MA 34100: Midterm 2

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1

Problem

Let $f : [a, b] \to \mathbb{R}$ be a constant function $f(x) = x_x$ Using the definition of Reimann integrable to prove that f is integrable on $[a, b]$ $\int_a^b f = x_0(b - a)$.

Answer

Because f is constant to some value $\alpha \in \mathbb{R}$ we know that $U(f, [a, b], \mathcal{P}) \leq M(f)(b-a)$ where $M(f) = \alpha$ and $L(f, [a, b], \mathcal{P}) \geq m(f)(b - a)$ where $m(f) = \alpha$ therefore $U(f, [a, b], \mathcal{P}) = L(f, [a, b], \mathcal{P})$ by lemma 3.5 which means $\int_{[a,b]}^{-} = \int_{[a,b]}^{+}$ and f is integrable.
Furthermore because of the inequality above we know that $M(f) = \alpha = supL(f, [a, b], \mathcal{P}$ and by the definition of Riemann integrable $\int_a^b f = \alpha(b - a)$ $\square$

2

Problem

Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. for $\alpha \in \mathbb{R}$, define $g(x) = f(x - \alpha)$. Prove that g is Riemman intergrable on $a + \alpha, b + \alpha]$ and $\int_{a+alpha}^{b+\alpha} g = \int_a^b f$.

Answer

$\int_{a+\alpha}^{b+\alpha} g(x) = \int_{a+\alpha}^{b+\alpha} f(x - \alpha) = F(b + \alpha - \alpha) - F(a + \alpha - \alpha) = F(b) - F(a) = \int_a^b f$ $\square$

3

Problem

Let $f : [-a, a] \to \mathbb{R}$ be an integrable funtion. Prove that if f is an even function (i.e $f(-x) = f(x)$ for all x) then $\int_{-a}^{a} = 2\int_0^a f$.

Answer

$\int_{-a}^{a} f = \int_{-a}^{0} f + \int_0^a f$
$\int_{-a}^{0} f = F(0) - F(-a) = -(F(-a) - F(0)) = -\int_0^{-a} f$
$\int_{-a}^{0} f = \int_0^a f(-x)$, by substitution
$\int_{-a}^{0} f = \int_0^a f$ therefor
$\int_{-a}^{a} f = 2\int_0^a f$ $\square$

4

Problem

Let $f : [-a, a] \to \mathbb{R}$ be an integrable function. Prove that if f is an odd function (i.e $f(-x) = -f(x)$ for all of x) then $\int_{-a}^{a} f = 0$.

Answer

$\int_{-a}^{a} f = \int_{-a}^{0} f + \int_{0}^{a} f$
$\int_{-a}^{0} f = -\int_{0}^{-a} f$
$\int_{-a}^{0} f = \int_{0}^{a} f(-x)$
$\int_{-a}^{0} f = -\int_{0}^{a} f$
$\int_{-a}^{a} f = -\int_{0}^{a} f + \int_{0}^{a} f \; \int_{-a}^{a} f = 0 \qquad \square$

5

Problem

Prove that if $f : [a, b] \to \mathbb{R}$ is Riemann integrable and $a = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = b$, then

$$\int_a^b f = \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} f.$$

Answer

$\int_a^b f = \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} f$
$\int_a^b f = \sum_{i=0}^{n} (F(x_{i+1}) - F(x_i))$
$\int_a^b f = (F(x_1) - F(x_0)) + (F(x_2) - F(x_1)) + \cdots + (F(x_n) - F(x_{n-1})) + (F(x_{n+1}) - F(x_n))$
$\int_a^b f = F(x_{n+1}) - F(x_0) = F(b) - F(a) = \int_a^b f \qquad \square$

6

Problem

Prove that if $f, g : [a, b] \to \mathbb{R}$ are Reiemann integrable, then $f + g$ is a Riemann ingegrable and

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

Answer

$U(f + g, [a, b], \mathcal{P}) = \sum_{i=0}^{n} sup(f + g)\delta(I_i)$
$U(f + g, [a, b], \mathcal{P}) \leq \sum sup(f)\delta(I_i) + \sum sup(g)\delta(I_i)$
$U(f + g, [a, b], \mathcal{P}) \leq U(f, [a, b], \mathcal{P}) + U(g, [a, b], \mathcal{P})$
By the definition of the upper sum there exists the partion p such that
$U(f, [a, b], \mathcal{P}) < \int_{[a,b]}^{+} f + \epsilon/2$ and $U(g, [a, b], \mathcal{P}) < \int_{[a,b]}^{+} g + \epsilon/2$ for $\epsilon > 0$
$\int_{[a,b]}^{+} f + g \leq U(f + g, [a, b], \mathcal{P}) \leq + \leq U(f, [a, b], \mathcal{P}) + U(g, [a, b], \mathcal{P}) \leq + \int_{[a,b]}^{+} f + \int_{[a,b]}^{+} g + \epsilon$
Therefore $\int_{[a,b]}^{+} f + g \leq \int_{[a,b]}^{+} f + \int_{[a,b]}^{+} g + \epsilon$
Following the same logic, which is difficult to type out on latex
$\int_{[a,b]}^{-} f + g \geq \int_{[a,b]}^{-} f + \int_{[a,b]}^{-} g - \epsilon$
$\int_{[a,b]}^{-} f + g \geq \int_{[a,b]}^{-} f + \int_{[a,b]}^{-} g$

By definition of f and g as integrable we know:
$\int_{[a,b]}^{-} f + \int_{[a,b]}^{-} g = \int_{[a,b]}^{+} f + \int_{[a,b]}^{+} g$
And with our previous inequalities above we can rearrange:
$\int_{[a,b]}^{+} f + g \leq \int_{[a,b]}^{+} f + \int_{[a,b]}^{+} g = \int_{[a,b]}^{-} f + \int_{[a,b]}^{-} g \leq \int_{[a,b]}^{-} f + g$
Which implies $\int_{[a,b]}^{+} f + g \leq \int_{[a,b]}^{-} f + g$ and $\int_{[a,b]}^{-} f + g \leq \int_{[a,b]}^{+} f + g$ by definition so the original presented integral is indeed integrable and it is equal to $\int_a^b f + \int_a^b g$ because the inequality above is equal throught due to abritrary choice of ϵ

7

Problem

Prove that if $f : [a,b] \to \mathbb{R}$ is Riemann integrable and $\lambda \in \mathbb{R}$, then λf is Riemann integrable and
$$\int_a^b \lambda f = \lambda \int_a^b f$$

Answer

$U(\lambda f, [a,b], \mathcal{P}) = \sum sup(\lambda f)\delta(I) = \lambda \sum sup(f)\delta(I) = \lambda U(f, [a,b], \mathcal{P})$
$\int_{[a,b]}^{+} \lambda f = inf(\lambda U(f, [a,b], \mathcal{P})) = \lambda inf(U(f, [a,b], \mathcal{P}))$
Similarily for $\int_{[a,b]}^{-} \lambda f = \lambda sup(L(f, [a,b], \mathcal{P}))$ Therefore $\int_{[a,b]}^{-} \lambda f = \lambda sup(L(f, [a,b], \mathcal{P})) = \lambda inf(U(f, [a,b], \mathcal{P})) = \int_{[a,b]}^{+} \lambda f$
Therefore it is Reimann integrable and the equality holds because $\lambda \int_a^b f = \lambda \int_{[a,b]}^{+} f = \lambda inf(U(f)) = \lambda sup(L(f)) = \lambda \int_{[a,b]}^{-} f = \int_a^b \lambda f$ $\square$