多智能体系统——基础

罗大会

2022 年 3 月 20 日

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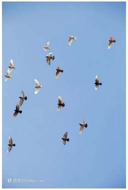
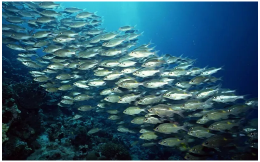
研究背景

- 自然界现象



(a) birds



(b) fish

(c) Ant

1) 鱼群/鸟群迁徙
2) 集体觅食
3) 躲避天敌

- 工程应用

(d) Robot Soccer

(e) Welding robot

(f) Plane

- 智能体的特点

1) 信息处理和执行能力有限

2) 传感和通信能力有限

3) 分布式

一致性问题的描述

一致性问题是多智能体协同控制中的典型问题之一,实际上也是根本问题.

- 聚集问题
- 同步问题
- 集群运动

- Boid模型
- Vicsek模型

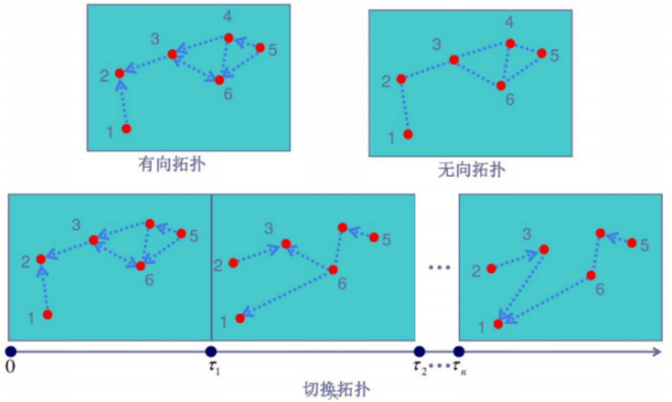
一致性问题的建模

- 智能体动态模型

表 1: 智能体动态模型

序号	模型	表达
1	线性系统模型	$\dot{x}_i = Ax_i + Bu_i$
	非线性系统模型	$\dot{x}_i = f(x_i, u_i)$
2	连续时间模型	$\dot{x}_i = Ax_i + Bu_i$
	离散时间模型	$\dot{x}_i(k+1) = Ax_i(k) + Bu_i(k)$
3	时变系统模型	$\dot{x}_i = A(t)x_i + B(t)u_i$
	时不变系统模型	$\dot{x}_i = Ax_i + Bu_i$
4	同构系统模型	$\dot{x}_i = Ax_i + Bu_i$
	异构系统模型	$\dot{x}_i = A_i x_i + B_i u_i$
5	低阶系统模型	一阶: $\dot{x}_i = u_i$; 二阶: $\dot{x}_{i1}(t) = x_{i2}(t), \dot{x}_{i2}(t) = u_i$
	高阶系统模型	$\dot{x}_i = Ax_i + Bu_i, A \in R^{n\times m}, B \in R^{n\times m}$

- 信息拓扑结构
  - 有向、无向
  - 固定、时变



Recommended Books:

W. Ren, R. W. Beard, *Distributed consensus in Multi-vehicle cooperative control theory and applications*, London: Spring-Verlag, 2008.

Y. D., Song, Y. J. ,Wang, *Cooprative control of nonlinear networked systems-infinite time and finite time design methods*, Switzerland: Springer, 2019.

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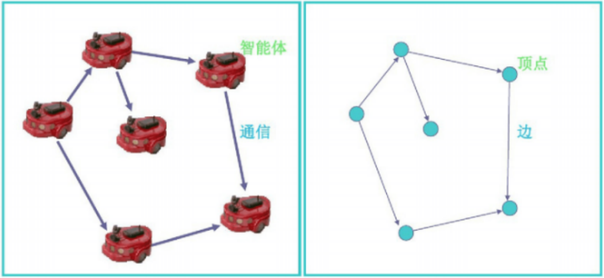
图论基础



