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Brief paper

Adaptive finite-time consensus control of a group of uncertain nonlinear mechanical systems[☆]

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ABSTRACT

In this paper we consider finite-time leaderless consensus control of multi-agent systems consisting of a group of nonlinear mechanical systems with parametric uncertainties. New adaptive finite time continuous distributed control algorithms are proposed for the multi-agent systems. It is shown that the states of the mechanical systems can reach a consensus within finite time under an undirected graph. Transient performances in terms of convergence rates and time are also analyzed. Finally simulation results illustrate and verify the effectiveness of the proposed schemes.

1. Introduction

Distributed coordination for a group of dynamic agents has attracted many researchers in recent years, due to the broad applications of multi-agent systems in many areas, such as cooperative control of unmanned air vehicles (UAVs) in DeLima, Toussaint, and York (2009), formation control in Do (2008), Dong and Farrell (2009) and Ren and Atkins (2007), flocking control in Do (2011) and Olfati-Saber (2006), distributed sensor networks in Cortes and Bullo (2005), attitude alignment for a cluster of satellites in Riyanto-Trilaksono, Syaichu-Rohman, and Eko-Poetro (2011), congestion control in communication networks in Ignaciuk and Bartoszewicz (2011), etc. The consensus of multi-agent systems means the states of all the agents converge to a common value by invoking some control schemes for each agent in the group. Consensus control problem is studied intensively since the state synchronization is often a basic requirement in many group behaviors, such as flocking. Related works on consensus control include, but not limited to, Dong (2012), Hong, Chen, and Bushnell (2008), Olfati-Saber and Murray (2004), Ren (2008), Ren and Beard (2005) and Yu and Xia (2012).

In Ren and Beard (2005) the problem of static information consensus among multi-agents in the presence of limited and unreliable information exchange with dynamically changing topologies is considered, which shows that static consensus under dynamically changing interaction topologies can be achieved asymptotically if the directed interaction graphs have a spanning tree frequently enough. Asymptotical time-varying trajectory consensus is rather difficult if only parts of the agents in the group have access to the desired trajectory. In Hong and Hu et al. (2006) multi-agent consensus problem with an active leader and variable interconnection topology is considered. To track the leader, a neighbor-based local controller together with a neighbor-based state observer is developed for each agent, while assuming leader's acceleration is partially known to all the agents in the group. Similar idea also appears in Hong et al. (2008), where distributed observer is designed for each agent using local information while assuming leader's acceleration is globally known. In Li, Liu, Ren, and Xie (2013) the unknown time-varying references can be compensated for by introducing discontinuous *sign* functions to cope with the unknown reference trajectories in generating the control signals, however chattering phenomenon of discontinuous input may affect the performance in practical implementation. In Dong

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(2011) consensus problem of multiple uncertain mechanical systems with a desired trajectory is considered. The desired trajectory is available to a portion of a group of mechanical systems. By assuming that the references are composed of a set of basic vector functions, which are available to all the agents, distributed adaptive control laws are proposed such that the state of each system asymptotically converges to the desired trajectory with information interchange between agents. In Mei, Ren, and Ma (2011) adaptive distributed leader-following control for a group of uncertain Euler–Lagrange systems is considered, where only a subset of the followers has access to the leader and the followers have only local interactions.

An important performance indicator for the consensus problem is the convergence rate. Most of the existing consensus control schemes for multi-agent systems so far achieve asymptotical convergence, namely the convergence rate at best is exponential, which means it needs infinite time for the tracking errors to converge to the origin. It is shown in Olfati-Saber and Murray (2004) that the second smallest eigenvalue of the interaction Laplacian matrix quantifies the rate of convergence. To get better convergence rate, several researchers try to find better interaction graph such as to get a larger 'second smallest eigenvalue'. Kim and Mesbahi (2006) consider the problem of finding the best vertex positional configuration so that the 'second smallest eigenvalue' of the associated interaction graph is maximized. In Xiao and Boyd (2006) by using semi-definite convex programming the weights among the agents are designed such that convergence rate is increased. However, all these efforts are only to choose proper interaction graphs, but not to find control schemes to achieve higher performance. In practice, it is often required that the consensus be reached in finite time. Thus several researchers invoke the finite-time control schemes to guarantee that the consensus can be reached within finite time. In addition, finite-time control schemes possess better disturbance rejection properties as shown in Bhat and Bernstein (2000). In Xiao, Wang, Chen, and Gao (2009) a finite-time formation control framework for multi-agent systems of first-order dynamics is developed. In Wang and Xiao (2010) a continuous finite time control scheme is developed for the state consensus problems for multi-agent systems of first-order dynamics. Li, Du, and Lin (2011) address the finite-time consensus problem for leaderless and leader–follower multi-agent systems of second-order double integrator dynamics. In Zhang and Yang (2013) the finite-time consensus tracking problem with one leader and the finite-time containment control problem with multiple leaders are considered.

Nonetheless, uncertainties do exist in practical systems. Adaptive control is one of the most effective ways to cope with parametric uncertainties by employing on-line parameter estimators. Great deal of works has been done for adaptive control of linear and nonlinear systems with unknown parameters, see for example Krstic, Kanellakopoulos, and Kokotovic (1995). It should be emphasized that for adaptive control systems it is very difficult to establish exponential convergence mainly due to the use of online parameter estimators that give rise to a highly nonlinear closed loop system, which hinders the potential applications of adaptive control. Therefore ensuring finite-time convergence will be more interesting and significant for adaptive control systems. However, achieving adaptive finite-time control is rather challenging. Certain key techniques employed in existing adaptive control literature cannot be applied. For example, Barbalat's lemma normally adopted for analyzing asymptotic convergence cannot be applied to the analysis of finite-time convergence. On the other hand, the finite-time convergence analysis tools adopted in systems without parametric uncertainties such as those in Bhat and Bernstein (1998) and Li et al. (2011) cannot be applied to adaptive finite time control directly. In Hong, Wang, and Cheng (2006) finite-time stabilization control for a class of single nonlinear systems with parametric uncertainties is investigated with a backstepping-like recursive control scheme, but the convergence time is not expressed explicitly. Furthermore, the existing finite-time consensus control schemes mainly focus on first-order integrator as in Wang and Xiao (2010) or second-order double-integrator as in Li et al. (2011). Note that adaptive consensus control for uncertain multi-agent nonlinear systems with an explicitly established convergence-time is still unavailable, to the best of the authors' knowledge.

In this paper, we address such an issue for a group of general nonlinear mechanical systems with parametric uncertainties. New continuous adaptive distributed finite-time controllers are proposed for each agent in the group of leaderless consensus control. In order to make the stability analysis method in Qian and Lin (2001a) applicable, we need to design suitable online parameter estimators to guarantee that the position errors and the virtual control errors converge to a pre-defined compact set within finite time, which is quite challenging. An adaptive distributed controller is designed such that all the positions of the agents converge to a consensus state with a configuration within finite-time. By assuming that only some of the agents have access to the static rendezvous location, the proposed adaptive distributed controllers ensure that all the agents converge to the rendezvous location with a configuration in finite-time. We also establish the transient performance for the tracking errors and the virtual control errors.

The remaining part of the paper is organized as follows. The problem of this paper is formulated and some useful preliminaries are presented in Section 2. In Section 3, distributed adaptive finite-time consensus design schemes are proposed and analyzed. Simulation results are given in Section 4 to validate the theoretical results. Finally, we conclude the paper in Section 5.

2. Problem formulation

In this paper, let $\mathbf{1} = [1, \dots, 1]^T$ and $\mathbf{0} = [0, \dots, 0]^T$. If P is a positive definite matrix, then let $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote its maximum and minimum eigenvalues, respectively. For a vector $\|\cdot\|$ denotes a standard Euclidean norm. For a matrix $\|\cdot\|_1$ denotes a standard column sum norm.

