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Prescribed-time control of uncertain strict-feedback-like systems

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Abstract

In this paper, we present a control method for achieving prescribed-time regulation of strict-feedback-like systems in the presence of unknown yet time-varying control gains and mismatched uncertainties. The developed control algorithms are capable of full-state regulation within pre-set time independent of initial condition and any other design parameter. The corresponding control action is continuous and bounded during the entire control process, which, for vanishing uncertainties/disturbances, reduces to zero as the states approach the equilibrium and remains zero thereafter, allowing for nonstop running of the system beyond the settling time. The efficiency of the proposed method is verified through numerical examples.

KEYWORDS

nonlinear control, uncertain systems, prescribed-time regulation

1 | INTRODUCTION

Finite-time control of nonlinear systems has drawn much attention during the past decades (e.g., References 1-5 and the related references cited therein). Most existing results literally are based on the special forms of the following model:

$$\begin{cases} \dot{x}_i = g_i(X,u,t)x_{i+1} + f_i(X,u,t), \quad i = 1, \ldots, n-1 \\ \dot{x}_n = g_n(X,u,t)u(t) + f_n(X,u,t), \end{cases} \tag{1}$$

where $X = [x_1, x_2, \ldots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are the system state vector and input, respectively, $f_i(X,u,t)$ is continuous and represents the uncertainty/disturbance in channel i, $g_i(X,u,t)(i=1,\ldots,n-1)$ and g_n denote the virtual and actual control gains of the system.

Note that the system model (1) covers a large class of nonlinear systems studied in the literature in the context of finite-time control. For instance, when $g_i(X,u,t)=1$ and $f_i(X,u,t)=0$ $(i=1,\ldots,n)$, one gets the simplest version of (1), known as a chain of integrators, on which most finite/fixed/prescribed-time control[3,5-9] results are established. When $g_i(X,u,t)=1$ $(i=1,\ldots,n)$, $f_i(X,u,t)=0$ $(i=1,\ldots,n-1)$ and $f_n(X,u,t)=f(X,t)\neq 0$, one gets from (1) a chain of integrators with matched disturbance. When $g_i(X,u,t)=1$, $f_i(X,u,t)=0$ $(i=1,\ldots,n-1)$, $g_n(X,u,t)=g(X,u,t)\neq 0$, $f_n(X,u,t)=f(X,u,t)$, one gets from (1) the normal-form-like nonlinear system, which is more general than the standard normal-form systems, the later of which has been extensively investigated under various issues such as nonlinear stabilization;[10] asymptotic control;[11] finite-time control;[12] or even prescribed-time control.[4,13,14] When each $g_i(X,u,t)$ and $f_i(X,u,t)$ $(i=1,\ldots,n)$ is a scalar real-valued function of $(x_1,x_2,\ldots,x_i,t)$, the model (1) represents the standard strict-feedback nonlinear system, which has been the main focus of various recent works on uniformly ultimately bounded stability and asymptotic stability as well as finite-time stability.[15] Thus far, very few has ever addressed the issue of fixed-time stability for strict-feedback systems with mismatched uncertainties, not to mention the issue of prescribed-time stability for strict-feedback-like systems (more general than the standard strict-feedback systems), where the uncertain functions appearing in $\dot{x}_i$ can involve the "later" state variables $x_{i+2}, \ldots, x_n$ and the control input u.[16]

In fact, it is nontrivial to tackle the corresponding control problem for such systems, and this is particularly true in the context of prescribed-time control. When the actual control gain $g_n(X, u, t) = g_n(X, t)$ and the virtual control gain $g_i(X, u, t) = g_i(X, t)$ $(i = 1, \ldots, n-1)$ all are precisely known and the disturbances $f_i(X, u, t) = f_i(X, t)$ $(i = 1, \ldots, n)$ are of some special forms and satisfy certain somewhat restrictive conditions, Krishnamurthy et al. propose a method to achieve prescribed-time regulation by using the dynamic high gain scaling and temporal scale transformation technologies,[16] which has been extended to output feedback control recently.[17]

The challenge associated with prescribed-time stabilization of such system is obvious and significant when the actual and virtual control gains are unknown yet time-varying and the uncertain nonlinear terms (involved in all the channels) do not satisfy the dominance conditions related to the control gains, which in fact makes the existing methods[1,3,13,16] inapplicable, rendering the underlying prescribed-time control an interesting open problem.

In this work, we address this problem by using a new approach based on scaling of states and virtual controls. Our method is partially motivated by the existing work,[13] but the difference as compared with Reference 13 is essential. In fact, the method in Reference 13 only applies to systems in normal-form with matched uncertainties, and the control scheme only works on finite time interval, whereas the system here is allowed to be in the strict-feedback-like form as described in (1), where all the control gains $g_i(X, u, t)$ $(i = 1, \ldots, n)$ and disturbances $f_i(X, u, t)$ $(i = 1, \ldots, n)$ are unknown yet time-varying and do not necessarily need to be in special form to meet the dominance conditions. Furthermore, the system (achieving regulation in prescribed time) is expected to operate over infinite time interval. Given the complex nature involved in the control design and stability analysis in the method by Krishnamurthy et al.[16] extension of which to address such scenario would be very difficult if not impossible. With the conditions as imposed in this work, we literally enlarge the classes of high-order uncertain nonlinear systems by relaxing the restrictions on the control gains and the mismatched uncertainties/disturbances, whose prescribed-time control problem has not been explored priori to this work. The main contributions of our work can be summarized as follows:

(1) Aiming at a more general yet more complex class of nonlinear systems, we establish a method for prescribed-time regulation. Different from the finite/fixed-time control that is based on fractional power of state feedback, our result provides a solution with settling time being fully independent of initial conditions;
(2) Unlike the prescribed performance control method,[18-20] our method makes all the system states converge to the origin precisely within a preset time without the need for barrier Lyapunov function or any other state transformation;
(3) In contrast to most existing results[13,14,16,17] that are only valid for $t \in [0, t_f)$, our control scheme, featured with simplicity and elegance, is fully functional for $t \in [0, +\infty)$ in that it makes each system state converge to zero (within user-chosen settling time) and remain zero thereafter, allowing for nonstop running of the system beyond the settling time t_f.

Notations: Throughout the paper, the notations used are standard. $\mathbb{R}$, $\mathbb{R}_{>k}$, and $\mathbb{R}^n$ denote the fields of real numbers, real numbers greater than k and n-dimensional Euclidean space, respectively. $\overline{g}(\cdot)$ and $\underline{g}(\cdot)$ are the upper and lower bounds of the function $g(\cdot)$, respectively. $|\bullet|$ denotes the absolute value of a real number (variable). $\bullet^{(j)}$ denotes the jth derivative of $\bullet$, and $\bullet^j$ denotes the jth power of $\bullet$. $[\bullet]'$ denotes the time derivative of $\bullet$. $\lim_{t \to t_f} f(\cdot)$ denotes the limit of $f(\cdot)$ as t approaches to t_f. A continuous function $V : [0, c) \to [0, +\infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $V(0) = 0$.

2 | ASSUMPTIONS AND PROBLEM FORMULATION

2.1 | Assumptions

For the strict-feedback-like system as described in (1) to admit a feasible prescribed-time control solution, the following conditions are imposed.

Assumption 1. The actual and virtual control gains $g_i(\cdot)(i = 1, \ldots, n)$ are unknown continuous functions, and the signs of $g_i(\cdot)s$ are known, and without loss of generality, it is assumed that $g_i(\cdot)s$ satisfy $0 < \underline{g}_i(\overline{x}_i) \leq g_i(X, u, t) \leq \overline{g}_i(\overline{x}_i) < +\infty$, with $\overline{x}_i = [x_1, x_2, \ldots, x_i]^{\mathrm{T}}$, and $\underline{g}_i(\overline{x}_i)$ as well as $\overline{g}_i(\overline{x}_i)$ are known scalar real-valued continuous functions.

