Adaptive Prescribed-Time Control for a Class of Uncertain Nonlinear Systems

Changchun Hua, Pengju Ning, and Kuo Li

***Abstract*—This article focuses on the problem of prescribed-time control for a class of uncertain nonlinear systems. First, a prescribed-time stability theorem is proposed by following the adaptive technology for the first time. Based on this theorem, a new state feedback control strategy is put forward by using the backstepping method for high-order nonlinear systems with unknown parameters to ensure the prescribed-time convergence. Moreover, the prescribed-time controller is obtained in the form of continuous time-varying feedback, which can render all system states converge to zero within the prescribed time. It should be noted that the prescribed time is independent of system initial conditions, which means that the prescribed time can be set arbitrarily within the physical limitations. Finally, two simulation examples are provided to illustrate the effectiveness of our proposed algorithm.**

***Index Terms*—Adaptive control, backstepping method, prescribed-time stability, uncertain nonlinear system.**

I. Introduction

Finite-Time control has attracted the attention of more and more scholars for its faster convergence rate over the past few decades [1]. The concept of finite-time control was introduced for the first time to achieve time-optimal control in [2]. Since then, nonsmooth feedback control methods have been used to achieve finite-time stabilization [3], [4]. Unfortunately, due to the existence of negative fractional powers and discontinuous terms, the singularity and chattering phenomenon appear in the operation process of the system [5], which is greatly limited the application of sliding mode finite-time control in practical systems, for example, missile guidance [6]. To overcome these limitations, a classical finite-time control theorem based on the "Lyapunov's differential inequality" was first proposed in [7], and further researched in [8]. The results in [7] and [8] have been extended in many systems, for example, double-integrator systems [9], time-varying dynamical systems [10], general nonlinear systems [11], stochastic systems [12], and multiagent systems [13]. The homogeneous approach was proposed in [14] and [15]. In addition, some important finite-time stabilization results on partial differential equations (PDEs) were shown in [16] and [17]. Although the finite-time control strategies were widely investigated, their settling time depends on the initial conditions [18]. In addition, in many practical applications, it is difficult or even impossible to obtain system initial values in advance, which makes it impossible to control the settling time. Therefore, the work in [5] proposed a fixed-time control algorithm, which allowed the upper bound of settling time independent of initial conditions. Then, the further results of fixed-time control for a class of linear and nonlinear systems were given in [1], [19], and [20]. The problem of fixed-time stabilization of reaction–diffusion PDEs was solved in [21] by means of a continuous boundary time-varying feedback control strategy. Although the existing fixed-time stabilization results can achieve the system state convergence to zero within the desired time by appropriately assigning control parameters, it will be an extremely difficult task to achieve convergence at arbitrary time [22]. Lately, the problem of preset the upper bound of the settling time in fixed-time stabilization, which can be referred as the predefined-time stabilization problem, was studied in [22]–[24]. In these results, the upper bound of settling time can be arbitrarily selected by suitably tuning control parameters. But, the true convergence time in predefined-time stabilization results still relies on initial conditions, which is different from the prescribed-time stabilization results [26]. The true convergence time in prescribed-time stabilization results is independent of initial conditions.

The prescribed-time stabilization problem has been studied in [25]–[35]. In [25] and [27], by scaling system state (when time toward the prescribed terminal time, time-varying function grows to infinity), global prescribed-time stabilization problem for nonlinear system in strict feedback form was resolved. The advantages of prescribed-time regulation over finite-time regulation were demonstrated in [28]. In [29], a sufficient condition was proposed for ensuring arbitrary time convergence and a new prescribed-time control strategy was put forward for linear systems. The prescribed-time stabilization for reaction–diffusion PDEs has been solved in [30] by employing the backstepping method. By using a decoupling backstepping approach, the work in [31] achieved prescribed-time trajectory tracking control for triangular systems of reaction–diffusion PDEs. The latest result of prescribed-time stabilization for reaction–diffusion PDEs was shown in [32]. In [33], some Lyapunov-like prescribed-time stability theorems were given and compared with some finite-time stability theorems. Based on the Lyapunov-like function, the work of [26] solved the global prescribed-time stabilization problem for a class of nonlinear systems. The prescribed-time stabilization of strict-back systems and stochastic systems (where the nonlinear function satisfying a linear growth condition) were investigated in [34] and [35], respectively.

As is well known, it is difficult to obtain a precisely known system model in practice. Adaptive control is a functional approach to handle nonlinear systems with uncertain parameters because it can estimate unknown parameters online [36]. Many well-known adaptive design methods for uncertain nonlinear systems have been proposed in many

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references, such as [37]–[39]. Due to the existence of estimation errors, Barbalat's lemma, which is widely used in adaptive control, cannot be directly applied to the stability analysis of finite/prescribed-time control [36]. The existing adaptive finite/prescribed-time control results are mainly divided into two types: one is that the system states converge to the neighborhood of the equilibrium point or defined compact within a finite-time (see [40], [41]); the other is that the system states converge to the equilibrium point in a finite interval (see [36], [42]). Obviously, compared with the former, the work of the latter will be more difficult but meaningful. Therefore, the latter problem will be studied in this article. In [36], global adaptive finite-time control is investigated for a class of nonlinear systems in p normal form with parametric uncertainties. From the abovementioned result, we know that the finite/fixed-time control methods based on "Lyapunov's differential inequality" in [7] and [5] cannot be directly used for the system with uncertain parameters. The work in [42] achieved adaptive prescribed-time stabilization for uncertain nonlinear systems based on the dynamic high-gain scaling technique. However, due to the introduction of the time-varying function to scale the states in all coordinate transformations, the computational burden for the derivative of the scaling function is greatly increased [35]. In addition, prescribed-time stabilization is of great significance to settle transient problems, and such excellent stabilization is consistent with many applications with strict time requirements, for example, missile interception. But most of the controlled plants require the system state to reach origin within a prescribed time while staying there, such as aircraft autonomous rendezvous and docking, and unmanned aerial vehicles arrive the assigned location. In this case, we should not only consider the finite-time interval but also ensure that the system state remains at the origin after the prescribed time. Based on the abovementioned discussions, the main contributions are summarized as follows.

