Rudin Textbook Notes

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It begins!

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The rational numbers are inadequate, both as a field and an unordered set.

Example 1.1. There is no rational number p such that $p^2 = 2$.

Proof. For the sake of contradiction, assume that $p^2 = 2$ has a rational solution, $\frac{a}{b}$,where a and b are integers such that $\gcd(a, b) = 1$. Therefore, we can take the square root of both sides as so:

$$\sqrt{2} = \frac{a}{b}$$

Through algebraic manipulation, we then get

$$2a^2 = b^2$$

This means b must be even, and so b^2 is divisible by 4. This means a must also be even. Contradiction, as we assumed that $\gcd(a, b) = 1$. $\square$

Now, something more interesting: let A be the set of positive rationals p such that $p^2 < 2$, and B be the set of positive rationals p such that $p^2 > 2$.

Proposition 1.1. There is NO largest element in A.

Proof. We have some rational p such that $p^2 < 2$. Now, define a new rational q such that $q = p - \frac{p^2-2}{p-2} = \frac{2(p+1)}{(p+2)}$. Why do we define a rational like so? Well, for one, $q > p$, because $\frac{p^2-2}{p+2}$ is less than zero. Also, $q^2 - 2 = \frac{2(p^2-2)}{(p+2)^2} < 0$. Therefore, for some arbitrary rational p such that $p^2 < 2$, we have found another rational q such that $q > p$ and $q^2 < 2$. $\square$

Proposition 1.2. There is NO smallest element in B.

Proof. Very similar proof. We have some rational p such that $p^2 > 2$. Now, define a new rational q such that $q = p - \frac{p^2-2}{p-2} = \frac{2(p+1)}{(p+2)}$. Why do we define a rational like so? Well, for one, $q < p$, because $\frac{p^2-2}{p+2}$ is greater than zero. Also, $q^2 - 2 = \frac{2(p^2-2}{(p+2)^2} > 0$. Therefore, for some arbitrary rational p such that $p^2 > 2$, we have found another rational q such that $q < p$ and $q^2 > 2$. $\square$

The reason we went through this whole process is to show that even though, say, there's a rational between any two rationals, there are still "gaps" that the rationals have. That's where the real numbers come into play!
To talk about these gaps, it's necessary to talk about bounds, first.

Definition 1.1. Suppose S is an ordered set, and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ $\forall x \in S$, then β is an **upper bound** of E. A similar definition for **lower bound**.

This is a good definition, but an upper/lower bound for a set is NOT unique. In other words, there can be multiple, if not an infinite number of upper/lower bounds for a set. If there a way to define a unique bound? Yes!

Definition 1.2. Suppose S is an ordered set, and $E \subset S$ is bounded above. Suppose there is an element $\alpha \in S$ with the following properties:

(i) α is an upper bound of E.

(ii) If $x < \alpha$, then x is **not** an upper bound of E.

Then α is the least upper bound of E, also known as the **supremum** of E, and we can say $\alpha = \sup E$.
A similar definition can be made for the greatest lower bound of E (assuming E is bounded below), or the **infimum** of E, so we can say $\gamma = \inf E$.