Assumption 2 (Bounds on vanishing disturbance f_i). There exist some unknown and possibly time-varying but bounded continuous function $d(t)$ $(0 \le d(t) < +\infty)$ and a known non-negative scalar real-valued continuous function $\phi_i\left(\bar{x}_i, t\right)$ $(i = 1, \ldots, n)$ with $\phi_i(0, t) = 0$ such that $|f_i(X, u, t)| \le d_i(t)\phi_i\left(\bar{x}_i, t\right)$. Both f_i and ϕ_i are bounded if $\bar{x}_i$ is bounded.

Remark 1. Assumption 1 is standard and necessary for the system to be fully controllable.[21] Assumption 2 is reasonable and commonly used, and with some crude information on $f_i(\cdot)$, one can readily derive (extract) $\phi_i(\cdot)$ from $f_i(\cdot)$.[22] For example, if $f(u, x_1, t) = (3 + \sin(u) + \cos(t))x_1$, we immediately get $\phi(\cdot) = |x_1|$ such that $|f(x_1, u, t)| \le d(t)\phi(x_1)$ holds, with $5 \le d(t) \le +\infty$, and $\phi(\cdot) \to 0$ as $x_1 \to 0$.

Remark 2. In the recent work,[16] the problem of prescribed-time control for a system model similar to (1) is studied. To see the distinction of the proposed control, it is worth comparing the conditions imposed in this work with that in Reference 16. Assumption 1 here merely requires that the upper and lower bounds on the control coefficients be known, whereas in Reference 16, the control coefficients g_i $(i = 1, \ldots, n)$ must be known precisely. Furthermore, in addition to other conditions, it is required in Reference 16 that the time-varying scalar real-valued uncertain functions f_i $(i = 1, \ldots, n)$ must be in special forms to meet the "cascading dominance" conditions associated with the control coefficient g_i $(i = 1, \ldots, n)$ in each channel. Evidently, the conditions imposed on g_i and f_i $(i = 1, \ldots, n)$ here are much more relaxed, enlarging the classes of the systems being handled by the proposed method.

2.2 | Scaling of states plus virtual controls

Similar to the work of Song et al.,[13] we define a monotonically increasing function as

$$\mu(t) = \left(\frac{t_f}{t_f - t}\right)^m, \quad t \in [0,\ t_f), \tag{2}$$

where $m \in \mathbb{R}_{\ge 2}$ is a positive integer and $t_f \in \mathbb{R}_{>0}$ is the settling time specifiable in advance irrespective of initial condition or any other design parameter. It is obvious that $\mu(0) = 1$ and $\mu(t_f) = +\infty$. To develop a new prescribed-time control method for the strict-feedback-like system, we employ this finite-time function, originated from Reference 13, to perform scaling of both states and virtual controls as follows (For reading and notational convenience, some arguments of functions are omitted if there is no confusion will occur.)

$$\begin{aligned} \eta_i &= \mu(t)\omega_i, \qquad i = 1, \ldots, n \\ \omega_i &= x_i + \psi_{i-1}(\bar{x}_i, t), \end{aligned} \tag{3}$$

where $\psi_0 = 0$ and $\psi_1, \ldots, \psi_{n-1}$ represent the virtual controls to be designed later.

It should be noted that as the system model (1) is more general than the standard strict-feedback one, the previous work by Reference 13 that is based on systems in normal-form is then inapplicable. To tackle this problem, we establish a set of new relations linking x_i with ω_i, η_i and μ through (2) and (3). Here, the scaling on ω_i (i.e., the state x_i plus the virtual control ψ_i) is performed, which is subtle as compared with the scaling used in Reference 13, yet quite effective in facilitating our subsequent design and analysis. It is such treatment that allows the development of the new prescribed time control algorithms applicable not only to the normal-form-like nonlinear systems, but also to the more general strict-feedback-like systems operational beyond the settling time, as detailed in the sequel. To proceed, the following lemma is useful for our later technical development.

Lemma 1 (See the work of Song et al.[13]). *Consider the function defined in (2). If a continuously differentiable function* $V : [0,\ t_f) \to [0,\ +\infty)$ *satisfies*

$$\dot{V}(t) \le -2k\mu(t)V(t) + \frac{\mu(t)}{4\lambda}d(t)^2, \tag{4}$$

for positive constants k, λ *and an unknown bounded function* $d(t)$, *then*

$$V(t) \le \zeta(t)^{2k}V(0) + \frac{\|d\|^2_{[0,t]}}{8k\lambda}, \quad \forall t \in \left[0,\ t_f\right), \tag{5}$$

where $\|d\|_{[0,t]} := \sup_{\tau\in[0,t)}|d(\tau)|$ *and* $\zeta(t)$ *is the monotonically decreasing function as*

$$\zeta(t) = \exp^{\frac{t_f}{m-1}\left(1-\left(\frac{t_f}{t_f-t}\right)^{m-1}\right)}, \tag{6}$$

with the properties that $\zeta(0) = 1$ *and* $\zeta(t_f) = 0$.

2.3 | Prescribed-time control objective

With $t_f > 0$, the prescribed-time control objective is to design a suitable control protocol $u(t)$ such that

1) The state X and the control $u(t)$ of the system are bounded and continuous for $t \in [0, +\infty)$;
2) Both X and $u(t)$ tends to zero as $t \to t_f$ and remain at zero for $t \geq t_f$.

Remark 3. The prescribed-control scheme pursued here is expected to be functional for $t \in [0, +\infty)$, allowing the system to operate over infinite time interval, which is in contrast to the solution by the existing works[13,16] that is invalid beyond the prescribed time interval $[0,\ t_f)$.

3 | MAIN RESULTS AND PROOF

We begin with the development of prescribed-time control scheme for strict-feedback-like systems in Section 3.1, and then we followed by detailed stability analysis in Section 3.2.

3.1 | Main results

Theorem 1. *Consider system (1) under Assumptions 1 and 2. If the following control scheme is applied*

$$u(t) = \begin{cases} -\frac{1}{\underline{g}_n(X)}\left(k_n\eta_n + \lambda\eta_n\Phi_n^2\right), & t \in [0,\ t_f) \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

with

$$\begin{aligned} \Phi_i &= \left|\mu^{-1}\dot{\mu}\omega_i\right| + \phi_i + \left|\frac{\partial\psi_i}{\partial t}\right| + \sum_{l=1}^{i}\left|\frac{\partial\psi_i}{\partial x_l}\right|(|x_{l+1}| + \phi_l), \quad (i = 1, \ldots, n) \\ \psi_j &= \frac{1}{\underline{g}_j}\left(k_j\eta_j + \lambda\eta_j\Phi_j^2\right), \ (j = 1, \ldots, n-1) \\ k_i &= k_0 + \frac{\overline{g}_{i-1} + \overline{g}_i}{2\mu}, \ (i = 2, \ldots, n-1) \\ k_1 &= k_0 + \frac{\overline{g}_1}{2\mu}, \ k_n = k_0 + \frac{\overline{g}_{n-1}}{2\mu}, \end{aligned} \tag{8}$$

where $k_0 \in \mathbb{R}_{>0}$ *is a user-design constant and* μ *and* $\omega_i,\ \eta_i\ (i = 1, \ldots, n)$ *are defined as in (2) and (3), respectively. Then the prescribed-time control objective as stated in Section 2.3 is achieved.*

Remark 4. It is interesting to note that although the control scheme is presented for two time intervals, the control signal $u(t)$ is continuous and bounded everywhere, that is $u(t)$ is continuous and bounded not only for $t \in [0,\ t_f)$, but also for $t \in [t_f, +\infty)$, as shown in next subsection.

3.2 | Prescribed time stability proof

The proof of Theorem 1 is quite involved that needs to be carried out in two steps: First, we show that under the proposed control as given in Section 3.1, the time derivative of Lyapunov function $V = \frac{1}{2}\sum_{i=1}^{n}\eta_i^2$ is governed by (4) as in *Lemma* 1, thus the boundedness of η_i $(i = 1, ..., n)$ is established; Second, we prove by contradiction that $\lim_{t\to t_f}\eta_i(t) = 0$ $(i = 1, \dots, n)$, then using this fact to establish that $\lim_{t\to t_f} x_i(t) = 0$ $(i = 1, \dots, n)$ and $\lim_{t\to t_f}\dot{x}_i(t) = 0$ $(i = 1, \dots, n)$ and $\lim_{t\to t_f} u(t) = 0$.