1) For a class of uncertain nonlinear systems, a novel adaptive prescribed-time stability theorem is proposed for the first time. Based on this theorem, a new state feedback control strategy is put forward for nonlinear systems by following adaptive technology and the backstepping method. Moreover, the prescribed time is independent of system initial conditions and can be preset according to practical requirements.
2) Different from the state scaling design method by introducing a time-varying function (which grows to infinity as time tends to the prescribed time), our approach does not use the time-varying function for state transformations, and the time-varying function is only suitably used to design controller. Even compared with the latest prescribed-time stabilization results, our proposed control strategy can greatly reduce the computational burden.
3) Based on the temporal transformation and state scaling design method, the time interval, where the closed-loop system is defined in previous prescribed-time stabilization results, is finite and cannot be extended to infinity. Different from these results, we designed an adaptive control strategy that could extend the time interval to infinity.

Notation: In this article, some notations are stated as follows. The arguments of the functions will be omitted or simplified when no confusion can arise from the context. For example, x or $x(\cdot)$ will be used to denote $x(t)$. $\mathcal{R}$ indicates the set of real number; $\mathcal{R}^+$ represents the set of nonnegative real number, and $\mathcal{R}^n$ denote the n-dimensional Euclidean space.

II. Preliminaries

Consider a nonlinear system with uncertain parameters as follows:

$$\dot{x}(t) = f(t, x(t), u(t), \theta) \tag{1}$$

where $x(t) \in \mathcal{R}^n$, $u(t) \in \mathcal{R}^m$, and $\theta \in \mathcal{R}^r$ are state vector, control input vector, and uncertain parameter vector, respectively; $x(0) = x_0$ denotes the system initial condition; $f(\cdot) : \mathcal{R}^+ \times \mathcal{R}^n \times \mathcal{R}^m \times \mathcal{R}^r \to \mathcal{R}^n$ satisfies the Lipschitz condition for x with $f(t, 0, 0, \theta) = 0$, and is continuous with respect to t. Then, the system (1) has a unique solution for all $x_0 \in \mathcal{R}^n$.

Definition 1: For the system (1), if there exists a controller with adaptive law

$$u(t) = u(t, x(t), \hat{\Theta}(t, x(t))),\ u(t, 0, \hat{\Theta}(t, 0)) = 0 \tag{2}$$

$$\dot{\hat{\Theta}}(t) = \psi(t, x(t), \hat{\Theta}(t, x(t))),\ \psi(t, 0, \hat{\Theta}(t, 0)) = 0 \tag{3}$$

where $\hat{\Theta}(t)$ is the estimate value of Θ with $\hat{\Theta}(0) = \hat{\Theta}_0$, Θ is an uncertain parameter or parameter vector and depends on θ, $\tilde{\Theta}(t) = \Theta - \hat{\Theta}(t)$ is estimate error, such that for any $x(0) \in \Phi_\delta := \{\|x\| \leq \delta\} \subset R^n$ (δ is a positive constant), the trajectories of $x(t)$ and $\hat{\Theta}(t)$ under controller (2) are bounded, and $x(t) = 0\ \forall t \geq T_p$ with the positive constant T_p, then the equilibrium point of system (1) is called as locally prescribed-time stable with the prescribed time T_p (denoted by locally T_p-PTS), which is independent of the system initial conditions, and Φ_δ is the attraction domain of system (1). In particular, if $x(0) \in R^n$, then the equilibrium point of system (1) is called as globally prescribed-time stable (denoted by globally T_p-PTS).

The useful lemmas are listed as follows.

Lemma 1 ([43]): For a continuous function $g(y) \geq 0$ defined on $y \in [a, b)$ with the flaw $y = b$, if it satisfies

$$\lim_{y \to b^-} (b - y) g(y) = d$$

where d is a positive constant or $+\infty$, then the improper integral $\int_a^b g(y)\, dy = +\infty$ is divergent.

Definition 2: If a continuous function satisfies

$$\mu(t) > 0 \quad \forall t \in [0, T_p)$$
$$\lim_{t \to T_p^-} (T_p - t)\mu(t) = \rho$$

where ρ is a positive constant or $+\infty$, then $\mu(t)$ is called as a prescribed-time adjustment (T_p-PTA) function.

Remark 1: According to Definition 2, one can verify that $\lim_{t \to T_p^-} \mu(t) = +\infty$. Moreover, it is easy to find that $\int_0^{T_p} \mu(t)\, dt = +\infty$, by using Lemma 1. Naturally, the T_p-PTA function satisfies all the conditions for the definition of the T-finite-time stable (T-FTS) function in [33], that is, a T_p-PTA function can also be regarded as a T-FTS function. There are many functions that satisfy the definition of the T_p-PTA function, for example, $\frac{1}{(T_p - t)^k}$ $(k \geq 1)$, $\frac{1}{\ln(1 + T_p - t)}$, etc.

Theorem 1: For the system (1) with controller (2), if there exist two positive continuous differentiable functions $V_1(x(t))$, $V_2(\tilde{\Theta}(t))$ and class $\mathcal{K}_\infty$ functions α_i $(i = 1, 2, 3, 4)$ satisfying

$$V(t) = V_1(x(t)) + V_2(\tilde{\Theta}(t)) \quad \forall x \in \mathcal{R}^n \tag{4}$$

$$\alpha_1(\|x\|) \leq V_1(x) \leq \alpha_2(\|x\|) \quad \forall x \in \mathcal{R}^n \tag{5}$$

$$\alpha_3(\|\tilde{\Theta}\|) \leq V_2(\tilde{\Theta}) \leq \alpha_4(\|\tilde{\Theta}\|) \tag{6}$$

$$\dot{V}(t) \leq -c\mu(t) V_1(x(t)) \quad \forall t \in [0, T_p) \tag{7}$$

where $\mu(t)$ is defined in Definition 2 and c is a positive constant, then the equilibrium point of the system (1) is globally prescribed-time stable.