2.1. Graph theory

In this paper, the communications among the n agents are represented by a graph $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{1, \dots, n\}$ denotes the set of indexes (or vertices) corresponding to each agent, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges between two distinct agents. If $\mathcal{G}$ is undirected, then the edge $(i, j) \in \mathcal{E}$ also means $(j, i) \in \mathcal{E}$ which indicates that agents i and j can obtain state and structure information from each other. In this case, agent j is called a neighbor of agent i, and vice versa. We denote the set of neighbors for agent i as $\mathcal{N}_i \triangleq \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. Note that self edges (i, i) are not allowed, thus $(i, i) \notin \mathcal{E}$ and $i \notin \mathcal{N}_i$. $\mathcal{G}$ is connected means that there is an undirected sequence of edges between every pair of distinct agents (Ren & Cao, 2010). The connectivity matrix $A = [a_{ij}] \in \Re^{n\times n}$ is defined that $a_{ij} \neq 0$ if $(i, j) \in \mathcal{E}$, and $a_{ij} = 0$ if $(i, j) \notin \mathcal{E}$. Clearly the diagonal elements $a_{ii} = 0$ and for an undirected graph A is symmetric. We introduce an in-degree matrix $\triangle$ such that $\triangle = \mathrm{diag}(\triangle_i) \in \Re^{n\times n}$ with $\triangle_i = \sum_{j\in\mathcal{N}_i} a_{ij}$ being the ith row sum of A. Then, the Laplacian matrix of $\mathcal{G}$ is defined as $\mathcal{L} = \triangle - A$.

2.2. Preliminaries

To investigate the finite-time stability, some basic concepts and definitions are firstly introduced.

Definition 1 (Bhat & Bernstein, 1998). Consider a dynamic system

$$\dot{x} = f(x,t), \qquad f(0,t) = 0, \quad x \in U_0 \subset R^n \tag{1}$$

where $f : U_0 \times R^+ \to R^n$ is continuous on an open neighborhood U_0 of the origin $x = 0$. The equilibrium $x = 0$ of the system is (locally) finite-time stable if it is Lyapunov stable and for any initial condition $x(t_0) = x_0 \in U$ where $U \subset U_0$, if there is a settling time $T > t_0$, such that every solution $x(t; t_0, x_0)$ of system (1) satisfies $x(t; t_0, x_0) \in U\backslash\{0\}$ for $t \in [t_0, T)$, and

$$\lim_{t\to T} x(t; t_0, x_0) = 0, \qquad x(t; t_0, x_0) = 0 \quad \forall t > T.$$

If $U = R^n$, then the origin $x = 0$ is a globally finite-time stable equilibrium.

The following lemmas are useful for establishing system stability.

Lemma 1 (Bhat & Bernstein, 1998). *Suppose there is a C^1 positive definite Lyapunov function $V(x,t)$ defined on $U \times R^+$ where $U \subset U_0$ is the neighborhood of the origin, and there are positive real constants $c > 0$ and $0 < \alpha < 1$, such that $\dot{V}(x,t) + cV^\alpha(x,t)$ is negative semidefinite on U. Then $V(x,t)$ is locally in finite-time convergent with a settling time*

$$T \le \frac{V^{1-\alpha}(x_0,t)}{c(1-\alpha)}$$

for any given initial condition $x(t_0)$ in the neighborhood of the origin in U.

Lemma 2 (Qian & Lin, 2001a). *If $0 < p = p_1/p_2 \le 1$, where $p_1 > 0$ and $p_2 > 0$ are positive odd integers, then $|x^p - y^p| \le 2^{1-p}|x-y|^p$.*

Lemma 3 (Hardy, Littlewood, & Polya, 1952). *For $x_i \in \Re$, $i = 1, \ldots, n$, $0 < p \le 1$, then*

$$\left(\sum_{i=1}^n |x_i|\right)^p \le \sum_{i=1}^n |x_i|^p \le n^{1-p}\left(\sum_{i=1}^n |x_i|\right)^p.$$

Lemma 4 (Qian & Lin, 2001a). *Let c and d be positive constants and $\gamma(x,y) > 0$ is a real value function. Then*

$$|x|^c|y|^d \le \frac{c\gamma(x,y)|x|^{c+d}}{c+d} + \frac{d\gamma^{-c/d}(x,y)|y|^{c+d}}{c+d}.$$

Lemma 5 (Olfati-Saber & Murray, 2004). *For a connected undirected graph G, the Laplacian matrix L has the following property:*(1) *L is semi-definite.* (2) *0 is a simple eigenvalue of L and* **1** *is the associated eigenvector.* (3) *Assuming the eigenvalue of L is denoted as 0, $\lambda_2, \ldots \lambda_n$ satisfying $0 \le \lambda_2 \le \cdots \le \lambda_n$, then the second smallest eigenvalue $\lambda_2 > 0$. Furthermore, if $\mathbf{1}^T x = 0$, then $x^T L x \ge \lambda_2 x^T x$.*

Consider $y = x^{\frac{p}{q}}$ where p is a positive integer and q is a positive odd integer. If we ignore all the complex roots, then obviously $y = \mathrm{sign}(x)|x|^{\frac{p}{q}}$ if p is an odd integer; otherwise $y = |x|^{\frac{p}{q}}$.

2.3. System model

We consider a class of multiple mechanical nonlinear systems

$$M_i\ddot{q}_i + C_i(q_i,\dot{q}_i)\dot{q}_i + D_i(q_i)\dot{q}_i = \tau_i \tag{2}$$

where $q_i = (q_{i1}, \ldots, q_{im})^T \in \Re^m$, $i = 1, \ldots, n$ is the state of the ith system, $\tau_i \in \Re^m$ is the control input vector, $M_i \subset R^{m\times m}$ is an inertia matrix, $C_i(q_i,\dot{q}_i)$ is the centripetal and Coriolis matrix and $D_i(q_i)\dot{q}_i$ denotes the friction terms. Denote $v_i = \dot{q}_i$. The following assumptions are needed for the design of finite-time adaptive consensus controllers.

Assumption 1. The graph $\mathcal{G}$ is connected.

Assumption 2. $M_i = diag\{f_{i1}, \ldots, f_{im}\}$ where f_{ik}, $k = 1, \ldots, m$ are unknown positive constants. $(C_i(q_i,v_i)v_i + D_i(q_i)v_i) = G_i(q_i,v_i)\theta_i$ where $\theta_i \in R^l$ denotes the vector of unknown parameters and $G_i(q_i,v_i) \in R^{m\times l}$ does not contain unknown parameters. Furthermore, $\|G_i(q_i,v_i)\|_1 \le \rho \mathrm{P}(\|v_i\|)$, where ρ is a known positive constant and $\mathrm{P}(\cdot)$ is a polynomial with no constant term, i.e. $P(0) = 0$. f_{ik}, $k = 1, \ldots, m$ and θ_i are in known compact set.

Remark 1.

- For a typical mechanical system, q_i appears in the centripetal force and Coriolis force. For a constant v_0, there exists a constant p such that $\|G_i(q_i,v_0)\| < p$. Taking the two-link robot manipulator as an example, q_i is the joint-variable vector, consisting of joint angles θ_i and a joint offset d_i. The joint angles appear in $G_i(q_i,v_i)$ in terms of $\sin(\theta_i)$ or $\cos(\theta_i)$, and a joint offset d_i is always bounded.
- Typical examples of mechanical system satisfying Assumption 2 include Newton–Euler rigid body and three-link cylindrical robot manipulator, as described in Lewis, Dawson, and Abdallah (2004).

3. Control design and system analysis

System (2) can be written as

$$\begin{aligned}&\dot{q}_i = v_i\\ &M_i\dot{v}_i + C_i(q_i,v_i)v_i + D_i(q_i)v_i = \tau_i\end{aligned} \tag{3}$$

where $v_i = [v_{i1}, \ldots, v_{im}]^T$, $i \in \mathcal{V}$ denotes the velocity. The designed control law should ensure that all the agents reach consensus in finite time without additional information. A distributed adaptive control law is designed for each agent such that the positions of all the agents converge to a consensus location based on its neighbor's information. The controller is designed using a 'backstepping-like' procedure. In the first step virtual control $v_i^\star$ for v_i, $i = 1, \ldots, n$ is designed such that the position error converge to zero in finite-time. In the second step adaptive controllers and parameter estimators are designed such that v_i converge to $v_i^\star$ in finite-time. These finite time convergence results are summarized in Theorem 1. The transient performance of the close-loop adaptive control systems is established and presented in Theorem 2, by showing that the position errors and the virtual control errors converge faster than an exponential rate.

