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Distributed Adaptive Finite-Time Approach for Formation–Containment Control of Networked Nonlinear Systems Under Directed Topology

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***Abstract*— This paper presents a distributed adaptive finite-time control solution to the formation–containment problem for multiple networked systems with uncertain nonlinear dynamics and directed communication constraints. By integrating the special topology feature of the new constructed symmetrical matrix, the technical difficulty in finite-time formation–containment control arising from the asymmetrical Laplacian matrix under single-way directed communication is circumvented. Based upon fractional power feedback of the local error, an adaptive distributed control scheme is established to drive the leaders into the prespecified formation configuration in finite time. Meanwhile, a distributed adaptive control scheme, independent of the unavailable inputs of the leaders, is designed to keep the followers within a bounded distance from the moving leaders and then to make the followers enter the convex hull shaped by the formation of the leaders in finite time. The effectiveness of the proposed control scheme is confirmed by the simulation.**

***Index Terms*— Directed interaction topology, finite-time control, formation–containment, nonlinear multiagent systems (MAS).**

I. INTRODUCTION

ALTHOUGH considerable amount of interesting works on cooperative control of networked multiagent systems (MAS) have been reported during the past few years, most existing works are concerned with consensus of MAS in the sense of controlling the agents to reach an agreement (see [1]–[4] and the references therein). In contrast to the consensus problem, a more interesting yet challenging one is the formation–containment problem where multiple leader agents and follower agents are involved. By formation–containment, it is referred to the scenario that multiple leaders are required to operate according to the prescribed trajectory and then keep a specific formation structure, and the followers are forced to enter the convex hull spanned by those of the leaders [5]. The formation–containment problem stems from numerous natural phenomena and potential applications. For instance, for a vehicle group moving to a target area, the leaders equipped with necessary sensors to detect the obstacles must converge to a desired configuration encoded by the relative interagent positions, and the followers (the other vehicles) are required to stay within the safe area formed by the leaders when close to the hazardous obstacles. Such formation–containment operation is frequently encountered in autonomous operation of underwater vehicles [6], formation control of unmanned aerial vehicles (UAVs) [7], and robot swarms [8].

Of particular interest is the finite-time formation–containment control in that the leader agents form a specified formation configuration in finite time, and the follower agents enter the convex hull shaped by the formation of the leaders also in finite time. It is well known that systems with finite-settling-time possess faster convergence rate, higher accuracy, better disturbance rejection properties, and robustness against uncertainties [9], because of which the finite-time control has been extensively developed in recent years. Considerable amount of research has been conducted on finite-time control of MAS modeled by first-order linear or feedback linearizable nonlinear dynamics, such as [10]–[15]. The work by Li *et al*. [16] extends the finite-time control for single integrators to the second-order case under an undirected topology by adding a power integrator technique [17]. This technique has also been applied in [18] for formation control based on the output feedback and in [19] for containment control based on convergent observer design and also in [20] for rotating formation–containment control based on polar coordinate, all for second-order MAS under an undirected topology. By using the discontinuous terminal sliding-mode method, He *et al*. [21] addresses the finite-time containment control for second-order MAS under a directed topology. However, all the aforementioned works are about first- or second-order linear or linearizable dynamics. Meng *et al*. [22] investigates finite-time attitude containment control for multiple rigid bodies and Yu *et al*. [23] addresses the finite-time leader–follower consensus problem of MAS with followers having nonidentical unknown nonlinear dynamics. It is worth

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mentioning that in both the works the information from each follower's neighbors is utilized for the controller design; the communication between the followers are undirected; and the nonlinearities are assumed to admit the homogeneous property, without which the finite-time algorithms are invalid.

Efforts on adaptive finite-time consensus of second-order MAS and containment of high-order MAS have been made in [24] and [25], respectively, both of which rely on the symmetry property of the undirected network graph. However, results on finite-time distributed control of MAS with second-order nonlinear dynamics under a directed topology are limited. Extending the adaptive finite-time control method for second-order nonlinear MAS under an undirected graph to leaderless consensus of MAS under a directed graph encounters significant technical challenge. This is because the asymmetric connection of the directed graph remains as an obstacle. This obstacle is removed in recent works [26], [27] by making use of an important property on the newly constructed Laplacian established from the eigenvalue theory. It is worth noting that, although capable of dealing with directed graphs, the consensus results in [26], [27] cannot be directly extended to the finite-time formation–containment case where the leaders are dynamically moving and the control inputs are unknown to the followers before the leaders achieving the desired formation as usually the case in practice, making the underlying problem rather challenging.

To our best knowledge, little effort has been made on finite-time formation–containment control of multiple networked uncertain nonlinear systems under directed communication constraints. The main technical barriers stem from several aspects: 1) in addressing the adaptive finite-time control, the well known Barbalat lemma and the theory of uniformly ultimately boundedness [28], [29] commonly used for analyzing asymptotic convergence of MAS are not applicable anymore; 2) due to the directed communication constraints, the existing finite-time consensus control methods in [24] and [25] are not applicable here; 3) compared with the leaderless or leader–follower consensus [24], [26], [27], the formation–containment problem is much more involved due to the fact that both the leaders and the followers must be simultaneously controlled to achieve the formation and containment, respectively, and furthermore, before reaching the desired formation, the leaders keep moving continuously and the control inputs are unknown to the followers. Consequently, the underlying formation–containment control problem poses a significant challenge, rendering most existing adaptive finite-time consensus control methods inapplicable directly.

In this paper, we attempt to provide a feasible finite-time control solution to the formation–containment problem for MAS in the presence of uncertain nonlinear dynamics and directed communication constraints among leaders and followers. The main features and contributions of this paper can be summarized as follows.

1) As the Laplacian matrices corresponding to the formation and containment controls exhibit different structural properties, special treatment is needed in control design and stability analysis. By introducing the neighborhood error for leaders/followers into a carefully selected Lyapunov function and integrating the useful feature of the newly constructed symmetrical matrix, the technical difficulty in finite-time formation–containment control under single-way directed communication is circumvented, allowing for the development of distributed adaptive finite-time formation–containment protocols for multiple leaders and followers under one-way directed communication constraints.
2) Based upon the fractional power state feedback control plus adaptive compensation, algorithms that steer the leaders into a prespecified formation in finite time are developed.
3) The special topology features on the followers are utilized to develop the algorithm: a) to ensure all the followers to stay close to the leaders during the period of time that the leaders are achieving their formation and b) to steer the followers to enter the convex hull formed by the formation of the leaders in finite time.
4) All the internal signals are ensured to be uniformly ultimately bounded and the formation–containment configuration error uniformly converges to zero in finite time.
5) The finite-time periods for the leaders to reach the required formation configuration and for the followers to achieve the containment are explicitly provided.

II. Problem Formulation and Preliminaries

Throughout this paper, the initial time t_0 is set as $t_0 = 0$ without loss of generality; $\lambda_{\max}(F)$ and $\lambda_{\min}(F)$ denote the maximum and minimum eigenvalue of matrix F, respectively; $1_N(0_N) \in R^N$ denotes a vector with each entry being 1(0); $\otimes$ denotes the Kronecker product; For a vector $X = [x_1, x_2, \ldots, x_N]^T$, $|X| = [|x_1|, |x_2|, \ldots, |x_N|]^T$ with $|\cdot|$ the absolute value of a real number, $X^h = [x_1^h, x_2^h, \ldots, x_N^h]^T$ with $h \in R$, $\|X\|$ denotes the Euclidean norm of the vector X, and $\operatorname{diag}\{X\} = \operatorname{diag}\{x_1, \ldots, x_N\}$.

A. Problem Formulation

We consider a group of agents modeled by

$$\begin{aligned} \dot{x}_k(t) &= v_k(t) \\ g_k \dot{v}_k(t) &= u_k(t) + F_k(x_k, v_k), \quad k \in J \end{aligned} \tag{1}$$

where $J = \{1, \ldots, N\}$, $x_k = [x_{k1}, \ldots, x_{ks}]^T \in R^s$, $v_k = [v_{k1}, \ldots, v_{ks}]^T \in R^s$, and $u_k = [u_{k1}, \ldots, u_{ks}]^T \in R^s$ represent, respectively, the position, velocity, and control input of the kth subsystem; $g_k = \operatorname{diag}\{g_{ki}\} \in R^{s\times s}$, with $g_{ki} > 0$, is an unknown and constant matrix; $F_k(\cdot)$ represents the system nonlinearities satisfying $F_k(\cdot) = \beta_k^T \phi_k(x_k, v_k)$, with $\beta_k \in R^{m\times s}$ being the unknown constant parameter matrix and $\phi_k(\cdot) \in R^m$ being a known basic function vector. Furthermore, $\|\phi_k(x_k, v_k)\|_1 \le a(x_k, v_k) p(\|v_k\|_1) \le a_0 p(\|v_k\|_1) = P(\|v_k\|_1)$, where $a(x_k, v_k) \le a_0$ and a_0 is bounded for all x_k and v_k, $P(\cdot)$ is a polynomial satisfying $P(0) = 0$, and β_k is assumed in a known compact set (for simplicity, we only consider the case of $s \le 3$, namely, the case of point-mass multiple agents).

Remark 1: The model (1) has been widely used in addressing consensus of MAS [24], [30]. In this paper, we explore the

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more general and obviously more complex problem involving directed topology-based formation and containment simultaneously. Also, note that although g_k is unknown, we can always provide the robust estimates on its upper bound, i.e., $g_{ki} \leq \bar{g} < \infty$ for some known constant $\bar{g} > 0$.

For formation–containment problem, we call an agent the leader if its neighbor(s) is (are) only the agent(s) coordinating with the neighbor(s) to achieve formation; we call an agent the follower if it has atleast one neighbor and it coordinates with their neighbor(s) to achieve containment. Let $\mathscr{A} = \{1, 2, \ldots, M\}$ and $\mathscr{B} = \{M+1, M+2, \ldots, N\}$ be the set of indices for the leaders and followers, respectively.

Definition 1: Multiagent system (1) is said to achieve formation–containment in finite time, if for any initial states there exists a finite time T^* such that for all $t \geq T^*$ and all $j, k \in \mathscr{A}$

$$(x_k(t) - x_j(t)) - (\varpi_k - \varpi_j) = 0_s \tag{2}$$

where $\varpi = [\varpi_1^T, \varpi_2^T, \ldots, \varpi_M^T]^T$ ($\varpi_k = [\varpi_{k1}, \ldots, \varpi_{ks}]^T$, $k \in \mathscr{A}$) denotes the desired formation structure for the M leaders, and meanwhile, there exist a group of nonnegative constants σ_j ($j \in \mathscr{A}$) satisfying $\sum_{j=1}^{M} \sigma_j = 1$ such that for all $t \geq T^*$ and for all $i \in \mathscr{B}$

$$x_i(t) - \sum_{j=1}^{M} \sigma_j x_j(t) = 0_s. \tag{3}$$

Physically, (2) and (3) imply that the leaders collaborate to form a specific formation and the followers coordinate with their neighbors to enter the formation and achieve the containment in finite time.