Proof: The proof is divided into two parts. A) The trajectories of $x(t)$ and $\hat{\Theta}(t)$ under controller (2) are bounded. B) The solution $x(t)$ of the system (1) converges to zero within the prescribed time T_p.

Part A: From (7), one has

$$\dot{V}(t) \leq 0 \quad \forall t \in [0, T_p)$$

so $V(t)$ is monotonically decreasing on $t \in [0, T_p)$, and from (4), one gets

$$\begin{aligned} V_1(x(t)) &\leq V(t) \leq V(0) \\ V_2(\tilde{\Theta}(t)) &\leq V(t) \leq V(0) \end{aligned} \tag{8}$$

Then, according to (5), (6), and (8), one has

$$\begin{aligned} \|x(t)\| &\leq \alpha_1^{-1}(V_1(x(t))) \leq \alpha_1^{-1}(V(0)) \\ \|\tilde{\Theta}(t)\| &\leq \alpha_3^{-1}(V_2(\tilde{\Theta}(t))) \leq \alpha_3^{-1}(V(0)) \end{aligned}$$

such that $x(t)$ and $\tilde{\Theta}(t)$ are bounded. Since $\hat{\Theta}(t) = \Theta - \tilde{\Theta}(t)$, it is easy to know that $\hat{\Theta}(t)$ is bounded for all $t \in [0, T_p)$.

Part B: Integrate both sides of (7) to get

$$\int_0^{T_p} \dot{V}\, dt \leq -\int_0^{T_p} c\mu V_1\, dt \tag{9}$$

then, it can be transformed into $\int_0^{T_p} c\mu V_1\, dt \leq C$, where $C = V(0) - V(T_p^-)$ is a positive constant. Hence, $\int_0^{T_p} c\mu V_1\, dt$ is bounded. In the following, the contradiction method will be applied to prove $x(t)$ converges to zero within the prescribed time T_p.

First, we assume that

$$\lim_{t \to T_p^-} V_1(x(t)) = \epsilon \neq 0 \tag{10}$$

where ϵ is a positive constant, owing to $\lim_{t \to T_p^-} \mu(t) = +\infty$, then $\int_0^{T_p} c\mu V_1\, dt$ becomes an improper integral of unbounded function with the flaw of T_p.

On the one hand, since $c\mu V_1 \geq 0$, $F(\tau) = \int_0^{\tau} c\mu V_1 dt$ is monotonically increasing on $[0, T_p)$, and $F(\tau) \leq C$, then $\int_0^{T_p} c\mu V_1 dt$ is convergent. On the other hand, since $\lim_{t \to T_p^-} (T_p - t)\sigma\mu V_1 = c\rho\epsilon > 0$, one can get that the improper integral $\int_0^{T_p} c\mu V_1 dt$ is divergent from Lemma 1.

By comparison, it is easy to find that the content of two aspects are contradictory. Therefore, the assumption (10) cannot hold, which implies $\lim_{t \to T_p^-} V_1(t) = \epsilon = 0$. From (5), one has $\lim_{t \to T_p^-} x(t) = 0$. With the aid of the existence and continuation properties of the solution $x(t)$, one can get that $x(T_p) = 0$, $u(T_p) = 0$. Further, since $f(t, x(t), u(t), \theta)$ vanish at the origin, and set $u(t) = 0\ \forall t \geq T_p$, we have $x(t) = 0$, $\tilde{\Theta}(t) = \tilde{\Theta}(T_p)$, and $\hat{\Theta}(t) = \hat{\Theta}(T_p)\ \forall t \geq T_p$. From Definition 1, Parts A and B, we can get that the equilibrium point of the system (1) is globally prescribed-time stable. The proof is completed.

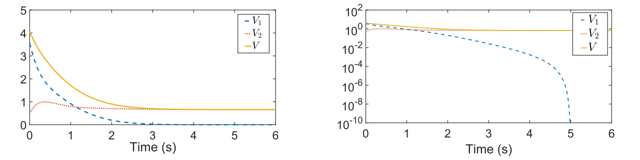
Remark 2: From the proof of Theorem 1, it is easy to find that V is monotonically decreasing and tends to a constant, V_1 converges to zero, V_2 is bounded, when $t \to T_p$. In addition, $V(t) = V_2(t), V_1(t) = 0$ for all $t \geq T_p$. The responses of V, V_1, and V_2 with $T_p = 5$ are shown in Fig. 1.

Remark 3: For the system (1), if the parameter θ is known, Theorem 1 is also valid, by setting $V_2 = 0$. Then, (7) turns into

$$\dot{V}(x) \leq -c\mu(t)V(x) \quad \forall t \in [0, T_p). \tag{11}$$

From Remark 2, we know that $\mu(t)$ can be regarded as a T-FTS function in [33]. Therefore, (11) is equivalent to [33, Eq. (7)]. In other words, for certain systems, Theorem 1 of this article is transformed into that in [33].

Remark 4: In the practical control system, since the actuator cannot generate infinite energy, the control signal $u(t)$ cannot be infinite [33]. Therefore, we should choose the appropriate T_p-PTA function to



(a) (b)

Fig. 1. Responses of V, V_1, and V_2. (a) General scale. (b) Logarithmic scale.

ensure the boundedness of controller $u(t)$. Take the first-order uncertain system

$$\dot{x} = u + \theta x^2 \tag{12}$$

as an example to illustrate the situation. The control signal with adaptive law of (12) can be designed as

$$\begin{aligned} u(t) &= -\mu x - \hat{\theta}x^2, \ \dot{\hat{\theta}}(t) = x^3, \ t \in [0, T_p) \\ u(t) &= 0, \qquad \dot{\hat{\theta}}(t) = 0, \qquad t \geq T_p. \end{aligned} \tag{13}$$

