MA 341: HW 2

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September 8, 2016

Problem 1

Question

Prove that if X is a set, $\mathcal{T} \subseteq \mathcal{P}(X)$ is a topology on X, and $I \subseteq \mathcal{T}$ is a finite set, then $(\bigcap\limits_{U \in I} U) \in \mathcal{T}$

Answer

If $I \subseteq \mathcal{T}$ then all $U \subseteq \mathcal{T}$ by transitivity of sets.
Using the second property of topologies $U_1, U_2 \in \mathcal{T}$ states $U_1 \cap U_2 \in \mathcal{T}$
Continuing to choose an arbitrary element $U_3 \in \mathcal{T}$ implies $\bigcap\limits_{U \in I} U \in \mathcal{T}$

Problem 2

Question

Under what conditions on a set X does X have a unique topology?

Answer

X has a unique topology when $X = \{\emptyset\} or X = \{x\}$
Every set X has the Indiscrete and Discrete Topologies, and the only way to set these equal to one another is to allow X to be the emptyset or the singleton set.

Problem 3

Let X be a set and define $\mathcal{T}_{co-fin} \subseteq \mathcal{P}(X)$ by $\mathcal{T}_{co-fin} = \{A \subseteq X | X - A \text{ is finite } \} \cup \{\emptyset\}$.

Question

a Prove that $(T)_{co-fin}$ is a topology on X

b Prove that if X is infinite, then $\mathcal{T}_{co-fin} \neq \mathcal{T}_{dis}$ where $\mathcal{T}_{dis}$ is the discrete topology on X.

c Prove that if $|X| > 1$, then $\mathcal{T}_{co-fin} \neq \mathcal{T}_{triv}$ where $\mathcal{T}_{triv}$ is the trivial topology.

Answer

a (i) $\emptyset \in \mathcal{T}_{co-fin}$ by construction.
The complement of X is $\emptyset$ which is in $\mathcal{T}_{co-fin}$ by definition, thus $X \in \mathcal{T}_{co-fin}$
(ii) For $U_1, U_2 \in \mathcal{T}_{co-fin}$ states each sets compliment is finite. Thus $(A - U_1) \cap (A - U_2)$ is finite and in A. Therefore $U_1 \cap U_2 \in \mathcal{T}_{co-fin}$
(iii)The compliment of I is finite by definition of $\mathcal{T}_{co-fin}$

Each compliment $U \in I$ is finite as well thus each U is closed.
The union of closed sets are all closed
$U \in \mathfrak{T}_{co-fin}$

b By construction the discrete space is Hausdorff
If $U_1, U_2 \in \mathfrak{T}_{co-fin}$ then $X - U_1$ and $X - U_2$ are both finite
$(X - U_1) \cup (X - U_2) = X - (U_1 \cap U_2)$ and $U_1 \cap U_2 \in \mathfrak{T}_{co-fin}$
Because X is infinite $U_1 \cap U_2 \neq \emptyset$
This implies $\mathfrak{T}_{co-fin}$ is not a Hausdorff space and $\mathfrak{T}_{co-fin} \neq \mathfrak{T}_{disc}$

c If $|X| > 1$, then $\exists x, y \in X$ such that $N_x \neq N_y$ for$N_x, N_y \in \mathfrak{T}_{co-fin}$ because each neighborhood is closed
Therefore the cofinite topology on X is at least $\mathbf{T}_0$ which the trivial topology cannot be. Therefore the are not equal.

Problem 4

Let $(X, \mathfrak{T})$ be a topological space and $A \subseteq X$

Question

a Prove that $\text{Int}(X - A) = X - \overline{A}$

b Prove that $\overline{X - A} = X -$ Int$A$

Answer

a $\text{Int}(X - A) = x \in U \in X$ and $x \ni A$
X - $\overline{A}$ removes all references to elements of A in X $A \cap X = \emptyset$
Therefore $\text{Int}(X - A) = X$ - $\text{Int}(A)$

b $x \in \overline{X - A}$ implies $x \in X$
A
X - $\text{Int}(A)$ = Elements of X which are not in any $U \in A$ such that $x \in U$
$= x \in U \in X$ and $xniA = x \in X$
A
$\overline{X - A} = X -$ Int$A$

Problem 5

Question

Let $(X, \mathfrak{T})$ be a topological space. Prove that for $C \subseteq X, C$ is a closed subset if and only if $C = \overline{C}$.

Answer

$\overline{C} = \bigcap_{A \subseteq U \subseteq X X - C \in \mathfrak{T}} U$ by definition C is closed, and the intersection of closed sets is closed, therefore $\overline{C}$ is closed.
If $C = \overline{C}$ then C is closed.

If C is closed then $C \in \bigcap_{A \subseteq U \subseteq X X - C \in \mathfrak{T}} U$
By set inclusion C is the minimal set thus $C = \overline{C}$

Problem 6

Question

Let $(X, \mathfrak{T})$ be a topological space and $A \subseteq X$. Prove that A is discrete if and only if $\mathfrak{T}_{A,X}$ is discrete where $\mathfrak{T}_{A,X}$ is the subspace topology on A

Answer

If $\mathfrak{T}_{A,X}$ is discrete then $\mathfrak{T}_{A,X}$ then A is necesarily discrete because it contains the intersection of all points that contain A in $\mathfrak{T}_{A,X}$ which is equal to the powerset of A If A is discrete then every point $x \in A$ there is an open subset where $U \cap A = \{x\}$.
This creates all subsets in A and it follows that since every subset in A is uniquely definited then the subspace topology which is the intersection of all these sets is the subset of all sets as well, and discrete.

Problem 7

Question

Let $(X, \mathfrak{T})$ be a topological space and $A \subseteq X$. Prove that A is dense if and only if for each non-empty open subset $U \in \mathfrak{T}$ that $A \cap U \neq \emptyset$.

Answer

Contrapositive: If A is not dense, $\overline{A} \neq X$
Given $\overline{A} \neq X$ then $X - \overline{A} \neq \emptyset$, which is $\mathfrak{T}$-open. Therefore there exists some $U \in A$ such that $A \cap U \neq \emptyset$
If $A \cap U = \emptyset$ then there exists a non-empty open set disjoint from A.
The complement of this open set is closed and contains A, but $A \neq X$ so $\overline{A} \neq X$

Problem 8

Question

Let $(X, \mathfrak{T})$ be a topological space such that $U \subseteq X$ is an open dense subset and $D \subseteq X$ is a dense subset. Prove that $U \cap D$ is dense.

Answer

Because U is open there exists all $x \in U$ and $x \in X$. Because U is dense then the closure of U is X.
$x \in \overline{D}$ because the closure of D is equal to X.
$x \in \overline{D} \cap \overline{U}$ therefore $\overline{D} \cap \overline{U} = X$ and is dense

Problem 9

Question

Let $X, \mathfrak{T}$ be a topological space. Prove that $A \subseteq X$ is nowhere dense if and only if $X - \overline{A}$ contains an open, dense subset of X. Prove that if A_1, A_2 are nowhere dense subsets of X, then $A_1 \cup A_2$ is a nowhere dense subset of X.

Answer

Given A is nowhere dense Int(closure of A) = $\emptyset$
X
$Int(\overline{A}$ = X $\overline{X\overline{X}}$ = X Therefore X - A is dense in X and contains a dense open subset