The objective of this paper is to establish distributed adaptive control schemes for the leaders and followers, respectively, such that the formation–containment can be achieved under one-way directed communication interactions in finite time. Before presenting the control algorithms, we make the following fundamental assumption.

Assumption 1: The communication topology between different leaders is directed and strongly connected, and the communication among different followers is directed. For each follower, there exists at least one leader that has a directed path to it.

Suppose that the communication network among the N agents is represented by a directed graph G, with the weighted adjacency matrix $A = [a_{ij}]$ and the Laplacian matrix L (see [3] for the details on graph theory). Under Assumption 1, the Laplatian L of the directed topology G can be represented as

$$L = \begin{bmatrix} L_1 & 0 \\ L_2 & L_3 \end{bmatrix}$$

where $L_1 \in R^{M \times M}$ (L_1 is the Laplacian associated with the interaction topology among leaders $G_{\mathscr{A}}$), $L_2 \in R^{(N-M)\times M}$, and $L_3 \in R^{(N-M)\times(N-M)}$. In [31] all the eigenvalues of L_3 have positive real parts. In addition, each entry of $-L_3^{-1}L_2$ is nonnegative, and each row of $-L_3^{-1}L_2$ has a sum equal to 1. By [32, Th.2.3, p. 134], there exists a matrix $\xi = \mathrm{diag}\{\xi_1, \ldots, \xi_{N-M}\}$ ($\xi_k > 0$) such that $(\xi L_3 + L_3^T \xi)$ is positive definite.

B. Some Useful Preliminaries

Before moving on, the following lemmas are needed.

Lemma 1 [33]: Suppose there exists a continuously differentiable function $V(x, t) : U_0 \times R^+ \to R$ ($U_0 \subset R^n$ is an open neighborhood of the origin), a real number $c > 0$ and $0 < \alpha < 1$, such that $V(x, t)$ is positive definite and $\dot{V}(x, t) + cV(x, t)^\alpha \leq 0$ on U ($U \subset U_0$), then $V(x, t)$ is locally in finite-time convergent with a finite settling time $T^* \leq (V(x(0))^{1-\alpha}/c(1-\alpha))$, such that for any given initial state $x(t_0) \in U/\{0\}$, $\lim_{t\to T^*} V(x, t) = 0$ and $V(x, t) = 0$ for $t \geq T^*$.

Lemma 2 [34]: For $x_i \in R$, $i = 1, 2, \ldots, N$, $0 < h \leq 1$, then $(\sum_{i=1}^{N} |x_i|)^h \leq \sum_{i=1}^{N} |x_i|^h \leq N^{1-h}(\sum_{i=1}^{N} |x_i|)^h$.

Lemma 3 [35]: If $h = h_2/h_1 \geq 1$, where $h_1, h_2 > 0$ are odd integers, then $|x - y|^h \leq 2^{h-1}|x^h - y^h|$. Reversely if $0 < h = h_1/h_2 \leq 1$, then $|x^h - y^h| \leq 2^{1-h}|x - y|^h$.

Lemma 4 [35]: For $x, y \in R$, if $c, d > 0$, then $|x|^c|y|^d \leq c/(c+d)|x|^{c+d} + d/(c+d)|y|^{c+d}$.

Definition 2 [36]: Let $\hat{\bullet}$ be the estimate of an unknown parameter $\bullet$ locating in a closed ball of known radius $r_{\mathscr{D}}$. The Lipschitz continuous projection algorithm $\mathrm{Proj}(\rho, \hat{\bullet})$ is defined as

$$\mathrm{Proj}(\varrho, \hat{\bullet}) = \begin{cases} \rho, & \text{if } \hbar(\hat{\bullet}) \leq 0 \text{ or } \hbar\ (\hat{\bullet}) \geq 0 \\ \text{and} & \dfrac{\partial \hbar(\hat{\bullet})}{\partial \hat{\bullet}} \rho \leq 0 \\ \rho - \hbar(\hat{\bullet})\rho, & \text{if } \hbar(\hat{\bullet}) > 0 \text{ and } \dfrac{\partial \hbar(\hat{\bullet})}{\partial \hat{\bullet}} \rho > 0 \end{cases}$$

where $\hbar(\hat{\bullet}) = ((\hat{\bullet}^2 - r_{\mathscr{D}}^2)/(\epsilon^2 + 2\epsilon r_{\mathscr{D}}))$, in which ϵ is an arbitrarily small positive constant.

III. Main Results

A. Finite-Time Formation Control for the Leader Agents

We first introduce the neighborhood error for the kth ($k \in \mathscr{A}$) leader as

$$e_{ki} = \sum_{j \in \mathscr{N}_k} a_{kj}(x_{ki} - \varpi_{ki} - x_{ji} + \varpi_{ji}), \quad i = 1, \ldots, s \tag{4}$$

where $\mathscr{N}_k$ is the kth agent's neighbor set, and ϖ_{ki} denotes the desired formation structure such that $x_{ki} - x_{ji} = \varpi_{ki} - \varpi_{ji}$ ($k, j \in \mathscr{A}$). Denote $x_{\mathscr{A}} = [x_{\mathscr{A}1}^T, x_{\mathscr{A}2}^T, \ldots, x_{\mathscr{A}s}^T]^T$, $\varpi = [\varpi_1^T, \varpi_2^T, \ldots, \varpi_s^T]^T$, and $E_{\mathscr{A}} = [e_{\mathscr{A}1}^T, e_{\mathscr{A}2}^T, \ldots, e_{\mathscr{A}s}^T]^T$, with $x_{\mathscr{A}i} = [x_{1i}, x_{2i}, \ldots, x_{Mi}]^T$, $\varpi_i = [\varpi_{1i}, \varpi_{2i}, \ldots, \varpi_{Mi}]^T$, and $e_{\mathscr{A}i} = [e_{1i}, e_{2i}, \ldots, e_{Mi}]^T$ ($i = 1, 2, \ldots, s$), then it holds that

$$E_{\mathscr{A}} = (I_s \otimes L_1)(x_{\mathscr{A}} - \varpi). \tag{5}$$

Since L_1 is the Laplacian of the strongly connected digraph $G_{\mathscr{A}}$, it is straightforward that $E_{\mathscr{A}} = (I_s \otimes L_1)(x_{\mathscr{A}} - \varpi) = 0$ if and only if $x_{1i} - \varpi_{1i} = \cdots = x_{Mi} - \varpi_{Mi}$ ($i = 1, \ldots, s$) according to [3, Lemma 1.3]. Thus with $E_{\mathscr{A}}$ being so defined the finite-time formation objective for the leaders is achieved if and only if $E_{\mathscr{A}} \to 0$ in finite time.

To develop the finite-time formation control scheme for the leaders, we introduce the following local virtual error:

$$\delta_{ki} = v_{ki}^{1/h} - v_{ki}^{*1/h}, \quad k \in \mathscr{A}, \ i = 1, \ldots, s \tag{6}$$

where $h = (2l-1)/(2l+1)$ with $l \in Z^+$, and v_{ki}^* is the virtual control of v_{ki} defined by $v_{ki}^* = -c_2 e_{ki}^h$, with $c_2 > 0$ being a design parameter.

The main idea for the controller design is to steer the neighborhood position error e_{ki} and the local virtual error δ_{ki} to converge to zero in finite time such that the formation is achieved by the leaders in finite time. Thus, the finite-time distributed control law for the kth leader is developed as

$$u_k = -c_1 \delta_k^{2h-1} - \hat{\beta}_k^T \phi_k, \quad k \in \mathscr{A} \tag{7}$$

with the updated law

$$\dot{\hat{\beta}}_k(j,i) = \mathrm{Proj}\big(\big(\Gamma_k \phi_k \delta_k^T\big)(j,i), \hat{\beta}_k(j,i)\big), \quad k \in \mathscr{A} \tag{8}$$

for $i = 1, \ldots, s$ and $j = 1, \ldots, m$, where $\delta_k^{2h-1} = [\delta_{k1}^{2h-1}, \ldots, \delta_{ks}^{2h-1}]^T$, $\bullet(j,i)$ denotes the (j,i)th element of $\bullet$, $c_1 > 0$ is a design parameter, $\Gamma_k = \mathrm{diag}\{\gamma_{k1}, \ldots, \gamma_{km}\} \in R^{m \times m} > 0$ is a design parameter matrix, $\hat{\beta}_k$ is the estimation of β_k, and $\phi_k(\cdot)$ is the scalar and readily computable function vector. The controller basically consists of two parts: $-c_1 \delta_k^{2h-1}$ to ensure finite-time convergence, and $\hat{\beta}_k^T \phi_k$ for nonlinear compensation.

We are ready to present the first main result and the proof.

Theorem 1: Consider a group of nonlinear systems consisting of N agents as described by (1) with M leaders and $N-M$ followers. Under Assumption 1, if the distributed control laws (7) and (8) are applied for the M leaders, then the leaders in system (1) are ensured to achieve the pregiven formation in finite time $T_{\mathscr{A}}^*$ as specified by

$$T_{\mathscr{A}}^* = T_{\mathscr{A}1}^* + T_{\mathscr{A}2}^* \tag{9}$$

in which $T_{\mathscr{A}1}^*$ and $T_{\mathscr{A}2}^*$ are given in (29) and (38), respectively.

Proof: The proof is somewhat involved and consists of four steps.