Step 1. Using (1) and the definition of η_i and ω_i $(i = 1, \dots, n)$ as introduced in (2) and (3), it is derived that for $t \in [0,\ t_f)$,

$$\begin{aligned}
\dot{\eta}_1 &= -\mu g_1\psi_1 + g_1\eta_2 + \mu L_1\\
\dot{\eta}_2 &= -\mu g_2\psi_2 + g_2\eta_3 + \mu L_2\\
&\ \vdots\\
\dot{\eta}_j &= -\mu g_j\psi_j + g_j\eta_{j+1} + \mu L_j, \quad j = 1, \dots, n-1\\
\dot{\eta}_n &= \mu g_n u + \mu L_n,
\end{aligned} \tag{9}$$

where $L_i = \mu^{-1}\dot{\mu}\omega_i + \dot{\psi}_{i-1} + f_i,\ i = 1, \dots, n$, are the lumped uncertainties. As uncertain terms are involved in channel i $(i=1, \dots, n)$, the time derivative of ψ_i is no longer computable (i.e., unavailable for control design), which is therefore treated as part of the lumped uncertainties. It is straightforward to show that

$$|\dot{\psi}_{i-1}| \le \Delta_i\left(\left|\frac{\partial\psi_{i-1}}{\partial t}\right| + \sum_{l=1}^{i-1}\left|\frac{\partial\psi_{i-1}}{\partial x_l}\right|(|x_{l+1}| + |\phi_l|)\right), \tag{10}$$

with

$$\Delta_i = \sup\{1, d_1, \dots,\ d_{i-1},\ g_1, \dots,\ g_i\}. \tag{11}$$

From *Assumption* 2, for $i=1, \dots, n$, we have $|f_i| \le d_i(t)\phi_i$, with which it follows that $|L_i| \le D_i\Phi_i$ with

$$\begin{aligned}
D_i &= \sup\{\Delta_i,\ d_i\}\\
\Phi_i &= |\mu^{-1}\dot{\mu}\omega_i| + \phi_i + \left|\frac{\partial\psi_{i-1}}{\partial t}\right| + \sum_{l=1}^{i-1}\left|\frac{\partial\psi_{i-1}}{\partial x_l}\right|(|x_{l+1}| + \phi_l),
\end{aligned} \tag{12}$$

where $\Phi_1, \dots, \Phi_n$ in (12) are explicitly computable, then from (9) we have

$$\begin{aligned}
\eta_j\dot{\eta}_j &= -\mu g_j\eta_j\psi_j + g_j\eta_j\eta_{j+1} + \mu\eta_j L_j,\ j = 1, \dots, n-1\\
\eta_n\dot{\eta}_n &= -\mu g_n\eta_n u + \mu\eta_n L_n.
\end{aligned} \tag{13}$$

Using Young's inequality with $\lambda > 0$ being any constant and the virtual control $\psi_1, \dots, \psi_{n-1}$ as in (8) as well as $u(t)$ as in (7), it is not difficult to show that

$$\mu\eta_i L_i \le \mu\lambda{\eta_i}^2{\Phi_i}^2 + \frac{\mu D_i^2}{4\lambda},\ (i = 1, \dots, n) \tag{14}$$

$$\begin{aligned}
-\mu g_j\eta_j\psi_j &= -\frac{g_j}{\underline{g}_j}k_j\mu\eta_j^2 - \frac{g_j}{\underline{g}_j}\lambda\mu\eta_j^2\Phi_j^2 \le -k_j\mu\eta_j^2 - \mu\lambda\eta_j^2\Phi_j^2,\ (j = 1, \dots, n-1)\\
-\mu g_n\eta_n u &\le -k_n\mu\eta_n^2 - \mu\lambda\eta_n^2\Phi_n^2.
\end{aligned} \tag{15}$$

Now choose the following Lyapunov function candidate

$$V = \frac{1}{2}\sum_{i=1}^{n}\eta_i^2, \tag{16}$$

then the time derivative of V along (9) is

$$\begin{aligned}\dot{V} &= \eta_1\dot{\eta}_1 + \eta_2\dot{\eta}_2 + \dots + \eta_n\dot{\eta}_n - \mu g_1\eta_1\psi_1 + g_1\eta_1\eta_2 + \mu\eta_1 L_1 \\ &\quad - \mu g_2\eta_2\psi_2 + g_2\eta_2\eta_3 + \mu\eta_2 L_2 - \dots - \mu g_n\eta_n u + \mu\eta_n L_n \\ &\le -\sum_{i=1}^{n} k_i\mu\eta_i^2 + \sum_{i=1}^{n-1} g_i\eta_i\eta_{i+1} + \sum_{i=1}^{n}\frac{\mu D_i^2}{4\lambda}.\end{aligned} \tag{17}$$

By using the definition for k_i $(i=1, \dots, n)$ as given in (8), we have

$$\begin{aligned}\sum_{i=1}^{n} k_i\mu\eta_i^2 &= k_1\mu\eta_1^2 + \sum_{i=2}^{n-1} k_i\mu\eta_i^2 + k_n\mu\eta_n^2 \\ &= \left(k_0 + \frac{\overline{g}_1}{2\mu}\right)\mu\eta_1^2 + \sum_{i=2}^{n-1}\left(k_0 + \frac{\overline{g}_{i-1}+\overline{g}_i}{2\mu}\right)\mu\eta_i^2 + \left(k_0 + \frac{\overline{g}_{n-1}}{2\mu}\right)\mu\eta_n^2 \\ &= \sum_{i=1}^{n} k_0\mu\eta_i^2 + \left(\frac{\overline{g}_1}{2}\eta_1^2 + \sum_{i=2}^{n-1}\frac{\overline{g}_{i-1}+\overline{g}_i}{2}\eta_i^2 + \frac{\overline{g}_{n-1}}{2}\eta_n^2\right).\end{aligned} \tag{18}$$

Using Young's inequality, it holds that

$$\sum_{i=1}^{n-1} g_i\eta_i\eta_{i+1} \le \sum_{i=1}^{n-1}\overline{g}_i\left(\frac{\eta_i^2+\eta_{i+1}^2}{2}\right) = \left(\frac{\overline{g}_1}{2}\eta_1^2 + \sum_{i=2}^{n-1}\frac{\overline{g}_{i-1}+\overline{g}_i}{2}\eta_i^2 + \frac{\overline{g}_{n-1}}{2}\eta_n^2\right). \tag{19}$$

Then subtracting (18) from (19), yields

$$-\sum_{i=1}^{n} k_i\mu\eta_i^2 + \sum_{i=1}^{n-1} g_i\eta_i\eta_{i+1} \le -\sum_{i=1}^{n} k_0\mu\eta_i^2. \tag{20}$$

Inserting (20) into (17) results in

$$\dot{V} \le -2k_0\mu V + \sum_{i=1}^{n}\frac{\mu D_i^2}{4\lambda}. \tag{21}$$

Step 2. Upon using *Lemma* 1, it is evident from (21) that

$$\begin{aligned} V(t) &\le V(0) + \sum_{i=1}^{n}\frac{\|D_i\|^2}{8k\lambda} \triangleq C \in \mathcal{L}_\infty \\ |\eta_i(t)| &\le \sqrt{2V(t)} \le \sqrt{2C} \in \mathcal{L}_\infty, \quad i = 1, \dots, n.\end{aligned} \tag{22}$$

Since $\omega_i = \mu^{-1}\eta_i$ $(i = 1, \dots, n)$, we have

$$\begin{aligned}\lim_{t\to t_f}\omega_i(t) &= \lim_{t\to t_f}\left(\frac{t_f - t}{t_f}\right)^m \eta_i(t) = 0, \qquad m \in \mathbb{R}_{\ge 2} \\ \lim_{t\to t_f}\dot{\mu}\mu^{-1}\omega_i(t) &= \lim_{t\to t_f}\frac{m\left(t_f - t\right)^{m-1}}{t_f}\eta_i(t) = 0, \\ \lim_{t\to t_f} x_1 &= \lim_{t\to t_f}\omega_1 = 0.\end{aligned} \tag{23}$$

