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Adaptive fault-tolerant PI tracking control with guaranteed transient and steady-state performance

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***Abstract*— It is a long lasting open problem to synthesize a general PI control for nonlinear systems with its gains analytically determined, yet ensuring stability and transient performance. The problem is further complicated if modeling uncertainties and external disturbances as well as actuation failures are involved in the systems. In this note, a generalized PI control with adaptively adjusting gains is presented, which gracefully avoids the adhoc and time-consuming "trial and error" process for determining the gains as involved in traditional PI control; collectively accommodates modeling uncertainties, undetectable disturbances and undetectable actuation failures that might occur in the systems; and dynamically maintains pre-specified transient and steady-state performance.**

***Index Terms*— Generalized adaptive PI control; Fault-tolerant; Nonlinear systems; Guaranteed transient performance.**

I. INTRODUCTION

The problem addressed in this work is: would PI (proportional and integral) control be applicable to uncertain nonlinear systems? Our interest in re-visiting PI control is largely motivated by the fact that, although various advanced control methods have been developed during the past decades, the preferred one in engineering practice is still the PID/PI control, due to its simplicity in structure and intuitiveness in concept, thus gained wide application in practical engineering systems [1]-[3]. However, the well-known PI control exhibits two major drawbacks that restrict its application to more general systems. The first one is the determination of the PI gains for a given system is an adhoc and painstaking process. Thus far there exists no systematic means to guide the determination of such gains that ensure system stability and performance, although various methods for tuning PI gains have been suggested in the literature [1], [4]-[7]. The second one is that although PI control has been demonstrated quite effective in dealing with certain linear time-invariant systems, its applicability to nonlinear systems remains unclear and lacks of theoretical insurance for closed-loop system stability and performance. While some efforts have been made in developing algorithms for tuning/adjusting PID/PI gains by utilizing generic algorithm, neural network and/or fuzzy system technique [8]-[12] (to just name a few), there still leave much to be desired in most existing methods in terms of simplicity, affordability and effectiveness. The interesting issues to address are therefore: would it be possible to construct PI-like control capable of dealing with nonlinear uncertain systems where the PI gains are systematically and adaptively determined by the control algorithm itself? Furthermore, is it possible to equip such PI scheme with adaptive and fault-tolerant capabilities yet guaranteeing transient performance? The purpose of this work is to present a solution to address these issues. The major contributions of this work can be summarized as follows.

1) New PI control design ensuring stable tracking control of nonlinear systems is presented.
2) Rigorous stability condition for nonlinear systems under the control of the proposed PI-like scheme is established.
3) The proposed PI-like control exhibits fault-tolerant capability without the need for FDD/FDI.
4) Transient performance is ensured with the proposed PI control despite system nonlinearities, modeling uncertainties and actuation faults.

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II. PROBLEM FORMULATION

Consider the following class of uncertain nonlinear systems,

$$\begin{aligned} \dot{x}_k &= x_{k+1}, \quad k = 1, 2, \cdots, n-1, \\ \dot{x}_n &= g(X, t)u_a + f(X, t), \end{aligned} \tag{1}$$

where $x_k \in R$ $(k = 1, \cdots, n)$ is the kth state with $x_1 = x$; $X = [x_1, \cdots, x_n]^T$; $u_a \in R$ is the actual control input of the system (the output of the actuator); $g(\cdot) \in R$ is the time-varying and uncertain control gain; $f(\cdot) \in R$ denotes the lumped uncertainties and external disturbances.

As unanticipated actuator faults may occur we additionally include such scenario in the model, where the actual control input u_a and the designed input u are no longer the same in that

$$u_a = \rho(t_\rho, t)u + u_r(t_r, t) \tag{2}$$

where $0 \leq \rho(\cdot) \leq 1$, known as "healthy indicator" [11], indicates the actuation effectiveness, $u_r(\cdot)$ is the uncontrollable portion of the control signal, t_ρ and t_r denote, respectively, the time instant at which the loss of actuation effectiveness fault and the additive actuation fault occur. In this work, we consider the case that $0 < \rho(\cdot) \leq 1$, i.e., although loosing its effectiveness the actuation is still functional such that u_a can be influenced by the control input u all the time. In addition, t_ρ and t_r are assumed completely unknown, this fact, together with the unknown and time varying ρ and u_r, literally implies that the occurrence instant and the magnitude of the actuation faults are unpredictable. The dynamic model considering actuation failures then becomes

$$\begin{aligned} \dot{x}_k &= x_{k+1}, \quad k = 1, 2, \cdots, n-1, \\ \dot{x}_n &= g(X, t)\rho(t_\rho, t)u + f(X, t) + g(X, t)u_r(t_r, t), \end{aligned} \tag{3}$$

The objective is to design a PI-like tracking controller for the system with lumped uncertainties and disturbances as well as actuator faults as described by (3) such that not only stable tracking is achieved, but also pre-described performance is ensured, yet all the internal signals are continuous and bounded. More specifically, the PI-like control ensures that: 1) the tracking error $E = X - X^* = [\epsilon_1, \epsilon_2, \cdots, \epsilon_n]^T$ $(\epsilon = \epsilon_1)$ converges to a small residual set containing the origin for any given desired trajectory $X^* = [x^*, \dot{x}^*, \cdots, x^{*(n-1)}]^T$; 2) the tracking error is confined within a pre-given bound all the time, i.e., there exists performance functions $\mu_{1k}(t)$ and $\mu_{2k}(t)$ such that $\mu_{1k}(t) \leq \epsilon_k(t) \leq \mu_{2k}(t)$ $(k = 1, \cdots, n)$ for all $t \geq 0$. In addition, the convergence rate is controlled by $e^{-a_0 t}$ for some pre-specified constant $a_0 > 0$; and 3) all the internal signals in the system are ensured to be continuous and bounded.

To proceed, the following assumptions are in order.

Assumption 1: The control gain $g(\cdot)$ is unknown and time-varying but bounded away from zero, i.e., there exist some unknown constants $\underline{g}$ and $\bar{g}$ such that $0 < \underline{g} \leq |g(\cdot)| \leq \bar{g} < \infty$, and $g(\cdot)$ is sign-definite (in this note $\text{sgn}(g) = +1$ is assumed without loss of generality).

Assumption 2: The desired state x^* and its derivative up to $(n-1)$th are assumed to be smooth and bounded. In addition, $x^{*(n)}$, the nth derivative of x^*, is bounded by an unknown constant x_m, i.e., $|x^{*(n)}| \le x_m < \infty, \forall t \ge t_0$.