多智能体网络　　　有向图

表 2: 图论基础

有向加权图或有向图	$\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$
$\mathcal{V} = \{v_1, v_2, \cdots, v_n\}$:	图的n个顶点
$\mathcal{E} \in \mathcal{V} \times \mathcal{V}$:	由节点对组成的边集合
$e_{ij} = (v_i, v_j) \in \mathcal{E}$:	如果存在从第i个顶点到第j个顶点的信息流， 则该节点对有连边
$\mathcal{A}$:	邻接矩阵，表示节点与边的关系
邻接矩阵	$\mathcal{A} = [a_{ij}]$, $a_{ij} = 1$ 如果$(v_i, v_j) \in \mathcal{E}$; 否则, $a_{ij} = 0$
加权邻接矩阵	$\mathcal{A} = [a_{ij}]$, $a_{ij} = \omega_{ij}$ 如果$(v_i, v_j) \in \mathcal{E}$; 否则, $a_{ij} = 0$
度矩阵	$\mathcal{D} = diag(deg(v_1), deg(v_2), \cdots, deg(v_n))$，$deg(v_i) = \sum_{i=1}^{n} a_{ij}$
图的Laplacian矩阵	$\mathcal{L} = \mathcal{D} - \mathcal{A}$

Consider information states with single-integrator dynamics given by

$$\dot{x}_i = u_i, i = 1, \cdots, N, \tag{1}$$

where $x_i \in R$ is the information state and $u_i \in R$ is the information control input of the ith agent

A continuous-time consensus algorithm is given by

$$u_i = -\sum_{j=1}^{N} a_{ij}(x_i - x_j), i = 1, \cdots, N. \tag{2}$$

With (2), if consensus is achieved or reached by the team of agents, for all $x_i(0)$ and all $i, j = 1, \cdots, N$, $\|\xi_i(t) - \xi_j\| \to 0$, as $t \to \infty$.

Denoted

$$
\begin{aligned}
e_i &= \sum_{j \in \mathcal{N}_i}^{N} a_{ij}(x_i - x_j) \\
&= \sum_{j \in \mathcal{N}_i}^{N} a_{ij} x_i - \sum_{j \in \mathcal{N}_i}^{N} a_{ij} x_j \\
&= -a_{i1}x_1 - \cdots - a_{i,i-1}x_{i-1} + \sum_{j \in \mathcal{N}_i}^{N} a_{ij} x_i - a_{i,i+1}x_{i+1} - \cdots - a_{iN}x_N \\
&= [-a_{i1}, \cdots, -a_{i,i-1}, \sum_{j \in \mathcal{N}_i}^{N} a_{ij}, -a_{i,i+1}, \cdots, -a_{iN}]X \qquad (3)
\end{aligned}
$$

where $X = [x_1, \cdots, x_N]^T$.

$$
\begin{aligned}
&E \\
=&\begin{bmatrix} \sum_{j\in\mathcal{N}_1}^{N} a_{ij} & -a_{12} & \cdots & -a_{1N} \\ -a_{21} & \sum_{j\in\mathcal{N}_2}^{N} a_{ij} & \cdots & -a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N1} & -a_{N2} & \cdots & \sum_{j\in\mathcal{N}_N}^{N} a_{ij} \end{bmatrix} X \\
=&\left(\begin{bmatrix} \sum_{j\in\mathcal{N}_1}^{N} a_{ij} & 0 & \cdots & 0 \\ 0 & \sum_{j\in\mathcal{N}_2}^{N} a_{ij} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{j\in\mathcal{N}_N}^{N} a_{ij} \end{bmatrix} - \begin{bmatrix} 0 & a_{12} & \cdots & a_{1N} \\ a_{21} & 0 & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & 0 \end{bmatrix} \right) X \\
=&(\mathcal{D}-\mathcal{A})X \\
=&\mathcal{L}X \qquad (4)
\end{aligned}
$$

where $E = [e_1, \cdots, e_N]^T$.

