CHAPTER 11

Recursive Description of Patterns

In the last chapter, we saw two equivalent ways to describe patterns. One was graph-theoretic, using the labels of paths in a kind of graph that we called an "automaton." The other was algebraic, using the regular expression notation. In this chapter, we shall see a third way to describe patterns, using a form of recursive definition called a "context-free grammar" ("grammar" for short).

One important application of grammars is the specification of programming languages. Grammars are a succinct notation for describing the syntax of typical programming languages; we shall see many examples in this chapter. Further, there is mechanical way to turn a grammar for a typical programming language into a "parser," one of the key parts of a compiler for the language. The parser uncovers the structure of the source program, often in the form of an expression tree for each statement in the program.

11.1 What This Chapter Is About

This chapter focuses on the following topics.

- Grammars and how grammars are used to define languages (Sections 11.2 and 11.3).
- Parse trees, which are tree representations that display the structure of strings according to a given grammar (Section 11.4).
- Ambiguity, the problem that arises when a string has two or more distinct parse trees and thus does not have a unique "structure" according to a given grammar (Section 11.5).
- A method for turning a grammar into a "parser," which is an algorithm to tell whether a given string is in a language (Sections 11.6 and 11.7).

- A proof that grammars are more powerful than regular expressions for describing languages (Section 11.8). First, we show that grammars are at least as descriptive as regular expressions by showing how to simulate a regular expression with a grammar. Then we describe a particular language that can be specified by a grammar, but by no regular expression.

11.2 Context-Free Grammars

Arithmetic expressions can be defined naturally by a recursive definition. The following example illustrates how the definition works. Let us consider arithmetic expressions that involve

1. The four binary operators, $+$, $-$, $*$, and $/$,
2. Parentheses for grouping, and
3. Operands that are numbers.

The usual definition of such expressions is an induction of the following form:

BASIS. A number is an expression.

INDUCTION. If E is an expression, then each of the following is also an expression:

a) (E). That is, we may place parentheses around an expression to get a new expression.

b) $E + E$. That is, two expressions connected by a plus sign is an expression.

c) $E - E$. This and the next two rules are analogous to (2), but use the other operators.

d) $E * E$.

e) E/E.

This induction defines a language, that is, a set of strings. The basis states that any number is in the language. Rule (a) states that if s is a string in the language, then so is the parenthesized string `(`s`)`; this string is s preceded by a left parenthesis and followed by a right parenthesis. Rules (b) to (e) say that if s and t are two strings in the language, then so are the strings s`+`t, s`-`t, s`*`t, and s`/`t.

Grammars allow us to write down such rules succinctly and with a precise meaning. As an example, we could write our definition of arithmetic expressions with the grammar shown in Fig. 11.1.

(1) <*Expression*> → *number*
(2) <*Expression*> → **(** <*Expression*> **)**
(3) <*Expression*> → <*Expression*> **+** <*Expression*>
(4) <*Expression*> → <*Expression*> **-** <*Expression*>
(5) <*Expression*> → <*Expression*> ***** <*Expression*>
(6) <*Expression*> → <*Expression*> **/** <*Expression*>

Fig. 11.1. Grammar for simple arithmetic expressions.

The symbols used in Fig. 11.1 require some explanation. The symbol

<Expression>

Syntactic category

is called a *syntactic category;* it stands for any string in the language of arithmetic expressions. The symbol → means "can be composed of." For instance, rule (2) in Fig. 11.1 states that an expression can be composed of a left parenthesis followed by any string that is an expression followed by a right parenthesis. Rule (3) states that an expression can be composed of any string that is an expression, the character `+`, and any other string that is an expression. Rules (4) through (6) are similar to rule (3).

Rule (1) is different because the symbol *number* on the right of the arrow is not intended to be a literal string, but a placeholder for any string that can be interpreted as a number. We shall later show how numbers can be defined grammatically, but for the moment let us imagine that *number* is an abstract symbol, and expressions use this symbol to represent any atomic operand.

The Terminology of Grammars

There are three kinds of symbols that appear in grammars. The first are "metasymbols," symbols that play special roles and do not stand for themselves.

Metasymbol

The only example we have seen so far is the symbol →, which is used to separate the syntactic category being defined from a way in which strings of that syntactic category may be composed. The second kind of symbol is a syntactic category, which as we mentioned represents a set of strings being defined. The third kind of symbol is called a *terminal.*

Terminal

Terminals can be characters, such as `+` or `(`, or they can be abstract symbols such as *number*, that stand for one or more strings we may wish to define at a later time.

Production

A grammar consists of one or more *productions.* Each line of Fig. 11.1 is a production. In general, a production has three parts:

1. A *head,* which is the syntactic category on the left side of the arrow,
2. The metasymbol →, and
3. A *body,* consisting of 0 or more syntactic categories and/or terminals on the right side of the arrow.

Head and body

For instance, in rule (2) of Fig. 11.1, the head is *<Expression>*, and the body consists of three symbols: the terminal `(`, the syntactic category *<Expression>*, and the terminal `)`.

◆ **Example 11.1.** We can augment the definition of expressions with which we began this section by providing a definition of *number*. We assume that numbers are strings consisting of one or more digits. In the extended regular-expression notation of Section 10.6, we could say

$$digit = [\mathbf{0\text{-}9}]$$
$$number = digit^{+}$$

However, we can also express the same idea in grammatical notation. We could write the productions

Notational Conventions

We denote syntactic categories by a name, in italics, surrounded by angular brackets, for example, <*Expression*>. Terminals in productions will either be denoted by a boldface **x** to stand for the string `x` (in analogy with the convention for regular expressions), or by an italicized character string with no angular brackets, for the case that the terminal, like *number*, is an abstract symbol.

We use the metasymbol ϵ to stand for an empty body. Thus, the production $<S> \rightarrow \epsilon$ means that the empty string is in the language of syntactic category $<S>$. We sometimes group the bodies for one syntactic category into one production, separating the bodies by the metasymbol |, which we can read as "or." For example, if we have productions

$$<S> \rightarrow B_1, <S> \rightarrow B_2, \ldots, <S> \rightarrow B_n$$

where the B's are each the body of a production for the syntactic category $<S>$, then we can write these productions as

$$<S> \rightarrow B_1 \mid B_2 \mid \cdots \mid B_n$$

<*Digit*> → **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9**

<*Number*> → <*Digit*>
<*Number*> → <*Number*> <*Digit*>

Note that, by our convention regarding the metasymbol |, the first line is short for the ten productions

<*Digit*> → **0**
<*Digit*> → **1**
...
<*Digit*> → **9**

We could similarly have combined the two productions for <*Number*> into one line. Note that the first production for <*Number*> states that a single digit is a number, and the second production states that any number followed by another digit is also a number. These two productions together say that any string of one or more digits is a number.

Figure 11.2 is an expanded grammar for expressions, in which the abstract terminal *number* has been replaced by productions that define the concept. Notice that the grammar has three syntactic categories, <*Expression*>, <*Number*> and <*Digit*>. We shall treat the syntactic category <*Expression*> as the *start symbol*; it generates the strings (in this case, well-formed arithmetic expressions) that we intend to define with the grammar. The other syntactic categories, <*Number*> and <*Digit*>, stand for auxiliary concepts that are essential, but not the main concept for which the grammar was written. ✦

Start symbol

✦ **Example 11.2.** In Section 2.6 we discussed the notion of strings of balanced parentheses. There, we gave an inductive definition of such strings that resembles, in an informal way, the formal style of writing grammars developed in this section.

Common Grammatical Patterns

Example 11.1 used two productions for <*Number*> to say that "a number is a string of one or more digits." The pattern used there is a common one. In general, if we have a syntactic category <X>, and Y is either a terminal or another syntactic category, the productions

$$<X> \rightarrow <X>Y \mid Y$$

say that any string of one or more Y's is an <X>. Adopting the regular expression notation, $<X> = Y^+$. Similarly, the productions

$$<X> \rightarrow <X>Y \mid \epsilon$$

tell us that every string of zero or more Y's is an <X>, or $<X> = Y^*$. A slightly more complex, but also common pattern is the pair of productions

$$<X> \rightarrow <X>ZY \mid Y$$

which say that every string of alternating Y's and Z's, beginning and ending with a Y, is an <X>. That is, $<X> = Y(ZY)^*$.

Moreover, we can reverse the order of the symbols in the body of the recursive production in any of the three examples above. For instance,

$$<X> \rightarrow Y<X> \mid Y$$

also defines $<X> = Y^+$.

(1) <*Digit*> → **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9**
(2) <*Number*> → <*Digit*>
(3) <*Number*> → <*Number*> <*Digit*>
(4) <*Expression*> → <*Number*>
(5) <*Expression*> → **(** <*Expression*> **)**
(6) <*Expression*> → <*Expression*> **+** <*Expression*>
(7) <*Expression*> → <*Expression*> **−** <*Expression*>
(8) <*Expression*> → <*Expression*> ***** <*Expression*>
(9) <*Expression*> → <*Expression*> **/** <*Expression*>

Fig. 11.2. Grammar for expressions with numbers defined grammatically.

We defined a syntactic category of "balanced parenthesis strings" that we might call <*Balanced*>. There was a basis rule stating that the empty string is balanced. We can write this rule as a production,

$$<Balanced> \rightarrow \epsilon$$

Then there was an inductive step that said if x and y were balanced strings, then so was $(x)y$. We can write this rule as a production

<*Balanced*> → **(** <*Balanced*> **)** <*Balanced*>

Thus, the grammar of Fig. 11.3 may be said to define balanced strings of parenthe-

$$<Balanced> \rightarrow \epsilon$$
$$<Balanced> \rightarrow (<Balanced>) <Balanced>$$

Fig. 11.3. A grammar for balanced parenthesis strings.

ses.

There is another way that strings of balanced parentheses could be defined. If we recall Section 2.6, our original motivation for describing such strings was that they are the subsequences of parentheses that appear within expressions when we delete all but the parentheses. Figure 11.1 gives us a grammar for expressions. Consider what happens if we remove all terminals but the parentheses. Production (1) becomes

$$<Expression> \rightarrow \epsilon$$

Production (2) becomes

$$<Expression> \rightarrow (<Expression>)$$

and productions (3) through (6) all become

$$<Expression> \rightarrow <Expression> <Expression>$$

If we replace the syntactic category $<Expression>$ by a more appropriate name, $<BalancedE>$, we get another grammar for balanced strings of parentheses, shown in Fig. 11.4. These productions are rather natural. They say that

1. The empty string is balanced,
2. If we parenthesize a balanced string, the result is balanced, and
3. The concatenation of balanced strings is balanced.

$$<BalancedE> \rightarrow \epsilon$$
$$<BalancedE> \rightarrow (<BalancedE>)$$
$$<BalancedE> \rightarrow <BalancedE> <BalancedE>$$

Fig. 11.4. A grammar for balanced parenthesis strings developed from the arithmetic expression grammar.

The grammars of Figs. 11.3 and 11.4 look rather different, but they do define the same set of strings. Perhaps the easiest way to prove that they do is to show that the strings defined by $<BalancedE>$ in Fig. 11.4 are exactly the "profile balanced" strings defined in Section 2.6. There, we proved the same assertion about the strings defined by $<Balanced>$ in Fig. 11.3. ✦

✦ **Example 11.3.** We can also describe the structure of control flow in languages like C grammatically. For a simple example, it helps to imagine that there are abstract terminals *condition* and *simpleStat*. The former stands for a conditional expression. We could replace this terminal by a syntactic category, say $<Condition>$.

The productions for <*Condition*> would resemble those of our expression grammar above, but with logical operators like `&&`, comparison operators like <, and the arithmetic operators.

The terminal *simpleStat* stands for a statement that does not involve nested control structure, such as an assignment, function call, read, write, or jump statement. Again, we could replace this terminal by a syntactic category and the productions to expand it.

We shall use <*Statement*> for our syntactic category of C statements. One way statements can be formed is through the while-construct. That is, if we have a statement to serve as the body of the loop, we can precede it by the keyword **while** and a parenthesized condition to form another statement. The production for this statement-formation rule is

<*Statement*> → **while** (*condition*) <*Statement*>

Another way to build statements is through selection statements. These statements take two forms, depending on whether or not they have an else-part; they are expressed by the two productions

<*Statement*> → **if** (*condition*) <*Statement*>
<*Statement*> → **if** (*condition*) <*Statement*> **else** <*Statement*>

There are other ways to form statements as well, such as for-, repeat-, and case-statements. We shall leave those productions as exercises; they are similar in spirit to what we have seen.

However, one other important formation rule is the block, which is somewhat different from those we have seen. A block is formed by curly braces `{` and `}`, surrounding zero or more statements. To describe blocks, we need an auxiliary syntactic category, which we can call <*StatList*>; it stands for a list of statements. The productions for <*StatList*> are simple:

<*StatList*> → ϵ
<*StatList*> → <*StatList*> <*Statement*>

That is, the first production says that a statement list can be empty. The second production says that if we follow a list of statements by another statement, then we have a list of statements.

Now we can define statements that are blocks as a statement list surrounded by `{` and `}`, that is,

<*Statement*> → { <*StatList*> }

The productions we have developed, together with the basis production that states that a statement can be a simple statement (assignment, call, input/output, or jump) followed by a semicolon is shown in Fig. 11.5. ❖

EXERCISES

11.2.1: Give a grammar to define the syntactic category <*Identifier*>, for all those strings that are C identifiers. You may find it useful to define some auxiliary syntactic categories like <*Digit*>.

<*Statement*> → **while** (*condition*) <*Statement*>
<*Statement*> → **if** (*condition*) <*Statement*>
<*Statement*> → **if** (*condition*) <*Statement*> **else** <*Statement*>
<*Statement*> → { <*StatList*> }
<*Statement*> → *simpleStat* **;**

<*StatList*> → ϵ
<*StatList*> → <*StatList*> <*Statement*>

Fig. 11.5. Productions defining some of the statement forms of C.

11.2.2: Arithmetic expressions in C can take identifiers, as well as numbers, as operands. Modify the grammar of Fig. 11.2 so that operands can also be identifiers. Use your grammar from Exercise 11.2.1 to define identifiers.

11.2.3: Numbers can be real numbers, with a decimal point and an optional power of 10, as well as integers. Modify the grammar for expressions in Fig. 11.2, or your grammar from Exercise 11.2.2, to allow reals as operands.