3.1. Controller design

- *Step* 1. Let $x_j = [q_{1j} - \delta_{1j}, \ldots, q_{nj} - \delta_{nj}]^T$, $x = [x_1^T, \ldots, x_m^T]^T$, $j = 1, \ldots, m$, where δ_{kj}, $k = 1, \ldots, n$ are constants denoting the final consensus configuration such that $q_{ij} - q_{kj} = \delta_{ij} - \delta_{kj}$, $i, k \in \mathcal{V}$. Define a Lyapunov function

$$\begin{aligned}V_1(t) &= \frac{1}{2}x^T(I_m \otimes L)x\\ &= \frac{1}{4}\sum_{j=1}^m\sum_{i=1}^n\sum_{k\in N_i} a_{ik}(q_{ij} - q_{kj} - \delta_{ij} + \delta_{kj})^2.\end{aligned} \tag{4}$$

Taking the derivative of V_1 yields

$$\begin{aligned}\dot{V}_1 &= \dot{x}^T(I_m \otimes L)x\\ &= \sum_{j=1}^m\sum_{i=1}^n\left[\sum_{k\in N_i} a_{ik}(q_{ij} - q_{kj} - \delta_{ij} + \delta_{kj})\right]\dot{q}_{ij}.\end{aligned}$$

Let

$$e_{ij} = \sum_{k\in N_i} a_{ik}(q_{ij} - q_{kj} - \delta_{ij} + \delta_{kj}). \tag{5}$$

Then

$$\dot{V}_1 = \sum_{j=1}^{m}\sum_{i=1}^{n} e_{ij}v_{ij}.$$

Denoting $v_{ij}^\star$ as the virtual control of v_{ij}, we have

$$\dot{V}_1 = \sum_{j=1}^{m}\sum_{i=1}^{n} e_{ij}v_{ij}^\star + \sum_{j=1}^{m}\sum_{i=1}^{n} e_{ij}(v_{ij} - v_{ij}^\star).$$

By choosing the virtual control as

$$v_{ij}^\star = -c_1 e_{ij}^{\frac{2\sigma-1}{2\sigma+1}} \tag{6}$$

where $\sigma \geq 2$ is a positive integer and c_1 is a positive constant to be designed, we get

$$\begin{aligned}\dot{V}_1 &= \sum_{j=1}^{m}\sum_{i=1}^{n} -c_1 e_{ij}^{\frac{4\sigma}{2\sigma+1}} + \sum_{j=1}^{m}\sum_{i=1}^{n} e_{ij}(v_{ij} - v_{ij}^\star) \\ &\leq -c_1\left(\sum_{j=1}^{m}\sum_{i=1}^{n} e_{ij}^2\right)^{\frac{2\sigma}{2\sigma+1}} + \sum_{j=1}^{m}\sum_{i=1}^{n} e_{ij}(v_{ij} - v_{ij}^\star).\end{aligned} \tag{7}$$

Let $e_j = [e_{1j}, \ldots, e_{nj}]^T$, then $e_j = Lx_j$. Thus $e_j^T e_j = x_j^T L^T L x_j = x_j^T L^2 x_j$. Since L is a diagonalizable symmetric semi-positive definite matrix, then it is easy to prove that $L^{1/2}$ is also a symmetric semi-positive definite matrix, and $L^2 = L^{1/2}LL^{1/2}$. Let $w = L^{1/2}\mathbf{1}$, we get $w^T w = (L^{1/2}\mathbf{1})^T(L^{1/2}\mathbf{1}) = \mathbf{1}^T L\mathbf{1} = 0$. Thus $w = \mathbf{0}$, which yields $w^T x_j = 0$, i.e. $\mathbf{1}^T L^{1/2} x_j = 0$. Then according to Lemma 5

$$e_j^T e_j = (L^{1/2}x_j)^T L (L^{1/2}x_j) \geq \lambda_2 x_j^T L x_j. \tag{8}$$

Let $e = [e_1^T, \ldots, e_m^T]^T$, then we get

$$\sum_{j=1}^{m}\sum_{i=1}^{n} e_{ij}^2 = e^T e \geq \lambda_2 x^T (I_m \otimes L)x = 2\lambda_2 V_1. \tag{9}$$

• *Step* 2. Let $g = \frac{2\sigma-1}{2\sigma+1}$. Invoke the *adding a power integrator* technique as in Qian and Lin (2001b) by defining another Lyapunov function

$$W_i = \frac{1}{2^{1-g}}\sum_{j=1}^{m} f_{ij}\int_{v_{ij}^\star}^{v_{ij}} (s^{1/g} - v_{ij}^{\star 1/g})^{2-g}\mathrm{d}s \tag{10}$$

and a new variable

$$\xi_{ij} = v_{ij}^{1/g} - v_{ij}^{\star 1/g} \tag{11}$$

where f_{ij} is defined in Assumption 2. From Qian and Lin (2001a), W_i is positive semi-definite and C^1. Taking the time derivative of W_i, we get

$$\begin{aligned}\dot{W}_i = \frac{1}{2^{1-g}}\sum_{j=1}^{m}\Bigg[& f_{ij}\xi_{ij}^{2-g}\dot{v}_{ij} \\ &- (2-g)f_{ij}\int_{v_{ij}^\star}^{v_{ij}} (s^{1/g} - v_{ij}^{\star 1/g})^{1-g}\mathrm{d}s \sum_{k\in\bar{N}_i}\frac{\partial v_{ij}^{\star 1/g}}{\partial q_{kj}}\dot{q}_{kj}\Bigg]\end{aligned}$$

where $\bar{N}_i = N_i \cup \{i\}$. Since

$$\begin{aligned}\left|\int_{v_{ij}^\star}^{v_{ij}} (s^{1/g} - v_{ij}^{\star 1/g})^{1-g}\mathrm{d}s\right| &\leq |v_{ij} - v_{ij}^\star||\xi_{ij}|^{1-g} \\ &= \left|\left(v_{ij}^{1/g}\right)^g - \left(v_{ij}^{\star 1/g}\right)^g\right||\xi_{ij}|^{1-g} \\ &\leq 2^{1-g}|\xi_{ij}^g||\xi_{ij}^{1-g}| = 2^{1-g}|\xi_{ij}|\end{aligned} \tag{12}$$

we have

$$\begin{aligned}\dot{W}_i &\leq \sum_{j=1}^{m}\frac{f_{ij}}{2^{1-g}}\xi_{ij}^{2-g}\dot{v}_{ij} + (2-g)\sum_{j=1}^{m} f_{ij}|\xi_{ij}|\sum_{k\in\bar{N}_i}\left|\frac{\partial v_{ij}^{\star 1/g}}{\partial q_{kj}}\right||v_{kj}| \\ &= \frac{1}{2^{1-g}}\xi_i^T\left[-(C_i(q_i, v_i)v_i + D_i v_i) + \tau_i\right] \\ &\quad + (2-g)\sum_{j=1}^{m} f_{ij}|\xi_{ij}|\sum_{k\in\bar{N}_i}\left|\frac{\partial v_{ij}^{\star 1/g}}{\partial q_{kj}}\right||v_{kj}|\end{aligned}$$

where

$$\xi_i = [\xi_{i1}^{2-g}, \ldots, \xi_{im}^{2-g}]^T. \tag{13}$$

Let $\hat{\theta}_i$, $i \in \mathcal{V}$, $j = 1, \ldots, m$ denote the estimate of θ_i and define $\tilde{\theta}_i = \hat{\theta}_i - \theta_i$. Then

$$\begin{aligned}\dot{W}_i &\leq \frac{1}{2^{1-g}}\xi_i^T\left(-G_i(q_i, v_i)\hat{\theta}_i + \tau_i\right) + \frac{1}{2^{1-g}}\xi_i^T G_i(q_i, v_i)\tilde{\theta}_i \\ &\quad + (2-g)c_1^{1/g}\sum_{j=1}^{m} f_{ij}|\xi_{ij}|\left(\mu|v_{ij}| + \eta\sum_{k\in N_i}|v_{kj}|\right)\end{aligned}$$

where $\mu = \max_{\forall i\in\mathcal{V}}\left\{\sum_{j\in N_i} a_{ij}\right\}$ and $\eta = \max_{\forall i,j\in\mathcal{V}}\{a_{ij}\}$. From Lemmas 2 and 4, we have

$$\begin{aligned}\sum_{j=1}^{m}\sum_{i=1}^{n} e_{ij}(v_{ij} - v_{ij}^\star) &\leq \sum_{j=1}^{m}\sum_{i=1}^{n} 2^{1-g}|e_{ij}||\xi_{ij}|^g \\ &\leq \frac{2^{1-g}}{1+g}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(e_{ij}^{1+g} + g\xi_{ij}^{1+g}\right).\end{aligned}$$