Assumption 3: For uncertain nonlinearities $f(\cdot)$, there exists an unknown constant $c_f \ge 0$ and a known scalar function $\varphi(X,t) \ge 0$ such that $|f(\cdot)| \le c_f\varphi(\cdot)$. If X is bounded, so is $\varphi(X,t)$.

Assumption 4: $\rho(\cdot)$ and $u_r(\cdot)$ are unknown, possibly fast time-varying and unpredictable, but bounded in that there exist some unknown constants ρ_m and $\bar{r}$ such that $0 < \rho_m \le \rho(\cdot) \le 1$ and $|u_r(\cdot)| \le \bar{r} < \infty$.

Remark 1: Assumptions 1-2 are commonly imposed in most existing works in addressing the tracking control problem of system (1) [9], [13], [15], [17]. Assumption 3 is related to the extraction of the core information from the nonlinearities of the system, which can be readily done for any practical system with only crude model information. As for Assumption 4, it is noted that most FDD/FDI based fault tolerant control implicitly assumes that the faults vary with time slowly enough to allow for timely fault identification and diagnosis [18]-[19] or that one has enough information on the faults to carry out parametric decomposition [15], while Assumption 4 imposes no such restriction, thus seems more practical.

Remark 2: Note that in practice it would be very difficult, if not impossible, to obtain the exact values of those bounds involved in Assumptions 1-4. The developed PI-like control in this work, however, is independent of those bound parameters, thus there is no need for analytical estimation of such bounds even though the fact that those bounds do exist is used in stability analysis.

III. Main Results

To help with the understand of the fundamental idea and the technical development of the proposed method, we start with controller design for the first-order nonlinear systems, followed by the extension to high-order case.

A. Generalized adaptive fault-tolerant PI control design for first-order nonlinear systems

In this subsection we develop the generalized PI control law for first-order nonlinear systems with actuation failures as described by (2). In this case (3) with (2) becomes

$$\dot{x}(t) = g(x,t)\rho(\cdot)u(t) + g(x,t)u_r(\cdot) + f(x(t),t) \tag{4}$$

where $x \in R$ denotes the system state. To facilitate the PI controller design, we first introduce a filtered variable s as,

$$s = \epsilon + \beta\int_0^t \epsilon d\tau \tag{5}$$

where $\epsilon = x - x^*$ is the tracking error, and $\beta > 0$ is a free parameter chosen by the designer.

To establish the main results, the following lemma is needed.

Lemma 1: Consider the filtered variable s defined in (5). If $lim_{t\to\infty} s = 0$, then $\epsilon(t)$ and $\int_0^t \epsilon d\tau$ converge asymptotically to zero as $t \to \infty$ with the same decreasing rate as that of s. In addition, if s is bounded, so are ϵ and $\int_0^t \epsilon d\tau$.

Proof: The proof can be readily done by using the L'Hopital's rule, so is omitted here. ■

The proposed generalized PI control is of the form

$$u = -(k_{p1} + \Delta k_{p1}(t))\epsilon(t) - (k_{I1} + \Delta k_{I1}(t))\int_0^t \epsilon(\tau)d\tau. \tag{6}$$

Different from the traditional PI control that involves constant gains, the PI gains here consist of two parts: 1) constant gains $k_{p1} > 0$ and $k_{I1} = \beta k_{p1} > 0$, with k_{p1} and β being chosen freely by the designer and 2) time-varying gains $\Delta k_{p1}(t)$ and $\Delta k_{I1}(t)$ determined automatically and adaptively by the following algorithm,

$$\Delta k_{p1} = \frac{\hat{c}\psi^2}{\psi|s| + \iota}, \quad \Delta k_{I1} = \beta\Delta k_{p1}, \tag{7}$$

with

$$\dot{\hat{c}} = -\sigma_1\gamma_1\hat{c} + \frac{\sigma_1\psi^2 s^2}{\psi|s| + \iota}, \tag{8}$$

where $\hat{c}$ is the estimation of c with c being a virtual parameter to be defined later, $\psi(\cdot) = 1 + \varphi(\cdot) + |\epsilon|$ is a scalar and readily computable function, β, σ_1 and γ_1 are positive design parameters chosen by the designer, $\iota > 0$ is a small constant.

At this point, it is worth stressing that the essential difference between the traditional PI control and the proposed one. Firstly, unlike the traditional PI control where the PI gains are case-based, hand-tuned by trial and error and remain constant during the entire system operation, here the PI gains have two components, one is constant and the other is time-varying. Furthermore, the constant part is determined by the designer quite flexibly and the time-varying part is consistently tuned automatically and adaptively by the algorithm. In addition, the P-gains (k_{p1}, Δk_{p1}) and I-gains (k_{I1}, Δk_{I1}) are determined correlatively through the parameter β, rather than independently as in traditional PI control. Most importantly, the proposed PI control with the gains so determined ensures system stability, despite modeling uncertainties and actuator faults, as stated in the following theorem.

Theorem 1: Consider the nonlinear uncertain system (4) with actuation failures as described by (2) under Assumptions 1-4. If controlled by the generalized PI controller (6) with the PI gains updated by (7) and (8), then modeling uncertainties and actuation faults are accommodated automatically without the need for fault detection and ultimately uniformly bounded stable tracking is ensured. Furthermore, all the internal signals in the system are guaranteed to be continuous and bounded everywhere. More specifically, u, ϵ, $\int_0^t \epsilon d\tau$, $\dot{\epsilon}$, s, $\dot{s}$, $\hat{c}$ and $\dot{\hat{c}}$ are bounded and continuous everywhere.