Step 1: Note that for directed communication topology the Laplacian matrix L_1 is no longer symmetric, which imposes technical challenge in constructing the Lyapunov function. To circumvent this difficulty, we introduce a diagonal matrix $P_{\mathscr{A}} = \mathrm{diag}\{p_{\mathscr{A}}\}$, with $p_{\mathscr{A}} = [p_1, p_2, \ldots, p_M]^T$ being the left eigenvector of L_1 associated with the zero eigenvalue, into the following Lyapunov function:

$$V_{\mathscr{A}1} = \frac{1}{(1+h)k_m}\Big(E_{\mathscr{A}}^{\frac{1+h}{2}}\Big)^T (I_s \otimes P_{\mathscr{A}}) E_{\mathscr{A}}^{\frac{1+h}{2}} \tag{10}$$

where $k_m > 0$ is a constant, which will be defined later. The time derivative of $V_{\mathscr{A}1}$ is computed as

$$\begin{aligned}\dot{V}_{\mathscr{A}1} &= \frac{1}{k_m}\big(E_{\mathscr{A}}^h\big)^T (I_s \otimes P_{\mathscr{A}})\dot{E}_{\mathscr{A}} \\ &= \frac{1}{k_m}\big(E_{\mathscr{A}}^h\big)^T (I_s \otimes (P_{\mathscr{A}}L_1))v_{\mathscr{A}}\end{aligned} \tag{11}$$

with $v_{\mathscr{A}} = \dot{x}_{\mathscr{A}}$. By inserting the virtual control $v_{\mathscr{A}}^* = -c_2 E_{\mathscr{A}}^h$ into (11), we have

$$\begin{aligned}\dot{V}_{\mathscr{A}1} &= -\frac{c_2}{k_m}\big(E_{\mathscr{A}}^h\big)^T (I_s \otimes (P_{\mathscr{A}}L_1))E_{\mathscr{A}}^h \\ &\quad + \frac{1}{k_m}\big(E_{\mathscr{A}}^h\big)^T (I_s \otimes (P_{\mathscr{A}}L_1))\big(v_{\mathscr{A}} - v_{\mathscr{A}}^*\big) \\ &= -\frac{c_2}{k_m}\big(E_{\mathscr{A}}^h\big)^T \left[I_s \otimes \left(\frac{1}{2}\big(P_{\mathscr{A}}L_1 + L_1^T P_{\mathscr{A}}\big)\right)\right] E_{\mathscr{A}}^h \\ &\quad + \frac{1}{k_m}\big(E_{\mathscr{A}}^h\big)^T (I_s \otimes (P_{\mathscr{A}}L_1))\big(v_{\mathscr{A}} - v_{\mathscr{A}}^*\big).\end{aligned} \tag{12}$$

Note that in (12) is involved a matrix of the form $Q = (1/2)(P_{\mathscr{A}}L_1 + L_1^T P_{\mathscr{A}})$, which exhibits the following crucial property (see [26] for detail proof): there exists a constant $k_m > 0$ such that:

$$\big(E_{\mathscr{A}}^h\big)^T (I_s \otimes Q) E_{\mathscr{A}}^h \geq k_m \big(E_{\mathscr{A}}^h\big)^T E_{\mathscr{A}}^h. \tag{13}$$

Upon using (13), it is deduced from (12) that

$$\begin{aligned}\dot{V}_{\mathscr{A}1} &\leq -c_2\big(E_{\mathscr{A}}^h\big)^T E_{\mathscr{A}}^h + \frac{1}{k_m} \\ &\quad \times \big(E_{\mathscr{A}}^h\big)^T (I_s \otimes (P_{\mathscr{A}}L_1))\big(v_{\mathscr{A}} - v_{\mathscr{A}}^*\big) \\ &= -c_2 \sum_{i=1}^{s}\sum_{k=1}^{M}(e_{ki})^{2h} \\ &\quad + \frac{1}{k_m}\sum_{i=1}^{s}\left[\sum_{k=1}^{M}\big(v_{ki} - v_{ki}^*\big)\sum_{j=1}^{M}\ell_{jk}(e_{ji})^h\right]\end{aligned} \tag{14}$$

where ℓ_{jk} is the (j,k)th element of $(P_{\mathscr{A}}L_1)^T$. By examining the second term of the right hand of (14), we get

$$\begin{aligned}&\frac{1}{k_m}\sum_{i=1}^{s}\left[\sum_{k=1}^{M}\big(v_{ki} - v_{ki}^*\big)\sum_{j=1}^{M}\ell_{jk}(e_{ji})^h\right] \\ &\leq \frac{1}{k_m}\sum_{i=1}^{s}\left[\sum_{k=1}^{M}2^{1-h}|\delta_{ki}|^h \sum_{j=1}^{M}|\ell_{jk}||e_{ji}|^h\right] \\ &\leq 2^{1-h}\ell_{\max}\frac{1}{k_m}\sum_{i=1}^{s}\left[\sum_{k=1}^{M}|\delta_{ki}|^h \sum_{j=1}^{M}|e_{ji}|^h\right] \\ &\leq 2^{1-h}\ell_{\max}\frac{1}{k_m}\sum_{i=1}^{s}\frac{1}{2}\left[\left(\sum_{k=1}^{M}|\delta_{ki}|^h\right)^2 + \left(\sum_{j=1}^{M}|e_{ji}|^h\right)^2\right] \\ &\leq 2^{-h}\ell_{\max}\frac{1}{k_m}\sum_{i=1}^{s}\left[M\sum_{k=1}^{M}|\delta_{ki}|^{2h} + M\sum_{k=1}^{M}|e_{ki}|^{2h}\right]\end{aligned} \tag{15}$$

in which the first inequality follows from the relation that $|v_{ki} - v_{ki}^*| \leq 2^{1-h}|\delta_{ki}|^h$ by Lemma 3, the second inequality is derived by setting $\ell_{\max} = \max_{j,k\in\mathscr{A}}|\ell_{jk}|$, the third inequality is established by using Young's inequality, and the fourth inequality follows from the relation that $(\sum_{i=1}^{M} x_i)^2 \leq M\sum_{i=1}^{M} x_i^2$. Upon substituting (15) into (14), we then arrive at:

$$\begin{aligned}\dot{V}_{\mathscr{A}1} &\leq -c_2\sum_{i=1}^{s}\sum_{k=1}^{M}(e_{ki})^{2h} \\ &\quad + 2^{-h}k_m^{-1}M\ell_{\max}\sum_{i=1}^{s}\sum_{k=1}^{M}[(\delta_{ki})^{2h} + (e_{ki})^{2h}].\end{aligned} \tag{16}$$

Step 2: Let $\bar{g}_{\mathscr{A}} = \max_{k\in\mathscr{A}}\{g_{ki}\}$. A new Lyapunov function is constructed as

$$V_{\mathscr{A}2} = V_{\mathscr{A}1} + \frac{1}{2^{1-h}\bar{g}_{\mathscr{A}}}\sum_{i=1}^{s}\sum_{k=1}^{M}g_{ki}\int_{v_{ki}^*}^{v_{ki}}\Big(\varsigma^{\frac{1}{h}} - \big(v_{ki}^*\big)^{\frac{1}{h}}\Big)d\varsigma \tag{17}$$

which is positive semidefinite and C^1 [17]. Note that the term containing the fractional power integrator is added to (17), which is motivated by the technique originated in [17] and is proven useful in finite time-related stability analysis such as [24] and [26]. Taking the derivative of $V_{\mathscr{A}2}(t)$ yields

$$\begin{aligned}
\dot{V}_{\mathscr{A}2} &= \dot{V}_{\mathscr{A}1} + \frac{1}{2^{1-h}\bar{g}_{\mathscr{A}}} \\
&\quad \times \sum_{i=1}^{s}\sum_{k=1}^{M}\Bigg[{}_{ki}\Big((v_{ki})^{\frac{1}{h}} - (v_{ki}^{*})^{\frac{1}{h}}\Big)\dot{v}_{ki} \\
&\qquad + g_{ki}(v_{ki} - v_{ki}^{*})\frac{d\Big(-(v_{ki}^{*})^{\frac{1}{h}}\Big)}{dt}\Bigg] \\
&= \dot{V}_{\mathscr{A}1} + \frac{1}{2^{1-h}\bar{g}_{\mathscr{A}}} \\
&\quad \times \sum_{i=1}^{s}\sum_{k=1}^{M}\Bigg[g_{ki}\delta_{ki}\dot{v}_{ki} \\
&\qquad + \; g_{ki}(v_{ki} - v_{ki}^{*})c_2^{1/h}\sum_{j\in\mathscr{N}_k} a_{kj}(v_{ki} - v_{ji})\Bigg] \\
&\leq \dot{V}_{\mathscr{A}1} + \frac{1}{2^{1-h}\bar{g}_{\mathscr{A}}}\sum_{i=1}^{s}\sum_{k=1}^{M} g_{ki}\delta_{ki}\dot{v}_{ki} \\
&\quad + \sum_{i=1}^{s}\sum_{k=1}^{M}|\delta_{ki}|^{h}c_2^{1/h}r_{\mathscr{A}}\sum_{j\in\mathscr{N}_k}(|v_{ki}| + |v_{ji}|) \qquad (18)
\end{aligned}$$

where $r_{\mathscr{A}} = \max_{\forall k,j\in\mathscr{A}}\{a_{kj}\}$.

To continue, let us examine the summation part of the second term on the right-hand side of (18). Define $\delta_k = [\delta_{k1}, \delta_{k2}, \ldots, \delta_{ks}]^T$. By applying the control law for the kth ($k \in \mathscr{A}$) leader given in (7), it holds that

$$\begin{aligned}
\sum_{i=1}^{s}\sum_{k=1}^{M} g_{ki}\delta_{ki}\dot{v}_{ki} &= \sum_{k=1}^{M}\delta_k^T g_k\dot{v}_k = \sum_{k=1}^{M}\delta_k^T(u_k + F_k) \\
&= \sum_{k=1}^{M}\delta_k^T\big(-c_1\delta_k^{2h-1} - \hat{\beta}_k^T\phi_k + \beta_k^T\phi_k\big) \\
&= -c_1\sum_{i=1}^{s}\sum_{k=1}^{M}\delta_{ki}^{2h} + \sum_{k=1}^{M}\delta_k^T(\beta_k - \hat{\beta}_k)^T\phi_k. \qquad (19)
\end{aligned}$$

