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A General Approach to Coordination Control of Mobile Agents with Motion Constraints

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***Abstract*—This paper proposes a general approach to design convergent coordination control laws for multi-agent systems subject to motion constraints. The main contribution of this paper is to prove in a constructive way that a gradient-descent coordination control law designed for single integrators can be easily modified to adapt for various motion constraints such as nonholonomic dynamics, linear/angular velocity saturation, and other path constraints while preserving the convergence of the entire multi-agent system. The proposed approach is applicable to a wide range of coordination tasks such as rendezvous and formation control in two and three dimensions. As a special application, the proposed approach solves the problem of distance-based formation control subject to nonholonomic and velocity saturation constraints.**

I. Introduction

Coordination control of multiple mobile agents has received tremendous research attention in recent years due to its great potentials in many application areas. The single-integrator model has been widely considered in distributed coordination control due to its simplicity. However, this model usually cannot well approximate real agent dynamics because the velocity of a single integrator can be arbitrarily assigned whereas the velocity of a real agent may be subject to various constraints such as nonholonomic dynamics and velocity saturations. If not handled properly, these constraints may undermine the system convergence and cause unpredictable system behaviors. Motivated by this, many researchers have studied distributed coordination control subject to various motion constraints such as nonholonomic constraints [1]–[11], velocity saturation [4], [8], [11]–[13], and obstacle avoidance [3], [4], [6], [10], [14], [15]. However, most of the existing approaches are merely applicable to unicycle agents moving in the plane and they are usually restricted to certain specific types of coordination tasks or motion constraints.

In this paper, we propose a general approach to handle multiple types of motion constraints while guaranteeing system convergence for a wide range of coordination control tasks in both two and three dimensions. Our approach starts from the observation that many motion constraints of a mobile agent can be viewed as constraints on the direction and magnitude of the agent velocity. For instance, a nonholonomic constraint may require the velocity direction of an agent to align with its heading vector; velocity saturation requires the velocity magnitude to be bounded; and obstacle avoidance requires an agent to turn its velocity direction away from any obstacles. Considering that gradient-descent control laws play an important role in the area of multi-agent coordination control (see [16] and the references therein), we suppose that a gradient control law designed for single integrators in a given coordination task has been obtained. In order to handle motion constraints, motivated by the above observation and a recent work in [17], we modify the gradient control law by introducing a time-varying orthogonal projection matrix and a time-varying scalar to adjust the velocity direction and magnitude, respectively.

Compared to the existing results, the proposed approach possesses the following novel features. First, the approach can handle multiple types of constraints such as nonholonomic constraints and linear/angular velocity saturations while guaranteeing system convergence. It also provides additional freedom to potentially fulfil other path constraints such as obstacle avoidance. Second, the proposed approach is applicable to a wide range of coordination tasks such as rendezvous and formation control. As a special yet important application, our approach successfully solves the problem of distance-based formation control with nonholonomic and velocity saturation constraints. This problem is still unsolved to a large extent up to now due to its highly nonlinear dynamics. This successful application demonstrates the usefulness of the proposed approach. Third, while most of the existing results are only applicable to unicycle agents in the plane, the proposed approach is applicable to nonholonomic agents moving in two- or three-dimensional spaces. Finally, the proposed approach establishes connections between single-integrator and nonholonomic models. These connections enhance the usefulness of the existing gradient coordination control laws designed for single-integrator models. The present paper is a significant generalization of our previous work in [18].

II. Problem Setup

Consider n agents in $\mathbb{R}^d$ where $n \geq 1$ and $d = 2$ or 3. Let $p_i \in \mathbb{R}^d$ be the position of agent $i \in \{1, \ldots, n\} := \mathcal{V}$ and $p = [p_1^T, \ldots, p_n^T]^T \in \mathbb{R}^{dn}$. The interaction among the agents is described by a graph $\mathcal{G} = (\mathcal{E}, \mathcal{V})$, which consists of the vertex set $\mathcal{V}$ and an edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. If $(i, j) \in \mathcal{E}$, agent i can receive information from agent j and agent j is a neighbor of agent i. The set of neighbors for agent i is denoted as $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$.

For a given motion coordination task, let $e(p)$ be the coordination error vector of appropriate dimensions so that $e(p) = 0$ when the coordination task is achieved. Let $V(e)$ be a continuously differentiable Lyapunov function satisfying $V(e) \geq 0$ for all e and $V(e) = 0 \Leftrightarrow e = 0$. The corresponding gradient control law is

$$\dot{p}_i = -\nabla_{p_i} V := f_i(e, p), \quad i \in \mathcal{V}. \tag{1}$$

Note that $\dot{V}(e) = \sum_{i\in\mathcal{V}} -f_i^T f_i \leq 0$ under the action of the gradient control law. The gradient control is distributed if $f_i(e, p)$ merely depends on the positions of agent i and its neighbors. The error dynamics of (1) is

$$\dot{e} = \frac{\partial e}{\partial p} f(e, p), \tag{2}$$

where $f = [f_1^T, \ldots, f_n^T]^T \in \mathbb{R}^{dn}$. Let $\Omega(r) = \{e : V(e) \leq r\}$ where $r \geq 0$ be the level set. The gradient control (1) is *convergent* if there exists $r_0 > 0$ such that the trajectory of (2) converges to $e = 0$ for any initial error $e_0 \in \Omega(r_0)$. In this case, $\Omega(r_0)$ is called the *attraction region*.

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The design of the gradient control law in (1) does not consider any motion constraints. When applied in practice, real agents may not be able to follow the gradient flow f_i exactly due to certain motion constraints such as nonholonomic dynamics and velocity saturation. As a result, the convergence of the entire coordination system may not be guaranteed. The objective of this paper is to modify the gradient control law to handle motion constraints while preserving the system convergence.

In this paper, we consider general coordination control tasks that satisfy the following mild assumption. Let $\|\cdot\|$ denote the Euclidian norm of a vector or the spectral norm of a matrix.

Assumption 1. *For a given coordination task, functions $V(e)$ and $e(p)$ satisfy the following conditions:*

(a) $\Omega(r)$ is compact for any $r \geq 0$;
(b) There exists $r_0 > 0$ such that $e = 0 \Leftrightarrow f = 0$ in $\Omega(r_0)$;
(c) $\|\partial e(p)/\partial p\|$ and $\|f(e,p)\|$ are bounded for bounded $\|e\|$;
(d) $f(e,p)$ is continuous in e and uniformly continuous[1] in p.

