Making Prescribed Time Tracking Control Practical and Global Under Mismatched and Non-vanishing Uncertainties

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***Abstract*—Current prescribed time control (PTC) normally relies on time-varying feedback control gain that grows unlimitedly with time and becomes infinite at the equilibrium, rendering it impractical for real time implementation. This paper proposes a method that does not involve infinite control gain anytime during system operation, yet is able to deal with non-vanishing and mismatched uncertainties, leading to a practical and global prescribed time tracking control solution for a larger class of nonlinear systems. The key design technique lies in the utilization of a time-varying scaling function that grows monotonically with time and maintains bounded at and beyond the settling time, making PTC practical and linking PTC with its practical version analytically. It is shown that, without using any prior information on the upper or lower bound of system control gain, the proposed control is able to settle the tracking error in the neighborhood of origin within prescribed time, in spite of mismatched and non-vanishing uncertainties. In addition, the solution is truly global, allowing the system to operate on the entire time interval.**

***Index Terms*—Prescribed time control; strict-feedback systems; adaptive control; tracking control.**

I. Introduction

Prescribed time control (PTC) has attracted considerable attention from control community during the past few years [1]–[14] mainly because of its ability to achieve closed loop system stability within finite time that is independent of system initial condition thus can be pre-specified arbitrarily [1]. As the convergence rate is one of the most important factors for any control system, PTC is of particular interest for time-critical systems, e.g., emergency braking [15], missile interception [16], spaceship docking [17], and so forth. Compared with traditional finite-time control [18]–[26] and fixed time control [27]–[31] methods where the settling time is not at user disposal, PTC has its superiority, thus has motivated numerous following up studies and extensions since its introduction, including the consensus tracking control of high-order multi-agent systems [2], prescribed-time stabilization for nonlinear strict feedback-like systems [3], tracking control of multi-input and multi-output nonlinear systems with non-vanishing uncertainties [4] and the inverse prescribed-time optimality control problem for stochastic strict-feedback nonlinear systems [5] and so on.

However, in the aforementioned PTC works ([1]–[5]), , infinity control gain is inevitably involved that might lead to numerical problem in the implementation of the controller, rendering PTC somewhat unpractical. Moreover, most PTC methods are invalid beyond the prescribed time interval.

Efforts have been made on making PTC functional beyond prescribed time. In [6], the prescribed-time consensus and containment control problem is addressed for multi-agent systems upon using piecewise continuous time-varying scaling function. This idea has been extended to consensus tracking control of nonlinear multi-agent systems in [7], to cluster lag consensus for second-order multi-agent systems in [8], and to cooperative guidance law of multiple missiles in [9].

It is worth noting that in most existing works on PCT, at least four major issues have not been adequately addressed: 1) practicality of PTC; 2) accommodation of completely unknown control gains; and 3) rejection of mismatched yet non-vanishing uncertainties and operational capability beyond the settling time. In this paper we present a method aiming at addressing those issues simultaneously. The main contributions of this paper can be summarized as follow:

1) Different from current prescribed time control (PTC) that normally relies on time-varying feedback control gain growing unlimitedly with time and becoming infinite at the equilibrium, the proposed method, with the aid of a time-varying scaling function that grows monotonically with time and maintains bounded at and beyond the settling time, does not involve infinite control gain anytime during system operation, making PTC practical and linking PTC with its practical version analytically;
2) Without using any prior information on the upper or lower bound of system control gain, the proposed control is able to settle the tracking error in the neighborhood of origin within prescribed time; and
3) The developed solution is truly global, allowing the system to operate on the entire time interval, yet is able to deal with non-vanishing and mismatched uncertainties, leading to a practical and global prescribed time tracking control solution for a larger class of nonlinear systems.

Notation: Throughout this paper, t_0 is set the initial time; R denotes the set of real numbers; $L_\infty := \{\chi(t)|\chi : R_+ \to R, \sup_{t\in R_+} |\chi(t)| < \infty\}$. We denote by $\chi^{(i)}$ the ith derivative of χ, and by χ^i the ith power of χ.

II. Problem formulation and preliminaries

A. Problem formulation

In this paper, we consider the following strict-feedback systems with non-vanishing uncertainties modeled by,

$$\begin{cases} \dot{x}_i(t) = g_i(\bar{x}_i,t)x_{i+1}(t) + f_i(\bar{x}_i,t) \\ \qquad i = 1,\cdots,n-1 \\ \dot{x}_n(t) = g_n(\bar{x}_n,t)u(t) + f_n(\bar{x}_n,t) \\ \qquad y = x_1 \end{cases} \tag{1}$$

where $x_i(t) \in R$ with $\bar{x}_i = [x_1,\cdots,x_i]^\top$ is the state vector, $u(t) \in R$ denotes the control input and y is the output, $f_i(\bar{x}_i,t) \in R$ denotes the lumped uncertainty, which is unknown smooth function, and $g_i \in R$ is the unknown time-varying control gain.

In this paper, we aim at developing a control strategy for system (1) with non-vanishing mismatched uncertainties such that the output signal $y(t)$ can synchronize with $x_r(t)$ in the prescribed time T^* making the tracking error $e(t) = y(t) - x_r(t)$ converge to a small residual set containing origin and maintain the synchronization after that prescribed time.

Assumption 1. *The control gain $g_i(t)(i = 1,\cdots,n)$ is unknown and time-varying yet bounded away from zero, namely, there exist unknown constants $\underline{g}_i$ and $\overline{g}_i$ such that $0 < \underline{g}_i \leq |g_i(t)| \leq \overline{g}_i < +\infty$ and the control direction is definite (without loss of generality, we assume that $sgn\big(g_i(t)\big) = 1$).*

Assumption 2. *For the non-vanishing uncertain term $f_i(\bar{x}_i,t)(i = 1,\cdots,n)$, there exists an unknown constant $\nu_i > 0$ and a known continuously differentiable scalar function $\vartheta_i(\bar{x}_i)$ such that*

$$|f_i(\bar{x}_i,t)| \leq \nu_i\vartheta_i(\bar{x}_i), \tag{2}$$

In addition, $\vartheta_i(\bar{x}_i)$ is bounded if and only if $\bar{x}_i$ is bounded.

Assumption 3. *For all $t \in [0,\infty)$, the desired trajectory $x_r(t) \in R$ and its pth$(p = 1,\cdots,n-1)$ order derivatives are assumed to be known, bounded and piecewise continuous.*

Remark 1. *With Assumption 1, the resultant control scheme becomes more practical and more elegant because it does not require the upper bounded or lower bound of the virtual or actual control gain g_i $(i = 1,\cdots,n)$, in contrast to most existing schemes that normally demand certain prior bound information on the virtual and/or the actual control gains ([1], [11], [32]). In Assumption 2, $\vartheta_i(\bar{x}_i)$ $(i = 1,\cdots,n)$ denotes a computable scalar function carrying core information of the system [33] that is independent of system parameters. Assumption 3 is a commonly required condition in addressing the tracking control problem (see [4], [13], [32]).*

B. The preliminaries

Inspired by [1], we put forward a time-varying function defined on the whole time interval as follows,

$$\mu(t) = \begin{cases} \left(\frac{T^*}{T^*+t_0-t}\right)^q, & t \in [t_0, T^*+t_0-\epsilon) \\ \left(\frac{T^*}{\epsilon}\right)^q & t \in [T^*+t_0-\epsilon, +\infty) \end{cases} \tag{3}$$

where $q \geq 2$ is any user-defined real number, $T^* > T_s > 0$ with T_s being the physically feasible time limitation, and $\epsilon > 0$ is regarded as the adjustable time error parameter. It is noted that $\mu(t_0) = 1$, $\mu(t) \to (\frac{T^*}{\epsilon})^q$ as $t \to (T^*+t_0-\epsilon)^-$ and $\mu(t) = (\frac{T^*}{\epsilon})^q$ as $t \geq T^*+t_0-\epsilon$. ϵ $(0 < \epsilon < T^*+t_0)$ is chosen as a non-zero positive constant so that μ make sense. The remarkable thing is that $\sup|\mu| \leq (\frac{T^*}{\epsilon})^q$, which guarantees boundedness of μ.

Lemma 1. *Let $\psi(t)$ be an unknown bound function and $\|\psi\|_{[t_0,t]} = \sup\limits_{\tau\in[t_0,t]}|\psi(\tau)|$, and $V : [t_0,\infty) \to R^+ \cap \{0\}$ be a continuously differentiable function, then*

1) if it satisfies

$$\dot{V} \leq -k\mu(t)V - \frac{2q}{T^*}\mu(t)^{\frac{1}{q}}V + \frac{\psi(t)}{\mu(t)}, \tag{4}$$

for $t \in [t_0, T^*+t_0-\epsilon)$, we have

$$V(t) \leq \frac{\zeta_1(t)}{\mu(t)^2}V(t_0) + \frac{\|\psi\|_{[t_0,t]}}{k\mu(t)^2}, \tag{5}$$

where $\zeta_1(t) = \exp\Big(-\frac{kT^*}{q-1}\Big(\big(\frac{T^*}{T^*+t_0-t}\big)^{q-1} - 1\Big)\Big)$ is a monotonically decreasing function with $\zeta_1(t_0) = 1$ and $\zeta_1(t) \to 0$ as $t \to (T^*+t_0-\epsilon)^-$;

2) if

$$\dot{V} \leq -\overline{k}\mu V - \frac{2q}{T^*}\mu^{\frac{1}{q}}V + \frac{\psi(t)}{\mu}, \tag{6}$$

for $t \in [T^*+t_0-\epsilon, +\infty)$, we have

$$\begin{aligned} V(t) \leq& \frac{\zeta_2(t)\zeta_1(T^*+t_0-\epsilon)}{\mu^2}V(t_0) \\ &+ \frac{\zeta_2(t)\|\psi\|_{[t_0,t]}}{k\mu^2} + \frac{\|\psi(t)\|_{[T^*+t_0-\epsilon,t]}}{\overline{k}\mu^2}, \end{aligned} \tag{7}$$

where $\zeta_2(t) = \exp\Big(-\overline{k}\big(\frac{T^*}{\epsilon}\big)^q(t-T^*-t_0+\epsilon)\Big)$ is a monotonically decreasing function with $\zeta_2(T^*+t_0-\epsilon) = 1$ and $\zeta_2(t) \to 0$ as $t \to +\infty$.

