Rudin Textbook Notes

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It begins!

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The rational numbers are inadequate, both as a field and an unordered set.

Example 1.1. There is no rational number p such that $p^2 = 2$.

Proof. For the sake of contradiction, assume that $p^2 = 2$ has a rational solution, $\frac{a}{b}$,where a and b are integers such that $\gcd(a, b) = 1$. Therefore, we can take the square root of both sides as so:

$$\sqrt{2} = \frac{a}{b}$$

Through algebraic manipulation, we then get

$$2a^2 = b^2$$

This means b must be even, and so b^2 is divisible by 4. This means a must also be even. Contradiction, as we assumed that $\gcd(a, b) = 1$. □

Now, something more interesting: let A be the set of positive rationals p such that $p^2 < 2$, and B be the set of positive rationals p such that $p^2 > 2$.

Proposition 1.1. There is NO largest element in A.

Proof. We have some rational p such that $p^2 < 2$. Now, define a new rational q such that $q = p - \frac{p^2-2}{p-2} = \frac{2(p+1)}{(p+2)}$. Why do we define a rational like so? Well, for one, $q > p$, because $\frac{p^2-2}{p+2}$ is less than zero. Also, $q^2 - 2 = \frac{2(p^2-2)}{(p+2)^2} < 0$. Therefore, for some arbitrary rational p such that $p^2 < 2$, we have found another rational q such that $q > p$ and $q^2 < 2$. □

Proposition 1.2. There is NO smallest element in B.

Proof. Very similar proof. We have some rational p such that $p^2 > 2$. Now, define a new rational q such that $q = p - \frac{p^2-2}{p-2} = \frac{2(p+1)}{(p+2)}$. Why do we define a rational like so? Well, for one, $q < p$, because $\frac{p^2-2}{p+2}$ is greater than zero. Also, $q^2 - 2 = \frac{2(p^2-2}{(p+2)^2} > 0$. Therefore, for some arbitrary rational p such that $p^2 > 2$, we have found another rational q such that $q < p$ and $q^2 > 2$. □

The reason we went through this whole process is to show that even though, say, there's a rational between any two rationals, there are still "gaps" that the rationals have. That's where the real numbers come into play!