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Brief paper

Smooth second-order sliding modes: Missile guidance application☆

Yuri B. Shtessel[a,*], Ilya A. Shkolnikov[b], Arie Levant[c]

[a]Department of Electrical and Computer Engineering, University of Alabama in Huntsville, Huntsville, AL 35899, USA
[b]Z/I Imaging Corporation, Intergraph Company, 230 Business Park Blvd., Madison, AL 35757, USA
[c]Applied Mathematics Department, School of Mathematical Sciences, Tel-Aviv University, Tel-Aviv 69978, Israel

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Abstract

A new smooth second-order sliding mode control is proposed and proved using homogeneity-based technique for a system driven by sufficiently smooth uncertain disturbances. The main target application of this technique—the missile-interceptor guidance system against targets performing evasive maneuvers is considered. The smooth second-order sliding mode control-based guidance law is designed and compared with augmented proportional navigation guidance law via computer simulations of a guided missile intercepting a maneuvering ballistic target.

Keywords: Higher (second) order sliding mode control; Missile guidance

1. Introduction

1.1. Motivation

A direct application of discontinuous sliding mode control law in the outer loop of a multi-loop control system, usually, is not possible, since the profile generated by the outer loop controller must be followed by the inner loop. Furthermore, even a simple PID control used in the inner loop requires smoothness of the tracking profile.

The idea discussed in this work is to achieve enhanced tracking accuracy with a finite-time convergence for the outer loop relative-degree one dynamics by employing second-order-sliding (2-sliding) mode (see Davila, Fridman, & Levant, 2005; Levant, 1998, 2003) via a novel *smooth* 2-sliding control (SSOSM). An SSOSM disturbance observer is employed, if necessary, to retain overall high accuracy of the closed-loop tracking. The reason for applying a disturbance cancellation term estimated via an observer is that proposed novel SSOSM control achieves accuracy of real 2-sliding in a sampled data control implementation only in absence of disturbance inputs.

This particular design was inspired by and is applied to the homing missile guidance problem where the accuracy of enforcing the desired closed-loop line-of-sight dynamics would be enhanced if SSOSM guidance were employed. This enhancement results in either decreasing the miss distance as an ultimate performance criterion of a hit-to-kill homing interceptor (see Garnell & East, 1977; Zarchan, 1998) or/and decreasing acceleration advantage ratio facilitating feasibility of implementation.

1.2. State of art

The main problem in using SMC (see Edwards & Spurgeon, 1998; Utkin, Guldner, & Shi, 1999) guidance laws (see Moon, Kim, & Kim, 2001; Zhou, Mu, & Xu, 1999) is in the necessity of smoothing high-frequency switching control for the price of loosing robustness.

Higher order sliding mode (HOSM) control developed in works of Davila et al. (2005), Floquet, Barbot, and Perruquetti (2003), Levant (1998, 2003), and Orlov (2005) mitigates the problems associated with SMC, i.e. HOSM is applicable to

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* Corresponding author. Tel.: +1 256 824 6164; fax: +1 256 824 6802.
E-mail addresses: shtessel@ece.uah.edu (Y.B. Shtessel), ishkolnikov@ziimaging.com (I.A. Shkolnikov), Levant@post.tau.ac.il (A. Levant).

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the systems with an arbitrary relative degree, and continuous control functions can be achieved while robustness is retained. The continuous 2-sliding control, in particular super-twisting control (Levant, 1998), and quasi-continuous high-order sliding mode control (Levant, 2005a), are robust to the disturbances but lack smoothness.

In this work, SSOSM control is studied and its finite-time convergence is proved using homogeneity-based technique (see Bacciotti & Rosier, 2001; Bhatt & Bernstein, 2000; Levant, 2005b; Orlov, 2005). The missile-interceptor *smooth* guidance law that is robust to the target evasive maneuvers is designed based on the proposed SSOSM control. Its superiority over the augmented proportional navigation (APN) guidance is verified via a computer simulation in a realistic missile-target engagement scenario.

2. Smooth second-order sliding mode control

2.1. Prescribed sliding variable dynamics

Consider SISO dynamics

$$\dot{\sigma} = g(t) + u, \quad \sigma \in \Re, \tag{1}$$

which will be further interpreted as the sliding variable dynamics calculated along the system trajectory. The condition $\sigma = 0$ defines the system motion on the sliding surface, $u \in \Re$ is a control input that needs to be smooth, and $g(t)$ is a sufficiently smooth uncertain function.

The problem that is addressed in this section is to design smooth control u that drives $\sigma, \dot{\sigma} \to 0$ in finite time.