Note that for all $k, j \in \mathscr{A}$, it can be deduced from Lemmas 3 and 4 that

$$\begin{aligned}
c_2^{1/h}r_{\mathscr{A}}|\delta_{ki}|^{h}|v_{ji}| &\leq c_2^{1/h}r_{\mathscr{A}}\big(|\delta_{ki}|^{h}|v_{ji} - v_{ji}^{*}| + |\delta_{ki}|^{h}|v_{ji}^{*}|\big) \\
&\leq c_2^{1/h}r_{\mathscr{A}}(2^{1-h}|\delta_{ki}|^{h}|\delta_{ji}|^{h} + c_2|\delta_{ki}|^{h}|e_{ji}|^{h}) \\
&\leq 2^{-h}c_2^{1/h}r_{\mathscr{A}}(|\delta_{ki}|^{2h} + |\delta_{ji}|^{2h}) \\
&\quad + c_2^{2(1+1/h)}r_{\mathscr{A}}^2|\delta_{ki}|^{2h} + \frac{1}{4}|e_{ji}|^{2h} \qquad (20)
\end{aligned}$$

with which the last term on the right-hand side of (18) can be written as

$$\begin{aligned}
&\sum_{i=1}^{s}\sum_{k=1}^{M}|\delta_{ki}|^{h}c_2^{1/h}r_{\mathscr{A}}\sum_{j\in\mathscr{N}_k}(|v_{ki}| + |v_{ji}|) \\
&\leq \sum_{i=1}^{s}\sum_{k=1}^{M}\sum_{j\in\mathscr{N}_k}\Big[2^{1-h}c_2^{1/h}r_{\mathscr{A}}|\delta_{ki}|^{2h} + c_2^{2(1+1/h)}r_{\mathscr{A}}^2|\delta_{ki}|^{2h} \\
&\qquad + \frac{1}{4}|e_{ki}|^{2h} + 2^{-h}c_2^{1/h}r_{\mathscr{A}}(|\delta_{ki}|^{2h} + |\delta_{ji}|^{2h}) \\
&\qquad + c_2^{2(1+1/h)}r_{\mathscr{A}}^2|\delta_{ki}|^{2h} + \frac{1}{4}|e_{ji}|^{2h}\Big] \\
&\leq \sum_{i=1}^{s}\sum_{k=1}^{M}\Big[\big(2^{2-h}c_2^{1/h}r_{\mathscr{A}} + 2c_2^{2(1+1/h)}r_{\mathscr{A}}^2\big)\bar{n}_{\mathscr{A}}\delta_{ki}^{2h} + \frac{\bar{n}_{\mathscr{A}}}{2}e_{ki}^{2h}\Big] \\
&= k_{10}\sum_{i=1}^{s}\sum_{k=1}^{M}\delta_{ki}^{2h} + k_{20}\sum_{i=1}^{s}\sum_{k=1}^{M}e_{ki}^{2h} \qquad (21)
\end{aligned}$$

where $\bar{n}_{\mathscr{A}}$ denotes the maximum number of the elements in $\mathscr{N}_k$ for all $k \in \mathscr{A}$, $k_{10} = (2^{2-h}c_2^{1/h}r_{\mathscr{A}} + 2c_2^{2(1+1/h)}r_{\mathscr{A}}^2)\bar{n}_{\mathscr{A}}$, and $k_{20} = (\bar{n}_{\mathscr{A}}/2)$.

Substituting (16), (19), and (21) into (18) yields that

$$\begin{aligned}
\dot{V}_{\mathscr{A}2} &\leq -k_1\sum_{i=1}^{s}\sum_{k=1}^{M}\delta_{ki}^{2h} - k_2\sum_{i=1}^{s}\sum_{k=1}^{M}e_{ki}^{2h} \\
&\quad + \sum_{k=1}^{M}\frac{1}{2^{1-h}\bar{g}_{\mathscr{A}}}\delta_k^T(\beta_k - \hat{\beta}_k)^T\phi_k \qquad (22)
\end{aligned}$$

with

$$\begin{aligned}
k_1 &= -2^{-h}k_m^{-1}M\ell_{\max} + \frac{c_1}{2^{1-h}\bar{g}_{\mathscr{A}}} - k_{10} \\
k_2 &= c_2 - 2^{-h}k_m^{-1}M\ell_{\max} - k_{20}. \qquad (23)
\end{aligned}$$

Step 3: The Lyapunov function candidate is chosen as

$$V_{\mathscr{A}3} = V_{\mathscr{A}2} + \sum_{k=1}^{M}\frac{\mathrm{tr}\{\tilde{\beta}_k^T\Gamma_k^{-1}\tilde{\beta}_k\}}{2^{2-h}\bar{g}_{\mathscr{A}}} \qquad (24)$$

where $\tilde{\beta}_k = \beta_k - \hat{\beta}_k$. By applying the updated law for $\hat{\beta}_k$ ($k \in \mathscr{A}$) given in (8), the derivative of $V_{\mathscr{A}3}$ is derived as

$$\begin{aligned}
\dot{V}_{\mathscr{A}3} &= \dot{V}_{\mathscr{A}2} + \sum_{k=1}^{M}\frac{\mathrm{tr}\{\tilde{\beta}_k^T\Gamma_k^{-1}(-\dot{\hat{\beta}}_k)\}}{2^{1-h}\bar{g}_{\mathscr{A}}} \\
&\leq -k_1\sum_{i=1}^{s}\sum_{k=1}^{M}\delta_{ki}^{2h} - k_2\sum_{i=1}^{s}\sum_{k=1}^{M}e_{ki}^{2h} \qquad (25)
\end{aligned}$$

which implies that $\dot{V}_{\mathscr{A}3} \leq 0$, and then $V_{\mathscr{A}3}(t) \leq V_{\mathscr{A}3}(0) < \infty$. Furthermore, it is readily derived, for all $k \in \mathscr{A}$, that

$$\begin{aligned}
\|\tilde{\beta}_k\|_\infty &\leq \sqrt{\mathrm{tr}\{\tilde{\beta}_k^T\tilde{\beta}_k\}} \leq 2^{2-h}\bar{g}_{\mathscr{A}}\bar{\gamma}V_{\mathscr{A}3}(t) \\
&\leq 2^{2-h}\bar{g}_{\mathscr{A}}\bar{\gamma}V_{\mathscr{A}3}(0) \qquad (26)
\end{aligned}$$

where we denote by $\bar{\gamma} = \max_{k\in\mathscr{A}}\{\gamma_{k1}, \ldots, \gamma_{km}\}$ ($\Gamma_k = \mathrm{diag}\{\gamma_{k1}, \ldots, \gamma_{km}\}$).

Step 4: We prove that there exists a finite time $T_{\mathscr{A}1}^{*} > 0$ such that the states enter a globally attractive region if the initial states are outside of it. Once the states enter this compact set, there exists another finite time $T^*_{\mathscr{A}2} > 0$ such that after this time $T^*_{\mathscr{A}2}$, we have $V_{\mathscr{A}2} = 0$. On the other hand, if initially the states are within this compact, it is straightforward that $V_{\mathscr{A}2}$ converges to zero within $T^*_{\mathscr{A}2}$.

Define a compact set

$$\Theta_{\mathscr{A}} = \{(x_{\mathscr{A}}, \upsilon_{\mathscr{A}}) : |\delta_{ki}| < \zeta_{\mathscr{A}1} \quad |e_{ki}| < \zeta_{\mathscr{A}2}\} \tag{27}$$

where $\zeta_{\mathscr{A}1}$ and $\zeta_{\mathscr{A}2}$ are chosen as

$$\zeta_{\mathscr{A}1} = \left(\frac{1}{2^{2-h}s}\right)^{1/h}, \quad \zeta_{\mathscr{A}2} = \left(\frac{1}{2c_2 s}\right)^{1/h}.$$

It is readily seen that with such $\zeta_{\mathscr{A}1}$ and $\zeta_{\mathscr{A}2}$ we have $|\upsilon_{ki}| \le (|\upsilon_{ki} - \upsilon^*_{ki}|) + |\upsilon^*_{ki}| \le 2^{1-h}|\delta_{ki}|^h + c_2|e_{ki}|^h \le 2^{1-h}\zeta^h_{\mathscr{A}1} + c_2\zeta^h_{\mathscr{A}2} \le 1/s)$ such that $\|\upsilon_k\|_1 \le \sum_{i=1}^s |\upsilon_{ki}| \le 1$ in the compact set $\Theta_{\mathscr{A}}$. Then from (25) we can deduce that if the initial states are outside a compact set $\Theta_{\mathscr{A}}$, then $\dot V_{\mathscr{A}3} \le 0$. Furthermore, there exists a positive constant $d_{\mathscr{A}\zeta}$ satisfying $d_{\mathscr{A}\zeta} \ge \min\{k_1 s M \zeta^{2h}_{\mathscr{A}1}, k_2 s M \zeta^{2h}_{\mathscr{A}2}\}$ such that

$$\dot V_{\mathscr{A}3} < -d_{\mathscr{A}\zeta}. \tag{28}$$

Let $\zeta_{\mathscr{A}}$ be the boundary value of $\Theta_{\mathscr{A}}$, namely, $\zeta_{\mathscr{A}} = \min_{(x_{\mathscr{A}},\upsilon_{\mathscr{A}})\in\Theta_{\mathscr{A}}}\{V_{\mathscr{A}3}(t)\}$. It thus can be concluded from (28) that there exists a finite time $T^*_{\mathscr{A}1}$ satisfying

$$T^*_{\mathscr{A}1} \le (V_{\mathscr{A}3}(0) - \zeta_{\mathscr{A}})/d_{\mathscr{A}\zeta} \tag{29}$$

such that the states enter $\Theta_{\mathscr{A}}$ before $t = T^*_{\mathscr{A}1}$. Once entering this set, δ_{ki} and e_{ki} will stay in it forever because $\dot V_{\mathscr{A}2} < 0$ (it will be proved later) in this set.