Proof: By utilizing the filtered variable s as given in (5), we re-express (4) as

$$\dot{s} = g\rho u + +gu_r + f - \dot{x}^* + \beta\epsilon(t). \tag{9}$$

Noting that we have purposely linked k_p and k_I through β by $k_I = \beta k_p$, which not only reduces the design degree of complexity from 2 to 1, but also allows for (6) to be expressed as $u = -(k_{p1} + \Delta k_{p1})s$, facilitating stability analysis as seen shortly. Defining $V_1 = \frac{1}{2}s^2$, and taking the time derivative of V_1 along (9) yields

$$\dot{V}_1 = -k_{p1}g\rho s^2 - g\rho\Delta k_{p1}s^2 + s(gu_r + f - \dot{x}^* + \beta\epsilon(t)).$$

It is straightforward from Assumptions 1-4 that $|gu_r - \dot{x}^* + f + \beta\epsilon(t)| \le \bar{g}\bar{r} + x_m + c_f\varphi(\cdot) + \beta|\epsilon| \le c\psi(\cdot)$ with $c = \max\{\bar{g}\bar{r} + x_m, c_f, \beta\} < \infty$ and $\psi(\cdot) = 1 + \varphi(\cdot) + |\epsilon|$. Here c is an unknown virtual (bearing no physical meaning) parameter. Upon inserting Δk_{p1} as given in (7), we get

$$\begin{aligned} \dot{V}_1 &\le -k_{p1}\underline{g}\rho_m s^2 + [-\frac{g\rho\hat{c}\psi^2 s^2}{\psi|s| + \iota} + |s|c\psi] \\ &\le -k_{p1}\underline{g}\rho_m s^2 + [-\underline{g}\rho_m\frac{\hat{c}\psi^2 s^2}{\psi|s| + \iota} + \frac{c\psi^2 s^2 + c\psi|s|\iota}{\psi|s| + \iota}] \\ &\le -k_p\underline{g}\rho_m s^2 + [(c - \underline{g}\rho_m\hat{c})\frac{\psi^2 s^2}{\psi|s| + \iota} + c\iota] \end{aligned} \tag{10}$$

where the facts that $\hat{c} \ge 0$ for any $\hat{c}(0) \ge 0$ and $\frac{\psi|s|}{\psi|s| + \iota} \le 1$ for any $\iota > 0$ have been used. Note that there appears a parameter estimation error of the form $\tilde{\bullet} = \bullet - \underline{g}\rho_m\hat{\bullet}$ in (10), we thus introduce the error of the form $\tilde{c} = c - \underline{g}\rho_m\hat{c}$, named virtual parameter estimation error here, and blend such error into the second part of the Lyapunov function candidate, $V_2 = \frac{1}{2\sigma_1 \underline{g}}\tilde{c}^2$. Such treatment allows the unknown and time-varying control gain $g(\cdot)$ and the unknown actuation effectiveness fault $\rho(\cdot)$ to be processed gracefully.

By considering the Lyapunov function candidate $V = V_1 + V_2$, it thus follows from (10) that

$$\dot{V} \leq -k_{p1}\underline{g}\rho_m s^2 + [(c - \underline{g}\rho_m\hat{c})(\frac{\psi^2 s^2}{\psi|s| + \iota} - \frac{\dot{\hat{c}}}{\sigma_1}) + c\iota].$$

By inserting the adaptive law for $\hat{c}$ given in (8), we then have

$$\dot{V} \leq -k_{p1}\underline{g}\rho_m s^2 + c\iota + \gamma_1\tilde{c}\hat{c}.$$

Note that $\tilde{c}\hat{c} = \tilde{c}\frac{1}{\underline{g}\rho_m}(c - \tilde{c}) \leq \frac{1}{2\underline{g}\rho_m}(c^2 - \tilde{c}^2)$, then we have

$$\dot{V} \leq -k_{p1}\underline{g}\rho_m s^2 - \frac{\gamma_1}{2\underline{g}\rho_m}\tilde{c}^2 + \frac{\gamma_1}{2\underline{g}\rho_m}c^2 + c\iota \leq -l_1 V + l_2 \quad (11)$$

where $l_1 = \min\{2k_{p1}\underline{g}\rho_m, \gamma_1\sigma_1\} > 0$, and $l_2 = \frac{\gamma_1 c^2}{2\underline{g}\rho_m} + c\iota < \infty$. From (11), it can be conclude that the set $\Omega = \{(s, \tilde{c})|V \leq \frac{l_2}{l_1}\}$ is globally attractive. Once $(s, \tilde{c}) \notin \Omega$, $\dot{V} < 0$. Therefore, there exists a finite time T_0 such that $(s, \tilde{c}) \in \Omega$ for $\forall t > T_0$. This further implies that $|s| \leq \sqrt{2V_1} \leq \sqrt{\frac{2l_2}{l_1}}$ for $\forall t > T_0$. That is, s is Ultimately Uniformly Bounded (UUB), and thus the tracking error ϵ is also UUB according to Lemma 1.

In the sequel we prove all the internal signals in the system are continuous and bounded. By solving (11), it is derived that, $V(t) \leq \exp^{-l_1 t}V(0) + \frac{l_2}{l_1} \in L_\infty$ for all $t \geq 0$, which then implies that $s \in L_\infty$ and $\hat{c} \in L_\infty$. According to Lemma 1, $s \in L_\infty$ implies that $\epsilon \in L_\infty$ and $\int_0^t \epsilon d\tau \in L_\infty$, where $\epsilon \in L_\infty$ further implies that $x \in L_\infty$ (because x^* is bounded from Assumption 2) and then $\varphi(x, t) \in L_\infty$ by Assumption 3. From the definition of $\psi(\cdot)$, i.e., $\psi(\cdot) = 1 + \varphi(\cdot) + |\epsilon|$, it follows that $\psi(\cdot) \in L_\infty$, which ensures that both Δk_{p1} and Δk_{I1} are bounded. Then $u \in L_\infty$ and $\dot{\hat{c}} \in L_\infty$ from (6)-(8). Finally, one can conclude from (9) that $\dot{s} \in L_\infty$. Therefore, all the internal signals in the system are continuous and bounded. ■

B. Generalized adaptive fault-tolerant PI control design with guaranteed transient performance for high-order nonlinear systems

In this subsection we extend the previous PI-like control to the high-order nonlinear system (3) involving modeling uncertainties and actuation faults. The goal is to develop a PI-like control that not only exhibits robust, adaptive and fault-tolerant capabilities, but also guarantees pre-specified transient and steady state performance.

To proceed, we introduce a filtered variable ω as follows,

$$\omega = \beta_1\epsilon_1 + \beta_2\epsilon_2 + \cdots + \beta_{n-1}\epsilon_{n-1} + \epsilon_n, \quad (12)$$

with $\epsilon_1 = \epsilon(t)$ and $\dot{\epsilon}_k = \epsilon_{k+1}$ $(k = 1, 2, \cdots, n-1)$, where β_k $(k = 1, \cdots, n-1)$ are some constants chosen by the designer such that the polynomial $s^{n-1} + \beta_{n-1}s^{n-2} + \cdots + \beta_1$ is Hurwitz.