In order to prove that $\lim_{t\to t_f} x_2 = 0$, let us first prove $\lim_{t\to t_f}\eta_1 = 0$ by contradiction. From (22) we know that η_1 is bounded for all $t \in [0,\ t_f)$, which has only three possibilities:

$$1)\ \lim_{t\to t_f}\eta_1 = 0,$$

$$2)\ \lim_{t\to t_f}\eta_1 = C_0, \quad C_0 \in R_{>0} \tag{24}$$

$$3)\ \lim_{t\to t_f}\eta_1 = -C_0.$$

Assume that $\lim_{t\to t_f}\eta_1 = C_0$, then from (8) and *Assumption* 2 ($f_1(X,u,t) \le d_1\phi_1(x_1,t)$ and $\phi_1(0,t) = 0$) we have

$$\begin{aligned}
&\lim_{t\to t_f}\phi_1 = 0,\ \lim_{t\to t_f}k_1\eta_1 = \lim_{t\to t_f}\left(k_0 + \frac{\overline{g}_1}{2\mu}\right)\eta_1 = k_0C_0\\
&\lim_{t\to t_f}\psi_1 = \lim_{t\to t_f}\frac{1}{\underline{g}_1}\left(k_1\eta_1 + \lambda\eta_1\left(\mu^{-1}\dot{\mu}\omega_1 + \phi_1\right)^2\right) = \lim_{t\to t_f}\frac{k_1\eta_1}{\underline{g}_1} = \frac{k_0C_0}{C_{\underline{g}_1}},\\
&\lim_{t\to t_f}x_2 = \lim_{t\to t_f}(\omega_2 - \psi_1) = -\lim_{t\to t_f}\psi_1 = -\frac{k_0C_0}{C_{\underline{g}_1}},
\end{aligned} \tag{25}$$

where $\underline{g}_1(0) = C_{\underline{g}_1} \le C_{g_1} \le C_{\overline{g}_1} = \overline{g}_1(0)$. As $\dot{x}_1 = g_1x_2 + f_1$, upon using L'Hôpital's rule, we get

$$\lim_{t\to t_f}\eta_1 = \lim_{t\to t_f}\left(\frac{t_f}{t_f - t}\right)^m x_1 = \lim_{t\to t_f}\frac{t_f^m\,(g_1x_2 + f_1)}{-m\left(t_f - t\right)^{m-1}}. \tag{26}$$

From Assumptions 1 and 2, we have that $\lim_{t\to t_f}|f_1| \le \lim_{t\to t_f}d_1\phi_1(x_1,t) = 0$ and $\lim_{t\to t_f}g_1(X,u,t) = C_{g_1}$, then: (Note that L'Hôpital's rule still holds when the limit is infinity.[23])

$$\lim_{t\to t_f}\eta_1 = \frac{t_f^m C_{g_1}k_0C_0}{mC_{\underline{g}_1}\left(t_f - t\right)^{m-1}} = +\infty. \tag{27}$$

This actually contradicts the already proven fact that η_1 is bounded. In addition, if we assume that $\lim_{t\to t_f}\eta_1 = -C_0$, using the analysis similar to that from (25) to (27), we have $\lim_{t\to t_f}\eta_1 = \frac{t_f^m C_{g_1}k_0C_0}{mC_{\underline{g}_1}\left(t_f-t\right)^{m-1}} = -\infty$, which also contradicts the fact that η_1 is bounded. So it must be true that

$$\lim_{t\to t_f}\eta_1 = \lim_{t\to t_f}\left(\frac{t_f}{t_f - t}\right)^m x_1 = 0, \quad m \in \mathbb{R}_{\ge 2}. \tag{28}$$

From (28) we can deduce that the rate of convergence of x_1 (to zero) is faster than the rate of $\left(t_f/\left(t_f - t\right)\right)^m$ (namely the time-varying function $\mu(t)$) growing to infinity as $t \to t_f$, which also means that even if we accelerate the growth of $\mu(t)$ by increasing m, it still holds that $\lim_{t\to t_f}\eta_1 \equiv 0$. Then

$$\begin{aligned}
&\lim_{t\to t_f}\psi_1 = \lim_{t\to t_f}\frac{1}{\underline{g}_1}\left(k_1\eta_1 + \lambda\eta_1\Phi_1^2\right) = 0,\\
&\lim_{t\to t_f}x_2 = \lim_{t\to t_f}(\omega_2 - \psi_1) = 0,\\
&\lim_{t\to t_f}\dot{x}_1 = \lim_{t\to t_f}(g_1x_2 + f_1) = 0.
\end{aligned} \tag{29}$$

To complete the proof, we need to show that $\lim_{t\to t_f}x_3 = 0$ because there involves some additional analysis in order to come to the conclusion. As we did previously, let us first prove $\lim_{t\to t_f}\eta_2 = 0$ by contradiction. From (3) and (22) we have $\lim_{t\to t_f}\eta_2 = \lim_{t\to t_f}\left(t_f/\left(t_f - t\right)\right)^m(x_2 + \psi_1) \le \sqrt{2C}$, which has only three possibilities:

$$\begin{aligned}
&1)\ \lim_{t\to t_f}\eta_2 = 0,\\
&2)\ \lim_{t\to t_f}\eta_2 = C_1, \quad C_1 \in \mathbb{R}_{>0}\\
&3)\ \lim_{t\to t_f}\eta_2 = -C_1.
\end{aligned} \tag{30}$$

Recall that

$$
\begin{aligned}
\lim_{t\to t_f}\left(\frac{t_f}{t_f-t}\right)^m x_1 &= 0,\\
\lim_{t\to t_f}\frac{t_f^m C_{g_1} x_2}{-mC_{\underline{g}_1}\left(t_f-t\right)^{m-1}} &= 0,\\
\lim_{t\to t_f}\mu\psi_1 = \lim_{t\to t_f}\frac{1}{\underline{g}_1}\left(k_1\mu^2 x_1+\lambda\mu^2 x_1\Phi_1^2\right) &= 0.
\end{aligned}
\tag{31}
$$

From Assumption 2 and (29) we have

$$
\begin{aligned}
\lim_{t\to t_f}\phi_2 &= 0.\\
\lim_{t\to t_f}\left|\frac{\partial\psi_1}{\partial t}\right| &= 0.\\
\lim_{t\to t_f}\left|\frac{\partial\psi_1}{\partial x_1}\right|(|x_2|+\phi_1) &= 0.
\end{aligned}
\tag{32}
$$

then it holds that

$$
\lim_{t\to t_f}\Phi_2 = \lim_{t\to t_f}\left(\left|\mu^{-1}\dot{\mu}\omega_2\right|+\phi_2+\left|\frac{\partial\psi_1}{\partial t}\right|+\left|\frac{\partial\psi_1}{\partial x_1}\right|(|x_2|+\phi_1)\right) = 0. \tag{33}
$$