11.2.4*: Operands of C arithmetic expressions can also be expressions involving pointers (the `*` and `&` operators), fields of a record structure (the `.` and `->` operators), or array indexing. An index of an array can be any expression.

a) Write a grammar for the syntactic category <*ArrayRef*> to define strings consisting of a pair of brackets surrounding an expression. You may use the syntactic category <*Expression*> as an auxiliary.

b) Write a grammar for the syntactic category <*Name*> to define strings that refer to operands. An example of a name, as discussed in Section 1.4, is `(*a).b[c][d]`. You may use <*ArrayRef*> as an auxiliary.

c) Write a grammar for arithmetic expressions that allow names as operands. You may use <*Name*> as an auxiliary. When you put your productions from (a), (b), and (c) together, do you get a grammar that allows expressions like `a[b.c][*d]+e`?

11.2.5*: Show that the grammar of Fig. 11.4 generates the profile-balanced strings defined in Section 2.6. *Hint*: Use two inductions on string length similar to the proofs in Section 2.6.

11.2.6*: Sometimes expressions can have two or more kinds of balanced parentheses. For example, C expressions can have both round and square parentheses, and both must be balanced; that is, every `(` must match a `)`, and every `[` must match a `]`. Write a grammar for strings of balanced parentheses of these two types. That is, you must generate all and only the strings of such parentheses that could appear in well-formed C expressions.

11.2.7: To the grammar of Fig. 11.5 add productions that define for-, do-while-, and switch-statements. Use abstract terminals and auxiliary syntactic categories as appropriate.

11.2.8*: Expand the abstract terminal *condition* in Example 11.3 to show the use of logical operators. That is, define a syntactic category <*Condition*> to take the

place of the terminal *condition*. You may use an abstract terminal *comparison* to represent any comparison expression, such as `x+1<y+z`. Then replace *comparison* by a syntactic category <*Comparison*> that expresses arithmetic comparisons in terms of the comparison operators such as < and a syntactic category <*Expression*>. The latter can be defined roughly as in the beginning of Section 11.2, but with additional operators found in C, such as unary minus and `%`.

11.2.9*: Write productions that will define the syntactic category <*SimpleStat*>, to replace the abstract terminal *simpleStat* in Fig. 11.5. You may assume the syntactic category <*Expression*> stands for C arithmetic expressions. Recall that a "simple statement" can be an assignment, function call, or jump, and that, technically, the empty string is also a simple statement.

11.3 Languages from Grammars

A grammar is essentially an inductive definition involving sets of strings. The major departure from the examples of inductive definitions seen in Section 2.6 and many of the examples we had in Section 11.2 is that with grammars it is routine for several syntactic categories to be defined by one grammar. In contrast, our examples of Section 2.6 each defined a single notion. Nonetheless, the way we constructed the set of defined objects in Section 2.6 applies to grammars. For each syntactic category $<S>$ of a grammar, we define a language $L(<S>)$, as follows:

BASIS. Start by assuming that for each syntactic category $<S>$ in the grammar, the language $L(<S>)$ is empty.

INDUCTION. Suppose the grammar has a production $<S> \rightarrow X_1X_2\cdots X_n$, where each X_i, for $i = 1, 2, \ldots, n$, is either a syntactic category or a terminal. For each $i = 1, 2, \ldots, n$, select a string s_i for X_i as follows:

1. If X_i is a terminal, then we may only use X_i as the string s_i.
2. If X_i is a syntactic category, then select as s_i any string that is already known to be in $L(X_i)$. If several of the X_i's are the same syntactic category, we can pick a different string from $L(X_i)$ for each occurrence.

Then the concatenation $s_1s_2\cdots s_n$ of these selected strings is a string in the language $L(<S>)$. Note that if $n = 0$, then we put ϵ in the language.

One systematic way to implement this definition is to make a sequence of rounds through the productions of the grammar. On each round we update the language of each syntactic category using the inductive rule in all possible ways. That is, for each X_i that is a syntactic category, we pick strings from $L(<X_i>)$ in all possible ways.

Example 11.4. Let us consider a grammar consisting of some of the productions from Example 11.3, the grammar for some kinds of C statements. To simplify, we shall only use the productions for while-statements, blocks, and simple statements, and the two productions for statement lists. Further, we shall use a shorthand that

condenses the strings considerably. The shorthand uses the terminals **w** (*while*), **c** (parenthesized *condition*), and **s** (*simpleStat*). The grammar uses the syntactic category <*S*> for statements and the syntactic category <*L*> for statement lists. The productions are shown in Fig. 11.6.

(1) <*S*> → **w c** <*S*>
(2) <*S*> → `{` <*L*> `}`
(3) <*S*> → **s** `;`
(4) <*L*> → <*L*> <*S*>
(5) <*L*> → ϵ

Fig. 11.6. Simplified grammar for statements.

Let L be the language of strings in the syntactic category <*L*>, and let S be the language of strings in the syntactic category <*S*>. Initially, by the basis rule, both L and S are empty. In the first round, only productions (3) and (5) are useful, because the bodies of all the other productions each have a syntactic category, and we do not yet have any strings in the languages for the syntactic categories. Production (3) lets us infer that `s;` is a string in the language S, and production (5) tells us that ϵ is in language L.

The second round begins with $L = \{\epsilon\}$, and $S = \{$`s;`$\}$. Production (1) now allows us to add `wcs;` to S, since `s;` is already in S. That is, in the body of production (1), terminals **w** and **c** can only stand for themselves, but syntactic category <*S*> can be replaced by any string in the language S. Since at present, string `s;` is the only member of S, we have but one choice to make, and that choice yields the string `wcs;`.

Production (2) adds string `{}`, since terminals `{` and `}` can only stand for themselves, but syntactic category <*L*> can stand for any string in language L. At the moment, L has only ϵ.

Since production (3) has a body consisting of a terminal, it will never produce any string other than `s;`, so we can forget this production from now on. Similarly, production (5) will never produce any string other than ϵ, so we can ignore it on this and future rounds.

Finally, production (4) produces string `s;` for L when we replace <*L*> by ϵ and replace <*S*> by `s;`. At the end of round 2, the languages are $S = \{$`s;`, `wcs;`, `{}`$\}$, and $L = \{\epsilon,$ `s;`$\}$.

On the next round, we can use productions (1), (2), and (4) to produce new strings. In production (1), we have three choices to substitute for <*S*>, namely `s;`, `wcs;`, and `{}`. The first gives us a string for language S that we already have, but the other two give us new strings `wcwcs;` and `wc{}`.

Production (2) allows us to substitute ϵ or `s;` for <*L*>, giving us old string `{}` and new string `{s;}` for language S. In production (4), we can substitute ϵ or `s;` for <*L*> and `s;`, `wcs;`, or `{}` for <*S*>, giving us for language L one old string, `s;`, and the five new strings `wcs;`, `{}`, `s;s;`, `s;wcs;`, and `s;{}`.[1]

[1] We are being extremely systematic about the way we substitute strings for syntactic categories. We assume that throughout each round, the languages L and S are fixed as they were defined at the end of the previous round. Substitutions are made into each of the production bodies. The bodies are allowed to produce new strings for the syntactic categories of the

The current languages are $S = \{$`s;`, `wcs;`, `{}`, `wcwcs;`, `wc{}`, `{s;}`$\}$, and

$L = \{\epsilon$, `s;`, `wcs;`, `{}`, `s;s;`, `s;wcs;`, `s;{}`$\}$

We may proceed in this manner as long as we like. Figure 11.7 summarizes the first three rounds. ❖

	S	L
Round 1:	`s;`	ϵ
Round 2:	`wcs;` `{}`	`s;`

Round 3:	`wcwcs;`
	`wc{}`
	`{s;}`
		`wcs;`
		`{}`
		`s;s;`
		`s;wcs;`
		`s;{}`

Fig. 11.7. New strings on first three rounds.

Infinite language

As in Example 11.4, the language defined by a grammar may be infinite. When a language is infinite, we cannot list every string. The best we can do is to *enumerate* the strings by rounds, as we started to do in Example 11.4. Any string in the language will appear on some round, but there is no round at which we shall have produced all the strings. The set of strings that would ever be put into the language of a syntactic category <S> forms the (infinite) language L(<S>).

EXERCISES

11.3.1: What new strings are added on the fourth round in Example 11.4?

11.3.2*: On the ith round of Example 11.4, what is the length of the shortest string that is new for either of the syntactic categories? What is the length of the longest new string for

a) <S>
b) <L>?

11.3.3: Using the grammar of

a) Fig. 11.3
b) Fig. 11.4

generate strings of balanced parentheses by rounds. Do the two grammars generate the same strings on the same rounds?

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heads, but we do not use the strings newly constructed from one production in the body of another production on the same round. It doesn't matter. All strings that are going to be generated will eventually be generated on some round, regardless of whether or not we immediately recycle new strings into the bodies or wait for the next round to use the new strings.

11.3.4: Suppose that each production with some syntactic category $<S>$ as its head also has $<S>$ appearing somewhere in its body. Why is $L(<S>)$ empty?

11.3.5*: When generating strings by rounds, as described in this section, the only new strings that can be generated for a syntactic category $<S>$ are found by making a substitution for the syntactic categories of the body of some production for $<S>$, such that *at least one substituted string was newly discovered on the previous round*. Explain why the italicized condition is correct.

11.3.6**: Suppose we want to tell whether a particular string s is in the language of some syntactic category $<S>$.

a) Explain why, if on some round, all the new strings generated for any syntactic category are longer than s, and s has not already been generated for $L(<S>)$, then s cannot ever be put in $L(<S>)$. *Hint*: Use Exercise 11.3.5.

b) Explain why, after some finite number of rounds, we must fail to generate any new strings that are as short as or shorter than s.

c) Use (a) and (b) to develop an algorithm that takes a grammar, one of its syntactic categories $<S>$, and a string of terminals s, and tells whether s is in $L(<S>)$.

11.4 Parse Trees

As we have seen, we can discover that a string s belongs to the language $L(<S>)$, for some syntactic category $<S>$, by the repeated application of productions. We start with some strings derived from basis productions, those that have no syntactic category in the body. We then "apply" productions to strings already derived for the various syntactic categories. Each application involves substituting strings for occurrences of the various syntactic categories in the body of the production, and thereby constructing a string that belongs to the syntactic category of the head. Eventually, we construct the string s by applying a production with $<S>$ at the head.

It is often useful to draw the "proof" that s is in $L(<S>)$ as a tree, which we call a *parse tree*. The nodes of a parse tree are labeled, either by terminals, by syntactic categories, or by the symbol ϵ. The leaves are labeled only by terminals or ϵ, and the interior nodes are labeled only by syntactic categories.

Every interior node v represents the application of a production. That is, there must be some production such that

1. The syntactic category labeling v is the head of the production, and
2. The labels of the children of v, from the left, form the body of the production.

Example 11.5. Figure 11.8 is an example of a parse tree, based on the grammar of Fig. 11.2. However, we have abbreviated the syntactic categories $<Expression>$, $<Number>$, and $<Digit>$ to $<E>$, $<N>$, and $<D>$, respectively. The string represented by this parse tree is `3*(2+14)`.

For example, the root and its children represent the production

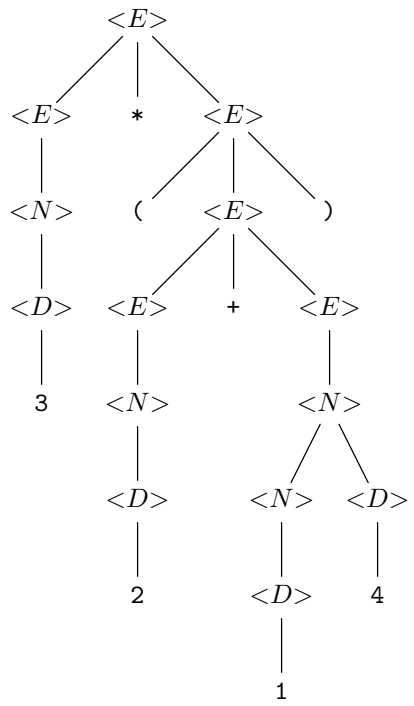


Fig. 11.8. Parse tree for the string `3*(2+14)` using the grammar from Fig. 11.2.

$$<E> \rightarrow <E> * <E>$$

which is production (6) in Fig. 11.2. The rightmost child of the root and its three children form the production $<E> \rightarrow (<E>)$, or production (5) of Fig. 11.2. ✥

Constructing Parse Trees

Yield of a tree

Each parse tree represents a string of terminals s, which we call the *yield* of the tree. The string s consists of the labels of the leaves of the tree, in left-to-right order. Alternatively, we can find the yield by doing a preorder traversal of the parse tree and listing only the labels that are terminals. For example, the yield of the parse tree in Fig. 11.8 is `3*(2+14)`.

If a tree has one node, that node will be labeled by a terminal or ϵ, because it is a leaf. If the tree has more than one node, then the root will be labeled by a syntactic category, since the root of a tree of two or more nodes is always an interior node. This syntactic category will always include, among its strings, the yield of the tree. The following is an inductive definition of the parse trees for a given grammar.

BASIS. For every terminal of the grammar, say `x`, there is a tree with one node labeled `x`. This tree has yield `x`, of course.

INDUCTION. Suppose we have a production $<S> \rightarrow X_1X_2 \cdots X_n$, where each of the X_i's is either a terminal or a syntactic category. If $n = 0$, that is, the production is really $<S> \rightarrow \epsilon$, then there is a tree like that of Fig. 11.9. The yield is ϵ, and

Fig. 11.9. Parse tree from production $<S> \rightarrow \epsilon$.

the root is $<S>$; surely string ϵ is in $L(<S>)$, because of this production.

Now suppose $<S> \rightarrow X_1X_2 \cdots X_n$ and $n \geq 1$. We may choose a tree T_i for each X_i, $i = 1, 2, \ldots, n$, as follows:

1. If X_i is a terminal, we must choose the 1-node tree labeled X_i. If two or more of the X's are the same terminal, then we must choose different one-node trees with the same label for each occurrence of this terminal.

2. If X_i is a syntactic category, we may choose any parse tree already constructed that has X_i labeling the root. We then construct a tree that looks like Fig. 11.10. That is, we create a root labeled $<S>$, the syntactic category at the head of the production, and we give it as children, the roots of the trees selected for $X_1, X_2, \ldots, X_n$, in order from the left. If two or more of the X's are the same syntactic category, we may choose the same tree for each, but we must make a distinct copy of the tree each time it is selected. We are also permitted to choose different trees for different occurrences of the same syntactic category.

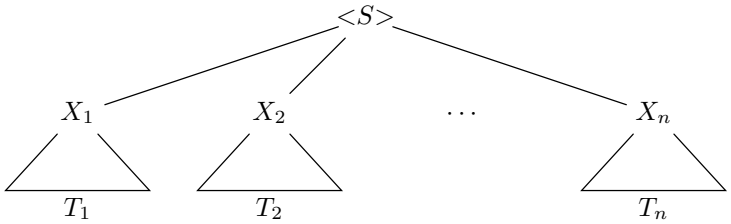


Fig. 11.10. Constructing a parse tree using a production and other parse trees.

