ therefore $\bigcup_{B\in J} f^{-1}(B) \subseteq f^{-1}(\bigcup_{B\in J} B)$
$f^{-1}(\bigcup_{B\in J} B) \subseteq \bigcup_{B\in J} f^{-1}(B)$ and $\bigcup_{B\in J} f^{-1}(B) \subseteq f^{-1}(\bigcup_{B\in J} B) \rightarrow f^{-1}(\bigcup_{B\in J} B) = \bigcup_{B\in J} f^{-1}(B)$

d Let $x \in f^{-1}(\bigcap A) \rightarrow f(x) \in \bigcap B$
$f(x) \in alB_i$
$x \in$ all $f^{-1}(B_i)$
$x \in \bigcap f^{-1}(B)$
Therefore $f^{-1}(\bigcap B) = \bigcap f^{-1}(B)$

Problem 4

Let X, Y, Z be sets and $f : X \mapsto Y, g : Y \mapsto Z$ be functions

Questions

a Prove that if f, g are one-to-one, then $g \circ f$ is one-to-one.

b Prove that if $g \circ f$ is one-to-one then f is one-to-one. Give an example where $g \circ f$ is one to one but g is not one-to-one.

c Prove that if f, g are onto, then $g \circ f$ is onto.

d Prove that if $g \circ f$ is onto, then g is onto. Give an example where $g \circ f$ is onto but f is not onto.

Answer

a Let $a, b \in X$. If $f(a) = f(b)$ because f is injective
$g(f(a)) = g(f(b))$ because g is injective. It then must be true that $g \circ f$ is injective.

b For $x, y \in X$ let $f(x) = g(y)$.
It follows that $g(f(x)) = f(g(y))$. Thus $x = y$ because $g \circ f$ is injective

For g is not one-to-one: Let X = {1}, Y = {2,3}, Z{4}
Let f(1) = 2, g(2) = 4, g(3) = 4.
Here the composition of $g \circ f$ is injective, but g is not because g(2) = g(3) and 2 $\neq$ 3

c Assume $g \circ f$ is not onto given f, g are onto. This would mean $\exists b \in Z$ such that for all elements $a \in X g(f(a)) \neq b$ this contradicts the surjective quality of g, so $g \circ f$ must be surjective.

d Since $g \circ f$ is surjective then for all $c \in Z$ there exists an $a \in X$ such that $c = g(f(a))$. Then there exists some b such that $b = f(a) \in B$ which satisfies $g(b) = c$. Therefore g is onto.

Let $X = \{1\}, Y = \{1, 2\}, Z = \{1\}$
$f(1) = 1, g(1) = 1, g(2) = 1.$
$g \circ f$ is surjective but f is not.

Problem 5

Question

Give an example of a function $f : X \mapsto Y$ and subsets $A, B \subseteq X$ such that $f(A \cap B) \neq f(A) \cap f(B)$.

Answer

$A = \{0\}, B = \{1, ..., n\}$, Let $f(n) = 1$ $f(A \cap B) = \emptyset \to \emptyset$, $f(A) = \{1\}, f(B) = \{1\}$
$f(A) \cap f(B) = \{1\} \to f(A \cap B) \neq f(A) \cap f(B)$

Problem 6

Question

Let X be a set and let P_1, P_2 be partitions of X such that P_1 is a refinement of P_2. Prove that for each $A_2 \in P_2$, the subset $P_{1,A_2} = \{A_1 \in P_1 : A_1 \subseteq A_2\}$ is a partition of A_2.

Answer

Let $P_2 = A_i$ such that $A_i \subset A_i$ and $A_i \cap A_j = \emptyset$ by definition of partition.
Let P_{1,A_i} contain some $B_i \subset A_i \cup A_j$
$B_i \cap B_j = \emptyset$ by definition of partition
$x' in \bigcup B_i$
$A_j \in \bigcup B_i =$ some set C
Since $x \in A_j$
Since P_1 is a partition $\exists B_i \in P_1$ such that $x \in B_i$. If $B_i \not\subset A_i, \exists A_j$ such that $B_i \subset A_j, x \in B_i \subset A_j, x \in A_j$
Which contradicts the definition of partition.

Problem 7

Question

Given an integer $d > 0$, prove that the sets $[j] = \{x \in \mathbb{Z} : d|x - j\}$ for $j = 0, ..., d - 1$ is a partition of $\mathbb{Z}$. Describe the equivalence relation associated to this partition in terms of d.

Answer

By the euclidian algorithm $x = dn + j$ such that $d|x - j, n$ times.
By definition of partition all sets must be disjoint, which can easily be shown by substituting in values for j into the algorithm.
$dn = (x - d - j)$ with $j < d$. $(n + 1)d = (x - j)$ which is a different subset of [j] thus the sets are disjoint.

Problem 8

Question

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $x_0 \in \mathbb{R}$. Prove that one of the subsets
$A_{f,+} = \{x \in \mathbb{R} : f(x) > x_0\}, A_{f.-} = \{x \in \mathbb{R} : f(x) < x_0\}, A_{f,0} = \{x \in \mathbb{R} : f(x) = x_0\}$
must be infinite

Answer

This can be shown, assuming I am understanding the problem correctly, for all but $A_{f,0}$. the cardinality of the other two is trivially $|\mathbb{R}| > x_0$ or $\mathbb{R} < x_0$. because x_0 is some real number it is non-infinite, so there lies an infitely large span of numbers greater and less than it.
Thus the subsets of $\mathbb{R}$ are infinite.