In the following, we will prove $\dot V_{\mathscr{A}2} < 0$ in $\Theta_{\mathscr{A}}$. With this in mind, we first examine the third term on the right hand of (22). Suppose the polynomial $P(\cdot)$ is of the form that $P(x) = \eta_1 x + \eta_2 x^2 + \cdots + \eta_{n_p} x^{n_p}$, where $n_p \ge 2$ is an integer and η_j $(j = 1, \ldots, n_p)$ are positive constants. Let $\bar\eta = \max\{\eta_1, \ldots, \eta_{n_p}\}$. It thus holds that $\|\phi_k(\cdot)\|_1 \le P(\|\upsilon_k\|_1) \le n_p\bar\eta\|\upsilon_k\|_1$, i.e., $\sum_{j=1}^m |\phi_{kj}| \le n_p\bar\eta\sum_{j=1}^s|\upsilon_{kj}|$, which, together with (26), yields that

$$\begin{aligned}\sum_{k=1}^M \frac{\delta_k^T\tilde\beta_k^T\phi_k}{2^{1-h}\bar g_{\mathscr{A}}} &\le \sum_{k=1}^M \frac{\|\tilde\beta_k\|_\infty}{2^{1-h}\bar g_{\mathscr{A}}}\sum_{i=1}^s|\delta_{ki}|\left(\sum_{j=1}^m|\phi_{kj}\right)\\ &\le 2\bar\gamma V_{\mathscr{A}3}(0)n_p\bar\eta\sum_{k=1}^M\sum_{i=1}^s|\delta_{ki}|\left(\sum_{j=1}^s|\upsilon_{kj}|\right).\end{aligned} \tag{30}$$

Following the similar line to that in the proof of (20), we get:

$$\begin{aligned}|\delta_{ki}||\upsilon_{kj}| &\le 2^{1-h}|\delta_{ki}||\delta_{kj}|^h + c_2|\delta_{ki}||e_{kj}|^h\\ &\le \frac{2^{1-h}}{1+h}(|\delta_{ki}|^{1+h} + h|\delta_{kj}|^{1+h})\\ &\quad+\frac{1}{1+h}\left(c_2^{1+h}|\delta_{ki}|^{1+h} + h|e_{kj}|^{1+h}\right)\end{aligned}$$

which by inserting into (30), we have

$$\begin{aligned}&\sum_{k=1}^M \frac{\delta_k^T\tilde\beta_k^T\phi_k}{2^{1-h}\bar g_{\mathscr{A}}}\\ &\le 2\bar\gamma V_{\mathscr{A}3}(0)n_p\bar\eta\sum_{k=1}^M\sum_{i=1}^s|\delta_{ki}|\left(\sum_{j=1}^s|\upsilon_{kj}|\right)\\ &\le 2\bar\gamma V_{\mathscr{A}3}(0)n_p\bar\eta\sum_{k=1}^M\sum_{i=1}^s\\ &\quad\times\left[\frac{2^{1-h}}{1+h}\left(\sum_{j=1}^s|\delta_{ki}|^{1+h} + h\sum_{j=1}^s|\delta_{kj}|^{1+h}\right)\right.\\ &\quad\left.+\frac{1}{1+h}\left(\sum_{j=1}^s c_2^{1+h}|\delta_{ki}|^{1+h} + h\sum_{j=1}^s|e_{kj}|^{1+h}\right)\right]\\ &\le k_3\sum_{k=1}^M\sum_{i=1}^s\delta_{ki}^{1+h} + k_4\sum_{k=1}^M\sum_{i=1}^s e_{ki}^{1+h}\end{aligned} \tag{31}$$

where

$$\begin{aligned}k_3 &= 2\bar\gamma V_{\mathscr{A}3}(0)n_p\bar\eta\left(2^{1-h}s + \frac{c_2^{1+h}s}{1+h}\right)\\ k_4 &= 2\bar\gamma V_{\mathscr{A}3}(0)n_p\bar\eta\frac{sh}{1+h}.\end{aligned} \tag{32}$$

With (31), (22) then becomes

$$\begin{aligned}\dot V_{\mathscr{A}2} \le &-\frac{k_1}{2}\sum_{i=1}^s\sum_{k=1}^M\delta_{ki}^{2h} - \frac{k_2}{2}\sum_{i=1}^s\sum_{k=1}^n e_{ki}^{2h}\\ &+\sum_{i=1}^s\sum_{k=1}^M\left(k_3\delta_{ki}^{1+h} - \frac{k_1}{2}\delta_{ki}^{2h}\right) + \sum_{i=1}^s\sum_{k=1}^M\left(k_4 e_{ki}^{1+h} - \frac{k_2}{2}e_{ki}^{2h}\right).\end{aligned} \tag{33}$$

Note that by choosing the design parameters c_1 and c_2 such that $k_1 > 2k_3$ and $k_2 > 2k_4$, and moreover, by $\zeta_{\mathscr{A}2} \le 1$, we then have

$$\dot V_{\mathscr{A}2} \le -\frac{k_1}{2}\sum_{i=1}^s\sum_{k=1}^M\delta_{ki}^{2h} - \frac{k_2}{2}\sum_{i=1}^s\sum_{k=1}^M e_{ki}^{2h}. \tag{34}$$

It should be mentioned that c_1 and c_2 can be chosen as $c_1 > 2^{1-h}\bar g_{\mathscr{A}}(2^{-h}k_m^{-1}M\ell_{\max} + 2^{2-h}c_2^{1/h}r_{\mathscr{A}}\bar n_{\mathscr{A}} + 2c_2^{2(1+1/h)}r^2_{\mathscr{A}}\bar n_{\mathscr{A}} + 4\bar\gamma V_{\mathscr{A}3}(0)n_p\bar\eta(2^{1-h}s + (c_2^{1+h}s/1+h))$ and $c_2 > \max\{1/2s, 2^{-h}k_m^{-1}M\ell_{\max} + (\bar n_{\mathscr{A}}/2) + 4\bar\gamma V_{\mathscr{A}3}(0)n_p\bar\eta(sh/1+h)$ such that $k_1 > 2k_3 > 0$ and $k_2 > 2k_4 > 0$.

Next, we will prove that the relation $\dot V_{\mathscr{A}2} + cV^{\alpha}_{\mathscr{A}2} \le 0$ holds for some constants $c > 0$ and $0 < \alpha < 1$. Note that

$$\begin{aligned}V_{\mathscr{A}2} &= V_{\mathscr{A}1} + \frac{1}{2^{1-h}\bar g_{\mathscr{A}}}\sum_{i=1}^s\sum_{k=1}^M g_{ki}\int_{\upsilon^*_{ki}}^{\upsilon_{ki}}\left(\varsigma^{\frac{1}{h}} - (\upsilon^*_{ki})^{\frac{1}{h}}\right)d\varsigma\\ &\le \frac{\lambda_{\max}(P_{\mathscr{A}})}{(1+h)k_m}\sum_{i=1}^s\sum_{k=1}^M e_{ki}^{1+h} + \frac{1}{2^{1-h}}\sum_{i=1}^s\sum_{k=1}^M|\upsilon_{ki} - \upsilon^*_{ki}||\delta_{ki}|\\ &\le \frac{\lambda_{\max}(P_{\mathscr{A}})}{(1+h)k_m}\sum_{i=1}^s\sum_{k=1}^M e_{ki}^{1+h} + \sum_{i=1}^s\sum_{k=1}^M\delta_{ki}^{1+h}\\ &\le k_{\mathscr{A}\upsilon}\sum_{i=1}^s\sum_{k=1}^M e_{ki}^{1+h} + k_{\mathscr{A}\upsilon}\sum_{i=1}^s\sum_{k=1}^M\delta_{ki}^{1+h}\end{aligned} \tag{35}$$

where $k_{\mathscr{A}v} = \max\{(\lambda_{\max}(P_{\mathscr{A}})/(1+h)k_m), 1\}$. Upon using Lemma 2, it thus follows from (35) that:

$$\begin{aligned} V_{\mathscr{A}2}^{\frac{2h}{1+h}} &\le \left(k_{\mathscr{A}v}\sum_{i=1}^{s}\sum_{k=1}^{M} e_{ki}^{1+h} + k_{\mathscr{A}v}\sum_{i=1}^{s}\sum_{k=1}^{M}\delta_{ki}^{1+h}\right)^{\frac{2h}{1+h}} \\ &\le k_{\mathscr{A}v}^{\frac{2h}{1+h}}\left(\sum_{i=1}^{s}\sum_{k=1}^{M} e_{ki}^{1+h}\right)^{\frac{2h}{1+h}} + k_{\mathscr{A}v}^{\frac{2h}{1+h}}\left(\sum_{i=1}^{s}\sum_{k=1}^{M}\delta_{ki}^{1+h}\right)^{\frac{2h}{1+h}} \\ &\le k_{\mathscr{A}v}^{\frac{2h}{1+h}}\left(\sum_{i=1}^{s}\sum_{k=1}^{M} e_{ki}^{2h} + \sum_{i=1}^{s}\sum_{k=1}^{M}\delta_{ki}^{2h}\right). \end{aligned} \tag{36}$$

Let $k_{\mathscr{A}d} = \min\{k_1/2, k_2/2\}$ and $\tilde{c}_{\mathscr{A}} = (\rho_1 k_{\mathscr{A}d}/(k_{\mathscr{A}v}^{(2h/1+h)}))$ with $0 < \rho_1 \le 1$. It thus follows from (34) and (36) that:

$$\dot{V}_{\mathscr{A}2}(t) + \tilde{c}_{\mathscr{A}} V_{\mathscr{A}2}^{\frac{2h}{1+h}}(t) \le 0. \tag{37}$$

Then after the states enter the globally attractive region $\Theta_{\mathscr{A}}$, we can conclude from (37) and Lemma 1 that there exists another finite time $T_{\mathscr{A}2}^* > 0$ such that $V_{\mathscr{A}2} = 0$ when $t \ge T_{\mathscr{A}1}^* + T_{\mathscr{A}2}^*$, in which

$$T_{\mathscr{A}2}^* \le \frac{V_{\mathscr{A}2}(0)^{\frac{1-h}{1+h}} k_{\mathscr{A}v}^{\frac{2h}{1+h}}(1+h)}{(1-\rho_2)\rho_1 k_{\mathscr{A}d}(1-h)} \tag{38}$$

with $0 < \rho_2 < 1$. That is, there exists a finite time $T_{\mathscr{A}}^*$ satisfying (9) such that $V_{\mathscr{A}2} = 0$ when $t \ge T_{\mathscr{A}}^*$. From the definition of $V_{\mathscr{A}2}(t)$ given in (17), one concludes that $E_{\mathscr{A}} = 0$, and then $x_{\mathscr{A}} - \varpi = 0$ from the definition of $E_{\mathscr{A}}$ in (5) under Assumption 1, which implies that the leaders form the desired formation structure specified by the formation vector ϖ described in Definition 1, in finite time $T_{\mathscr{A}}^*$. ■

B. Finite-Time Containment Control for Followers

In this section, we develop the adaptive containment control schemes for the $(N-M)$ followers so that the states of the followers are steered to enter the convex hull formed by those of the leaders in finite time.

