MA 35300: HW 9

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Problem 1

Problem

Let the rows of $A \in M_{nxn}(F)$ be $a_1, a_2, ..., a_n$, and let B be the matrix in which the rows are $a_n, a_{n-1}, ..., a_1$. Calculate $\det(B)$ in terms of $\det(A)$.

Answer

The matrix B is formed by independent row changes where a_n swaps with a_1, a_{n-1} swaps with a_2, etc...
These swaps occur $\frac{n}{2}$ Times, and by Theorem 4.5 introduce a multiplication of -1 each time
Therefore $det(B) = (-1)^{\frac{n}{2}} det(A)$

Problem 2

Problem

Use Cramer's Rule to solve the given system of linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned} \qquad \text{where } a_{11}a_{22} - a_{12}a_{21} \neq 0$$

Answer

Let $\quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

Using Cramer's Rule, namely $x_k = \frac{det(M_k)}{det(A)}$ we can form the linear equation:

$$x_1 = \frac{det\begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}}{det(A)} = \frac{b_1a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}},$$

$$x_2 = \frac{det\begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}}{det(A)} = \frac{b_1a_{11} - a_{21}b_2}{a_{11}a_{22} - a_{12}a_{21}}.$$

Problem 3

Problem

Use Theorem 4.8 to prove a result analogous to Theorem 4.3 (page 212), but for columns:

(a) The determinant of an n x n matrix is a linear function of each column when the remaining columns are held fixed

(b) If $A \in M_{nxn}(F)$ has a column cosisting entirely of zeroes, then det(A) = 0.

Answer

Theorem 4.8: $\det(A^t) = \det(A)$
Theorem 4.3: The determinant of an n x n matrix is a linear function of each row when the remaining rows are held fixed. That is, for $1 \leq r \leq n$, we have

$$\det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u+kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k\,\det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

(a) The detirminant of an n x n matrix is a linear function of each column when the remaning columns are held fixed.
When you take A^t the columns of A become the rows of of A^t.
Because of Theorem 4.8, Theorem 4.3 can be applied again towards A^t which makes the above statement true.

(b) If $A \in M_{nxn}(F)$ has a column consisting entirely of zeroes, then det(A) = 0
A column of zeroes in A means a row of zeroes in A^t
Theorem 4.4 states if a matrix, A^t, has a row of zeroes det(A) = 0
By Theorem 4.8: $\det(A^t) = \det(A) = 0$

Problem 4

Problem

Prove that an upper triangular n x n matrix is invertible if and only if all of its diagonal entries are nonzero.

Answer

By the proof on the final page of Monday's notes. The determinant of a matrix is equal to the product of the entries along the diagonal
That is: $\det(A) = \prod_{k=1}^{n} a_{kk}$ where $a_{kk} \in A$
If one of the entries a_{kk} for $1 \leq k \leq n$ is 0, then the det(A) = 0
Therefore A is only invertible if the diagonal entries are nonzero.
For the only if portion, if none of the entries along the diagonal are zero then the determinant is nonzero
So A is invertible.

Problem 5

Problem

Let $\beta = \{u_1, u_2, ..., u_n\}$ be a subset of F^n consisting of n distinct vectors, and let B be the matrix in $M_{nxn}(F)$ having u_j as colum j. Prove that β is a basis for F^n if and only if det(B) $\neq 0$.

Answer

If the set β is a basis then β is an indenpendent set of size n.
This is the same as the set of columns of B being independent
If the columns are independent then the rank(B) = n
rank(B) = n if and only if det(B) $\neq 0$.
Therefore β is a basis for F^n if and only if det(B) $\neq 0$

Problem 6

Problem

Prove that if $A, B \in M_{nxn}(F)$ are similar, then $det(A) = det(B)$.

Answer

$B = Q^{-1}AQ$
$det(B) = det(Q^{-1}AQ) = det(Q^{-1})det(A)det(Q) = det(Q)^{-1}det(A)det(Q) = det(A)$