The following Lemma is needed in order to establish the boundedness relation between the filtered error ω and the tracking error ϵ_k $(k = 1, \cdots, n)$.

Lemma 2: Consider the filtered error ω given in (12), if $\omega \to 0$ as $t \to \infty$, then the tracking error $\epsilon(t)$ and its derivatives ϵ_k $(k = 2, \cdots, n)$ converge asymptotically to zero as $t \to \infty$ with the same decreasing rate as that of ω. Furthermore, if ω is preserved within a pre-specified performance bound all the time, i.e., there exist performance functions $\mu_1(t)$ and $\mu_2(t)$ such that

$$\mu_1(t) \leq \omega(t) \leq \mu_2(t), \forall t \geq 0, \quad (13)$$

then similar bounds also hold for ϵ_k $(k = 1, \cdots, n)$.

Proof: The proof is given in the Appendix. ■

According to Lemma 2, to guarantee that the tracking errors ϵ_k $(k = 1, \cdots, n)$ satisfy the pre-described performance requirements, it is sufficient to design a PI-like control to ensure that the filtered error $\omega(t)$ is confined within the bound (13) all the time. With this in mind, we choose the performance functions $\mu_1(t)$ and $\mu_2(t)$ as in the earlier work [16]-[17],

$$\mu_1(t) = -\underline{\delta}\eta(t), \quad \mu_2(t) = \bar{\delta}\eta(t), \quad (14)$$

where $0 < \underline{\delta} \leq \bar{\delta}$ are lower-upper bound constants and $\eta(t)$ is a strictly decreasing smooth function of time that determines the convergence rate of $\omega(t)$.

The rate function $\eta(t)$ in this work is chosen as $\eta(t) = (\eta_0 - \eta_\infty)e^{-a_0 t} + \eta_\infty$, where $\eta_0 > \eta_\infty > 0$ and a_0 is an arbitrarily positive constant. It is noted that $\bar{\delta}\eta_0$ and $-\underline{\delta}\eta_0$ characterize the upper bound of the overshoot of $\omega(t)$ and the lower bound of the undershoot of $\omega(t)$ respectively. In addition, the transient performance can be adjusted and improved by changing the design parameters η_0, η_∞, a_0, $\underline{\delta}$ and $\bar{\delta}$ properly.

We know from (13) that both pre-required transient performance and steady-state control precision can be achieved if the tracking error is controlled to behave and evolve according to (13). On the other hand, however, technical challenge arises in control design and stability analysis if we directly deal with (13) due to such constraint. To circumvent this difficulty, we carry out the following performance transformation

$$\omega(t) = T(\nu)\eta(t) \quad (15)$$

where ν is the new transformed error and $T(\nu)$ is a smooth and strictly increasing function which is thus invertible. It is interesting to note that if $T(\nu)$ exhibits the following properties: i).$-\underline{\delta} < T(\nu) < \bar{\delta}$; ii). $lim_{\nu\to-\infty}T(\nu) = -\underline{\delta}$, $lim_{\nu\to\infty}T(\nu) = \bar{\delta}$; iii). $lim_{\nu\to 0}T(\nu) = 0$, in particular, $T(0) = 0$, then it can be readily verified that (13) is naturally ensured as long as ν is controlled to be bounded for $t \geq 0$ (because $T(\nu)$ in (15) is thus bounded according to its properties). Therefore, the problem of ensuring pre-specified performance bound can be addressed by stabilizing ν. To this end, we explicitly express ν as defined in (15) as

$$\nu = T^{-1}(\frac{\omega(t)}{\eta(t)}) \quad (16)$$

which is well defined because $T(\nu)$ is invertible and $\eta(t)$ is strictly positive. It should be mentioned that the inversion in (16) can be implemented if and only if $-\underline{\delta} < \frac{\omega(t)}{\eta(t)} < \bar{\delta}$ is satisfied for all $t \geq 0$. To ensure this, we need to select initial value $\eta(0)$, $-\underline{\delta}$ and $\bar{\delta}$ such that $-\underline{\delta} < \frac{\omega(0)}{\eta(0)} < \bar{\delta}$ for any arbitrary initial tracking error $\omega(0)$, and also we must design an appropriate control to ensure signal ν satisfying $\nu(t) \in L_\infty$, and therefore $T(\nu)$ is confined within $(-\underline{\delta}, \bar{\delta})$, then from (15), $-\underline{\delta} < \frac{\omega(t)}{\eta(t)} < \bar{\delta}$ is satisfied all the time.

The above analysis indicates that the problem of guaranteeing prescribed performance boils down to selecting $\eta(0)$ properly according to the maximum possible initial tracking error $\omega(0)$ and to designing control u to steer $\nu(t)$ into a compact set and confine $\nu(t)$ within such set ever after (i.e., $\nu(t)$ is UUB). As the first step, we choose $T(\nu)$ as [17],

$$T(\nu) = \frac{\bar{\delta}e^{(\nu+r)} - \underline{\delta}e^{-(\nu+r)}}{e^{(\nu+r)} + e^{-(\nu+r)}}, \quad (17)$$

with $r = \frac{1}{2}ln(\underline{\delta}/\bar{\delta})$. Note that $T(\nu)$ satisfies the above conditions (i)-(iii), and the variable r is well defined because $\underline{\delta}/\bar{\delta}$ is always positive.

The important implication of the above transformation is that if $T(\nu)$ is ensured to be bounded such that $-\underline{\delta} < T(\nu) < \bar{\delta}$, the lower and upper bounds imposed on $\omega(t)$ as in (13) are satisfied thanks to (15). In other words, the constrained tracking error relation (13)

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can be converted into the unconstrained relation (15) as long as the signal ν is controlled to be bounded, all the time. Therefore, we only need to focus on stabilizing ν such that ν is ensured to be UUB. To this end, note that

$$\nu = T^{-1}(\frac{\omega(t)}{\eta(t)}) = \frac{1}{2}\ln(\bar{\delta}\lambda_\omega(t) + \bar{\delta}\underline{\delta}) - \frac{1}{2}\ln(\bar{\delta}\underline{\delta} - \underline{\delta}\lambda_\omega(t)), \quad (18)$$

with $\lambda_\omega(t) = \frac{\omega(t)}{\eta(t)}$. It then follows that

$$\dot{\nu} = \frac{\partial T^{-1}}{\partial \lambda_\omega}\dot{\lambda}_\omega = k_\omega(\dot{\omega} - \frac{\omega\dot{\eta}}{\eta}), \quad (19)$$

where

$$k_\omega = \frac{1}{2\eta}(\frac{1}{\lambda_\omega + \underline{\delta}} - \frac{1}{\lambda_\omega - \bar{\delta}}). \quad (20)$$

Note that k_ω is strictly positive as $\eta \in (\eta_0, \eta_\infty)$ and $\lambda_\omega \in (-\underline{\delta}, \bar{\delta})$. This is crucial to the following controller design.