Then the Lyapunov function $V = V_1 + V_2$ satisfies $\dot{V} \leq -2\mu(t)V_1$ $\forall t \in [0, T_p)$, where $V_1 = x^2$ and $V_2 = \tilde{\theta}^2$. To facilitate analysis, we choose $\mu(t) = \frac{\sigma}{(T-t)^k}$, $k \geq 1$. From (12), (13), and Theorem 1, the solution $x(t)$ of (12) satisfies

$$\begin{aligned} \lim_{t \to T_p^-} x(t) &= \lim_{t \to T_p^-} \frac{(T_p - t)^\sigma}{T_p^\sigma} x(0), \qquad k = 1 \\ \lim_{t \to T_p^-} x(t) &= \lim_{t \to T_p^-} e^{\frac{\sigma[(T_p - t)^{1-k} - T_p^{1-k}]}{1-k}} x(0), \ k > 1 \end{aligned}$$

and the control input $u(t)$ satisfies

$$\begin{aligned} \lim_{t \to T_p^-} u(t) &= \lim_{t \to T_p^-} \frac{\sigma(T_p - t)^{\sigma-1}}{T_p^\sigma} x(0), \qquad k = 1 \\ \lim_{t \to T_p^-} u(t) &= \lim_{t \to T_p^-} \frac{\sigma e^{\frac{\sigma[(T_p - t)^{1-k} - T_p^{1-k}]}{1-k}}}{(T_p - t)^k} x(0), \ k > 1. \end{aligned} \tag{14}$$

From (14), to ensure the boundedness and continuity of $u(t)$, the T_p-PTA function $\mu(t)$ must satisfy $\sigma > 1$ for $k = 1$, and $\sigma > 0$ for $k > 1$. It should be pointed out that with increase of k, the computation burden for derivative of $\mu(t)$ increases greatly. This shortcoming is especially obvious when the system order is high. Therefore, we should choose a smaller k as much as possible under the premise of ensuring the prescribed-time stability.

III. Main Results

A. Problem Formulation

The uncertain system is considered as follows:

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \theta^T f_i(\bar{x}_i), \quad i = 1, \ldots, n-1 \\ \dot{x}_n &= u + \theta^T f(\bar{x}_n) \end{aligned} \tag{15}$$

where $x_j \in \mathcal{R}$ $(j = 1, \ldots, n)$, $u \in \mathcal{R}$, and $\theta \in \mathcal{R}^r$ are system state, control input, and uncertain parameter vector, respectively; $\bar{x}_i = [x_1, \ldots, x_i]^T \in \mathcal{R}^i$; $x(t) = \bar{x}_n(t)$; $f_i : \mathcal{R}^i \to \mathcal{R}^r$ are known smooth functions with $f_i(0) = 0$ $i = 1, \ldots, n$.

B. Controller Design

In the following, we will propose an adaptive prescribed-time controller for any given time T_p. The design procedure is divided into two cases : $0 \leq t < T_p$ and $t \geq T_p$.

Case 1 $(0 \leq t < T_p)$*:* The state transformation is presented as follows:

$$
\begin{aligned}
z_1 &= x_1 \\
z_i &= x_i - \alpha_{i-1}, \quad i = 2, \ldots, n
\end{aligned} \tag{16}
$$

where α_{i-1} is the virtual controller designed latter.

Before designing controller, the Lyapunov function is chosen as

$$V = V_1 + V_2 \tag{17}$$

where V_1 and V_2 are defined as

$$V_1 = \sum_{j=1}^{n} z_j^2, \; V_2 = \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \tag{18}$$

where $\tilde{\theta} = \theta - \hat{\theta}, \hat{\theta}$ is the estimate value of θ, and $\Gamma \in \mathcal{R}^{r \times r}$ is a positive definite matrix. From (17) and (18), one can verify that V, V_1, and V_2 satisfy the conditions in Theorem 1.

In the following, a prescribed-time controller will be designed based on the backstepping method. It contains n steps:

Step 1: The derivative of z_1 satisfies

$$\dot{z}_1 = z_2 + \alpha_1 + \theta^T f_1. \tag{19}$$

From (19), the virtual controller α_1 can be designed as

$$\alpha_1 = -\frac{\sigma_1 z_1}{T_p - t} - \hat{\theta}^T f_1 \tag{20}$$

where $\sigma_1 > n$ is a design parameter. Then, the derivative of z_1 and $W_1 = z_1^2 + V_2$ satisfy

$$
\begin{aligned}
\dot{z}_1 &= -\frac{\sigma_1 z_1}{T_p - t} + \Omega_1 \\
\dot{W}_1 &= -\frac{2\sigma_1 z_1^2}{T_p - t} + 2z_1 z_2 + 2\tilde{\theta}^T(\tau_1 - \Gamma^{-1}\dot{\hat{\theta}})
\end{aligned} \tag{21}
$$

where $\Omega_1 = z_2 + \tilde{\theta}^T f_1$ and $\tau_1 = z_1 f_1$.

Step 2: From (16) and (20), the derivative of z_2 satisfies

$$
\begin{aligned}
\dot{z}_2 = z_3 + \alpha_2 + \theta^T f_2 - \frac{\partial \alpha_1}{\partial \hat{\theta}} \dot{\hat{\theta}} \\
- \frac{\partial \alpha_1}{\partial x_1}(x_2 + \theta^T f_1) - \frac{\partial \alpha_1}{\partial t}
\end{aligned} \tag{22}
$$

Based on (21) and (22), the virtual controller α_2 is designed as

$$
\begin{aligned}
\alpha_2 = & -\frac{\sigma_2 z_2}{T_p - t} - z_1 - \hat{\theta}^T f_2 + \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma \tau_2 \\
& + \frac{\partial \alpha_1}{\partial x_1}(x_2 + \hat{\theta}^T f_1) + \frac{\partial \alpha_1}{\partial t}
\end{aligned} \tag{23}
$$

where $\tau_2 = \tau_1 + z_2(f_2 - \frac{\partial \alpha_1}{\partial x_1} f_1)$ and $\sigma_2 > n - 1$ is a design parameter.