The drift term $g(t)$ is to be cancelled by means of a special observer to be developed further. The prescribed compensated σ-dynamics in (1) is chosen as

$$\begin{cases} \dot{x}_1 = -\alpha_1 |x_1|^{(p-1)/p} \operatorname{sign}(x_1) + x_2, \\ \dot{x}_2 = -\alpha_2 |x_1|^{(p-2)/p} \operatorname{sign}(x_1), \quad \sigma = x_1. \end{cases} \tag{2}$$

Definition. We call a system finite-time stable (see Bacciotti & Rosier, 2001; Bhatt & Bernstein, 2000) if it is asymptotically stable with a finite settling time for any solution and initial conditions.

Lemma 1. *Let $p \geqslant 2$, $\alpha_1, \alpha_2 > 0$. Then the system* (2) *is finite-time stable with the settling time being a continuous function of the initial conditions, vanishing at the origin.*

Proof. See Appendix A.

2.2. Nonlinear disturbance observer/differentiator

The sliding variable dynamics (1) is sensitive to the unknown bounded term $g(t)$. Let the variables $\sigma(t)$ and $u(t)$ be available in real time, $g(t)$ be $m-1$ times differentiable, so that $g^{(m-1)}(t)$ has a known Lipshitz constant $L > 0$. The control function $u(t)$ is Lebesgue-measurable. Eq. (1) is understood in the Filippov sense (Filippov, 1988), which means in particular that $\sigma(t)$ is an absolutely continuous function defined $\forall t \geqslant 0$.

Consider the following observer:

$$\begin{cases} \dot{z}_0 = v_0 + u, \\ v_0 = -\lambda_0 L^{1/(m+1)} |z_0 - \sigma|^{m/(m+1)} \operatorname{sign}(z_0 - \sigma) + z_1, \\ \dot{z}_1 = v_1, \\ v_1 = -\lambda_1 L^{1/m} |z_1 - v_0|^{(m-1)/m} \operatorname{sign}(z_1 - v_0) + z_2, \\ \vdots \\ \dot{z}_{m-1} = v_{m-1}, \\ v_{m-1} = -\lambda_{m-1} L^{1/2} |z_{m-1} - v_{m-2}|^{1/2} \operatorname{sign}(z_{m-1} - v_{m-2}) + z_m, \\ \dot{z}_m = -\lambda_m L \operatorname{sign}(z_m - v_{m-1}). \end{cases} \tag{3}$$

Lemma 2. *Suppose that $\sigma(t)$ and $u(t)$ are measured with some Lebesgue-measurable noises bounded, respectively, by $\varepsilon > 0$ and $k\varepsilon^{(m-1)/m}$, $k > 0$, is any fixed constant. The parameters λ_i being chosen sufficiently large in the reverse order, the following inequalities are established in finite time for some positive constants μ_i, η_i depending exclusively on k and the choice of parameters*:

$$\begin{cases} |z_0 - \sigma(t)| \leqslant \mu_0 \varepsilon, \\ \vdots \\ |z_i - g^{(i-1)}(t)| \leqslant \mu_i \varepsilon^{(m-i+1)/(m+1)}, \quad i = 1, \ldots, m, \\ |v_j - g^{(j)}(t)| \leqslant \eta_j \varepsilon^{(m-j)/(m+1)}, \quad j = 0, \ldots, m-1. \end{cases} \tag{4}$$

Proof. See Appendix B.

In particular, in the absence of input noises the exact equalities are established in a finite time:

$$z_0 = \sigma(t),$$
$$z_1 = g(t), \ldots, z_i = v_{i-1} = g^{(i-1)}, \quad i = 1, \ldots, m. \tag{5}$$

Remark 1. Parameters λ_i can be chosen recursively so that parameters $\lambda_0, \ldots, \lambda_k$ which are valid for $m = k$ can serve (after changing the notation) as $\lambda_1, \ldots, \lambda_{k+1}$ with $m = k + 1$, which means that only λ_0 is to be assigned. The simulation-checked set 8, 5, 3, 2, 1.5, 1.1 is sufficient for the observer design with $m \leqslant 5$ (Levant, 2003).

2.3. Disturbance cancellation

Let the sliding variable dynamics be of the form (1), with $g(t)$ being $(m-1)$-smooth with a known Lipshitz constant $L > 0$ of $g^{(m-1)}(t)$. The prescribed compensated σ-dynamics (2) with $p = m + 1$, $m \geqslant 1$, is easily provided using the observer (3) via control u

$$\begin{cases} u = -z_1 - \alpha_1 |\sigma|^{m/(m+1)} \operatorname{sign}(\sigma) + w, \\ \dot{w} = -\alpha_2 |\sigma|^{(m-1)/(m+1)} \operatorname{sign}(\sigma). \end{cases} \tag{6}$$

When exact measurements are available, z_1 becomes equal to $g(t)$ in a finite time, and the σ-dynamics are described by the finite-time stable system (2) thereafter.