Assumption 1 implies that $e = 0$ is asymptotically stable and $\Omega(r_0)$ is the attraction region according to the invariance principle [19, Thm 4.4]. The attraction region may be the entire space or a sufficiently small neighborhood of $e = 0$. If the attraction region is the entire space, then the coordination system is globally stable; otherwise, the system is locally stable.

Assumption 1 is satisfied by a wide range of coordination control laws such as the distance-based formation control law as shown below. More examples are given in the appendix. In these examples, the underlying graphs are assumed to be bidirectional and connected. If the graph is not bidirectional, the control laws may still work, but they may not be gradient control laws. For the sake of simplicity, suppose the weight for each edge to be one and let $m = |\mathcal{E}|/2$ denote the number of undirected edges.

Example 1 (**Distance-Based Formation Control**). *The objective of distance-based formation control is to steer a group of agents from some initial positions to a desired geometric pattern defined by constant inter-neighbor distances $\{\ell_{ij}\}_{(i,j)\in\mathcal{E}}$. Consider the Lyapunov function*

$$V = \frac{1}{8}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}\left(\|p_i - p_j\|^2 - \ell_{ij}^2\right)^2.$$

Then $V = 0$ if and only if the inter-neighbor distances satisfy the constraints. The gradient control law

$$\dot{p}_i = f_i = \sum_{j\in\mathcal{N}_i}\left(\|p_i - p_j\|^2 - \ell_{ij}^2\right)(p_j - p_i) \tag{3}$$

is the distance-based formation control law studied in [20]–[24]. We next show that all the conditions in Assumption 1 are satisfied. Consider any oriented graph and define the error state as $e_k = \|q_k\|^2 - \ell_k^2$ where $q_k = p_i - p_j$ and $\ell_k = \ell_{ij}$ with $k = 1,\dots,m$. Let $e = [e_1,\dots,e_m]^T \in \mathbb{R}^m$ and $q = [q_1^T,\dots,q_m^T]^T \in \mathbb{R}^{dm}$. We have $q = (H\otimes I)p$ where $H \in \mathbb{R}^{m\times n}$ is the incidence matrix of the oriented graph [21], $\otimes$ denotes the Kronecker product, and I is the identity matrix with appropriate dimensions. Then, $V(e) = 1/4\sum_{k=1}^m \|e_k\|^2$, $\partial e/\partial p = 2\mathrm{diag}(q_1^T,\dots,q_m^T)(H\otimes I)$ is bounded when e is bounded, f is uniformly continuous in both e and p, and $\|f_i\|$ is bounded when $\|e\|$ is bounded. Let $R \in \mathbb{R}^{m\times dn}$ be the rigidity matrix of the network (see the definition in [21]). Then, $R = \mathrm{diag}(q_1^T,\dots,q_m^T)(H\otimes I)$ and $\dot{p} = f = -R^Te$. A sufficient (but not necessary) condition for R to have full row rank is that the network is minimally infinitesimally rigid [20], [21]. Under this condition, $f = 0 \Leftrightarrow e = 0$ holds in a sufficiently small neighborhood of $e = 0$ [20], [21].

[1] A function $f(x)$ is uniformly continuous in x if for any $\epsilon > 0$ there exists $\delta > 0$ such that $\|f(x_1) - f(x_2)\| < \epsilon$ for every pair of x_1 and x_2 satisfying $\|x_1 - x_2\| < \delta$. A sufficient (yet not necessary) condition for uniform continuity is that if a function is differentiable and its derivative is bounded, then the function is uniformly continuous. This sufficient condition will be frequently used in the proof of Theorem 3

III. Nonholonomic Constraints

In this section, we modify the gradient control law in (1) to handle the nonholonomic constraint that the velocity direction of each agent must align with its heading vector.

A. A Modified Gradient Control Law

Let $h_i(t) \in \mathbb{R}^d$ be the unit-length heading vector of agent i. The proposed modified gradient control law is

$$\begin{aligned}\dot{p}_i &= h_i h_i^T f_i,\\ \dot{h}_i &= w_i \times h_i, \quad i \in \mathcal{V},\end{aligned} \tag{4}$$

where $\times$ denotes the cross product and $w_i \in \mathbb{R}^3$ is the angular velocity to be designed. In this control law, since $h_i h_i^T$ is an orthogonal projection matrix, the velocity $\dot{p}_i$ is the orthogonal projection of f_i onto h_i. As a result, the velocity is aligned with the heading vector h_i and the nonholonomic constraint is satisfied. The magnitude of h_i is invariant since $w_i \times h_i$ is always orthogonal to h_i.

Our objective is to design w_i so that the entire multi-agent system remains convergent in the sense that $V \to 0$. To this end, design

$$w_i = h_i \times f_i. \tag{5}$$

The geometric interpretation of (5) is that w_i attempts to rotate h_i to align with f_i (see Figure 1 for an illustration). Denote $[\cdot]_\times$ as the skew-symmetric matrix of a vector. For any $x = [x_1, x_2, x_3]^T \in \mathbb{R}^3$,

$$[x]_\times := \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

Then we have $x \times y = [x]_\times y$ for any $x, y \in \mathbb{R}^3$. Substituting (5) into (4) gives

$$\dot{h}_i = -[h_i]_\times w_i = -[h_i]_\times^2 f_i = (I - h_i h_i^T) f_i,$$

where the last equability follows from the fact that $-[x]_\times^2 = I - xx^T$ for any unit vector $x \in \mathbb{R}^3$ [25, Thm 2.11]. Then, the modified gradient control law (4) becomes

$$\begin{aligned}\dot{p}_i &= h_i h_i^T f_i,\\ \dot{h}_i &= (I - h_i h_i^T) f_i, \quad i \in \mathcal{V}.\end{aligned} \tag{6}$$

Note that $I - h_i h_i^T$ is an orthogonal projection matrix that projects any vector onto the orthogonal complement of h_i. Although derived in $\mathbb{R}^3$, control law (6) is also valid in $\mathbb{R}^2$ because the case of $\mathbb{R}^2$ can be viewed as a special case of $\mathbb{R}^3$ by treating the plane spanned by h_i and f_i as the x–y plane in $\mathbb{R}^3$.

The convergence of (6) is analyzed below.

Theorem 1 (**Modified Gradient Control Law**). *Under Assumption 1, the modified gradient coordination control law* (6) *is convergent with the same attraction region as* (1).

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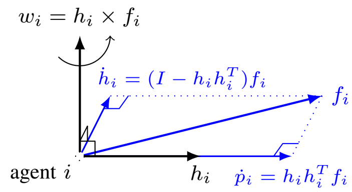

Fig. 1: An illustration of the modified gradient control law in (6).