✦ **Example 11.6.** Let us follow the construction of the parse tree in Fig. 11.8, and see how its construction mimics a proof that the string `3*(2+14)` is in $L(<E>)$. First, we can construct a one-node tree for each of the terminals in the tree. Then the group of productions on line (1) of Fig. 11.2 says that each of the ten digits is a string of length 1 belonging to $L(<D>)$. We use four of these productions to create the four trees shown in Fig. 11.11. For instance, we use the production $<D> \rightarrow$**1** to create the parse tree in Fig. 11.11(a) as follows. We create a tree with a single node labeled `1` for the symbol **1** in the body. Then we create a node labeled $<D>$ as the root and give it one child, the root (and only node) of the tree selected for **1**.

Our next step is to use production (2) of Fig. 11.2, or $<N> \rightarrow <D>$, to discover that digits are numbers. For instance, we may choose the tree of Fig. 11.11(a) to substitute for $<D>$ in the body of production (2), and get the tree of Fig. 11.12(a). The other two trees in Fig. 11.12 are produced similarly.

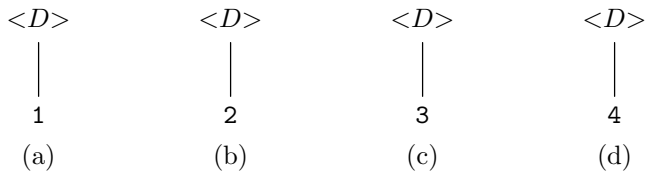


Fig. 11.11. Parse trees constructed using production $<D> \rightarrow \mathbf{1}$ and similar productions.

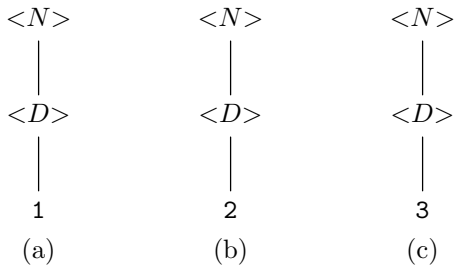


Fig. 11.12. Parse trees constructed using production $<N> \rightarrow <D>$.

Now we can use production (3), which is $<N> \rightarrow <N><D>$. For $<N>$ in the body we shall select the tree of Fig. 11.12(a), and for $<D>$ we select Fig. 11.11(d). We create a new node labeled by $<N>$, for the head, and give it two children, the roots of the two selected trees. The resulting tree is shown in Fig. 11.13. The yield of this tree is the number `14`.

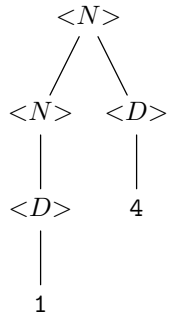


Fig. 11.13. Parse trees constructed using production $<N> \rightarrow <N><D>$.

Our next task is to create a tree for the sum `2+14`. First, we use the production (4), or $<E> \rightarrow <N>$, to build the parse trees of Fig. 11.14. These trees show that `3`, `2`, and `14` are expressions. The first of these comes from selecting the tree of Fig. 11.12(c) for $<N>$ of the body. The second is obtained by selecting the tree of Fig. 11.12(b) for $<N>$, and the third by selecting the tree of Fig. 11.13.

Then we use production (6), which is $<E> \rightarrow <E>+<E>$. For the first $<E>$ in the body we use the tree of Fig. 11.14(b), and for the second $<E>$ in the body we use the tree of Fig. 11.14(c). For the terminal $+$ in the body, we use a one-node tree with label `+`. The resulting tree is shown in Fig. 11.15; its yield is `2+14`.

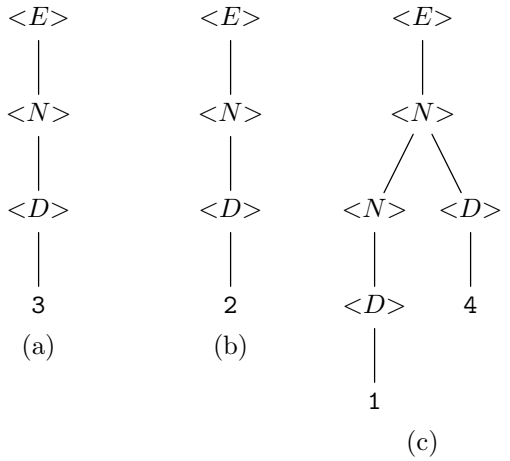


Fig. 11.14. Parse trees constructed using production $<E> \rightarrow <N>$.

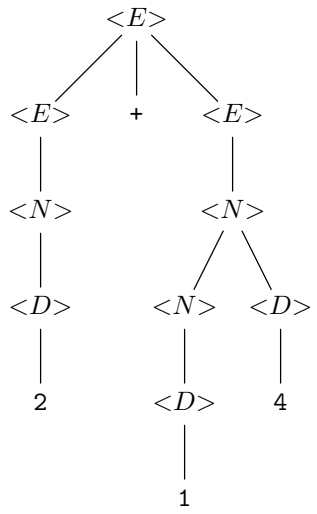


Fig. 11.15. Parse tree constructed using production $<E> \rightarrow <E>+<E>$.

We next use production (5), or $<E> \rightarrow (<E>)$, to construct the parse tree of Fig. 11.16. We have simply selected the parse tree of Fig. 11.15 for the $<E>$ in the body, and we select the obvious one-node trees for the terminal parentheses.

Lastly, we use production (8), which is $<E> \rightarrow <E> * <E>$, to construct the parse tree that we originally showed in Fig. 11.8. For the first $<E>$ in the body, we choose the tree of Fig. 11.14(a), and for the second we choose the tree of Fig. 11.16. ✦

Why Parse Trees "Work"

The construction of parse trees is very much like the inductive definition of the strings belonging to a syntactic category. We can prove, by two simple inductions, that the yields of the parse trees with root $<S>$ are exactly the strings in $L(<S>)$, for any syntactic category $<S>$. That is,

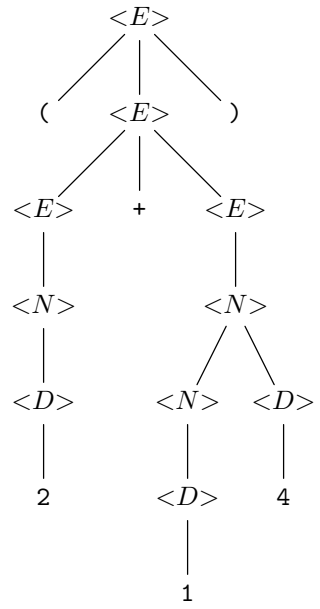


Fig. 11.16. Parse tree constructed using production <E> → (<E>).

1. If T is a parse tree with root labeled <S> and yield s, then string s is in the language $L(<S>)$.

2. If string s is in $L(<S>)$, then there is a parse tree with yield s and root labeled <S>.

This equivalence should be fairly intuitive. Roughly, parse trees are assembled from smaller parse trees in the same way that we assemble long strings from shorter ones, using substitution for syntactic categories in the bodies of productions. We begin with part (1), which we prove by complete induction on the height of tree T.

BASIS. Suppose the height of the parse tree is 1. Then the tree looks like Fig. 11.17, or, in the special case where $n = 0$, like the tree of Fig. 11.9. The only way we can construct such a tree is if there is a production $<S> \to \mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_n$, where each of the $\mathbf{x}$'s is a terminal (if $n = 0$, the production is $<S> \to \epsilon$). Thus, $\mathtt{x}_1\mathtt{x}_2 \cdots \mathtt{x}_n$ is a string in $L(<S>)$.

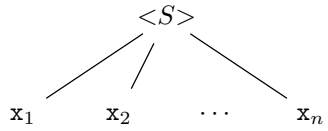


Fig. 11.17. Parse tree of height 1.

INDUCTION. Suppose that statement (1) holds for all trees of height k or less. Now consider a tree of height $k+1$ that looks like Fig. 11.10. Then each of the subtrees T_i, for $i = 1, 2, \ldots, n$, can be of height at most k. For if any one of the subtrees had height $k+1$ or more, the entire tree would have height at least $k+2$. Thus, the inductive hypothesis applies to each of the trees T_i.

By the inductive hypothesis, if X_i, the root of the subtree T_i, is a syntactic category, then the yield of T_i, say s_i, is in the language $L(X_i)$. If X_i is a terminal, let us define string s_i to be X_i. Then the yield of the entire tree is $s_1 s_2 \cdots s_n$.

We know that $<S> \rightarrow X_1 X_2 \cdots X_n$ is a production, by the definition of a parse tree. Suppose that we substitute string s_i for X_i, whenever X_i is a syntactic category. By definition, X_i is s_i if X_i is a terminal. It follows that the substituted body is $s_1 s_2 \cdots s_n$, the same as the yield of the tree. By the inductive rule for the language of $<S>$, we know that $s_1 s_2 \cdots s_n$ is in $L(<S>)$.

Now we must prove statement (2), that every string s in a syntactic category $<S>$ has a parse tree with root $<S>$ and s as yield. To begin, let us note that for each terminal $\mathtt{x}$, there is a parse tree with both root and yield $\mathtt{x}$. Now we use complete induction on the number of times we applied the inductive step (described in Section 11.3) when we deduced that s is in $L(<S>)$.

BASIS. Suppose s requires one application of the inductive step to show that s is in $L(<S>)$. Then there must be a production $<S> \rightarrow \mathtt{x}_1 \mathtt{x}_2 \cdots \mathtt{x}_n$, where all the $\mathtt{x}$'s are terminals, and $s = \mathtt{x}_1 \mathtt{x}_2 \cdots \mathtt{x}_n$. We know that there is a one node parse tree labeled $\mathtt{x}_i$ for $i = 1, 2, \ldots, n$. Thus, there is a parse tree with yield s and root labeled $<S>$; this tree looks like Fig. 11.17. In the special case that $n = 0$, we know $s = \epsilon$, and we use the tree of Fig. 11.9 instead.

INDUCTION. Suppose that any string t found to be in the language of any syntactic category $<T>$ by k or fewer applications of the inductive step has a parse tree with t as yield and $<T>$ at the root. Consider a string s that is found to be in the language of syntactic category $<S>$ by $k+1$ applications of the inductive step. Then there is a production $<S> \rightarrow X_1 X_2 \cdots X_n$, and $s = s_1 s_2 \cdots s_n$, where each substring s_i is either

1. X_i, if X_i is a terminal, or
2. Some string known to be in $L(X_i)$ using at most k applications of the inductive rule, if X_i is a syntactic category.

Thus, for each i, we can find a tree T_i, with yield s_i and root labeled X_i. If X_i is a syntactic category, we invoke the inductive hypothesis to claim that T_i exists, and if X_i is a terminal, we do not need the inductive hypothesis to claim that there is a one-node tree labeled X_i. Thus, the tree constructed as in Fig. 11.10 has yield s and root labeled $<S>$, proving the induction step.

EXERCISES

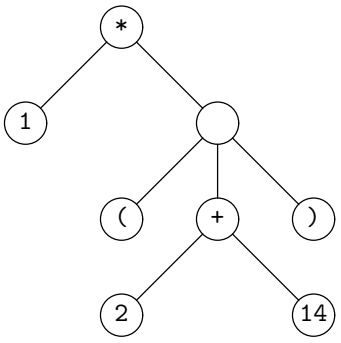
11.4.1: Find a parse tree for the strings

Syntax Trees and Expression Trees

Often, trees that look like parse trees are used to represent expressions. For instance, we used *expression trees* as examples throughout Chapter 5. *Syntax tree* is another name for "expression tree." When we have a grammar for expressions such as that of Fig. 11.2, we can convert parse trees to expression trees by making three transformations:

1. Atomic operands are condensed to a single node labeled by that operand.
2. Operators are moved from leaves to their parent node. That is, an operator symbol such as `+` becomes the label of the node above it that was labeled by the "expression" syntactic category.
3. Interior nodes that remain labeled by "expression" have their label removed.

For instance, the parse tree of Fig. 11.8 is converted to the following expression tree or syntax tree:



a) `35+21`
b) `123-(4*5)`
c) `1*2*(3-4)`

according to the grammar of Fig. 11.2. The syntactic category at the root should be $<E>$ in each case.

11.4.2: Using the statement grammar of Fig. 11.6, find parse trees for the following strings:

a) `wcwcs;`
b) `{s;}`
c) `{s;wcs;}`.

The syntactic category at the root should be $<S>$ in each case.

11.4.3: Using the balanced parenthesis grammar of Fig. 11.3, find parse trees for the following strings:

a) `(()())`
b) `((()))`

c) `((())())`.

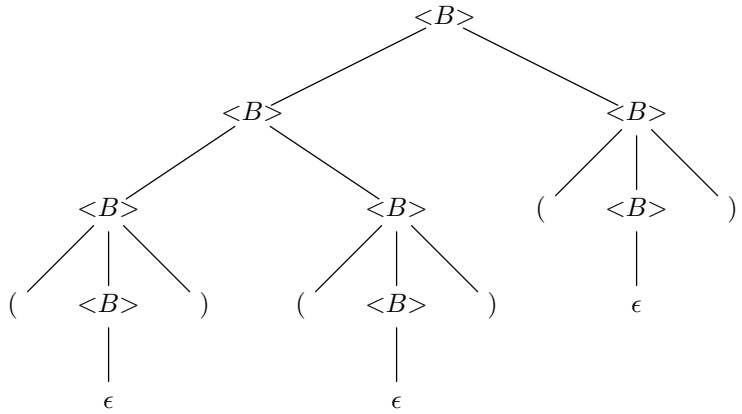
11.4.4: Find parse trees for the strings of Exercise 11.4.3, using the grammar of Fig. 11.4.

11.5 Ambiguity and the Design of Grammars

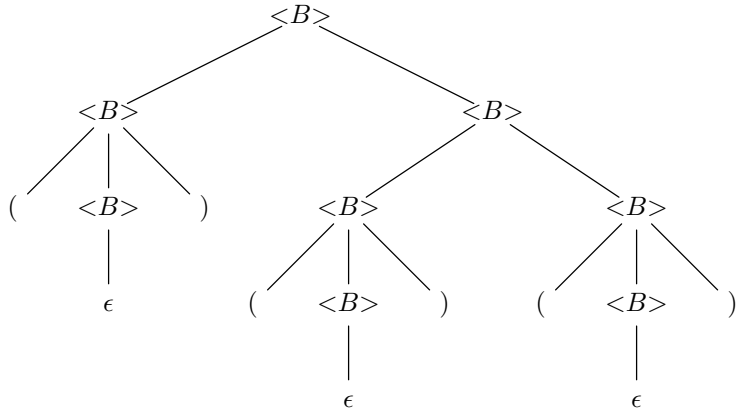
Let us consider the grammar for balanced parentheses that we originally showed in Fig. 11.4, with syntactic category $<B>$ abbreviating $<Balanced>$:

$$<B> \rightarrow (<B>) \mid <B><B> \mid \epsilon \tag{11.1}$$

Suppose we want a parse tree for the string `()()()`. Two such parse trees are shown in Fig. 11.18, one in which the first two pairs of parentheses are grouped first, and the other in which the second two pairs are grouped first.