Theorem. *Let $m \geqslant 1$, $\alpha_1, \alpha_2 > 0$, $g(t)$ be $m-1$ times differentiable, $g^{(m-1)}(t)$ having a known Lipshitz constant $L > 0$. Then the closed-loop system* (1), (3), (6) *is finite-time stable. If $\sigma(t)$ is measured with some Lebesgue-measurable noises bounded, respectively, by $\varepsilon \geqslant 0$ and $k\varepsilon^{(m-1)/m}$, then the inequality $|\sigma| \leqslant \gamma\varepsilon$*

is established in finite time for some positive constant γ depending exclusively on the parameters of the controller and the constant k.

Proof. See Appendix D.

Remark 2. Control (6) can be interpreted as a smooth second-order sliding mode (SSOSM) control, since being smooth it provides a finite-time convergence $\sigma, \dot{\sigma} \to 0$.

Remark 3. The sliding variable in (1) and its derivative could also be stabilized at zero in the presence of the disturbance $g(t)$ with a bounded derivative $\dot{g}(t)$ in a finite time using super-twisting control law (Levant, 2003)

$$\begin{cases} u = -\alpha_1 |\sigma|^{1/2} \operatorname{sign}(\sigma) + w, \\ \dot{w} = -\alpha_2 \operatorname{sign}(\sigma). \end{cases}$$

However, the super-twisting control law is not smooth.

3. Homing guidance: smooth second-order sliding mode approach

3.1. Introduction to sliding mode guidance and problem formulation

Recent application of SMC (see Utkin et al., 1999) to the homing missile guidance (see Moon et al., 2001; Shkolnikov, Shtessel, Lianos, & Thies, 2000; Zhou et al., 1999), resulted in a series of algorithms achieving smaller acceleration ratio required for intercept of weaving targets as compared to proportional navigation (PN) and APN guidance laws. Since the guidance law must be smooth, all SMC guidance laws include smoothing procedures that yield partial loss of robustness to disturbances. Here the problem is addressed using SSOSM studied in Section 2.

3.2. Intercept strategy: geometric approach

In polar coordinate system, the missile-target relative position is presented by $\mathbf{R} = (r, \lambda)$, where r is the range along Line-Of-Sight (LOS), and $\lambda =$ LOS angle. The following state model (see Shkolnikov et al., 2000) of missile-target engagement process is used

$$\begin{cases} \dot{r} = V_r, \\ \dot{V}_r = V_\lambda^2/r + A_{T,r} - \sin(\lambda - \gamma_M) n_L, \\ \dot{\lambda} = V_\lambda / r, \\ \dot{V}_\lambda = -V_r V_\lambda / r + A_{T,\lambda} - \cos(\lambda - \gamma_M) n_L, \end{cases} \tag{7}$$

where $\dot{\lambda} = \omega_\lambda$ (rad/s^2) is LOS rate, $V_\lambda = r\omega_\lambda$ (m/s) is a transversal component of relative velocity in the reference frame rotating with LOS, n_L is missile normal acceleration, $A_{T,r}$, $A_{T,\lambda}$ (disturbances) are projections of bounded target acceleration along and orthogonal to LOS. The unit of accelerations n_L and $A_{T,r}$, $A_{T,\lambda}$ is m/s^2.

It is well known that for achieving a direct hit, i.e. $r = 0$, it is necessary to keep $V_r < 0$. Direct hit can be achieved (see, for instance, Moon et al., 2001; Shkolnikov et al., 2000) if

$$V_\lambda = c_0 \sqrt{r}, \tag{8}$$

where $c_0 \geqslant 0$ is a constant.

Unlike that in the work Shkolnikov et al. (2000), where various continuous approximations of SMC guidance laws were studied, the SSOSM guidance law in a format of Eqs. (3) and (6) with $p = 3$ and $m = 2$ is to be derived in terms of the normal acceleration command n_{Lc} for stabilizing the manifold

$$\sigma = V_\lambda - c_0 \sqrt{r} = 0. \tag{9}$$

This command is to be followed by the flight control system (autopilot). The expected advantages are in increasing robustness and accuracy of hit-to-kill intercept.

