MA 34100: Midterm 2

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Problem 1

Problem

(a) For a function $f : (X, \mathcal{T}_x) \to (Y, \mathcal{T}_y)$ to be continuous at $x_0 \in X$

(b) For a function $f : (a, b) \to \mathbb{R}$ to be differentiable at x_0

Answer

(a) Given topological spaces $(X, \mathcal{T}_x), (Y, \mathcal{T}_y)$, we say that a function $f : X \to Y$ is continuous at $x \in X$ if for each open neighborhood $V \subseteq Y$ there exists an open neighborhood $U \subseteq X$ of x such that $f(U) \subset V$.

(b) Given a function $f : U \to \mathbb{R}$ where $U \subseteq \mathbb{R}$ is an open subset, we say that the function f is differentiable at $x \in X$ if the limit:
$\lim\limits_{t \to 0} \frac{f(x+t)-f(x)}{t}$
exists.

Problem 2

Problem

(a) If $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b), then f is continuous on (a, b)

(b) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $A \subset \mathbb{R}$ is connected, then $f^{-1}(A)$ is connected

(c) If $f : [a, b] \to \mathbb{R}$ is a function and f^2 is differentiable, then f is differentiable.

(d) If $f : \mathbb{R} \to \mathbb{R}$ is continuous at $C \subset \mathbb{R}$ is compact, then $f(C)$ is compact.

Answer

(a) True

(b) False

(c) False

(d) True

Problem 3

Problem

(a) $\lim_{x\to\infty}(1+\frac{3}{x})^x$

(b) $\lim_{x\to 0}(\frac{3x+sin(x)}{2x})$

(c) $\lim_{x\to\infty}(\frac{4x}{4x+1})^{4x-2}(\frac{4x\sqrt{2}}{4x-3})$

Answer

(a) Let $m=\frac{x}{3}$ and substitute it for x.
This makes the limit $\lim_{x\to\infty}(1+\frac{1}{m})^{3m}$
By the notes the limit $\lim_{x\to\infty}(1+\frac{1}{m})^m = e$
Therefore the limit here is e^3

(b) Using Algebra
$\frac{3x+sin(x)}{2x} = \frac{3x}{2x}+\frac{sin(x)}{2x}$
Using Lemma 4.3
$= \lim_{x\to 0}(\frac{3}{2})+\frac{1}{2}\lim_{x\to 0}(\frac{sin(x)}{x})$ $= \frac{3}{2}+\frac{1}{2}*1 = 2$ by the class hw for $\frac{sin(x)}{x}$

(c) Using Algebra
$(\frac{4x}{4x+1})^{4x-2}(\frac{4x\sqrt{2}}{4x-3}) = \frac{(\frac{4x}{4x+1})^{4x}}{(\frac{4x}{4x+1})^{-2}}(\frac{4x\sqrt{2}}{4x-3})$
Using Lemma 4.2, We can apply the limit to each product/quotient independently then reassemble.
Massaging the numerator $(\frac{4x}{4x+1})^{4x} = (1+\frac{1}{4x})^{-4x}$
Substitute $m=\frac{x}{4}$ and we get $= \lim_{x\to\infty}(1+\frac{1}{m})^{-m}$ which by the class homework $= e^{-1}$
The denominator $\lim_{x\to\infty}(\frac{4x}{4x+1})^{-2}$ is in indeterminant form so L'Hopitals can be applied to the inside
$\lim_{x\to\infty}(\frac{4x}{4x+1})^{-2} \stackrel{H}{=} \lim_{x\to\infty}(\frac{4}{4}) = 1,\ 1^{-2}=1$
The second product is also in indeterminant form so L'Hopitals can be applied again.
$\lim_{x\to\infty}(\frac{4x\sqrt{2}}{4x-3}) \stackrel{H}{=} \lim_{x\to\infty}(\frac{4\sqrt{2}}{4}) = \sqrt{2}$
Combining them all we get:
$\lim_{x\to\infty}(\frac{4x}{4x+1})^{4x-2}(\frac{4x\sqrt{2}}{4x-3}) = \frac{e^{-1}}{1}\sqrt{2} = \frac{\sqrt{2}}{e}$

Problem 4

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^4 + 7x^3 - 9$.

Problem

(a) Prove that f is continuous using the definition of continuity.

(b) Prove that there exists $x_0, x_{x1} \in \mathbb{R}$ with $x_0 \neq x_1$ such that $f(x_0) = f(x_1) = 0$

Answer

(a) Too lazy to write out, all polynomials are continuous by the homework

(b) Using the intermediate value theorem, I will show the existance of two unique zeroes to the function f
$f(0) = -9$
$f(2) = 63$, therefore there exists an $x_0 \in (0, 2) = s_1$ where $f(x_0) = 0$
$f(-1) = -15$
$f(-8) = 503$, therefore there exists an $x_1 \in (-1, -8) = s_2$ where $f(x_1) = 0$
Because $s_1 \cap s_2 = \emptyset$ $x_0 \neq x_0$, so the proposition above holds.