Note that in (15) the filtered error $\omega(t)$ rather the tracking error $\epsilon(t)$ itself is used, it is such treatment that allows the problem of guaranteeing transient performance for the high-order nonlinear system to be addressed gracefully, as seen shortly.

As discussed earlier, UUB control of $\omega(t)$ is sufficient to ensure prescribed performance bounded control of $\epsilon_k(t)$ $(k = 1, \cdots, n)$. We now present the following PI-like control to stabilize and meanwhile ensure the pre-specified transient performance of system (3),

$$u = -(k_{p2} + \Delta k_{p2})\nu(t) - (k_{I2} + \Delta k_{I2})\int_0^t \nu(\tau)d\tau. \quad (21)$$

Here the PI gains consist of two parts: 1) k_{p2} and k_{I2} given by

$$k_{p2} = k_\omega^{-1}\bar{k}_{p2}, \quad k_{I2} = \beta k_{p2}, \quad (22)$$

where k_ω is given as in (20) and $\bar{k}_{p2} > 0$ is user-defined constant, and 2) Δk_{p2} and Δk_{I2}, adjusted automatically and adaptively by the following algorithm

$$\Delta k_{p2} = \frac{\hat{b}\chi^2}{\chi|\nu + \beta\int_0^t \nu(\tau)d\tau| + k_\omega^{-1}\iota}, \quad \Delta k_{I2} = \beta\Delta k_{p2}, \quad (23)$$

with the adaptive law

$$\dot{\hat{b}} = -\sigma_2\gamma_2\hat{b} + \frac{\sigma_2 k_\omega\chi^2(\nu + \beta\int_0^t \nu(\tau)d\tau)^2}{\chi|\nu + \beta\int_0^t \nu(\tau)d\tau| + k_\omega^{-1}\iota}, \quad (24)$$

where $\hat{b}$ is the estimation of the virtual parameter b to be defined later, $\chi(\cdot) = \varphi(\cdot) + 1 + |\epsilon_2| + \cdots + |\epsilon_n| + |\frac{\dot{\eta}}{\eta}\omega| + k_\omega^{-1}|\nu(t)|$ is a scalar and readily computable function, β, σ_2 and γ_2 are positive design parameters chosen by the designer, and $\iota > 0$ is a small constant. It is seen that in constructing the PI gains the rate function $\eta(t)$ and the tracking error $\omega(t)$ are explicitly utilized and both k_{p2} and k_{I2} are now time-varying rather than constant.

Theorem 2: Consider the high-order uncertain nonlinear system with actuation failures as described by (3). Suppose that Assumptions 1-4 hold. If the control algorithm (21)-(23) with the adaptive law (24) is applied, then it is established that 1) UUB stable tracking is ensured; 2) the tracking errors ϵ_k $(k = 1, \cdots, n)$ converge to small residual sets containing the origin at the rate of $e^{-a_0 t}$; 3) the tracking errors ϵ_k $(k = 1, \cdots, n)$ remain within the bounds related to (13) all the time; and 4) all the internal signals are bounded and continuous everywhere.

Proof: From the definition of the filtered error ω, the high-order system (3) can be re-written as

$$\dot{\omega} = g\rho u + f + gu_r - \dot{x}_n^* + \beta_1\epsilon_2 + \cdots + \beta_{n-1}\epsilon_n, \quad (25)$$

which, upon using (19), can be further expressed in terms of the transformed variable ν as

$$\dot{\nu} = k_\omega(g\rho u + L(\cdot)) \quad (26)$$

where $L(\cdot) = f + gu_r - \dot{x}_n^* + \beta_1\epsilon_2 + \cdots + \beta_{n-1}\epsilon_n - \frac{\dot{\eta}}{\eta}\omega$. Define a filtered variable S as

$$S = \nu + \beta\int_0^t \nu(\tau)d\tau. \quad (27)$$

Then the control law given in (21) becomes $u = -k_\omega^{-1}\bar{k}_{p2}S - \Delta k_{p2}S$, and the error dynamic (26) is rewritten as

$$\dot{S} = k_\omega(g\rho u + L(\cdot) + k_\omega^{-1}\beta\nu(t)). \quad (28)$$

From Assumptions 1-4, it is readily verified that $|L(\cdot) + k_\omega^{-1}\beta\nu(t)| \leq c_f\varphi(\cdot) + \xi + \beta_1|\epsilon_2| + \cdots + \beta_{n-1}|\epsilon_n| + |\frac{\dot{\eta}}{\eta}\omega| + k_\omega^{-1}\beta|\nu(t)| \leq b\chi(\cdot)$, where $b = max\{c_f, \zeta, \beta_1, \cdots, \beta_{n-1}, 1, \beta\} < \infty$ with $\zeta = \bar{g}\bar{r} + x_m < \infty$, and $\chi(\cdot) = \varphi(\cdot) + 1 + |\epsilon_2| + \cdots + |\epsilon_n| + |\frac{\dot{\eta}}{\eta}\omega| + k_\omega^{-1}|\nu(t)|$ (a computable and implementable signal). A newly defined virtual parameter estimate error is introduced as, $\tilde{b} = b - \underline{g}\rho_m\hat{b}$, with which the Lyapunov function candidate is constructed as, $V = V_3 + V_4$, with $V_3 = \frac{1}{2}S^2$ and $V_4 = \frac{1}{2\sigma_2\underline{g}\rho_m}\tilde{b}^2$. Taking the time derivative of V_3 along (28) yields

$$\dot{V}_3 = S\dot{S} = Sk_\omega(g\rho u + L(\cdot) + k_\omega^{-1}\beta\nu(t)). \quad (29)$$