Proof: Solving the differential inequality (4) on $t \in [t_0, T^*+t_0-\epsilon)$ gives that

$$\begin{aligned} &V(t) \\ \leq& \exp\left(\int_{t_0}^t \Big(-k\mu(\tau) - \frac{2q}{T^*}\mu^{\frac{1}{q}}\Big)d\tau\right)V(t_0) \\ &+ \int_{t_0}^t \exp\left(\int_\tau^t \Big(-k\mu(s) - \frac{2q}{T^*}\mu^{\frac{1}{q}}\Big)ds\right)\frac{\psi(\tau)}{\mu(\tau)}d\tau \\ \triangleq& \phi_1(t)V(t_0) + \phi_2(t). \end{aligned} \tag{8}$$

where $\phi_1(t) = \exp\Big(\int_{t_0}^t\Big(-k\mu(\tau) - \frac{2q}{T^*}\mu^{\frac{1}{q}}\Big)d\tau\Big)$ and $\phi_2(t) = \int_{t_0}^t \exp\Big(\int_\tau^t\Big(-k\mu(s) - \frac{2q}{T^*}\mu^{\frac{1}{q}}\Big)ds\Big)\frac{\psi(\tau)}{\mu(\tau)}d\tau$.

Firstly, by (3), the function $\phi_1(t)$ can be rewritten as,

$$\begin{aligned} &\phi_1(t) \\ =& \exp\left(-k\int_{t_0}^t \mu(\tau)d\tau - \frac{2q}{T^*}\int_{t_0}^t \frac{T^*\,d\tau}{T^*+t_0-\tau}\right) \\ =& \exp\left(-k\int_{t_0}^t \mu(\tau)d\tau\right)\exp\left(-2q\int_{t_0}^t \frac{d\tau}{T^*+t_0-\tau}\right) \\ =& \exp\left(-\frac{kT^*}{q-1}\big(\mu(t)^{1-\frac{1}{q}} - 1\big)\right)\exp\left(-2q\int_{t_0}^t \frac{d\tau}{T^*+t_0-\tau}\right) \\ =& \exp\left(-\frac{kT^*}{q-1}\big(\mu(t)^{1-\frac{1}{q}} - 1\big)\right)\left(\frac{T^*+t_0-t}{T^*}\right)^{2q} \end{aligned}$$

$$=\mu(t)^{-2}\exp\left(-\frac{kT^*}{q-1}\big(\mu(t)^{1-\frac{1}{q}}-1\big)\right), \tag{9}$$

which is a monotonically decreasing function.

Next, the function $\phi_2(t)$ is computed as

$$\begin{aligned}\phi_2(t)=&\int_{t_0}^{t}\exp\left(-k\int_{\tau}^{t}\mu(s)ds\right)\\&\times\exp\left(-2q\int_{\tau}^{t}\frac{ds}{T^*+t_0-s}\right)\frac{\psi(\tau)}{\mu(\tau)}d\tau\\=&\int_{t_0}^{t}\exp\left(-k\int_{t_0}^{t}\mu(s)ds+k\int_{t_0}^{\tau}\mu(s)ds\right)\\&\times\left(\frac{T^*+t_0-t}{T^*+t_0-\tau}\right)^{2q}\frac{\psi(\tau)}{\mu(\tau)}d\tau\\\le&\frac{\|\psi\|_{[t_0,t]}}{\mu(t)^2}\exp\Big(-k\int_{t_0}^{t}\mu(s)ds\Big)\\&\times\int_{t_0}^{t}\exp\left(k\int_{t_0}^{\tau}\mu(s)ds\right)\mu(\tau)d\tau\\=&\frac{\|\psi\|_{[t_0,t]}}{\mu(t)^2}\exp\Big(-k\int_{t_0}^{t}\mu(s)ds\Big)\\&\times\int_{t_0}^{t}\exp\left(k\int_{t_0}^{\tau}\mu(s)ds\right)d\left(\int_{t_0}^{\tau}\mu(s)ds\right)\\=&\frac{\|\psi\|_{[t_0,t]}}{\mu(t)^2}\exp\Big(-k\int_{t_0}^{t}\mu(s)ds\Big)\\&\times\frac{1}{k}\exp\left(k\int_{t_0}^{\tau}\mu(s)ds\right)\Bigg|_{t_0}^{t}\\=&\frac{\|\psi\|_{[t_0,t]}}{k\mu(t)^2}\left(1-\exp\Big(-\frac{kT^*}{q-1}\big(\mu(t)^{1-\frac{1}{q}}-1\big)\Big)\right)\\\le&\frac{\|\psi\|_{[t_0,t]}}{k\mu(t)^2}.\end{aligned} \tag{10}$$

By inserting (9) and (10) into (8), we then can derive (5).

Now, we consider the case of $t\in[T^*+t_0-\epsilon,+\infty)$.

From (5), we get $V(T^*+t_0-\epsilon)=\lim\limits_{t\to(T^*+t_0-\epsilon)^-}V(t)=0$ on $t\in[t_0,T^*+t_0-\epsilon)$ as $\epsilon\to 0$. When $t\in[T^*+t_0-\epsilon,+\infty)$, it is noted that $\mu=\left(\frac{T^*}{\epsilon}\right)^q$ is constant, and in this case,

$$\dot V(t)\le-\bar{k}\mu V-\frac{2q}{T^*}\mu^{\frac{1}{q}}V+\frac{\psi(t)}{\mu}.$$

Obviously,

$$\begin{aligned}&V(t)\\\le&\exp\left(\Big(-\bar{k}\mu-\frac{2q}{T^*}\mu^{\frac{1}{q}}\Big)(t-T^*-t_0+\epsilon)\right)V(T^*+t_0-\epsilon)\\&+\int_{T^*+t_0-\epsilon}^{t}\exp\left(\Big(-\bar{k}\mu-\frac{2q}{T^*}\mu^{\frac{1}{q}}\Big)(t-\tau)\right)\frac{\psi}{\mu}d\tau\\\le&\exp\left(\Big(-\bar{k}\mu-\frac{2q}{T^*}\mu^{\frac{1}{q}}\Big)(t-T^*-t_0+\epsilon)\right)V(T^*+t_0-\epsilon)\\&+\frac{\|\psi(t)\|_{[T^*+t_0-\epsilon,t]}}{\bar{k}\mu^2+\frac{2q}{T^*}\mu^{\frac{1}{q}+1}}\\&\times\left(1-\exp\left(\Big(-\bar{k}\mu-\frac{2q}{T^*}\mu^{\frac{1}{q}}\Big)(t-T^*-t_0+\epsilon)\right)\right)\\\le&\zeta_2(t)V(T^*+t_0-\epsilon)+\frac{\|\psi(t)\|_{[T^*+t_0-\epsilon,t]}}{\bar{k}\mu^2}.\end{aligned} \tag{11}$$

By combining (11) with (5), we have

$$\begin{aligned}V(t)\le&\zeta_2(t)\left(\frac{\zeta_1(T^*+t_0-\epsilon)}{\mu^2}V(t_0)+\frac{\|\psi\|_{[t_0,t]}}{k\mu^2}\right)\\&+\frac{\|\psi(t)\|_{[T^*+t_0-\epsilon,t]}}{\bar{k}\mu^2}\\=&\frac{\zeta_2(t)\zeta_1(T^*+t_0-\epsilon)}{\mu^2}V(t_0)\\&+\frac{\zeta_2(t)\|\psi\|_{[t_0,t]}}{k\mu^2}+\frac{\|\psi(t)\|_{[T^*+t_0-\epsilon,t]}}{\bar{k}\mu^2},\end{aligned} \tag{12}$$

which is consistent with (7). By noting that if $\epsilon\to 0$, $\mu\to\infty$, and in such case we can inferred that $\lim\limits_{t\to(T^*+t_0-\epsilon)^-}V(t)=\lim\limits_{t\to(T^*+t_0-\epsilon)^-}\Big(\frac{\zeta_1(t)}{\mu(t)^2}V(t_0)+\frac{\|\psi\|_{[t_0,t]}}{k\mu(t)^2}\Big)=0$, and $\lim\limits_{t\to(T^*+t_0-\epsilon)^+}V(t)=\lim\limits_{t\to(T^*+t_0-\epsilon)^+}\Big(\frac{\zeta_2(t)\zeta_1(T^*+t_0-\epsilon)}{\mu^2}V(t_0)+\frac{\zeta_2(t)\|\psi\|_{[t_0,t]}}{k\mu^2}+\frac{\|\psi(t)\|_{[T^*+t_0-\epsilon,t]}}{\bar{k}\mu^2}\Big)=0$, and meanwhile, $\lim\limits_{t\to\infty,\epsilon\to 0}V(t)=0$. ■

Remark 2. *Lemma 1 is introduced to facilitate the global stability analysis of the proposed practical PTC scheme for strict-feedback systems with unmatched non-vanishing uncertainties existing on the whole time interval, making it different from the existing prescribed-time work [12] in which the unmatched uncertainties are also involved. However, in [12] the system uncertainties are vanishing and needed to satisfy certain conditions and both the control input and adaptive law are designed to be zero beyond the prescribed settling time, making it unpractical.*

Remark 3. *According to* (3) *and Lemma 1, the practical executable time "$T^*+t_0-\epsilon$" previous to the prescribed time T^*. Let $T=T^*-\epsilon$ be prescribed time,* (3) *can be written as*

$$\mu(t)=\begin{cases}\left(\frac{T+\epsilon}{T+\epsilon+t_0-t}\right)^q, & t\in[t_0,t_0+T)\\ \left(\frac{T+\epsilon}{\epsilon}\right)^q & t\in[t_0+T,+\infty)\end{cases} \tag{13}$$

where $T\ge T_s>0,\epsilon>0,q\ge 2$. The practical executable time "$T+\epsilon$" lags behind the prescribed time T. The relevant discussion in Lemma 1 still holds. Distinguishing the work in [11], the parameter $q\ge 2$ in (13) *rather than $q=1$. With the help of Lemma 1 under $q\ge 2$, control signal is guaranteed to be bounded as following theoretical analysis. To guarantee that the system strictly achieves tracking target within prescribed time, the time-varying strategy* (3) *is the analytical tool in this paper.*

Lemma 2. *Let $\phi(t)$ be an unknown bound function and $\|\phi\|_{[t_0,t]}=\sup\limits_{\tau\in[t_0,t]}|\phi(\tau)|$. If a continuously differentiable function $\chi(t)$ satisfies*

$$\dot\chi(t)\le-k\mu(t)\chi(t)+\mu(t)\phi(t) \tag{14}$$

with $k>0$, then

1) for $t \in [t_0, T^* + t_0 - \epsilon)$

$$\chi(t) \leq \exp\left(\frac{kT^*}{q-1}(1-\mu(t)^{1-\frac{1}{q}})\right)\chi(t_0) + \frac{\|\phi\|_{[t_0,t]}}{k}, \quad (15)$$

2) for $t \in [T^* + t_0 - \epsilon, +\infty)$

$$\begin{aligned}\chi(t) \leq& \exp\Big(\frac{kT^*}{q-1}\big(1-\mu(T^*+t_0-\epsilon)^{1-\frac{1}{q}}\big)\Big)\chi(t_0)\\ &+\frac{\|\phi\|_{[t_0,T^*+t_0-\epsilon]}}{k}+\frac{\|\phi\|_{[T^*+t_0-\epsilon,t]}}{k\mu^2}. \end{aligned} \quad (16)$$

Proof: 1) For $t \in [t_0, T^* + t_0 - \epsilon) \subseteq [t_0, T^* + t_0)$, the proof is similar that in Lemma 1 of [1]. To help with the understanding, the proof flow is given as follows.