Define a new Lyapunov function

$$V_2 = V_1 + \sum_{i=1}^{n} W_i. \tag{14}$$

Then

$$\begin{aligned}\dot{V}_2 &\leq -c_1\sum_{j=1}^{m}\sum_{i=1}^{n} e_{ij}^{\frac{4\sigma}{2\sigma+1}} + \frac{2^{1-g}}{1+g}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(e_{ij}^{1+g} + g\xi_{ij}^{1+g}\right) \\ &\quad + \frac{1}{2^{1-g}}\sum_{i=1}^{n}\xi_i^T\left(-G_i(q_i, v_i)\hat{\theta}_i + \tau_i\right) + \frac{1}{2^{1-g}}\sum_{i=1}^{n}\xi_i^T G_i(q_i, v_i)\tilde{\theta}_i \\ &\quad + (2-g)c_1^{1/g}\sum_{i=1}^{n}\sum_{j=1}^{m} f_{ij}|\xi_{ij}|\left(\mu|v_{ij}| + \eta\sum_{k\in N_i}|v_{kj}|\right).\end{aligned} \tag{15}$$

Based on Lemmas 2–4 and (11),

$$\begin{aligned}&(2-g)c_1^{1/g}\sum_{i=1}^{n}\sum_{j=1}^{m} f_{ij}|\xi_{ij}|\Big(\mu|v_{ij}| + \eta\sum_{k\in N_i}|v_{kj}|\Big) \\ &\leq \sum_{i=1}^{n}\sum_{j=1}^{m}\Bigg(\frac{f_{ij}\mu g + \eta n_i g}{1+g}|e_{ij}|^{1+g} + (2-g)c_1^{1/g}\Big(f_{ij}\mu \\ &\quad + \frac{f_{ij}c_1^2(2-g)c_1^{1/g} + \eta n_i + \eta c_1^2(2-g)c_1^{1/g} + \eta n_i g}{1+g}\Big)|\xi_{ij}|^{1+g}\Bigg)\end{aligned}$$

where $n_i = \dim N_i$. Since $F_i^{-1} = \mathrm{diag}(1/f_{i1}, \ldots, 1/f_{im})$, the torque is designed as

$$\tau_i = G(q_i, v_i)\hat{\theta}_i - c_2\xi_i^* \tag{16}$$

where $\xi_i^* = [\xi_{i1}^{2g-1}, \ldots, \xi_{im}^{2g-1}]^T$ and c_2 is a positive constant to be chosen. From (15) and (16), we get

$$\dot{V}_2 \le -k_1 \sum_{j=1}^{m}\sum_{i=1}^{n} e_{ij}^{\frac{4\sigma}{2\sigma+1}} - k_2 \sum_{j=1}^{m}\sum_{i=1}^{n} \xi_{ij}^{\frac{4\sigma}{2\sigma+1}} + \frac{1}{2^{1-g}} \sum_{i=1}^{n} \xi_i^T G_i(q_i, v_i)\tilde{\theta}_i \tag{17}$$

where

$$k_1 = c_1 - \frac{2^{1-g}}{1+g} - \frac{f_{\max}\mu g + \eta n_i g}{1+g}$$
$$k_2 = c_2 - \frac{g2^{1-g}}{1+g} - (2-g)c_1^{1/g}\Big(f_{\max}\mu + \frac{f_{\max}c_1^2(2-g)c_1^{1/g} + \eta n_i + \eta c_1^2(2-g)c_1^{1/g} + \eta n_i g}{1+g}\Big) \tag{18}$$

with $f_{\max} \ge \max_{j=1,\ldots,m}(f_{1j}, \ldots, f_{nj})$ being a positive constant known from Assumption 2. Thus we can find c_1 and c_2 such that $k_1 > 0$ and $k_2 > 0$.

Consider the following Lyapunov function

$$V_3 = V_2 + \frac{1}{2}\sum_{i=1}^{n} \tilde{\theta}_i^T \Gamma_i^{-1}\tilde{\theta}_i \tag{19}$$

where Γ_i is a positive diagonal matrix. Then

$$\dot{V}_3 \le -k_1 \sum_{j=1}^{m}\sum_{i=1}^{n} e_{ij}^{\frac{4\sigma}{2\sigma+1}} - k_2 \sum_{j=1}^{m}\sum_{i=1}^{n} \xi_{ij}^{\frac{4\sigma}{2\sigma+1}} + \sum_{i=1}^{n} \tilde{\theta}_i^T \Gamma_i^{-1}\left(\dot{\hat{\theta}}_i + \frac{1}{2^{1-g}}\Gamma_i G_i(q_i, v_i)^T \xi_i\right).$$

The parameter update law for θ_i is

$$\dot{\hat{\theta}}_{ik} = \mathrm{Proj}\Big(\beta_i(k), \hat{\theta}_{ik}\Big), \quad k = 1, \ldots, l \tag{20}$$

where $\beta_i(k)$ is the kth element of $\beta_i = -\frac{1}{2^{1-g}}\Gamma_i G_i(q_i, v_i)^T \xi_i$. The operator $\mathrm{Proj}(\cdot,\cdot)$ is a Lipschitz continuous projection algorithm in Pomet and Praly (1992) which is defined as follows:

$$\mathrm{Proj}(a, \hat{b}) = \begin{cases} a & \text{if } \mu(\hat{b}) \le 0 \\ a & \text{if } \mu(\hat{b}) \ge 0 \text{ and } \mu'(\hat{b})a \le 0 \\ (1-\mu(\hat{b}))a & \text{if } \mu(\hat{b}) > 0 \text{ and } \mu'(\hat{b})a > 0 \end{cases} \tag{21}$$

where $\mu(\hat{b}) = \frac{\hat{b}^2 - b_M^2}{\epsilon^2 + 2\epsilon b_M}$, $\mu'(\hat{b}) = \frac{\partial \mu(\hat{b})}{\hat{b}}$, ϵ is an arbitrarily small positive constant, b_M is a positive constant satisfying $|b| < b_M$.

Then from (17), (19), (20) and based on the property of the parameter projection in Krstic et al. (1995), we get

$$\dot{V}_3 \le -k_1 \sum_{j=1}^{m}\sum_{i=1}^{n} e_{ij}^{\frac{4\sigma}{2\sigma+1}} - k_2 \sum_{j=1}^{m}\sum_{i=1}^{n} \xi_{ij}^{\frac{4\sigma}{2\sigma+1}}. \tag{22}$$

3.2. Stability analysis

We are now at the position to state our first result in the following theorem.

Theorem 1. *Consider the undirected leaderless multi-agent uncertain systems (2), under the control of the distributed adaptive controllers (16) with parameter estimators in (20). If Assumptions 1 and 2 are satisfied, then the positions of the group of mechanical systems will reach consensus with specified configurations δ_{ij} in finite time T satisfying*

$$T \le \frac{\varrho_1^{1-\kappa}}{k_x(1-\kappa)} + \frac{V_3(t_0) - \varrho_1}{d_{v_3}} \tag{23}$$

where κ, k_x, ϱ_1 and d_{v_3} are computable.

Proof. From (22) we get $\dot{V}_3 \le 0$. Therefore e_{ij} and $\tilde{\theta}_i$ are bounded and it can be easily checked that $\lim_{t\to\infty} e_{ij} = 0$. Since we cannot use $\dot{V}_3 + cV_3^{\alpha} \le 0$ to establish the finite-time stability of e_{ij} and ξ_{ij} due to the existence of the terms $\tilde{\theta}_i^T\tilde{\theta}_i$, we will prove that there exist two positive constants k_x and $0 < \kappa < 1$ such that $\dot{V}_2 + k_x V_2^{\kappa} \le 0$ which enable us to show that the systems are finite-time stable.