By using the control law u as equivalently expressed as $u = -k_\omega^{-1}\bar{k}_{p2}S - \Delta k_{p2}S$ and plugging in (23), one readily gets that

$$\begin{aligned}\dot{V}_3 &= -\bar{k}_{p2}g\rho S^2 + Sk_\omega(-g\rho\Delta k_{p2}S + L(\cdot) + k_\omega^{-1}\beta\nu(t)) \\ &\leq -\bar{k}_p\underline{g}\rho_m S^2 + [-\frac{k_\omega g\rho\hat{b}\chi^2 S^2}{\chi|S| + k_\omega^{-1}\iota} + |S|k_\omega b\chi] \\ &\leq -\bar{k}_{p2}\underline{g}\rho_m S^2 + [(b - \underline{g}\rho_m\hat{b})\frac{k_\omega\chi^2 S^2}{\chi|S| + k_\omega^{-1}\iota} + b\iota]. \quad (30)\end{aligned}$$

The time derivative of V_4 is computed, by using the adaptive law for $\hat{b}$ given in (24), as

$$\dot{V}_4 = (b - \underline{g}\rho_m\hat{b})(\gamma_2\hat{b} - \frac{k_\omega\chi^2 S^2}{\chi|S| + k_\omega^{-1}\iota}). \quad (31)$$

Combining (30) and (31) yields that

$$\dot{V} = \dot{V}_3 + \dot{V}_4 \leq -\bar{k}_{p2}\underline{g}\rho_m S^2 + \gamma_2\tilde{b}\hat{b} + b\iota. \quad (32)$$

By using the fact that $\tilde{b}\hat{b} \leq \frac{1}{2\underline{g}\rho_m}(b^2 - \tilde{b}^2)$, one gets

$$\dot{V} \leq -\bar{k}_{p2}\underline{g}\rho_m S^2 - \frac{\gamma_2}{2\underline{g}\rho_m}\tilde{b}^2 + \frac{\gamma_2}{2\underline{g}\rho_m}b^2 + b\iota \leq -l_3 V + l_4, \quad (33)$$

with $l_3 = \min\{2\bar{k}_{p2}\underline{g}\rho_m, \gamma_2\sigma_2\} > 0$, and $l_4 = \frac{\gamma_2 b^2}{2\underline{g}\rho_m} + b\iota < \infty$. By following the same line as in the proof of Theorem 1, it is readily derived from (33) that S is Ultimately Uniformly Bounded (UUB) and then the transformed error $\nu(t)$ is also UUB according to Lemma 1. It thus follows from the definition of $T(\nu)$ given in (17) that $T(\nu)$ is UUB. Note that $\omega(t) = T(\nu)\eta(t)$, it is then concluded that the filtered error $\omega(t)$ converges to a small residual set containing the origin with the rate of $e^{-a_0 t}$ as defined in η. According to Lemma 2, the tracking error ϵ_k $(k = 1, 2, \cdots, n)$ have the same decreasing rate as that of ω and similar bounds also hold for ϵ_k if $\omega(t)$ is bounded. Thus it is established that ϵ_k $(k = 1, 2, \cdots, n)$ converges to a small residual set containing the origin at the rate controlled by $\eta(t)$. Namely, the transient and steady-state performance is guaranteed with the proposed PI-like control. Furthermore, the maximum overshoot of the tracking error ϵ_k $(k = 1, 2, \cdots, n)$ is confined within a pre-given bound and the transient performance and steady-state tracking precision can be improved by adjusting the design parameters of $\eta(t)$, $\underline{\delta}$ and $\bar{\delta}$.

In the following, we prove that all the internal signals are bounded and continuous everywhere. According to (33), it can be derived that $V \in L_\infty$, and then $S \in L_\infty$ and $\hat{b} \in L_\infty$. By the definition of S as in (27), it follows from Lemma 1 that $\nu \in L_\infty$ and $\int_0^t \nu d\tau \in L_\infty$, thus $T(\nu)$ is bounded, which further implies that $\omega(t)$ is bounded. From Assumption 2 and the definition of $\omega(t)$, it follows that x_k $(k = 1, \cdots, n)$ is bounded, then $\chi(t)$ is bounded because $\varphi(t)$ is bounded according to Assumption 3. Thus Δk_{p2} and Δk_{I2} are bounded, and therefore, u is bounded. It then can be concluded that all the internal signals are bounded and continuous everywhere. ■

Remark 3: It is seen that in the proposed PI-like control, there is no need for adhoc process for PI gains determination and the transient performance is guaranteed without the need for for FDD/FDI to provide the fault occurrence instant, fault type, or fault magnitude.

Remark 4: In developing the control schemes, a number of virtual parameters b, c and upper/lower bounds such as ρ_m, $\bar{g}$, $\underline{g}$ etc. are defined and used in stability analysis, but these parameters are not involved in the control algorithms, thus analytical estimation of those parameters (a nontrivial task) is not needed in setting up and implementing the proposed PI-like control strategies.

IV. Numerical Simulations

To verify the effectiveness of the proposed PI-like control, we use the well-known inverted pendulum example taken from [9]:

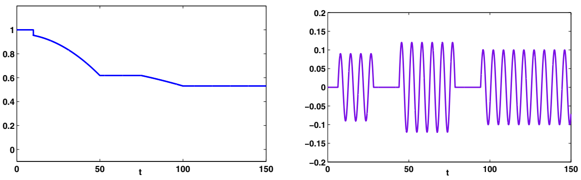
$$\dot{x}_1 = x_2,\ \dot{x}_2 = g(x_1, x_2)u_a(t) + f(x_1, x_2) + d(t), \tag{34}$$

with $f(x_1, x_2) = \frac{9.8 \sin x_1 - m l x_2^2 \cos x_1 \sin x_1/(m_c+m)}{l[4/3 - m \cos^2 x_1/(m_c+m)]}$, and $g(x_1, x_2) = \frac{\cos x_1/(m_c+m)}{l[4/3 - m\cos^2 x_1/(m_c+m)]}$, where m is the mass of the pole, m_c is the mass of the cart, l is the half length of the pole, and u_a is the actual control input. The simulation is conducted with $m = 0.1$kg, $m_c = 1$kg and $l = 0.5$m; $d(t) = 5\cos(2t)$; $u_a = \rho(\cdot)u(t) + u_r(\cdot)$ with $\rho(\cdot)$ and $u_r(\cdot)$ being as shown in Fig.1; $x^* = \frac{\pi}{30}\sin(t)$.