Now to prove by contradiction that $\eta_2(t)$ tends to zero as t goes to t_f, let us assume that $\lim_{t\to t_f}\eta_2 = C_1$, from (3) and (23), we have

$$
\begin{aligned}
\lim_{t\to t_f}\psi_2 &= \lim_{t\to t_f}\frac{1}{\underline{g}_2}\left(k_2\eta_2+\lambda\eta_2\Phi_2^2\right) = \frac{k_0C_1}{C_{\underline{g}_2}}\\
\lim_{t\to t_f}x_3 &= \lim_{t\to t_f}(\omega_3-\psi_2) = -\lim_{t\to t_f}\psi_2 = -\frac{k_0C_1}{C_{\underline{g}_2}},
\end{aligned}
\tag{34}
$$

where $\underline{g}_2(0) = C_{\underline{g}_2} \le C_{g_2} \le C_{\bar{g}_2} = \bar{g}_2(0)$. As $\dot{x}_2 = g_2x_3+f_2$, then by using L'Hôpital's rule, it is evident that

$$
\begin{aligned}
\lim_{t\to t_f}\eta_2 &= \lim_{t\to t_f}\left(\frac{t_f}{t_f-t}\right)^m(x_2+\psi_1) = \lim_{t\to t_f}\frac{t_f^m\left(g_2x_3+f_2\right)}{-m\left(t_f-t\right)^{m-1}}+\lim_{t\to t_f}\mu\psi_1\\
&= \frac{t_f^m C_{g_2}k_0C_1}{mC_{\underline{g}_2}\left(t_f-t\right)^{m-1}} = +\infty,
\end{aligned}
\tag{35}
$$

which contradicts the previously proven fact that η_2 is bounded (if we assume that $\lim_{t\to t_f}\eta_2 = -C_1$, we will have $\lim_{t\to t_f}\eta_2 = \frac{t_f^m C_{g_2}k_0C_1}{mC_{\underline{g}_2}(t_f-t)^{m-1}} = -\infty$, which also contradict the previously established fact). So it must be true that

$$
\lim_{t\to t_f}\eta_2 = \lim_{t\to t_f}\left(\frac{t_f}{t_f-t}\right)^m(x_2+\psi_1) = 0. \tag{36}
$$

From (33) and (36) we have

$$
\begin{aligned}
\lim_{t\to t_f}\psi_2 &= \lim_{t\to t_f}\frac{1}{\underline{g}_i}\left(k_2\eta_2+\lambda\eta_2\Phi_2^2\right) = 0,\\
\lim_{t\to t_f}x_3 &= \lim_{t\to t_f}(\omega_3-\psi_2) = 0,\\
\lim_{t\to t_f}\dot{x}_2 &= \lim_{t\to t_f}(g_2x_3+f_2) = 0.
\end{aligned}
\tag{37}
$$

By following the above analysis procedures, it is not difficult to finally establish that

$$\lim_{t\to t_f} \eta_i = 0,\quad \lim_{t\to t_f} x_i = 0,\quad \lim_{t\to t_f} \dot{x}_i = 0,\quad (i = 1, \ldots, n)$$
$$\lim_{t\to t_f} \psi_i = 0,\quad \lim_{t\to t_f} u = 0.\quad (i = 1, \ldots, n-1) \tag{38}$$

Thus, from (22), (28), (36), and (38), we can conclude that η_i $(i = 1, \ldots, n)$ are bounded over $[0,\ t_f)$ and converge to zero as $t \to t_f$. Furthermore, when $t \in [t_f, +\infty)$, from the control input setup in (7), we have $u(t) \equiv 0$. From (38) we have $\lim_{t\to t_f} u = 0$, which shows that $u(t)$ is continuous at $t = t_f$ and $u(t)$ becomes zero at $t = t_f$ and thereafter. As this point we have shown that $u(t)$ is bounded and continuous everywhere because each term in $u(t)$ is bounded and continuous for $t \in [0, +\infty)$. Since both $x_i(t) \to 0$ and $\dot{x}_i(t) \to 0$ $(i = 1, \ldots, n)$ as $t \to t_f$, then it is concluded that each state $x_i(t)$ converges to zero within prescribed time t_f and remains zero thereafter $(t \geq t_f)$, that is prescribed time control objective is achieved, which completes the proof. ■

Remark 5. It is important to note that in the proposed prescribed-time control method, the control action $u(t)$ reduces to zero as the system approaches its equilibrium (as $t \to t_f$), and remains zero for $t \geq t_f$, the system is therefore allowed to continue operating beyond t_f, which is in contrast to the previous methods.[9,13,16]

Remark 6. To close this section, it is worth mentioning that in (7), we have used $u(t) = 0$ for $t \geq t_f$, which seems involving a control switching at $t = t_f$, however, as we have shown that $\lim_{t\to t_f} u(t) = 0$ thus this action does not cause discontinuity to the control signal at $t = t_f$. In fact $u(t)$ is always bounded and continuous for $t \in [0, +\infty)$, including at $t = t_f$, and reduces to zero at $t = t_f$ and remains zero thereafter. This is also verified through the simulations.

4 | APPLICATIONS

In this section, we apply the proposed prescribed-time control method to three other special forms of system (1), namely, a pure chain of integrators, a chain of integrators with disturbance, and normal-form-like systems, leading to prescribed-time stabilization solutions that are different from those obtained in previous works.[1,3,13]

4.1 | Application to a pure chain of integrators

Consider the following nth order integrator

$$\begin{cases} \dot{x}_i = x_{i+1}, & i = 1, \ldots, n-1 \\ \dot{x}_n = u \end{cases}, \tag{39}$$

which is the simplest version of (1) with $g_i(X, u, t) = 1,\ f_i(X, u, t) = 0$ $(i = 1, \ldots, n)$, here $X = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$ is the system state vector and $u \in \mathbb{R}$ is the control input.

Corollary 1. *Consider system (39), with the following control scheme*

$$u(t) = \begin{cases} -\psi_n\left(t, X(t)\right), & t \in \left[0, t_f\right) \\ 0, & \text{otherwise} \end{cases}, \tag{40}$$

with

$$\psi_i = k\eta_i + \mu^{-1}\left(\dot{\mu}\omega_i + \eta_{i-1}\right) + \dot{\psi}_{i-1}, \quad i = 1, \ldots, n$$
$$\dot{\psi}_j = \sum_{l=1}^{j} \frac{\partial \psi_j}{\partial x_l} x_{l+1} + \frac{\partial \psi_j}{\partial t}, \quad (j = 1, \ldots, n-1) \tag{41}$$
$$\psi_0 = 0,\ \eta_0 = 0,\ k > 0,$$

where μ and ω_i, η_i $(i = 1, \dots, n)$ are defined as in (2) and (3), respectively. Then the prescribed-time control objective as stated in Section 2.3 is achieved.

Proof of Corollary 1. For the pure chain of integrators as described in (39), with the controller being designed in (40), by choosing $V = \frac{1}{2}\sum_{i=1}^{n}\eta_i^2$, and carrying out the analysis similar to *Step* 1, it can be shown that

$$\dot{V} \leq -2k\mu V.$$

The rest of the proof can be done by following the procedures as in Step 2. ■

Remark 7. For the pure chain of integrators as in (39), Pal et al. recently propose a prescribed-time control,[3] where the feedback signal x is "unevenly" processed through the factor $1 - e^{-x}$, which maps any feedback $x \in (-\infty, +\infty)$ into $(-\infty, 1)$, leading to a control scheme capable of ensuring prescribed time stability for (39). Although Trinh et al. pointed out that there are errors in their stability analysis,[24] this issue has been addressed.[25] To see the difference between the proposed method and the one by Reference 3, it is worth comparing the control structures when applying to second-order integrator $\ddot{x}(t) = u(t)$. The control scheme from Reference 3 takes the form

$$u_{\mathrm{Pal}}(t) = \begin{cases} -x_1 - \dot{\Psi}_1(t, x_1) - \Psi_2(t, \bar{x}_2), & t \in [0,\ t_f) \\ 0, & \text{otherwise} \end{cases}, \tag{42}$$

where $\Psi_1(t, x_1)$ and $\Psi_2(t, \bar{x}_2)$ in (42) are defined as

$$\begin{aligned} \Psi_1(t, x_1) &= \frac{m}{t_f - t}\left(1 - e^{-x_1}\right), \\ \Psi_2(t, \bar{x}_2) &= \frac{m}{t_f - t}\left(1 - e^{-x_2 - \Psi_1}\right). \end{aligned} \tag{43}$$

It can be seen that Ψ_1 increases exponentially with $|x_1|$ (roughly) if $x_1 < 0$ and Ψ_2 grows exponentially with $|x_2 + \Psi_1|$ (roughly) if $x_2 + \Psi_1 < 0$, which would cause the control effort excessively (or even prohibitively) too large (especially at the initial moment), as observed in simulation in Section 5, whereas such issue is not involved in the proposed method. In addition, Basin also discusses the issue and pointed out "lower initial values of the control law may be confusing for correct evaluation of the required control magnitude."[26]

4.2 | Application to a chain of integrators with disturbance

Consider the following nth order integrator with disturbance

$$\begin{cases} \dot{x}_i = x_{i+1}, & i = 1, \dots, n-1 \\ \dot{x}_n = u + f(X, u, t) \end{cases}, \tag{44}$$

which is a special case of (1), with $g_i(X, u, t) = 1$, $(i = 1, \dots, n)$, $f_i(X, u, t) = 0$ $(i = 1, \dots, n-1)$, and $f_n(X, u, t) = f(X, u, t) \neq 0$, where $X = [x_1, \dots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$ is the system state vector and $u \in R$ is the control input.

