A Unified Event-Triggered Control Approach for Uncertain Pure-Feedback Systems With or Without State Constraints

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***Abstract*—Existing schemes for systems with state constraints require the bounds of the constraints for controller design and may result in conservativeness or even become invalid when they are applied to systems without such constraints. In this paper, we study the problem of event-triggered control for a class of uncertain nonlinear systems by considering the cases with or without state constraints in a unified manner. By introducing a new universal-constrained function and using certain transformation techniques, the original-constrained system is converted into an equivalent totally unconstrained one. Then, an event-triggered adaptive neural-network (NN) controller is designed to stabilize the unconstrained system and compensate for the control sampling errors caused by event-triggered transmission of control signals. Unlike some existing control schemes developed for systems with state constraints, which need to check whether each virtual control meets certain feasibility conditions at every design step, our proposed unified method enables such feasibility conditions to be relaxed. In addition, a suitable event-triggering rule is designed to determine when to transmit control signals. It is theoretically shown that the designed controller can achieve the desired tracking ability and reduce the communication burden from the controller to the actuator at the same time. Simulation verification also confirms the effectiveness of the proposed approach.**

***Index Terms*—Event-triggered control, pure-feedback systems, state constraints.**

I. Introduction

THE PROBLEM for output constraint and state constraints of control systems is a topic of great importance in both practical applications and fundamental control theories. For safety-critical systems, such as autonomous vehicles, chemical plant, and robotic systems, both human operator and the process itself might at risk whenever certain unsafe states are reached [1]. Consequently, the designed controller must ensure the closed-loop system complies with state constraints while controlling it. Many effective approaches have been developed to address this problem, including using set invariance control [2], model predictive control [3], and reference governors [4]. Recently, the utilization of barrier Lyapunov function (BLF) or integral BLF (iBLF) for control of nonlinear systems subject to output and state constraints has received increasing attention [5]–[10]. In [5] and [6], BLF-based controllers are presented for strict-feedback systems with static output constraint and asymmetric time-varying output constraint, respectively. In [7] and [8], two adaptive backstepping control schemes are presented by using BLF at each recursive step for strict-feedback systems and pure-feedback systems with full-state constraints, respectively. In [9], the problem of tracking control is investigated for state-constrained strict-feedback systems and new iBLFs are constructed to handle the unknown control gains and state constraints simultaneously. The control design of strict-feedback/pure-feedback systems with state constraints is more difficult than output constraint because current BLF-based and iBLF-based control solutions rely on feasibility conditions for virtual controllers, that is, the virtual controller $\alpha_{i-1}(i = 2, \ldots, n)$ should satisfy that $-F_{i1} < \alpha_{i-1} < F_{i2}$ for all time, where F_{i1} and F_{i2} are constraint boundaries. Thus, offline optimization method is used to select a set of optimal design parameters to meet the feasibility of every virtual controller at each recursive step, which increases the computational cost and is undesirable in practical applications. Moreover, if the states are to be constrained in small sets, no optimal parameters could be identified to satisfy the feasibility condition, resulting in the designed controller inapplicable, as pointed out in [9] and [10]. Later, a non-BLF- and non-iBLF-based approach is proposed to cope with full-state constraints in strict-feedback systems without involving feasibility conditions in [11]. Although the obstacle caused by state constraints can be handled gracefully, the tracking performance would be affected and it is difficult to analyze the size of the compact set of tracking error even at steady-state phase. Thus, new techniques need to be developed to tackle the full-state constraints problem for strict-feedback/pure-feedback systems with less restrictive conditions.

Moreover, note that all the above-mentioned results such as [5]–[11] cannot be applied to address the case without

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constraint requirement. This is mainly because the BLF functions in [5]–[10] and constrained functions in [11] are constructed specially for the bounded constraint. When the constraint boundary is infinite, the state can be regarded as unconstrained. However, the employed Lyapunov functions and constrained functions in the above-mentioned references are not equivalent to the general unconstrained case, so that the corresponding methodologies for the design and analysis of these control schemes would fail. On the other hand, although we can artificially add constraints to the unconstrained situation and then regard them as constrained cases uniformly, this will bring unnecessary conservativeness such as excessive control effort. Besides, if the initial values of the system states are not within the added constraint boundaries, these methods are also ineffective. In practical applications, many systems operate under the alternation of constrained and unconstrained cases. One example is that a mobile robot starts from a freely moving area for some time and then enters a narrower (constrained) area for the purpose of, for instance, collision avoidance or target hiding. Therefore, it is meaningful to consider both the constrained and unconstrained cases in a unified way without bring unnecessary conservativeness.

On the other hand, event-triggered control is recently emerged as an alternate control paradigm to reduce unnecessary network communication [12], [13]. Different from the traditional time-triggered control, system data are transmitted only when a specific event is triggered. As a consequence, the transmission load and computation burden can be largely reduced. Some representative results of event-triggered control for nonlinear systems can be found in [14]–[20] and the references therein. In [14] and [15], the event-triggered control schemes are based on the assumption that the closed-loop systems are input-to-state stability (ISS) with respect to measurement error caused by the event-triggering rules. In [16], a new event-triggered control design method is developed for nonlinear uncertain systems in the strict-feedback form from the small-gain point of view. It is particularly shown that the ISS gain with the sampling error as the input can be designed to satisfy the above-mentioned assumption. In [17], three different event-based controller update strategies are proposed for strict-feedback systems with unknown parameters and the ISS assumption is removed by co-designing the adaptive controller and the controller triggering mechanism at the same time. In [18] and [19], neural network (NN)-based adaptive event-triggered controllers are presented for nonlinear continuous-time systems satisfying the Brunovsky canonical form or input-to-state linearizable conditions, respectively. In [20], event-triggered output feedback controllers are developed for a class of uncertain nonlinear systems and a new one-bit signal transmission rule is designed to additionally decrease the communication burden. However, all the above-mentioned results can only guarantee the stability of the closed-loop systems. When the states of the system have constraint requirements, how to design event-based control schemes becomes much harder. So far, to the best of our knowledge, there is still no result investigating state constraints problem with event-triggered control strategy.

Motivated by the above discussions, we will address this problem for a class of uncertain pure-feedback systems. First, in order to satisfy time-varying asymmetric constraints for system states without involving feasibility conditions, we introduce a new universal-constrained function, which is able to address both constrained and unconstrained cases in a unified manner. Then, by using certain transformation techniques, a new unconstrained system is obtained by incorporating the constraint boundaries into the original nonlinear system. However, this results in a circular design problem for the transformed pure-feedback systems, which makes the design and stability analysis still a challenging issue. To address it, we employ a dynamic surface control technique within a general framework of adaptive NN backstepping design approach. In addition, to deal with the event-based input, we introduce an intermediate variable to the system such that we can co-design the controller and the event-triggering mechanism. Then, a novel adaptive controller is designed to compensate for the control sampling errors caused by event-triggered input. The main contributions of this paper are summarized as follows.

1) The design and analysis for both constrained and unconstrained cases are unified without changing the control structure, which have never been developed in other output/state constraints-related literature.
2) An event-triggered control scheme is proposed for systems with state constraints such that not only the required constraints are obeyed for all time but also the communication burden from the controller to the actuator is reduced simultaneously.
3) Different from [11], where the tracking performance would be affected by the constrained states and the size of the compact set of tracking error cannot be analyzed even at steady-state phase, both tracking problem and state constraints can be handled gracefully at the same time here. It is proved that the designed controller has the desired tracking ability, while ensuring system stability and that all the states satisfy the given constraints, if any, without involving feasibility conditions.

II. Problem Formulation and Preliminaries

A. System Description

Consider a class of pure-feedback nonlinear systems given by

$$\begin{aligned} \dot{x}_i &= f_i(\bar{x}_i, x_{i+1}), \quad i = 1, \ldots, n-1 \\ \dot{x}_n &= f_n(\bar{x}_n, u) \\ y &= x_1 \end{aligned} \tag{1}$$

where $\bar{x}_i = [x_1, \ldots, x_i]^T \in \mathbb{R}, i = 1, \ldots, n$ are the state variables; $u \in \mathbb{R}$ is the control input; $y \in \mathbb{R}$ is the system output; $f_i(\bar{x}_i, x_{i+1}), i = 1, \ldots, n-1$, and $f_n(\bar{x}_n, u)$ are unknown nonlinear continuous functions; and $f_n(\bar{x}_n, u)$ is continuously differentiable with respect to u.

The control objective is to design an adaptive NN control law and its event-triggered mechanism such that the following criteria are satisfied.

1) All the signals in the closed-loop system remain uniformly ultimately bounded.

2) The output $y(t)$ follows a desired trajectory $y_d(t)$ and all states are required to remain in the open sets

$$\Omega_{x_1} = \{x_1 \in \mathbb{R} : y_d(t) - F_{11}(t) < x_1(t) < y_d(t) + F_{12}(t)\}$$
$$\Omega_{x_i} = \{x_i \in \mathbb{R} : -F_{i1}(t) < x_i(t) < F_{i2}(t), i = 2, \ldots, n\} \quad (2)$$

at the same time, where $F_{i1}(t)$ and $F_{i2}(t)$ are time-varying strictly positive and C^1 functions.