Laplacian矩阵的部分性质：

- 0是Laplacian矩阵的特征值，$\mathbf{1}_n = [1, 1, \ldots, 1]^T$ 为属于特征值0的右特征向量；
- 如果有向图$\mathcal{G}$的阶数为n, Laplacian矩阵为L, 如果$\mathcal{G}$是强连通的，那么有$rank(L) = n - 1$
- 如果$\mathcal{G}$ 是连通的且对称，那么L是对称、半正定的，并且所有的特征值都是实数且非负，可以写成

$$0 = \lambda_1(L) < \lambda_2(L) \leq \cdots \leq \lambda_n(L) \tag{5}$$

Properties of real symmetric matrices：

- 实对称矩阵的特征值为实数;
- 设$\lambda_1, \cdots, \lambda_n (n \geq 2)$ 是实对称矩阵A的n个特征值,$p_1, \cdots, p_n$ 是对应的特征向量, 若$\lambda_1 \neq \cdots \neq \lambda_n$, 则$p_1, \cdots, p_n$ 两两正交;
- 设A为n阶实对称阵, 则必有正交矩阵P $(P^{-1} = P^T)$, 使$P^{-1}AP = \Lambda$, 其中Λ是以A 的n个特征值为对角元素的对角矩阵.

Leaderless

Assumption

The directed communication network $\mathcal{G}$ is strongly connected.

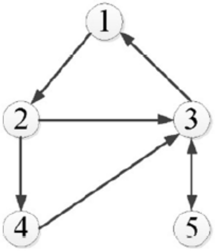


图 1: Strongly connected graph

Lemma

Suppose that the digraph is strongly connected. Let $\omega_l = [\omega_1, \cdots, \omega_N]^T$ be the left eigenvector of $\mathcal{L}$ associated with eigenvalue $\lambda = 0$ (i.e., ω_l is the first left eigenvector of $\mathcal{L}$). Define

$$\begin{aligned} \Omega &= diag\{\omega_i\} \in R^{N\times N}, \\ Q &= \Omega L + L^T\Omega. \end{aligned} \tag{6}$$

Then

$$V = \sum_{i,j=1}^{N} \omega_i a_{ij}(x_j - x_i)^2 = x^T Q x \tag{7}$$

Moreover, $\Omega > 0$ and $Q \geq 0$.

H. W. Zhang, F. L. Lewis & Z. H. Qu, "Lyapunov, adaptive, and optimal design techniques for coopperative systems on directed communication graph", *IEEE Trans. Ind. Electron.*, Vol. 59, No. 7, PP. 3026-3041, Jul. 2012.

Lemma

If $Q = Q^T$ *and* $Q \geq 0$ *or* $Q \leq 0$*, the the null space* $Null(Q) = \{x | x^T Qx = 0\}$.

Lemma

Let the digraph be strongly connected and Q *is defined as* (6)*; then,* $Null(Q) = Null(L) = span\{\mathbf{1}_N\}$.

Consider a group of N agents with single integrator dynamics

$$\dot{x}_i = u_i, i \in \mathcal{N}, \tag{8}$$

where $x_i \in R$, $u_i \in R$.

Consider a common linear consensus protocol

$$u_i = -\sum_{j=1}^{N} a_{ij}(x_i - x_j). \tag{9}$$

Let $x = [x_1, \cdots, x_N]^T$; the closed-loop system can be written collectively as

$$\dot{x} = -Lx. \tag{10}$$

Theorem 1

For strongly connected digraphs, consensus in (8) can be reached using the consensus protocol (9)

Proof: Consider the Lyapunov function candidate as

$$V = x^T \Omega x \tag{11}$$

Then

$$\dot{V} = 2x^T \Omega \dot{x} = -2x^T \Omega L x = -x^T (\Omega L + L^T \Omega) x \tag{12}$$

Due to $\Omega L + L^T \Omega \geq 0$, $\dot{V} \leq 0$. By LaSalle invariance principle, the trajectories converge to the largest invariant set $S = \{x \in R^N | \dot{V} = 0\}$.

It is straightforward that $S = \{x^* \in R^N | x^* = \alpha \mathbf{1}_N, \forall \alpha \in R\}$.

Therefore $x(t) \to \alpha \mathbf{1}_N$ for all some $\alpha \in R$ as $t \to \infty$.

Keep in mind that Theorem 1 doesn't guarantee whether the group decision value α is equal to **Ave(x(0))**, or not. In other words, Theorem does not necessarily address the average consensus problem.

Considering that the solution of (10) with fixed topology is given by

$$x(t) = exp(-Lt)x(0) \tag{13}$$

by explicit calculation of $exp(-Lt)$, one can obtain the group decision value for a general digraph.