Proof. The error dynamics corresponding to (6) is $\dot{e} = (\partial e/\partial p) M f$ where $M = \mathrm{diag}(h_1 h_1^T, \ldots, h_n h_n^T) \in \mathbb{R}^{dn}$. The time derivative of V is

$$\dot{V} = -\sum_{i\in\mathcal{V}} f_i^T \dot{p}_i = -\sum_{i\in\mathcal{V}} f_i^T h_i h_i^T f_i \le 0.$$

It follows that $\Omega(V(e_0)) \subseteq \Omega(r_0)$ is positively invariant for any $e_0 \in \Omega(r_0)$. Let $\mathcal{M} = \{e : \dot{V}(e) = 0\}$. Then, the system trajectory starting from any point in $\Omega(V(e_0))$ converges to the largest invariant set in $\mathcal{M} \cap \Omega(V(e_0))$ by the invariance principle [19, Thm 4.4]. For any point in $\mathcal{M}$, we have $h_i^T f_i = 0$ for all i, which indicates either (i) $f_i = 0$ for all i or (ii) $h_i \perp f_i$ but $f_i \neq 0$ for certain i. In the first case, it follows that $e = 0$ by condition (b) in Assumption 1. As a result, the error converges to zero and the theorem is proved. The second case is impossible. To see that, assume $h_i \perp f_i$ but $f_i \neq 0$. Then, $\dot{p}_i = h_i h_i^T f_i = 0$ for all i, which indicates that all the agents are stationary. As a result, f_i is time-invariant for all i. However, it follows from $h_i \perp f_i$ that $\dot{h}_i = (I - h_i h_i^T) f_i = f_i \neq 0$. As a result, h_i is rotating. It is impossible to maintain $h_i \perp f_i$ if f_i is time-invariant while h_i is rotating. Hence the system trajectory will escape from $\mathcal{M}$. □

Theorem 1 indicates that if $\Omega(r_0)$ is the attraction region of the gradient system (1), then it remains an attraction region for the modified gradient system (6). As a result, if the original gradient control is globally (respectively, locally) stable, then the modified one is also globally (respectively, locally) stable. The initial values of the heading vectors, $\{h_i(0)\}_{i\in\mathcal{V}}$, do not affect the convergence. The final values $\{h_i(\infty)\}_{i\in\mathcal{V}}$ are not specified.

B. Application to Unicycle Models

Considering that unicycle models have been widely considered in multi-agent coordination control, we apply (6) to derive the specific control law for unicycle agents moving in the plane. It is, however, worth noting that (6) is applicable to agents moving in both two and three dimensions.

Let $p_i = [x_i, y_i]^T \in \mathbb{R}^2$ and $\theta_i \in \mathbb{R}$ be the position coordinate and heading angle of agent i, respectively. The motion of agent i is governed by the unicycle model

$$\begin{aligned} \dot{x}_i &= v_i \cos\theta_i, \\ \dot{y}_i &= v_i \sin\theta_i, \\ \dot{\theta}_i &= w_i, \end{aligned} \tag{7}$$

where $v_i \in \mathbb{R}$ and $w_i \in \mathbb{R}$ are the linear and angular velocities. We propose the following control law for the unicycle model,

$$\begin{aligned} v_i &= [\cos\theta_i, \sin\theta_i] f_i, \\ w_i &= [-\sin\theta_i, \cos\theta_i] f_i. \end{aligned} \tag{8}$$

The convergence of the control law is proved below.

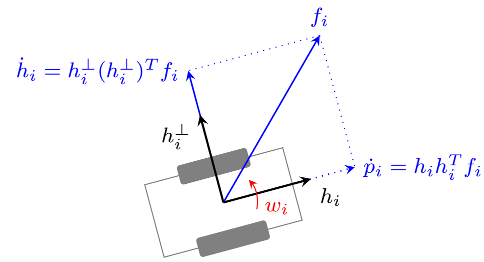

Fig. 2: The geometric interpretation of the control law in (8). Note that $\dot{p}_i$ is the orthogonal projection of f_i onto h_i and $\dot{h}_i$ is the orthogonal projection of f_i onto $h_i^{\perp}$. The angular velocity aims to turn h_i to align with f_i.

Theorem 2 (**Control Law for Unicycle Agents**)**.** *Under Assumption 1, control law* (8) *designed for the unicycle model in* (7) *is convergent with the same attraction region as* (1)*.*

Proof. Let $h_i = [\cos\theta_i, \sin\theta_i]^T$ and $h_i^{\perp} = [-\sin\theta_i, \cos\theta_i]^T$. Note that $h_i \perp h_i^{\perp}$. Substituting control law (8) into the unicycle model yields $\dot{p}_i = h_i h_i^T f_i$ and $\dot{h}_i = h_i^{\perp}(h_i^{\perp})^T f_i$. Since $h_i^{\perp}(h_i^{\perp})^T = I - h_i h_i^T$ for any $h_i \in \mathbb{R}^2$, the closed-loop system has the same expression as (6). The convergence property then follows from Theorem 1. □

The geometric interpretation of the control law in (8) is illustrated in Figure 2. The initial values of the heading angles, $\{\theta_i(0)\}_{i\in\mathcal{V}}$, do not affect the convergence. The final values $\{\theta_i(\infty)\}_{i\in\mathcal{V}}$ are not specified. We next apply (8) to derive a displacement-based formation control law for unicycles.

Example 2 (**Displacement-Based Formation Control of Unicycles**)**.** *Consider the displacement-based formation control law* $\dot{p}_i = f_i = \sum_{j\in\mathcal{N}_i}(p_j - p_i - p_j^* + p_i^*)$ *(details are given in Example 3 in the appendix). Substituting* f_i *into* (8) *yields*

$$\begin{aligned} v_i &= [\cos\theta_i, \sin\theta_i] \sum_{j\in\mathcal{N}_i} \left(p_j - p_i - p_j^* + p_i^*\right), \\ w_i &= [-\sin\theta_i, \cos\theta_i] \sum_{j\in\mathcal{N}_i} \left(p_j - p_i - p_j^* + p_i^*\right). \end{aligned} \tag{9}$$

Another well-known formation control law for unicycles proposed in [1, Eq. (1)] is

$$\begin{aligned} v_i &= [\cos\theta_i, \sin\theta_i] \sum_{j\in\mathcal{N}_i} \left(p_j - p_i - p_j^* + p_i^*\right), \\ w_i &= \cos t. \end{aligned} \tag{10}$$

The two control laws in (9) *and* (10) *have the same linear velocity. They, however, have different angular velocities. The angular velocity in* (10)*,* $w_i = \cos t$*, will cause periodical rotation of the unicycle. As a comparison, the control law in* (9) *is more reasonable in the sense that it avoids unnecessary periodical rotations by turning the heading vector to align with the gradient flow.*

IV. Nonholonomic and Velocity Saturation Constraints

In this section, we generalize (6) to propose a flexible control law to simultaneously handle nonholonomic and linear/angular velocity saturation constraints.