(a) Parse tree that groups from the left.



(b) Parse tree that groups from the right.

Fig. 11.18. Two parse trees with the same yield and root.

It should come as no surprise that these two parse trees exist. Once we establish that both `()` and `()()` are balanced strings of parentheses, we can use the production $<B> \rightarrow <B><B>$ with `()` substituting for the first $<B>$ in the body and `()()` substituting for the second, or vice-versa. Either way, the string `()()()` is discovered to be in the syntactic category $<B>$.

Ambiguous grammar

A grammar in which there are two or more parse trees with the same yield and the same syntactic category labeling the root is said to be *ambiguous.* Notice that not every string has to be the yield of several parse trees; it is sufficient that there be even one such string, to make the grammar ambiguous. For example, the string `()()()` is sufficient for us to conclude that the grammar (11.1) is ambiguous. A grammar that is not ambiguous is called *unambiguous.* In an unambiguous grammar, for every string s and syntactic category $<S>$, there is at most one parse tree with yield s and root labeled $<S>$.

An example of an unambiguous grammar is that of Fig. 11.3, which we reproduce here with $<B>$ in place of $<Balanced>$,

$$<B> \rightarrow (<B>)<B> \mid \epsilon \tag{11.2}$$

A proof that the grammar is unambiguous is rather difficult. In Fig. 11.19 is the unique parse tree for string `()()()`; the fact that this string has a unique parse tree does not prove the grammar (11.2) is unambiguous, of course. We can only prove unambiguity by showing that *every* string in the language has a unique parse tree.

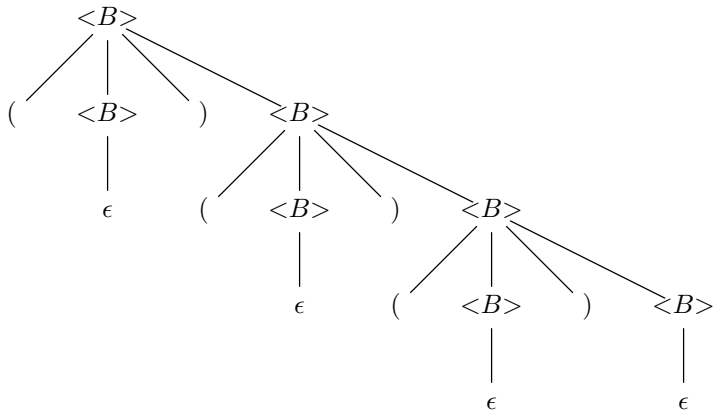


Fig. 11.19. Unique parse tree for the string `( ) ( ) ( )` using the grammar (11.2).

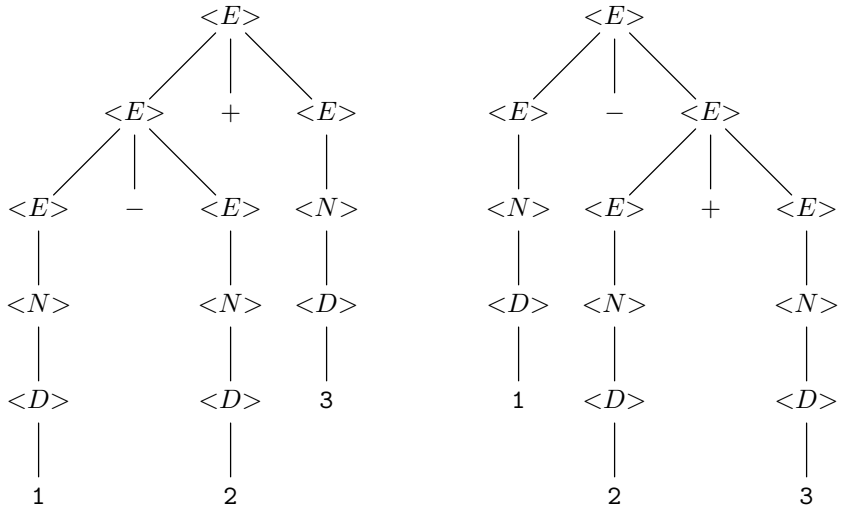
Ambiguity in Expressions

While the grammar of Fig. 11.4 is ambiguous, there is no great harm in its ambiguity, because whether we group several strings of balanced parentheses from the left or the right matters little. When we consider grammars for expressions, such as that of Fig. 11.2 in Section 11.2, some more serious problems can occur. Specifically, some parse trees imply the wrong value for the expression, while others imply the correct value.

Why Unambiguity Is Important

The parser, which constructs parse trees for programs, is an essential part of a compiler. If a grammar describing a programming language is ambiguous, and if its ambiguities are left unresolved, then for at least some programs there is more than one parse tree. Different parse trees for the same program normally impart different meanings to the program, where "meaning" in this case is the action performed by the machine language program into which the original program is translated. Thus, if the grammar for a program is ambiguous, a compiler cannot properly decide which parse tree to use for certain programs, and thus cannot decide what the machine-language program should do. For this reason, compilers must use specifications that are unambiguous.

◆ **Example 11.7.** Let us use the shorthand notation for the expression grammar that was developed in Example 11.5. Then consider the expression `1-2+3`. It has two parse trees, depending on whether we group operators from the left or the right. These parse trees are shown in Fig. 11.20(a) and (b).



(a) Correct parse tree. (b) Incorrect parse tree.

Fig. 11.20. Two parse trees for the expression $1 - 2 + 3$.

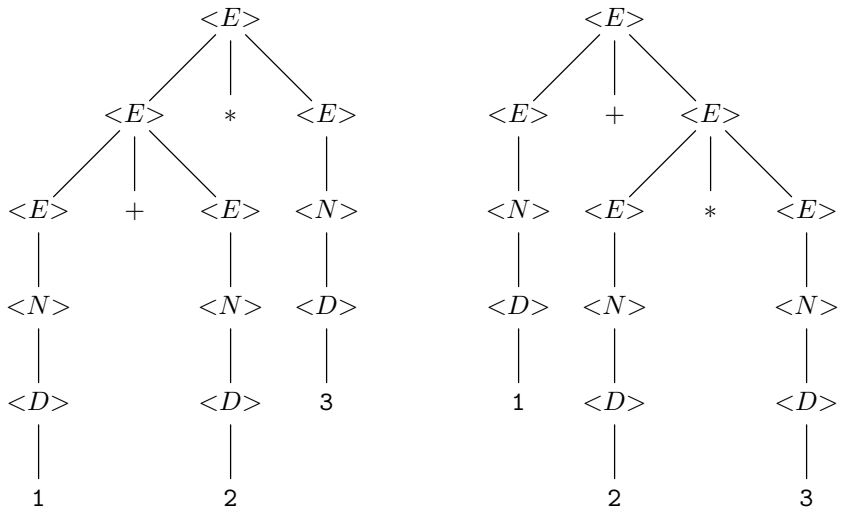
The tree of Fig. 11.20(a) associates from the left, and therefore groups the operands from the left. That grouping is correct, since we generally group operators at the same precedence from the left; `1-2+3` is conventionally interpreted as `(1-2)+3`, which has the value 2. If we evaluate the expressions represented by subtrees, working up the tree of Fig. 11.20(a), we first compute $1 - 2 = -1$ at the leftmost child of the root, and then compute $-1 + 3 = 2$ at the root.

On the other hand, Fig. 11.20(b), which associates from the right, groups our expression as `1-(2+3)`, whose value is -4. This interpretation of the expression is unconventional, however. The value -4 is obtained working up the tree of Fig.

11.20(b), since we evaluate $2 + 3 = 5$ at the rightmost child of the root, and then $1 - 5 = -4$ at the root. ◆

Associating operators of equal precedence from the wrong direction can cause problems. We also have problems with operators of different precedence; it is possible to group an operator of low precedence before one of higher precedence, as we see in the next example.

◆ **Example 11.8.** Consider the expression `1+2*3`. In Fig. 11.21(a) we see the expression incorrectly grouped from the left, while in Fig. 11.21(b), we have correctly grouped the expression from the right, so that the multiplication gets its operands grouped before the addition. The former grouping yields the erroneous value 9, while the latter grouping produces the conventional value of 7. ◆



(a) Incorrect parse tree. (b) Correct parse tree.

Fig. 11.21. Two parse trees for the expression `1+2*3`.

Unambiguous Grammars for Expressions

Just as the grammar (11.2) for balanced parentheses can be viewed as an unambiguous version of the grammar (11.1), it is possible to construct an unambiguous version of the expression grammar from Example 11.5. The "trick" is to define three syntactic categories, with intuitive meanings as follows.

1. <*Factor*> generates expressions that cannot be "pulled apart," that is, a factor is either a single operand or any parenthesized expression.

2. <*Term*> generates a product or quotient of factors. A single factor is a term, and thus is a sequence of factors separated by the operators $*$ or /. Examples of terms are `12` and `12/3*45`.

3. <*Expression*> generates a sum or difference of one or more terms. A single term is an expression, and thus is a sequence of terms separated by the operators + or −. Examples of expressions are `12`, `12/3*45`, and `12+3*45-6`.

Figure 11.22 is a grammar that expresses the relationship between expressions, terms, and factors. We use shorthands <*E*>, <*T*>, and <*F*> for <*Expression*>, <*Term*>, and <*Factor*>, respectively.

(1) $<E> \rightarrow <E> + <T> \mid <E> - <T> \mid <T>$

(2) $<T> \rightarrow <T> * <F> \mid <T> / <F> \mid <F>$

(3) $<F> \rightarrow (<E>) \mid <N>$

(4) $<N> \rightarrow <N><D> \mid <D>$

(5) $<D> \rightarrow \mathbf{0} \mid \mathbf{1} \mid \cdots \mid \mathbf{9}$

Fig. 11.22. Unambiguous grammar for arithmetic expressions.

For instance, the three productions in line (1) define an expression to be either a smaller expression followed by a `+` or `-` and another term, or to be a single term. If we put these ideas together, the productions say that every expression is a term, followed by zero or more pairs, each pair consisting of a `+` or `-` and a term. Similarly, line (2) says that a term is either a smaller term followed by `*` or `/` and a factor, or it is a single factor. That is, a term is a factor followed by zero or more pairs, each pair consisting of a `*` or a `/` and a factor. Line (3) says that factors are either numbers, or expressions surrounded by parentheses. Lines (4) and (5) define numbers and digits as we have done previously.

The fact that in lines (1) and (2) we use productions such as

$$<E> \rightarrow <E> + <T>$$

rather than the seemingly equivalent $<E> \rightarrow <T> + <E>$, forces terms to be grouped from the left. Thus, we shall see that an expression such as `1-2+3` is correctly grouped as `(1-2)+3`. Likewise, terms such as `1/2*3` are correctly grouped as `(1/2)*3`, rather than the incorrect `1/(2*3)`. Figure 11.23 shows the only possible parse tree for the expression `1-2+3` in the grammar of Fig. 11.22. Notice that `1-2` must be grouped as an expression first. If we had grouped `2+3` first, as in Fig. 11.20(b), there would be no way, in the grammar of Fig. 11.22, to attach the `1-` to this expression.

The distinction among expressions, terms, and factors enforces the correct grouping of operators at different levels of precedence. For example, the expression `1+2*3` has only the parse tree of Fig. 11.24, which groups the subexpression `2*3` first, like the tree of Fig. 11.21(b) and unlike the incorrect tree of Fig. 11.21(a), which groups `1+2` first.

As for the matter of balanced parentheses, we have not proved that the grammar of Fig. 11.22 is unambiguous. The exercises contain a few more examples that should help convince the reader that this grammar is not only unambiguous, but gives the correct grouping for each expression. We also suggest how the idea of this grammar can be extended to more general families of expressions.

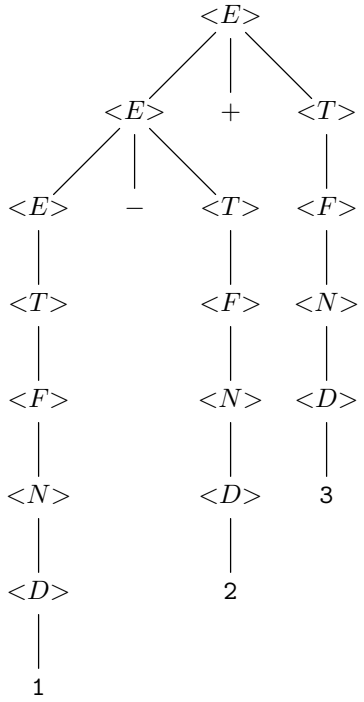


Fig. 11.23. Parse tree for the expression 1 − 2 + 3 in the unambiguous grammar of Fig. 11.22.

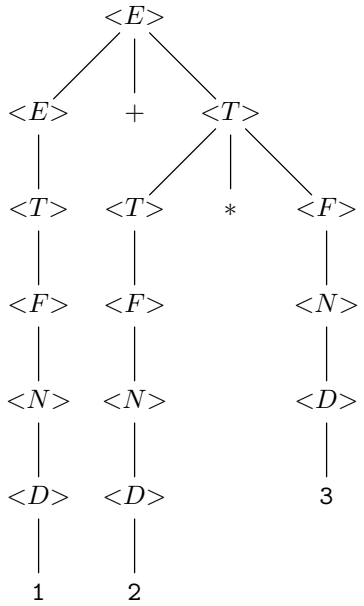


Fig. 11.24. Parse tree for 1 + 2 ∗ 3 in the unambiguous grammar of Fig. 11.22.

EXERCISES

11.5.1: In the grammar of Fig. 11.22, give the unique parse tree for each of the following expressions:

a) `(1+2)/3`
b) `1*2-3`
c) `(1+2)*(3+4)`

11.5.2*: The expressions of the grammar in Fig. 11.22 have two levels of precedence; + and − at one level, and ∗ and / at a second, higher level. In general, we can handle expressions with k levels of precedence by using $k + 1$ syntactic categories. Modify the grammar of Fig. 11.22 to include the exponentiation operator `^`, which is at a level of precedence higher than `*` and `/`. As a hint, define a *primary* to be an operand or a parenthesized expression, and redefine a *factor* to be one or more primaries connected by the exponentiation operator. Note that exponentiation groups from the right, not the left, and `2^3^4` means `2^(3^4)`, rather than `(2^3)^4`. How do we force grouping from the right among primaries?