Theorem

Assume $\mathcal{G}$ is a strongly connected digraph with Laplacian L satisfying $L\omega_r = 0$, $\omega_l^T L = 0$, and $\omega_l^T \omega_r = 1$. Then

$$R = \lim_{t\to\infty} exp(-Lt) = \omega_r \omega_l^T \in R^{N\times N} \tag{14}$$

Proof: Let $A = -L$ and let J be the Jordan form associated with A, i.e., $A = SJS$. We have

$$\begin{aligned} \exp(At) &= \exp(SJS^{-1}t) = S\exp(Jt)S^{-1} \\ &= S\exp\left(\begin{bmatrix} -\Lambda_1 & 0 & \cdots & 0 \\ 0 & -\Lambda_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\Lambda_k \end{bmatrix} t\right) S^{-1}, \end{aligned} \tag{15}$$

where $\Lambda_m, m = 1, \cdots, k$ are standard Jordan blocks, and $\Lambda_1 = 0$.

then

$$\exp(Jt) \to \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix} = \widetilde{Q}, \text{as} \;\; t \to \infty. \tag{16}$$

Notice that $R = S\widetilde{Q}S^{-1}$. Since $AS = SJ$, the first column of S is ω_r. Similarly, $S^{-1}A = JS^{-1}$ that means the first row of S^{-1} is ω_l^T. Due to the fact that $S^{-1}S = I_N$, ω_l satisfies the property $\omega_l^T\omega_r = 1$ as stated in the equation. Therefore,

$$R = \begin{bmatrix} \omega_{r1} & \overline{\omega}_{12} & \cdots & \overline{\omega}_{1N} \\ \omega_{r2} & \overline{\omega}_{22} & \cdots & \overline{\omega}_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ \omega_{rN} & \cdots & \overline{\omega}_{N,N-1} & \overline{\omega}_{N,N} \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_{l1} & \omega_{l2} & \cdots & \omega_{lN} \\ \widetilde{\omega}_{21} & \widetilde{\omega}_{22} & \cdots & \widetilde{\omega}_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ \widetilde{\omega}_{N1} & \cdots & \widetilde{\omega}_{N,N-1} & \widetilde{\omega}_{N,N} \end{bmatrix} \tag{17}$$

R. O., Saber & R. M. Murray, "Consensus problem in nerworks of agents with switching topology and time-delays", *IEEE Trans. Auto. Contr.*, Vol. 49, No.9, PP. 1520–1533, Sep. 2004.

$$R=\begin{bmatrix}\omega_{r1} & 0 & \cdots & 0\\ \omega_{r2} & 0 & \cdots & 0\\ \vdots & \ddots & \ddots & \vdots\\ \omega_{rN} & \cdots & 0 & 0\end{bmatrix}\begin{bmatrix}\omega_{l1} & \omega_{l2} & \cdots & \omega_{lN}\\ \widetilde{\omega}_{21} & \widetilde{\omega}_{22} & \cdots & \widetilde{\omega}_{2N}\\ \vdots & \ddots & \ddots & \vdots\\ \widetilde{\omega}_{N1} & \cdots & \widetilde{\omega}_{N,N-1} & \widetilde{\omega}_{N,N}\end{bmatrix}$$

$$=\begin{bmatrix}\omega_{r1}\omega_{l1} & \omega_{r1}\omega_{l2} & \cdots & \omega_{r1}\omega_{lN}\\ \omega_{r2}\omega_{l1} & \omega_{r2}\omega_{l2} & \cdots & \omega_{r2}\omega_{lN}\\ \vdots & \ddots & \ddots & \vdots\\ \omega_{rN}\omega_{l1} & \cdots & \omega_{rN}\omega_{l,N-1} & \omega_{rN}\omega_{l,N}\end{bmatrix}=\omega_r\omega_l^T \tag{18}$$

- $\mathcal{G}$ is undirected graph: $\omega_r=\omega_l=\frac{1}{\sqrt{N}}\mathbf{1}_N$, therefore $R=\begin{bmatrix}\frac{1}{N} & \cdots & \frac{1}{N}\\ \vdots & \ddots & \vdots\\ \frac{1}{N} & \cdots & \frac{1}{N}\end{bmatrix}$,

$\lim\limits_{i\to\infty} x_i(t)=\frac{1}{N}\sum_{i=1}^{N} x_i(0)$, the average consensus is achieved.