Then, the derivatives of z_2 and $W_2 = W_1 + z_2^2$ satisfy

$$
\begin{aligned}
\dot{z}_2 = & -\frac{\sigma_2 z_2}{T_p - t} + \Omega_2 \\
\dot{W}_2 = & -\frac{2}{T_p - t} \sum_{j=1}^{2} \sigma_j z_j^2 + 2\tilde{\theta}^T(\tau_2 - \Gamma^{-1}\dot{\hat{\theta}}) \\
& + 2\frac{\partial \alpha_1}{\partial \hat{\theta}} z_2(\Gamma \tau_2 - \dot{\hat{\theta}}) + 2z_2 z_3
\end{aligned} \tag{24}
$$

where $\Omega_2 = z_3 - z_1 + \tilde{\theta}^T f_2 + \frac{\partial \alpha_1}{\partial \hat{\theta}}(\Gamma \tau_2 - \dot{\hat{\theta}}) - \frac{\partial \alpha_1}{\partial x_1} \tilde{\theta}^T f_1$.

Step i $(i = 3, \ldots, n - 1)$*:* The derivative of z_i satisfies

$$
\begin{aligned}
\dot{z}_i = z_{i+1} + \alpha_i + \theta^T f_i - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} - \frac{\partial \alpha_{i-1}}{\partial t} \\
- \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}(x_{j+1} + \theta^T f_j).
\end{aligned} \tag{25}
$$

Based on (25), the virtual controller α_i is designed as

$$
\begin{aligned}
\alpha_i = & -\frac{\sigma_i z_i}{T_p - t} - z_{i-1} - \hat{\theta}^T f_i + \frac{\partial \alpha_{i-1}}{\partial t} \\
& + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}(x_{j+1} + \hat{\theta}^T f_j) + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \Gamma \tau_i \\
& + \left(f_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} f_j\right) \Gamma \sum_{j=2}^{i-1} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} z_j
\end{aligned} \tag{26}
$$

where $\tau_i = \tau_{i-1} + z_i(f_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} f_j)$ and $\sigma_i > n - i + 1$ is a design parameter.

Then, the derivatives of z_i and $W_i = W_{i-1} + z_i^2$ satisfy

$$
\begin{aligned}
\dot{z}_i = & -\frac{\sigma_i z_i}{T_p - t} + \Omega_i \\
\dot{W}_i = & -\frac{2}{T_p - t} \sum_{j=1}^{i} \sigma_j z_j^2 + 2\tilde{\theta}^T(\tau_i - \Gamma^{-1}\dot{\hat{\theta}}) \\
& + 2\sum_{j=2}^{i} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} z_j(\Gamma \tau_i - \dot{\hat{\theta}}) + 2z_i z_{i+1}
\end{aligned} \tag{27}
$$

where $\Omega_i = z_{i+1} - z_{i-1} + \tilde{\theta}^T f_i + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}(\Gamma \tau_i - \dot{\hat{\theta}}) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \tilde{\theta}^T f_j + (f_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} f_j)\Gamma \sum_{j=2}^{i-1} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} z_j$.

Step n: From (16) and (26), the derivative of z_n satisfies

$$
\begin{aligned}
\dot{z}_n = u + \theta^T f_n - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} - \frac{\partial \alpha_{n-1}}{\partial t} \\
- \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j}(x_{j+1} + \theta^T f_j).
\end{aligned} \tag{28}
$$

The controller u and the adaptive law can be designed as

$$u = -\hat{\theta}^T f_n + \Phi(\bar{x}_n, t) \quad \dot{\hat{\theta}} = \Gamma \tau_n \tag{29}$$

where $\tau_n = \tau_{n-1} + z_n(f_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} f_j)$, $\sigma_n > 1$ is a design parameter, and

$$
\begin{aligned}
\Phi = & -\frac{\sigma_n z_n}{T_p - t} - z_{n-1} + \frac{\partial \alpha_{n-1}}{\partial t} + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \Gamma \tau_n \\
& + \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j}(x_{j+1} + \hat{\theta}^T f_j) \\
& + \left(f_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} f_j\right) \Gamma \sum_{j=2}^{n-1} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} z_j.
\end{aligned} \tag{30}
$$

Then, we have

$$
\begin{aligned}
\dot{z}_n &= -\frac{\sigma_n z_n}{T_p - t} + \Omega_n \\
\dot{V} &= \dot{W}_n \leq -2\mu(t) V_1
\end{aligned} \tag{31}
$$

where $\Omega_n = -z_{n-1} + \tilde{\theta}^T f_n + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}(\Gamma \tau_n - \dot{\hat{\theta}}) - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \tilde{\theta}^T f_j + (f_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} f_j)\Gamma \sum_{j=2}^{n-1} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} z_j$, $\mu(t) = \frac{\sigma}{T_p - t}$ is a T_p-PTA function and $\sigma = \min\{\sigma_i\} > 1$ $(i = 1, \ldots, n)$ is a positive parameter.

Case 2 $(t \geq T_p)$*:* For this case, the control signal and adaptive law can be designed as

$$u = 0, \quad \dot{\hat{\theta}} = 0. \tag{32}$$

Although u and $\dot{\hat{\theta}}$ are piecewise functions, they are continuous for $t \in R^+$, it will be proved in the following.

C. Stability Analysis

Theorem 2: There is a continuous time-varying adaptive controller defined by (29) and (32), such that the state $x(t)$ of the system (15) converges to the origin within the prescribed-time T_p while maintaining the boundedness of $\hat{\theta}$ and $\tilde{\theta}$.

Proof: From (31) and Theorem 1, one can get that $z_i(t)$ and $\hat{\theta}(t)$ are bounded for all $t \in [0, T_p)$ and $\lim_{t\to T_p^-} V_1(t) = 0, \lim_{t\to T_p^-} z_i(t) = 0$ $(i = 1, \ldots, n)$. In the following, we prove the prescribed-time stability for $\dot{V} = -\frac{2\sigma V_1}{T_p - t}$, which will then imply the prescribed-time stability for $\dot{V} < -\frac{2\sigma V_1}{T_p - t}$, due to the comparison lemma.