11.5.3*: Extend the unambiguous expression grammar to allow the comparison operators, `=`, `<=`, and so on, which are all at the same level of precedence and left-associative. Their precedence is below that of + and −.

11.5.4: Extend the expression grammar of Fig. 11.22 to include the unary minus sign. Note that this operator is at a higher precedence than the other operators; for instance, `-2*-3` is grouped `(-2)*(-3)`.

11.5.5: Extend your grammar of Exercise 11.5.3 to include the logical operators `&&`, `||`, and `!`. Give `&&` the precedence of `*`, `||` the precedence of `+`, and `!` a higher precedence than unary −. `&&` and `||` are binary operators that group from the left.

11.5.6*: Not every expression has more than one parse tree according to the ambiguous grammar of Fig. 11.2 in Section 11.2. Give several examples of expressions that have unique parse trees according to this grammar. Can you give a rule indicating when an expression will have a unique parse tree?

11.5.7: The following grammar defines the set of strings (other than ϵ) consisting of `0`'s and `1`'s only.

$$<String> \rightarrow <String><String> \mid \mathbf{0} \mid \mathbf{1}$$

In this grammar, how many parse trees does the string `010` have?

11.5.8: Give an unambiguous grammar that defines the same language as the grammar of Exercise 11.5.7.

11.5.9*: How many parse trees does grammar (11.1) have for the empty string? Show three different parse trees for the empty string.

11.6 Constructing Parse Trees

Grammars are like regular expressions, in that both notations describe languages but do not give directly an algorithm for determining whether a string is in the

language being defined. For regular expressions, we learned in Chapter 10 how to convert a regular expression first into a nondeterministic automaton and then into a deterministic one; the latter can be implemented directly as a program.

There is a somewhat analogous process for grammars. We cannot, in general, convert a grammar to a deterministic automaton at all; the next section discusses some examples of when that conversion is impossible. However, it is often possible to convert a grammar to a program that, like an automaton, reads the input from beginning to end and renders a decision whether the input string is in the language of the grammar. The most important such technique, called "LR parsing" (the LR stands for left-to-right on the input), is beyond the scope of this book.

Recursive-Descent Parsing

What we shall give instead is a simpler but less powerful parsing technique called "recursive descent," in which the grammar is replaced by a collection of mutually recursive functions, each corresponding to one of the syntactic categories of the grammar. The goal of the function S that corresponds to the syntactic category $<S>$ is to read a sequence of input characters that form a string in the language $L(<S>)$, and to return a pointer to the root of a parse tree for this string.

A production's body can be thought of as a sequence of goals — the terminals and syntactic categories — that must be fulfilled in order to find a string in the syntactic category of the head. For instance, consider the unambiguous grammar for balanced parentheses, which we reproduce here as Fig. 11.25.

(1) $<B> \rightarrow \epsilon$
(2) $<B> \rightarrow$ ($<B>$) $<B>$

Fig. 11.25. Grammar for balanced parentheses.

Production (2) states that one way to find a string of balanced parentheses is to fulfill the following four goals in order.

1. Find the character `(`, then
2. Find a string of balanced parentheses, then
3. Find the character `)`, and finally
4. Find another string of balanced parentheses.

In general, a terminal goal is satisfied if we find that this terminal is the next input symbol, but the goal cannot be satisfied if the next input symbol is something else. To tell whether a syntactic category in the body is satisfied, we call a function for that syntactic category.

The arrangement for constructing parse trees according to a grammar is suggested in Fig. 11.26. Suppose we want to determine whether the sequence of terminals $X_1X_2\cdots X_n$ is a string in the syntactic category $<S>$, and to find its parse tree if so. Then on the input file we place $X_1X_2\cdots X_n$ `ENDM`, where `ENDM` is a special symbol that is not a terminal.[2] We call `ENDM`, the *endmarker,* and its purpose is to

Endmarker

[2] In real compilers for programming languages, the entire input might not be placed in a file at once, but terminals would be discovered one at a time by a preprocessor called a "lexical analyzer" that examines the source program one character at a time.

indicate that the entire string being examined has been read. For example, in C programs it would be typical to use the end-of-file or end-of-string character for the endmarker.

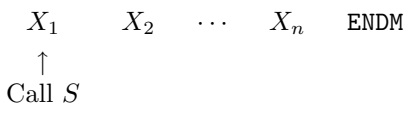


Fig. 11.26. Initializing the program to discover an <S> on the input.

Input cursor

An *input cursor* marks the terminal to be processed, the *current* terminal. If the input is a string of characters, then the cursor might be a pointer to a character. We start our parsing program by calling the function S for the starting syntactic category <S>, with the input cursor at the beginning of the input.

Each time we are working on a production body, and we come to a terminal **a** in the production, we look for the matching terminal a at the position indicated by the input cursor. If we find a, we advance the input cursor to the next terminal on the input. If the current terminal is something other than a, then we fail to match, and we cannot find a parse tree for the input string.

On the other hand, if we are working on a production body and we come to a syntactic category <T>, we call the function T for <T>. If T "fails," then the entire parse fails, and the input is deemed not to be in the language being parsed. If T succeeds, then it "consumes" some input, but moving the input cursor forward zero or more positions on the input. All input positions, from the position at the time T was called, up to but not including the position at which T leaves the cursor, are consumed. T also returns a tree, which is the parse tree for the consumed input.

When we have succeeded with each of the symbols in a production body, we assemble the parse tree for the portion of the input represented by that production. To do so, we create a new root node, labeled by the head of the production. The root's children are the roots of the trees returned by successful calls to functions for the syntactic categories of the body and leaves created for each of the terminals of the body.

A Recursive-Descent Parser for Balanced Parentheses

Let us consider an extended example of how we might design the recursive function B for the syntactic category <B> of the grammar of Fig. 11.25. B, called at some input position, will consume a string of balanced parentheses starting at that position and leave the input cursor at the position immediately after the balanced string.

The hard part is deciding whether to satisfy the goal of finding a <B> by using production (1), <B> $\rightarrow \epsilon$, which succeeds immediately, or by using production (2), that is,

$$<B> \rightarrow \texttt{(}<B>\texttt{)}<B>$$

The strategy we shall follow is that whenever the next terminal is (, use production (2); whenever the next terminal is) or the endmarker, use production (1).

The function B is given in Fig. 11.27(b). It is preceded by important auxiliary elements in Fig. 11.27(a). These elements include:

```
#define FAILED NULL

typedef struct NODE *TREE;
struct NODE {
    char label;
    TREE leftmostChild, rightSibling;
};

TREE makeNode0(char x);
TREE makeNode1(char x, TREE t);
TREE makeNode4(char x, TREE t1, TREE t2, TREE t3, TREE t4);
TREE B();

TREE parseTree; /* holds the result of the parse */
char *nextTerminal; /* current position in input string */

void main()
{
    nextTerminal = "()()"; /* in practice, a string
        of terminals would be read from input */
    parseTree = B();
}

TREE makeNode0(char x)
{
    TREE root;

    root = (TREE) malloc(sizeof(struct NODE));
    root->label = x;
    root->leftmostChild = NULL;
    root->rightSibling = NULL;
    return root;
}

TREE makeNode1(char x, TREE t)
{
    TREE root;

    root = makeNode0(x);
    root->leftmostChild = t;
    return root;
}

TREE makeNode4(char x, TREE t1, TREE t2, TREE t3, TREE t4)
{
    TREE root;

    root = makeNode1(x, t1);
    t1->rightSibling = t2;
    t2->rightSibling = t3;
    t3->rightSibling = t4;
    return root;
}
```

Fig. 11.27(a). Auxiliary functions for recursive-descent parser.

```
     TREE B()
     {
(1)      TREE firstB, secondB;
(2)      if(*nextTerminal == '(') /* follow production 2 */ {
(3)          nextTerminal++;
(4)          firstB = B();
(5)          if(firstB != FAILED && *nextTerminal == ')') {
(6)              nextTerminal++;
(7)              secondB = B();
(8)              if(secondB == FAILED)
(9)                  return FAILED;
                 else
(10)                 return makeNode4('B',
                         makeNode0('('),
                         firstB,
                         makeNode0(')'),
                         secondB);
             }
             else /* first call to B failed */
(11)             return FAILED;
         }
         else /* follow production 1 */
(12)         return makeNode1('B', makeNode0('e'));
     }
```

Fig. 11.27(b). Function to construct parse trees for strings of balanced parentheses.

1. Definition of a constant `FAILED` to be the value returned by B when that function fails to find a string of balanced parentheses on the input. The value of `FAILED` is the same as `NULL`. The latter value also represents an empty tree. However, the parse tree returned by B could not be empty if B succeeds, so there is no possible ambiguity in this definition of `FAILED`.

2. Definitions of the types `NODE` and `TREE`. A node consists of a label field, which is a character, and pointers to the leftmost child and right sibling. The label may be `'B'` to represent a node labeled B, `'('` and `')'` to represent nodes labeled with left- or right-parentheses, respectively, and `'e'` to represent a node labeled ϵ. Unlike the leftmost-child-right-sibling structure of Section 5.3, we have elected to use `TREE` rather than `pNODE` as the type of pointers to nodes since most uses of these pointers here will be as representations of trees.

3. Prototype declarations for three auxiliary functions to be described below and the function B.

4. Two global variables. The first, `parseTree`, holds the parse tree returned by the initial call to B. The second, `nextTerminal`, is the input cursor and points to the current position on the input string of terminals. Note that it is important for `nextTerminal` to be global, so when one call to B returns, the place where it left the input cursor is known to the copy of B that made the call.

5. The function `main`. In this simple demonstration, `main` sets `nextTerminal` to point to the beginning of a particular test string, `()()`, and the result of a call to B is placed in `parseTree`.

6. Three auxiliary functions that create tree nodes and, if necessary, combine subtrees to form larger trees. These are:

   a) Function *makeNode*0(x) creates a node with zero children, that is, a leaf, and labels that leaf with the symbol x. The tree consisting of this one node is returned.

   b) Function *makeNode*1(x, t) creates a node with one child. The label of the new node is x, and the child is the root of the tree t. The tree whose root is the created node is returned. Note that `makeNode1` uses `makeNode0` to create the root node and then makes the root of tree t be the leftmost child of the root. We assume that all leftmost-child and right-sibling pointers are `NULL` initially, as they will be because they are all created by `makeNode0`, which explicitly `NULL`'s them. Thus, it is not mandatory that `makeNode1` to store `NULL` in the `rightSibling` field of the root of t, but it would be a wise safety measure to do so.

   c) Function *makeNode*4(x, t_1, t_2, t_3, t_4) creates a node with four children. The label of the node is x, and the children are the roots of the trees t_1, t_2, t_3, and t_4, from the left. The tree whose root is the created node is returned. Note that `makeNode4` uses `makeNode1` to create a new root and attach t_1 to it, then strings the remaining trees together with right-sibling pointers.

Now we can consider the program of Fig. 11.27(b) line by line. Line (1) is the declaration of two local variables, `firstB` and `secondB`, to hold the parse trees returned by the two calls to B in the case that we elect to try production (2). Line (2) tests if the next terminal on the input is `(`. If so, we shall look for an instance of the body of production (2), and if not, then we shall assume that production (1) is used, and that ϵ is the balanced string.

At line (3), we increment `nextTerminal`, because the current input `(` has matched the `(` in the body of production (2). We now have the input cursor properly positioned for a call to B that will find a balanced string for the first $<B>$ in the body of production (2). That call occurs at line (4), and the tree returned is stored in variable `firstB` to be assembled later into a parse tree for the current call to B.

At line (5) we check that we are still capable of finding a balanced string. That is, we first check that the call to B on line (4) did not fail. Then we test that the current value of `nextTerminal` is `)`. Recall that when B returns, `nextTerminal` points to the next input terminal to be formed into a balanced string. If we are to match the body of production (2), and we have already matched the `(` and the first $<B>$, then we must next match the `)`, which explains the second part of the test. If either part of the test fails, then the current call to B fails at line (11).

If we pass the test of line (5), then at lines (6) and (7) we advance the input cursor over the right parenthesis just found and call B again, to match the final $<B>$ in production (2). The tree returned is stored temporarily in `secondB`.

If the call to B on line (7) fails, then `secondB` will have value `FAILED`. Line (8) detects this condition, and the current call to B also fails.

Line (10) covers the case in which we have succeeded in finding a balanced

string. We return a tree constructed by `makeNode`4. This tree has a root labeled `'B'`, and four children. The first child is a leaf labeled `(`, constructed by `makeNode`0. The second is the tree we stored in `firstB`, which is the parse tree produced by the call to B at line (4). The third child is a leaf labeled `)`, and the fourth is the parse tree stored in `secondB`, which was returned by the second call to B at line (7).

Line (11) is used only when the test of line (5) fails. Finally, line (12) handles the case where the original test of line (1) fails to find `(` as the first character. In that case, we assume that production (1) is correct. This production has the body ϵ, and so we consume no input but return a node, created by `makeNode`1, that has the label `B` and one child labeled ϵ.

◆ **Example 11.9.** Suppose we have the terminals `( ) ( ) ENDM` on the input. Here, `ENDM` stands for the character `'\0'`, which marks the end of character strings in C. The call to B from `main` in Fig. 11.27(a) finds `(` as the current input, and the test of line (2) succeeds. Thus, `nextTerminal` advances at line (3), and at line (4) a second call to B is made, as suggested by "call 2" in Fig. 11.28.

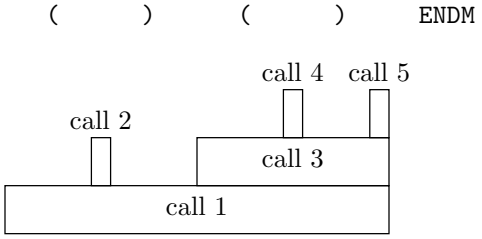


Fig. 11.28. Calls made while processing the input ( ) ( ) `ENDM`.

In call 2, the test of line (2) fails, and we thus return the tree of Fig. 11.29(a) at line (12). Now we return to call 1, where we are at line (5), with `)` pointed to by `nextTerminal` and the tree of Fig. 11.29(a) in `firstB`. Thus, the test of line (5) succeeds. We advance `nextTerminal` at line (6) and call B at line (7). This is "call 3" in Fig. 11.28.