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A. A Flexible Coordination Control Law

The proposed flexible coordination control law is

$$\begin{aligned}\dot{p}_i &= \kappa_i h_i h_i^T f_i,\\ \dot{h}_i &= (I - h_i h_i^T) h_i^{\mathrm{d}},\end{aligned} \tag{11}$$

第二个方程其实是为了让h_i逼近h_i^d的方向，因为平衡处他们是平行的

where $\kappa_i(t) > 0$ and $h_i^{\mathrm{d}}(t) \in \mathbb{R}^d$ are time-varying. The variable κ_i can be used to adjust the velocity magnitude to fulfil the linear velocity saturation constraint. The desired heading vector h_i^{d} can be used to adjust the velocity direction to fulfil the angular velocity saturation constraint. The vector h_i^{d} also provides additional freedom to fulfil other path constraints such as obstacle avoidance. The magnitude of h_i is invariant for arbitrary h_i^{d} because $\dot{h}_i$ is always orthogonal to h_i. In the special case of $\kappa_i = 1$ and $h_i^{\mathrm{d}} = f_i$, control law (11) degenerates to (6).

The convergence of (11) is analyzed below. Since system (11) is nonautonomous, we use Barbalat's Lemma [19, Lem 8.2] to derive the convergence result.

Theorem 3 (Flexible Coordination Control Law). *Under Assumption 1, the control law in* (11) *is convergent with the same attraction region as* (1) *if $\kappa_i(t)$ and $h_i^{\mathrm{d}}(t)$ satisfy the following conditions:*

(a) $\kappa_i(t)$ is uniformly continuous in t and bounded with $0 < \kappa_{\min} \le \kappa_i(t) \le \kappa_{\max}$ for all i and all t;

(b) $\phi_i^{\mathrm{d}}(t)$ is bounded with $0 \le \phi_i^{\mathrm{d}}(t) \le \phi_{\max}^{\mathrm{d}} < \pi/2$ for all i and all t, where $\phi_i^{\mathrm{d}}(t)$ is the angle between h_i^{d} and f_i;

(c) $\|h_i^{\mathrm{d}}(t)\|$ is bounded with $0 \le \|h_i^{\mathrm{d}}(t)\| \le \mu_{\max}^{\mathrm{d}}$ and $\|h_i^{\mathrm{d}}(t)\| = 0$ only if $\|f_i\| = 0$ for all i and all t.

Proof. The error dynamics corresponding to (11) is $\dot{e} = (\partial e/\partial p) M f$ where $M = \mathrm{diag}(\kappa_1 h_1 h_1^T, \dots, \kappa_n h_n h_n^T) \in \mathbb{R}^{dn}$. The derivative of V is

$$\dot{V} = -\sum_{i\in\mathcal{V}} f_i^T \dot{p}_i = -\sum_{i\in\mathcal{V}} \kappa_i f_i^T h_i h_i^T f_i \le 0.$$

Since $\dot{V} \le 0$, for any initial condition $e_0 \in \Omega(r_0)$, the set $\Omega(V(e_0)) \subseteq \Omega(r_0)$ is positively invariant. Since V is nonincreasing and bounded from below, V converges as $t \to \infty$.

We next prove that $\dot{V}$ is uniformly continuous in t by showing that h_i, f_i, and κ_i are all uniformly continuous in t. *Step (i):* Since $\|\dot{h}_i\| = \|(I - h_i h_i^T) h_i^{\mathrm{d}}\| \le \|h_i^{\mathrm{d}}\| \le \mu_{\max}^{\mathrm{d}}$, h_i is uniformly continuous in t because it is differentiable and its derivative is bounded. *Step (ii):* Since e is bounded on $\Omega(V(e_0))$, f_i and $\partial e/\partial p$ are also bounded according to condition (c) in Assumption 1. It follows from the boundedness of f_i and h_i as well as $\kappa_i \le \kappa_{\max}$ that $\|\dot{p}_i\| = \|\kappa_i h_i h_i^T f_i\|$ is bounded. As a result, p_i is uniformly continuous in t because it is differentiable and its derivative is bounded. Moreover, since $\dot{p}_i$ is bounded for all i, we know that $\dot{e} = (\partial e/\partial p)\dot{p}$ is bounded and hence e is uniformly continuous in t. *Step (iii):* Since $f_i(e,p)$ is continuous in e and e is bounded on $\Omega(V(e_0))$, we know that f_i is uniformly continuous in e. Together with condition (d) in Assumption 1, it is implied that $f(e,p)$ is uniformly continuous in both e and p. It then follows from the uniform continuity of e and p as shown in Step (ii) that $f_i(e,p)$ is uniformly continuous in t. Finally, since κ_i is uniformly continuous as assumed, we conclude that $\dot{V} = -\sum_{i\in\mathcal{V}} \kappa_i f_i^T h_i h_i^T f_i$ is uniformly continuous in t and hence $\dot{V} \to 0$ as $t \to \infty$ by Barbalat's Lemma [19, Lem 8.2].

Because $\kappa_i \ge \kappa_{\min}$, $\dot{V} \to 0$ implies $h_i^T f_i$ converges to zero for all $i \in \mathcal{V}$. It follows that either (i) $\|f_i\| = 0$ for all i or (ii) $h_i \perp f_i$ but $f_i \neq 0$ for certain i. In the first case, the system trajectory converges to $e = 0$ according to condition (b) in Assumption 1. The second case is impossible. To see that, assume $h_i \perp f_i$ but $f_i \neq 0$ for certain i. Since $h_i^T f_i = 0$ for all i, we have $\dot{p}_i = \kappa_i h_i h_i^T f_i = 0$ for all i and hence all the agents are stationary. Since f_i is continuous in p, f_i is time-invariant. However, $\|\dot{h}_i\| = \|(I - h_i h_i^T) h_i^{\mathrm{d}}\| \ge \|h_i^{\mathrm{d}}\| \cos\phi_i^{\mathrm{d}} \ge \|h_i^{\mathrm{d}}\| \cos\phi_{\max}^{\mathrm{d}}$. Since $\|f_i\| \neq 0 \Rightarrow \|h_i^{\mathrm{d}}\| \neq 0$ as assumed, we know $\|\dot{h}_i\| \neq 0$ and consequently h_i will keep rotating. It is impossible to maintain $h_i \perp f_i$ if f_i is time-invariant but h_i is rotating. □

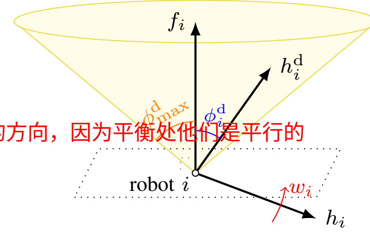

Fig. 3: An illustration of the control law in (11).