The solution of the differential inequality (14) on $t \in [t_0, T^* + t_0 - \epsilon)$ is derived as

$$\begin{aligned}\chi(t) \leq& \exp\left(-k\int_{t_0}^{t}\mu(\tau)d\tau\right)\chi(t_0)\\ &+\int_{t_0}^{t}\exp\left(-k\int_{\tau}^{t}\mu(s)ds\right)\mu(\tau)|\phi(\tau)|d\tau\\ \leq& \exp\left(\frac{kT^*}{q-1}\big(1-\mu(t)^{1-\frac{1}{q}}\big)\right)\chi(t_0)\\ &+\|\phi\|_{[t_0,t]}\exp\left(-k\int_{t_0}^{t}\mu(s)ds\right)\\ &\times\int_{t_0}^{t}\exp\left(k\int_{t_0}^{\tau}\mu(s)ds\right)d\Big(\int_{t_0}^{\tau}\mu(s)ds\Big)\\ \leq& \exp\left(\frac{kT^*}{q-1}\big(1-\mu(t)^{1-\frac{1}{q}}\big)\right)\chi(t_0)+\frac{\|\phi\|_{[t_0,t]}}{k}. \end{aligned} \quad (17)$$

2) Upon using (14) and (3), we can obtain that

$$\dot{\chi}(t) \leq -k\mu\chi(t) + \mu\phi(t), \quad t \in [T^* + t_0 - \epsilon, +\infty)$$

which yields

$$\begin{aligned}\chi(t) \leq& \exp\big(-k\mu(t-T^*-t_0+\epsilon)\big)\chi(T^*+t_0-\epsilon)\\ &+\frac{\|\phi\|_{[T^*+t_0-\epsilon,t]}}{k\mu^2}\Big(1-\exp\big(-k\mu(t-T^*-t_0+\epsilon)\big)\Big)\\ \leq& \exp\big(-k\mu(t-T^*-t_0+\epsilon)\big)\\ &\times\chi(T^*-t_0+\epsilon)+\frac{\|\phi\|_{[T^*+t_0-\epsilon,t]}}{k\mu^2}\\ \leq& \exp\big(-k\mu(t-T^*-t_0+\epsilon)\big)\\ &\times\bigg(\exp\Big(\frac{kT^*}{q-1}\big(1-\mu(T^*-t_0+\epsilon)^{1-\frac{1}{q}}\big)\Big)\chi(t_0)\\ &+\frac{\|\phi\|_{[t_0,T^*-t_0+\epsilon]}}{k}\bigg)+\frac{\|\phi\|_{[T^*-t_0+\epsilon,t]}}{k\mu^2}\\ \leq& \exp\Big(\frac{kT^*}{q-1}\big(1-\mu(T^*+t_0-\epsilon)^{1-\frac{1}{q}}\big)\Big)\chi(t_0)\\ &+\frac{\|\phi\|_{[t_0,T^*-t_0+\epsilon]}}{k}+\frac{\|\phi\|_{[T^*-t_0+\epsilon,t]}}{k\mu^2}. \end{aligned} \quad (18)$$

It is noted that $\chi(t)$ is bounded from (18). ■

Lemma 3. *([13]) Consider the following one-order differential equation:*

$$\dot{\chi}_1(t) = -a\chi_1(t) + b\chi_2(t) \quad (19)$$

where $\chi_2(t)$ is a nonnegative function, $a > 0$, $b > 0$. Then, for any given positive initial state $\chi_1(t_0) \geq 0$, the associated solution $\chi_1(t) \geq 0$ holds for $\forall t \geq t_0$.

Lemma 4. *([7]) For any given vectors $\eta_1, \eta_2 \in R^m$, the following inequality (i.e. Young's Inequality) holds:*

$$\|\eta_1\|\|\eta_2\| \leq \frac{\mu\gamma\|\eta_1\|^2}{2} + \frac{\|\eta_2\|^2}{2\mu\gamma},$$

where μ is defined as in (3) and $\gamma > 0$ is a user-design constant.

Remark 4. *The inequality (14) in Lemma 2, although similar to that in [1], allows the design variable "μ" to be bounded and well defined on $[0, +\infty)$. Lemma 2 is crucial to guarantee the boundedness of the updated parameter utilized in the proposed adaptive control scheme. With the aid of Lemma 3, such estimated parameter is also ensured to be non-negative. In developing Lemma 4, we purposely introduce the design variable μ through Young's inequality, which plays an important role in our later stability analysis.*

III. MAIN RESULT

A. Controller Design

Now, the adaptive control law is designed by Backstepping technique [34]. The error surface $z_i (i = 1, \cdots, n)$ is introduced as

$$\begin{aligned} z_1(t) =& y(t) - x_r(t) = x_1(t) - x_r(t)\\ z_i(t) =& x_i(t) - \alpha_{i-1}(t), \quad i = 2, \cdots, n, \end{aligned} \quad (20)$$

where $\alpha_{i-1}(t) \in R$ is the virtual control.

Step 1. From (1) and the definition of α_1 in (20), we can obtain that

$$\dot{x}_1 = g_1 x_2 + f_1 = g_1 z_2 + g_1\alpha_1 + f_1. \quad (21)$$

Then, the time derivative of $\frac{1}{2\underline{g}_1} z_1^2$ along (20) is

$$\begin{aligned}\frac{1}{\underline{g}_1}z_1\dot{z}_1 =& \frac{1}{\underline{g}_1}z_1(\dot{x}_1 - \dot{x}_r)\\ =& \frac{1}{\underline{g}_1}z_1(g_1 z_2 + g_1\alpha_1 + f_1(\bar{x}_1, t) - \dot{x}_r). \end{aligned} \quad (22)$$

With the help of Lemma 4, it is not difficult to get that

$$\frac{1}{\underline{g}_1}z_1 g_1 z_2 \leq \frac{1}{\underline{g}_1}|z_1||g_1||z_2| \leq \frac{\mu z_1^2 z_2^2}{2} + \frac{\overline{g}_1^2}{2\underline{g}_1^2\mu}; \quad (23)$$

$$\frac{1}{\underline{g}_1}z_1 f_1(\bar{x}_1, t) \leq \frac{1}{\underline{g}_1}|z_1|\nu_1\vartheta_1 \leq \frac{\gamma_1\mu z_1^2\nu_1^2\vartheta_1^2}{2} + \frac{1}{2\gamma_1\underline{g}_1^2\mu}; \quad (24)$$

$$-\frac{1}{\underline{g}_1}z_1\dot{x}_r \leq \frac{1}{\underline{g}_1}|z_1||\dot{x}_r| \leq \frac{r_1\mu|\dot{x}_r|^2 z_1^2}{2} + \frac{1}{2r_1\underline{g}_1^2\mu}. \quad (25)$$

By (23), (24) and (25), (22) can be rewritten as

$$\begin{aligned}\frac{1}{\underline{g}_1}z_1\dot{z}_1 \leq& \frac{g_1}{\underline{g}_1}z_1\alpha_1 + \frac{\omega_1\gamma_1\psi_1\mu z_1^2}{2} + \frac{r_1\mu\phi_1 z_1^2}{2}\\ &+\frac{\overline{g}_1^2}{2\underline{g}_1^2\mu} + \frac{1}{2\gamma_1\underline{g}_1^2\mu} + \frac{1}{2r_1\underline{g}_1^2\mu} + \frac{\mu z_1^2 z_2^2}{2}, \end{aligned} \quad (26)$$

where $\omega_1=\max\{\nu_1^2\}$ is the unknown virtual constant, $\psi_1=\vartheta_1^2$ and $\phi_1=|\dot{x}_r|^2$ are the computable bounded functions.

Define the first Lyapunov function as

$$V_1=\frac{1}{2\underline{g}_1}z_1^2+\frac{\widetilde{\omega}_1^2}{2\mu^2}. \tag{27}$$

where $\widetilde{\omega}_1=\omega_1-\widehat{\omega}_1$ is the parameter estimation error, $\widehat{\omega}_1$ is the estimation value of parameter ω_1 and μ is defined as in (3).

By mean of (20)-(27), the time derivative of (27) is ~~given as~~

$$\begin{aligned}\dot{V}_1=&\frac{1}{\underline{g}_1}z_1\dot{z}_1+\frac{1}{\mu^2}\widetilde{\omega}_1\dot{\widetilde{\omega}}_1-\frac{\dot{\mu}}{\mu^3}\widetilde{\omega}_1^2\\ \leq&\frac{g_1}{\underline{g}_1}z_1\alpha_1+\frac{\omega_1\gamma_1\mu z_1^2\psi_1}{2}+\frac{r_1\mu\phi_1z_1^2}{2}\\ &+\frac{\overline{g}_1^2}{2\underline{g}_1^2\mu}+\frac{\mu z_1^2z_2^2}{2}+\frac{1}{2r_1\underline{g}_1^2\mu}+\frac{1}{2\gamma_1\underline{g}_1^2\mu}\\ &-\frac{1}{\mu^2}\widetilde{\omega}_1\dot{\widehat{\omega}}_1-\frac{\dot{\mu}}{\mu^3}\widetilde{\omega}_1^2.\end{aligned} \tag{28}$$

The virtual control α_1 is designed as

$$\begin{aligned}&\alpha_1(x_1,x_r,\mu,\widehat{\omega}_1)\\ =&-\Big(k_{11}+\frac{\gamma_1\widehat{\omega}_1\psi_1}{2}+\frac{r_1\phi_1}{2}\Big)\mu z_1-\frac{k_{12}q^2}{T^{*2}}\mu^{1+\frac{2}{q}}z_1^3,\end{aligned} \tag{29}$$

~~with~~ adaptive law

$$\dot{\widehat{\omega}}_1=-k_{\omega_1}\mu\widehat{\omega}_1+\frac{\gamma_1\mu^3}{2}\psi_1z_1^2,\quad \widehat{\omega}_1(t_0)>0, \tag{30}$$

where $k_{11}>0$, $k_{12}\geq\frac{1}{2}$, $\gamma_1>0$, $r_1>0$ and k_{ω_1} is designed as $k_{\omega_1}=\overline{k}_{\omega_1}+2\frac{q}{T^*}\big(\frac{T^*}{\epsilon}\big)^{1-q}$ with $\overline{k}_{\omega_1}>0$. From Lemma 3, it is confirmed that $\widehat{\omega}_1\geq 0$.