From $\dot{V} = -\frac{2\sigma V_1}{T_p - t} = -\frac{2\sigma V}{T_p - t} + \frac{2\sigma V_2}{T_p - t}$, we can obtain

$$V(t) = \left(1 - \frac{t}{T_p}\right)^{2\sigma} V_1(0) + V_2(t) - (T_p - t)^{2\sigma} \int_0^t \frac{\dot{V}_2(s)}{(T_p - s)^{2\sigma}}\, ds.$$

Obviously, it can be inferred that

$$V_1(t) = \left(1 - \frac{t}{T_p}\right)^{2\sigma} V_1(0) - (T_p - t)^{2\sigma} \int_0^t \frac{\dot{V}_2(s)}{(T_p - s)^{2\sigma}}\, ds. \tag{33}$$

Then, we have

$$\lim_{t\to T_p^-} V_1(t) = \lim_{t\to T_p^-} -\frac{(T_p - t)\dot{V}_2(t)}{2\sigma} = 0. \tag{34}$$

Naturally, it is easy to get

$$\lim_{t\to T_p^-} (T_p - t)\dot{\hat{\theta}} = 0. \tag{35}$$

Similarly, from (21), (24), (27), and (31), we can obtain

$$z_i = \left(1 - \frac{t}{T_p}\right)^{\sigma_i} z_i(0) + (T_p - t)^{\sigma_i} \int_0^t \frac{\Omega_i(s)}{(T_p - s)^{\sigma_i}}\, ds \tag{36}$$

for $i = 1, \ldots, n$. Then, it is easy to prove that

$$\lim_{t\to T_p^-} \frac{z_i(t)}{(T_p - t)^k} = \lim_{t\to T_p^-} \frac{\Omega_i(t)}{(\sigma_i - k)(T_p - t)^{k-1}} \tag{37}$$

where $0 < k < \sigma_i$ is a positive constant.

Based on (35) and (37), it is easy to prove that

$$\lim_{t\to T_p^-} \Omega_1 = 0 \Rightarrow \lim_{t\to T_p^-} \frac{z_1}{T_p - t} = 0, \lim_{t\to T_p^-} \dot{z}_1 = 0$$

$$\Rightarrow \lim_{t\to T_p^-} \alpha_1 = 0, \lim_{t\to T_p^-} x_2 = 0, \lim_{t\to T_p^-} f_2 = 0 \Rightarrow$$

$$\lim_{t\to T_p^-} \Omega_2 = 0, \lim_{t\to T_p^-} \frac{z_2}{T_p - t} = 0 \Rightarrow \lim_{t\to T_p^-} \frac{\Omega_1}{T_p - t} = 0$$

$$\lim_{t\to T_p^-} \frac{z_1}{(T_p - t)^2} = 0, \lim_{t\to T_p^-} \frac{\dot{z}_1}{T_p - t} = 0 \Rightarrow$$

$$\lim_{t\to T_p^-} \alpha_2 = 0, \lim_{t\to T_p^-} x_3 = 0 \Rightarrow \cdots \Rightarrow \lim_{t\to T_p^-} x_n = 0$$

$$\lim_{t\to T_p^-} f_n = 0 \Rightarrow \lim_{t\to T_p^-} \dot{\hat{\theta}} = 0, \lim_{t\to T_p^-} u = 0. \tag{38}$$

From (32) and (38), we can easily get $u(T_p) = \lim_{t\to T_p^-} u(t) = 0$ and $\dot{\hat{\theta}}(T_p) = \lim_{t\to T_p^-} \dot{\hat{\theta}}(t) = 0$. So $u(t)$ and $\dot{\hat{\theta}}(t)$ are continuous and bounded for all $t \in \mathcal{R}^+$. Then, one has $\tilde{\theta}$ and $\hat{\theta}$ are bounded for all $t \in \mathcal{R}^+$. From (15) and the continuity of $u(t)$ for all $t \in \mathcal{R}^+$, it is easy to verify that $x_i(t), i = 1, \ldots, n$ is differentiable for all $t \in \mathcal{R}^+$. With the help of continuation properties of the $x(t)$, we know that $x(T_p) = 0$. Further, by the fact that $f_i(\cdot)$ vanish at the origin and $u(t) = 0$ for any $t \geq T_p$, it can be concluded that $x(t) = 0$ for all $t \geq T_p$. This proof is finished. ■

Remark 5: The system (15) is in the form of the lower triangular structure, and some practical systems, such as missile guidance, spacecraft, can be transformed into this form. In missile guidance, the missile needs to hit the target or intercept the enemy's missile within a given time [6]. However, the existing works [1]–[24], [40], [41] can only achieve finite/fixed-time control for a class of linear or nonlinear systems, and the true settling time of them is related to the system initial conditions and control parameters. Although the control strategy in [25]–[34] and [42] can be used to realize the objective of prescribed-time control, the computation burden for the derivative of $\mu(t)$ is largely increase, since the time-varying function $\mu(t)$ is used to scale the states in all the transformations. A new nonscaling design framework was put forward in [35], which can largely reduce computation burden by designing virtual controller as $\alpha_i = -k_i \mu^{\delta_i}(t) z_i$ where $\mu(t) = \frac{T_p}{(T_p - t)^m}$, $m \geq 2$, $\delta_1 = 1$, $\delta_i = 3 \cdot 5^{i-2}$, k_i are positive constants, but our results can more reduce computational burden and save control effort compared to [35], because $\mu(t) = \frac{\sigma_i}{T_p - t}$ is part of the virtual controller.

Remark 6: In this article, all the closed-loop signals, including the control input, actually converge to zero as the time tends to the prescribed time T_p. Therefore, this article can extend the solution beyond terminal time. This is essentially different from the related literature (e.g., [25], [27], and [28]), where only the boundedness of the control input and the scaled state are guaranteed. Therefore, they consider that the time interval for defining the closed-loop system is finite and cannot be extended to infinity.