Corollary 2. *Consider system (44) with $f(X, u, t)$ being vanishing disturbance satisfying $|f(X, u, t)| \leq d\phi(X, t)$ with $\phi(0, t) = 0$, if the following control scheme is applied*

$$u(t) = \begin{cases} -\psi_n(t, X(t)) - \lambda\eta_n\phi^2, & t \in [0,\ t_f) \\ 0, & \text{otherwise} \end{cases}, \tag{45}$$

with

$$\begin{aligned} \psi_i &= k\eta_i + \mu^{-1}(\dot{\mu}\omega_i + \eta_{i-1}) + \dot{\psi}_{i-1},\ (i = 1, \dots, n) \\ \dot{\psi}_j &= \sum_{l=1}^{j}\frac{\partial\psi_j}{\partial x_l}x_{l+1} + \frac{\partial\psi_j}{\partial t}, \quad (j = 1, \dots, n-1) \\ \psi_0 &= 0,\ \eta_0 = 0, \end{aligned} \tag{46}$$

where $k > 0$ and $\lambda > 0$ are user-design constants and μ and ω_i, η_i $(i = 1, \dots, n)$ are defined as in (2) and (3), respectively. Then the prescribed-time control objective as stated in Section 2.3 is achieved.

Proof of Corollary 2. For the chain of integrators with disturbance as described in (44), with the controller being designed in (45) and by choosing $V = \frac{1}{2}\sum_{i=1}^{n}\eta_i^2$, it can be derived that

$$\dot{V} \leq -2k\mu V + \frac{\mu d^2}{4\lambda},$$

implying that $\eta_i (i = 1, \dots, n)$ is bounded. Then following the procedures in Step 2, the proof is completed. ■

Remark 8. Pal et al.[3] recently develop a prescribed-time control method for (44) by using terminal sliding mode control to compensate for the external disturbance. However, for normal-form systems with $g_n(\cdot)$ being unknown yet time-varying, the impact of the disturbances/uncertainties cannot be completely and effectively compensated, rendering the method therein inapplicable to such systems.

4.3 | Application to normal-form-like systems

Consider the following nth order normal-form-like nonlinear system

$$\begin{cases} \dot{x}_i = x_{i+1}, \quad i = 1, \dots, n-1 \\ \dot{x}_n = g(X, u, t)u + f(X, u, t), \end{cases} \tag{47}$$

where $g(X, u, t)$, $f(X, u, t)$ are unknown scalar real-valued continuous functions of X, u and t with $X = [x_1, \dots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$ being the system state vector and $u \in \mathbb{R}$ being the control input.

Corollary 3. *Consider system (47) with $f(X, u, t)$ being vanishing disturbance satisfying $|f(X, u, t)| \leq d\phi(X, t)$ with $\phi(0, t) = 0$ and $0 < \underline{g}(X) \leq g(X, u, t)$, If the following control scheme is applied*

$$u(t) = \begin{cases} -\frac{1}{\underline{g}(X)}\left(k\eta_n + \lambda\eta_n\Phi^2\right), & t \in [0,\ t_f) \\ 0, & \text{otherwise}, \end{cases} \tag{48}$$

with

$$\begin{aligned} &\Phi = |\mu^{-1}(\dot{\mu}\omega_n + \eta_{n-1})| + \phi(X) + |\dot{\psi}_{n-1}|, \\ &\psi_i = k\eta_i + \mu^{-1}(\dot{\mu}\omega_i + \eta_{i-1}) + \dot{\psi}_{i-1}, \ (i = 1, \dots, n-1) \\ &\dot{\psi}_j = \sum_{l=1}^{j}\frac{\partial\psi_j}{\partial x_l}x_{l+1} + \frac{\partial\psi_j}{\partial t}, \quad (j = 1, \dots, n-2) \\ &\psi_0 = 0, \ \eta_0 = 0, \end{aligned} \tag{49}$$

where $k > 0$ and $\lambda > 0$ are user-design constants and μ, ω_i, η_i $(i = 1, \dots, n)$ are defined as in (2) and (3), respectively. Then the prescribed-time control objective as stated in Section 2.3 is achieved.

Proof of Corollary 3. For the normal form nonlinear systems as described in (47), if the controller is designed as in (48), then by choosing $V = \frac{1}{2}\sum_{i=1}^{n}\eta_i^2$, it holds that

$$\dot{V} \leq -2k\mu V + \frac{\mu d^2}{4\lambda},$$

thus the result follows by using the analysis similar to that in Step 2. ■

Remark 9. Although system (47) is in normal-form, its control gain $g(\cdot)$ and nonlinear term $f(\cdot)$ are functions of X, u, and t, and are unknown yet time-varying, consequently, the methods by References 3 and 16 are actually inapplicable. The method by Reference 13, although applicable to this case, only works over finite time interval $[0,\ t_f)$. While the control algorithms (48) and (49) can guarantee that $u(t) \to 0$ as $t \to t_f$ and $u(t) = 0$ for all $t > t_f$, allowing the system to be operational over the entire time interval $[0, +\infty)$. To further compare the difference between the proposed method and the one by Reference 13, let us examine the case of $\ddot{x} = g(\cdot)u + f(\cdot)$, for which the control scheme from Reference 13 takes the form

$$u_{\mathrm{Song}}(t) = -\frac{1}{\underline{g}(X)}\left(k + \theta + \lambda\phi(X)^2\right) z(X), \tag{50}$$

with

$$X = [x,\ \dot{x}]^T, \quad z = k_1\mu x + \dot{\mu}x + \mu\dot{x}, \tag{51}$$

where $k > 0$, $k_1 > 0$ and $\lambda > 0$ are user-design constants, $\underline{g}(X)$ and $\phi(X)$ satisfy Assumptions 1 and 2, respectively. Note that there involves a scalar real-valued function θ in (50), which is quite complex to determine (see the works by Song et al.[13] for detail). Whereas the proposed control for this case takes the form

$$u(t) = \begin{cases} -\frac{1}{\underline{g}(X)}\left(k\mu(\dot{x} + \psi_1) + \lambda\mu(\dot{x} + \psi_1)\Phi^2\right), & t \in [0,\ t_f) \\ 0, & \text{otherwise} \end{cases}, \tag{52}$$

with

$$\begin{aligned} &\Phi = |\mu^{-1}\dot{\mu}(\dot{x} + \psi_1)| + |x| + \phi + |\dot{\psi}_1|, \\ &\psi_1 = k\mu x + \mu^{-1}\dot{\mu}x, \quad \dot{\psi}_1 = \frac{\partial\psi_1}{\partial x_1}x_2 + \frac{\partial\psi_1}{\partial t}, \end{aligned} \tag{53}$$

where $k > 0$ and $\lambda > 0$ are user-design constants, μ is defined as in (2), $\underline{g}$ and ϕ satisfy Assumptions 1 and 2, respectively. The difference in control structure leads to distinction in control performance, as seen in Section 5 in simulation verification.

Remark 10. To close this section, it is worth pointing out that the main benefit of this approach to finite-time stability is that it provides a way for the designer to specify the convergence time at will by presetting the time parameter t_f, and just like traditional finite-time controllers, such controllers have better robustness in the presence of uncertainties and disturbances. However, the drawback of our approach is also obvious: the controller (7) was developed for nonlinear systems without measurement disturbances, which is limiting in practice. Because of this, the presence of measurement disturbances will degrade the stabilization of the state to a nonzero neighborhood of the origin and the controller only works in a finite time interval $[0,\ t_f)$. In addition, the control input remains bounded and converges to zero as $t \to t_f$ although the time-varying function $\mu(t)$ tends to infinity as $t \to t_f$. The use of control gain that grows unbounded can be traced back to many previous works, such as optimal control over time intervals, and funnel control[27,28] (the control gain becomes sufficiently large as the tracking error tends to the funnel boundary).

