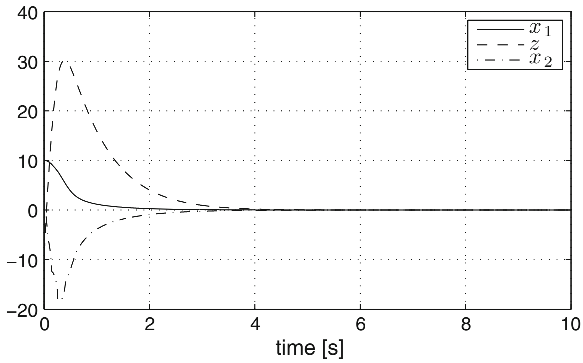

Fig. 6.3 Profile of state trajectories of the closed-loop system in Example 6.2

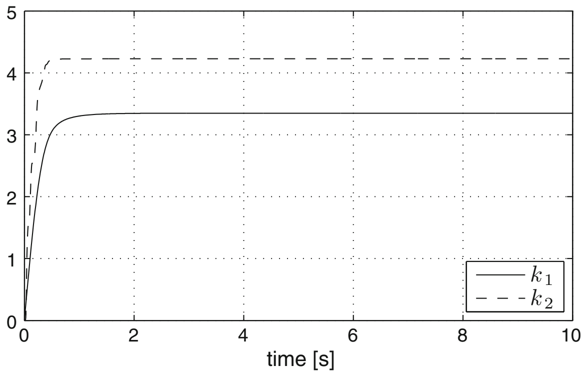

Fig. 6.4 Profile of dynamic gains of the closed-loop system in Example 6.2

The performance of the controller is simulated with parameters $w_1 = -2$, $w_2 = 2$, $w_3 = 2$, and $w_4 = 2$, and initial conditions $x_1(0) = 10$, $z(0) = -10$, $x_2(0) = 0$, and $k_1(0) = k_2(0) = 0$. Let $\lambda_1 = \lambda_2 = 0.1$. The results are shown in Figs. 6.3 and 6.4. The plant state asymptotically converges to the equilibrium point $[x_1, x_2, z]^\mathsf{T} = 0$, while the two dynamic gains asymptotically approach some constants, respectively.

6.3 Unknown Control Direction

So far, we have assumed that the control direction, i.e., the sign of b, is known. In this section, we will study the scenario where the control direction is unknown using the Nussbaum gain technique. For this purpose, we first introduce some technical tools.

For any function $v : \mathbb{R} \mapsto \mathbb{R}$, denote its positive and negative truncated functions by $v^+(s)$ and $v^-(s)$, i.e.,

$$v^+(s) = \max\{0, v(s)\}, \;\; v^-(s) = \min\{0, v(s)\}.$$

Obviously, the truncated functions satisfy the following properties

$$\begin{aligned} &v^+(s) \geq 0 \\ &v^-(s) \leq 0 \\ &v(s) = v^+(s) + v^-(s). \end{aligned}$$

Definition 6.2 A continuous function $v : \mathbb{R} \mapsto \mathbb{R}$ is called a class $\mathcal{N}$ function, denoted by $v \in \mathcal{N}$, if

$$\liminf_{k\to\infty} \frac{k - \int_0^k v^-(s)ds}{\int_0^k v^+(s)ds} = 0, \tag{6.56}$$

$$\liminf_{k\to\infty} \frac{k + \int_0^k v^+(s)ds}{-\int_0^k v^-(s)ds} = 0. \tag{6.57}$$

Lemma 6.3 *If $v \in \mathcal{N}$, then*

$$\limsup_{k\to\infty} \frac{1}{k} \int_0^k v(s)ds = +\infty, \tag{6.58}$$

$$\liminf_{k\to\infty} \frac{1}{k} \int_0^k v(s)ds = -\infty. \tag{6.59}$$

Proof From the property (6.56), there exists a sequence $k_1 < k_2 < \ldots$, with $\lim_{i\to\infty} k_i = +\infty$, such that,

$$\lim_{i\to\infty} \frac{k_i - \int_0^{k_i} v^-(s)ds}{\int_0^{k_i} v^+(s)ds} = 0.$$

which is equivalent to

$$\lim_{i\to\infty}\frac{1}{k_i}\int_0^{k_i} v^+(s)ds = +\infty \tag{6.60}$$

and

$$\lim_{i\to\infty}\frac{-\int_0^{k_i} v^-(s)ds}{\int_0^{k_i} v^+(s)ds} = 0. \tag{6.61}$$

From (6.61), one has

$$\lim_{i\to\infty}\frac{\int_0^{k_i} v(s)ds}{\int_0^{k_i} v^+(s)ds} = \lim_{i\to\infty}\frac{\int_0^{k_i} v^+(s)ds + \int_0^{k_i} v^-(s)ds}{\int_0^{k_i} v^+(s)ds} = 1,$$

which, together with (6.60), implies

$$\lim_{i\to\infty}\frac{1}{k_i}\int_0^{k_i} v(s)ds = +\infty. \tag{6.62}$$

Hence, the equation (6.58) is proved. The equation (6.59) can be proved similarly and is left for readers. □

Remark 6.2 A function satisfying the properties (6.58) and (6.59) is called a *Nussbaum function*. Therefore, a class $\mathcal{N}$ function is a type of Nussbaum function.

