MA 35300: HW 2

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Problem 1

Question

An m x n matrix A is called *upper triangular* if all entries lying below the diagonal entries are zero, that is, $A_{ij} = 0$ if $i > j$. Prove that the upper triangular matrices form a subspace of $\mathbf{M}_{mxn}(F)$.

Answer

Let $\mathbf{M}_{mxn}(F)$ be denoted as the vector space $\mathbf{V}$ of all m x n matrices
$\mathbf{W} = \{A \in \mathbf{V} | A_{ij} = 0 \text{ when } i > j\}$ defines the *upper triangular* matrices.

$\mathbf{W}$ is a vector subspace if and only if: (a) the m x n zero matrix $0_{mxn} \in \mathbf{W}$, (b) $A + B \in \mathbf{W}$ whenever $A, B \in \mathbf{W}$, and (c) $cA \in \mathbf{W}$ whenever $c \in F$ and $A \in \mathbf{W}$

a 0_{mxn} exists natively in $\mathbf{W}$ because it follows the A_{ij} set rule

b Let $A, B \in W$: For A_{ij}, B_{ij} when $j \geq i$ is simply the addition defined under the field $A_{ij} + B_{ij}$ which does not violate the set rule.
For times when $j < i$ it is $0 + 0 = 0$ which maintains the set rule, so $\mathbf{W}$ is closed under addition

c For all $c \in F$ and $A \in \mathbf{W}$, cA takes the form of cA_{ij} for some $i, j \in \mathbb{N}$.
When $j \geq i$ it follows the multiplication under the field and remains a member of $\mathbf{W}$.
When $i > j$ it is $c0 = 0$ by the fact that $\mathbf{V}$ is a vector space.

Therefore $\mathbf{W}$ is a vector space.

Problem 2

Question

Show that if

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric 2 x 2 matrices.

Answer

$\mathbf{V} = \mathbf{M}_{2x2}(F)$ is the vector space of 2 x 2 matrices
$\mathbf{W} = A \in \mathbf{V} | A^t = A$ is the collection of symmetric 2 x 2 matrices
span($\{M_1, M_2, M_3\} \subseteq \mathbf{W}$) because each matrices exists natively in $\mathbf{W}$ as the transpose does not affect their arrangment, and when added in any combination $aM_1 + bM_2 + cM_3 = dM_4$ such that M_4 is also unaffected by the transpose.
Conversely, by the definition of transpose, $A_{ij} = A_{ji}$, we have few cases under 2 x 2 matrices. $A_{11}, A_{12}, A_{21}, A_{22}$.
The transpose does not affect the order of the matrices except in the case of $A_{12} = A_{21}$ so it follows that

$A^t = A$ if and only if $A_{12} = A_{21}$.
In our circumstance we can form any matrices $A = A^t$ with:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = A_{11}M_1 + A_{22}M_2 + A_{12}M_3$$

As the A stated above is the definition of $A^t = A$ and A is the definition of the span($\{M_1, M_2, M_3\}$ then $\mathbf{W} \subseteq A \to \mathbf{W} \subseteq$ span($\{M_1.M_2, M_3\}$
Thus span($\{M_1.M_2, M_3\} = W$ the set of all symmetric 2 x 2 matrices.