Noting the term "$-\frac{1}{\mu^2}\widetilde{\omega}_1\dot{\widehat{\omega}}_1$" in (28) with $\dot{\widehat{\omega}}_1$ in (30), and $\widetilde{\omega}_1\widehat{\omega}_1=\widetilde{\omega}_1(\omega_1-\widetilde{\omega}_1)=-\widetilde{\omega}_1^2+\widetilde{\omega}_1\omega_1$, it is derived that

$$\begin{aligned}-\frac{1}{\mu^2}\widetilde{\omega}_1\dot{\widehat{\omega}}_1=&-\frac{1}{\mu^2}\widetilde{\omega}_1\Big(-k_{\omega_1}\mu\widehat{\omega}_1+\frac{\gamma_1\mu^3}{2}\psi_1z_1^2\Big)\\ =&-\frac{k_{\omega_1}\mu}{\mu^2}\widetilde{\omega}_1^2+\frac{k_{\omega_1}\mu}{\mu^2}\widetilde{\omega}_1\omega_1-\frac{1}{2}\mu\gamma_1\widetilde{\omega}_1\psi_1z_1^2\\ \leq&-\frac{k_{\omega_1}\mu}{\mu^2}\widetilde{\omega}_1^2+\frac{k_{\omega_1}\mu}{\mu^2}\Big(\frac{\widetilde{\omega}_1^2}{2}+\frac{\omega_1^2}{2}\Big)-\frac{1}{2}\mu\gamma_1\widetilde{\omega}_1\psi_1z_1^2\\ =&-\frac{k_{\omega_1}\mu}{2\mu^2}\widetilde{\omega}_1^2+\frac{k_{\omega_1}\omega_1^2}{2\mu}-\frac{1}{2}\mu\gamma_1\widetilde{\omega}_1\psi_1z_1^2.\end{aligned} \tag{31}$$

By noting that

$$\begin{aligned}&-\frac{k_{12}q^2}{T^{*2}}\frac{g_1}{\underline{g}_1}\mu^{1+\frac{2}{q}}z_1^4\\ \leq&-\frac{k_{12}q^2}{T^{*2}}\mu^{1+\frac{2}{q}}z_1^4\\ =&-k_{12}\Big(\frac{q^2}{T^{*2}}\mu^{1+\frac{2}{q}}z_1^4+\frac{1}{\underline{g}_1^2\mu}\Big)+\frac{k_{12}}{\underline{g}_1^2\mu},\end{aligned} \tag{32}$$

with the aid of Young's inequality, the inside of parentheses in (32) holds that

$$\begin{aligned}\frac{q^2}{T^{*2}}\mu^{1+\frac{2}{q}}z_1^4+\frac{1}{\underline{g}_1^2\mu}\geq&2\Big(\frac{q}{T^*}\sqrt{\mu}\mu^{\frac{1}{q}}z_1^2\Big)\times\Big(\frac{1}{\underline{g}_1\sqrt{\mu}}\Big)\\ =&\frac{2q}{\underline{g}_1T^*}\mu^{\frac{1}{q}}z_1^2.\end{aligned} \tag{33}$$

Inserting (33) into (32), we have

$$-\frac{k_{12}q^2}{T^{*2}}\frac{g_1}{\underline{g}_1}\mu^{1+\frac{2}{q}}z_1^4\leq-\frac{4k_{12}}{2\underline{g}_1}\frac{q}{T^*}\mu^{\frac{1}{q}}z_1^2+\frac{k_{12}}{\underline{g}_1^2\mu}. \tag{34}$$

Recalling (29), (31) and (34), (28) holds that

$$\begin{aligned}\dot{V}_1(t)\leq&-k_{11}\mu z_1^2-\frac{4k_{12}}{2\underline{g}_1}\frac{q}{T^*}\mu^{\frac{1}{q}}z_1^2\\ &+\frac{\mu z_1^2z_2^2}{2}-\frac{k_{\omega_1}\mu}{2\mu^2}\widetilde{\omega}_1^2-\frac{\dot{\mu}}{\mu^3}\widetilde{\omega}_1^2+\frac{\Phi_1}{\mu},\end{aligned} \tag{35}$$

where $\Phi_1=\frac{\overline{g}_1^2}{2\underline{g}_1^2}+\frac{1}{2\gamma_1\underline{g}_1^2}+\frac{1}{2r_1\underline{g}_1^2}+\frac{k_{\omega_1}\omega_1^2}{2}+\frac{k_{12}}{\underline{g}_1^2}$ is an unknown finite positive constant.

Step 2: The time derivative of $z_2=x_2-\alpha_1$ is

$$\begin{aligned}\dot{z}_2=&\dot{x}_2-\dot{\alpha}_1\\ =&f_2+g_2\alpha_2+g_2z_3-\dot{\alpha}_1\end{aligned} \tag{36}$$

where $\dot{\alpha}_1=\frac{\partial\alpha_1}{\partial x_1}(g_1x_2+f_1)+\sum_{j=0}^{1}\frac{\partial\alpha_1}{\partial x_r^{(j)}}x_r^{(j+1)}+\frac{q}{T^*}\mu^{1+\frac{1}{q}}\frac{\partial\alpha_1}{\partial\mu}+\frac{\partial\alpha_1}{\partial\widehat{\omega}_1}\dot{\widehat{\omega}}_1$.

The virtual controller α_2 is designed as

$$\begin{aligned}\alpha_2=&-\Big(k_{21}+\frac{\gamma_2\widehat{\omega}_2\psi_2}{2}+\frac{r_2\phi_2}{2}\Big)\mu z_2\\ &-\frac{k_{22}q^2}{T^{*2}}\mu^{1+\frac{2}{q}}z_2^3-\frac{\mu z_1^2z_2}{2},\end{aligned} \tag{37}$$

with the adaptive law

$$\dot{\widehat{\omega}}_2=-k_{\omega_2}\mu\widehat{\omega}_2+\frac{1}{2}\gamma_2\mu^3\psi_2z_2^2,\widehat{\omega}_2>0, \tag{38}$$

where $\widehat{\omega}_2$ is the estimation value of parameter $\omega_2=\max\{\nu_1^2,\nu_2^2\}$, $k_{21}>0$, $k_{22}\geq\frac{1}{2}$, $k_{\omega_2}=\overline{k}_{\omega_2}+2\frac{q}{T^*}\big(\frac{T^*}{\epsilon}\big)^{1-\frac{1}{q}}$, $\overline{k}_{\omega_2}>0$.

~~Consider the~~ second Lyapunov function ~~as~~

$$V_2=V_1+\frac{1}{2\underline{g}_2}z_2^2+\frac{\widetilde{\omega}_2^2}{2\mu^2} \tag{39}$$

where $\widetilde{\omega}_2=\omega_2-\widehat{\omega}_2$ is the parameter estimation error. ~~It follows that~~

$$\begin{aligned}\dot{V}_2=&\dot{V}_1+\frac{1}{\underline{g}_2}(z_2f_2+g_2z_2z_3+g_2z_2\alpha_2-z_2\dot{\alpha}_1)\\ &-\frac{1}{\mu^2}\widetilde{\omega}_2\dot{\widehat{\omega}}_2-2\mu^{\frac{1}{q}}\frac{\widetilde{\omega}_2^2}{2\mu^2}.\end{aligned} \tag{40}$$

By using Lemma 4 and following the similar procedure to (23) and (24), we have

$$\begin{aligned}&\frac{1}{\underline{g}_2}(z_2f_2+g_2z_2z_3-z_2\dot{\alpha}_1)\\ \leq&\frac{\mu z_2^2z_3^2}{2}+\frac{\overline{g}_2^2}{2\mu\underline{g}_2^2}+\frac{z_2f_2}{\underline{g}_2}-\frac{z_2}{\underline{g}_2}\Big(\frac{\partial\alpha_1}{\partial x_1}(g_1x_2+f_1)\\ &+\sum_{j=0}^{1}\frac{\partial\alpha_1}{\partial x_r^{(j)}}x_r^{(j+1)}+\frac{\partial\alpha_1}{\partial\mu}\Big(\frac{q}{T^*}\mu^{1+\frac{1}{q}}\Big)+\frac{\partial\alpha_1}{\partial\widehat{\omega}_1}\dot{\widehat{\omega}}_1\Big)\\ \leq&\frac{\mu z_2^2z_3^2}{2}+\frac{\overline{g}_2^2}{2\mu\underline{g}_2^2}+\frac{\gamma_2\omega_2\mu z_2^2\psi_2}{2}+\frac{r_2\mu\phi_2z_2^2}{2}\end{aligned}$$

$$+\frac{2}{2\gamma_2\underline{g}_2^2\mu}+\frac{\overline{g}_1^2+4}{2\mu r_2\underline{g}_2^2}, \tag{41}$$

where $\psi_2 = \left(\frac{\partial\alpha_1}{\partial x_1}\vartheta_1\right)^2 + \vartheta_2^2$ and $\phi_2 = \left(\frac{\partial\alpha_1}{\partial x_1}x_2\right)^2 + \left(\frac{q}{T^*}\mu^{1+\frac{1}{q}}\frac{\partial\alpha_1}{\partial\mu}\right)^2 + \left(\frac{\partial\alpha_1}{\partial\widehat{\omega}_1}\dot{\widehat{\omega}}_1\right)^2 + \sum_{j=0}^{1}\left(\frac{\partial\alpha_1}{\partial x_r^{(j)}}x_r^{(j+1)}\right)^2$ are calculable variables.

From (37) and (38), we obtain that

$$\begin{aligned}-\frac{g_2}{\underline{g}_2}z_2\alpha_2 \le &-\frac{g_2}{\underline{g}_2}\Big(k_{21}\mu z_2^2+\frac{k_{22}q^2}{T^{*2}}\mu^{1+\frac{2}{q}}z_2^4\\ &+\frac{\gamma_2\mu\widehat{\omega}_2\psi_2 z_2^2}{2}+\frac{\mu z_1^2z_2^2}{2}+\frac{r_2\mu\phi_2}{2}z_2^2\Big)\\ \le &-k_{21}\mu z_2^2-\frac{4k_{22}}{2\underline{g}_2}\frac{q}{T^*}\mu^{\frac{1}{q}}z_2^2-\frac{r_2\phi_2}{2}\mu z_2^2\\ &-\frac{\gamma_2\mu\widehat{\omega}_2\psi_2 z_2^2}{2}-\frac{\mu z_1^2z_2^2}{2}+\frac{k_{22}}{\underline{g}_2^2\mu}.\end{aligned} \tag{42}$$

Substituting (41)-(42) into (40), it follows that

$$\begin{aligned}\dot{V}_2 \le &-\sum_{j=1}^{2}\left(2k_{j1}\underline{g}_j\right)\mu\frac{z_j^2}{2\underline{g}_j}-\sum_{j=1}^{2}\frac{4k_{j2}}{2\underline{g}_j}\frac{q}{T^*}\mu^{\frac{1}{q}}z_2^2+\frac{\mu z_2^2z_3^2}{2}\\ &-\sum_{j=1}^{2}k_{\omega_j}\mu\frac{\widetilde{\omega}_j^2}{2\mu^2}-2\frac{q}{T^*}\mu^{\frac{1}{q}}\sum_{j=1}^{2}\frac{\widetilde{\omega}_j^2}{2\mu^2}+\frac{\Phi_2}{\mu},\end{aligned} \tag{43}$$

where $\Phi_2 = \sum_{j=1}^{2}\left(\frac{\overline{g}_j^2}{2\underline{g}_j^2}+\frac{k_{\omega_j}\omega_j^2}{2}+\frac{k_{j2}}{\underline{g}_j^2}+\frac{j}{2\gamma_j\underline{g}_j^2}\right)+\left(\frac{1}{2r_1\underline{g}_1^2}+\frac{\overline{g}_1^2+4}{2r_2\underline{g}_2^2}\right)$.