3.3. Smooth second-order sliding mode guidance

In order to derive the guidance law n_L that runs $\sigma \to 0$ in finite time, the input–output (σ) dynamics of the relative degree 1 is identified based on Eqs. (7) and (9) as

$$\dot{\sigma} = g(V_r(t), V_\lambda(t), r(t), A_{T,\lambda}(t)) - \cos(\lambda - \gamma_M) n_L, \tag{10}$$

where

$$g(V_r(t), V_\lambda(t), r(t), A_{T,\lambda}(t)) = -V_r V_\lambda / r + A_{T,\lambda} - c_0 V_r / (2\sqrt{r}),$$

$$\dot{g}(V_r(t), V_\lambda(t), r(t), A_{T,\lambda}(t)) = -\frac{(\dot{V}_r V_\lambda + V_r \dot{V}_\lambda) r - V_r^2 V_\lambda}{r^2} + \dot{A}_{T,\lambda} - \frac{c_0(\dot{V}_r r - V_r^2)}{2r\sqrt{r}}.$$

Remark 4. It is assumed that the target acceleration transversal to the LOS $A_{T,\lambda}(t)$ is differentiable. Then the function $g(V_r(t), V_\lambda(t), r(t), A_{T,\lambda}(t))$ is also differentiable, and the function $\dot{g}(V_r(t), V_\lambda(t), r(t), A_{T,\lambda}(t))$ is continuous everywhere except for $r = 0$. This singularity point occurs when intercept by impact happens. However, technically, the intercept by impact ("hit-to-kill") happens when $r \neq 0$ but belongs to the interval $r \in [r_{\min}, r_{\max}] = [0.1, 0.25]$ m (see Garnell & East, 1977; Zarchan, 1998). This fact is due to a certain size of the ballistic target, and a particular intercept value of $r^0 \in [0.1, 0.25]$ m, named "zero intercept", depends on this size. Therefore, the function $g(V_r(t), V_\lambda(t), r(t), A_{T,\lambda}(t))$ is differentiable and the function $\dot{g}(V_r(t), V_\lambda(t), r(t), A_{T,\lambda}(t))$ is continuous everywhere until hit-to-kill "zero intercept" happens.

Since the function $g(V_r(t), V_\lambda(t), r(t), A_{T,\lambda}(t))$ is differentiable $\forall r \geqslant r^0$ its derivative has a Lipshitz constant. Assume the following inequalities hold $|\dot{A}_{T,\lambda}| \leqslant \dot{A}_{T,\lambda}^{\mathrm{LIM}}$, $|A_{T,\lambda}| \leqslant A_{T,\lambda}^{\mathrm{LIM}}$, $|A_{T,r}| \leqslant A_{T,r}^{\mathrm{LIM}}$, $|V_\lambda(t)| \leqslant V_\lambda^{\mathrm{LIM}}$, $V_r(0) = M \ll 0$, $M \leqslant V_r(t) \leqslant 0$, $|\sin(\lambda - \gamma_M)| < c_1 < 1$, $|\cos(\lambda - \gamma_M)| < c_2 < 1$ in a reasonable flight domain. Then the Lipshitz constant L for

$\dot{g}(V_r(t), V_\lambda(t), r(t), A_{T,\lambda}(t))$ can be estimated as

$$\begin{aligned}
&|\dot{g}(V_r(t), V_\lambda(t), r(t), A_{T,\lambda}(t))| \\
&\leqslant \dot{A}_{T,\lambda}^{\mathrm{LIM}} + \frac{|(\dot{V}_r V_\lambda + V_r \dot{V}_\lambda) r - V_r^2 V_\lambda|}{r^2} + \frac{c_0 |(\dot{V}_r r - V_r^2)|}{2r\sqrt{r}} \\
&\leqslant \dot{A}_{T,\lambda}^{\mathrm{LIM}} + \frac{1}{(r^0)^2}\Bigg[(V_\lambda^{\mathrm{LIM}})^3 + 2M^2 V_\lambda^{\mathrm{LIM}} \\
&\quad + \frac{c_0\sqrt{r^0}}{2}(V_\lambda^{\mathrm{LIM}} + M^2)\Bigg] \\
&\quad + \frac{1}{r^0}[(A_{T,\lambda}^{\mathrm{LIM}} + c_1 n_L^{\mathrm{LIM}}) V_\lambda^{\mathrm{LIM}} + |M|(A_{T,\lambda}^{\mathrm{LIM}} + c_2 n_L^{\mathrm{LIM}}) \\
&\quad + \frac{c_0\sqrt{r^0}}{2}(A_{T,\lambda}^{\mathrm{LIM}} + c_1 n_L^{\mathrm{LIM}})] = L.
\end{aligned}$$

Since $0 < r^0 \leqslant 1$ then

$$L \approx \dot{A}_{T,\lambda}^{\mathrm{LIM}} + \frac{V_\lambda^{\mathrm{LIM}}}{(r^0)^2}[(V_\lambda^{\mathrm{LIM}})^2 + 2M^2]. \tag{11}$$