It is interesting to note that the time-varying constraints (2) involve the following three cases.

Case 1: $F_{i1}(t)$ and $F_{i2}(t)$ are bounded functions, that is, there exist positive constants $\underline{M}_i$ and $\overline{M}_i$ such that $\underline{M}_i < F_{ij}(t) < \overline{M}_i, j = 1, 2$. In this case, the state x_i should meet the constraint requirements *for all time*. One example is that in order for an automatically driving unmanned ground vehicle not to hit other objects including moving vehicles on both sides or not to collide with the front vehicle or be hit by the rear vehicle, its position and velocity must be controlled to strictly obey the corresponding constraints during the entire operation.

Case 2: $F_{i1}(t) = F_{i2}(t) \equiv +\infty$. In this case, there is no constraint requirement on x_i. For example, an unmanned aerial vehicle flying in an open space.

Case 3: $(0 < F_{ij}(t) \le +\infty, j = 1, 2)$. In this case, the state x_i has no constraint requirement over certain time periods. For instance, $F_{ij}(t) = (1/(t - t_0)) + 1$ with t_0 being the initial time instant, which means that the constraint boundaries monotonically decrease from infinity to 1 continuously and there is no constraint requirement on x_i at the initial time. The example of a mobile robot mentioned in the introduction belongs to such a case.

Remark 1: In the existing literature with output and state constraints such as [5]–[11], $F_{ij}(t)$ need to be bounded all the time, that is, only case 1 is considered. If there exists some time instants t_m such that $F_{ij}(t_m) = +\infty$, the proposed control schemes would fail. This is mainly because the designed controllers depend on the boundedness of $F_{ij}(t)$ so that these controllers are only applicable to constrained case but not suitable for unconstrained situation. Whereas, in this paper, by allowing $F_{i1}(t)$ and $F_{i2}(t)$ to be unbounded, more general cases are considered in a unified approach. However, this is at an expense that the problem becomes much more complicated in constructing BLF functions/constrained functions and designing controllers.

To achieve the above-mentioned control objectives, we make the following assumptions.

Assumption 1: The desired trajectory y_d is continuous, its first-order derivative $\dot{y}_d$ and second-order derivative $\ddot{y}_d$ are bounded and available, that is, there exists an unknown constant $B_0 > 0$ such that

$$\Omega_d = \left\{\left[y_d, \dot{y}_d, \ddot{y}_d\right]^T : y_d^2 + \dot{y}_d^2 + \ddot{y}_d^2 \le B_0\right\} \subset \mathbb{R}^3. \quad (3)$$

For simplicity of presentation, we define $g_n(\bar{x}_n, u) = [(\partial f_n(\bar{x}_n, u))/\partial u]$. According to the mean value theorem [22], there exists a constant $0 < \alpha < 1$ such that

$$f_n(\bar{x}_n, u) = f_n(\bar{x}_n, 0) + g_n(\bar{x}_n, \alpha u)u. \quad (4)$$

Assumption 2: The function $g_n(\bar{x}_n, u)$ is bounded, that is, there exist unknown constants $0 < \underline{g}_n < \overline{g}_n < \infty$ such that $\underline{g}_n \le |g_n(\cdot)| \le \overline{g}_n$. Without loss of generality, this paper assumes that $0 < \underline{g}_n \le g_n(\cdot) \le \overline{g}_n$.

Remark 2: Compared with [21], [23], and [24], where the partial derivatives $\partial f_i(\bar{x}_i, x_{i+1})/\partial x_{i+1} (i = 1, \ldots, n-1)$ are also required to be either strictly positive or strictly negative, Assumption 2 relaxes this condition.

B. Neural Networks and Function Approximation

In this paper, radial basis function NNs (RBFNNs) are used to approximate the unknown functions [25], [26]. In other words, any continuous scalar function $\Pi(x)$ in a compact set can be written as

$$\Pi(x) = W^T S(x) + \varepsilon(x) \quad (5)$$

where $W \in \mathbb{R}^N$ is the ideal weight vector with N being the number of nodes, $S(x) = [S_1(x), \ldots, S_N(x)]^T \in \mathbb{R}^N$ is the basis function vector, and $\varepsilon(x)$ denotes the approximation error. We choose the Gaussian functions as $S_i(x)$, which have the form

$$S_i(x) = \exp\left[-\frac{(x - \tau_i)^T (x - \tau_i)}{\psi^2}\right], i = 1, \ldots, N \quad (6)$$

where $\tau_i \in \mathbb{R}^n$ and $\psi \in \mathbb{R}$ are constants called the center and width of the basis function, respectively. Before moving on, we introduce the following assumption.

Assumption 3: The NN approximation error $\varepsilon(x)$ satisfies

$$|\varepsilon(x)| \le \varepsilon_1 \quad (7)$$

where ε_1 is an unknown non-negative constants.

III. Main Results

A. System Transformation

In this section, we first construct the following functions $h_j : \mathbb{R}^2 \to [0, 1], (j = 1, 2)$ to represent the distance between $x(t)$ and the constraint boundaries $-F_1(t)$ and $F_2(t)$.

$$h_1(x, F_1) = \begin{cases} -\left(\frac{x(t)}{F_1(t)}\right)^{2p} + 1, & -F_1(t) < x(t) \le 0 \\ 1, & 0 < x(t) < F_2(t) \end{cases} \quad (8)$$

$$h_2(x, F_2) = \begin{cases} 1, & -F_1(t) < x(t) \le 0 \\ -\left(\frac{x(t)}{F_2(t)}\right)^{2p} + 1, & 0 < x(t) < F_2(t) \end{cases} \quad (9)$$

where $F_1(t)$ and $F_2(t)$ are strictly positive C^1 functions, p is a positive integer satisfying $2p > n$ so as to ensure differentiability of the virtual controller designed later. Note that $\lim_{x\to 0^-} h_j(x, F_j)^{(k)} = \lim_{x\to 0^+} h_j(x, F_j)^{(k)} = 0$ with $j = 1, 2$ and $k = 1, \ldots, n$, thus $h_j(x, F_j)$ are bounded C^n functions.

To carry out full-state constraints, the new constrained function is defined as

$$\xi(x) = \frac{x(t)}{h_1(x, F_1) h_2(x, F_2)}. \quad (10)$$

It is interesting to note that $\xi(x)$ exhibits the following properties: 1) $\xi = 0$ if and only if $x = 0$; 2) $\xi(x) \to \infty$ as $x(t) \to -F_1(t)$ or $x(t) \to F_2(t)$; and 3) when there is no constraint requirement on x, it follows from $F_1 = F_2 = +\infty$ and $h_1(x, F_1) = h_2(x, F_2) = 1$ that $\xi = x$.

Based on these properties of $\xi(x)$, we have the following lemma, which is useful to establish constraint satisfaction.

Lemma 1: For any initial condition satisfying $-F_1(0) < x(0) < F_2(0)$, if $\xi(x) \in L_\infty$, then $-F_1(t) < x(t) < F_2(t)$ holds for all $t \in [0, +\infty)$.

Proof: We give the proofs by contradiction. Suppose that $x(t) = F_3$ for $t = t_1$, where $F_3 \leq -F_1(t_1)$ or $F_3 \geq F_2(t_1)$. Since $-F_1(0) < x(0) < F_2(0)$ and $x(t)$ is a continuous function, according to the intermediate value theorem as in [27], there exists a time instant $0 < t_2 < t_1$ such that $x(t_2) = -F_1(t_2)$ or $x(t_2) = F_2(t_2)$. As a result $h_1(x(t_2), F_1(t_2)) = 0$ or $h_2(x(t_2), F_2(t_2)) = 0$, and thus $\xi(x(t_2)) = \infty$, which leads to a contradiction for the boundedness of $\xi(x)$. Therefore, we know $-F_1(t) < x(t) < F_2(t), \forall t \in [0, +\infty)$. This completes the proof. ■

Remark 3: When $F_1 = F_2 = +\infty$, it follows from the property 3) of (10) that the constructed constrained function $\xi(x)$ is x, which is equivalent to the general unconstrained situation. Thus, the newly introduced constrained function (10) can also be applied to systems with unbounded constraints, which enables us to consider the cases both with and without state constraints in a unified manner.