In call 3 we succeed at line (2), advance `nextTerminal` at line (3), and call B at line (4); this call is "call 4" in Fig. 11.28. As with call 2, call 4 fails the test of line (2) and returns a (distinct) tree like that of Fig. 11.29(a) at line (12).

We now return to call 3, with `nextTerminal` still pointing to `)`, with `firstB` (local to this call of B) holding a tree like Fig. 11.29(a), and with control at line (5). The test succeeds, and we advance `nextTerminal` at line (6), so it now points to `ENDM`. We make the fifth call to B at line (7). This call has its test fail at line (2) and returns another copy of Fig. 11.29(a) at line (12). This tree becomes the value of `secondB` for call 3, and the test of line (8) fails. Thus, at line (10) of call 3, we construct the tree shown in Fig. 11.29(b).

At this point, call 3 returns successfully to call 1 at line (8), with `secondB` of call 1 holding the tree of Fig. 11.29(b). As in call 3, the test of line (8) fails, and at line (10) we construct a tree with a new root node, whose second child is a copy of the tree in Fig. 11.29(a) — this tree was held in `firstB` of call 1 — and whose fourth child is the tree of Fig. 11.29(b). The resulting tree, which is placed in `parseTree` by `main`, is shown in Fig. 11.29(c). ◆

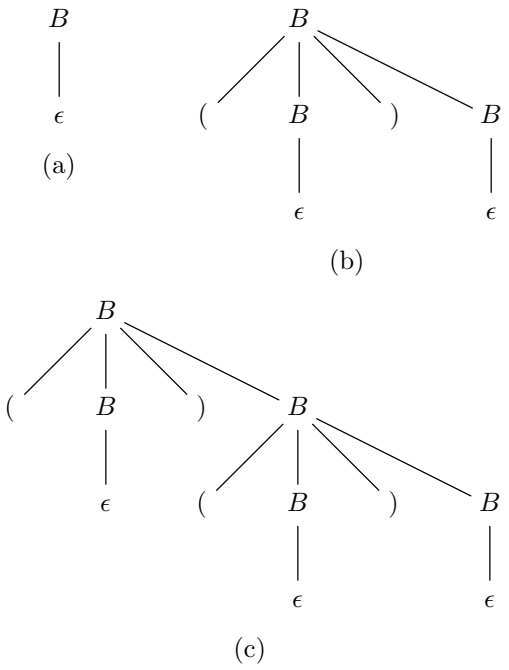


Fig. 11.29. Trees constructed by recursive calls to B.

Constructing Recursive-Descent Parsers

We can generalize the technique used in Fig. 11.27 to many grammars, although not to all grammars. The key requirement is that for each syntactic category <S>, if there is more than one production with <S> as the head, then by looking at only the current terminal (often called the *lookahead* symbol), we can decide on the one production for <S> that needs to be tried. For instance, in Fig. 11.27, our decision strategy is to pick the second production, with body `(`<B>`)`<B>, whenever the lookahead symbol is `(`, and to pick the first production, with body ϵ, when the lookahead symbol is `)` or `ENDM`.

Lookahead symbol

It is not possible to tell, in general, whether there is an algorithm for a given grammar that will always make the right decision. For Fig. 11.27, we claimed, but did not prove, that the strategy stated above will work. However, if we have a decision strategy that we believe will work, then for each syntactic category <S>, we can design the function S to do the following:

1. Examine the lookahead symbol and decide which production to try. Suppose the chosen production has body $X_1X_2\cdots X_n$.
2. For $i = 1, 2, \ldots, n$ do the following with X_i.
   a) If X_i is a terminal, check that the lookahead symbol is X_i. If so, advance the input cursor. If not, then this call to S fails.
   b) If X_i is a syntactic category, such as <T>, then call the function T corresponding to this syntactic category. If T returns with failure, then the call to S fails. If T returns successfully, store away the returned tree for use later.

If we have not failed after considering all the X_i's, then assemble a parse tree to return by creating a new node, with children corresponding to $X_1, X_2, \ldots, X_n$, in order. If X_i is a terminal, then the child for X_i is a newly created leaf with label X_i. If X_i is a syntactic category, then the child for X_i is the root of the tree that was returned when a call to the function for X_i was completed. Figure 11.29 was an example of this tree construction.

If the syntactic category $<S>$ represents the language whose strings we want to recognize and parse, then we start the parsing process by placing the input cursor at the first input terminal. A call to the function S will cause a parse tree for the input to be constructed if there is one and will return failure if the input is not in the language $L(<S>)$.

EXERCISES

11.6.1: Show the sequence of calls made by the program of Fig. 11.27 on the inputs

a) `(())`
b) `(()())`
c) `())(`

in each case followed by the endmarker symbol `ENDM`.

11.6.2: Consider the following grammar for numbers.

$<Number> \rightarrow <Digit><Number> \mid \epsilon$
$<Digit> \rightarrow \mathbf{0} \mid \mathbf{1} \mid \cdots \mid \mathbf{9}$

Design a recursive-descent parser for this grammar; that is, write a pair of functions, one for $<Number>$ and the other for $<Digit>$. You may follow the style of Fig. 11.27 and assume that there are functions like $makeNode1$ that return trees with the root having a specified number of children.

11.6.3**: Suppose we had written the productions for $<Number>$ in Exercise 11.6.2 as

$$<Number> \rightarrow <Digit><Number> \mid <Digit>$$

or as

$$<Number> \rightarrow <Number><Digit> \mid \epsilon$$

Would we then be able to design a recursive-descent parser? Why or why not?

(1) $<L> \rightarrow$ **(** $<E>$ $<T>$
(2) $<T> \rightarrow$ **,** $<E>$ $<T>$
(3) $<T> \rightarrow$ **)**
(4) $<E> \rightarrow <L>$
(5) $<E> \rightarrow atom$

Fig. 11.30. Grammar for list structures.

11.6.4*: The grammar in Fig. 11.30 defines nonempty lists, which are elements separated by commas and surrounded by parentheses. An element can be either an atom or a list structure. Here, <E> stands for element, <L> for list, and <T> for "tail," that is, either a closing `)`, or pairs of commas and elements ended by `)`. Write a recursive-descent parser for the grammar of Fig. 11.30.

11.7 A Table-Driven Parsing Algorithm

As we have seen in Section 6.7, recursive function calls are normally implemented by a stack of activation records. As the functions in a recursive-descent parser do something very specific, it is possible to replace them by a single function that examines a table and manipulates a stack itself.

Remember that the function S for a syntactic category <S> first decides what production to use, then goes through a sequence of steps, one for each symbol in the body of the selected production. Thus, we can maintain a stack of grammar symbols that roughly corresponds to the stack of activation records. However, both terminals and syntactic categories are placed on the stack. When a syntactic category <S> is on top of the stack, we first determine the correct production. Then we replace <S> by the body of the selected production, with the left end at the top of the stack. When a terminal is at the top of the stack, we make sure it matches the current input symbol. If so, we pop the stack and advance the input cursor.

To see intuitively why this arrangement works, suppose that a recursive-descent parser has just called S, the function for syntactic category <S>, and the selected production has body **a**<B><C>. Then there would be four times when this activation record for S is active.

1. When it checks for `a` on the input,
2. When it makes the call to B,
3. When that call returns and C is called, and
4. When the call to C returns and S is finished.

If, in the table-driven parser, we immediately replace <S> by the symbols of the body, **a**<B><C> in this example, then the stack will expose these symbols at the same points on the input when control returns to the corresponding activation of S in the recursive-descent parser.

1. The first time, **a** is exposed, and we check for `a` on the input, just as function S would.
2. The second time, which occurs immediately afterward, S would call B, but we have <B> at the top of the stack, which will cause the same action.
3. The third time, S calls C, but we find <C> on top of the stack and do the same.
4. The fourth time, S returns, and we find no more of the symbols by which <S> was replaced. Thus, the symbol below the point on the stack that formerly held <S> is now exposed. Analogously, the activation record below S's activation record would receive control in the recursive-descent parser.

Parsing Tables

As an alternative to writing a collection of recursive functions, we can construct a *parsing table*, whose rows correspond to the syntactic categories, and whose columns correspond to the possible lookahead symbols. The entry in the row for syntactic category $<S>$ and lookahead symbol X is the number of the production with head $<S>$ that must be used to expand $<S>$ if the lookahead is X.

Certain entries of the parse table are left blank. Should we find that syntactic category $<S>$ needs to be expanded, and the lookahead is X, but the entry in the row for $<S>$ and the column for X is blank, then the parse has failed. In this case, we can be sure that the input is not in the language.

✥ **Example 11.10.** In Fig. 11.31 we see the parsing table for the grammar of Fig. 11.25, the unambiguous grammar for balanced parentheses. This parsing table is rather simple, because there is only one syntactic category. The table expresses the same strategy that we used in our running example of Section 11.6. Expand by production (2), or $<B> \rightarrow (<B>)<B>$, if the lookahead is `(`, and expand by production (1), or $<B> \rightarrow \epsilon$, otherwise. We shall see shortly how parsing tables such as this one are used. ✥

	(	)	ENDM
$<B>$	2	1	1

Fig. 11.31. Parsing table for the balanced parentheses grammar.

✥ **Example 11.11.** Figure 11.32 is another example of a parsing table. It is for the grammar of Fig. 11.33, which is a variant of the statement grammar of Fig. 11.6.

	w	c	{	}	s	;	ENDM
$<S>$	1		2		3		
$<T>$	4		4	5	4		

Fig. 11.32. Parsing table for the grammar of Fig. 11.33.

The grammar of Fig. 11.33 has the form it does so that it can be parsed by recursive descent (or equivalently, by the table-driven parsing algorithm we are describing). To see why this form is necessary, let us consider the productions for $<L>$ in the grammar of Fig. 11.6:

$$<L> \rightarrow <L><S> \mid \epsilon$$

(1) $<S> \rightarrow$ **w c**$<S>$
(2) $<S> \rightarrow$ `{` $<T>$
(3) $<S> \rightarrow$ **s ;**

(4) $<T> \rightarrow <S><T>$
(5) $<T> \rightarrow$ `}`

Fig. 11.33. Grammar for simple statements, parsable by recursive descent.

If the current input is a terminal like **s** that begins a statement, we know that $<L>$ must be expanded at least once by the first production, whose body is $<L><S>$. However, we cannot tell how many times to expand until we examine subsequent inputs and see how many statements there are in the statement list.

Our approach in Fig. 11.33 is to remember that a block consists of a left bracket followed by zero or more statements and a right bracket. Call the zero or more statements and the right bracket the "tail," represented by syntactic category $<T>$. Production (2) in Fig. 11.33 says that a statement can be a left bracket followed by a tail. Productions (4) and (5) say that a tail is either a statement followed by a tail, or just a right bracket.

We can decide whether to expand a $<T>$ by production (4) or (5) quite easily. Production (5) only makes sense if a right bracket is the current input, while production (4) only makes sense if the current input can start a statement. In our simple grammar, the only terminals that start statements are **w**, `{`, and **s**. Thus, we see in Fig. 11.32 that in the row for syntactic category $<T>$ we choose production (4) on these three lookaheads and choose production (5) on the lookahead `}`. On other lookaheads, it is impossible that we could have the beginning of a tail, so we leave the entries for other lookaheads blank in the row for $<T>$.

Similarly, the decision for syntactic category $<S>$ is easy. If the lookahead symbol is **w**, then only production (1) could work. If the lookahead is `{`, then only production (2) is a possible choice, and on lookahead **s**, the only possibility is production (3). On any other lookahead, there is no way that the input could form a statement. These observations explain the row for $<S>$ in Fig. 11.32. ✥

How the Table-Driven Parser Works

Table driver

All parsing tables can be used as data by essentially the same program. This *driver* program keeps a stack of grammar symbols, both terminals and syntactic categories. This stack can be thought of as goals that must be satisfied by the remaining input; the goals must be met in order, from the top to the bottom of the stack.

1. We satisfy a terminal goal by finding that terminal as the lookahead symbol of the input. That is, whenever a terminal **x** is on top of the stack, we check that the lookahead is **x**, and if so, we both pop **x** from the stack and read the next input terminal to become the new lookahead symbol.

2. We satisfy a syntactic category goal $<S>$ by consulting the parsing table for the entry in the row for $<S>$ and the column for the lookahead symbol.

    a) If the entry is blank, then we have failed to find a parse tree for the input. The driver program fails.

b) If the entry contains production i, then we pop $<S>$ from the top of the stack and push each of the symbols in the body of production i onto the stack. The symbols of the body are pushed rightmost first, so that at the end, the first symbol of the body is at the top of the stack, the second symbol is immediately below it, and so on. As a special case, if the body is ϵ, we simply pop $<S>$ off the stack and push nothing.

Suppose we wish to determine whether string s is in $L(<S>)$. In that case, we start our driver with the string s `ENDM` on the input,[3] and read the first terminal as the lookahead symbol. The stack initially only consists of the syntactic category $<S>$.

◆ **Example 11.12.** Let us use the parsing table of Fig. 11.32 on the input

```
{w c s ; s ; }ENDM
```

Figure 11.34 shows the steps taken by the table-driven parser. The stack contents are shown with the top at the left end, so that when we replace a syntactic category at the top of the stack by the body of one of its productions, the body appears in the top positions of the stack, with its symbols in the usual order.

	STACK	LOOKAHEAD	REMAINING INPUT
1)	$<S>$	{	wcs;s;}ENDM
2)	{$<T>$	{	wcs;s;}ENDM
3)	$<T>$	w	cs;s;}ENDM
4)	$<S><T>$	w	cs;s;}ENDM
5)	**wc**$<S><T>$	w	cs;s;}ENDM
6)	**c**$<S><T>$	c	s;s;}ENDM
7)	$<S><T>$	s	;s;}ENDM
8)	**s;**$<T>$	s	;s;}ENDM
9)	**;**$<T>$	;	s;}ENDM
10)	$<T>$	s	;}ENDM
11)	$<S><T>$	s	;}ENDM
12)	**s;**$<T>$	s	;}ENDM
13)	**;**$<T>$	;	}ENDM
14)	$<T>$	}	ENDM
15)	}	}	ENDM
16)	ϵ	ENDM	ϵ

Fig. 11.34. Steps of a table-driven parser using the table of Fig. 11.32.

[3] Sometimes the endmarker symbol `ENDM` is needed as a lookahead symbol to tell us that we have reached the end of the input; other times it is only to catch errors. For instance, `ENDM` is needed in Fig. 11.31, because we can always have more parentheses after a balanced string, but it is not needed in Fig. 11.32, as is attested to by the fact that we never put any entries in the column for `ENDM`.