By using the parameter projection operation and based on Assumption 2, we know there exists a positive constant S_i such that $\|\tilde{\theta}_i\| \le S_i$. Then from (10) and (12) we have

$$W_i \le \sum_{j=1}^{m} f_{ij}\xi_{ij}^2 \le f_{\max}\sum_{j=1}^{m}\xi_{ij}^2. \tag{24}$$

Thus from (9) we obtain

$$V_2 \le \frac{1}{2\lambda_2}\sum_{i=1}^{n}\sum_{j=1}^{m} e_{ij}^2 + f_{\max}\sum_{i=1}^{n}\sum_{j=1}^{m}\xi_{ij}^2. \tag{25}$$

Suppose $P(\cdot)$ in Assumption 2 is

$$P(x) = p_1 x + p_2 x^2 + \cdots + p_h x^h \tag{26}$$

where $h \ge 2$ is an integer and p_k, $k = 1, \ldots, h$ are positive constants. From (11)

$$|v_{ij}| \le (|\xi_{ij}| + |v_{ij}^{\star}|^{1/g})^g \le |\xi_{ij}|^g + c_1|e_{ij}|^g. \tag{27}$$

Define a compact set

$$\Xi = \{\xi_{ij}, e_{ij} : |e_{ij}| < r_{ij}, |\xi_{ij}| < d_{ij}\} \tag{28}$$

in the neighborhood of $e_{ij} = 0$, $\xi_{ij} = 0$ where r_{ij} and d_{ij} are positive constants chosen to be

$$r_{ij} = \left(\frac{1}{2c_1}\sqrt{\frac{1}{m}}\right)^{1/g}, \qquad d_{ij} = \left(\frac{1}{2}\sqrt{\frac{1}{m}}\right)^{1/g}. \tag{29}$$

From (27) it is easy to check that in this set we have $\|v_i\| \le 1$. Thus from Assumption 2 we get $\|G_i(q_i, v_i)\|_1 \le \rho h p_{\max}\|v_i\|$ where $p_{\max} = \max(p_1, \ldots, p_h)$. Furthermore, $|G_i(q_i, v_i)(j,k)| < \rho h p_{\max}|v_{ij}|$, $k = 1, \ldots, l$ since $|v_{ij}| \le 1$. Then from (13), (17) and (27)

$$\frac{1}{2^{1-p}}\sum_{i=1}^{n}\xi_i^T G_i(q_i, v_i)\tilde{\theta}_i \le \frac{1}{2^{1-p}}\sum_{i=1}^{n}\Big\||\xi_i|^T|G_i(q_i, v_i)|\Big\|\|\tilde{\theta}_i\|$$
$$\le \frac{S_i}{2^{1-p}}\sum_{i=1}^{n}\sum_{j=1}^{m}\rho h p_{\max}|\xi_{ij}^{2-g}|\Big(|\xi_{ij}|^g + c_1|e_{ij}|^g\Big)$$
$$= k_3\sum_{i=1}^{n}\sum_{j=1}^{m} e_{ij}^2 + k_4\sum_{i=1}^{n}\sum_{j=1}^{m}\xi_{ij}^2 \tag{30}$$

where $|\xi_i|$ and $|G_i(q_i, v_i)|$ are taken element-wisely and k_3 and k_4 are

$$k_3 = \frac{S_{\max}\rho h p_{\max} g}{2^{2-p}}, \qquad k_4 = \frac{S_{\max}\rho h p_{\max}}{2^{1-p}}\left(1 + \frac{2-g}{2c_1^2}\right) \tag{31}$$

in which $S_{\max}$ is defined as $S_{\max} = \max(S_1, \ldots, S_n)$. Thus from (17) we get

$$\dot{V}_2 \le -\frac{k_1}{2}\sum_{j=1}^{m}\sum_{i=1}^{n} e_{ij}^{\frac{4\sigma}{2\sigma+1}} - \frac{k_2}{2}\sum_{j=1}^{m}\sum_{i=1}^{n} \xi_{ij}^{\frac{4\sigma}{2\sigma+1}} + \sum_{j=1}^{m}\sum_{i=1}^{n}\Big(k_3 e_{ij}^2 - \frac{k_1}{2}e_{ij}^{\frac{4\sigma}{2\sigma+1}}\Big) + \sum_{j=1}^{m}\sum_{i=1}^{n}\Big(k_4 \xi_{ij}^2 - \frac{k_2}{2}\xi_{ij}^{\frac{4\sigma}{2\sigma+1}}\Big). \tag{32}$$

Now we will establish the finite-time stability in two cases. In the first case we will prove that e_{ij} and ξ_{ij} will converge to zero in finite-time if $e_{ij}(t_0) \in \Xi$ and $\xi_{ij}(t_0) \in \Xi$. In the second case we will show that e_{ij} and ξ_{ij} will converge to Ξ within finite-time and thus converge to zero in finite-time if $e_{ij}(t_0) \notin \Xi$ and/or $\xi_{ij}(t_0) \notin \Xi$.

• Case 1: The initial conditions of e_{ij} and ξ_{ij} satisfy

$e_{ij}(t_0) \in \Xi,\ \xi_{ij}(t_0) \in \Xi.$

From (18) and (31) we can choose the design parameters c_1 and c_2 such that $k_1 > 2k_3$ and $k_2 > 2k_4$. Since $|e_{ij}| < 1$ and $|\xi_{ij}| < 1$, it is not hard to see that $\sum_{j=1}^{m}\sum_{i=1}^{n}\Big(k_3 e_{ij}^2 - \frac{k_1}{2}e_{ij}^{\frac{4\sigma}{2\sigma+1}}\Big) + \sum_{j=1}^{m}\sum_{i=1}^{n}\Big(k_4 \xi_{ij}^2 - \frac{k_2}{2}\xi_{ij}^{\frac{4\sigma}{2\sigma+1}}\Big) < 0$ if we choose c_1 and c_2 in this way. Thus we have

$$\dot{V}_2 \le -\frac{k_1}{2}\sum_{j=1}^{m}\sum_{i=1}^{n} e_{ij}^{\frac{4\sigma}{2\sigma+1}} - \frac{k_2}{2}\sum_{j=1}^{m}\sum_{i=1}^{n} \xi_{ij}^{\frac{4\sigma}{2\sigma+1}}. \tag{33}$$

From (25) and Lemma 3 we get $V_2^{\kappa} \le \frac{1}{2^{\kappa}\lambda_2^{\kappa}}\sum_{i=1}^{n}\sum_{j=1}^{m} e_{ij}^{\frac{4\sigma}{2\sigma+1}} + f_{\max}^{\kappa}\sum_{i=1}^{n}\sum_{j=1}^{m}\xi_{ij}^{\frac{4\sigma}{2\sigma+1}}$ where $\kappa = \frac{2\sigma}{2\sigma+1} \in (0, 1)$. Hence

$$\dot{V}_2 + k_x V_2^{\kappa} \le 0 \tag{34}$$

where $k_x < \min\Big\{2^{\kappa-1}\lambda_2^{\kappa}k_1, \frac{k_2}{2f_{\max}^{\kappa}}\Big\}$ is a positive constant. Since $0 < g < 1$, then based on Lemma 1, V_2 will converge to zero in a finite time T_1 which satisfies

$$T_1 \le \frac{V_2(t_0)^{1-\kappa}}{k_x(1-\kappa)}. \tag{35}$$

• Case 2: The initial values of e_{ij} and/or ξ_{ij} are outside the set Ξ. For the period of time when e_{ij} and/or ξ_{ij} are outside of the set Ξ, from (22), we know

$$\dot{V}_3 \le -d_{v_3} \tag{36}$$

where $d_{v_3} \ge \min\Big(k_1\sum_{i=1}^{n}\sum_{j=1}^{m} r_{ij}^{\frac{4\sigma}{2\sigma+1}}, k_2\sum_{i=1}^{n}\sum_{j=1}^{m} d_{ij}^{\frac{4\sigma}{2\sigma+1}}\Big)$ is a positive constant.