From $z_1 = x_1 - y_d$, the constraint of x_1 in (2) is equal to the constraint of z_1, that is, $-F_{11}(t) < z_1 < F_{12}(t)$. Now we introduce a new variable

$$\xi_1 = \frac{z_1}{h_{11}(z_1, F_{11}) h_{12}(z_1, F_{12})} \tag{11}$$

where $h_{11}(z_1, F_{11}) = h_1(z_1, F_{11})$ and $h_{12}(z_1, F_{12}) = h_2(z_1, F_{12})$. Throughout this paper, for ease notation, we abbreviate $h_{11}(z_1, F_{11})$ and $h_{12}(z_1, F_{12})$ by h_{11} and h_{12}, respectively. With the aid of Lemma 1, it can be concluded that for any initial condition satisfying $y_d(0) - F_{11}(0) < x_1(0) < y_d(0) + F_{12}(0)$, the constraint of z_1 is naturally ensured if we are able to keep ξ_1 bounded for $t \in [0, +\infty)$. Differentiating (11) with respect to time, we obtain

$$\dot{\xi}_1 = \mu_1 \big[f_1(x_1, x_2) - \dot{y}_d \big] + \nu_1 \tag{12}$$

where

$$\begin{aligned}
\mu_1 &= \frac{h_{11} h_{12} - b_{11} z_1}{h_{11}^2 h_{12}^2} \\
b_{11} &= -\frac{2p z_1^{2p-1} c(z_1)}{F_{11}^{2p}} - \frac{2p z_1^{2p-1} (1 - c(z_1))}{F_{12}^{2p}} \\
\nu_1 &= -\frac{b_{12} z_1}{h_{11}^2 h_{12}^2} \\
b_{12} &= \frac{2p z_1^{2p} \dot{F}_{11} c(z_1)}{F_{11}^{2p+1}} + \frac{2p z_1^{2p} \dot{F}_{12} (1 - c(z_1))}{F_{12}^{2p+1}} \\
c(z_1) &= \begin{cases} 1, & \text{if} - F_{11}(t) \leq z_1(t) \leq 0 \\ 0, & \text{if } \ 0 < z_1(t) \leq F_{12}(t). \end{cases}
\end{aligned} \tag{13}$$

Similarly, to further cope with the state constraints of $x_i (i = 2, \ldots, n)$, we introduce new transformed states

$$\xi_i = \frac{x_i}{h_{i1}(x_i, F_{i1}) h_{i2}(x_i, F_{i2})}, \quad i = 2, \ldots, n \tag{14}$$

where $h_{i1}(x_i, F_{i1}) = h_1(x_i, F_{i1})$ and $h_{i2}(x_i, F_{i2}) = h_2(x_i, F_{i2})$. For ease notation, we abbreviate $h_{i1}(x_i, F_{i1})$ and $h_{i2}(x_i, F_{i2})$ by h_{i1} and h_{i2}, respectively. According to Lemma 1, the problem of satisfying prespecified constraint bound boils down to ensure the boundedness of ξ_i for all $t > 0$. After some calculation, a new transformed system dynamic is provided by

$$\begin{aligned}
\dot{\xi}_1 &= \mu_1 \big[\Gamma_1(x_1, x_2, \xi_2) + \xi_2 - \dot{y}_d \big] + \nu_1 \\
\dot{\xi}_i &= \mu_i \big[\Gamma_i(\bar{x}_{i+1}, \xi_{i+1}) + \xi_{i+1} \big] + \nu_i, \quad i = 2, \ldots, n-1 \\
\dot{\xi}_n &= \mu_n \big[\Gamma_n(\bar{x}_n) + g_n u \big] + \nu_n
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
\Gamma_i(\bar{x}_{i+1}, \xi_{i+1}) &= f_i(\bar{x}_i, x_{i+1}) - \xi_{i+1}, \quad i = 1, \ldots, n-1 \\
\Gamma_n(\bar{x}_n) &= f_n(\bar{x}_n), \ \ \mu_i = \frac{h_{i1} h_{i2} - b_{i1} x_i}{h_{i1}^2 h_{i2}^2} \\
b_{i1} &= -\frac{2p x_i^{2p-1} c(x_i)}{F_{i1}^{2p}} - \frac{2p x_i^{2p-1} (1 - c(x_i))}{F_{i2}^{2p}} \\
\nu_i &= -\frac{b_{i2} x_i}{h_{i1}^2 h_{i2}^2} \\
b_{i2} &= \frac{2p x_i^{2p} \dot{F}_{i1} c(x_i)}{F_{i1}^{2p+1}} + \frac{2p x_i^{2p} \dot{F}_{i2} (1 - c(x_i))}{F_{i2}^{2p+1}} \\
c(x_i) &= \begin{cases} 1, & \text{if} - F_{i1}(t) \leq x_i(t) \leq 0 \\ 0, & \text{if } \ 0 < x_i(t) \leq F_{i2}(t). \end{cases}
\end{aligned} \tag{16}$$

In (16), both μ_i and ν_i are computable and continuously differentiable, and thus they can be included in the control design. Clearly, the original-constrained system (1) is transformed into the new unconstrained system (15), whose stability can ensure the asymmetric time-varying state constraints given by (2). Now, we only need to focus on designing controller u to stabilize the transformed system (15).

B. Control Design and Stability Analysis

In this section, we will combine the NN-based adaptive dynamic surface control with event-triggered control strategy. To facilitate the control design, we introduce the following transformation of coordination:

$$\zeta_1 = \xi_1, \quad \zeta_i = \xi_i - \alpha_{if}, i = 2, \ldots, n \tag{17}$$

where α_{if} is the output of a first-order filter

$$l_i \dot{\alpha}_{if} + \alpha_{if} = \alpha_{i-1} \tag{18}$$

with l_i being a positive design parameter and α_{i-1} as the input. Note that α_{i-1} is an intermediate control which shall be developed at the $(i-1)$th step of the design procedure to be presented. Finally, the overall control law u and corresponding event-triggering rules are constructed at step n. For simplicity of presentation, we define $y_i = \alpha_{if} - \alpha_{i-1}$. Also denote $\bar{y}_d^{(2)} = [y_d, \dot{y}_d, \ddot{y}_d]^T$, $\bar{\zeta}_i = [\zeta_1, \ldots, \zeta_i]$, $\bar{x}_i = [x_1, \ldots, x_i]$, $\bar{y}_i = [y_1, \ldots, y_i]$, and $\hat{\bar{\vartheta}}_i = [\hat{\vartheta}_1, \ldots, \hat{\vartheta}_i]$.

Step 1: According to the definition of ζ_i and y_i, we have

$$\xi_i = \zeta_i + y_i + \alpha_{i-1}, i = 2, \ldots, n. \tag{19}$$

Differentiating ζ_1 with respect to time, it follows that

$$\dot{\zeta}_1 = \mu_1\big[\Gamma_1(x_1, x_2, \xi_2) + \zeta_2 + y_2 + \alpha_1 - \dot{y}_d\big] + \nu_1. \quad (20)$$

Then, the derivative of $(1/2)\zeta_1^2$ along (20) is

$$\zeta_1\dot{\zeta}_1 = \mu_1\big[\zeta_1\Gamma_1(x_1, x_2, \xi_2) + \zeta_1\zeta_2 + \zeta_1 y_2 + \zeta_1\alpha_1 - \zeta_1\dot{y}_d\big] + \zeta_1\nu_1. \quad (21)$$

Since $\Gamma_1(x_1, x_2, \xi_2)$ is an unknown and continuous function, we employ RBFNNs to approximate it as

$$\begin{aligned}\Gamma_1(x_1, x_2, \xi_2) &= W_1^T S(X_1) + \varepsilon_1(X_1)\\ &\le \left\|W_1^T\right\| \cdot \|S(X_1)\| + \varepsilon_{11}\\ &\le a_1\phi_1(X_1)\end{aligned} \quad (22)$$

where $X_1 = [x_1, x_2, \xi_2]^T$, $|\varepsilon_1(X_1)| \le \varepsilon_{11}$, $a_1 = \max\{\|W_1^T\|, \varepsilon_{11}\}$, $\phi_1(X_1) = \|S(X_1)\| + 1$. By using the Young's inequality as in [28], we have

$$\begin{aligned}\mu_1\zeta_1\Gamma_1 &\le \mu_1|\zeta_1|a_1\phi_1 \le r_1 a_1^2\mu_1^2\phi_1^2\zeta_1^2 + \frac{1}{4r_1}\\ \mu_1\zeta_1 y_2 &\le \mu_1^2\zeta_1^2 + \frac{y_2^2}{4}.\end{aligned} \quad (23)$$

Substituting (23) into (21), it becomes

$$\begin{aligned}\zeta_1\dot{\zeta}_1 \le{}& r_1\vartheta_1\mu_1^2\phi_1^2\zeta_1^2 + \mu_1^2\zeta_1^2 + \mu_1\big[\zeta_1\zeta_2 + \zeta_1\alpha_1 - \zeta_1\dot{y}_d\big]\\ &+ \zeta_1\nu_1 + \frac{y_2^2}{4} + \frac{1}{4r_1}\end{aligned} \quad (24)$$

where $\vartheta_1 = a_1^2$. According to (13), we have $\mu_1 > 0$. Hence, the virtual control law α_1 and the adaption law can be constructed as

$$\begin{aligned}\alpha_1 &= -k_1\frac{\zeta_1}{\mu_1} - r_1\hat{\vartheta}_1\mu_1\phi_1^2\zeta_1 - \mu_1\zeta_1 + \dot{y}_d - \frac{\nu_1}{\mu_1}\\ \dot{\hat{\vartheta}}_1 &= -\sigma_1\hat{\vartheta}_1 + r_1\mu_1^2\phi_1^2\zeta_1^2\end{aligned} \quad (25)$$

where $k_1 > 0$, $\sigma_1 > 0$, and $r_1 > 0$ are the positive design parameters and $\hat{\vartheta}_1$ is the estimation of ϑ_1. Now we define a Lyapunov function candidate

$$V_1 = \frac{1}{2}\zeta_1^2 + \frac{1}{2}\tilde{\vartheta}_1 + \frac{1}{2}y_2^2 \quad (26)$$