To proceed, we first define the local neighborhood state error of the kth ($k \in \mathscr{B}$) follower as

$$e_{ki} = \sum_{j\in\mathscr{N}_k} a_{kj}(x_{ki} - x_{ji}), \quad k \in \mathscr{B}. \tag{39}$$

Let $x_{\mathscr{B}i} = [x_{M+1,i}, x_{M+2,i}, \ldots, x_{Ni}]^T \in R^{N-M}$, $e_{\mathscr{B}i} = [e_{M+1,i},\ e_{M+2,i}, \ldots, e_{Ni}]^T \in R^{N-M}$ for $i = 1, 2, \ldots, s$, $x_{\mathscr{B}} = [x_{\mathscr{B}1}^T, x_{\mathscr{B}2}^T, \ldots,\ x_{\mathscr{B}s}^T]^T \in R^{(N-M)s}$, and $E_{\mathscr{B}} = [e_{\mathscr{B}1}^T, e_{\mathscr{B}2}^T, \ldots, e_{\mathscr{B}s}^T]^T \in R^{(N-M)s}$. Then it holds that

$$\begin{aligned} E_{\mathscr{B}} &= (I_s \otimes L_2)x_{\mathscr{A}} + (I_s \otimes L_3)x_{\mathscr{B}} \\ &= (I_s \otimes L_3)\big[x_{\mathscr{B}} - \big(-I_s \otimes \big(L_3^{-1}L_2\big)\big)x_{\mathscr{A}}\big]. \end{aligned} \tag{40}$$

From (40), it is deduced that the followers enter the convex hull formed by the leaders if $E_{\mathscr{B}} \to 0$. Indeed, according to [31] each entry of $-L_3^{-1}L_2$ is nonnegative and each row of $-L_3^{-1}L_2$ has a sum equal to 1.

Define the local virtual error for the kth ($k \in \mathscr{B}$) follower as $\delta_{ki} = v_{ki}^{1/h} - v_{ki}^{*1/h}$ ($i = 1, \ldots, s$), with $v_{ki}^* = -c_4 e_{ki}^h$ ($c_4 > 0$ is a design parameter).

From the definition of the neighborhood state error e_{ki} and local virtual error δ_{ki} for the kth ($k \in \mathscr{B}$) follower, we know that the finite-time containment control objective for the MAS under consideration is achieved if $e_{ki} \to 0$ and $\delta_{ki} \to 0$ in finite time. Thus, the finite-time containment control law for the the kth ($k \in \mathscr{B}$) follower is designed as

$$u_k = -c_3\delta_k^{2h-1} - \hat{\beta}_k^T\phi_k, \quad k \in \mathscr{B} \tag{41}$$

with the updated law

$$\dot{\hat{\beta}}_k(j,i) = \mathrm{Proj}\big(\big(\Sigma_k\phi_k\delta_k^T\big)(j,i), \hat{\beta}_k(j,i)\big), \quad k \in \mathscr{B} \tag{42}$$

for $j = 1, \ldots, m$, $i = 1, \ldots, s$, where $c_3 > 0$ is a design parameter, $\Sigma_k = \mathrm{diag}\{\sigma_{k1}, \ldots, \sigma_{km}\} \in R^{m\times m} > 0$ is a design parameter matrix, and β_k and $\phi_k(\cdot)$ are given the same as before. Note that in the proposed containment control scheme the negative fraction power feedback term, $-c_3\delta_k^{2h-1}$, is to ensure the finite-time containment error convergence and the adaptive term, $\hat{\beta}_k^T\phi_k$, is to compensate the nonlinearities in the systems.

Theorem 2: Consider the MAS (1) under Assumption 1. If the $N-M$ followers are controlled by the distributed control law (41) and (42), then the states of the followers are ensured to converge to the convex hull shaped by those of the leaders in finite time T^* as specified by

$$T^* = T_{\mathscr{A}}^* + T_{\mathscr{B}}^* \tag{43}$$

in which $T_{\mathscr{A}}^*$ and $T_{\mathscr{B}}^*$ are given in (9) and (72), respectively.

Proof: Note that before the leaders achieve the desired formation, which will take time $T_{\mathscr{A}}^*$ to achieve, the leaders are dynamically moving and the control inputs are unknown to the followers. In order to force the followers to enter into convex hull formed by the formation of the leaders, we first prove that the states of the followers remain bounded around those of the leaders for $t \in [0, T_{\mathscr{A}}^*)$, and then prove that the containment is achieved in finite time T^* ($T^* > T_{\mathscr{A}}^*$). Thus we need to explore two cases: $t \in [0, T_{\mathscr{A}}^*)$ and $t \ge T_{\mathscr{A}}^*$.

Case 1 ($t \in [0, T_{\mathscr{A}}^*)$): According to [32, Th. 2.3, p. 134], there exists a matrix $\xi = \mathrm{diag}\{\xi_1, \ldots, \xi_{N-M}\}$ ($\xi_k > 0$) such that $(1/2)(\xi L_3 + L_3^T\xi)$ is positive definite. Denote by λ_m the minimum eigenvalue of $(1/2)(\xi L_3 + L_3^T\xi)$. We then construct the following Lyapunov function for the containment control under the directed topology as:

$$V_{\mathscr{B}1} = \frac{1}{(1+h)\lambda_m}\Big(E_{\mathscr{B}}^{\frac{1+h}{2}}\Big)^T(I_s\otimes\xi)E_{\mathscr{B}}^{\frac{1+h}{2}} \tag{44}$$

the time derivative of which is computed as

$$\begin{aligned} &\dot{V}_{\mathscr{B}1} \\ &= \frac{1}{\lambda_m}\big(E_{\mathscr{B}}^h\big)^T(I_s\otimes\xi)(I_s\otimes L_3)\big[v_{\mathscr{B}} - \big(-I_s\otimes\big(L_3^{-1}L_2\big)\big)v_{\mathscr{A}}\big] \\ &= \frac{1}{\lambda_m}\big(E_{\mathscr{B}}^h\big)^T(I_s\otimes(\xi L_3))v_{\mathscr{B}} + \frac{1}{\lambda_m}\big(E_{\mathscr{B}}^h\big)^T(I_s\otimes(\xi L_2))v_{\mathscr{A}} \\ &\le \frac{1}{\lambda_m}\big(E_{\mathscr{B}}^h\big)^T(I_s\otimes(\xi L_3))v_{\mathscr{B}} + \frac{1}{2\lambda_m}\big(E_{\mathscr{B}}^h\big)^T E_{\mathscr{B}}^h \\ &\quad + \frac{1}{2\lambda_m}[(I_s\otimes(\xi L_2))v_{\mathscr{A}}]^T[(I_s\otimes(\xi L_2))v_{\mathscr{A}}] \end{aligned} \tag{45}$$

in which $v_{\mathscr{B}} = \dot{x}_{\mathscr{B}}$.

Now we examine the first term on the right hand of (45). Recalling that the virtual control $\upsilon^*_{\mathscr{B}} = -c_4 E^h_{\mathscr{B}}$ and $\upsilon_{\mathscr{B}} = \dot{x}_{\mathscr{B}}$, then we have

$$\begin{aligned}
&\frac{1}{\lambda_m}\left(E^h_{\mathscr{B}}\right)^T (I_s \otimes (\xi L_3))\upsilon_{\mathscr{B}} \\
&= -\frac{c_4}{\lambda_m}\left(E^h_{\mathscr{B}}\right)^T (I_s \otimes (\xi L_3)) E^h_{\mathscr{B}} \\
&\quad + \frac{1}{\lambda_m}\left(E^h_{\mathscr{B}}\right)^T (I_s \otimes (\xi L_3))\left(\upsilon_{\mathscr{B}} - \upsilon^*_{\mathscr{B}}\right) \\
&\le -c_4\left(E^h_{\mathscr{B}}\right)^T E^h_{\mathscr{B}} + \frac{1}{\lambda_m}\left(E^h_{\mathscr{B}}\right)^T (I_s \otimes (\xi L_3))\left(\upsilon_{\mathscr{B}} - \upsilon^*_{\mathscr{B}}\right) \\
&= -c_4 \sum_{i=1}^{s} \sum_{k=M+1}^{N} e^{2h}_{ki} \\
&\quad + \frac{1}{\lambda_m}\sum_{i=1}^{s}\left[\sum_{k=M+1}^{N}\left(\upsilon_{ki} - \upsilon^*_{ki}\right)\sum_{j=M+1}^{N} l_{jk} e^h_{ji}\right] \qquad (46)
\end{aligned}$$

where l_{jk} is the (j,k)th element of $(\xi L_3)^T$, and the fact that $(E^h_{\mathscr{B}})^T (I_s \otimes (\xi L_3)) E^h_{\mathscr{B}} \ge \lambda_m (E^h_{\mathscr{B}})^T E^h_{\mathscr{B}}$ has been used.

By following the same line as in the proof of (15) and (16) in Theorem 1, one gets:

$$\begin{aligned}
\frac{1}{\lambda_m}\left(E^h_{\mathscr{B}}\right)^T (I_s \otimes (\xi L_3))\upsilon_{\mathscr{B}} &\le -c_4 \sum_{i=1}^{s}\sum_{k=M+1}^{N} e^{2h}_{ki} \\
&\quad + 2^{-h}\lambda_m^{-1}(N-M) l_{\max}\sum_{i=1}^{s}\sum_{k=M+1}^{N}\left(\delta^{2h}_{ki} + e^{2h}_{ki}\right) \quad (47)
\end{aligned}$$

where $l_{\max} = \max_{j,k\in\mathscr{B}} |l_{jk}|$.

Let $\bar{g}_{\mathscr{B}} = \max_{k\in\mathscr{B}}\{g_{ki}\}$, we then define two Lyapunov functions as

$$V_{\mathscr{B}2} = \frac{1}{2^{1-h}\bar{g}_{\mathscr{B}}}\sum_{i=1}^{s}\sum_{k=M+1}^{N} g_{ki}\int_{\upsilon^*_{ki}}^{\upsilon_{ki}}\left(\varsigma^{\frac{1}{h}} - (\upsilon^*_{ki})^{\frac{1}{h}}\right)d\varsigma \quad (48)$$

$$V_{\mathscr{B}3} = \sum_{k=M+1}^{N}\frac{\mathrm{tr}\left\{\tilde{\beta}_k^T \Sigma_k^{-1}\tilde{\beta}_k\right\}}{2^{2-h}\bar{g}_{\mathscr{B}}} \quad (49)$$

respectively. By applying the control law u_k ($k \in \mathscr{B}$) given in (41) and the adaptive law for $\hat{\beta}_k$ given in (42) and following the same line as in the proof of (18) and (25), we arrive at:

$$\begin{aligned}
&\frac{1}{\lambda_m}\left(E^h_{\mathscr{B}}\right)^T (I_s \otimes (\xi L_3))\upsilon_{\mathscr{B}} + \dot{V}_{\mathscr{B}2} + \dot{V}_{\mathscr{B}3} \\
&\quad \le -k_5\sum_{i=1}^{s}\sum_{k=M+1}^{N}\delta^{2h}_{ki} - k_6\sum_{i=1}^{s}\sum_{k=M+1}^{N} e^{2h}_{ki} \quad (50)
\end{aligned}$$

where

$$\begin{aligned}
k_5 &= -2^{-h}\lambda_m^{-1}(N-M)l_{\max} + \frac{c_3}{2^{1-h}\bar{g}_{\mathscr{B}}} \\
&\quad - \left(2^{2-h}c_4^{1/h} r_{\mathscr{B}} + 2c_4^{2(1+1/h)} r^2_{\mathscr{B}}\right)\bar{n}_{\mathscr{B}} \\
k_6 &= c_4 - 2^{-h}\lambda_m^{-1}(N-M)l_{\max} - \frac{\bar{n}_{\mathscr{B}}}{2} \quad (51)
\end{aligned}$$

with $r_{\mathscr{B}} = \max_{\forall k,j\in\mathscr{B}}\{a_{kj}\}$, and $\bar{n}_{\mathscr{B}}$ denotes the maximum number of the elements in $\mathscr{N}_k$ for all $k \in \mathscr{B}$, where c_3 and c_4 can be chosen such that $k_5 > 0$ and $k_6 > 0$.