To determine the time T_2 that e_{ij} and ξ_{ij} reach the set Ξ from the initials, we have to calculate a ϱ_1 such that $V_3 \ge \varrho_1$ when $e_{ij}(t) \notin \Xi$ and/or $\xi_{ij}(t) \notin \Xi$. Then T_2 will not be longer than $(V_3(t_0) - \varrho_1)/d_{v_3}$. From (14) and (19), such a ϱ_1 is calculated as

$$\varrho_1 = \min\left\{V_2|_{(e_{ij}=r_{ij},\xi_{ij}=0)}, \left(\sum_{i=1}^{n} W_i\right)\Bigg|_{(\xi_{ij}=d_{ij},|e_{ij}|<r_{ij})}\right\}.$$

From (4) we know

$$\begin{aligned} V_2|_{(e_{ij}=r_{ij},\xi_{ij}=0)} &= V_1|_{e_{ij}=r_{ij}} \\ &\ge \frac{\zeta}{4\eta^2}\sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{k\in N_i}\eta^2(q_{ij}-q_{kj}-\delta_{ij}+\delta_{kj})^2 \\ &\ge \frac{\zeta}{4\eta^2 n_{\max}}\sum_{j=1}^{m}\sum_{i=1}^{n}\Big[\sum_{k\in N_i} a_{ik}(q_{ij}-q_{kj}-\delta_{ij}+\delta_{kj})\Big]^2 \\ &= \frac{\zeta}{4\eta^2 n_{\max}}\sum_{j=1}^{m}\sum_{i=1}^{n} r_{ij}^2 \end{aligned} \tag{37}$$

where $\zeta = \min_{\forall i,j\in\mathcal{V}}\{a_{ij}\}$, $n_{\max} = \max\{n_1, \ldots, n_n\}$. $\eta = \max_{\forall i,j\in\mathcal{V}}\{a_{ij}\}$, $n_i = \dim\{N_i\}$. It can be checked that

$$\begin{aligned} \int_{v_{ij}^{\star}}^{v_{ij}} (s^{1/g} - v_{ij}^{\star 1/g})^{2-g} ds &\ge \frac{2\sigma-1}{2^{\frac{(1-g)(2-g)}{g}}(4\sigma+2)}\left|v_{ij} - v_{ij}^{\star}\right|^{\frac{4\sigma+2}{2\sigma-1}} \\ &\ge \frac{2\sigma-1}{2^{\frac{(1-g)(2-g)}{g}}(4\sigma+2)}\Big((d_{ij}+r_{ij})^g - r_{ij}^g\Big)^{\frac{4\sigma+2}{2\sigma-1}} := s_{ij}. \end{aligned} \tag{38}$$

This is because for a fixed ξ_{ij}, $v_{ij} - v_{ij}^{\star}$ decreases monotonously as e_{ij} increases. Thus from (10)

$$\left(\sum_{i=1}^{n} W_i\right)\Bigg|_{\xi_{ij}=d_{ij},|e_{ij}|<r_{ij}} = \sum_{i=1}^{n}\sum_{j=1}^{m}\frac{f_{ij}}{2^{1-g}}s_{ij} \ge \frac{f_{\min}}{2^{1-g}}\sum_{i=1}^{n}\sum_{j=1}^{m} s_{ij}$$

where $f_{\min} \le \min_{\forall i,j\in\mathcal{V}} f_{ij}$ is a positive constant known from Assumption 2.

So with ϱ_1, e_{ij} and ξ_{ij} will reach the set within a time T_2 which satisfies

$$T_2 \le \frac{V_3(t_0) - \varrho_1}{d_{v_3}}. \tag{39}$$

After reaching the set, e_{ij} and ξ_{ij} will stay in this set forever because $\dot{V}_2 < 0$ in this set, and it will take a time less than $T_3 = \frac{\varrho_1^{1-\kappa}}{k_x(1-\kappa)}$ to arrive at the origin. Thus in this case the system will be finite time stable with T satisfying (23). In summary, the closed-loop system is globally finite-time convergent. This completes the proof. □

Remark 2. From (25) and (36) we know k_x and d_{v_3} will be larger if we choose larger controller parameters c_1 and c_2. Since κ is a constant, thus from (18) and (22) we know (23) can be adjusted arbitrarily small, but it will result in larger control effort.

Remark 3. From the proof of Theorem 1 we can see that local finite-time stability and global asymptotical stability yield global finite-time stability.

3.3. *Transient performance analysis*

As mentioned in the introduction, for adaptive control of nonlinear systems it is very difficult to establish exponential convergence rate. Nevertheless we explore the transient performance of the resulting closed-loop adaptive control system here. The following theorem is established.

Theorem 2. *Tracking errors e_{ij} and virtual control errors ξ_{ij} converge to the origin with a speed faster than an exponential rate when $e_{ij} \in \Xi$ and $\xi_{ij} \in \Xi$.*

Proof. Based on (34) which holds when $(e_{ij}, \xi_{ij}) \in \Xi$, we get $\dot{V}_2 \le -k_x V_2^{\kappa} \le 0$. Thus $V_2 \le V_2(t_0)$, and we can find a positive constant

k_p such that

$$\dot V_2 + k_p V_2 \le 0 \tag{40}$$

where $k_p = k_x^{1/\kappa} V_2(t_0)^{\frac{\kappa-1}{\kappa}}$. Therefore

$$V_2(t) \le V_2(t_0)e^{-k_p t}. \tag{41}$$

From (37) we get $V_2 \ge \frac{\zeta}{4\eta^2 n_{\max}} \sum_{j=1}^m \sum_{i=1}^n e_{ij}^2$. Thus the tracking errors satisfy

$$\|e(t)\|^2 = \sum_{j=1}^m \sum_{i=1}^n e_{ij}(t)^2 \le \frac{4\eta^2 n_{\max}}{\zeta} V_2(t_0)e^{-k_p t}. \tag{42}$$

From (38) we have

$$\sum_{j=1}^m \sum_{i=1}^n \left| v_{ij} - v_{ij}^\star \right|^{\frac{4\sigma+2}{2\sigma-1}} \le \vartheta_0 V_2(t_0)e^{-k_p t} \tag{43}$$

where $\vartheta_0 = \frac{2^{\frac{(1-g)(2-g)}{g}}(4\sigma+2)}{2\sigma-1}$. Based on (6), (42) and (43), we get

$$|v_{ij}| \le |v_{ij} - v_{ij}^\star| + |v_{ij}^\star| \le \vartheta_1 e^{-k_{v1}t} + \vartheta_2 e^{-k_{v2}t} \tag{44}$$

where $\vartheta_1 = (\vartheta_0 V_2(t_0))^{\frac{2\sigma-1}{4\sigma+2}}$, $k_{v1} = k_p \frac{2\sigma-1}{4\sigma+2}$, $k_{v2} = \frac{k_p g}{2}$, $\vartheta_2 = c_1\left(\frac{4\eta^2 n_{\max}}{\zeta} V_2(t_0)\right)^{\frac{g}{2}}$. From (44) we get $|v_{ij}| \le \vartheta_1 + \vartheta_2$, and $|v_{ij}^\star| < \vartheta_2$. According to the differential mean value theorem, there exists a ψ with $|\psi| < \vartheta_1 + \vartheta_2$, such that $|v_{ij}^{1/g} - v_{ij}^{\star 1/g}| = \frac{1}{g}|\psi|^{\frac{1-g}{g}}|v_{ij} - v_{ij}^\star|$. Since function $y = |x|^{\frac{1-g}{g}}$ is monotonously decreasing or increasing when $x < 0$ or $x \ge 0$ respectively, thus there exists a positive constant k_σ, such that

$$|v_{ij}^{1/g} - v_{ij}^{\star 1/g}| < k_\sigma |v_{ij} - v_{ij}^\star| \tag{45}$$

where $k_\sigma = \frac{1}{g}(\vartheta_1 + \vartheta_2)^{\frac{1-g}{g}}$. Then from Lemma 3, (43) and (45) we obtain

$$\begin{aligned} \|\xi(t)\|^2 &= \sum_{j=1}^m \sum_{i=1}^n \xi_{ij}(t)^2 \le \sum_{j=1}^m \sum_{i=1}^n k_\sigma^2 |v_{ij} - v_{ij}^\star|^2 \\ &\le k_\sigma^2 \sqrt{mn}\Big(\sum_{j=1}^m \sum_{i=1}^n |v_{ij} - v_{ij}^\star|^4\Big)^{\frac{1}{2}} \\ &\le k_\sigma^2 \sqrt{mn}\Big(\sum_{j=1}^m \sum_{i=1}^n |v_{ij} - v_{ij}^\star|^{\frac{4\sigma+2}{2\sigma-1}}\Big)^{\frac{1}{2}\frac{8\sigma-4}{4\sigma+2}} \le k_\xi e^{-k_s t} \end{aligned} \tag{46}$$

where $k_\xi = k_\sigma^2 \sqrt{mn}\Big(\vartheta_0 V_2(t_0)\Big)^{\frac{4\sigma-2}{4\sigma+2}}$ and $k_s = k_p \frac{4\sigma-2}{4\sigma+2}$. This completes the proof. □