where $\tilde{\vartheta}_1 = \hat{\vartheta}_1 - \vartheta_1$. Differentiating (26) and combining (24) and (25), we have

$$\begin{aligned}\dot{V}_1 \le{}& -k_1\zeta_1^2 - r_1\tilde{\vartheta}_1\mu_1^2\phi_1^2\zeta_1^2 + \mu_1\zeta_1\zeta_2 + \tilde{\vartheta}_1\dot{\hat{\vartheta}}_1 + y_2\dot{y}_2\\ &+ \frac{y_2^2}{4} + \frac{1}{4r_1}\\ \le{}& -k_1\zeta_1^2 - \frac{\sigma_1}{2}\tilde{\vartheta}_1^2 + \frac{\sigma_1}{2}\vartheta_1^2 + \frac{y_2^2}{4} + \frac{1}{4r_1}\\ &+ \mu_1\zeta_1\zeta_2 + y_2\dot{y}_2\end{aligned} \quad (27)$$

and the second inequality holds due to the result $-\tilde{\vartheta}_1\hat{\vartheta}_1 = -\tilde{\vartheta}_1(\tilde{\vartheta}_1 + \vartheta) \le -\tilde{\vartheta}_1^2 + (1/2)\tilde{\vartheta}_1^2 + (1/2)\vartheta_1^2 \le -(1/2)\tilde{\vartheta}_1^2 + (1/2)\vartheta_1^2$. Recalling the definition of y_i and the first-order filter (18), it is easy to obtain that

$$\dot{y}_2 = -\frac{y_2}{l_2} - \dot{\alpha}_1 = -\frac{y_2}{l_2} + w_2(\cdot) \quad (28)$$

where $w_2(\bar{\zeta}_3, \bar{x}_3, \bar{y}_3, \bar{\hat{\vartheta}}_2, \bar{y}_d^{(2)}) = -[(\partial\alpha_1)/(\partial\zeta_1)]\dot{\zeta}_1 - [(\partial\alpha_1)/(\partial\hat{\vartheta}_1)]\dot{\hat{\vartheta}}_1 - [(\partial\alpha_1)/(\partial\phi_1)]\dot{\phi}_1 - [(\partial\alpha_1)/(\partial\mu_1)]\dot{\mu}_1 - [(\partial\alpha_1)/(\partial\nu_1)]\dot{\nu}_1 - [(\partial\alpha_1)/(\partial\dot{y}_d)]\ddot{y}_d$ is a continuous function. Then, it follows that

$$y_2\dot{y}_2 = \frac{y_2^2}{l_2} + w_2 y_2 \le -\frac{y_2^2}{l_2} + \frac{y_2^2}{4} + w_2^2. \quad (29)$$

By choosing $(1/l_2) > l_2^* + (1/2)$ with l_2^* being a positive design parameter and using (29), (27) becomes

$$\dot{V}_1 \le -k_1\zeta_1^2 - \frac{\sigma_1}{2}\tilde{\vartheta}_1^2 - l_2^* y_2^2 + \mu_1\zeta_1\zeta_2 + w_2^2 + \Theta_1 \quad (30)$$

where $\Theta_1 = (\sigma_1/2)\vartheta_1^2 + (1/4r_1)$ and $\mu_1\zeta_1\zeta_2$ will be handled in the next step.

Remark 4: Now we pause to stress that when NNs are used to approximate the unknown nonaffine function $\Gamma_1(\bar{x}_2, \zeta_2)$ at the first step, it usually lead to a circular design problem at the second step as pointed out in [23], which is due to the repeated differentiations of virtual control α_1 in the backstepping design. Here, by using the dynamic surface control technique, the operation of differentiation is replaced by the simpler algebraic operation in (18) such that both the circular design problem and the explosion of complexity problem for control laws are eliminated at the same time.

Step i ($i = 2, \ldots, n-1$): The time derivative of $\zeta_i = \xi_i - \alpha_{if}$ is

$$\dot{\zeta}_i = \mu_i\big[\Gamma_i(\bar{x}_{i+1}, \xi_{i+1}) + \zeta_{i+1} + y_{i+1} + \alpha_i\big] + \nu_i - \dot{\alpha}_{if}. \quad (31)$$

Since $\Gamma_i(\bar{x}_{i+1}, \xi_{i+1})$ is a continuous function, there exists an ideal NN approximation such that

$$\Gamma_i(\bar{x}_{i+1}, \xi_{i+1}) = W_i^T S(X_i) + \varepsilon_i(X_i) \le a_i\phi_i(X_i) \quad (32)$$

where $X_i = [\bar{x}_{i+1}, \xi_{i+1}]^T$, $|\varepsilon_i(X_i)| \le \varepsilon_{i1}$, $a_i = \max\{\|W_i^T\|, \varepsilon_{i1}\}$, and $\phi_i(X_i) = \|S(X_i)\| + 1$. By using the Young's inequality and following the analysis similar to (24), we have

$$\begin{aligned}\zeta_i\dot{\zeta}_i \le{}& r_i\vartheta_i\mu_i^2\phi_i^2\zeta_i^2 + \mu_i^2\zeta_i^2 + \mu_i\zeta_i\zeta_{i+1} + \mu_i\zeta_i\alpha_i + \zeta_i\nu_i\\ &- \zeta_i\dot{\alpha}_{if} + \frac{y_{i+1}^2}{4} + \frac{1}{4r_i}.\end{aligned} \quad (33)$$

Then, the virtual control law α_i and the adaption law are designed as

$$\begin{aligned}\alpha_i &= -k_i\frac{\zeta_i}{\mu_i} - r_i\hat{\vartheta}_i\mu_i\phi_i^2\zeta_i - \mu_i\zeta_i - \frac{\nu_i}{\mu_i} + \frac{\dot{\alpha}_{if}}{\mu_i} - \frac{\mu_{i-1}}{\mu_i}\zeta_{i-1}\\ \dot{\hat{\vartheta}}_i &= -\sigma_i\hat{\vartheta}_i + r_i\mu_i^2\phi_i^2\zeta_i^2\end{aligned} \quad (34)$$

where $k_i > 0$, $\sigma_i > 0$, and $r_i > 0$ are the positive design parameters and $\hat{\vartheta}_i$ is the estimation of $\vartheta_i = a_i^2$. Define a positive definite Lyapunov function

$$V_i = \frac{1}{2}\zeta_i^2 + \frac{1}{2}\tilde{\vartheta}_i^2 + \frac{1}{2}y_{i+1}^2 \quad (35)$$

where $\tilde{\vartheta}_i = \hat{\vartheta}_i - \vartheta_i$. It follows that

$$\begin{aligned}\dot{V}_i \le{}& -k_i\zeta_i^2 - \frac{\sigma_i}{2}\tilde{\vartheta}_i^2 + \mu_i\zeta_i\zeta_{i+1} - \mu_{i-1}\zeta_{i-1}\zeta_i + y_{i+1}\dot{y}_{i+1}\\ &+ \frac{y_{i+1}^2}{4} + \frac{1}{4r_i} + \frac{\sigma_i}{2}\vartheta_i^2.\end{aligned} \quad (36)$$

Note that $\dot{y}_{i+1} = \dot{\alpha}_{i+1,f} - \dot{\alpha}_i = -[(y_{i+1})/(l_{i+1})] + w_{i+1}(\cdot)$, where $w_{i+1}(\cdot) = -[(\partial\alpha_i)/(\partial\zeta_i)]\dot{\zeta}_i - [(\partial\alpha_i)/(\partial\hat{\vartheta}_i)]\dot{\hat{\vartheta}}_i - [(\partial\alpha_i)/(\partial\phi_i)]\dot{\phi}_i - [(\partial\alpha_i)/(\partial\mu_i)]\dot{\mu}_i - [(\partial\alpha_i)/(\partial\mu_{i-1})]\dot{\mu}_{i-1} - [(\partial\alpha_i)/(\partial\nu_i)]\dot{\nu}_i$, then we have

$$y_{i+1}\dot{y}_{i+1} = -\frac{y_{i+1}^2}{l_{i+1}} + w_{i+1}y_{i+1} \le -\frac{y_{i+1}^2}{l_{i+1}} + \frac{y_{i+1}^2}{4} + w_{i+1}^2. \tag{37}$$

Substituting (37) into (36) generates

$$\begin{aligned}\dot{V}_i \le& -k_i\zeta_i^2 - \frac{\sigma_i}{2}\tilde{\vartheta}_i^2 + \mu_i\zeta_i\zeta_{i+1} - \mu_{i-1}\zeta_{i-1}\zeta_i \\ &- \left(\frac{1}{l_{i+1}} - \frac{1}{2}\right)y_{i+1}^2 + \frac{1}{4r_i} + \frac{\sigma_i}{2}\vartheta_i^2 + w_{i+1}^2. \end{aligned}\tag{38}$$

By choosing $[1/(l_{i+1})] > l_{i+1}^* + (1/2)$ with l_{i+1}^* being a positive design parameter, (38) can be rewritten as

$$\begin{aligned}\dot{V}_i \le& -k_i\zeta_i^2 - \frac{\sigma_i}{2}\tilde{\vartheta}_i^2 - l_{i+1}^* y_{i+1}^2 + \mu_i\zeta_i\zeta_{i+1} - \mu_{i-1}\zeta_{i-1}\zeta_i \\ &+ \Theta_i + w_{i+1}^2\end{aligned}\tag{39}$$

where $\Theta_i = (1/4r_i) + (\sigma_i/2)\vartheta_i^2$.