For comparison, both traditional PI control with constant PI gains and the proposed PPB based PI control with adaptive gains as given in Theorem 2 are tested. The procedure for the proposed PPB based PI controller design is as follows: 1) choose free design parameters: $\sigma_2 = 0.1$, $\gamma_2 = 1$, and $\beta_1 = 1$ (defined in (12)); 2) derive the computable scalar function $\varphi(\cdot)$ in Assumption 3 to get $\varphi(\cdot) = 1 + x_2^2$; 3) pre-specify the performance function $\mu_1(t)$ and $\mu_2(t)$ given in (14) with $\underline{\delta} = 0.25$, $\bar{\delta} = 0.5$ and $\eta(t) = 0.385e^{-2t} + 0.015$; and 4) determine the gains: the first part for 'P' gain is chosen as $k_{p2} = k_\omega^{-1}\bar{k}_{p2}$ with $\bar{k}_{p2} = 10$; the adaptive part for 'P' gain, i.e., Δk_{p2}, is computed automatically by the adaptive algorithm (23)-(24) with no need for "trial and error" process; the gains for 'I' are calculated by $k_{I2} = \beta k_{p2}$ and $\Delta k_{I2} = \beta \Delta k_{p2}$. In addition, $\hat{b}(0) = 0$. For the traditional PI control with constant gains, to guess the "right" PI gains, by referring the proposed PI gains self-tuning within 1 to 20 for 'P' and 10 to 500 for 'I', we set $k_p = 10$ and $k_I = 100$ for the traditional PI control in the simulation.

The results are shown in Figures 2-5. It is clearly seen that the proposed PPB based PI control has much better transient and steady-state performance as compared with the traditional one. In particular, the PPB based PI-like control ensures that the tracking error is confined within the pre-specified bound, as theoretically predicted. Furthermore, by comparing the control results in the simulation as reflected by the tracking error ϵ_1, ϵ_2, and the control inputs u_i as well as the PI gains k_p and k_I, it is seen that under the same condition the proposed PI control with adaptive self-tuning (time-varying) PI gains leads to much better control results as compared with the traditional PI control with constant PI gains.

In addition, the tracking precision is heavily dependent on the parameter β: larger β leads to better tracking precision, this is because larger β makes stronger integral action from the control scheme, thus better capability for disturbance rejection and higher steady-state control precision are achieved (fairly good tracking performance is obtained with β taking value from 1 to 500, plots not shown here due to page limit). Other parameters such as γ_i, σ_i etc can affect the control performance but not the stability. Furthermore, these parameters can be chosen quite arbitrarily in a clear direction, not like groping in the dark as with the traditional PI control.



(a) ρ (b) u_r

Fig. 1. The actuation faults simulated

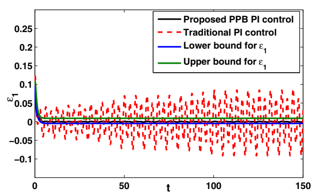


Fig. 2. Position tracking performance comparison between traditional PI and the proposed one.

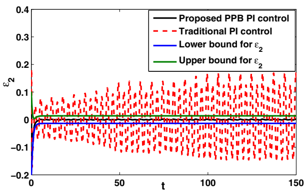


Fig. 3. Velocity tracking performance comparison between traditional PI and the proposed one.

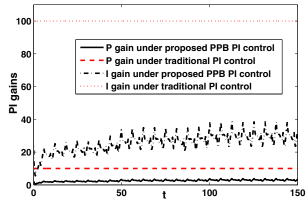


Fig. 4. The evolution of self-tuning P and I gains.

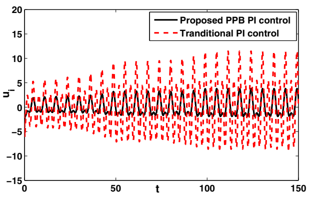


Fig. 5. Control signals.

V. Conclusion

Although PI/PID control has been widely utilized in practice, no framework for analytically specifying the "correct" PI/PID gains for general nonlinear systems is available. In this work, PI-like tracking control design for nonlinear systems subject to uncertainties and external disturbances as well as actuation faults is studied. The PI gains are analytically derived from stability and performance consideration. Furthermore, modeling uncertainties, external disturbances and actuation faults are collectively accommodated with the proposed PI-like control law. It should be pointed out that to ensure the prescribed performance bound, certain information on system initial condition must be known a priori. Removing such constraint and extending the results to MIMO nonlinear systems or even nonaffine systems represent an interesting topic for future research.

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APPENDIX

Proof of Lemms 2: To facilitate the proof, we first transform (12) into the following form:

$$\begin{aligned} \omega(t) &= \dot{\omega}_{n-1} + \alpha_{n-1}\omega_{n-1}, \\ \omega_{n-1} &= \dot{\omega}_{n-2} + \alpha_{n-2}\omega_{n-2}, \\ &\vdots \quad \vdots \quad \vdots \\ \omega_2 &= \dot{\omega}_1 + \alpha_1\omega_1 = \dot{\epsilon}_1 + \alpha_1\epsilon_1, \\ \omega_1 &= \epsilon_1. \end{aligned} \tag{35}$$

Solving the first differentiate equation in (35) yields

$$\omega_{n-1}(t) = e^{-\alpha_{n-1}t}\omega_{n-1}(0) + e^{-\alpha_{n-1}t}\int_0^t \omega(\tau)e^{\alpha_{n-1}\tau}d\tau. \tag{36}$$

From (36) it is not hard to check that if $\int_0^t \omega(\tau)e^{\alpha_{n-1}\tau}d\tau$ is bounded, then $\omega_{n-1} \to 0$ as $t \to \infty$. If, however, $\int_0^t \omega(\tau)e^{\alpha_{n-1}\tau}d\tau$ is unbounded, we then apply the L‘Hopital’s rule to (36) and obtain

$$\lim_{t\to\infty} \omega_{n-1}(t) = 0 + \lim_{t\to\infty} \frac{\omega(t)e^{\alpha_{n-1}t}}{\alpha_{n-1}e^{\alpha_{n-1}t}} = \lim_{t\to\infty} \frac{\omega(t)}{\alpha_{n-1}}, \tag{37}$$

which implies that if $\omega(t) \to 0$ as $t \to \infty$, then $\omega_{n-1} \to 0$ as $t \to \infty$ with the same decreasing rate as ω, and so is $\dot{\omega}_{n-1}$ by the first equation in (35). Similarly, from the second equation in (35), we have

$$\lim_{t\to\infty} \omega_{n-2}(t) = \lim_{t\to\infty} \frac{\omega_{n-1}(t)}{\alpha_{n-2}} = \lim_{t\to\infty} \frac{\omega(t)}{\alpha_{n-1}\alpha_{n-2}}. \tag{38}$$