Problem 5

Problem

(a) Let $f : (0, \infty) \to \mathbb{R}$ be differentiable and that $\lim_{x\to\infty} f(x) + f'(x) = L$. Prove that $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} f'(x) = 0$

(b) Let $a_1, ..., a_n \in \mathbb{R}$ and define
$$f(x) = \sum_{i=1}^{n} (a_i - x)^2$$
Prove that f has a unique absolute minimum point x_0 and find x_0

(c) Let $a > b > 0$ and $n \in \mathbb{N}$ with $n \geq 2$. Prove that
$$a^{1/n} - b^{1/n} < (a - b)^{1/n}$$

Answer

(a) Assume $\lim_{x\to\infty} = L$
This means $\lim\limits_{x\to\infty} \frac{L}{x} = 0$
But if we replace $\lim\limits_{x\to\infty} \frac{f(x)}{x}$ we have an indeterminant form so we can apply L'Hopitals
$\overset{H}{=} \lim\limits_{x\to\infty} f'(x)$
Therefore $\lim\limits_{x\to\infty} f'(x) = 0$

(b) $\sum\limits_{i=1}^{n} (a_i - x)^2 = \sum\limits_{i=1}^{n} (a_i^2 - 2a_i x + x^2)$
$\sum\limits_{i=1}^{n} (a_i^2) - \sum\limits_{i=1}^{n} 2a_i x + \sum\limits_{i=1}^{n} x^2$ Taking the derivative and setting it equal to zero $f'(x) = 0 - \sum\limits_{i=1}^{n} 2a_i - 2nx$
$x = \frac{\sum_{i=1}^{n} 2a_i}{2n} = \frac{1}{n} \sum\limits_{i=1}^{n} a_i$

(c) Use a substitution $c = (\frac{a}{b})^{1/n}$
Because of the definition of a and b, $c \in (0, 1)$
Subbing c into the equation we get:

$$x^n * (x^{1/n} - y^{1/n}) = (x-y)^{1/n} * x * n$$
$= 1 - c < (1 - c^n)$
$(1-c)^n < (1-c)$
This is necessarily true because $c < 1$.

Problem 6

Problem

(a) Let $f : [0,1] \to \mathbb{R}$ be a continuous function that $f(0) = f(1)$. Prove that there exists $c \in [0, 1/2]$ such that $f(c) = f(c + 1/2)$

(b) Let $f : [0,1] \to \mathbb{R}$ be such that for each $\alpha]in\mathbb{R}$ that $|f^{-1}(\alpha)| = 0$ or 2. Prove that f cannot be continuous at every $x \in [0,1]$.

(c) Let $f, g : [a,b] \to \mathbb{R}$ be continuous at $x_0 \in (a,b)$. Using the definition of continuity, prove that fg is continuous at x_0

Answer

(a) Define some function $g = f(x) - f(x + 1/2)$
Because $f(0) = f(1)$ we can solve:
$g(0) = f(0) - f(1/2)$ and $g(1/2) = f(1/2) - f(1)$.
to $g(0) = -g(1/2)$
We can then describe 3 cases for g:

  (a) $g(0) > 0 > g(1/2)$: Because of the intermediate value theorem, we know there exists some value c such that $g(c) = 0 \to f(c) = f(c + 1/2)$

  (b) $g(0) < 0 < g(1/2)$: Again using the intermediate value theorem, $\exists c \in (0, 1/2)$ s.t. $g(c) = 0$ which again implies $f(c) = f(c + 1/2)$

  (c) The final case is $g(0) = 0 = g(1/2)$ which is a trivial answer to the problem.

(b) Pre-image of $|f^{-1}(\alpha)|$ aside, the function f cannot be a continuous function into R based upon the definition of it's domain.

Assume f is continuous. Because [0,1] is a compact subset we can apply the Extreme Value Theorem.
This states there is some $x_{min}, x_{max} \in [0,1]$ such that $f(x_{min}) \leq f(x) \leq f(x_{max})$ for all $x \in \mathbb{R}$.
Ths is untrue in $\mathbb{R}$ by the definition of $\mathbb{R}$ as the unbounded set $(-\infty, \infty)$
Therefore f is non-continuous.

(c) For $x \in X$, it suffices to prove that for each $B(f(x)g(x), r) \subseteq \mathbb{R}^n$, there exists an open neighborhood $U \subseteq X$ of x such that $(fg)(U) \subseteq B(f(x)g(x), r)$. For simplicity, we will assume that $r < 1$ and let $M_f = |f(x)| + 1, M_g = |g(x)| + 1$. Since f is continuous, there exists an open neighborhood $U_{f,1} \subseteq X of x such that f(U) \subseteq B(f(x), 1)$. As f,g are continuous, there exist open neighborhoods $U_{f,2}, U g of x such that f(U_f) \subseteq B(f(x), r/2M_g)$ and $g(U_g) \subseteq B(g(x), r/2M_f)$. Let $U_f = U_{f,1} \cap U_{f,2}$ and note that $|f(x'|)|M_f$ for every $x' \in U_f$. Now, for all $x_0 \in U = U_f \cap U_g$ we have:

$$\begin{aligned}
d(f(x), g(x), f(x')g(x'))) &\leq d(f(x)g(x), f(x')g(x)) + d(f(x')g(x), f(x')g(x')) \\
&= |g(x)| d(f(x), f(x')) + |f(x')| d(g(x), g(x')) \\
&< |g(x)| \tfrac{r}{2M_g} + |f(x')| \tfrac{r}{2M_f} \\
&< r/2 + r/2 = r
\end{aligned}$$