Remark 6.3 From the proof of Lemma 6.3, we can see that (6.56) and (6.57) are equivalent to

$$\limsup_{k\to\infty}\frac{1}{k}\int_0^{k} v^+(s)ds = +\infty, \quad \liminf_{k\to\infty}\frac{-\int_0^{k} v^-(s)ds}{\int_0^{k} v^+(s)ds} = 0. \tag{6.63}$$

and, respectively,

$$\limsup_{k\to\infty} \frac{-1}{k}\int_0^k v^-(s)ds = +\infty,\ \ \liminf_{k\to\infty} \frac{\int_0^k v^+(s)ds}{-\int_0^k v^-(s)ds} = 0. \tag{6.64}$$

Lemma 6.4 *For $v \in \mathcal{N}$, let $\hat{v}(s) = av^+(s) + bv^-(s)$ for two constants a and b satisfying $ab > 0$. Then, $\hat{v} \in \mathcal{N}$.*

Proof We only prove the case with $a, b > 0$. Denote $\hat{v}(s) = \hat{v}^+(s) + \hat{v}^-(s)$ where $\hat{v}^+(s) = av^+(s)$ and $\hat{v}^-(s) = bv^-(s)$. Clearly, $v(s)$ satisfies (6.63) and (6.64) if and only if $\hat{v}(s)$ satisfies (6.63) and (6.64). By Remark 6.3, $v \in \mathcal{N}$ if and only if $\hat{v} \in \mathcal{N}$. □

Example 6.3 The function

$$v(s) = \sin(as)\exp(bs^2),\ \ a, b > 0$$

is a class $\mathcal{N}$ function. The verification is given below.

Denote $k_i = i\pi/a$ for an integer i. Let

$$P_i^+ = \int_{k_{2i-2}}^{k_{2i-1}} \sin(as)\exp(bs^2)ds$$

$$P_i^- = \int_{k_{2i-1}}^{k_{2i}} \sin(as)\exp(bs^2)ds.$$

It is noted that

$$\int_0^{k_{2i-1}} v^+(s)ds = P_i^+ + \cdots + P_1^+$$

$$\int_0^{k_{2i-1}} v^-(s)ds = P_{i-1}^- + \cdots + P_1^-.$$

Consider the sequence k_{2i-1}, $i = 1, 2, \ldots$. One has

$$
\begin{aligned}
&\lim_{i\to\infty} \frac{-\int\limits_0^{k_{2i-1}} v^-(s)ds}{\int\limits_0^{k_{2i-1}} v^+(s)ds} \\
&= \lim_{i\to\infty} \frac{-P_{i-1}^- - \cdots - P_1^-}{P_i^+ + \cdots + P_1^+} \le \lim_{i\to\infty} \frac{-(i-1)P_{i-1}^-}{P_i^+} \\
&= \lim_{i\to\infty} \frac{-(i-1)\int\limits_{k_{2i-3}}^{k_{2i-2}} \sin(as)\exp(bs^2)ds}{\int\limits_{k_{2i-2}}^{k_{2i-1}} \sin(as)\exp(bs^2)ds} \\
&= \lim_{i\to\infty} \frac{(i-1)\int\limits_0^{\pi/a} \sin(as)\exp(b(s+k_{2i-3})^2)ds}{\int\limits_0^{\pi/a} \sin(as)\exp(b[(s+k_{2i-3})^2 + 2(s+k_{2i-3})\pi/a + (\pi/a)^2])ds} \\
&\le \lim_{i\to\infty} \frac{i-1}{\exp(2bk_{2i-3}\pi/a)} \\
&= \lim_{i\to\infty} \frac{i-1}{\exp((2i-3)2b(\pi/a)^2)} = 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{i\to\infty} \frac{k_{2i-1}}{\int\limits_0^{k_{2i-1}} v^+(s)ds} \le \lim_{i\to\infty} \frac{(2i-1)\pi/a}{P_i^+} \\
&\le \lim_{i\to\infty} \frac{(2i-1)\pi/a}{\int\limits_0^{\pi/a} \sin(as)\exp(bs^2)ds\exp((2i-3)2b(\pi/a)^2)} = 0.
\end{aligned}
$$

As a result,

$$
\lim_{i\to\infty} \frac{k_{2i-1} - \int\limits_0^{k_{2i-1}} v^-(s)ds}{\int\limits_0^{k_{2i-1}} v^+(s)ds} = 0
$$

which implies (6.56). The equation (6.57) can be verified in a similar way and is left for readers.