- $\mathcal{G}$ is directed graph

  $\omega_r = \frac{1}{\sqrt{N}}\mathbf{1}_N$, therefore

$$R = \begin{bmatrix} \frac{\omega_{l1}}{\sqrt{N}} & \frac{\omega_{l2}}{\sqrt{N}} & \cdots & \frac{\omega_{lN}}{\sqrt{N}} \\ \frac{\omega_{l1}}{\sqrt{N}} & \frac{\omega_{l2}}{\sqrt{N}} & \cdots & \frac{\omega_{lN}}{\sqrt{N}} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\omega_{l1}}{\sqrt{N}} & \frac{\omega_{l2}}{\sqrt{N}} & \cdots & \frac{\omega_{lN}}{\sqrt{N}} \end{bmatrix}, \tag{19}$$

  $\lim_{t\to\infty} x_i(t) = \sum_{j=1}^{N} \frac{\omega_{lj}}{\sqrt{N}} x_i(0) = \sum_{j=1}^{N} \xi_i x_i(0)$, the weight average consensus is achieved. Due to $\omega_l^T \omega_r = 1$, $\sum_{j=1} \omega_{lj} = \sqrt{N}$, then

$$\sum_{j=1}^{N} \xi_j = \sum_{j=1}^{N} \frac{\omega_{lj}}{\sqrt{N}} = \frac{\sum_{j=1}^{N} \omega_{lj}}{\sqrt{N}} = \frac{\sqrt{N}}{\sqrt{N}} = 1. \tag{20}$$

Assumption

The digraph $\widetilde{\mathcal{G}}$ contains a spanning tree and the root v_{ir} can can get information from the leader node.

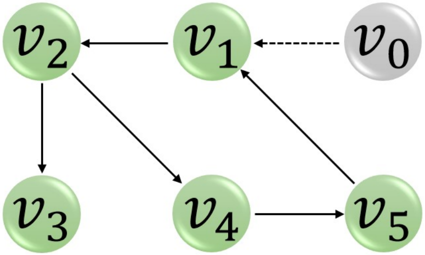


图 2: Directed graph containing a spanning tree

One leader

Denote the adjacency matrix of the subgraph$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ formed by the followers by

$$\mathcal{A} = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1N} \\ a_{21} & 0 & \cdots & a_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ a_{N-1,1} & \cdots & 0 & a_{N-1,N} \\ a_{N,1} & \cdots & a_{N-1,1} & 0 \end{bmatrix} \in R^{N \times N}.$$

Let $\mathcal{D} = diag\{d_1, d_2, \cdots, d_N\}$, $d_i = \sum_{j=1}^{N} a_{ij}$. The Laplacian matrix of subgraph $\overline{\mathcal{G}}$ can be defined as

$$\mathcal{L} = \mathcal{D} - \mathcal{A} \tag{21}$$

Thus, for the digraph $\widetilde{\mathcal{G}} = (\widetilde{\mathcal{V}}, \widetilde{\mathcal{E}})$, we have

$$\widetilde{\mathcal{L}} = \begin{bmatrix} 0 & 0_{1\times N} \\ -b & \mathcal{L}_1 = \mathcal{L} + B \end{bmatrix}. \tag{22}$$

If the graph $\mathcal{G}$ has a spanning tree, then $Rank(\widetilde{\mathcal{L}}) = N$.

In light of (22), it follows that $Rank[-b, \mathcal{L}_1] = N$ and $\mathcal{L}_1 \mathbf{1}_N = b$. Thus, $Rank(\mathcal{L} + B) = N$. An important special case is that $\widetilde{\mathcal{G}}$ is strongly connected. In this situation $\mathcal{L}_1$ is irreducible. Then, $\mathcal{L}_1$ is irreducibly diagonally dominant as long as gain$b_i \neq 0$ and, hence, nonsingular.

G. Chen, F. L. Lewis, "Distributed adaptive tracking control for synchronization of unkrown networked Lagrangian systems", *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, Vol. 41, No. 3, PP. 805-816, Jun. 2011.

Lemma

There exists a positive diagonal matrix $P = diag\{p_1, \cdots, p_N\} \in R^{N\times N}$ such that

$$Q = P\mathcal{L}_1 + \mathcal{L}_1^T P > 0. \tag{23}$$

One such P is given by $diag\{p_1, \cdots, p_N\}$, where $[p_1, \cdots, p_N]^T = (\mathcal{L}_1)^{-1}\mathbf{1}_N$.