TABLE I: The design of adaptive backstepping controller on $t \in [t_0, T^*-t_0+\epsilon)$

The time-varying gain function
$\mu = \left(\frac{T^*}{T^*+t_0-t}\right)^q$, $q \ge \max\{n, 2\}$
Introducing error variables:
$z_1 = x_1 - x_r, \quad i = 1$
$z_i = x_i - \alpha_{i-1}, \quad i = 2, \cdots, n$
Control laws:
$i = 1$
$\alpha_1 = -k_{11}\mu z_1 - \frac{k_{12}q^2}{T^{*2}}\mu^{1+\frac{2}{q}}z_1^3 - \frac{\gamma_1\widehat{\omega}_1\psi_1}{2}\mu z_1 - \frac{r_1\phi_1}{2}\mu z_1$,
$\psi_1 = \vartheta_1^2$,
$\phi_1 = (\dot{x}_r)^2$,
where $k_{11} > 0$, $k_{12} \ge \frac{1}{2}$, $\gamma_1 > 0$, $r_1 > 0$.
$i = 2, \cdots, n$
$\alpha_i = -k_{i1}\mu z_i - \frac{k_{i2}q^2}{T^{*2}}\mu^{1+\frac{2}{q}}z_i^3 - \frac{\gamma_i\widehat{\omega}_i\psi_i}{2}\mu z_i - \frac{\mu z_{i-1}^2 z_i}{2} - \frac{r_i\phi_i}{2}\mu z_i$,
$\psi_i = \sum_{j=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial x_j}\vartheta_j\right)^2 + \vartheta_i^2$,
$\phi_i = \sum_{j=1}^{i-1}\left(\left(\frac{\partial\alpha_{i-1}}{\partial x_j}x_{j+1}\right)^2 + \left(\frac{\partial\alpha_{i-1}}{\partial\widehat{\omega}_j}\dot{\widehat{\omega}}_j\right)^2\right)$
$\quad + \sum_{j=0}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial x_r^{(j)}}x_r^{(j+1)}\right)^2 + \left(\frac{q}{T^*}\mu^{1+\frac{1}{q}}\frac{\partial\alpha_{i-1}}{\partial\mu}\right)^2$,
where $k_{i1} > \iota_i > 0$, $k_{i2} \ge \frac{1}{2}$, $\gamma_i > 0$, $r_i > 0$.
Parameter adaptive laws:
$\dot{\widehat{\omega}}_i = -k_{\omega_i}\mu\widehat{\omega}_i + \frac{1}{2}\gamma_i\mu^3\psi_i z_i^2, \quad i = 1, \cdots, n$
where $k_{\omega_i} = \overline{k}_{\omega_i} + 2\frac{q}{T^*}\left(\frac{T^*}{\epsilon}\right)^{1-q}$, $\overline{k}_{\omega_i} > 0$, $\widehat{\omega}_i(t_0) > 0$,
$\widehat{\omega}_i$ is the estimation of $\omega_i = \max_{1\le j\le i}\{\nu_j^2\}$
Final controller
$u = \alpha_n$

Step i $(i = 3, \cdots, n)$: ~~We choose the~~ ith Lyapunov function ~~as~~

$$V_i = V_{i-1} + \frac{1}{2\underline{g}_i}z_i^2 + \frac{\widetilde{\omega}_i}{2\mu^2} \tag{44}$$

TABLE II: The design of adaptive backstepping controller on $t \in [T^*-t_0+\epsilon, +\infty)$

The gain function
$\mu = \left(\frac{T^*}{\epsilon}\right)^q$, $q \ge \max\{n, 2\}$,
Introducing error variables:
$z_1 = x_1 - x_r, \quad i = 1$
$z_i = x_i - \overline{\alpha}_i, \quad i = 2, \cdots, n$
Control laws:
$i = 1$
$\overline{\alpha}_1 = -k_{11}\mu z_1 - \frac{k_{12}q^2}{T^{*2}}\mu^{1+\frac{2}{q}}z_1^3 - \frac{\gamma_1\widehat{\omega}_1\overline{\psi}_1}{2}\mu z_1 - \frac{r_1\overline{\phi}_1}{2}\mu z_1$,
$\overline{\psi}_1 = \vartheta_1^2$,
$\overline{\phi}_1 = (\dot{x}_r)^2$,
where $k_{11} > 0$, $k_{12} \ge \frac{1}{2}$, $\gamma_1 > 0$, $r_1 > 0$.
$i = 2, \ldots, n$
$\overline{\alpha}_i = -k_{i1}\mu z_i - \frac{k_{i2}q^2}{T^{*2}}\mu^{1+\frac{2}{q}}z_i^3 - \frac{\gamma_i\widehat{\varpi}_i\overline{\psi}_i}{2}\mu z_i$
$\quad - \frac{\mu z_{i-1}^2 z_i}{2} - \frac{r_i\overline{\phi}_i}{2}\mu z_i + \delta_i(T^*-t_0+\epsilon)$,
$\overline{\psi}_i = \sum_{j=1}^{i-1}\left(\frac{\partial\overline{\alpha}_{i-1}}{\partial x_j}\vartheta_j\right)^2 + \vartheta_i^2, \quad i = 2, \cdots, n$
$\overline{\phi}_i = \sum_{j=1}^{i-1}\left(\left(\frac{\partial\overline{\alpha}_{i-1}}{\partial x_j}x_{j+1}\right)^2 + \left(\frac{\partial\overline{\alpha}_{i-1}}{\partial\widehat{\varpi}_j}\dot{\widehat{\omega}}_j\right)^2\right)$
$\quad + \sum_{j=0}^{i-1}\left(\frac{\partial\overline{\alpha}_{i-1}}{\partial x_r^{(j)}}x_r^{(j+1)}\right)^2$
$\delta_i(T^*-t_0+\epsilon) = \left(\frac{\gamma_i\left(\widehat{\omega}_i\overline{\psi}_i - \widehat{\omega}_i\psi_i\right)}{2}\mu z_i + \frac{r_i\left(\overline{\phi}_i - \phi_i\right)}{2}\mu z_i\right)\Big\vert_{t=T^*-t_0+\epsilon}$,
where $k_{i1} > \iota_i > 0$, $k_{i2} \ge \frac{1}{2}$, $\gamma_i > 0$, $r_i > 0$,
and $\delta_i(T^*-t_0+\epsilon)$ is control action softening unit.
Parameter update laws:
$\dot{\widehat{\omega}}_i = -k_{\omega_i}\mu\widehat{\omega}_i + \frac{1}{2}\gamma_i\mu^3\overline{\psi}_i z_i^2, \quad i = 1, \cdots, n$
where $k_{\omega_i} = \overline{k}_{\omega_i} + 2\frac{q}{T^*}\left(\frac{T^*}{\epsilon}\right)^{1-q}$, $\overline{k}_{\omega_i} > 0$, $\widehat{\omega}_i > 0$,
$\widehat{\omega}_i$ is the estimation of $\omega_i = \max_{1\le j\le i}\{\nu_j^2\}$.
Final controller
$u = \overline{\alpha}_n$

where $\widetilde{\omega}_i = \omega_i - \widehat{\omega}_i$ is the parameter estimation error with the ith unknown factor $\omega_i = \max_{1\le j\le i}\{\nu_j^2\}$. The ~~virtual control and actual control as well as~~ the adaptive law can be ~~recursively obtained~~ by following the standard backstepping procedure, which are given in Table I and ~~Table~~ II. By inserting the virtual and actual ~~control~~ as well as the adaptive law into the derivative of the Lyapunov function, we then arrive at

$$\begin{aligned}\dot{V}_m \le &-\sum_{j=1}^{m}\left(2k_{j1}\underline{g}_j\right)\mu\frac{z_j^2}{2\underline{g}_j}-\sum_{j=1}^{m}\frac{4k_{j2}}{2\underline{g}_j}\frac{q}{T^*}\mu^{\frac{1}{q}}z_j^2+\frac{\mu z_m^2z_{m+1}^2}{2}\\ &-\sum_{j=1}^{m}k_{\omega_j}\mu\frac{\widetilde{\omega}_j^2}{2\mu^2}-2\frac{q}{T^*}\mu^{\frac{1}{q}}\sum_{j=1}^{m}\frac{\widetilde{\omega}_j^2}{2\mu^2}+\frac{\Phi_m}{\mu},\end{aligned} \tag{45}$$

for $m = 3, \cdots, n-1$ and

$$\begin{aligned}\dot{V}_n \le &-\sum_{j=1}^{n}\left(2k_{j1}\underline{g}_j\right)\mu\frac{z_j^2}{2\underline{g}_j}-\sum_{j=1}^{n}\frac{4k_{j2}}{2\underline{g}_j}\frac{q}{T^*}\mu^{\frac{1}{q}}z_j^2\\ &-\sum_{j=1}^{n}k_{\omega_j}\mu\frac{\widetilde{\omega}_j^2}{2\mu^2}-2\frac{q}{T^*}\mu^{\frac{1}{q}}\sum_{j=1}^{n}\frac{\widetilde{\omega}_j^2}{2\mu^2}+\frac{\Phi_n}{\mu},\end{aligned} \tag{46}$$

where $\Phi_m = \sum_{j=1}^{m}\left(\frac{\overline{g}_j^2}{2\underline{g}_j^2}+\frac{k_{\omega_j}\omega_j^2}{2}+\frac{k_{j2}}{\underline{g}_j^2}+\frac{j}{2\gamma_j\underline{g}_j^2}\right)+\left(\frac{1}{2r_1\underline{g}_1^2}+\sum_{j=2}^{m}\frac{\sum_{k=1}^{m-1}\overline{g}_k^2+2\cdot j}{2r_j\underline{g}_j^2}\right)$ and $\Phi_n = \sum_{j=1}^{n-1}\left(\frac{\overline{g}_j^2}{2\underline{g}_j^2}\right)+\sum_{j=1}^{n}\left(\frac{k_{\omega_j}\omega_j^2}{2}+\frac{k_{j2}}{\underline{g}_j^2}+\frac{j}{2\gamma_j\underline{g}_j^2}\right)+\left(\frac{1}{2r_1\underline{g}_1^2}+\sum_{j=2}^{n}\frac{\sum_{k=1}^{n-1}\overline{g}_k^2+2\cdot j}{2r_j\underline{g}_j^2}\right)$.