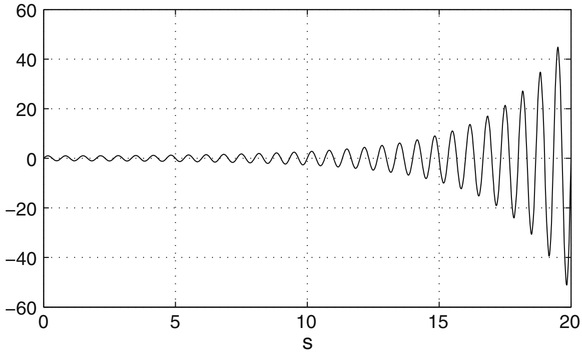

Fig. 6.5 Profile of the class $\mathcal{N}$ function $v(s) = \sin(3\pi s)\exp(0.01s^2)$

The profile of the function $v(s)$ is depicted in Fig. 6.5 with $a = 3\pi$ and $b = 0.01$.

Example 6.4 The function

$$v(s) = \sin(as)s^2, \ a > 0$$

is not a class $\mathcal{N}$ function, but it still satisfies (6.58) and (6.59). The verification is given below.

As in Example 6.3, denote $k_i = i\pi/a$ for an integer i. Using the identity

$$\int \sin(as)s^2 ds = \frac{(2 - a^2s^2)\cos as}{a^3} + \frac{2s\sin as}{a^2},$$

one has

$$P_i^+ = \int\limits_{k_{2i-2}}^{k_{2i-1}} \sin(as)s^2 ds = \frac{-4 + [(2i-1)^2 + (2i-2)^2]\pi^2}{a^3}$$

$$P_i^- = \int\limits_{k_{2i-1}}^{k_{2i}} \sin(as)s^2 ds = \frac{4 - [(2i)^2 + (2i-1)^2]\pi^2}{a^3}.$$

It is noted that

$$\int\limits_0^{k_{2i-1}} v^+(s)ds = P_i^+ + \cdots + P_1^+$$

$$\int_0^{k_{2i-1}} v^-(s)ds = P_{i-1}^- + \cdots + P_1^-.$$

Consider the sequence k_{2i-1}, $i = 1, 2, \ldots$. One has

$$\lim_{i\to\infty} \frac{-\int_0^{k_{2i-1}} v^-(s)ds}{\int_0^{k_{2i-1}} v^+(s)ds} = \lim_{i\to\infty} \frac{-P_{i-1}^- - \cdots - P_1^-}{P_i^+ + \cdots + P_1^+} = 1.$$

It is easy to see that

$$\begin{aligned}\liminf_{k\to\infty} \frac{k - \int_0^k v^-(s)ds}{\int_0^k v^+(s)ds} &\geq \liminf_{k\to\infty} \frac{-\int_0^k v^-(s)ds}{\int_0^k v^+(s)ds} \\ &= \lim_{i\to\infty} \frac{-\int_0^{k_{2i-1}} v^-(s)ds}{\int_0^{k_{2i-1}} v^+(s)ds} = 1.\end{aligned}$$

So, $v(s)$ is not a class $\mathcal{N}$ function.

On the other hand, it is noted that

$$\lim_{i\to\infty} \frac{1}{k_{2i-1}} \int_0^{k_{2i-1}} v(s)ds = \lim_{i\to\infty} \frac{(P_i^+ + \cdots + P_1^+) + (P_{i-1}^- + \cdots + P_1^-)}{(2i-1)\pi/a} = +\infty,$$

which implies (6.58). The verification of (6.59) is similar and left for readers.

Lemma 6.5 *Consider two continuously differentiable functions* $V : [0, \infty) \mapsto \mathbb{R}^+$, $k : [0, \infty) \mapsto \mathbb{R}$. *Let* $b : [0, \infty) \mapsto [\underline{b}, \bar{b}]$ *for two constants* $\underline{b}$ *and* $\bar{b}$. *If* $0 \notin [\underline{b}, \bar{b}]$ *and*

$$\begin{aligned}\dot{V}(t) &\leq (b(t)v(k(t)) + v^*)\dot{k}(t), \\ \dot{k}(t) &\geq 0, \quad \forall t \geq 0\end{aligned} \tag{6.65}$$

for a constant v^* *and a function* $v \in \mathcal{N}$, *then* $V(t)$ *and* $k(t)$ *are bounded over* $[0, \infty)$.