To continue, we estimate the second term on the right-hand side of (45). Note that

$$\begin{aligned}
V_{\mathscr{B}1} &\ge \frac{\lambda_{\min}(\xi)}{(1+h)\lambda_m}\left(E_{\mathscr{B}}^{\frac{1+h}{2}}\right)^T E_{\mathscr{B}}^{\frac{1+h}{2}} \\
&= \frac{\lambda_{\min}(\xi)}{(1+h)\lambda_m}\sum_{i=1}^{s}\sum_{k=M+1}^{N} e^{1+h}_{ki} \quad (52)
\end{aligned}$$

which then implies that

$$\begin{aligned}
\frac{1}{2\lambda_m}\left(E^h_{\mathscr{B}}\right)^T E^h_{\mathscr{B}} &= \frac{1}{2\lambda_m}\sum_{i=1}^{s}\sum_{k=M+1}^{N}\left(e^{1+h}_{ki}\right)^{\frac{2h}{1+h}} \\
&\le \frac{[s(N-M)]^{1-\frac{2h}{1+h}}}{2\lambda_m} \\
&\quad \times\left(\sum_{i=1}^{s}\sum_{k=M+1}^{N} e^{1+h}_{ki}\right)^{\frac{2h}{1+h}} \le k_{\mathscr{B}} V_{\mathscr{B}1}^{\frac{2h}{1+h}} \quad (53)
\end{aligned}$$

with $k_{\mathscr{B}} = (1/2)(s(N-M)/\lambda_m)^{((1-h)/(1+h))}((1+h)/\lambda_{\min}(\xi))^{(2h/(1+h))}$.

At last, we examine the third term on the right hand of (45) to get

$$\frac{1}{2\lambda_m}[(I_s \otimes (\xi L_2))\upsilon_{\mathscr{A}}]^T[(I_s \otimes (\xi L_2))\upsilon_{\mathscr{A}}] < \zeta \quad (54)$$

for some bounded positive constant ζ. To this end we denote by $\chi_{j,k}$ the (j,k)th element of (ξL_2), then we have

$$\begin{aligned}
&[(I_s \otimes (\xi L_2))\upsilon_{\mathscr{A}}]^T[(I_s \otimes (\xi L_2))\upsilon_{\mathscr{A}}] \\
&= \sum_{i=1}^{s}\left[\sum_{j=M+1}^{N}\left(\sum_{k=1}^{M}\chi_{jk}\upsilon_{ki}\right)^2\right] \\
&\le \chi_{\max}\sum_{i=1}^{s}\left[\sum_{j=M+1}^{N}\left(\sum_{k=1}^{M}|\upsilon_{ki}\right)^2\right] \\
&= (N-M)\chi_{\max}\sum_{i=1}^{s}\left(\sum_{k=1}^{M}|\upsilon_{ki}|\right)^2 \quad (55)
\end{aligned}$$

where $\chi_{\max} = \max_{j,k\in\mathscr{A}\cup\mathscr{B}}\{|\chi_{jk}|\}$.

To show the boundedness of $\sum_{i=1}^{s}(\sum_{k=1}^{M}|\upsilon_{ki}|)^2$ in (55), we recall (37) to get $\dot{V}_{\mathscr{A}2}(t) \le -\tilde{c}_{\mathscr{A}} V_{\mathscr{A}2}(t)^{(2h/1+h)}$, which implies that $V_{\mathscr{A}2}(t) \le V_{\mathscr{A}2}(0)$. Let $c_{\mathscr{A}} = \tilde{c}_{\mathscr{A}} V_{\mathscr{A}2}(0)^{(2h/1+h)-1}$, and then $\dot{V}_{\mathscr{A}2}(t) \le -c_{\mathscr{A}} V_{\mathscr{A}2}(t)$, which further yields

$$V_{\mathscr{A}2}(t) \le V_{\mathscr{A}2}(t_0)e^{-c_{\mathscr{A}}t}. \quad (56)$$

On one hand, by Lemma 3, it is readily seen that

$$|\varsigma^{\frac{1}{h}} - (\upsilon^*_{ki})^{\frac{1}{h}}| \ge \left(2^{h-1}|\varsigma - \upsilon^*_{ki}|\right)^{\frac{1}{h}}. \quad (57)$$

Then from (17) and (57), if $\upsilon_{ki} \ge \upsilon^*_{ki}$, it holds that

$$\begin{aligned}
&\frac{1}{2^{1-h}\bar{g}_{\mathscr{A}}}\sum_{i=1}^{s}\sum_{k=1}^{M} g_{ki}\int_{\upsilon^*_{ki}}^{\upsilon_{ki}}\left(\varsigma^{\frac{1}{h}} - (\upsilon^*_{ki})^{\frac{1}{h}}\right)d\varsigma \\
&\quad \ge \frac{h\underline{g}_{\mathscr{A}}}{2^{1/h-h}\bar{g}_{\mathscr{A}}(1+h)}\sum_{i=1}^{s}\sum_{k=1}^{M}\left(\upsilon_{ki} - \upsilon^*_{ki}\right)^{1+1/h} \quad (58)
\end{aligned}$$

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where $\underline{g}_{\mathscr{A}} = \min_{k,i\in\mathscr{A}}\{g_{ki}\}$, and we can also derive (58) by the similar line if $v_{ki} < v_{ki}^*$. Let $k_0 = (h\underline{g}_{\mathscr{A}}/2^{1/h-h}\bar{g}_{\mathscr{A}}(1+h))^{-1}$, then by combining (17), (56), and (58), we have

$$\sum_{i=1}^{s}\sum_{k=1}^{M}\left(v_{ki}-v_{ki}^*\right)^{1+1/h} \le k_0 V_{\mathscr{A}2}(t_0)e^{-c_{\mathscr{A}}t}. \tag{59}$$

By using Lemma 2 twice, it then follows from (59) that:

$$\begin{aligned}\sum_{i=1}^{s}\sum_{k=1}^{M}|v_{ki}-v_{ki}^*| &= \sum_{i=1}^{s}\sum_{k=1}^{M}\left(|v_{ki}-v_{ki}^*|^{\frac{1+h}{h}}\right)^{\frac{h}{1+h}}\\ &\le (sM)^{1-\frac{h}{1+h}}\left(\sum_{i=1}^{s}\sum_{k=1}^{M}|v_{ki}-v_{ki}^*|^{\frac{1+h}{h}}\right)^{\frac{h}{1+h}}\\ &\le \mu_1 e^{-k_{\mu 1}t}\end{aligned} \tag{60}$$

where $\mu_1 = (sM)^{1-(h/1+h)}(k_0V_{\mathscr{A}2}(0))^{(h/1+h)}$ and $k_{\mu 1} = (h/1+h)c_{\mathscr{A}}$. On the other hand, in view of Lemma 2, it is readily seen that

$$\begin{aligned}\sum_{i=1}^{s}\sum_{k=1}^{M}\left|v_{ki}^*\right| &= \sum_{i=1}^{s}\sum_{k=1}^{M}\left|c_2e_{ki}^h\right| = c_2\sum_{i=1}^{s}\sum_{k=1}^{M}\left|e_{ki}^{1+h}\right|^{\frac{h}{1+h}}\\ &\le c_2(sM)^{\frac{1}{1+h}}\left(\sum_{i=1}^{s}\sum_{k=1}^{M}\left|e_{ki}^{1+h}\right|\right)^{\frac{h}{1+h}}.\end{aligned} \tag{61}$$

From the definition of $V_{\mathscr{A}1}$ in (10), it follows that $\sum_{i=1}^{s}\sum_{k=1}^{M}e_{ki}^{1+h} = (E_{\mathscr{A}}^{(1+h/2)})^T E_{\mathscr{A}}^{(1+h/2)} \le (1+h)k_m p_m^{-1}V_{\mathscr{A}1} \le (1+h)k_m p_m^{-1}V_{\mathscr{A}2} \le (1+h)k_m p_m^{-1}V_{\mathscr{A}2}(t_0)e^{-c_{\mathscr{A}}t}$ ($p_m = \min\{p_1,\ldots,p_M\}$), which together with (61) yields that:

$$\sum_{i=1}^{s}\sum_{k=1}^{M}\left|v_{ki}^*\right| \le \mu_2 e^{-k_{\mu 2}t} \tag{62}$$

where $\mu_2 = c_2(sM)^{(1/1+h)}[(1+h)k_m p_m^{-1}V_{\mathscr{A}2}(0)]^{(h/1+h)}$ and $k_{\mu 2} = (h/1+h)c_{\mathscr{A}}$.

Recalling (60) and (62), it then holds that

$$\begin{aligned}\sum_{i=1}^{s}\left(\sum_{k=1}^{M}|v_{ki}|\right)^2 &\le \left(\sum_{i=1}^{s}\sum_{k=1}^{M}|v_{ki}|\right)^2\\ &\le \left(\sum_{i=1}^{s}\sum_{k=1}^{M}|v_{ki}-v_{ki}^*| + \sum_{i=1}^{s}\sum_{k=1}^{M}|v_{ki}^*|\right)^2\\ &\le (\mu_1e^{-k_{\mu1}t}+\mu_2e^{-k_{\mu2}t})^2 \le (\mu_1+\mu_2)^2.\end{aligned} \tag{63}$$

By inserting (63) into (55), it is readily seen that (54) holds with $\zeta = (N-M/2\lambda_m)\chi_{\max}(\mu_1+\mu_2)^2$.