Step n: Now we are able to design the event-triggering rule as well as the NN-based adaptive controller. Inspired by the relative threshold strategy in [17], we will design a varying threshold for the triggering event which ensures more precise control signal applied to the system. Based on this consideration, we co-design the control law and the event-triggered strategy as follows:

$$\begin{aligned}u_s(t) =& -k_n(1+\delta_1)\frac{\zeta_n}{\mu_n} - r_n\hat{\vartheta}_n\mu_n\phi_n^2\zeta_n - \left(\frac{m_1}{1-\delta_1}\right)^2\mu_n\zeta_n \\ &- \frac{\nu_n^2\zeta_n}{\mu_n} - \frac{\dot{\alpha}_{nf}^2\zeta_n}{\mu_n} - \frac{\mu_{n-1}^2}{\mu_n}\zeta_{n-1}^2\zeta_n \end{aligned}\tag{40}$$

$$\dot{\hat{\vartheta}}_n = -\sigma_n\hat{\vartheta}_n + \frac{r_n}{1+\delta_1}\mu_n^2\phi_n^2\zeta_n^2 \tag{41}$$

$$u(t) = u_s(t_k), \forall t \in [t_k, t_{k+1}) \tag{42}$$

$$t_{k+1} = \inf\{t \in \mathbb{R}^+ |\ |u_s(t) - u(t)| \ge \delta_1|u(t)| + m_1\} \tag{43}$$

where $k_n, \sigma_n, r_n, 0 < \delta_1 < 1, m_1 > 0$ are all positive design parameters, $\hat{\vartheta}_n$ is the estimation of $\vartheta_n = a_n^2$, and $t_{k+1}, k \in \mathbb{Z}$ is the triggering time. Whenever (43) is triggered, the control value is updated to $u_s(t_{k+1})$ and the updated control signal is transmitted from the controller to the actuator. For all $t \in (t_k, t_{k+1})$, the control signal holds as a constant $u_s(t_k)$ and no signal is transmitted from the controller to the actuator.

Theorem 1: Consider the uncertain pure-feedback nonlinear system (1) subject to asymmetric time-varying full-state constraint (2). Suppose that Assumptions 1–3 hold, if the controller (40)–(42) with event-triggering rule (43) is applied, then for any initial condition satisfying $x_i(0) \in \Omega_{x_i} (i = 1, \dots, n)$ in (2), there exist design parameters $k_i > 0, r_i > 0, p > 0, \sigma_i > 0 (i = 1, \dots, n)$, and $l_j > 0 (j = 2, \dots, n)$, such that the following objectives are achieved.

$\mathscr{O}_1$) The closed-loop system is stable and all the signals are uniformly bounded.

$\mathscr{O}_2$) The tracking error $z_1(t)$ is bounded by $-F_{11}(t) < z_1(t) < F_{12}(t)$ for all time and enters the set $\Omega_{z_1} = \{|z_1| |z_1| \le \sqrt{(2\Lambda/\gamma)}\}$ as time goes by, where $F_{11}(t)$ and $F_{12}(t)$ represent the transient performance bounds.

$\mathscr{O}_3$) The asymmetric time-varying state constraints are satisfied at all times, that is, $x_i(t) \in \Omega_{x_i} (i = 1, \dots, n)$, $\forall t \ge 0$.

$\mathscr{O}_4$) The Zeno behavior is avoided in the sense that there exists a time $t^* > 0$ such that the interexecution intervals $t_{k+1} - t_k$ are lower bounded by $t^*, \forall k \in \mathbb{Z}^+$.

Proof: Since the value of control signal is updated to $u_s(t_{k+1})$ at the triggering time t_{k+1}, then from the event-triggered condition (43), we have $|u_s(t) - u(t)| \le \delta_1|u(t)| + m_1$ holds for all t. In the case that $u(t) \ge 0$, we get $-\delta_1 u(t) - m_1 \le u_s(t) - u(t) \le \delta_1 u(t) + m_1$. This further implies that there exist time-varying parameters satisfying $|\lambda_1(t)| \le 1$ and $|\lambda_2(t)| \le 1$ such that $u_s(t) - u(t) = \lambda_1(t)\delta_1 u(t) + \lambda_2(t)m_1$. Similar results can also be obtained for $u(t) < 0$, thus

$$u_s(t) = (1 + \lambda_1(t)\delta_1)u(t) + \lambda_2(t)m_1, \quad \forall t \in [t_k, t_{k+1}]. \tag{44}$$

It follows that

$$u(t) = \frac{u_s(t)}{1+\lambda_1(t)\delta_1} - \frac{\lambda_2(t)m_1}{1+\lambda_1(t)\delta_1}. \tag{45}$$

Now with the help of (15) and the definition that $\zeta_n = \xi_n - \alpha_{nf}$, we have

$$\begin{aligned}\dot{\zeta}_n =& \mu_n\left[\Gamma_n(\bar{x}_n) + \frac{g_n}{1+\lambda_1(t)\delta_1}u_s - \frac{\lambda_2(t)m_1}{1+\lambda_1(t)\delta_1}g_n\right] \\ &+ \nu_n - \dot{\alpha}_{nf}.\end{aligned}\tag{46}$$

Since $\Gamma_n(\bar{x}_n)$ is a continuous function, there exists an ideal NN approximation such that

$$\Gamma_n(\bar{x}_n) = W_n^T S(X_n) + \varepsilon_i(X_n) \le a_n\phi_n(X_n) \tag{47}$$

where $X_n = [\bar{x}_n]^T$, $|\varepsilon_n(X_n)| \le \varepsilon_{n1}$, $a_n = \max\{\|W_n^T\|, \varepsilon_{n1}\}$, and $\phi_n(X_n) = \|S(X_n)\| + 1$. Define a positive definite Lyapunov function

$$V_n = \frac{1}{2}\zeta_n^2 + \frac{\underline{g}_n}{2}\tilde{\vartheta}_n^2 \tag{48}$$

where $\tilde{\vartheta}_n = \hat{\vartheta}_n - \vartheta_n$. Then, the time derivative of V_n is

$$\begin{aligned}\dot{V}_n =& \mu_n\Gamma_n(\bar{x}_n)\zeta_n - \frac{\lambda_2(t)m_1}{1+\lambda_1(t)\delta_1}\mu_n g_n\zeta_n + \frac{g_n}{1+\lambda_1(t)\delta_1}\mu_n u_s\zeta_n \\ &+ \nu_n\zeta_n - \dot{\alpha}_{nf}\zeta_n + \underline{g}_n\tilde{\vartheta}_n\dot{\hat{\vartheta}}_n.\end{aligned}\tag{49}$$

By using the Young's inequality, we have

$$\begin{aligned}\mu_n\Gamma_n\zeta_n &\le \mu_n a_n\phi_n|\zeta_n| \le \frac{\underline{g}_n}{1+\delta_1}r_n\vartheta_n\mu_n^2\phi_n^2\zeta_n^2 + \frac{1+\delta_1}{4r_n\underline{g}_n} \\ \nu_n\zeta_n &\le \frac{\underline{g}_n}{1+\delta_1}\nu_n^2\zeta_n^2 + \frac{1+\delta_1}{4\underline{g}_n} \\ -\dot{\alpha}_{nf}\zeta_n &\le \frac{\underline{g}_n}{1+\delta_1}\dot{\alpha}_{nf}^2\zeta_n^2 + \frac{1+\delta_1}{4\underline{g}_n}.\end{aligned}\tag{50}$$

According to (40), it is obvious that $\mu_n u_s\zeta_n \le 0$. With the aid of $|\lambda_i(t)| \le 1 (i = 1, 2)$, we obtain

$$\begin{aligned}&\frac{g_n}{1+\lambda_1(t)\delta_1}\mu_n u_s\zeta_n \le \frac{\underline{g}_n}{1+\delta_1}\mu_n u_s\zeta_n \le -k_n\underline{g}_n\zeta_n^2 \\ &-\frac{\underline{g}_n}{1+\delta_1}r_n\hat{\vartheta}_n\mu_n^2\phi_n^2\zeta_n^2 - \frac{\underline{g}_n}{1+\delta_1}\left(\frac{m_1}{1-\delta_1}\right)^2\mu_n^2\zeta_n^2 \\ &-\frac{\underline{g}_n}{1+\delta_1}\dot{\alpha}_{nf}^2\zeta_n^2 - \frac{\underline{g}_n}{1+\delta_1}\mu_{n-1}^2\zeta_{n-1}^2\zeta_n^2 - \frac{\underline{g}_n}{1+\delta_1}\nu_n^2\zeta_n^2\end{aligned}\tag{51}$$

$$-\frac{\lambda_2(t)m_1}{1+\lambda_1(t)\delta_1}\mu_n g_n \zeta_n \leq \mu_n \bar{g}_n |\zeta_n| \frac{m_1}{1-\delta_1} \leq \frac{\underline{g}_n}{1+\delta_1}\left(\frac{m_1}{1-\delta_1}\right)^2 \mu_n^2 \zeta_n^2 + \frac{\bar{g}_n^2(1+\delta_1)}{4\underline{g}_n}. \tag{52}$$