Proof Let

$$\hat{v}(s) = \bar{b}v^{+}(s) + \underline{b}v^{-}(s).$$

For $\bar{b}\underline{b} > 0$, by Lemma 6.4, $\hat{v}(s) \in \mathcal{N}$. It is noted that, for all $\tau \geq 0$,

$$\begin{aligned} b(\tau)v(k(\tau)) &= b(\tau)v^{+}(k(\tau)) + b(\tau)v^{-}(k(\tau)) \\ &\leq \bar{b}v^{+}(k(\tau)) + \underline{b}v^{-}(k(\tau)) = \hat{v}(k(\tau)). \end{aligned}$$

Integrating the first inequality of (6.65) gives, for all $t \geq 0$,

$$\begin{aligned} 0 \leq V(t) &\leq \int_0^t (b(\tau)v(k(\tau)) + v^*)\dot{k}(\tau)d\tau + V(0) \\ &= \int_0^t b(\tau)v(k(\tau))\dot{k}(\tau)d\tau + \int_0^t v^*\dot{k}(\tau)d\tau + V(0) \\ &\leq \int_{k(0)}^{k(t)} \hat{v}(s)ds + \int_0^t v^*\dot{k}(\tau)d\tau + V(0) \\ &= \int_0^{k(t)} \hat{v}(s)ds - \int_0^{k(0)} \hat{v}(s)ds + v^*k(t) - v^*k(0) + V(0) \end{aligned} \tag{6.66}$$

Denote a constant $c(0) = \int_0^{k(0)} \hat{v}(s)ds + v^*k(0) - V(0)$, one has

$$\int_0^{k(t)} \hat{v}(s)ds + v^*k(t) \geq c(0). \tag{6.67}$$

As $\hat{v} \in \mathcal{N}$, by (6.59) of Lemma 6.3, there exists $k^* > 1$ such that

$$\frac{1}{k^*}\int_0^{k^*} \hat{v}(s)ds < -|c(0)| - v^*.$$

If $k(t)$ is not bounded over $[0, \infty)$, then there exists $t^* > 0$ such that $k(t^*) = k^*$. Thus,

$$\int_0^{k(t^*)} \hat{v}(s)ds < -|c(0)|k(t^*) - v^*k(t^*) < c(0) - v^*k(t^*)$$

which contradicts (6.67).

As $k(t)$ is bounded over $[0, \infty)$, so is $V(t)$ by (6.66). □

Remark 6.4 If b is a constant, $v \in \mathcal{N}$ in Lemma 6.5 can be replaced by a Nussbaum function v satisfying (6.58) and (6.59). In fact, for a constant b, $\hat{v}(s) = bv(s)$ is also a Nussbaum function satisfying (6.58) and (6.59). Then, the proof of Lemma 6.5 simply follows.

Next, we will show how a class $\mathcal{N}$ function can be used to deal with control systems with unknown control direction. For convenience, we use the system (2.46) of relative degree one as a case study. Because the control direction is unknown, Assumption 2.2 used in Theorem 2.8 will be weakened to the following one.

Assumption 6.5 The function $b(d)$ is away from zero, i.e., $b(d) \neq 0, \ \forall d \in \mathbb{D}$.

Since $b(d)$ is continuous in d, we can assume that $b(d) \in [\underline{b}, \bar{b}], \ \forall d \in \mathbb{D}$ for two unknown constants $\underline{b}$ and $\bar{b}$ and $0 \notin [\underline{b}, \bar{b}]$.

Theorem 6.3 *Consider the system* (2.46) *with any unknown compact set* $\mathbb{D}$*. Under Assumptions* 6.5 *and* 2.5*, let* v *be a continuously differentiable class* $\mathcal{N}$ *function, then there exists a controller*

$$
\begin{aligned}
u &= v(k)\rho(x)x \\
\dot{k} &= \lambda\rho(x)x^2, \ \lambda > 0,
\end{aligned}
\tag{6.68}
$$

that solves the GASP of the system (2.46)*. In particular, the function* ρ *is given in Algorithm* 6.5*.*

Proof By Corollary 2.3, for any sufficient smooth function $\Delta(z) > 0$, there exists a continuously differentiable function $V'(z)$ satisfying $\underline{\alpha}'(\|z\|) \leq V'(z) \leq \bar{\alpha}'(\|z\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}'$ and $\bar{\alpha}'$, such that, along the trajectory of $\dot{z} = q(z, x, d)$,

$$
\dot{V}'(z) \leq -\Delta(z)\|z\|^2 + p'\varkappa(x)x^2 \tag{6.69}
$$

for some unknown constant p' and some known smooth function $\varkappa(x) \geq 1$.

Also, note that

$$
|f(z, x, d)| \leq cm_1(z)\|z\| + cm_2(x)|x|, \ \forall d \in \mathbb{D}. \tag{6.70}
$$

for some unknown real number $c > 0$ and two known smooth functions $m_1(z)$ and $m_2(x)$.

Let

$$
U(z, x) = V'(z) + x^2/2.
$$

Then, along the trajectory of the closed-loop system,