The vector $h_i^{\mathrm{d}}(t)$ is not required to be continuous. Even if $h_i^{\mathrm{d}}(t)$ is discontinuous, h_i may be still uniformly continuous as long as $\dot{h}_i$ is bounded. As a result, nonsmooth stability analysis tools [26] are not desired to analyze the system convergence. The conditions on $\kappa_i(t)$ and $h_i^{\mathrm{d}}(t)$ in Theorem 3 are mild. We may choose $\kappa_{\min}$ to be arbitrarily small, $\kappa_{\max}$ arbitrarily large, the angle $\phi_{\max}^{\mathrm{d}}$ arbitrarily close to $\pi/2$ so that $\kappa_i(t)$ and $h_i^{\mathrm{d}}(t)$ may vary within broad intervals. This provides great flexibility to design $\kappa_i(t)$ and $h_i^{\mathrm{d}}(t)$.

B. Application to Unicycles subject to Velocity Saturation

We now apply (11) to derive the specific control law for unicycle agents subject to both linear and angular velocity saturation constraints. It is worth noting that (11) is applicable to agents moving in two- and three-dimensional spaces.

Consider the unicycle model in (7). Here $v_i > 0$ indicates that the agent moves forward, and $v_i < 0$ backward; and $w_i > 0$ indicates that the agent turns its heading vector to the left (i.e., counterclockwise), and $w_i < 0$ to the right (i.e., clockwise). Suppose v_i and w_i are constrained by

$$\begin{aligned}-\mathrm{v}_i^{\mathrm{b}} \le\, & v_i \le \mathrm{v}_i^{\mathrm{f}},\\ -\mathrm{w}_i^{\mathrm{r}} \le\, & w_i \le \mathrm{w}_i^{\mathrm{l}},\end{aligned}$$

where $\mathrm{v}_i^{\mathrm{f}}, \mathrm{v}_i^{\mathrm{b}} > 0$ are the maximum forward and backward linear speeds, respectively, and $\mathrm{w}_i^{\mathrm{r}}, \mathrm{w}_i^{\mathrm{l}} > 0$ are the maximum left-turn and right-turn angular speeds, respectively. Define the saturation functions for the linear and angular speeds for agent i as

$$\mathrm{sat}_{\mathrm{v}_i}(x) = \begin{cases} -\mathrm{v}_i^{\mathrm{b}}, & x \in (-\infty, -\mathrm{v}_i^{\mathrm{b}}),\\ x, & x \in [-\mathrm{v}_i^{\mathrm{b}}, \mathrm{v}_i^{\mathrm{f}}],\\ \mathrm{v}_i^{\mathrm{f}}, & x \in (\mathrm{v}_i^{\mathrm{f}}, +\infty),\end{cases}$$

$$\mathrm{sat}_{\mathrm{w}_i}(x) = \begin{cases} -\mathrm{w}_i^{\mathrm{r}}, & x \in (-\infty, -\mathrm{w}_i^{\mathrm{r}}),\\ x, & x \in [-\mathrm{w}_i^{\mathrm{r}}, \mathrm{w}_i^{\mathrm{l}}],\\ \mathrm{w}_i^{\mathrm{l}}, & x \in (\mathrm{w}_i^{\mathrm{l}}, +\infty).\end{cases}$$

Note that the saturation bounds $\mathrm{v}_i^{\mathrm{f}}, \mathrm{v}_i^{\mathrm{b}}, \mathrm{w}_i^{\mathrm{r}}, \mathrm{w}_i^{\mathrm{l}}$ may differ for different agents. The proposed control law for unicycle i is

$$\begin{aligned} v_i &= \mathrm{sat}_{\mathrm{v}_i}\Big\{[\cos\theta_i, \sin\theta_i] f_i\Big\},\\ w_i &= \mathrm{sat}_{\mathrm{w}_i}\Big\{[-\sin\theta_i, \cos\theta_i] f_i\Big\}.\end{aligned} \tag{12}$$

The convergence of the control law is proved below.

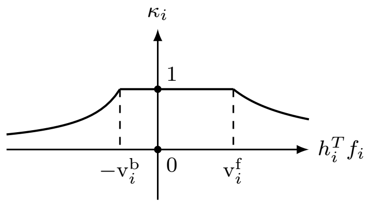

Fig. 4: An illustration of κ_i in (14). Here $h_i^T f_i$ is treated as one single variable.

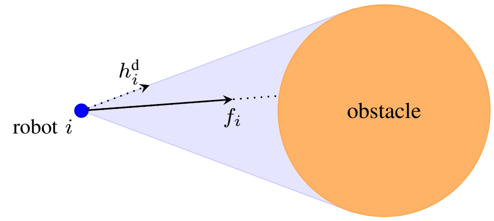

Fig. 5: An illustration of the proposed strategy for obstacle avoidance.

Theorem 4 (**Linear and Angular Velocity Saturation**). *Under Assumption 1, the control law in* (12) *applied to* (7) *renders the close-loop system convergent with the same attraction region as* (1).

Proof. The control law in (12) can be rewritten as $v_i = \mathrm{sat}_{\mathrm{v}_i}(h_i^T f_i)$ and $w_i = \mathrm{sat}_{\mathrm{w}_i}((h_i^\perp)^T f_i)$. Substituting into the unicycle model in (7) yields

$$\begin{aligned}\dot{p}_i &= h_i \mathrm{sat}_{\mathrm{v}_i}(h_i^T f_i),\\ \dot{h}_i &= h_i^\perp \mathrm{sat}_{\mathrm{w}_i}((h_i^\perp)^T f_i).\end{aligned} \tag{13}$$

The idea of the proof is to rewrite (13) as the expression of (11) and the convergence result follows from Theorem 3.