Substituting (41) and (50)–(52) into (49), it follows that

$$\begin{aligned}\dot{V}_n &\leq -k_n \underline{g}_n \zeta_n^2 - \frac{\underline{g}_n}{1+\delta_1} r_n \tilde{\vartheta}_n \mu_n^2 \phi_n^2 \zeta_n^2 + \frac{1+\delta_1}{4 r_n \underline{g}_n} + \frac{1+\delta_1}{2\underline{g}_n} \\ &\quad + \frac{\bar{g}_n^2(1+\delta_1)}{4\underline{g}_n} + \underline{g}_n \tilde{\vartheta}_n \dot{\hat{\vartheta}}_n \\ &\leq -k_n \underline{g}_n \zeta_n^2 - \frac{1}{2}\underline{g}_n \sigma_n \tilde{\vartheta}_n^2 - \frac{\underline{g}_n}{1+\delta_1}\mu_{n-1}^2 \zeta_{n-1}^2 \zeta_n^2 + \Theta_n \end{aligned} \tag{53}$$

where $\Theta_n = [(1+\delta_1)/(4 r_n \underline{g}_n)] + [(3(1+\delta_1))/(4\underline{g}_n)] + [(\bar{g}_n^2(1+\delta_1))/(4\underline{g}_n)]$. Now we consider the following Lyapunov function:

$$V = V_1 + \cdots + V_n. \tag{54}$$

It is easy to obtain that

$$\begin{aligned}\dot{V} &\leq -\sum_{i=1}^{n-1} k_i \zeta_i^2 - \sum_{i=1}^{n-1}\frac{\sigma_i}{2}\tilde{\vartheta}_i^2 - \sum_{i=2}^{n} l_i^* y_i^2 + \sum_{i=2}^{n} w_i - k_n \underline{g}_n \zeta_n^2 \\ &\quad - \frac{1}{2}\underline{g}_n \sigma_n \tilde{\vartheta}_n^2 - \frac{\underline{g}_n}{1+\delta_1}\mu_{n-1}^2 \zeta_{n-1}^2 \zeta_n^2 + \mu_{n-1}\zeta_{n-1}\zeta_n + \Theta \\ &\leq -\sum_{i=1}^{n-1} k_i \zeta_i^2 - \sum_{i=1}^{n-1}\frac{\sigma_i}{2}\tilde{\vartheta}_i^2 - \sum_{i=2}^{n} l_i^* y_i^2 - k_n \underline{g}_n \zeta_n^2 \\ &\quad - \frac{1}{2}\underline{g}_n \sigma_n \tilde{\vartheta}_n^2 + \sum_{i=2}^{n} w_i + \frac{1+\delta_1}{4\underline{g}_n} + \Theta \end{aligned} \tag{55}$$

where $\Theta = \Theta_1 + \cdots + \Theta_n$ and the second inequality holds due to the result that $\mu_{n-1}\zeta_{n-1}\zeta_n \leq [(\underline{g}_n)/(1+\delta_1)]\mu_{n-1}^2\zeta_{n-1}^2\zeta_n^2 + [(1+\delta_1)/(4\underline{g}_n)]$.

Now we define the following compact sets $\Omega_V = \{\sum_{i=1}^{n}\zeta_i^2 + \sum_{i=1}^{n}\tilde{\vartheta}_i + \sum_{i=2}^{n} y_i \leq 2p\} \subset \mathbb{R}^{3n-1}$, $\Omega_d = \{[y_d, \dot{y}_d, \ddot{y}_d]^T : y_d^2 + \dot{y}_d^2 + \ddot{y}_d^2 \leq B_0\} \subset \mathbb{R}^3$, then there exists positive constant Π_i such that $|w_i| \leq \Pi_i$ on compact set $\Omega_V \times \Omega_d$. Hence,

$$\begin{aligned}\dot{V} &\leq -\sum_{i=1}^{n-1} k_i \zeta_i^2 - \sum_{i=1}^{n-1}\frac{\sigma_i}{2}\tilde{\vartheta}_i^2 - \sum_{i=2}^{n} l_i^* y_i^2 - k_n \underline{g}_n \zeta_n^2 \\ &\quad - \frac{1}{2}\underline{g}_n \sigma_n \tilde{\vartheta}_n^2 + \sum_{i=2}^{n}\Pi_i + \frac{1+\delta_n}{4\underline{g}_n} + \Theta \\ &\leq -\gamma V + \Lambda \end{aligned} \tag{56}$$

where $\gamma = \min_{i=1,\ldots,n-1}\{\sigma_i, \sigma_n, (k_i/2), [(k_n \underline{g}_n)/2], [(l_{i+1}^*)/2]\}$ and $\Lambda = \sum_{i=2}^{n}\Pi_i + [(1+\delta_n)/(4\underline{g}_n)] + \Theta$. Let $\gamma > (\Lambda/p)$, then $\dot{V} \leq 0$ on $V = p$. Therefore, $V \leq p$ is an invariant set, that is, if $V(0) \leq p$, then $V(t) \leq p$ for $t \geq 0$. Now we are ready to establish the results of the theorem as follows.

First, we show that objective $\mathscr{O}_1$ is achieved. It can be concluded from (56) that V enters and remains in the set $\Omega = \{V||V| \leq (\Lambda/\gamma)\}$ after a finite time T_0, thus the signals ζ_i and $\tilde{\vartheta}_i$ are ultimately uniformly bounded (UUB) [29]. By integrating both sides of the inequality (56) we obtain that

$$V(t) \leq e^{-\gamma t}V(0) + \frac{\Lambda}{\gamma}\left(1 - e^{-\gamma t}\right) \leq V(0) + \frac{\Lambda}{\gamma}. \tag{57}$$

Therefore, $V \in L_\infty$ for any bounded initial conditions, which indicates that $\zeta_i \in L_\infty$, $\tilde{\vartheta}_i \in L_\infty$, and $y_i \in L_\infty$. Based on this result, we can conclude that all the internal signals are bounded.

Second, we prove that objective $\mathscr{O}_2$ is obtained. According to the condition that $y_d(0) - F_{11}(0) < x_1(0) < y_d(0) + F_{12}(0)$, we have $-F_{11}(0) < z_1(0) < F_{12}(0)$. Note that $\zeta_1(t) = \xi_1(t) = [z_1(t)/(h_{11}(z_1, F_{11})h_{12}(z_1, F_{12}))] \in L_\infty$, then from Lemma 1, it follows that $-F_{11}(t) < z_1(t) < F_{12}(t)$. On the other hand, one can conclude from (57) that ζ_1 enters the set $\Omega_{\zeta_1} = \{|\zeta_1||\zeta_1| \leq \sqrt{(2\Lambda/\gamma)}\}$ as time goes by. According to the fact that $h_{11}(z_1, F_{11})h_{12}(z_1, F_{12}) \leq 1$, it can be concluded that z_1 also enters the set $\Omega_{z_1} = \{|z_1||z_1| \leq \sqrt{(2\Lambda/\gamma)}\}$ as time goes by.

Next, we show that objective $\mathscr{O}_3$ is derived. Since $\zeta_1 \in L_\infty$ and $-F_{11}(t) < z_1(t) < F_{12}(t)$, it follows that Φ_1, μ_1, $\hat{\vartheta}$, and α_1 are bounded. As y_2 is bounded, then from (18) and the definition $y_2 = \alpha_{2f} - \alpha_1$, we have that α_{2f} and $\dot{\alpha}_{2f}$ are bounded. According to the definition of ζ_2 in (17), we obtain ξ_2 is bounded. Together with Lemma 1, we infer that $-F_{21}(t) < x_2(t) < F_{22}(t)$ for any initial condition satisfying $-F_{21}(0) < x_2(0) < F_{22}(0)$. Similarly, it is also easy to obtain that $x_i(t) \in \Omega_{x_i} (i = 3, \ldots, n)$ for $x_i(0) \in \Omega_{x_i} (i = 3, \ldots, n)$. Therefore, the asymmetric time-varying state constraints are satisfied at all times.

Finally, we show that objective $\mathscr{O}_4$ is achieved. By recalling the control sampling error $e(t) = u_s(t) - u(t), \forall t \in [t_k, t_{k+1})$, we have

$$\frac{d}{dt}|e| = \frac{d}{dt}(e * e)^{\frac{1}{2}} = \mathrm{sign}(e)\dot{e} \leq |\dot{u}_s|. \tag{58}$$

Based on (1), $\dot{x}_i$ is a smooth function of bounded signals, which further implies $\dot{x}_i \in L_\infty$, $\dot{\alpha}_i \in L_\infty$, and $\dot{\zeta}_i \in L_\infty$. Since $\dot{u}_s$ is a continuous function of all bounded closed-loop signals, it follows that $\dot{u}_s \in L_\infty$. Therefore, there must exist a constant $\varsigma_1 > 0$ such that $|\dot{u}_s| \leq \varsigma_1$. By noting that $e(t_k) = 0$ and $\lim_{t\to t_{k+1}} e(t) = \delta_1|u(t)| + m_1 > m_1$, it can be obtained that the lower bound of interexecution intervals t^* must satisfy $t^* \geq (m_1/\varsigma_1)$ such that the Zeno behavior is successfully excluded. ■

Remark 5: For the event-triggered controller in Theorem 1, when to transmit the control signal $u(t)$ is determined by the events (43). In this way, the control signals allow to be intermittently sent to the actuator and the communication burden is reduced subsequently. A key design technique is to make the controller have the property that $\mu_n \zeta_n u_s \leq 0$, which enables (51) to be obtained. In summary, the control sampling errors caused by the event-triggering rules are handled in a robust way.