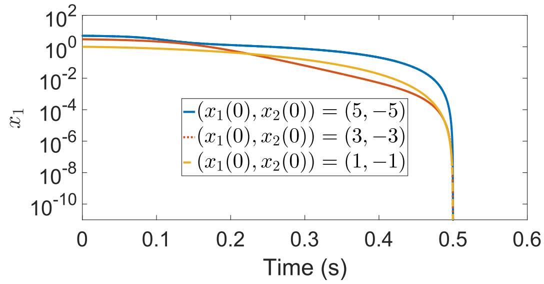
Remark 7: The dynamics of infinite-dimensional systems and distributed parameter systems are usually modeled by PDEs, including hyperbolic PDEs, parabolic PDEs, and elliptic PDEs. Many practical systems can be transformed into this form, such as reaction–diffusion systems, axially moving systems, and industrial moving strip systems. Therefore, the study of PDEs has important theoretical and application significance. There are many excellent finite/fixed-time and prescribed-time stabilization results on PDEs, for example, [16], [17], [21], [30]–[32]. These works are mainly for linear PDEs, and the control algorithm in this article can provide a new idea for the prescribed-time stabilization of nonlinear PDEs with uncertain parameters.

IV. Simulation Results

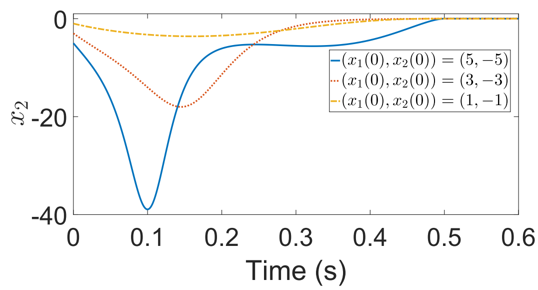
Example 1: A single-link manipulator with uncertain parameter was considered as

$$M\ddot{q} + \omega\dot{q} + \frac{1}{2}mgl\sin q = \tau \tag{39}$$

where q, $\dot{q}$, and $\ddot{q}$ stand for the link angular position, velocity, and acceleration, respectively. M is the mechanical inertial, ω represents the viscous friction coefficient at the joint, which is difficult to accurately



(a)



(b)

Fig. 2. Responses of x with different initial conditions. (a) Logarithmic scale with x_1. (b) General scale with x_2.

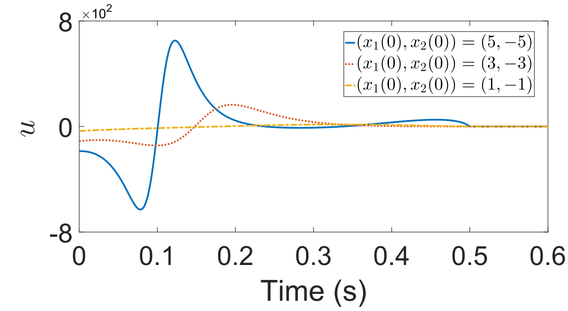


Fig. 3. Responses of u with different initial conditions.

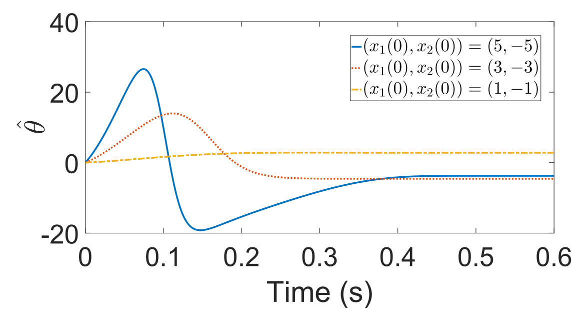
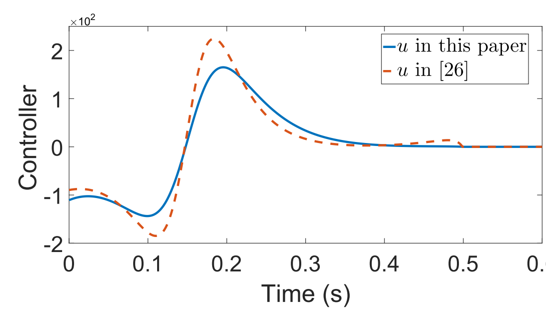
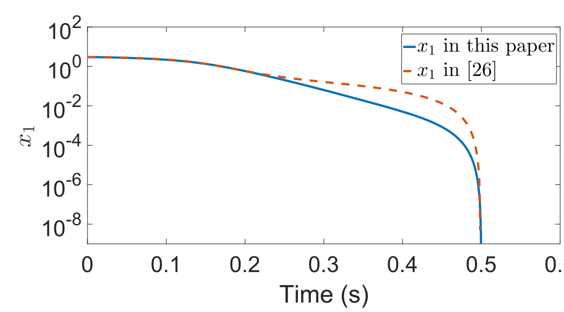


Fig. 4. Responses of $\hat{\theta}$ with different initial conditions.



(a)



(b)

Fig. 5. Comparison with [25] under $(x_1(0), x_2(0)) = (1, -1)$. (a) General scale Controller. (b) Logarithmic scale with x_1

determine in practice, and τ denotes the control torque of the link. m, l, and g stand for the link mass, link length, and acceleration of gravity, respectively. To simplify the following analysis, we let $x_1 = Mq, x_2 = M\dot{q}, u = \tau$, and $\theta = \frac{\omega}{M}$. Then, the system (39) can be converted into

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u - \theta x_2 - \frac{1}{2}mgl\sin\frac{x_1}{M}. \end{cases} \tag{40}$$

The controller is designed as follows:

$$u(t) = -\frac{\sigma_2 z_2}{T_p - t} + \Phi_1,\ \dot{\hat{\theta}}(t) = -2z_2x_2, \quad t \in [0, T_p)$$
$$u(t) = 0, \qquad \dot{\hat{\theta}}(t) = 0, \qquad t \geq T_p \tag{41}$$

where $\Phi_1 = -x_1 + \hat{\theta}x_2 + \frac{1}{2}mgl\sin\frac{x_1}{M} - \frac{\sigma_1 x_2}{T_p - t} - \frac{\sigma_1 x_1}{(T_p - t)^2}$.