Next, assuming the variables $V_r(t), V_\lambda(t), r(t), \lambda, \gamma_m$ measured, $A_{T,\lambda}(t)$ is estimated by the observer (3) with $m = 2$, $\lambda_0 = 2$, $\lambda_1 = 1.5$ and $\lambda_2 = 1.1$ (see Davila et al., 2005). This is

$$\begin{aligned}
&\dot{z}_0 = v_0 - \cos(\lambda - \gamma_M) n_L - V_r V_\lambda - c_0 V_r/(2\sqrt{r}), \\
&v_0 = -2L^{1/3}|z_0 - \sigma|^{2/3}\,\mathrm{sign}(z_0 - f) + z_1, \\
&\dot{z}_1 = v_1, \quad v_1 = -1.5L^{1/2}|z_1 - v_0|^{1/2}\,\mathrm{sign}(z_1 - v_0) + z_2, \\
&\dot{z}_2 = -1.1L\,\mathrm{sign}(z_2 - v_1), \\
&\hat{A}_{T,\lambda} = z_1.
\end{aligned} \tag{12}$$

In the absence of input noises, we obtain $\hat{A}_{T,\lambda} = A_{T,\lambda}$ after finite-time transient process in observer (12).

The guidance law is designed in terms of a control input n_L using SSOSM control (6), (12) with $p = 3$ and $m = 2$:

$$\begin{aligned}
n_{Lc} = \frac{1}{\cos(\lambda - \gamma_M)}\Bigg(&\alpha_1|\sigma|^{2/3}\,\mathrm{sign}\,\sigma + \alpha_2\int|\sigma|^{1/3}\,\mathrm{sign}(\sigma)\,\mathrm{d}\tau \\
&-N'\frac{V_r V_\lambda}{r} - \frac{c_0 V_r}{2\sqrt{r}} + \hat{A}_{T,\lambda}\Bigg), \quad N' = 1.
\end{aligned} \tag{13}$$

Remark 5. The term $-N'\frac{V_r V_\lambda}{r} = -N' V_r \omega_\lambda$ is known as PN guidance (see Zarchan, 1998). Rewriting Eq. (13) as

$$n_{Lc} = \frac{1}{\cos(\lambda - \gamma_M)}\left(-N'\frac{V_r V_\lambda}{r} + U_d + \hat{A}_{T,\lambda}\right),$$

$$U_d = \alpha_1|\sigma|^{2/3}\,\mathrm{sign}\,\sigma + \alpha_2\int|\sigma|^{1/3}\,\mathrm{sign}(\sigma)\,\mathrm{d}\tau - \frac{c_0 V_r}{2\sqrt{r}}, \tag{14}$$

one can interpret the SSOSM guidance law as a pseudo PN guidance with a robust finite-time convergence term U_d.

3.4. Analysis of internal dynamics

Eq. (10) shows that the fourth-order system (7), (9) has a relative degree 1. Therefore, it is supposed to have the third-order internal dynamics. This is

$$\begin{cases}
\dot{r} = V_r, \\
\dot{V}_r = (\sigma + c_0\sqrt{r})^2/r + A_{T,r} - \sin(\lambda - \gamma_M) n_L, \\
\dot{\lambda} = (\sigma + c_0\sqrt{r})/r.
\end{cases} \tag{15}$$

Suppose that the sliding variable (9) is stabilized at zero with its derivative in finite time via the SSOSM guidance (14). Therefore, the internal dynamics (15) become forced zero dynamics in the 2-sliding mode.

$$\begin{cases}
\dot{r} = V_r, \\
\dot{V}_r = c_0^2 + A_{T,r} - \sin(\lambda - \gamma_M) n_{Lc}, \\
\dot{\lambda} = c_0/\sqrt{r}.
\end{cases} \tag{16}$$

Studying Eq. (16) we have to find out if there exists a moment $t = t_{\mathrm{int}}$ at which zero intercept happens, i.e. $|r(t_{\mathrm{int}})| = r^0$. Eq. (16) evolution is irrelevant $\forall t > t_{\mathrm{int}}$.

Assuming $|A_{T,r}| \leqslant A_{T,r}^{\mathrm{LIM}}$ the following inequality holds

$$r(t) \leqslant r(0) + Mt + \frac{c_0^2 + A_{T,r}^{\mathrm{LIM}} + c_1 n_L^{\mathrm{LIM}}}{2} t^2. \tag{17}$$

The minimal value of $r(t)$ is identified

$$r(t^*) \leqslant r(0) - \frac{M^2}{2(c_0^2 + A_{T,r}^{\mathrm{LIM}} + c_1 n_L^{\mathrm{LIM}})} \tag{18}$$

and is achieved at

$$t^* = -\frac{M}{c_0^2 + A_{T,r}^{\mathrm{LIM}} + c_1 n_L^{\mathrm{LIM}}}. \tag{19}$$

The parameters $V_r(0) = M < 0$ and $c_0 > 0$ can be easily selected to meet the condition $|r(t^*)| \leqslant r^0$ that implies the intercept at $t_{\mathrm{int}} \leqslant t^*$ provided by SSOSM guidance (14).