Remark 6: The tracking error can be decreased by increasing the values of k_i and r_i and decreasing the values of l_{i+1} and δ_1. However, higher tracking accuracy might need lager control action and more triggering events which would increase more frequent transmission of control signal, thus certain tradeoff should be made in balancing system performance and control action/the numbers of triggered events. In addition, in the absence of output constraint, the proposed control scheme allows shaping of the transient tracking error trajectory by choosing the performance boundaries $F_{11}(t)$ and $F_{12}(t)$ as required.

In the case that $F_{i1}(t)$ and $F_{i2}(t)$ are unbounded for all t, system (1) has no constraint requirement, for which we have the following results.

Corollary 1: Consider the uncertain pure-feedback nonlinear system (1). Suppose that Assumptions 1–3 hold, if the controller (40)–(42) with event-triggering rule (43) and $\dot{F}_{ij} = 0, i = 1, \ldots, n, j = 1, 2$ is applied, then for any initial condition, all the results except $\mathscr{O}_3$ in Theorem 1 still hold.

C. Discussions on the Merits of the Design Approach

To close this section, we compare the proposed control scheme and the existing methods of handling state constraints. For fair comparisons, we only discuss the case of symmetric constraints, namely, the state x is required to remain in the set $\Omega_x = \{x| - F(t) < x(t) < F(t)\}$, where $F(t) > 0$ is the constraint boundary.

1) Although various control methods are developed to deal state/output constraints, all of them can only be applied to situations where the states/output have the constraint requirement *for all time*. In most existing literatures using BLF-based control scheme such as [5]–[8] and [10], the following BLF function is commonly used
$$V_b = \frac{1}{2}\ln\left(\frac{F^2}{F^2 - x^2}\right) \tag{59}$$
where $\ln(\bullet)$ denotes the natural logarithm of $\bullet$. In [11], the constrained function is constructed as
$$\xi = \frac{x}{(F+x)(F-x)}. \tag{60}$$
Note that $\lim_{F\to+\infty} V_b = 0$ and $\lim_{F\to+\infty} \xi = 0$, which are not equivalent to the general unconstrained case, thus (59) and (60) have no generic property stated in Remark 3, that is, $\lim_{F\to+\infty} \xi = x$. This means that if the system has no constraint requirement or the constraint boundary grows continuously to infinity, the control schemes in the above-mentioned literature would fail. Whereas, in this paper, by constructing a new universal-constrained function (10), both constrained and unconstrained cases can be handled together without changing the control structure.
2) For BLF-based control method in [7]–[10], they usually convert the original state constraints into new bounds on tracking errors, which will bring undesirable feasibility conditions for virtual controls, as pointed out in Section I. In this paper, we employ a constrained function (10) to impose constraints on $x_i (i = 2, \ldots, n)$ directly, and then transform the original-constrained system into a new unconstrained one by incorporating the constraint boundaries into the original nonlinear system. In this way, the feasibility conditions and the selection of optimal parameters are avoided gracefully.
3) In [11], although the obstacle caused by state constraints can be handled without using the feasibility conditions, the tracking performance would be affected. According to [11, eq. (55)], we have $z_1 = (\delta_1/\delta_2)e$, where $\delta_1 = (F+x)(F-x)(F+y_d)(F-y_d), \delta_2 = F^2 + xy_d > 0$, and z_1 is the tracking error. It should be noted that only $z_1 \in L_\infty$ can be derived according to $e \in L_\infty$ and $(\delta_1/\delta_2) \in L_\infty$, and the size of the compact set of tracking error z_1 cannot be analyzed even at steady-state phase. Besides, when δ_2 approaches zero, z_1 tends to infinity, rendering a poor tracking effect. However, as presented in Theorem 1, with the proposed control scheme the tracking error z_1 converges to the compact set Ω_{z_1} at steady-state phase that can be decreased by choosing control parameter appropriately as stated in Remark 6.
4) Different from the existing results in output constraint and state constraint field, our proposed event-triggered control law allows the control signals to be intermittently sent to the actuator, instead of transmitting it continuously or in a periodic sampled manner. Such a feature can reduce the communication burden and computational cost.

In summary, the proposed event-triggered control scheme is able to accommodate the general cases of state constraints without feasibility conditions, achieve desired tracking ability and reduce communication burden at the same time.

IV. Simulation Results

In this section, we present a simulation study to illustrate and verify the effectiveness of proposed event-triggered control scheme. Consider the following system
$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 + x_2^2\\ \dot{x}_2 &= x_1x_2 + \left(1 + x_2^2\right)u + 0.7\cos(u)\\ y &= x_1.\end{aligned} \tag{61}$$
The objective is for $y(t)$ to track the desired trajectory $y_d(t) = 0.3\sin(t)$ subject to the following asymmetric state constraints
$$\begin{aligned}-0.3 - 0.2 * 2^{-0.3t} + y_d < x_1(t) < 0.3 + 0.3 * 2^{-0.3t} + y_d\\ -0.5 - 0.1\sin(t) < x_2(t) < 0.5 + \frac{1}{t}.\end{aligned} \tag{62}$$
It is interesting to note that the constraint boundary $0.5 + (1/t)$ is infinity at the initial time so that the other schemes in the existing literature cannot cope with such constraints. However, it can be handled gracefully with the proposed control scheme in Theorem 1. For simulation, the initial conditions are selected as $x(0) = [x_1(0), x_2(0)]^T = [0.1, 0.01]^T$. The NNs employed in this paper contain 41 nodes with center $\tau_i = 0$, width $\psi^2 = 9$ and are used to approximate the unknown functions. The controller is determined by Theorem 1 with control parameters $k_1 = k_2 = 15, \sigma_1 = \sigma_2 = 1, r_1 = r_2 = 0.05, l_1 = l_2 = 0.1$ and triggering parameters $\delta_1 = 0.1, m_1 = 0.05$.

The system state trajectories are shown in Figs. 1 and 2. It can be seen that the system output x_1 is able to track the desired signal y_d, and both states x_1 and x_2 are within the predefined time-varying asymmetric constraints for all time. The tracking error is presented in Fig. 3 to show the good tracking performance. The boundedness of control input and adaptive parameters are shown in Figs. 4 and 5, respectively. The time interval of triggered control signal transmission are

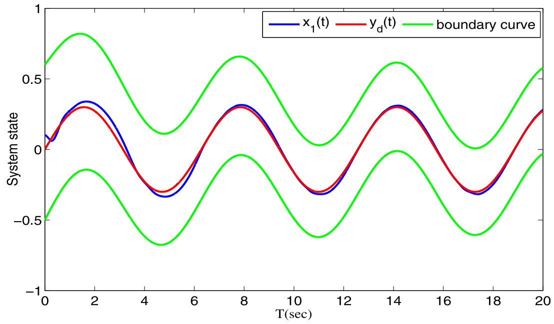

Fig. 1. System state $x_1(t)$ remains in constrained space.

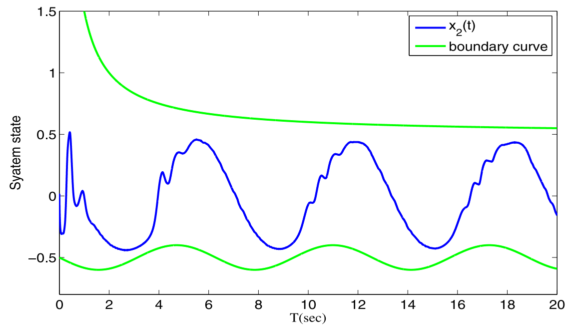

Fig. 2. System state $x_2(t)$ remains in constrained space.

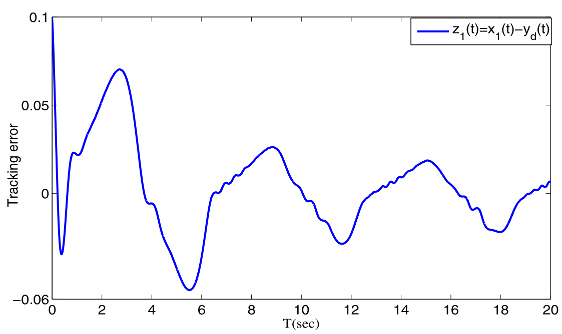

Fig. 3. Tracking error $z_1(t)$.