Line (1) of Fig. 11.34 shows the initial situation. As <S> is the syntactic category in which we want to test membership of the string `{wcs;s;}`, we start with the stack holding only <S>. The first symbol of the given string, `{`, is the lookahead symbol, and the remainder of the string, followed by `ENDM` is the remaining input.

If we consult the entry in Fig. 11.32 for syntactic category <S> and lookahead `{`, we see that we must expand <S> by production (2). The body of this production is `{`<T>, and we see that this sequence of two grammar symbols has replaced <S> at the top of the stack when we get to line (2).

Now there is a terminal, `{`, at the top of the stack. We thus compare it with the lookahead symbol. Since the stack top and the lookahead agree, we pop the stack and advance to the next input symbol, `w`, which becomes the new lookahead symbol. These changes are reflected in line (3).

Next, with <T> on top of the stack and `w` the lookahead symbol, we consult Fig. 11.32 and find that the proper action is to expand by production (4). We thus pop <T> off the stack and push <S><T>, as seen in line (4). Similarly, the <S> now on top of the stack is replaced by the body of production (1), since that is the action decreed by the row for <S> and the column for lookahead `w` in Fig. 11.32; that change is reflected in line (5). After lines (5) and (6), the terminals on top of the stack are compared with the current lookahead symbol, and since each matches, they are popped and the input cursor advanced.

The reader is invited to follow lines (7) through (16) and check that each is the proper action to take according to the parsing table. As each terminal, when it gets to the top of the stack, matches the then current lookahead symbol, we do not fail. Thus, the string `{wcs;s;}` is in the syntactic category <S>; that is, it is a statement. ◆

Constructing a Parse Tree

The algorithm described above tells whether a given string is in a given syntactic category, but it doesn't produce the parse tree. There is, however, a simple modification of the algorithm that will also give us a parse tree, when the input string is in the syntactic category with which we initialize the stack. The recursive-descent parser described in the previous section builds its parse trees bottom-up, that is, starting at the leaves and combining them into progressively larger subtrees as function calls return.

For the table-driven parser, it is more convenient to build the parse trees from the top down. That is, we start with the root, and as we choose a production with which to expand the syntactic category at the top of the stack, we simultaneously create children for a node in the tree under construction; these children correspond to the symbols in the body of the selected production. The rules for tree construction are as follows.

1. Initially, the stack contains only some syntactic category, say, <S>. We initialize the parse tree to have only one node, labeled <S>. The <S> on the stack *corresponds to* the one node of the parse tree being constructed.

2. In general, if the stack consists of symbols $X_1X_2\cdots X_n$, with X_1 at the top, then the current parse tree's leaves, taken from left to right, have labels that form a string s of which $X_1X_2\cdots X_n$ is a suffix. The last n leaves of the parse tree *correspond to* the symbols on the stack, so that each stack symbol X_i corresponds to a leaf with label X_i.

3. Suppose a syntactic category <S> is on top of the stack, and we choose to replace <S> by the body of a production <S> $\rightarrow Y_1Y_2\cdots Y_n$. We find the leaf of the parse tree corresponding to this <S> (it will be the leftmost leaf that has a syntactic category for label), and give it n children, labeled $Y_1, Y_2, \ldots, Y_n$, from the left. In the special case that the body is ϵ, we instead create one child, labeled ϵ.

✦ **Example 11.13.** Let us follow the steps of Fig. 11.34 and construct the parse tree as we go. To begin, at line (1), the stack consists of only <S>, and the corresponding tree is the single node shown in Fig. 11.35(a). At line (2) we expanded <S> using the production

<S> → `{`<T>

and so we give the leaf of Fig. 11.35(a) two children, labeled `{` and <T>, from the left. The tree for line (2) is shown in Fig. 11.35(b).

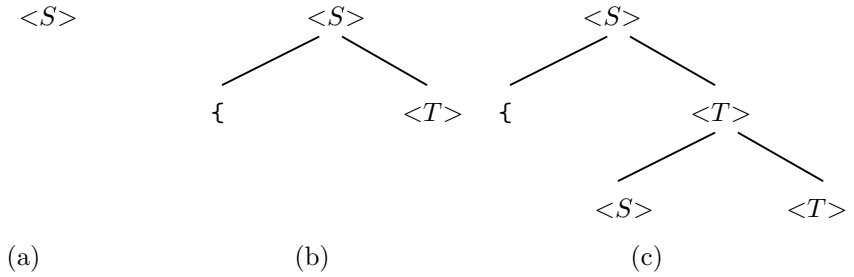


Fig. 11.35. First steps in construction of the parse tree.

Line (3) results in no change in the parse tree, since we match terminals and do not expand a syntactic category. However, at line (4) we expand <T> into <S><T>, and so we give the leaf labeled <T> in Fig. 11.35(b) two children with these symbols as labels, as shown in Fig. 11.35(c). Then at line (5) the <S> is expanded to **wc**<S>, which results in the leaf labeled <S> in Fig. 11.35(c) being given three children. The reader is invited to continue this process. The final parse tree is shown in Fig. 11.36. ✦

Making Grammars Parsable

As we have seen, many grammars require modification in order to be parsable by the recursive-descent or table-driven methods we have learned in this section and the previous. While we cannot guarantee that any grammar can be modified so that these methods work, there are several tricks worth learning because they are often effective in making grammars parsable.

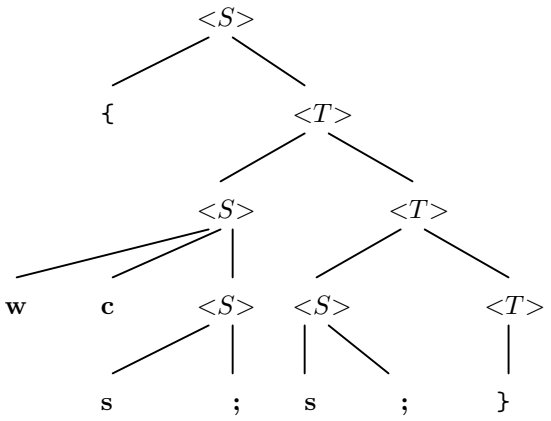


Fig. 11.36. Complete parse tree for the parse of Fig. 11.34.

Left recursion elimination

The first trick is to *eliminate left recursion*. We pointed out in Example 11.11 how the productions

$$<L> \rightarrow <L><S> \mid \epsilon$$

could not be parsable by these methods because we could not tell how many times to apply the first production. In general, whenever a production for some syntactic category $<X>$ has a body that begins with $<X>$ itself, we are going to get confused as to how many times the production needs to be applied to expand $<X>$. We call this situation *left recursion*. However, we can often rearrange symbols of the body of the offending production so $<X>$ comes later. This step is *left recursion elimination*.

✦ **Example 11.14.** In Example 11.11 discussed above, we can observe that $<L>$ represents zero or more $<S>$'s. We can therefore eliminate the left-recursion by reversing the $<S>$ and $<L>$, as

$$<L> \rightarrow <S><L> \mid \epsilon$$

For another example, consider the productions for numbers:

$$<Number> \rightarrow <Number> \ <Digit> \mid <Digit>$$

Given a digit as lookahead, we do not know how many times to use the first production to expand $<Number>$. However, we observe that a number is one or more digits, allowing us to reorder the body of the first production, as:

$$<Number> \rightarrow <Digit> \ <Number> \mid <Digit>$$

This pair of productions eliminates the left recursion. ✦

Left factoring

Unfortunately, the productions of Example 11.14 are still not parsable by our methods. To make them parsable, we need the second trick, which is *left factoring*. When two productions for a syntactic category $<X>$ have bodies that begin with

the same symbol C, we cannot tell which production to use, whenever the lookahead is something that could come from that common symbol C.

To *left factor* the productions, we create a new syntactic category $<T>$ that represents the "tail" of either production, that is, the parts of the body that follow C. We then replace the two productions for $<X>$ by a production

$$<X> \rightarrow C\ <T>$$

and two productions with head $<T>$. These two productions have bodies that are the "tails," that is, whatever follows C in the two productions for $<X>$.

✥ **Example 11.15.** Consider the productions for $<Number>$ that we developed in Example 11.14:

$$<Number> \rightarrow <Digit>\ <Number> \mid <Digit>$$

These two productions begin with a common symbol, $<Digit>$. We thus cannot tell which to use when the lookahead is a digit. However, we can defer the decision if we left factor them into

$$<Number> \rightarrow <Digit>\ <Tail>$$
$$<Tail> \rightarrow <Number> \mid \epsilon$$

Here, the two productions for $<Tail>$ allow us to choose the tail of the first production for $<Number>$, which is $<Number>$ itself, or the tail of the second production for $<Number>$, which is ϵ.

Now, when we have to expand $<Number>$ and see a digit as the lookahead, we replace $<Number>$ by $<Digit>\ <Tail>$, match the digit, and then can choose how to expand tail, seeing what follows that digit. That is, if another digit follows, then we expand by the first choice for $<Tail>$, and if something other than a digit follows, we know we have seen the whole number and replace $<Tail>$ by ϵ. ✥

EXERCISES

11.7.1: Simulate the table-driven parser using the parsing table of Fig. 11.32 on the following input strings:

a) `{s;}`
b) `wc{s;s;}`
c) `{{s;s;}s;}`
d) `{s;s}`

11.7.2: For each of the parses in Exercise 11.7.1 that succeeds, show how the parse tree is constructed during the parse.

11.7.3: Simulate the table-driven parser, using the parsing table of Fig. 11.31, on the input strings of Exercise 11.6.1.

11.7.4: Show the construction of the parse trees during the parses of Exercise 11.7.3.

11.7.5*: The following grammar

(1) $<Statement> \rightarrow$ **if** (*condition*)
(2) $<Statement> \rightarrow$ **if** (*condition*) $<Statement>$
(3) $<Statement> \rightarrow$ *simpleStat* **;**

represents selection statements in C. It is not parsable by recursive descent (or equivalently, by a table-driven parser), because with lookahead **if**, we cannot tell which of the first two productions to use. Left-factor the grammar to make it parsable by the algorithms of this section and Section 11.6. *Hint*: When you left-factor, you get a new syntactic category with two productions. One has body ϵ; the other has a body that begins with **else**. Evidently, when **else** is a lookahead, the second production can be the choice. For no other lookahead can it be the choice. But, if we examine on what lookaheads it might make sense to expand by the production with body ϵ, we discover that these lookahead symbols include **else**. However, we may arbitrarily decide never to expand to ϵ when the lookahead is **else**. That choice corresponds to the rule that "an **else** matches the previous unmatched **then**." It is thus the "correct" choice. You might wish to find an example of an input where expanding to ϵ with lookahead **else** allows the parser to complete the parse. You will discover that in any such parse, the constructed parse tree matches an **else** with the "wrong" **then**.

11.7.6**: The following grammar

$<Structure> \rightarrow$ **struct** { $<FieldList>$ }
$<FieldList> \rightarrow$ *type* $fieldName$ **;** $<FieldList>$
$<FieldList> \rightarrow \epsilon$

requires some modification in order to be parsable by the methods of this section or the previous. Rewrite the grammar to make it parsable, and construct the parsing table.

11.8 Grammars Versus Regular Expressions

Both grammars and regular expressions are notations for describing languages. We saw in Chapter 10 that the regular expression notation was equivalent to two other notations, deterministic and nondeterministic automata, in the sense that the set of languages describable by each of these notations is the same. Is it possible that grammars are another notation equivalent to the ones we have seen previously?

The answer is "no"; grammars are more powerful than the notations such as regular expressions that we introduced in Chapter 10. We shall demonstrate the expressive power of grammars in two steps. First, we shall show that every language describable by a regular expression is also describable by a grammar. Then we shall exhibit a language that can be described by a grammar, but not by any regular expression.

Simulating Regular Expressions by Grammars

The intuitive idea behind the simulation is that the three operators of regular expressions — union, concatenation, and closure — can each be "simulated" by one or two productions. Formally, we prove the following statement by complete induction on n, the number of operator occurrences in the regular expression R.

STATEMENT For every regular expression R, there is a grammar such that for one of its syntactic categories $<S>$, we have $L(<S>) = L(R)$.

That is, the language denoted by the regular expression is also the language of the syntactic category $<S>$.

BASIS. The basis case is $n = 0$, where the regular expression R has zero operator occurrences. Either R is a single symbol — say, $\mathbf{x}$ — or R is ϵ or $\emptyset$. We create a new syntactic category $<S>$. In the first case, where $R = \mathbf{x}$, we also create the production $<S> \rightarrow \mathbf{x}$. Thus, $L(<S>) = \{\mathtt{x}\}$, and $L(R)$ is the same language of one string. If R is ϵ, we similarly create the production $<S> \rightarrow \epsilon$ for $<S>$, and if $R = \emptyset$, we create no production at all for $<S>$. Then $L(<S>)$ is $\{\epsilon\}$ when R is ϵ, and $L(<S>)$ is $\emptyset$ when R is $\emptyset$.

INDUCTION. Suppose the inductive hypothesis holds for regular expressions with n or fewer occurrences of operators. Let R be a regular expression with $n+1$ operator occurrences. There are three cases, depending on whether the last operator applied to build R is union, concatenation, or closure.

1. $R = R_1 \mid R_2$. Since there is one operator occurrence, $\mid$ (union), that is part of neither R_1 nor R_2, we know that neither R_1 nor R_2 have more than n operator occurrences. Thus, the inductive hypothesis applies to each of these, and we can find a grammar G_1 with a syntactic category $<S_1>$, and a grammar G_2 with a syntactic category $<S_2>$, such that $L(<S_1>) = L(R_1)$ and $L(<S_2>) = L(R_2)$. To avoid coincidences when the two grammars are merged, we can assume that as we construct new grammars, we always create syntactic categories with names that appear in no other grammar. As a result, G_1 and G_2 have no syntactic category in common. We create a new syntactic category $<S>$ that appears neither in G_1, in G_2, nor in any other grammar that we may have constructed for other regular expressions. To the productions of G_1 and G_2 we add the two productions

   $$<S> \rightarrow <S_1> \mid <S_2>$$

   Then the language of $<S>$ consists of all and only the strings in the languages of $<S_1>$ and $<S_2>$. These are $L(R_1)$ and $L(R_2)$, respectively, and so

   $$L(<S>) = L(R_1) \cup L(R_2) = L(R)$$

   as we desired.

2. $R = R_1 R_2$. As in case (1), suppose there are grammars G_1 and G_2, with syntactic categories $<S_1>$ and $<S_2>$, respectively, such that $L(<S_1>) = L(R_1)$ and $L(<S_2>) = L(R_2)$. Then create a new syntactic category $<S>$ and add the production

   $$<S> \rightarrow <S_1><S_2>$$

   to the productions of G_1 and G_2. Then $L(<S>) = L(<S_1>)L(<S_2>)$.