4. Simulation example

An incoming *orthogonal intercept* by a homing missile is simulated. Missile speed in projection to the plane of engagement is 473 m/s. Initial conditions are taken as: $r = 30\,000$ m, $V_r = -636.4$ m/s, $\lambda = 0.785$ rad, $\omega_\lambda = 0.0141$ 1/s, also $n_{Lc}^{\mathrm{LIM}} = 10G$, $G = 9.81$ m/s^2. Missile seeker that measures LOS, λ, and the LOS rate is taken as a first-order lag system with a time constant 0.1 s with the additive gaussian noise with standard deviation of 1.0 mrad. The LOS rate ω_λ provided by the seeker from noisy angular measurement is filtered by a first-order filter with a time constant of 0.05 s and is fed to the guidance law as the LOS rate estimate $\hat{\omega}_\lambda$. The missile has the initial heading error of 0.785 rad. In 20 s after the beginning of engagement

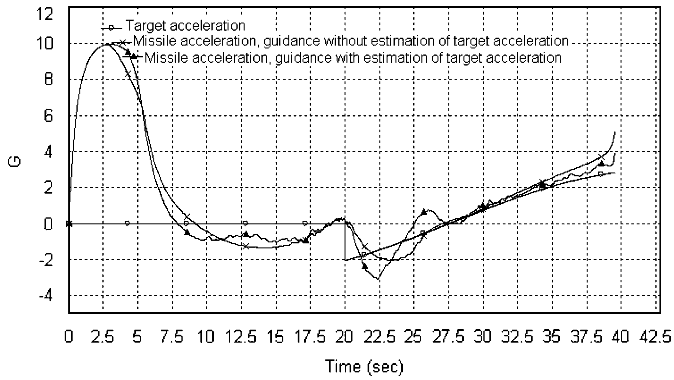

Fig. 1. Missile ($\times$, $\triangle$) and target ($\circ$) accelerations transversal to LOS with ($\times$) and without ($\triangle$) estimating $A_{T,\lambda}$.

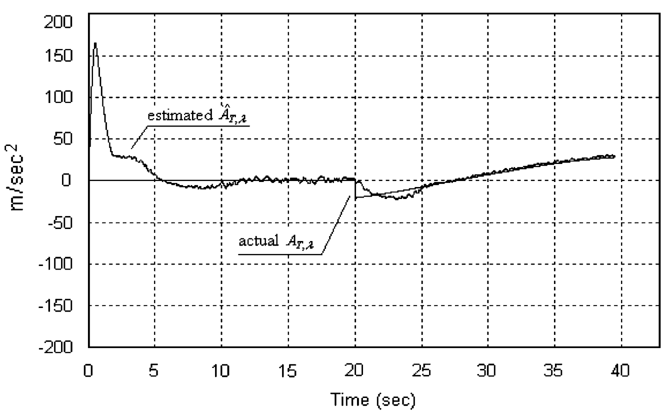

Fig. 2. Estimate $\hat{A}_{T,\lambda}$ versus actual $A_{T,\lambda}$.

the target executed a constant $3g$-turn evasive maneuver. Here, we assume the perfect knowledge of the range r and the range rate V_r. Simulation uses the Euler method with a time step of 0.0001 s. Guidance update rate is 100 Hz. An airframe is taken as a first-order lag block with a time constant equal to 0.5 s (see Zarchan, 1998).

4.1. Simulation of APN guidance with ideal information on target acceleration

APN guidance (see Zarchan, 1998) is taken as

$$n_{Lc} = -N' V_r \hat{\omega}_\lambda + 0.5 A_{T,\lambda}, \quad N' = 4 \tag{20}$$

and $A_{T,\lambda}$, is assumed perfectly known. This engagement resulted in *a miss distance* of 76 m even for a perfectly known $A_{T,\lambda}$.

4.2. Simulation of smooth 2-sliding guidance with and without estimation of target acceleration

In this experiment, SSOSM guidance (9), (14) is used with $N' = 4$, $c_0 = 0.1$ and *without estimating* $A_{T,\lambda}$, i.e. it is assumed $\hat{A}_{T,\lambda} = 0$ in (14), in order to demonstrate the effect of the robust finite-time convergence term U_d.