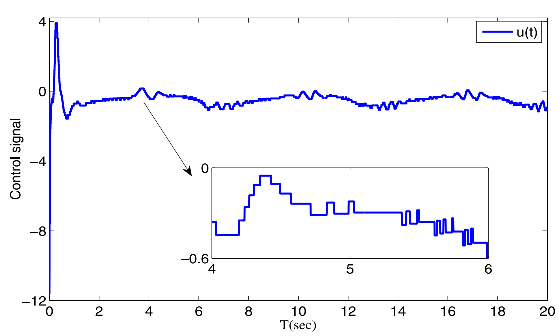

Fig. 4. Control input signal $u(t)$.

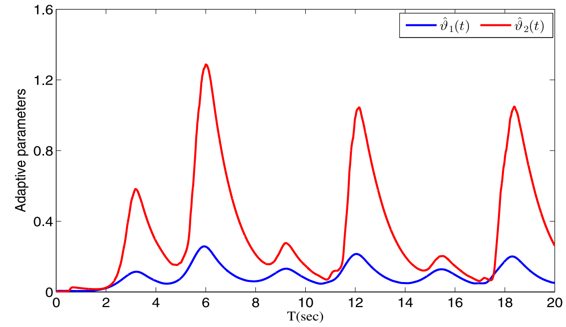

Fig. 5. Adaptive parameters $\hat{\vartheta}_1(t)$ and $\hat{\vartheta}_2(t)$.

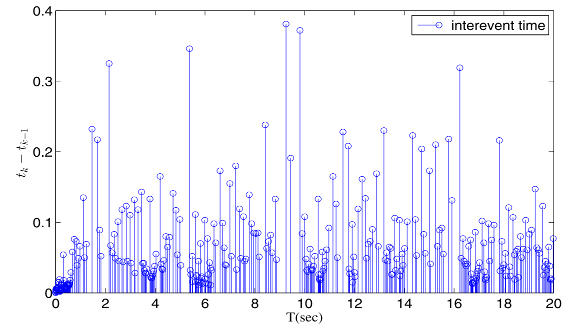

Fig. 6. Time interval of triggered transmission of control signal.

presented in Fig. 6. It is calculated that the number of triggered control signal transmission within 20 seconds is 352. From the simulation results, it is easy to observe that not only the asymmetric time-varying state constraints are satisfied but also the communication burden is reduced with the proposed event-triggered control scheme. In addition, since the feasibility conditions are not required to be met for virtual control α_1, the parameter selection is more flexible and easy to implement without using the parameter optimization method as in [8].

V. Conclusion

In this paper, a new event-based NN adaptive control scheme is presented for a class of uncertain pure-feedback systems possibly subject to state constraints. Compared with the existing works, the proposed control schemes are able to deal with more general cases of state constraints, remove the need of feasibility conditions and reduce communication burden. On the other hand, addressing event-triggering effect in the sensor-to-controller channel is an important problem. However, it is a challenging issue when backstepping technique is used for controller design, due to the noncontinuity of virtual control signals which depend on the intermittently transmitted signals from the sensors. Thus, it is still an open, yet interesting and important topic for the future research.

References

[1] M. Z. Romdlony and B. Jayawardhana, "Stabilization with guaranteed safety using control Lyapunov–Barrier function," *Automatica*, vol. 66, no. 66, pp. 39–47, 2016.

[2] M. Burger and M. Guay, "Robust constraint satisfaction for continuous-time nonlinear systems in strict feedback form," *IEEE Trans. Autom. Control*, vol. 55, no. 11, pp. 2597–2601, Nov. 2010.

[3] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert, "Survey constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, no. 6, pp. 789–814, 2000.

[4] A. Bemporad, "Reference governor for constrained nonlinear systems," *IEEE Trans. Autom. Control*, vol. 43, no. 3, pp. 415–419, Mar. 1998.

[5] K. P. Tee, S. S. Ge, and E. H. Tay, "Barrier Lyapunov functions for the control of output-constrained nonlinear systems," *Automatica*, vol. 45, no. 4, pp. 918–927, 2009.

[6] K. P. Tee, B. Ren, and S. S. Ge, "Control of nonlinear systems with time-varying output constraints," *Automatica*, vol. 47, no. 11, pp. 2511–2516, 2011.
[7] Y.-J. Liu, J. Li, S. Tong, and C. L. P. Chen, "Neural network control-based adaptive learning design for nonlinear systems with full-state constraints," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 27, no. 7, pp. 1562–1571, Jul. 2016.
[8] Y.-J. Liu and S. Tong, "Barrier Lyapunov functions-based adaptive control for a class of nonlinear pure-feedback systems with full state constraints," *Automatica*, vol. 64, no. 64, pp. 70–75, 2016.
[9] Z.-L. Tang, S. S. Ge, K. P. Tee, and W. He, "Robust adaptive neural tracking control for a class of perturbed uncertain nonlinear systems with state constraints," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 46, no. 12, pp. 1618–1629, Dec. 2016.
[10] B. S. Kim and S. J. Yoo, "Approximation-based adaptive control of uncertain non-linear pure-feedback systems with full state constraints," *IET Control Theory Appl.*, vol. 8, no. 17, pp. 2070–2081, Nov. 2014.
[11] K. Zhao and Y. D. Song "Removing the feasibility conditions imposed on tracking control designs for state-constrained strict-feedback systems," *IEEE Trans. Autom. Control*, vol. 64, no. 3, pp. 1265–1272, Mar. 2019. doi: 10.1109/TAC.2018.2845707.
[12] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *IEEE Trans. Autom. Control*, vol. 52, no. 9, pp. 1680–1685, Sep. 2007.
[13] J. Lunze and D. Lehmann, "A state-feedback approach to event-based control," *Automatica*, vol. 46, no. 1, pp. 211–215, 2010.
[14] R. Postoyan, P. Tabuada, D. Nešić, and A. Anta, "A framework for the event-triggered stabilization of nonlinear systems," *IEEE Trans. Autom. Control*, vol. 60, no. 4, pp. 982–996, Apr. 2015.
[15] X. Wang and M. Lemmon, "On event design in event-triggered feedback systems," *Automatica*, vol. 47, no. 10, pp. 2319–2322, 2011.
[16] T. Liu and Z.-P. Jiang, "A small-gain approach to robust event-triggered control of nonlinear systems," *IEEE Trans. Autom. Control*, vol. 60, no. 8, pp. 2072–2085, Aug. 2015.
[17] L. Xing, C. Wen, Z. Liu, H. Su, and J. Cai, "Event-triggered adaptive control for a class of uncertain nonlinear systems," *IEEE Trans. Autom. Control*, vol. 62, no. 4, pp. 2071–2076, Apr. 2017.
[18] A. Sahoo, H. Xu, and S. Jagannathan, "Neural network-based adaptive event-triggered control of nonlinear continuous-time systems," in *Proc. IEEE Int. Symp. Intell. Control*, 2013, pp. 35–40.
[19] A. Sahoo, H. Xu, and S. Jagannathan, "Neural network-based event-triggered state feedback control of nonlinear continuous-time systems," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 27, no. 3, pp. 497–509, Mar. 2016.
[20] L. Xing, C. Wen, Z. Liu, H. Su, and J. Cai, "Event-triggered output feedback control for a class of uncertain nonlinear systems," *IEEE Trans. Autom. Control*, vol. 64, no. 1, pp. 290–297, Jan. 2019. doi: 10.1109/tac.2018.2823386.
[21] T. P. Zhang and S. S. Ge, "Adaptive dynamic surface control of nonlinear systems with unknown dead zone in pure feedback form," *Automatica*, vol. 44, no. 7, pp. 1895–1903, 2008.
[22] T. M. Apostol, *Mathematical Analysis*. Reading, MA, USA: Addison-Wesley, 1963.
[23] C. Wang, D. J. Hill, S. S. Ge, and G. Chen, "An ISS-modular approach for adaptive neural control of pure-feedback systems," *Automatica*, vol. 42, no. 5, pp. 723–731, 2006.
[24] L. Chen and Q. Wang, "Adaptive dynamic surface control for unknown pure feedback non-affine systems with multiple constraints," *Nonlin. Dyn.*, vol. 90, no. 2, pp. 1191–1207, 2017.
[25] R. M. Sanner and J.-J. E. Slotine, "Gaussian networks for direct adaptive control," *IEEE Trans. Neural Netw.*, vol. 3, no. 6, pp. 837–863, Nov. 1992.
[26] Y.-D. Song, X. C. Huang, and Z.-J. Jia, "Dealing with the issues crucially related to the functionality and reliability of NN-associated control for nonlinear uncertain systems," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 28, no. 11, pp. 2614–2625, Nov. 2017.
[27] V. Kostrykin and A. Oleynik, "An intermediate value theorem for monotone operators in ordered Banach spaces," *Fixed Point Theory Appl.*, p. 211, 2012. doi: 10.1186/1687-1812-2012-211.
[28] H. Deng and M. Krstic, "Stochastic nonlinear stabilization—I: A backstepping design," *Syst. Control Lett.*, vol. 32, no. 3, pp. 143–150, 1997.
[29] S. S. Ge and C. Wang, "Adaptive neural control of uncertain MIMO nonlinear systems," *IEEE Trans. Neural Netw.*, vol. 15, no. 3, pp. 674–692, May 2004.

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