B. Stability Analysis

The main result is given in the following Theorem.

Theorem 1. *~~Under Assumption 1-3, the uncertain nonlinear system (1) with~~ the control strategies α_i, $\overline{\alpha}_i$ and adaptive laws $\widehat{\omega}_i$ on Table I-II~~, achieves~~ the following objectives.*

(1) The tracking error is forced to enter a compact set within the prescribed time T^*. More significantly, for all $t \in [t_0, T^* + t_0 - \epsilon)$,

$$\begin{aligned}&|z_i|\\ \le&\frac{1}{\mu}\sqrt{\zeta_1(t)\sum_{j=1}^{n}\left(\frac{\overline{g}_i}{\underline{g}_j}z_j^2(t_0)+\overline{g}_i\widetilde{\omega}_j^2(t_0)\right)}+\frac{1}{\mu}\sqrt{\frac{2\overline{g}_i\Phi}{k}},\end{aligned} \tag{47}$$

with $\zeta_1(t) = \exp\left(-\frac{kT^*}{q-1}(\mu(t)^{1-\frac{1}{q}}-1)\right)$.

(2) ~~The system (1) with the proposed control given in Table I and II maintains~~ tracking ~~with~~ zero-error ~~convergence~~ after the prescribed time T^*. Particularly, for $t \in [T^* + t_0 - \epsilon, \infty)$,

$$\begin{aligned}&|z_i|\\ \le&\frac{1}{\mu}\sqrt{\zeta_2(t)\zeta_1(T^*+t_0-\epsilon)\sum_{j=1}^{n}\left(\frac{\overline{g}_i}{\underline{g}_j}z_j^2(t_0)+\overline{g}_i\widetilde{\omega}_j^2(t_0)\right)}\\ &+\frac{1}{\mu}\left(\sqrt{2\overline{g}_i\zeta_2(t)\Phi/k}+\sqrt{2\overline{g}_i\overline{\Phi}/k}\right)\end{aligned} \tag{48}$$

where $\zeta_2(t) = \exp\left(-\overline{k}\left(\frac{T^*}{\epsilon}\right)^q(t - T^* - t_0 + \epsilon)\right)$.

(3) All the internal signals, including the virtual control α_i, the control input u and z_i are continuous and remain uniformly bounded for all $t \in [t_0, \infty)$.

Proof: Let $T = T^* - t_0 + \epsilon$. The proof is divided into two stages: $t \in [t_0, T)$ and $t \in [T, \infty)$.

Stage 1. $t \in [t_0, T)$

We first prove that the output signal $y(t)$ can track the desired reference trajectory $x_r(t)$ within the prescribed time T^*. ~~Consider the following~~ Lyapunov function candidate ~~as~~

$$V = \sum_{i=1}^{n}\frac{z_i^2}{2\underline{g}_i} + \sum_{i=1}^{n}\frac{\omega_i^2}{2\mu^2}, \tag{49}$$

whose time derivative along (35), (43) and (46) is

$$\begin{aligned}\dot{V}\le&-\sum_{i=1}^{n}2k_{i1}\underline{g}_i\mu\frac{z_i^2}{2\underline{g}_i}-\sum_{i=1}^{n}4k_{i2}\frac{q}{T^*}\mu^{\frac{1}{q}}\frac{z_i^2}{2\underline{g}_i}\\ &-\sum_{i=1}^{n}k_{\omega_i}\mu\frac{\widetilde{\omega}_i^2}{2\mu^2}-2\frac{q}{T^*}\mu^{\frac{1}{q}}\sum_{i=1}^{n}\frac{\widetilde{\omega}_i^2}{2\mu^2}+\frac{\Phi}{\mu}\\ \le&-k\mu V-2\frac{q}{T^*}\mu^{\frac{1}{q}}V+\frac{\Phi}{\mu}\end{aligned} \tag{50}$$

where $k = \min_{1\le i\le n}\{2k_{i1}\underline{g}_i, k_{\omega_i}\}$, $k_{i1} > 0$, $k_{i2} \ge \frac{1}{2}$, $k_{\omega_i} = \overline{k}_{\omega_i} + 2\frac{q}{T^*}(\frac{T^*}{\epsilon})^{1-q}$, $\overline{k}_{\omega_i} > 0$ and $\Phi = \sum_{j=1}^{n-1}\left(\frac{\overline{g}_j^2}{2\underline{g}_j^2}\right) + \sum_{j=1}^{n}\left(\frac{k_{\omega_j}\omega_j^2}{2} + \frac{k_{j2}}{\underline{g}_j^2} + \frac{j}{2\gamma_j\underline{g}_j^2}\right) + \left(\frac{1}{2r_1\underline{g}_1^2} + \sum_{j=2}^{n}\frac{\sum_{k=1}^{n-1}\overline{g}_k^2+2\cdot j}{2r_j\underline{g}_j^2}\right)$.

According to Lemma 1, we derive from (50) that

$$V(t) \le \frac{\zeta_1}{\mu(t)^2}V(t_0) + \frac{\Phi}{k\mu(t)^2}. \tag{51}$$

In light of (49), we can obtain that

$$|z_i| \le \sqrt{2\overline{g}_i V(t)}, \quad (i = 1, \cdots, n)$$

and

$$|\widetilde{\omega}_i| \le \sqrt{2\mu^2 V(t)}, \quad (i = 1, \cdots, n).$$

~~Combining with~~ (51), it is not difficult to derive that

$$\begin{aligned}&|z_i|\\ \le&\frac{1}{\mu}\sqrt{\zeta_1(t)\sum_{j=1}^{n}\left(\frac{\overline{g}_i}{\underline{g}_j}z_j^2(t_0)+\overline{g}_i\widetilde{\omega}_j^2(t_0)\right)}+\frac{1}{\mu}\sqrt{\frac{2\overline{g}_i\Phi}{k}},\end{aligned} \tag{52}$$

and

$$|\widetilde{\omega}_i| \le \sqrt{\zeta_1(t)\sum_{j=1}^{n}\left(\frac{z_j^2(t_0)}{\underline{g}_j}+\omega_j^2(t_0)\right)}+\sqrt{\frac{2\Phi}{k}}, \tag{53}$$

both of which guarantees the boundedness of z_i and $\widetilde{\omega}_i$.

Under Assumption 2 and 3, we have $\psi_1 = \vartheta_1^2 \in L_\infty$, $\phi_1 = |\dot{x}_r|^2 \in L_\infty$. By recalling that the function μ in (3) is bounded on the whole time interval $[t_0, +\infty)$, it can be seen that the term "μz_1" in (52) is bounded on $[t_0, +\infty)$. Upon using Lemma 2, we can conclude that $\widehat{\omega}_1 \in L_\infty$ is ensured to be bounded.

By recalling that $0 < \mu^{\frac{2}{q}-2} \le 1$ and $0 < \zeta_1(t) < 1$, we have

$$\begin{aligned}&|\alpha_1(x_1, x_r, \mu, \widehat{\omega}_1)|\\ \le&\left(k_{11}+\frac{\gamma_1\widehat{\omega}_1\psi_1}{2}+\frac{r_1\phi_1}{2}\right)\mu|z_1|+\frac{k_{12}q^2}{T^{*2}}\mu^{\frac{2}{q}-2}(\mu|z_1|)^3\\ \le&\left(k_{11}+\frac{\gamma_1\widehat{\omega}_1\psi_1}{2}+\frac{r_1\phi_1}{2}\right)\\ &\times\left(\sqrt{\zeta_1(t)\sum_{j=1}^{n}\left(\frac{\overline{g}_i}{\underline{g}_j}z_j^2(t_0)+\overline{g}_i\omega_j^2(t_0)\right)}+\sqrt{\frac{2\overline{g}_i\Phi}{k}}\right)\\ &+\frac{k_{12}q^2}{T^{*2}}\left(\sqrt{\zeta_1(t)\sum_{j=1}^{n}\left(\frac{\overline{g}_i}{\underline{g}_j}z_j^2(t_0)+\overline{g}_i\omega_j^2(t_0)\right)}+\sqrt{\frac{2\overline{g}_i\Phi}{k}}\right)^3\\ \in&L_\infty.\end{aligned}$$

It is noted that ϕ_i and ψ_i $(i = 2, \cdots, n)$ associated with μ in Table I may be large yet bounded ~~computable quantities~~. In the controller design the tunable parameters γ_i and r_i are added to allow the computation to develop on schedule. According to Lemma 2, $\widehat{\omega}_i$ is bounded. Note that $0 < \frac{1}{\mu} \le 1$, we then have

$$\begin{aligned}&|\alpha_i|\\ \le&\left(k_{i1}+\frac{\gamma_i\widehat{\omega}_i\psi_i}{2}+\frac{r_i\phi_i}{2}\right.\\ &\left.+\frac{1}{2}\left(\sqrt{\zeta_1(t)\sum_{j=1}^{n}\left(\frac{\overline{g}_{i-1}}{\underline{g}_j}z_j^2(t_0)+\overline{g}_{i-1}\omega_j^2(t_0)\right)}+\sqrt{\frac{2\overline{g}_{i-1}\Phi}{k}}\right)^2\right)\\ &\times\left(\sqrt{\zeta_1(t)\sum_{j=1}^{n}\left(\frac{\overline{g}_i}{\underline{g}_j}z_j^2(t_0)+\overline{g}_i\omega_j^2(t_0)\right)}+\sqrt{\frac{2\overline{g}_i\Phi}{k}}\right)\end{aligned}$$

$$
+\left(\frac{k_{i2}q^2}{T^{*2}}+\frac{1}{2}\right)
\times\left(\sqrt{\zeta_1(t)\sum_{j=1}^{n}\left(\frac{\overline{g}_i}{\underline{g}_j}z_j^2(t_0)+\overline{g}_i\omega_j^2(t_0)\right)}+\sqrt{\frac{2\overline{g}_i\Phi}{k}}\right)^3
\in L_\infty,
$$

from which we see that $\alpha_i \in L_\infty$, $(i = 2, \cdots, n)$ and then the control input u is bounded.

Stage 2. $t \in [T, \infty)$

We prove that the output error enters and remains in the compact beyond finite time T.