Rewrite the saturation function as $\mathrm{sat}_{\mathrm{v}_i}(h_i^T f_i) = \kappa_i h_i^T f_i$, where

$$\kappa_i = \begin{cases} \dfrac{\mathrm{v}_i^{\mathrm{b}}}{-h_i^T f_i}, & h_i^T f_i \in (-\infty, -\mathrm{v}_i^{\mathrm{b}}),\\ 1, & h_i^T f_i \in [-\mathrm{v}_i^{\mathrm{b}}, \mathrm{v}_i^{\mathrm{f}}],\\ \dfrac{\mathrm{v}_i^{\mathrm{f}}}{h_i^T f_i}, & h_i^T f_i \in (\mathrm{v}_i^{\mathrm{f}}, +\infty).\end{cases} \tag{14}$$

The value of κ_i in (14) is depicted in Figure 4. With the notation of κ_i, we have $\dot{p}_i = h_i \mathrm{sat}_{\mathrm{v}_i}(h_i^T f_i) = \kappa_i h_i h_i^T f_i$. Similarly, rewrite $\mathrm{sat}_{\mathrm{w}_i}((h_i^\perp)^T f_i) = \rho_i (h_i^\perp)^T f_i$, where

$$\rho_i = \begin{cases} \dfrac{\mathrm{w}_i^{\mathrm{r}}}{-(h_i^\perp)^T f_i}, & (h_i^\perp)^T f_i \in (-\infty, -\mathrm{w}_i^{\mathrm{r}}),\\ 1, & (h_i^\perp)^T f_i \in [-\mathrm{w}_i^{\mathrm{r}}, \mathrm{w}_i^{\mathrm{l}}],\\ \dfrac{\mathrm{w}_i^{\mathrm{l}}}{(h_i^\perp)^T f_i}, & (h_i^\perp)^T f_i \in (\mathrm{w}_i^{\mathrm{l}}, +\infty).\end{cases}$$

Then, we have $\dot{h}_i = h_i^\perp (h_i^\perp)^T (\rho_i f_i) = (I - h_i h_i^T)(\rho_i f_i)$.

First, as shown in Figure 4, κ_i in (14) is uniformly continuous in $h_i^T f_i$ by definition though κ_i is not differentiable. Similar to the proof of Theorem 3, we know that f_i and h_i are uniformly continuous in t. Thus, κ_i is uniformly continuous in t. Second, for any initial error e_0, the set $\Omega(V(e_0)) \subseteq \Omega(r_0)$ is compact and positively invariant. Since $\|f_i\|$ is bounded over the compact set $\Omega(V(e_0))$, there exists a constant γ such that $\|f_i\| \leq \gamma$ and hence $|h_i^T f_i| \leq \|f_i\| \leq \gamma$ for all t. Then, $1 \geq \kappa_i \geq \min\{\mathrm{v}_i^{\mathrm{b}}/\gamma, \mathrm{v}_i^{\mathrm{f}}/\gamma\} = \kappa_{\min}$. Therefore, κ_i is bounded from both below and above for all t and condition (a) in Theorem 3 is satisfied. Similarly, we have $1 \geq \rho_i \geq \min\{\mathrm{w}_i^{\mathrm{r}}/\gamma, \mathrm{w}_i^{\mathrm{l}}/\gamma\}$. It follows that $\|\rho_i f_i\|$ is bounded from above and $\rho_i f_i = 0$ if and only if $f_i = 0$. Then, the convergence result follows directly from Theorem 3. □

V. Application to Distance-Based Formation Control

In this section, we consider the problem of distance-based formation control of unicycle agents subject to linear and angular velocity saturations. This problem is challenging to analyze because distance-based formation control laws are nonlinear. It is still an unsolved problem to a large extent up to now. We show that this problem can be successfully solved by our proposed approach. In the meantime, we demonstrate how to apply the proposed approach to achieve obstacle avoidance.

A. Proposed Control Law and Obstacle Avoidance Strategy

The distance-based formation control law for unicycles is

$$\begin{aligned} v_i &= \mathrm{sat}_{\mathrm{v}_i}\Big\{[\cos\theta_i, \sin\theta_i] f_i\Big\},\\ w_i &= \mathrm{sat}_{\mathrm{w}_i}\Big\{[-\sin\theta_i, \cos\theta_i] h_i^{\mathrm{d}}\Big\},\end{aligned} \tag{15}$$

where f_i is the distance-based formation control law designed for the single-integrator model as shown in (3). It is noted that (15) would become (12) if h_i^{d} is replaced by f_i. Here h_i^{d} can be designed to potentially achieve obstacle avoidance as shown below. When there are no obstacles, design

$$h_i^{\mathrm{d}}(t) = \begin{cases} f_i, & \|f_i\| \leq \alpha,\\ \dfrac{f_i}{\|f_i\|}\alpha, & \|f_i\| > \alpha,\end{cases}$$

so that h_i^{d} is aligned with f_i and satisfies $\|h_i^{\mathrm{d}}\| \leq \alpha$ where $\alpha > 0$ is a constant control gain. When the distance between agent i and an obstacle is less than a predefined threshold and the gradient flow f_i points towards the obstacle, agent i must change its velocity direction; otherwise, the agent will collide with the obstacle. As shown in Figure 5, the obstacle and the agent form a cone with the agent as the vertex. We may choose h_i^{d} to be a vector along the edge of the cone. In terms of magnitude, we may choose $\|h_i^{\mathrm{d}}\| = \alpha$.

If the angle between h_i^{d} and f_i is always less than $\pi/2$, then the system convergence is guaranteed because all the conditions in Theorem 3 are satisfied. However, if there are multiple obstacles, we may not be able to find h_i^{d} satisfying the angle condition. In this case, the convergence may not be guaranteed. Indeed, obstacle avoidance subject to control saturation is a very challenging research problem. Even if an obstacle can be successfully detected, the agent may still collide to the obstacle due to the lack of sufficient maneuverability. To tackle this problem, more complicated strategies may be designed based on other theoretical tools such as reciprocal velocity obstacles [27], [28] or game theory [29].

B. Simulation Results

To demonstrate the control law in (15) and the obstacle avoidance strategy, a simulation example is shown in Figure 6. In this example, there are three agents and the underlying graph is complete. The target formation is an equilateral triangle with side length equal to four meters. The maximum forward and backward linear speeds are $\mathrm{v}_i^{\mathrm{f}} = 1$ m/s and $\mathrm{v}_i^{\mathrm{b}} = 0.5$ m/s for all i. The maximum angular speeds are $\mathrm{w}_i^{\mathrm{l}} = \mathrm{w}_i^{\mathrm{r}} = \pi/4$ rad/s for all i. For obstacle avoidance, α is chosen to be equal to 1. Agent i triggers obstacle avoidance mechanism when the gradient flow points to an obstacle and the

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distance from agent i to any point on the obstacle is less than two meters.