5 | NUMERICAL SIMULATION

To verify the efficiency of the proposed method, we conduct simulation on three examples with comparison.

Example 1. Consider the second-order system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = u. \end{cases} \tag{54}$$

with $X = [x_1,\ x_2]^{\mathrm{T}}$ and u being the states and control of the system, respectively. We compare the controller as designed in (40) and the one by Pal et al.[3] as given in (42). The initial states are set as $(x_1(0), x_2(0)) = (-3, 2)$, the other control parameters are chosen as $t_f = 5s$, $m = 2$ and $k = 1$. The state response and the control effort are shown in Figure 1. From which it is seen that each state (x_1 and x_2) and the control signal $u(t)$ converge to zero within the prescribed time $t_f = 5s$. However, the proposed control achieves faster convergence with much less control effort as compared with that by Pal et al.[3]

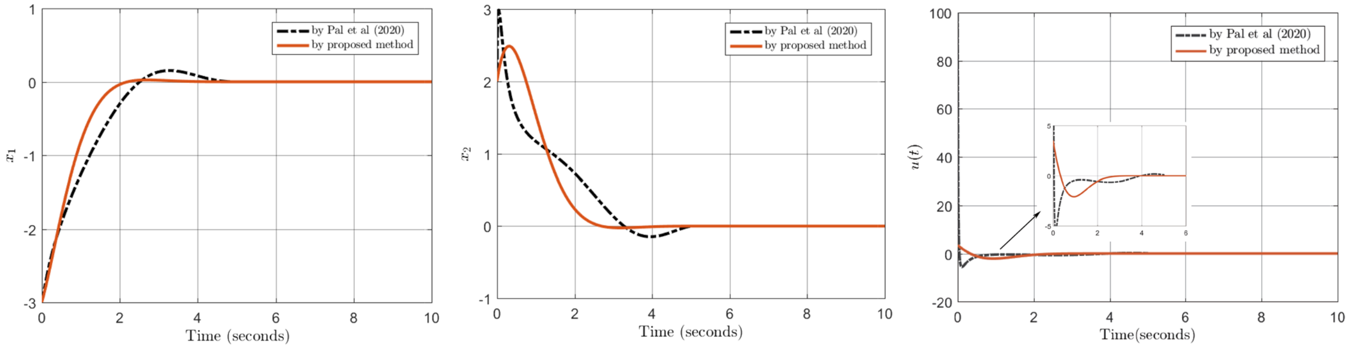
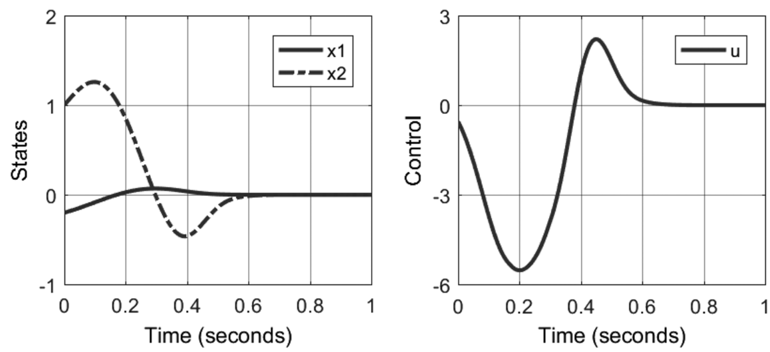


FIGURE 1 Comparison of the control performance of the proposed method and the one by Pal et al.[3] [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 2 Scheme by Song et al.[13] with $\overline{t_f} = 1.02$ s



Example 2. The dynamics of the model of the "wink-rock" unstable motion in high-performance aircraft is described as follows[13]

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = g(\cdot)u + f(\cdot) \end{cases}, \tag{55}$$

with

$$\begin{aligned} g(x, t) &= 2 + 0.4\sin(t) \\ f(\cdot) &= a_1(t)x_1 + a_2(t)x_2 + a_3|x_1|x_2 + a_4|x_2|x_2 + a_5x_1^3, \end{aligned} \tag{56}$$

where $a_1(t) = \cos(\varpi_1 t)$, $a_2(t) = 2\sin(\varpi_2 t)$, $a_3 = 2$, $a_4 = 3, a_5 = 1$ and $\varpi_1 = \varpi_2 = 1$. For the system under consideration, it is readily verified that all the assumptions and conditions are satisfied. And $\phi(\cdot)$ is obtained as $\phi(\cdot) = |x_1| + |x_2| + |x_1x_2| + x_2^2 + |x_1|^3$ that satisfies $|f(\cdot)| \leq d(t)\phi(\cdot)$ in Assumption 2. The controller for system (55) is designed as in (52), where the following control parameters are used $\underline{g} = 1$, $m = 2$, $k = 3$, $\lambda = 0.1$. The controller designed by Song et al.[13] has been given in (50) with the following control parameters $\underline{g} = 1.6$, $m = 4$, $k = 4$, $\theta = 8$, $\lambda = 0.1$. The initial condition are $(x_1(0), x_2(0)) = (-0.2,\ 1)$, To avoid numerical issue involved in the previous work,[13] the settling time $\overline{t_f} = 1.02$ s is used, which is slightly larger than $t_f = 1$ s. Such treatment, however, is not needed in the proposed control. The state response and control effort are shown in Figures 2 and 3, respectively, which confirm the effectiveness of the proposed prescribed-time control method.

In fact, it is seen that the proposed control, as compared with that by Song et al.[13] results in better transient and steady-state control performance with less control effort. And our control is able to operate continuously over $[0, +\infty)$, while the scheme in the work by Song et al.[13] can only operate on $[0,\ 1)$, confirming the theoretical prediction.

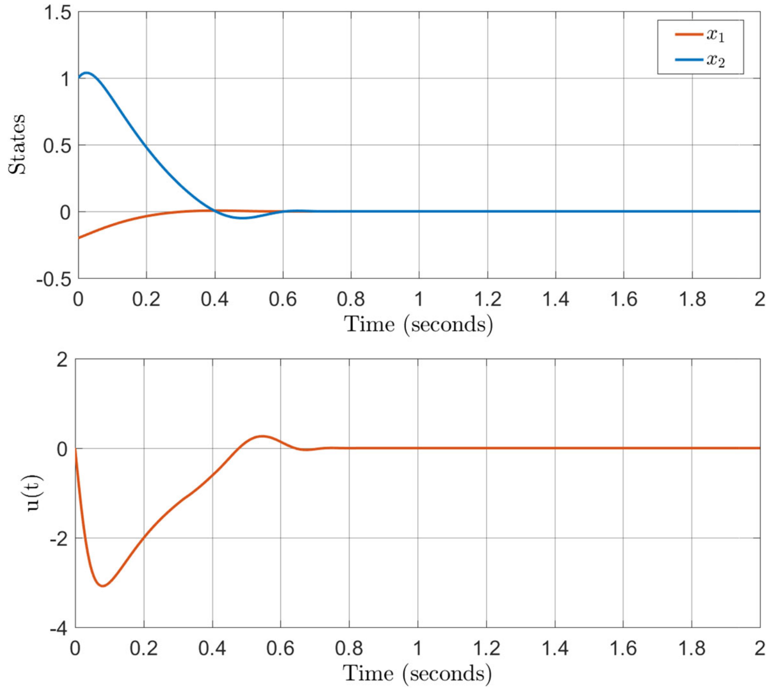


FIGURE 3 Scheme in this paper with $t_f = 1$ s [Colour figure can be viewed at wileyonlinelibrary.com]