Remark 4.
- From (42) and (46) we know the tracking errors and the virtual control errors converge with exponential rate at least and will converge to the origin within finite time if $(e_{ij}, \xi_{ij}) \in \Xi$.
- On the other hand, if $e_{ij}(t_0) \notin \Xi$ and/or $\xi_{ij}(t_0) \notin \Xi$, both e_{ij} and ξ_{ij} will enter Ξ within T_2 defined in (39). From (9) we know $\|e(t_0)\| \ge 2\lambda_2 V_1(t_0)$. However at $t = T_2$, $\|e\| < mnr_{ij}^2$. Thus for $t \in [t_0, T_2]$, $\|e(t)\|$ drops from $2\lambda_2 V_1(t_0)$ to mnr_{ij}^2. For ξ_{ij}, suppose $v_{ij}(t_0) = 0$, then for $t = t_0$, $\|\xi\| \ge 2c_1^{2/g} V_1(t_0)$, and for $t = T_2$, $\|\xi(T_2)\| \le mnd_{ij}^2$. As discussed in Remark 2, T_2 can be made arbitrarily small by adjusting the control parameters. Therefore, any pre-specified convergence time can be met by choosing appropriate design parameters for any initial values of tracking errors and virtual control errors.
- If the upper bound of θ is unknown, then from (22) we can still choose the control parameters c_1 and c_2 by incorporating $|\tilde\theta| \le \sqrt{2V_3(t_0)}$ such that $\frac{1}{2^{1-g}} \sum_{i=1}^n \xi_i^T G_i(q_i, v_i)\tilde\theta_i \le \frac{k_1}{2} \sum_{j=1}^m \sum_{i=1}^n e_{ij}^{\frac{4\sigma}{2\sigma+1}} + \frac{k_2}{2} \sum_{j=1}^m \sum_{i=1}^n \xi_{ij}^{\frac{4\sigma}{2\sigma+1}}$. This will ensure that e_{ij} and ξ_{ij} converge to Ξ and then converge to the origin within finite time. However this is not a global result since the initial conditions of system states must be incorporated into the controllers.

In order to consider the rendezvous problem, we now define the following set of error variables

$$z_{ij} = \sum_{k\in N_i} a_{ik}\Big(q_{ij} - q_{kj} - \delta_{ij} + \delta_{kj}\Big) + b_i(q_{ij} - \delta_{ij} - Y_j) \tag{47}$$

where $Y = \{Y_1, \ldots, Y_m\}$ is a rendezvous location which is only available to part of the agents. Let $z_j = [z_{1j}, \ldots, z_{nj}]^T$, Then we have

$$z_j = (L + B)(\underline{x}_j - \underline{Y}_j) \tag{48}$$

where $z_j = [z_{1j}, \ldots, z_{nj}]^T$, $\underline{x}_j = [q_{1j} - \delta_{1j}, \ldots, q_{nj} - \delta_{nj}]^T$ and $\underline{Y}_j = [Y_j, \ldots, Y_j]^T \in R^n$. Define a Lyapunov function

$$V_4 = \frac{1}{2} z^T \Big(I_m \otimes (L + B)\Big)^{-1} z \tag{49}$$

where $z = [z_1^T, \ldots, z_m^T]^T$. Taking the derivative of V_4 yields

$$\dot V_4 = z^T v = \sum_{i=1}^n \sum_{j=1}^m z_{ij} v_{ij}$$

where $v = [v_1^T, \ldots, v_m^T]^T$ with $v_j = [v_{1j}, \ldots, v_{nj}]$, $j = 1, \ldots, m$. Choose the virtual control of v_{ij} as

$$v_{ij}^* = -c_3 z_{ij}^g. \tag{50}$$

Then we get

$$\dot V_4 = -c_3 \sum_{i=1}^n \sum_{j=1}^m z_{ij}^{1+g} + \sum_{i=1}^n \sum_{j=1}^m (v_{ij} - v_{ij}^*).$$

Also from (49) we know

$$z^T z = \sum_{i=1}^n \sum_{j=1}^m z_{ij}^2 \ge \frac{2}{\lambda_{\max}\Big(I_m \otimes (L + B)\Big)^{-1}} V_4. \tag{51}$$

Define a new Lyapunov function

$$V_5 = V_4 + \sum_{i=1}^n \Delta_i \tag{52}$$

where

$$\Delta_i = \frac{1}{2^{1-g}} \sum_{j=1}^m f_{ij} \int_{v_{ij}^*}^{v_{ij}} (s^{1/g} - v_{ij}^{*1/g})^{2-g} \mathrm{d}s \tag{53}$$

and design the controller τ_i as

$$\tau_i = -G_i(q_i, v_i)\hat\theta_i - c_2 \chi_i^* \tag{54}$$

where $\chi_{ij} = v_{ij}^{1/g} - v_{ij}^{*1/g}$ and $\chi_i^* = [\chi_{i1}^{2g-1}, \ldots, \chi_{im}^{2g-1}]$. Then

$$\begin{aligned} \dot V_5 \le& -k_1 \sum_{j=1}^m \sum_{i=1}^n z_{ij}^{\frac{4\sigma}{2\sigma+1}} - k_2 \sum_{j=1}^m \sum_{i=1}^n \chi_{ij}^{\frac{4\sigma}{2\sigma+1}} \\ &+ \frac{1}{2^{1-g}} \sum_{i=1}^n \chi_i^T G_i(q_i, v_i)\tilde\theta_i. \end{aligned} \tag{55}$$

Thus by defining

$$V_6 = V_5 + \frac{1}{2}\sum_{i=1}^{n}\tilde{\theta}_i^T \Gamma^{-1}\tilde{\theta}_i, \tag{56}$$

and choosing the parameter update law

$$\dot{\hat{\theta}}_{ik} = \mathrm{Proj}\Big(\gamma_i(k), \hat{\theta}_{ik}\Big), \quad k = 1, \ldots, l \tag{57}$$

where $\gamma_i = -\frac{1}{2^{1-g}}\Gamma_i G_i(q_i, v_i)^T \chi_i$ and $\gamma_i(k)$ is the kth element of γ_i, we get

$$\dot{V}_6 \le -k_1 \sum_{j=1}^{m}\sum_{i=1}^{n} z_{ij}^{\frac{4\sigma}{2\sigma+1}} - k_2 \sum_{j=1}^{m}\sum_{i=1}^{n} \chi_{ij}^{\frac{4\sigma}{2\sigma+1}}. \tag{58}$$

Then from (58) the following corollary is established.

Corollary 1. *Consider multi-agent systems (2) under the control of distributed adaptive controllers (50) and (54) with parameter estimators (57). If Assumptions 1 and 2 are satisfied, then all the positions of the agents converge to Y with specified configurations δ_{ij} within finite time T satisfying*

$$T \le \frac{\varrho_2^{1-\kappa}}{k_y(1-\kappa)} + \frac{V_6(t_0) - \varrho_2}{d_{v6}}. \tag{59}$$

Proof. Based on (58) and following a similar procedure from (24) to (39), the conclusion can be established. □

4. Simulation illustrations

In this section we use four cylindrical robot arms chosen from Section 3.2 in Lewis et al. (2004) to demonstrate the effectiveness of our proposed finite-time control schemes. The robot parameters are: $M = [m_1\ 0; 0\ m_2]$, $C = [\cos(q_1)\ c_1\dot{q}_2; c_2\dot{q}_2\ \sin(q_2)]$, $D = [d_1\ 0; 0\ d_2]$ with $q = [q_1\ q_2]^T$, $m_1 = m_2 = 5$, $c_1 = c_2 = 2$, $d_1 = d_2 = 3$. The control parameters are chosen as $c_1 = 5$, $c_2 = 13$, $c_{11} = c_{21} = c_{31} = 3$, $c_{12} = c_{22} = c_{32} = 10$, $\Gamma_i = \Gamma = I_4$. The initial values of the estimates are selected as 60% of their true values, respectively. The connection graph is shown in Fig. 1. The initial positions of the robots are at (20, 54), (4, 20), (44, 1) and (−2, −29) respectively. Fig. 2 shows the positions of the robots. For rendezvous seeking, Fig. 1 is also the connection graph and only robot 1 has access to the rendezvous point, which is given to be (43, 65). The initial positions are at (0, 4), (4, 20), (14, 1) and (12, −29). The results on positions of the robots are shown in Fig. 3.