For simulation, we select the system parameters $M = 1.0\,\mathrm{kg}\cdot\mathrm{m}^2$, $\omega = 1\ \mathrm{N}\cdot\mathrm{m}\cdot\mathrm{s/rad}$, $m = 0.5\,\mathrm{kg}$, $l = 1\,\mathrm{m}$, and $g = 9.81\ \mathrm{m/s}^2$. From the proposed control algorithm, we take the design parameters as $\sigma_1 = \sigma_2 = 3$, and the prescribed time is selected as $T_p = 0.5$. The initial conditions of the system are given by three different conditions $(x_1(0), x_2(0)) = (5, -5), (3, -3), (1, -1)$, and the initial value of the uncertain parameter estimate value is given $\hat{\theta}(0) = 0$.

The simulation results are shown in Figs. 2–5. On the basis of Fig. 2, we can see that the system states converge to zero within the prescribed time under different initial conditions. From Fig. 3–4, it is easy to see that the control signals are continuous and steer to zero within the prescribed time, and the parameter estimates $\hat{\theta}$ are bounded. The simulation result compared with the work in [25] is shown in Fig. 5 under the initial condition $(x_1(0), x_2(0)) = (1, -1)$; we can see that our control strategy can achieve the same control effect, or even better.

Example 2: Considering the following system:

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta x_1 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u + \theta x_3^2 \end{aligned} \tag{42}$$

according to the design procedure proposed in Section III, we can design the prescribed-time adaptive controller as

$$u(t) = -\frac{\sigma_3 z_3}{T_p - t} + \Phi_2, \qquad \dot{\hat{\theta}}(t) = \tau, \quad t \in [0, T_p)$$
$$u(t) = 0, \qquad \dot{\hat{\theta}}(t) = 0, \quad t \geq T_p \tag{43}$$

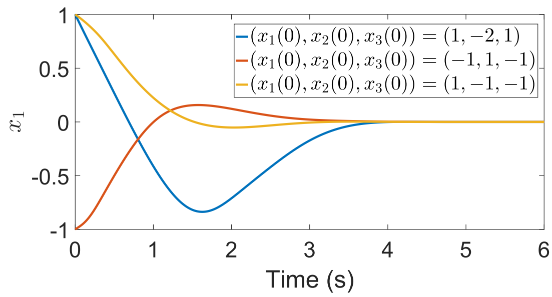


Fig. 6. Responses of x_1 with different initial conditions.

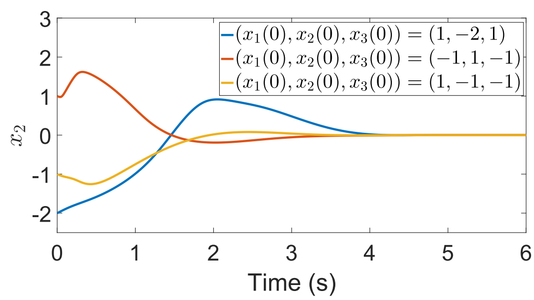


Fig. 7. Rresponses of x_2 with different initial conditions.

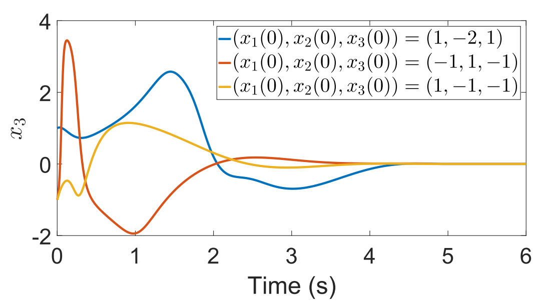


Fig. 8. Responses of x_3 with different initial conditions.

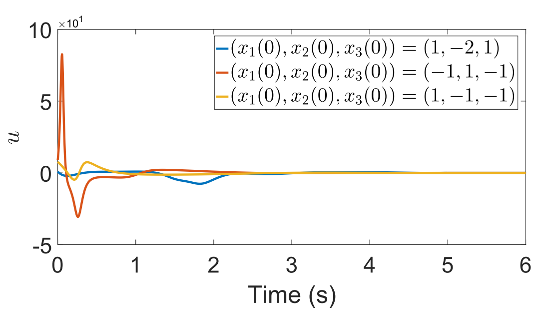


Fig. 9. Responses of u with different initial conditions.

where $\Phi_2 = -z_2 - \hat{\theta}x_3^2 + \frac{\partial \alpha_2}{\partial x_1}(x_2 + \hat{\theta}x_1) + \frac{\partial \alpha_2}{\partial x_2}x_3 + \frac{\partial \alpha_2}{\partial t} + \frac{\partial \alpha_2}{\partial \hat{\theta}}\tau - z_2\frac{\partial \alpha_1}{\partial \hat{\theta}}(x_3 - \frac{\partial \alpha_2}{\partial x_1}x_1)$ and $\tau = z_1x_1 - z_2\frac{\partial \alpha_1}{\partial x_1}x_1 + z_3(x_3^2 - \frac{\partial \alpha_2}{\partial x_1}x_1)$. For simulation, the uncertain parameter is selected as $\theta = 0.5$. The initial conditions are given by $(x_1(0), x_2(0), x_3(0)) = (1, -2, 1), (-1, 1, -1), (1, -1, -1)$, $\hat{\theta}(0) = 0$, the prescribed time is chosen as $T_p = 5$ s, and the design parameters were chosen as $\sigma_1 = \sigma_2 = \sigma_3 = 5$. The simulation results are shown in Figs. 6 and 9, from which we can see that all system states converge to zero within the prescribed time under different initial conditions. Moreover, the adaptive controller is continuous and converges to zero within the prescribed time.

V. Conclusion

In this article, the adaptive prescribed-time control problem is studied for a class of uncertain nonlinear systems. First, the definition of prescribed-time stable is given. Then, to achieve adaptive prescribed-time control, a novel adaptive prescribed-time convergence theorem is proposed for the first time. Based on the theorem, a time-varying state feedback controller for an uncertain nonlinear systems is designed, which can guarantee that all system states converge to zero within a prescribed time. Moreover, this prescribed time is independent of system initial conditions and can be set arbitrarily. Future work will concentrate on the problem of prescribed-time control for more general uncertain nonlinear systems.

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