Z. K. Li, G. H. Wen, Z. S. Duan & W. Ren, "Designing fully distributed consensus protocols for linear multi-agent systems with directed graphs", *IEEE Trans. Auto. Contr.*, Vol. 60, No. 4, PP. 1152-1157, Apr. 2015.

Denote $\delta_i = x_i - x_0$,

Controller: $u_i = \sum_{j \in N_i} a_{ij}(x_j - x_i) + b_i(x_0 - x_i)$, $U = -L_1\delta$

Choose the Lyapunov function candidate that

$$V = \delta^T P \delta, \tag{24}$$

where $\delta = [\delta_1, \cdots, \delta_N]^T$. Noted that

$$L_1\delta = L_1(X - \mathbf{1}_N * x_0) = L_1X - L_1\mathbf{1}_N * x_0 = L_1X. \tag{25}$$

Taking derivation for (24), we have

$$\begin{aligned} \dot{V} &= 2\delta^T P\dot{\delta} = 2\delta^T P\dot{X} = 2\delta^T PU = -2\delta^T P(L_1X) \\ &= -2\delta^T P(L_1\delta) = -\delta^T(PL_1 + L_1^T P)\delta = -X^T QX \leq -\lambda_1 X^T X \\ &\leq -\lambda_1 \frac{\delta^T \delta}{X^T PX}\delta^T P\delta \leq -\frac{\lambda_1}{\max\{p_i\}} V \end{aligned} \tag{26}$$

Therefore, $\|\delta\| \leq \exp(-\frac{\lambda_1}{\max\{p_i\}}t)\sqrt{\frac{\max\{p_i\}}{\min\{p_i\}}}\|\delta(0)\|$.

Multiple leaders

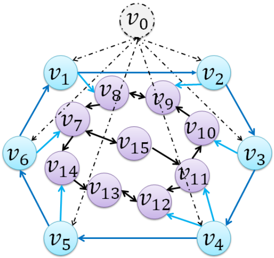


图 3: Directed graph containing a spanning tree

表 3: 合围(Containment control)研究现状

No.	Year	Main work
1	2008	proposed containment control; Stop-Go a collection of mobile robots to a given target destination
2	2009	Defined containment control; fixed/ switching digraph; continuous/discrete: $\dot{x}_i(t) = u_i(t)$ stationary or dynamic leaders
3	2010	finite-time attitude containment control; stationary/dynamic leaders; distributed sliding-mode estimator ; Lagrange system

M. Ji, G. Ferrari-Trecate, M. Egerstedt and A. Buffa, "Containment control in mobile networks," *IEEE Trans. Auto. Control.*, Vol. 53, No. 8, PP. 1972-1975, Sep., 2008.

Y. C. Cao, W. Ren, "Containment Control with Multiple Stationary or Dynamic Leaders Under a Directed Interaction Graph," *48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, PP. 3014-3019, Dec., 2009.

Z. Y. Meng, W. Ren and Z. You, "Distributed finite-time attitude containment control for multiple rigid bodies," *Automatica*, PP. 2092-2099, 2010.

表 4: 合围(Containment control)研究现状

No.	Year	Main work
4	2011	general linear dynamics; continuous/discrete-time the leader have no neigebors
5	2012	general linear dynamics; internal/boundary agents; a pinning control stratege

Z. K. Li, W. Ren, X. D. Liu and M. Y. Fu, "Distributed containment control of multi-agent systems with general linear dynamics in the presence of multiple leaders," *International Journal of Robust and Nonlear Control*, Vol. 23, No. 5 PP. 534-547, Dec., 2011.

H. Y. Liu, G. M. Xie and L. Wang, "Containment of linear multi-agent systems under general interaction topologies," *Systems & Control Letters*, Vol. 61, PP. 528-534, Mar., 2012.

X. Y. Dong, Z. Y. Shi, G. Lu and Y. S. Zhong, "Formation-containment control for high-order linear time-invariant multi-agent systems," *Proceedings of the 33rd Chinese Control Conference*, PP. 28-30, Jul. 2014.