Remark 6. The guidance coefficient N' that is equal to 1 in Eq. (14) is taken equal to 4 in this experiment in order to have a better comparison of the SSOSM guidance law given by Eqs. (9) and (14) and the traditional APN guidance law given by Eq. (37). In this case the matched disturbance unaccounted for by control becomes $A_{T,\lambda} + 3V_r\omega_\lambda$.

Next, SOSM guidance (9), (14) is taken with the estimation term $\hat{A}_{T,\lambda}$ as in Eq. (12) and $N' = 4$, $c_0 = 0.1$. Missile and target accelerations transversal to LOS are presented in Fig. 1, which shows that the intercept happens with less acceleration ratio in the end game: $|n_L/A_{T,\lambda}| = 1.4$ for the SSOSM with $A_{T,\lambda}$ estimation versus $|n_L/A_{T,\lambda}| = 1.75$ without $A_{T,\lambda}$ estimation. Both engagements result in a *direct hit*, which means a miss distance achieved is less than 0.1 m. The high accuracy performance of the observer (12) is shown in Fig. 2.

5. Conclusions

Novel smooth second-order sliding mode (SSOSM) control is studied and its finite-time convergence proved using homogeneity-based technique for a system driven by sufficiently smooth uncertain disturbances. SSOSM control was applied to designing the guidance law for the missile-interceptor against a target performing evasive maneuvers. A hit-to-kill accuracy is achieved while the missile guided by the APN guidance law missed the target in a realistic missile-target engagement scenario. It is demonstrated via simulations that the hit-to-kill accuracy can be achieved with a reduced acceleration ratio.

Appendix A. Proof of Lemma 1

Proof. Consider the following Lyapunov function candidate:

$$\begin{aligned} V &= \frac{x_2^2}{2} + \int_0^{x_1} \alpha_2 |z|^{(p-2)/p} \operatorname{sign}(z)\, \mathrm{d}z \\ &= \frac{x_2^2}{2} + \frac{p}{2p-2} \alpha_2 x_1^{(2p-2)/p}. \end{aligned}$$

Its derivative is

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x} \dot{x} \\ &= [\alpha_2 |x_1|^{(p-2)/p} \operatorname{sign} x_1, x_2] \cdot \begin{bmatrix} x_2 - \alpha_1 |x_1|^{(p-1)/p} \operatorname{sign} x_1 \\ -\alpha_2 |x_1|^{(p-2)/p} \operatorname{sign} x_1 \end{bmatrix} \\ &= \alpha_2 x_2 x_1^{(p-2)/p} \operatorname{sign}(x_1) - \alpha_1 \alpha_2 |x_1|^{(p-2)/p + (p-1)/p} \\ &\quad - \alpha_2 x_2 x_1^{(p-2)/p} \operatorname{sign}(x_1) \\ &= -\alpha_1 \alpha_2 |x_1|^{(2p-3)/p}. \end{aligned}$$

Apply the LaSalle theorem. The set $x : \{\dot{V}(x) = 0\}$ consists of the axis $x_1 = 0$, and the only invariant set inside $x_1 = 0$ is the origin $x_1 = x_2 = 0$. Thus, the asymptotic convergence of x_1 and x_2 to zero is assured. It is easy to see that system

(2) is homogeneous (see Bacciotti & Rosier, 2001; Bhatt & Bernstein, 2000) with the dilation $d_k:(x_1, x_2)\mapsto(k^p x_1, k^{p-1}x_2)$ and the negative homogeneity degree equal to -1. Following (Bacciotti & Rosier, 2001; Bhatt & Bernstein, 2000; Levant, 2005b), the asymptotical stability implies here the finite-time stability and the continuity (and the homogeneity) of the settling time function. Lemma 1 is proven. □

Appendix B. Proof of Lemma 2

Proof. The proof is similar to the one of the differentiator convergence presented in the work of Levant (2003). Introduce the notation

$$\sigma_0 = z_0 - \sigma(t), \quad \sigma_1 = z_1 - g(t), \ldots, \sigma_m = z_m - g^{(m-1)}(t).$$

With $\varepsilon = 0$ exact measurements' case is obtained and any solution of the system satisfies the differential inclusion

$$\dot\sigma_0 = -\lambda_0 L^{1/(m+1)}|\sigma_0|^{m/(m+1)}\operatorname{sign}(\sigma_0) + \sigma_1,$$

$$\dot\sigma_1 = -\lambda_1 L^{1/m}|\sigma_1 - \dot\sigma_0|^{(m-1)/m}\operatorname{sign}(\sigma_1 - \dot\sigma_0) + \sigma_2,$$

$$\vdots$$

$$\dot\sigma_{m-1} = -\lambda_{m-1} L^{1/2}|\sigma_{m-1} - \dot\sigma_{m-2}|^{1/2}\operatorname{sign}(\sigma_{m-1} - \dot\sigma_{m-2}) + \sigma_m,$$

$$\dot\sigma_m \in -\lambda_m L \operatorname{sign}(\sigma_m - \dot\sigma_{m-1}) + [-L, L].$$