As can be seen, the Lyapunov function converges to zero, which indicates that the target formation is successfully achieved. The linear and angular speed saturation constraints are both satisfied. It is notable that the velocity control resembles bang-bang control within the first 18 seconds. That is because the gradient control term f_i may be extremely large when the distance errors are large ($\|f_i\|$ may reach 10^4 in this simulation example). Moreover, the angular speed for each agent may be discontinuous due to the discontinuous switch of h_i^{d} to avoid obstacles. Of course, one may design a continuous version of h_i^{d} to obtain a continuous angular velocity if needed.

VI. Conclusions

This paper proposed a general approach to design coordination control laws for multi-agent systems subject to motion constraints. It has been shown that a distributed gradient control law designed for single-integrator dynamics can be easily modified to accommodate heterogeneous motion constraints such as nonholonomic dynamics and velocity saturation while preserving the system convergence. The proposed approach also provides additional flexibility to handle path constraints such as obstacle avoidance. The proposed approach is applicable to a wide range of coordination tasks such as rendezvous and formation control in two- and three-dimensional spaces. Acceleration saturation is a common constraint that real mobile robots are subject to. It is meaningful to study if the proposed approach can be generalized to handle acceleration saturation constraints in the future.

Appendix

A. Examples Satisfying Assumption 1

Example 3 (Displacement-Based Formation control). *The objective of displacement-based formation control is to steer the agents from some initial positions to converge to a desired geometric pattern defined by constant relative positions* $\{p_i^*-p_j^*\}_{(i,j)\in\mathcal{E}}$. *This formation control problem degenerates to the rendezvous problem when* $p_i^*=p_j^*$ *for all* $i,j\in\mathcal{V}$. *Consider the Lyapunov function*

$$V=\frac{1}{4}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}\left\|(p_i-p_j)-(p_i^*-p_j^*)\right\|^2.$$

The target formation is achieved if and only if $V=0$ *since the graph is bidirectional and connected. The gradient control law*

$$\dot{p}_i=f_i=\sum_{j\in\mathcal{N}_i}\left[(p_j-p_i)-(p_j^*-p_i^*)\right]$$

is the displacement-based formation control law [24], [30]. Consider any oriented graph and define the error state as $e_k=p_i-p_j-(p_i^*-p_j^*)$ *with* $k=1,\dots,m$ *and* $e=(H\otimes I)(p-p^*)$. *Then,* $V(e)=1/2\sum_{i=1}^m\|e_k\|^2$, $\partial e/\partial p=H\otimes I$ *is constant,* f *is continuous in* e, *and* $\|f\|$ *is bounded when* $\|e\|$ *is bounded. Since* $V=1/2(p-p^*)^T(L\otimes I)(p-p^*)$ *and* $\dot{p}=f=-(L\otimes I)(p-p^*)$, *we have* $f=0\Leftrightarrow V=0\Leftrightarrow e=0$ *and the attraction region* $\Omega(r_0)$ *is the entire space* $\mathbb{R}^{dm}$. *Therefore, all the conditions in Assumption 1 are satisfied.*

Example 4 (Bearing-Based Formation Control). *The objective of bearing-based formation control is to steer the agents from some initial positions to converge to a desired geometric pattern defined by constant inter-neighbor bearings* $\{g_{ij}^*\}_{(i,j)\in\mathcal{E}}$. *Consider the Lyapunov function*

$$V=\frac{1}{4}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}\|P_{g_{ij}^*}(p_i-p_j)\|^2,$$

where $P_{g_{ij}^*}=I-g_{ij}^*(g_{ij}^*)^T$. *The gradient control law*

$$\dot{p}_i=f_i=\sum_{j\in\mathcal{N}_i}P_{g_{ij}^*}(p_j-p_i)$$

is the bearing-based formation control law proposed in [31]. For any oriented graph, define the error state as $e_k=P_{g_{ij}^*}(p_i-p_j)$ *with* $k=1,\dots,m$. *Then,* $V(e)=1/2\sum_{k=1}^m\|e_k\|^2$, $\partial e/\partial p=\mathrm{diag}(P_{g_1^*},\dots,P_{g_m^*})(H\otimes I)$ *is constant,* f *is uniformly continuous in* e, *and* $\|f\|$ *is bounded when* $\|e\|$ *is bounded. Let* $\mathcal{B}\in\mathbb{R}^{dn\times dn}$ *be the bearing Laplacian (see the definition in [32, Sec 3]). Then,* $V=1/2p^T\mathcal{B}p$ *and* $\dot{p}=f=-\mathcal{B}p$. *As a result,* $f=0\Leftrightarrow V=0\Leftrightarrow e=0$ *and the attraction region* $\Omega(r_0)$ *is the entire space* $\mathbb{R}^{dm}$. *Therefore, all the conditions in Assumption 1 are satisfied.*