The control strategies are proved to be continuous at $t = T$. From Table I, it is clear that $\overline{\alpha}_i(i = 1, \cdots, n)$ is right continuous at T. In light of the definition of control action softening unit $\delta_i(T)$ in Table II, we see that

$$
\begin{aligned}
\overline{\alpha}_i(T) =& -k_{i1}\mu(T)z_i(T)-\frac{k_{i2}q^2}{T^2}\mu(T)^{1+\frac{2}{q}}z_i(T)^3 \\
&-\frac{\gamma_i\mu(T)\widehat{\omega}_i(T)\overline{\psi}_i(T)}{2}z_i(T)-\frac{r_i\mu\overline{\phi}_i}{2}z_i \\
&+\frac{\gamma_i\mu(T)\widehat{\omega}_i(T)\big(\overline{\psi}_i(T)-\psi_i(T)\big)}{2}z_i \\
&+\mu(T)\big(\overline{\phi}_i(T)-\phi_i(T)\big)z_i(T) \\
=& -k_{i1}\mu(T)z_i(T)-\frac{k_{i2}q^2}{T^2}\mu(T)^{1+\frac{2}{q}}z_i(T)^3 \\
&-\frac{\gamma_i\mu(T)\widehat{\omega}_i\psi_i}{2}z_i-\mu(T)\phi_i(T)z_i(T) \\
=& \lim_{t\to T^-}\alpha_i(t).
\end{aligned}
\tag{54}
$$

At the same time, α_i, α_n and $\overline{\alpha}_n$ can also be ensured to be continuous at the breakpoint $t = T$. It is easily concluded that control action softening unit $\delta_i(T)$ is a finite constant. Adding such term to the virtual control $\overline{\alpha}_i$ to ensure the continuity of the control action does not impact the subsequent time derivative of the virtual control in step $i+1$ because this term is a constant value.

In the following, we analyze the system stability on $[T, \infty)$. Here, μ is a constant.

Consider the Lyapunov function for the overall system as

$$
V=\sum_{j=1}^{n}\frac{1}{\underline{g}_j}z_j^2+\sum_{j=1}^{n}\frac{1}{2\mu^2}\widetilde{\omega}_j^2. \tag{55}
$$

By noting that $\frac{g_j}{\underline{g}_j}z_j\delta_j(T) \le \frac{g_j}{\underline{g}_j}|z_j||\delta_j(T)| \le \frac{\iota_j\mu z_j^2}{2}+\frac{\overline{g}_j^2\delta_j^2}{2\iota_j\mu\underline{g}_j^2}$ (ι_j is any given positive constant, $j = 2, \cdots, n$) and the controller strategies from Table II, the derivative of V is computed as

$$
\begin{aligned}
\dot{V} \le& -\sum_{j=1}^{n}2(k_{j1}-\iota_j)\underline{g}_j\mu\frac{z_j^2}{2\underline{g}_j}-\sum_{j=1}^{n}4k_{j2}\frac{q}{T^*}\mu^{\frac{1}{q}}\frac{z_j^2}{2\underline{g}_j} \\
&-\sum_{i=1}^{n}\overline{k}_{\omega_j}\mu\frac{\widetilde{\omega}_j^2}{2\mu^2}-2\frac{q}{T^*}\mu^{\frac{1}{q}}\sum_{j=1}^{n}\frac{\widetilde{\omega}_j^2}{2\mu^2}+\frac{\overline{\Phi}}{\mu} \\
\le& -\overline{k}\mu V-2\frac{q}{T^*}\mu^{\frac{1}{q}}V+\frac{\overline{\Phi}}{\mu}
\end{aligned}
\tag{56}
$$

where $\overline{k} = \min\{2(k_{j1} - \iota_j)\underline{g}_j, \overline{k}_{\omega_j}\}$, $k_{j1} > 0$, $k_{j2} \ge \frac{1}{2}$ $(j = 1, \cdots, n)$, $\overline{\Phi} = \sum_{j=1}^{n-1}\left(\frac{\overline{g}_j^2}{2\underline{g}_j^2}\right)+\sum_{j=1}^{n}\left(\frac{k_{\omega_j}\varpi_j^2}{2}+\frac{k_{j2}}{\underline{g}_j^2}+\frac{j}{2\gamma_j\underline{g}_j^2}\right)+\sum_{j=2}^{n}\left(\frac{\sum_{k=1}^{n-1}\overline{g}_k^2+2\cdot j-1}{2r_j\underline{g}_j^2}+\frac{\overline{g}_j^2\delta_j^2}{2\iota_j\underline{g}_j^2}\right)+\frac{1}{2r_1\underline{g}_1^2}$, $\iota_1 = 0$, and $k_{j1} > \iota_j > 0$.

On the basis of Lemma 1, we have

$$
V(t) \le \frac{\zeta_2(t)\zeta_1(T)}{\mu^2}V(t_0)+\frac{\zeta_2(t)\Phi}{k\mu^2}+\frac{\overline{\Phi}}{\overline{k}\mu^2}
$$

from which we derive that

$$
\begin{aligned}
&|z_i| \\
\le& \frac{1}{\mu}\sqrt{\zeta_2(t)\zeta_1(T^*+t_0-\epsilon)\sum_{j=1}^{n}\left(\frac{\overline{g}_i}{\underline{g}_j}z_j^2(t_0)+\overline{g}_i\widetilde{\omega}_j^2(t_0)\right)} \\
&+\frac{1}{\mu}\left(\sqrt{2\overline{g}_i\zeta_2(t)\Phi/k}+\sqrt{2\overline{g}_i\overline{\Phi}/\overline{k}}\right)
\end{aligned}
\tag{57}
$$

and

$$
\begin{aligned}
&|\widetilde{\omega}_i| \\
\le& \sqrt{\zeta_2(t)\Big(\zeta_1(T^*-t_0+\epsilon)\sum_{i=1}^{n}\big(z_i^2(t_0)/\underline{g}_i+\widetilde{\omega}_i^2(t_0)\big)\Big)} \\
&+\sqrt{2\zeta_2(t)\Phi/k}+\sqrt{2\overline{\Phi}/\overline{k}}.
\end{aligned}
\tag{58}
$$

Therefore, $z_i \in L_\infty$ and $\widehat{\omega}_i \in L_\infty$ for any initial conditions. Especially, $z_i \to 0$ as $t \to \infty$ and $\epsilon \to 0$. From (57), it is straightforward that "μz_i" is bounded, which provides a strong assurance for the boundedness analysis of controllers. Note that ψ_1 and ϕ_1 are bounded, it follows from (29) and Assumption 3 that $\alpha_1 \in L_\infty$. The boundedness of α_i $(i = 2, \cdots, n-1)$ and u are also able to be established by following the similar procedure to the above analysis. This completes the proof.

■

Remark 5. *With Table I and II, the virtual and actual controls* $(\alpha_i, i = 1, \cdots, n)$ *are ensured to continue in switch point "*$T = T^* + t_0 - \epsilon$*" because it depends on the calculable control action softening unit* $\big(\delta_i(T^* - t_0 + \epsilon), i = 1, \cdots, n\big)$. *This feature originates from [35] and is verified in* (54). *This follows from the fact that the derivative of given time-varying gain bounded function in* (3) *is piecewise continuous. Without this compensation technique, the discontinuity of the control will cause severe oscillations in the neighbourhood of the switching time, seriously impairing the system's ability to operate.*

Remark 6. *It is worth mentioning that the proposed method is valid on the whole time interval, making it different from those existing PTC results such as ([1], [3]–[5], [10]) where the control gains grow unbounded towards the finite terminal time and thus the control schemes are only valid on* $[t_0, T^*)$ *rather than on* $[t_0, \infty)$ *that would substantially limit their application. It is noted that the work by [12] addresses the PTC problem on the whole time interval, however, only regulation rather than tracking is addressed, and further, the nonlinear function "f" is assumed to be known and vanishing. In addition, the upper* *and lower bound information of $\underline{g}_i$ and $\overline{g}_i$ are not needed in this work, distinguishing itself from most existing PTC works ([1], [11], [32]) where the bound information of g_i is needed to be known. Moreover, different from the existing PTC works ([10], [12], [13]) where the control scheme is discontinues especially at the prescribed time T^*, the proposed one remains continue and bounded throughout the whole time interval.*

IV. Numerical simulation

To verify the effectiveness of the proposed control method, two simulation examples are given.

A. Example 1

To compare the performance of proposed control strategy with that developed by [32], we employ the same model as that in [32] for the numerical simulation:

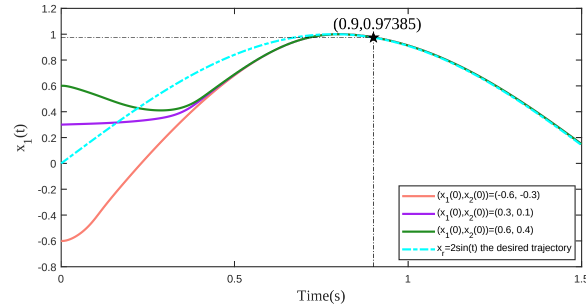
$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = g(\cdot)u + f(\cdot), \end{cases} \tag{59}$$

which represents the "wing-rock" unstable motion of some high-performance aircraft [36]. As in [32], for simulation purpose, we still consider $f(\cdot) = 1 + \cos(t)x_1 + 2\sin(2t)x_2 + 2|x_1|x_2 + 3|x_2|x_2 + x_1^3$ and $g(\cdot) = 2 + 0.4\sin(t)$. The desired trajectory is $x_r(t) = \sin(2t)$. Obviously, $g(t)$, $f(t)$ and $x_r(t)$ satisfy Assumption 1, 2 and 3, respectively. It is straightforward that $|f(\cdot)| \le \nu\vartheta(x_1, x_2)$ with $\vartheta(x_1, x_2) = 1 + |x_1| + |x_2| + |x_1x_2| + x_2^2 + |x_1|^3$, which satisfies Assumption 2.

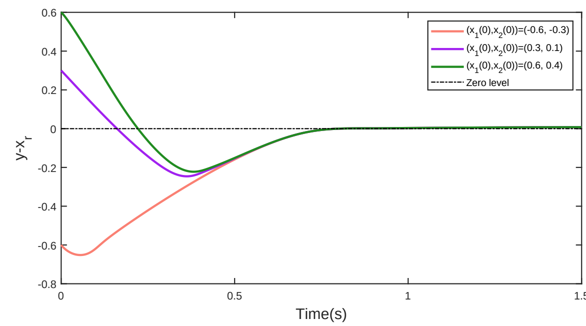
The control scheme is directly taken from Table I and II, where $k_{11} = 1$, $k_{12} = 0.5$, $k_{21} = 0.5$, $k_{22} = 0.5$, $\overline{k}_{\omega_2} = 0.1$, $q = 4$, $\gamma_2 = 0.2$, $r_1 = 0.00001$, $r_2 = 0.0001$, $t_0 = 0, T^* = 1.2$, $\epsilon = 0.3, t_{end} = 1.5$. Then $T^* + t_0 - \epsilon = 0.9$ and $k_{\omega_2} = \overline{k}_{\omega_2} + 2\frac{q}{T^*}\left(\frac{T^*}{\epsilon}\right)^{1-q} = 0.2042$. The initial conditions of the system are given by three different conditions $[x_1(0), x_2(0)] = [-0.6, -0.3], [0.3, 0.1], [0.6, 0.4]$, and the initial value of adaptive law is taken as $\widehat{\omega}(0) = 0.1$.