Derivatives of σ_i are easily excluded here from the right-hand side (Levant, 2003). The resulting differential inclusion is understood in the Filippov sense (Filippov, 1988; Levant, 2005b). Note that the obtained inclusion does not "remember" anything on the unknown signals $g(t)$ and $u(t)$ and coincides with the inclusion appearing in the proof of Theorem 5 (Levant, 2003). Thus the proof and the choice of the parameters are also the same. The inclusion is invariant with respect to the homogeneity transformation $G_\kappa : (t, \sigma_i) \mapsto (\kappa t, \kappa^{m-i+1}\sigma_i)$, $i = 0, \ldots, m$, i.e. it is homogeneous with the homogeneity degree -1. The measurement noises corresponding to σ and u belong to $[-\varepsilon, \varepsilon]$ and $[-k\varepsilon^{(m-1)/m}, k\varepsilon^{(m-1)/m}]$, respectively. The rest of the proof actually does not differ from (Levant, 2003) and can be also immediately obtained as a consequence of Theorem 2 (Levant, 2005b). □

Appendix C. Proof of Theorem

Proof. With exact measurements the statement of the Theorem is a consequence of Lemmas 1 and 2. Introduce the variables

$$\sigma_0 = z_0 - \sigma(t), \quad \sigma_1 = z_1 - g(t), \ldots, \sigma_m = z_m - g^{(m-1)}(t).$$

Similarly to the proof of Lemma 2 in the presence of noises the closed-loop system is reduced to an autonomous vector differential inclusion with the state variables $w, \sigma_0, \sigma_1, \ldots, \sigma_m$. The inclusion is invariant with respect to the homogeneity transformation

$$\sigma_i \mapsto \kappa^{m-i+1}\sigma_i, \;\; i = 0, \ldots, m; \quad (t, w) \mapsto (\kappa t, \kappa^m w).$$

As follows Lemmas 1 and 2, the system (1), (3) and (6) is finite-time stable. The rest of the proof follows from Theorem 2 (Levant, 2005b). □

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Dr. Yuri B. Shtessel received the M.S. and Ph.D. degrees in Electrical Engineering with concentration in Automatic Control from the Chelyabinsk State Technical University, Chelyabinsk, Russia in 1971 and 1978, respectively. From 1979 to 1991 he was with the Department of Applied Mathematics at Chelyabinsk State Technical University. From 1991 to 1993, he was with the Electrical and Computer Engineering Department and the Department of Mathematics at the University of South Carolina, Columbia, SC. Since 1993, he has been with the Electrical and Computer Engineering Department, The University of Alabama in Huntsville, where his present position is Professor. His current research interests include sliding mode control and observation with applications to missile guidance and control, launch vehicle control, and control of electric power converters.

He is a recipient of the IEEE Third Millennium Medal for the outstanding contribution to control systems engineering, 2000. He is Associate Fellow of AIAA and Senior Member of IEEE.

Ilya A. Shkolnikov received a B.S. degree in Aeromechanics and an M.S. degree in Applied Mathematics with concentration in Automatic Control from the Moscow Institute of Physics and Technology, Russia, in 1992 and 1994, respectively. His Ph.D. degree in Electrical Engineering has been completed at The University of Alabama in Huntsville in 2001. His research interests include nonlinear systems, sliding mode control, missile guidance and control, and robust numerical methods of scientific computing. He is currently employed with Intergraph Corp., Huntsville, AL, where he is working on the signal processing problems of analytical digital photogrammetry and Earth imaging.

Dr. Arie Levant (formerly L. V. Levantovsky) received his M.S. degree in Differential Equations from the Moscow State University, USSR, in 1980, and his Ph.D. degree in Control Theory from the Institute for System Studies (ISS) of the USSR Academy of Sciences (Moscow) in 1987. From 1980 until 1989 he was with ISS (Moscow). In 1990–1992 he was with the Mechanical Engineering and Mathematical Depts. of the Ben-Gurion University (Beer-Sheva, Israel). From 1993 until 2001 he was a Senior Analyst at the Institute for Industrial Mathematics (Beer-Sheva, Israel). Since 2001 he is a Senior Lecturer at the Applied Mathematics Dept. of the Tel-Aviv University (Israel).

His professional activities have been concentrated in nonlinear control theory, stability theory, singularity theory of differentiable mappings, image processing and numerous practical research projects in these and other fields. His current research interests are in high-order sliding-mode control and its applications, real-time robust exact differentiation and nonlinear robust output-feedback control.