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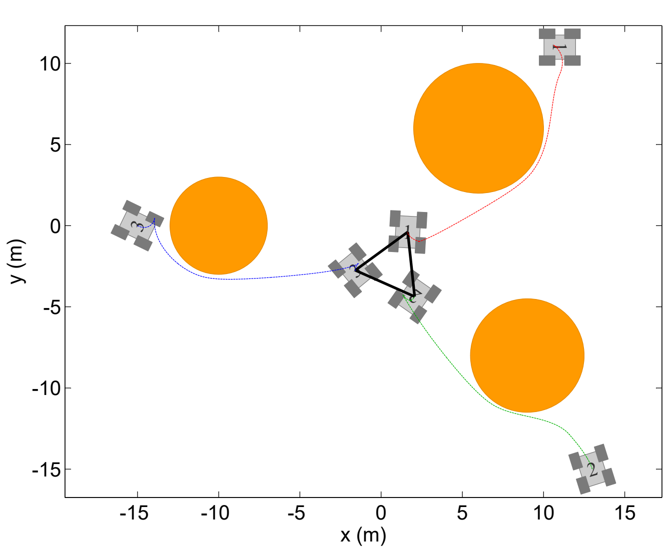
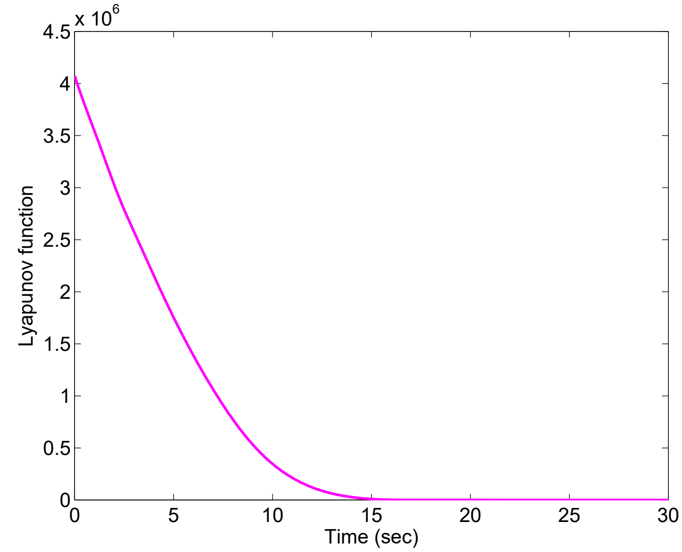
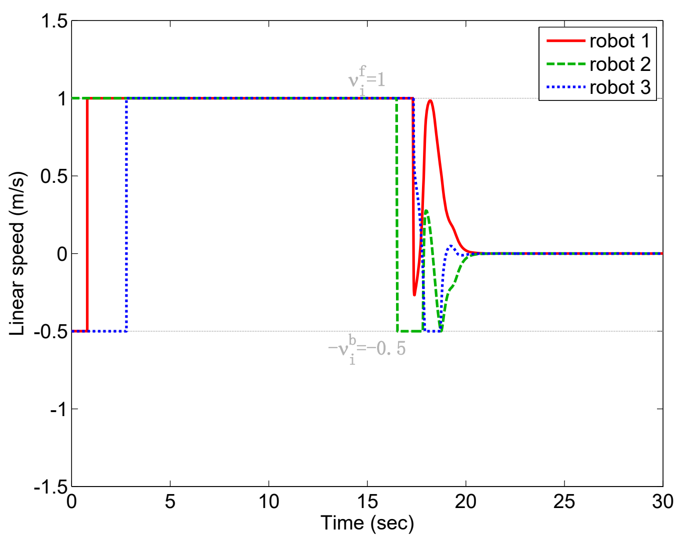


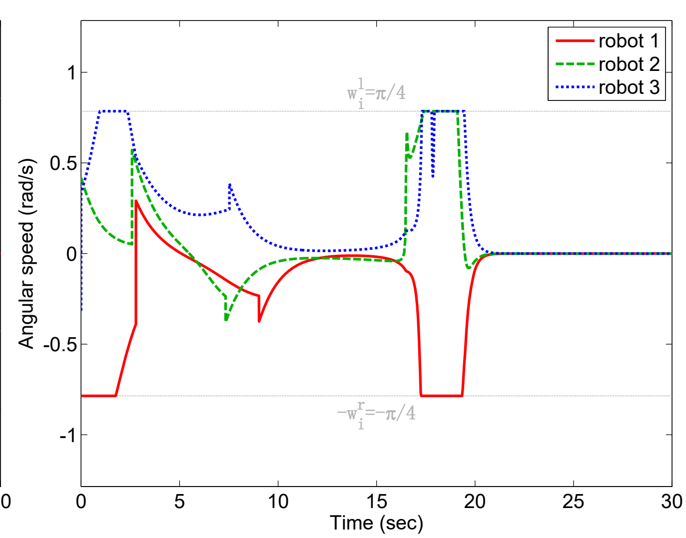
(a) Trajectory



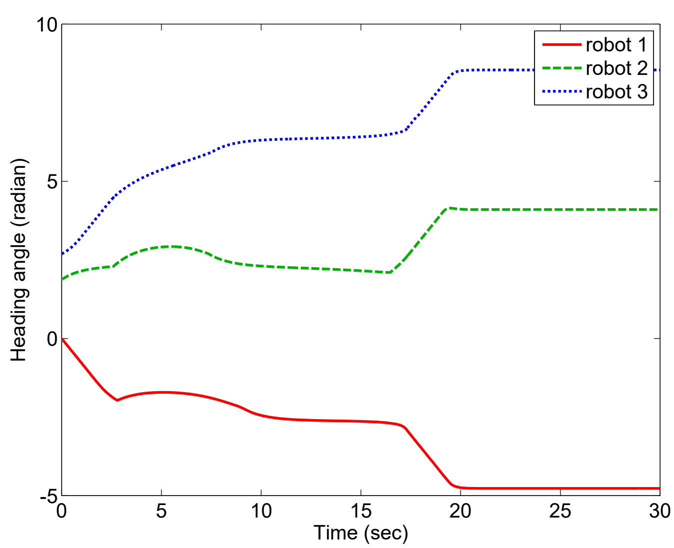
(b) The Lyapunov function converges to zero.



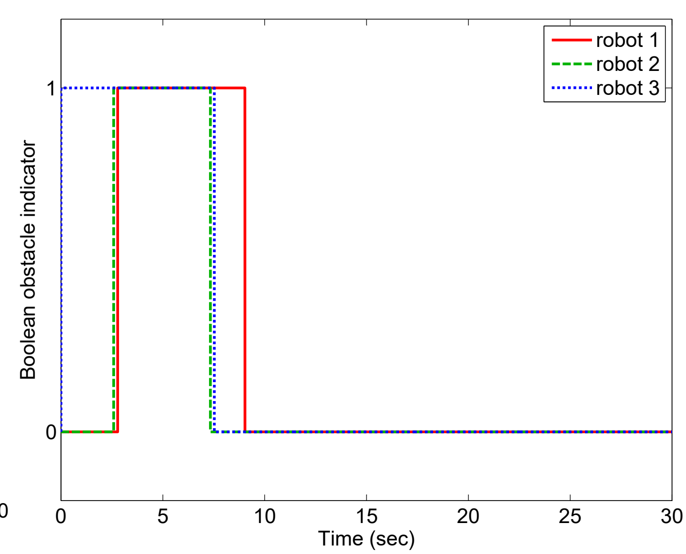
(c) The linear velocity saturation is satisfied.



(d) The angular velocity saturation is satisfied.



(e) Heading angle θ_i



(f) Obstacle indicator (one means obstacle detected and zero no obstacle)

Fig. 6: Simulation results of distance-based formation control of unicycle agents with motion constraints.

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