The simulation results are shown in Fig 1. From Fig 1(a), we can see that the system states can track the desired trajectory within the time $0.9s$ under different initial conditions, as well as precise tracking can be maintained after prescribed-time. Not that the system can remain operating beyond the finite time T^*, distinguishing itself from the method in [1]. From Fig. 2, it is observed that the control signal is continuous throughout time interval. From Fig. 3, it is shown that the parameter estimate $\widehat{\omega}$ is also bounded for any initial state.

The simulation results compared with that in [32] (all parameters are consistent) are shown in Fig. 4 and 5. The error between state x_1 and x_r is shown in Fig. 4(a) and the error between x_2 and $\dot{x}_r$ is shown in Fig. 4(b). The performance of $x_1(t)$ is shown in Fig. 5, from which we can see that the output can track the desired trajectory within the prescribed time $T^* = 0.8$. This is the same as that in [32], and further, the precise tracking can be maintained after the prescribed time T^*. From the simulation results we can see that the proposed control strategy is more straightforward and less computational burden with good tracking effect.



(a) state x_1



(b) error: $y(t) - x_r(t)$

Fig. 1: The performances of state x with different initial conditions

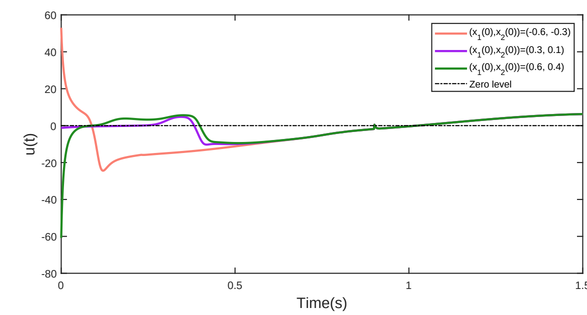


Fig. 2: The performances of u with different initial conditions

B. Example 2

We conduct the simulation on the following strict-feedback nonlinear systems under Assumption 1 and 2:

$$\begin{cases} \dot{x}_1 = g_1(x_1)x_2 + f_1(x_1) \\ \dot{x}_2 = g_2(\overline{x}_2)u + f_2(\overline{x}_2) \end{cases} \tag{60}$$

in which $g_1(x_1) = 2 + 0.2\sin(x_1)$, $g_2 = 1 + 0.5\cos(x_1x_2)$, $f_1(\cdot) = 0.2x_1^2\sin(0.5x_1)$, $f_2(\cdot) = x_2^2\cos(0.1x_1)$. Obviously, $\vartheta_1 = x_1^2$ and $\vartheta_2 = x_2^2$. The desired signal is chosen as $x_r = 2\sin(t)$, which meets Assumption 3. Meanwhile, the design parameters are selected as: $t_0 = 0$, $\epsilon = 0.6$, $T^* = 2.6$, $t_{end} = 5$, $T^* + t_0 - \epsilon = 2$, $q = 4$, $k_{11} = 1$, $k_{12} = \frac{1}{2}$, $\gamma_1 = 0.4$, $r_1 = 0.001$, $\overline{k}_{\omega_1} = 1$, $k_{\omega_1} = 1.0378$, $k_{21} = \frac{1}{2}$, $k_{22} = \frac{1}{2}$, $\gamma_2 = 0.2$, $r_2 = 0.001$, $\overline{k}_{\omega_2} = 1$, $k_{\omega_2} = 1.0378$. $[x_1(0), x_2(0)] = [-1, -0.5]$, $[0, 0], [1, 0.5]$. The simulation results are depicted in Fig. 6-8. The state x_1 under different

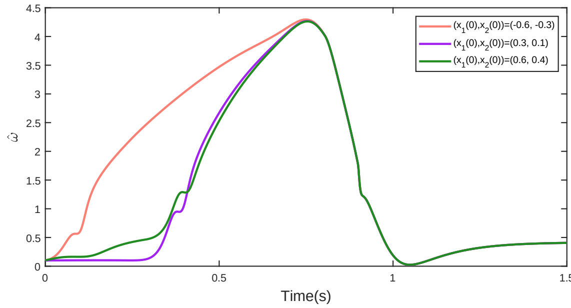
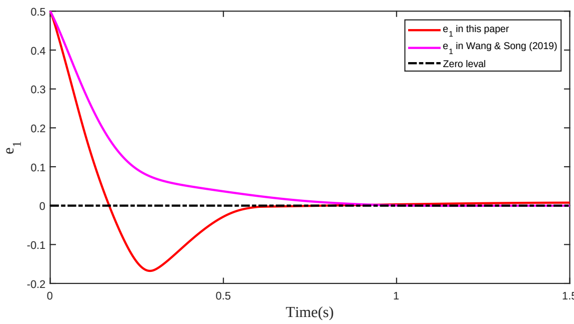
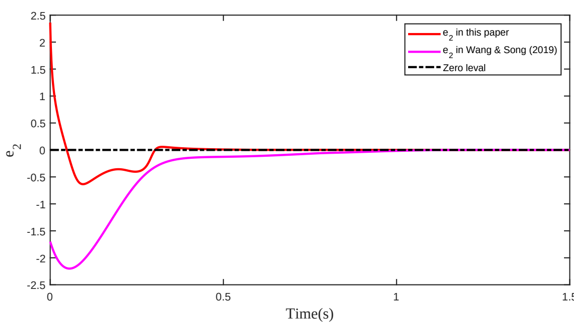


Fig. 3: The performances of $\hat{\omega}$ with different initial conditions



(a) error $e_1 = x_1 - x_r$



(b) error $e_2 = x_2 - \dot{x}_r$

Fig. 4: The tracking error e under different control schemes

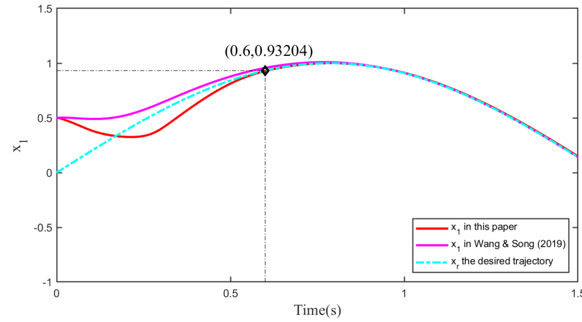


Fig. 5: Compare with [32], the performances of x_1, $\big([x_1(t_0), x_2(t_0)] = [0.5, 0.3], \epsilon = 0.2, T^* = 0.8, t_{end} = 1.5\big)$

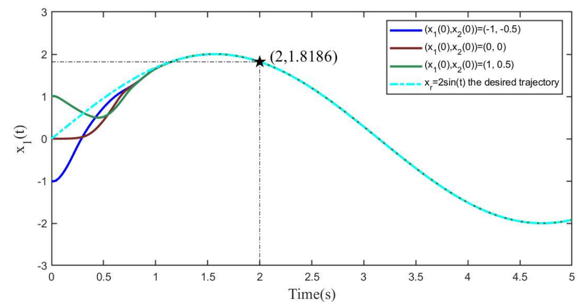


Fig. 6: The performances of x_1 with different initial conditions in Example IV-B

initial condition is shown in Fig. 6, from which we see that the system x_1 is capable of tracking the desired trajectory x_r within the finite time $2s$ and the precise tracking can be maintained after the prescribed time $2s$. Obviously, the prescribed time T^* is independent of the initial state and other design parameters of systems. The control signal is shown in Fig. 7. The evolution of adaptive law is presented in Fig. 8, from which we see that the adaptive parameters are bounded.

V. Conclusion

In this paper, a practical prescribed time tracking control method is presented for a class of nonlinear systems with mismatched yet non-vanishing uncertainties. The method proposed here, by means of a time-varying scaling function that increases monotonically with time and remains bounded at and beyond the prescribed time, avoids infinite control gain anytime during system operation, thus makes PTC practical and bridges PTC and its executable version analytically. It is shown that, without the need for any prior control gain information of system, the tracking error between the output of systems and desired trajectory is settled in the neighborhood of origin within pre-assigned time regardless of the initial condition and other design parameters. The developed solution is truly practical and global, allowing the systems to operate throughout the whole time interval. Extension of the proposed PTC method to cooperative control of multi-agent systems with unmatched uncertainties under directed topology represents an interesting topic for future research.

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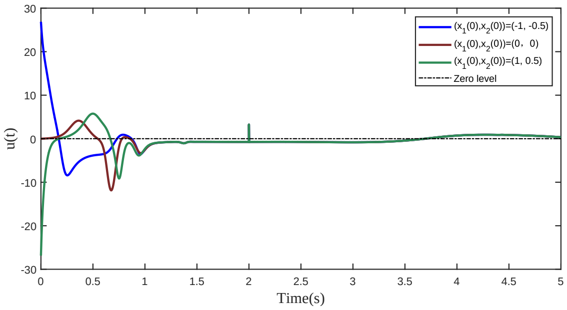
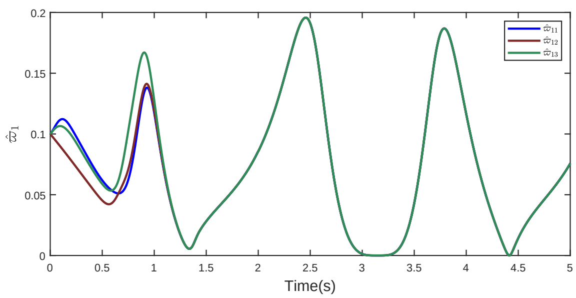
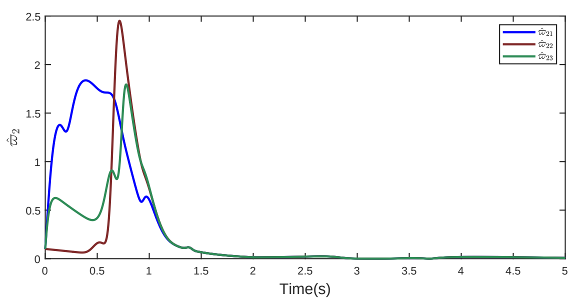


Fig. 7: The performances of u with different initial conditions in Example IV-B



(a) adaptive parameter $\hat{\varpi}_1$



(b) adaptive parameter $\hat{\varpi}_2$

Fig. 8: The performances of adaptive parameter $\hat{\varpi}$ with different initial conditions

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