3. $R = R_1{}^*$. Let G_1 be a grammar with a syntactic category $<S_1>$ such that $L(<S_1>) = L(R_1)$. Create a new syntactic category $<S>$ and add the productions

$$<S> \rightarrow <S_1><S> \mid \epsilon$$

Then $L(<S>) = L(<S_1>)^*$ because $<S>$ generates strings of zero or more $<S_1>$'s.

◆ **Example 11.16.** Consider the regular expression $\mathbf{a} \mid \mathbf{bc}^*$. We may begin by creating syntactic categories for the three symbols that appear in the expression.[4] Thus, we have the productions

$$<A> \rightarrow \mathbf{a}$$
$$<B> \rightarrow \mathbf{b}$$
$$<C> \rightarrow \mathbf{c}$$

According to the grouping rules for regular expressions, our expression is grouped as $\mathbf{a} \mid \big(\mathbf{b}(\mathbf{c})^*\big)$. Thus, we must first create the grammar for $\mathbf{c}^*$. By rule (3) above, we add to the production $<C> \rightarrow \mathbf{c}$, which is the grammar for regular expression $\mathbf{c}$, the productions

$$<D> \rightarrow <C><D> \mid \epsilon$$

Here, syntactic category $<D>$ was chosen arbitrarily, and could have been any category except for $<A>$, $<B>$, and $<C>$, which have already been used. Note that

$$L(<D>) = \big(L(<C>)\big)^* = c^*$$

Now we need a grammar for $\mathbf{bc}^*$. We take the grammar for $\mathbf{b}$, which consists of only the production $<B> \rightarrow \mathbf{b}$, and the grammar for $\mathbf{c}^*$, which is

$$<C> \rightarrow \mathbf{c}$$
$$<D> \rightarrow <C><D> \mid \epsilon$$

We shall create a new syntactic category $<E>$ and add the production

$$<E> \rightarrow <B><D>$$

This production is used because of rule (2) above, for the case of concatenation. Its body has $<B>$ and $<D>$ because these are the syntactic categories for the regular expressions, $\mathbf{b}$ and $\mathbf{c}^*$, respectively. The grammar for $\mathbf{bc}^*$ is thus

$$<E> \rightarrow <B><D>$$
$$<D> \rightarrow <C><D> \mid \epsilon$$
$$<B> \rightarrow \mathbf{b}$$
$$<C> \rightarrow \mathbf{c}$$

and $<E>$ is the syntactic category whose language is the desired one.

Finally, to get a grammar for the entire regular expression, we use rule (1), for union. We invent new syntactic category $<F>$, with productions

$$<F> \rightarrow <A> \mid <E>$$

Note that $<A>$ is the syntactic category for the subexpression $\mathbf{a}$, while $<E>$ is the syntactic category for the subexpression $\mathbf{bc}^*$. The resulting grammar is

---

[4] If one of these symbols appeared two or more times, it would not be necessary to make a new syntactic category for each occurrence; one syntactic category for each symbol would suffice.

$<F> \rightarrow <A> \mid <E>$
$<E> \rightarrow <B><D>$
$<D> \rightarrow <C><D> \mid \epsilon$
$<A> \rightarrow \mathbf{a}$
$<B> \rightarrow \mathbf{b}$
$<C> \rightarrow \mathbf{c}$

and $<F>$ is the syntactic category whose language is that denoted by the given regular expression. ✥

A Language with a Grammar but No Regular Expression

We shall now show that grammars are not only as powerful as regular expressions, but more powerful. We do so by exhibiting a language that has a grammar but has no regular expression. The language, which we shall call E, is the set of strings consisting of one or more 0's followed by an equal number of 1's. That is,

$$E = \{01, 0011, 000111, \ldots\}$$

To describe the strings of E there is a useful notation based on exponents. Let s^n, where s is a string and n is an integer, stand for $ss\cdots s$ (n times), that is, s concatenated with itself n times. Then

$$E = \{0^1 1^1, 0^2 1^2, 0^3 1^3, \ldots\}$$

or using a set-former,

$$E = \{0^n 1^n \mid n \geq 1\}$$

First, let us convince ourselves that we can describe E with a grammar. The following does the job.

(1) $<S> \rightarrow \mathbf{0}<S>\mathbf{1}$
(2) $<S> \rightarrow \mathbf{01}$

One use of the basis production (2) tells us that 01 is in $L(<S>)$. On the second round, we can use production (1), with 01 in place of $<S>$ in the body, which yields $0^2 1^2$ for $L(<S>)$. Another application of (1) with $0^2 1^2$ in place of $<S>$ tells us $0^3 1^3$ is in $L(<S>)$, and so on. In general, $0^n 1^n$ requires one use of production (2) followed by $n-1$ uses of production (1). As there are no other strings we can produce from these productions, we see that $E = L(<S>)$.

A Proof That E Is Not Defined by Any Regular Expression

Now we need to show that E cannot be described by a regular expression. It turns out to be easier to show that E is not described by any deterministic finite automaton. That proof also shows that E has no regular expression, because if E were the language of a regular expression R, we could convert R to an equivalent deterministic finite automaton by the techniques of Section 10.8. That deterministic finite automaton would then define the language E.

Thus, let us suppose that E is the language of some deterministic finite automaton A. Then A has some number of states, say, m states. Consider what happens when A receives inputs $000\cdots$. Let us refer to the initial state of the unknown automaton A by the name s_0. A must have a transition from s_0 on input 0 to some state we shall call s_1. From that state, another 0 takes A to a state we

Fig. 11.37. Feeding 0's to the automaton A.

The Pigeonhole Principle

The proof that language E has no deterministic finite automaton used a technique known as the *pigeonhole principle,* which is usually stated as,

> "If $m + 1$ pigeons fly into m pigeonholes, there must be at least one hole with two pigeons."

In this case, the pigeonholes are the states of automaton A, and the pigeons are the m states that A is in after seeing zero, one, two, and up to m `0`'s.

Notice that m must be finite to apply the pigeonhole principle. The story of the infinite hotel in Section 7.11 tells us that the opposite can hold for infinite sets. There, we saw a hotel with an infinite number of rooms (corresponding to pigeonholes) and a number of guests (corresponding to pigeons) that was one greater than the number of rooms, yet it was possible to accommodate each guest in a room, without putting two guests in the same room.

shall call s_2, and so on. In general, after reading i `0`'s A is in state s_i, as suggested by Fig. 11.37.[5]

Now A was assumed to have exactly m states, and there are $m+1$ states among $s_0, s_1, \ldots, s_m$. Thus, it is not possible that all of these states are different. There must be some distinct integers i and j in the range 0 to m, such that s_i and s_j are really the same state. If we assume that i is the smaller of i and j, then the path of Fig. 11.37 must have at least one loop in it, as suggested by Fig. 11.38. In practice, there could be many more loops and repetitions of states than is suggested by Fig. 11.38. Also notice that i could be 0, in which case the path from s_0 to s_i suggested in Fig. 11.38 is really just a single node. Similarly, s_j could be s_m, in which case the path from s_j to s_m is but a single node.

The implication of Fig. 11.38 is that the automaton A cannot "remember" how many `0`'s it has seen. If it is in state s_m, it might have seen exactly m `0`'s, and so it must be that if we start in state m and feed A exactly m `1`'s, A arrives at an accepting state, as suggested by Fig. 11.39.

However, suppose that we fed A a string of $m + j - i$ `0`'s. Looking at Fig. 11.38, we see that i `0`'s take A from s_0 to s_i, which is the same as s_j. We also see that $m - j$ `0`'s take A from s_j to s_m. Thus, $m - j + i$ `0`'s take A from s_0 to s_m, as suggested by the upper path in Fig. 11.39.

Hence, $m - j + i$ `0`'s followed by m `1`'s takes A from s_0 to an accepting state. Put another way, the string $\mathtt{0}^{m-j+i}\mathtt{1}^m$ is in the language of A. But since j is greater

[5] The reader should remember that we don't really know the names of A's states; we only know that A has m states for some integer m. Thus, the names $s_0, \ldots, s_m$ are not A's names for its states, but rather our names for its states. That is not as odd as it might seem. For example, we routinely do things like create an array s, indexed from 0 to m, and store in $s[i]$ some value, which might be the name of a state of automaton A. We might then, in a program, refer to this state name as $s[i]$, rather than by its own name.

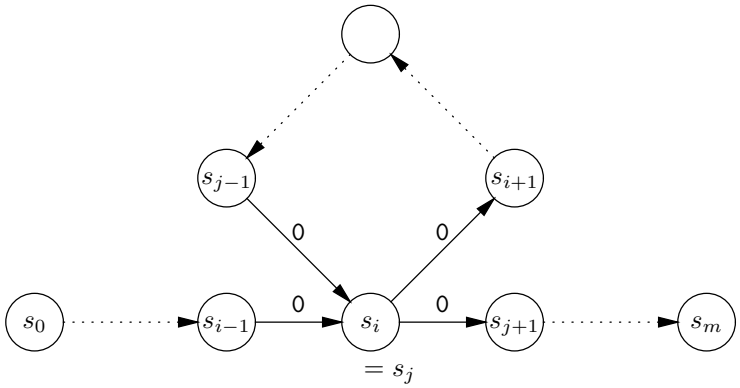


Fig. 11.38. The path of Fig. 11.37 must have a loop.

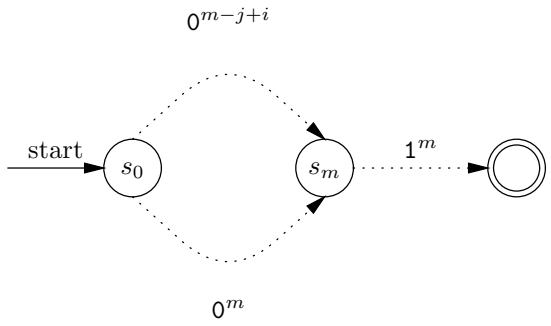


Fig. 11.39. Automaton A cannot tell whether it has seen m 0's or $m - j + i$ 0's.

than i, this string has more 1's than 0's, and is not in the language E. We conclude that A's language is not exactly E, as we had supposed.

As we started by assuming only that E had a deterministic finite automaton and wound up deriving a contradiction, we conclude that our assumption was false; that is, E has no deterministic finite automaton. Thus, E cannot have a regular expression either.

The language $\{0^n1^n \mid n \geq 1\}$ is just one of an infinite number of languages that can be specified by a grammar but not by any regular expression. Some examples are offered in the exercises.

EXERCISES

11.8.1: Find grammars that define the languages of the following regular expressions.

a) (**a** | **b**)*__a__
b) **a*** | **b*** | (**ab**)*
c) **a*b*c***

Languages a Grammar Cannot Define

We might ask whether grammars are the most powerful notation for describing languages. The answer is "not by a long shot." There are simple languages that can be shown not to have a grammar, although the proof technique is beyond the scope of this book. An example of such a language is the set of strings consisting of equal numbers of 0's, 1's, and 2's, in order, that is,

$$\{012, 001122, 000111222, \ldots\}$$

As an example of a more powerful notation for describing languages, consider C itself. For any grammar, and any of its syntactic categories <S>, it is possible to write a C program to tell whether a string is in $L(<S>)$. Moreover, a C program to tell whether a string is in the language described above is not hard to write.

Undecidable problems

Yet there are languages that C programs cannot define. There is an elegant theory of "undecidable problems" that can be used to show certain problems cannot be solved by any computer program. The theory of undecidability, with some examples of undecidable problems, is discussed briefly in Section 14.10.

11.8.2*: Show that the set of strings of balanced parentheses is not defined by any regular expression. *Hint*: The proof is similar to the proof for the language E above. Suppose that the set of balanced strings had a deterministic finite automaton of m states. Feed this automaton m ('s, and examine the states it enters. Show that the automaton can be "fooled" into accepting a string with unbalanced parentheses.

11.8.3*: Show that the language consisting of strings of the form $\mathtt{0}^n\mathtt{1}\mathtt{0}^n$, that is, two equal-length runs of 0's separated by a single 1, is not defined by any regular expression.

11.8.4*: One sometimes sees *fallacious* assertions that a language like E of this section is described by a regular expression. The argument is that for each n, $\mathbf{0}^n\mathbf{1}^n$ is a regular expression defining the language with one string, $\mathtt{0}^n\mathtt{1}^n$. Thus,

$$\mathbf{01} \mid \mathbf{0}^2\mathbf{1}^2 \mid \mathbf{0}^3\mathbf{1}^3 \mid \cdots$$

is a regular expression describing E. What is wrong with this argument?

11.8.5*: Another fallacious argument about languages claims that E has the following finite automaton. The automaton has one state a, which is both the start state and an accepting state. There is a transition from a to itself on symbols 0 and 1. Then surely string $\mathtt{0}^i\mathtt{1}^i$ takes state a to itself, and is this accepted. Why does this argument not show that E is the language of a finite automaton?

11.8.6**: Show that each of the following languages cannot be defined by a regular expression.

Palindromes

a) $\{ww^R \mid w$ is a string of a's and b's and w^R is its reversal$\}$

b) $\{\mathtt{0}^i \mid i$ is a perfect square$\}$

c) $\{\mathtt{0}^i \mid i$ is a prime$\}$

Which of these languages can be defined by a grammar?

11.9 Summary of Chapter 11

After reading this chapter, the reader should be aware of the following points:

- How a (context-free) grammar defines a language
- How to construct a parse tree to represent the grammatical structure of a string
- What ambiguity is and why ambiguous grammars are undesirable in the specification of programming languages
- A technique called recursive-descent parsing that can be used to construct parse trees for certain classes of grammars
- A table-driven way of implementing recursive-descent parsers
- Why grammars are a more powerful notation for describing languages than are regular expressions or finite automata

11.10 Bibliographic Notes for Chapter 11

Context-free grammars were first studied by Chomsky [1956] as a formalism for describing natural languages. Similar formalisms were used to define the syntax of two of the earliest important programming languages, Fortran (Backus et al. [1957]) and Algol 60 (Naur [1963]). As a result, context-free grammars are often referred to as Backus-Naur Form (BNF, for short). The study of context-free grammars through their mathematical properties begins with Bar-Hillel, Perles, and Shamir [1961]. For a more thorough study of context-free grammars and their applications see Hopcroft and Ullman [1979] or Aho, Sethi, and Ullman [1986].

Recursive-descent parsers have been used in many compilers and compiler-writing systems (see Lewis, Rosenkrantz, and Stearns [1974]). Knuth [1965] was the first to identify LR grammars, the largest natural class of grammars that can be deterministically parsed, scanning the input from left to right.

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