Suppose that there are M leaders and $N-M$ followers in the directed graph $\mathcal{G}$, where a leader is an agent that has no in-neighbor and a follower is an agent that has at least one in-neighbor. Let $L=\{1,2,\cdots,M\}$ and $F=\{M+1,\cdots,N\}$ is the leader set and the follower set, respectively.

The model is given that

$$\dot{x}_i(t)=u_i(t), \quad x_i\in\mathbb{R}. \tag{27}$$

The Laplacian Matrix is represented as

$$\widetilde{\mathcal{L}}=\begin{bmatrix}\mathcal{L}_{LL} & 0_{M\times(N-M)}\\ \mathcal{L}_{FL} & \mathcal{L}_{FF}\end{bmatrix}. \tag{28}$$

where $\mathcal{L}_{LL}\in R^{M\times M}$, $\mathcal{L}_{FL}\in R^{(N-M)\times M}$ and $\mathcal{L}_{FF}\in R^{(N-M)\times(N-M)}$.

Here, $\mathcal{L}_{LL}=0$.

Lemma

There exists a positive diagonal matrix $P_F = diag\{p_{M+1}, \cdots, p_N\} \in R^{(N-M)\times(N-M)}$ *such that*

$$Q_F = P_F \mathcal{L}_{FF} + \mathcal{L}_{FF}^T P_F > 0 \tag{29}$$

in which $p_{M+1}, \cdots, p_N$ *are chosen as* $[p_{M+1}, \cdots, p_N]^T = (L_{FF}^T)^{-1} 1_{N-M}$. *In addition, each entry of* $-\mathcal{L}_{FF}^{-1}\mathcal{L}_{FL}$ *is nonnegative, and each row of* $-\mathcal{L}_{FF}^{-1}\mathcal{L}_{FL}$ *has a sum equal to one.*

Noted: $\begin{bmatrix} \mathcal{L}_{FL} & \mathcal{L}_{FF} \end{bmatrix} \begin{bmatrix} \mathbf{1}_M \\ \mathbf{1}_{N-M} \end{bmatrix} = \mathbf{0} \Rightarrow \mathcal{L}_{FL}\mathbf{1}_M + \mathcal{L}_{FF}\mathbf{1}_{N-M} = \mathbf{0}$

$\Rightarrow -\mathcal{L}_{FF}^{-1}\mathcal{L}_{FL}\mathbf{1}_{N-M} = \mathbf{1}_M$.

Y,J. Wang, Y.D. Song, D.J. Hill, M. Kristic, "Prescribed-time consensus and containment control of networked multi-agent systems," *IEEE Trans. Cybern.*, Vol. 49, No. 4, PP. 1138-1147, Apr. 2019.

Z. Y. Meng, W. Ren and Z. You, "Distributed finite-time attitude containment control for multiple rigid bodies," *Automatica*, PP. 2092-2099, 2010.

Example:

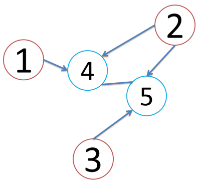


图 4: Graph

The Laplacian Matrix is represented as

$$\widetilde{\mathcal{L}} = \begin{bmatrix} \mathcal{L}_{LL} & 0_{M\times(N-M)} \\ \mathcal{L}_{FL} & \mathcal{L}_{FF} \end{bmatrix} = \begin{bmatrix} 0_{3\times 3} & 0_{3\times 2} \\ -1, -1, 0 & 3, -1 \\ 0, -1, -1 & -1, 3 \end{bmatrix}. \tag{30}$$

$$-\mathcal{L}_{FF}^{-1}\mathcal{L}_{FL} = -\begin{bmatrix} \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{8} \end{bmatrix} \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{4}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{4}{8} & \frac{3}{8} \end{bmatrix},$$

and then $rank(-\mathcal{L}_{FF}^{-1}\mathcal{L}_{FL}) = 2$.

Definition: We say MAS(27) achieves containment if for any initial states, there exist non-negative constant $\beta_j (j \in L)$ satisfying $\sum_{j=1}^{M} \beta_j = 1$ and $i \in F$, $x_i - \sum_{j=1}^{M} \beta_j x_j = 0_m$.