This means that $\omega_{n-2}(t) \to 0$ as $t \to \infty$ with the same decreasing rate as $\omega(t)$, and so is $\dot{\omega}_{n-2}(t)$. By carrying out the same procedure for the rest of the equations in (35), we can conclude that both $\omega_k(t)$ and $\dot{\omega}_k(t)$ $(k = 1, \cdots, n-1)$ converge to zero at the same decreasing rate as $\omega(t)$. From the definition of $\omega_k(t)$ $(k = 1, \cdots, n-1)$ in (35), it is straightforward that $\epsilon_k(t) \to 0$ $(k = 1, \cdots, n)$ at the same decreasing rate as $\omega(t)$ when $\omega(t) \to 0$ as $t \to \infty$. Further, if there exist performance functions $\mu_1(t) = -\underline{\delta}\eta(t)$ and $\mu_2(t) = \bar{\delta}\eta(t)$ such that $-\underline{\delta}\eta(t) < \omega(t) < \bar{\delta}\eta(t)$, then we get from (35) that

$$\begin{aligned} &\frac{\alpha_{n-1}\omega_{n-1}(0) + \underline{\delta}\eta_0}{\alpha_{n-1}e^{\alpha_{n-1}t}} - \frac{\underline{\delta}\eta_0}{\alpha_{n-1}} < \omega_{n-1}(t) \\ &< \frac{\alpha_{n-1}\omega_{n-1}(0) - \bar{\delta}\eta_0}{\alpha_{n-1}e^{\alpha_{n-1}t}} + \frac{\bar{\delta}\eta_0}{\alpha_{n-1}}, \end{aligned} \tag{39}$$

in which we have used the fact that the maximum of $\eta(t)$ is η_0. Note that $\dot{\omega}_{n-1} = \omega(t) - \alpha_{n-1}\omega_{n-1}$ from the first equation in (35). To establish the lower bound for $\dot{\omega}_{n-1}$, we use the lower bound on $\omega(t)$ $(-\underline{\delta}\eta_0)$ minus the upper bound on $\alpha_{n-1}\omega_{n-1}$ (which is obtained from the right hand side of (39)). And to establish the upper bound for $\dot{\omega}_{n-1}$, we use the upper bound on $\omega(t)$ $(\bar{\delta}\eta_0)$ minus the lower bound on $\alpha_{n-1}\omega_{n-1}$ (which is obtained from the left hand side of (39)). Thus we have

$$\begin{aligned} &-\underline{\delta}\eta_0 - \left(\frac{\alpha_{n-1}\omega_{n-1}(0) - \bar{\delta}\eta_0}{e^{\alpha_{n-1}t}} + \bar{\delta}\eta_0\right) < \dot{\omega}_{n-1}(t) \\ &< \bar{\delta}\eta_0 - \left(\frac{\alpha_{n-1}\omega_{n-1}(0) + \underline{\delta}\eta_0}{e^{\alpha_{n-1}t}} - \underline{\delta}\eta_0\right). \end{aligned} \tag{40}$$

Denote the minimum and maximum bounds of ω by $\omega_{\min}$ and $\omega_{\max}$, and the minimum and maximum bounds of ω_k by $\omega_{k,\min}$ and $\omega_{k,\max}$ $(k = 1, 2, \cdots, n)$ respectively. By using the relation of $\omega_{\min} \le \omega \le \omega_{\max}$ and $\omega_{k,\min} \le \omega_k \le \omega_{k,\max}$, we can establish the following general lower and upper bounds for ω_{n-1} from (39):

$$\begin{aligned} &\frac{\alpha_{n-1}\omega_{n-1}(0) - \omega_{\min}}{\alpha_{n-1}e^{\alpha_{n-1}t}} + \frac{\omega_{\min}}{\alpha_{n-1}} < \omega_{n-1}(t) \\ &< \frac{\alpha_{n-1}\omega_{n-1}(0) - \omega_{\max}}{\alpha_{n-1}e^{\alpha_{n-1}t}} + \frac{\omega_{\max}}{\alpha_{n-1}}, \end{aligned} \tag{41}$$

and the lower and upper bounds for $\dot{\omega}_{n-1}$ from (40):

$$\omega_{\min} - \alpha_{n-1}\omega_{n-1,\max} < \dot{\omega}_{n-1}(t) < \omega_{\max} - \alpha_{n-1}\omega_{n-1,\min}. \tag{42}$$

Then the lower and upper bounds for ω_k and $\dot{\omega}_k$ $(k = 1, \cdots, n-1)$ are established by computing step by step:

$$\begin{aligned} &\frac{\alpha_k\omega_k(0) - \omega_{k+1,\min}}{\alpha_k e^{\alpha_k t}} + \frac{\omega_{k+1,min}}{\alpha_k} < \omega_k(t) \\ &< \frac{\alpha_k\omega_k(0) - \omega_{k+1,\max}}{\alpha_k e^{\alpha_k t}} + \frac{\omega_{k+1,\max}}{\alpha_k}, \\ &\omega_{k+1,\min} - \alpha_k\omega_{k,\max} < \dot{\omega}_k(t) < \omega_{k+1,\max} - \alpha_k\omega_{k,\min}. \end{aligned} \tag{43}$$

In final, the bounds for ϵ_k $(k = 1, \cdots, n)$ can be readily derived as

$$\begin{aligned} &\frac{\alpha_1\epsilon_1(0) - \omega_{2,\min}}{\alpha_1 e^{\alpha_1 t}} + \frac{\omega_{2,\min}}{\alpha_1} < \epsilon_1 < \frac{\alpha_1\epsilon_1(0) - \omega_{2,\max}}{\alpha_1 e^{\alpha_1 t}} + \frac{\omega_{2,\max}}{\alpha_1}, \\ &\omega_{2,\min} - \alpha_1\epsilon_{1,\max} < \epsilon_2(t) < \omega_{2,\max} - \alpha_1\epsilon_{1,\min}, \end{aligned} \tag{44}$$

and the bounds for other ϵ_k $(k = 3, \cdots, n)$ can be obtained by applying the previous method to the equations in (35) step by step.