To see the effects of control parameters on the convergence time, we consider two group of parameters: $c_1 = 20$, $c_2 = 5$ and $c_1 = 5$, $c_2 = 5$, respectively. In Figs. 4 and 5, $\|e(t)\|$ and the torque of Robot 1 are presented. These results confirm our discussions in Remark 2.

We also make a comparison on convergence time between our proposed scheme and a typical non-finite-time based adaptive scheme that was proposed in Mei et al. (2011), where the controller is given as

$$v_i^\star = -c_1 e_i,\ \tau_i = -c_2(v_i - v_i^\star) + Y_i(q_i, v_i)\hat{\Theta}_i,$$
$$\dot{\hat{\Theta}}_i = -\Lambda_i Y_i(q_i, v_i)^T e_i \tag{60}$$

where $v_i^\star$ is the virtual control for the velocity v_i, $\hat{\Theta}_i$ is the estimate of unknown system parameter Θ_i, e_i is the position error defined similarly to (5). Both controllers are applied to the same group of robots with the same unknown parameters mentioned above for leaderless consensus. Fig. 6 shows the comparison results with the same controller parameters $c_1 = 5$ and $c_2 = 5$, the same initial positions and the same initial parameter estimates. It can be observed that $\|e(t)\|$ converge to 0.0001 in 0.06 s with our proposed controllers whereas it takes 0.25 s to converge to 0.0024 when the controller in Mei et al. (2011) is used.

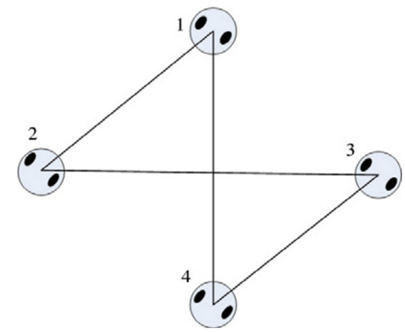

Fig. 1. Undirected graph topology for leaderless consensus and rendezvous seeking.

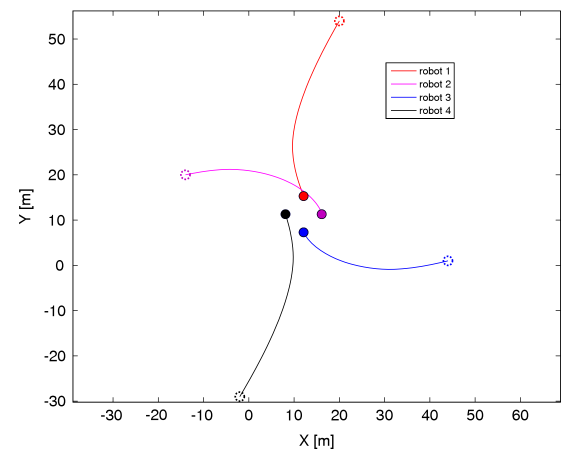

Fig. 2. Positions of four mechanical robots in leaderless consensus.

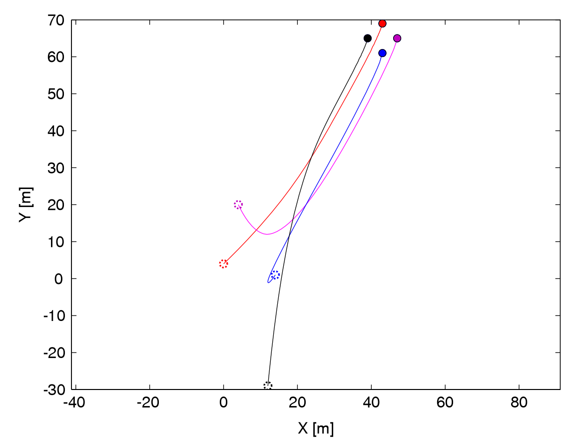

Fig. 3. Positions of four mechanical robots in location seeking.

5. Conclusion

In this paper we investigate the finite-time consensus control for a group of nonlinear mechanical systems with parametric uncertainties. New continuous distributed controllers are proposed for the multi-agent systems. Firstly for the leaderless multi-agent systems, it is shown that the states of the mechanical systems can reach a consensus within finite time. We also prove that with our control schemes all the systems can reach a static rendezvous location in finite time under a directed graph when only part of the agents have accesses to the rendezvous location. Transient performances in terms of convergence rates and time are also analyzed and established.

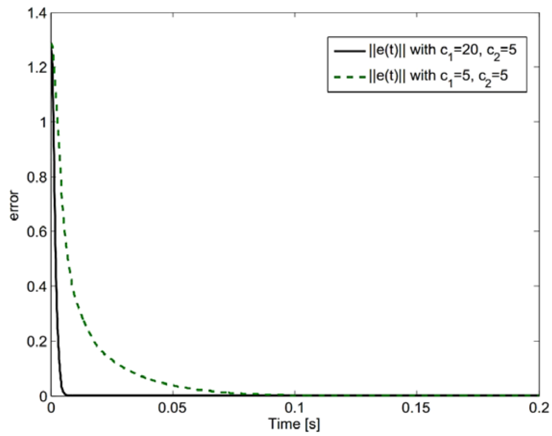

Fig. 4. Convergence rates with different control parameters for the case of leaderless consensus. Black solid line: $c_1 = 20, c_2 = 5$. Green dash line: $c_1 = 5$, $c_2 = 5$.

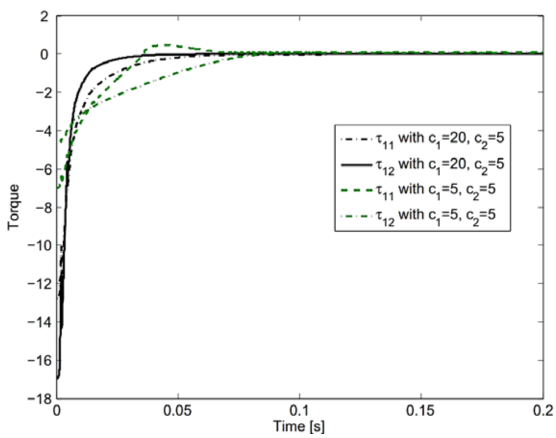

Fig. 5. Robot 1's torque with different control parameters for the leaderless consensus. Black lines: $c_1 = 20, c_2 = 5$. Green lines: $c_1 = 5, c_2 = 5$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

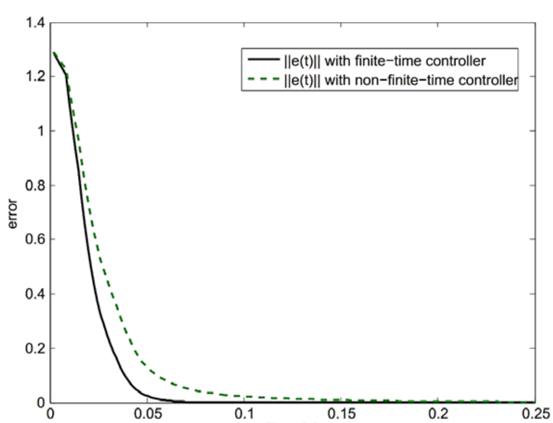

Fig. 6. Convergence time and rates with different controllers for leaderless consensus. Green dash line: non-finite-time controllers in Mei et al. (2011); Black solid line: finite-time controllers.

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