Let $E_L = [e_1^T, \cdots, e_M^T]^T$, $E_F = [e_{M+1}^T, \cdots, e_N^T]^T$, $X_L = [x_1^T, \cdots, x_M^T]^T$, $X_F = [x_{M+1}^T, \cdots, x_N^T]^T$

The error of neighbor is

$$e_i = \sum_{j \in \mathcal{N}_F} a_{ij}(x_i - x_j) + \sum_{k \in \mathcal{N}_L} a_{ik}(x_i - x_k), i \in F. \tag{31}$$

Then it holds

$$\begin{aligned} E_F =& \mathcal{L}_{FF} X_F + \mathcal{L}_{FL} X_L \\ =& \mathcal{L}_{FF} \left(X_F - (-\mathcal{L}_{FF}^{-1} \mathcal{L}_{FL}) X_L \right) \end{aligned} \tag{32}$$

Let $Z_F = X_F - (-\mathcal{L}_{FF}^{-1} \mathcal{L}_{FL}) X_L = \mathcal{L}_{FF}^{-1} E_F$.

Theorem

Consider system (27), the control scheme Controller

$$u_i = -e_i \tag{33}$$

or $U = -E_F$, the containment is achieved.

Proof: Choosing the Lyapunov function candidate as

$$V_F = E_F^T P_F E_F \tag{34}$$

According to , we have

$$\begin{aligned}\dot{E}_F =& \mathcal{L}_{FF}(\dot{X}_F - (-\mathcal{L}_{FF}^{-1}\mathcal{L}_{FL})\dot{X}_L) \\ =& \mathcal{L}_{FF}\dot{X}_F \\ =& -\mathcal{L}_{FF}E_F\end{aligned} \tag{35}$$

Then

$$
\begin{aligned}
\dot{V}_F =& 2E_F^T P_F \dot{E}_F \\
=& -2E_F^T P_F \mathcal{L}_{FF} E_F \\
=& -E_F^T (P_F \mathcal{L}_{FF} + \mathcal{L}_{FF}^F P_F) E_F \\
=& -E_F^T Q_F E_F \\
\leq& -\delta_{min}(Q_F) E_F^T E_F \\
\leq& -\frac{\delta_{min}(Q_F) E_F^T E_F}{E_F^T P_F E_F} \cdot V_F \\
\leq& -\frac{\delta_{min}(Q_F) E_F^T E_F}{\lambda_{max}(P_F) E_F^T E_F} V_F \\
\leq& -\frac{\delta_{min}(Q_F)}{\lambda_{max}(P_F)} V_F,
\end{aligned} \tag{36}
$$

which implies

$$V_F \leq \exp\left(-\frac{\delta_{min}(Q_F)}{\lambda_{max}(P_F)}(t-t_0)\right)V_F(t_0) \tag{37}$$

and then

$$\begin{aligned}\lambda_{min}(P_F)\|E_F\|^2 \leq& V_F\\ &\leq \exp\left(-\frac{\delta_{min}(Q_F)}{\lambda_{max}(P_F)}(t-t_0)\right)V_F(t_0)\\ &\leq \exp\left(-\frac{\delta_{min}(Q_F)}{\lambda_{max}(P_F)}(t-t_0)\right)\lambda_{\max}(P_F)\|E_F(t_0)\|^2, \end{aligned}\tag{38}$$

further

$$\|E_F\|^2 \leq \exp\left(-\frac{\delta_{min}(Q_F)}{\lambda_{max}(P_F)}(t-t_0)\right)\frac{\lambda_{\max}(P_F)}{\lambda_{\min}(P_F)}\|E_F(t_0)\|^2 \tag{39}$$

and then

$$\|Z_F\| = \|\mathcal{L}_{FF}^{-1}E_F\| \leq \exp\left(-\frac{\delta_{min}(Q_F)}{2\lambda_{max}(P_F)}(t-t_0)\right)\sqrt{\frac{\lambda_{\max}(P_F)}{\lambda_{\min}(P_F)}}\|E_F(t_0)\| \tag{40}$$

From (40), we have

$$\|Z_F\| \to 0 \quad \text{as} \quad t \to \infty, \tag{41}$$

that is $X_F \to (-\mathcal{L}_{FF}^{-1}\mathcal{L}_{FL})X_L$ as $t \to \infty$.