Example 3. Consider the following nonlinear systems

$$\begin{cases} \dot{x}_1 = \left(a + b\sin(u) + x_1^2\right)x_2 + \cos\left(cx_2^5\right)(1 - e^{-x_1}), \\ \dot{x}_2 = \left(a + x_1^4 + b\cos(u)\right)u + (1 + \sin(tu + h))\,e^{x_1}x_2 \end{cases} \tag{57}$$

where $a,\ b,\ c$, and h are uncertain parameters set as (for simulation purpose)

$$\begin{aligned} &a = 1.5 + 0.5\sin(t), \quad b = 0.25 + 0.25\sin(t), \\ &c \in [0,\ 5], \quad h \in [-3,\ 3]. \end{aligned} \tag{58}$$

Apparently, the controller proposed by Song et al.[13] is no longer applicable as the systems is not in normal form. The method by Pal et al.[3] is not applicable either because of the existence of the uncertain gain and disturbance in each channel. It is noted that according to the setting, the method by Krishnamurthy et al.[16] also becomes inapplicable because the required conditions therein are not satisfied. In addition, the system is expected to operate beyond the prescribed time, not just within $[0,\ t_f)$, which would make the method of Reference[13] and[16] invalid immediately. It is interesting to see that for the system under consideration, the proposed method can be directly applied because all the conditions imposed are trivially satisfied. The control scheme (7) for this case takes the following form

$$u(t) = \begin{cases} -\frac{1}{\underline{g}_2}(k_2\eta_2 + \lambda\eta_2\Phi_2^2), & t \in [0,\ t_f) \\ 0, & \text{otherwise} \end{cases}, \tag{59}$$

where $k_2,\ \eta_2, \Phi_1$, and Φ_2 are defined as (8). In view of the condition $|f_i(\bar{x}_i)| \le d_i\phi_i(\bar{x}_i)$ in Assumption 2, it is readily obtained that $\phi_1(x) = |1 - e^{-x_1}|$ and $\phi_2(x) = |e^{x_1}x_2|$, respectively. In the simulation the following control parameters are used:

$$\begin{aligned} &\underline{g}_1 = 0.5 + x_1^2, \quad \bar{g}_1 = 2.5 + x_1^2, \\ &\underline{g}_2 = 0.5 + x_1^4, \quad \lambda = 0.1. \end{aligned}$$

The results are shown in Figure 4, which again confirm the effectiveness of the proposed prescribed-time control method, in particular the controller is functional beyond the t_f. To study the robustness of the proposed controller, as

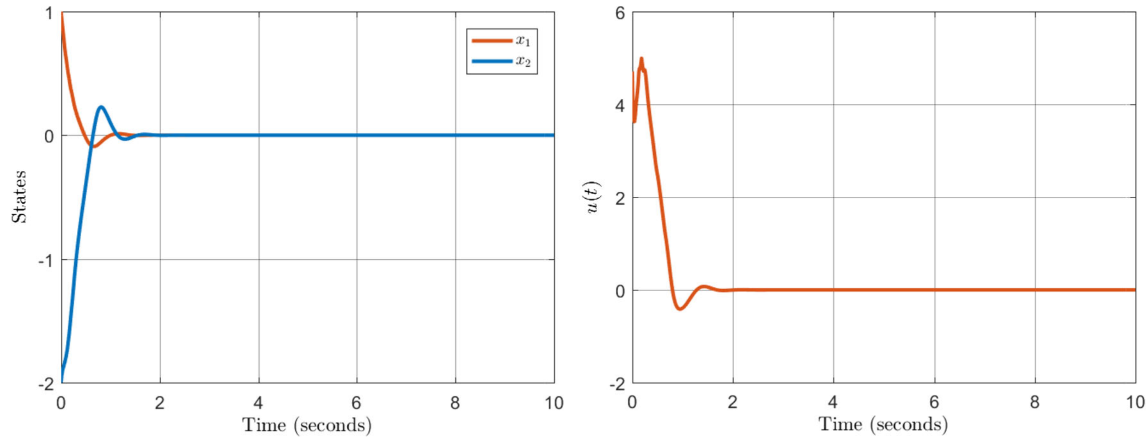


FIGURE 4 Simulations for closed-loop system with $t_f = 6\,s$, $(x_1(0), x_2(0)) = (1, -2)$, $m = 2$, $k_0 = 0.5$ [Colour figure can be viewed at wileyonlinelibrary.com]

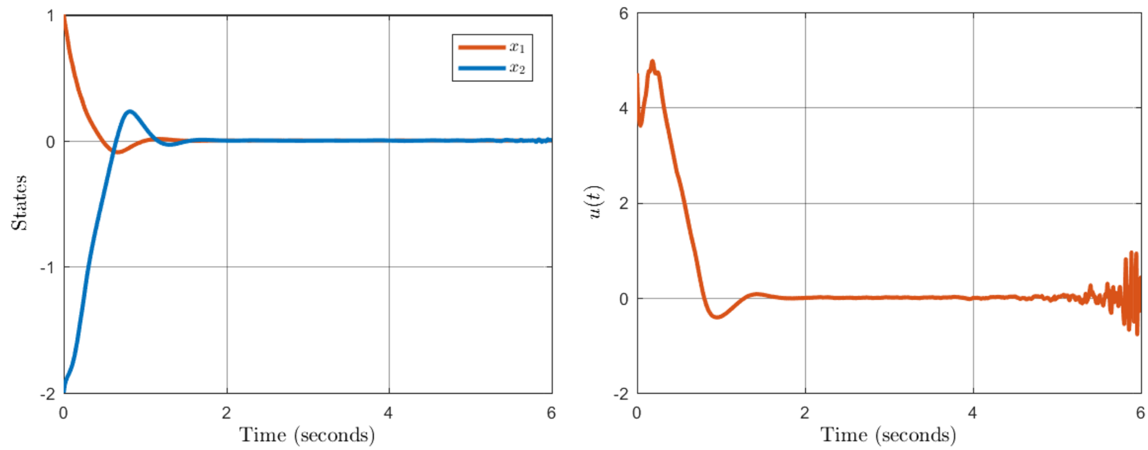


FIGURE 5 Simulations for closed-loop system with measurement noise with $\overline{t_f}$ = 6.4 s [Colour figure can be viewed at wileyonlinelibrary.com]

considered in Reference 16, the measurement of x_1 was corrupted with an additive noise of zero mean, SD of 0.02, and bandwidth of 10 Hz. The state response and control effort are shown in Figure 5. It is seen that the control input remains bounded and the states convergence to a nonzero neighborhood of the origin in the presence of the measurement noise.

6 | DISCUSSION AND CONCLUSIONS

The problem of prescribed-time regulation for a class of uncertain nonlinear systems in strict-feedback-like form is studied in this work. New control algorithms are developed to achieve full state regulation within the prescribed time (t_f) that is user-assignable in advance irrespective of system initial condition or any other parameter, and the corresponding control is fully functional (applicable) with bounded and continuous action over the whole time interval $[0, +\infty)$. To close this paper, we make the following comments:

1) If the uncertainty/disturbance $f_i(X, u, t)$ $(i = 1, \ldots, n)$ does not vanish at t_f and thereafter, the required control action will no longer disappear either. In this case it still holds that all the states converge to zero as t goes to t_f and u remains bounded for $t \in [0, t_f)$. Namely, nonzero control action is needed to fight consistently against the persistent (nonvanishing) disturbances/uncertainties. This does not become an issue for systems that stop operating when

t approaches t_f and beyond (for instance, missile interception and target attacking, payload delivering, and parts assembly etc.).

2) For systems in strict-feedback-like form that involve nonvanishing uncertainties/disturbances yet need to operate beyond t_f, excessively large control action might be required, how to address such issue remains unclear. One possible solution (the convergence precision will be sacrificed under this solution) is to pursue the so-called practical prescribed-time control (in which prescribed control with sufficient precision is reached within prespecified time) using, for instance, funnel control[27,28] and prescribed performance control.[18-20]
3) Another three possible extensions are the prescribed-time tracking control of MIMO strict-feedback-like systems, output feedback control of pure-feedback systems in prescribed-time setting, and adaptive prescribed-time control of uncertain nonlinear systems.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this article.

DATA AVAILABILITY STATEMENT

The data that supports the findings of this study are available in Section 5 of this article.

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