Let $V_{\mathscr{B}} = V_{\mathscr{B}1}+V_{\mathscr{B}2}+V_{\mathscr{B}3}$. Recalling (45), (50), (53), and (54), if $V_{\mathscr{B}1} \ge 1$, it is readily shown that

$$\dot{V}_{\mathscr{B}} \le k_{\mathscr{B}}V_{\mathscr{B}1}^{\frac{2h}{1+h}} + \zeta \le k_{\mathscr{B}}V_{\mathscr{B}} + \zeta \tag{64}$$

which implies, for $t \in [0, T_{\mathscr{A}}^*)$, that

$$\begin{aligned}V_{\mathscr{B}}(t) &\le \left(V_{\mathscr{B}}(0)+k_{\mathscr{B}}^{-1}\zeta\right)e^{k_{\mathscr{B}}t} - k_{\mathscr{B}}^{-1}\zeta\\ &\le \left(V_{\mathscr{B}}(0)+k_{\mathscr{B}}^{-1}\zeta\right)e^{k_{\mathscr{B}}T_{\mathscr{A}}^*} - k_{\mathscr{B}}^{-1}\zeta\end{aligned} \tag{65}$$

and otherwise, $V_{\mathscr{B}1} < 1$, both of which mean that $V_{\mathscr{B}1}(t)$ is bounded during $t \in [0, T_{\mathscr{A}}^*)$. Hence

$$|e_{ki}| \le \left[\frac{(1+h)\lambda_m}{\lambda_{\min}(\xi)}V_{\mathscr{B}1}\right]^{\frac{1}{1+h}} \tag{66}$$

for $k \in \mathscr{B}$, then it is deduced from the the definition of the neighborhood error in (39) and (66) that, the states of the kth follower are bounded around those of the leaders during $[0, T_{\mathscr{A}}^*)$ for all $k \in \mathscr{B}$.

Case 2. ($t \ge T_{\mathscr{A}}^*$):

Note that the leaders have formed the desired formation structure after $T_{\mathscr{A}}^*$, i.e., for all $k \in \mathscr{A}$, $E_{\mathscr{A}} = 0$, and moreover, $v_{\mathscr{A}} = 0$ when $t \ge T_{\mathscr{A}}^*$. The derivative of $V_{\mathscr{B}1}(t)$ defined in (44) then becomes

$$\begin{aligned}\dot{V}_{\mathscr{B}1} &= \frac{1}{\lambda_m}\left(E_{\mathscr{B}}^h\right)^T(I_s\otimes\xi)(I_s\otimes L_3)\\ &\quad\times\left[v_{\mathscr{B}}-(-I_s\otimes(L_3^{-1}L_2))v_{\mathscr{A}}\right]\\ &= \frac{1}{\lambda_m}\left(E_{\mathscr{B}}^h\right)^T(I_s\otimes(\xi L_3))v_{\mathscr{B}}.\end{aligned} \tag{67}$$

Recalling (50), it thus follows from (67) that:

$$\dot{V}_{\mathscr{B}} \le -k_5\sum_{i=1}^{s}\sum_{k=M+1}^{N}\delta_{ki}^{2h} - k_6\sum_{i=1}^{s}\sum_{k=M+1}^{N}e_{ki}^{2h} \tag{68}$$

where k_5 and k_6 are given in (51). It thus follows that $\dot{V}_{\mathscr{B}} \le 0$, and then $V_{\mathscr{B}}(t) \le V_{\mathscr{B}}(0) < \infty$. Furthermore, it is readily derived, for $k \in \mathscr{B}$, that

$$\|\tilde{\beta}_k\|_\infty \le 2^{2-h}\bar{g}_{\mathscr{B}}\bar{\sigma}V_{\mathscr{B}}(t) \le 2^{2-h}\bar{g}_{\mathscr{B}}\bar{\sigma}V_{\mathscr{B}}(0) \tag{69}$$

in which $\bar{\sigma} = \max_{k\in\mathscr{B}}\{\sigma_{k1},\ldots,\sigma_{km}\}$ ($\Sigma_k = \mathrm{diag}\{\sigma_{k1},\ldots,\sigma_{km}\}$).

Define a set

$$\Theta_{\mathscr{B}} = \{(x_{\mathscr{B}}, v_{\mathscr{B}}) : |\delta_{ki}| < \zeta_{\mathscr{B}1}|e_{ki}| < \zeta_{\mathscr{B}2}\} \tag{70}$$

where $\zeta_{\mathscr{B}1}$ and $\zeta_{\mathscr{B}2}$ are chosen as $\zeta_{\mathscr{B}1} = (1/2^{2-h}s)^{1/h}$ and $\zeta_{\mathscr{B}2} = ((1/2c_4s))^{1/h}$, respectively. By applying the same step as in the proof of (28) and (29), we then conclude that there exists a finite time $T_{\mathscr{B}1}^*$ satisfying

$$T_{\mathscr{B}1}^* \le (V_{\mathscr{B}}(0)-\zeta_{\mathscr{B}})/d_{\mathscr{B}\zeta} \tag{71}$$

where $\zeta_{\mathscr{B}} = \min_{(x_{\mathscr{B}},v_{\mathscr{B}})\in\Theta_{\mathscr{B}}}\{V_{\mathscr{B}}(t)\}$ and $d_{\mathscr{B}\zeta}$ satisfies $d_{\mathscr{B}\zeta} \ge \min\{k_3s(N-M)\zeta_{\mathscr{B}2}^{2h}, k_4s(N-M)\zeta_{\mathscr{B}1}^{2h}\}$, such that the states enter into $\Theta_{\mathscr{B}}$ before $t = T_{\mathscr{B}1}^*$. Once entering into this set, δ_{ki} and e_{ki} ($k \in \mathscr{B}$, $i = 1,\ldots,s$) will stay in it forever.

By following the same step as in the proof of (30) and (38), we can also derive that there exists a finite time $T_{\mathscr{B}}^* > 0$ such that $\bar{V}_{\mathscr{B}2} = V_{\mathscr{B}1}+V_{\mathscr{B}2} = 0$ when $t \ge T_{\mathscr{B}}^*$, where

$$T_{\mathscr{B}}^* = T_{\mathscr{B}1}^* + T_{\mathscr{B}2}^* \tag{72}$$

with

$$T_{\mathscr{B}2}^* \le \frac{\bar{V}_{\mathscr{B}2}(0)^{\frac{1-h}{1+h}}k_{\mathscr{B}v}^{\frac{2h}{1+h}}(1+h)}{(1-\rho_2)\rho_1k_{\mathscr{B}d}(1-h)} \tag{73}$$

where $0 < \rho_1 \le 1$ and $0 < \rho_2 < 1$, $k_{\mathscr{B}v} = \max\{(\lambda_{\max}(\xi)/(1+h)\lambda_m), 1\}$ and $k_{\mathscr{B}d} = \min\{k_5/2, k_6/2\}$. From the definition of $V_{\mathscr{B}1}(t)$ given in (44), one concludes

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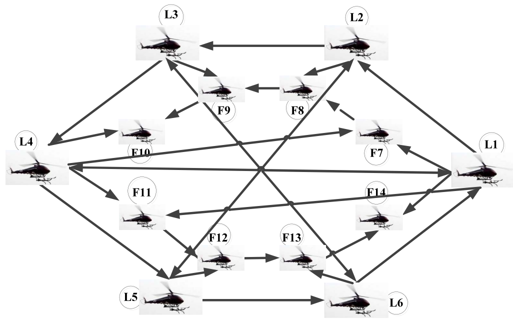


Fig. 1. Directed communication among leader UAVs and follower UAVs.

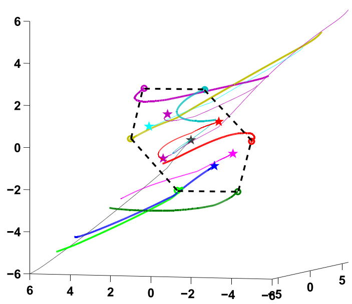


Fig. 2. Trajectory of each UAV from the initial position to the final position.

that $E_{\mathscr{B}} = 0$, implying that the containment for the followers is achieved in the finite time T^*. ■

Remark 2: It is noted that the proposed formation–containment control schemes involve the design parameters c_1 (c_3) and c_2 (c_4) that need to be specified to ensure $k_1 > 2k_3 > 0$ and $k_2 > 2k_4 > 0$. We have derived the condition for choosing the control parameters c_1 and c_2 in Theorem 1 below (34), but this condition is sufficiently not necessary. In fact, from the proof of (21), it can be seen that using Young's inequality flexibly, the expression of k_{10} and k_{20} can be derived differently. In other words, the expression of k_{10} and k_{20} is not unique. Note that the coefficients k_1 and k_2 in (23) depend on k_{10} and k_{20}, thus the condition derived for choosing the control parameters c_1 and c_2, which depend on the expression of k_{10} and k_{20}, can be different accordingly. We can draw the same conclusion for the containment control that the condition for choosing the control parameters c_3 and c_4 is sufficient but not necessary. As a result, there exist a wide range of choice for the design parameters involved in the control algorithms, making it straightforward and flexible to design the controller in practical applications.

IV. Numerical Simulations

To verify the effectiveness of the proposed finite-time control algorithms, numerical simulation on a group of 14 networked UAVs with nonlinear dynamics are conducted.

The dynamics of the kth ($k = 1, \ldots, 14$) UAV are modeled by

$$\begin{bmatrix} m_{x,k} & 0 & 0 \\ 0 & m_{y,k} & 0 \\ 0 & 0 & m_{z,k} \end{bmatrix} \cdot \begin{bmatrix} \dot{v}_{x,k} \\ \dot{v}_{y,k} \\ \dot{v}_{z,k} \end{bmatrix} = \begin{bmatrix} f_{x,k} \\ f_{y,k} \\ f_{z,k} \end{bmatrix} + \begin{bmatrix} u_{x,k} \\ u_{y,k} \\ u_{z,k} \end{bmatrix} \quad (74)$$

in which $f_{x,k} = (A_k + B_k|v_{x,k}|)v_{xk} - m_{y,k}v_{z,k}v_{yk}$, $f_{y,k} = [m_{x,k}v_{z,k}v_{x,k} + (C_k + D_k|v_{y,k}|)v_{yk}$, and $f_{z,k} = (E_k + F_k|v_{z,k}|)v_{z,k}$, where $M_k = \mathrm{diag}\{m_{x,k}, m_{y,k}, m_{z,k}\}$ denotes the mass matrix; $r_k = [x_k, y_k, z_k]^T$, $\dot{r}_k = v_k = [v_{x,k}, v_{y,k}, v_{z,k}]^T$, and $u_k = [u_{x,k}, u_{y,k}, u_{z,k}]^T$ denote the position, velocity, and control input vector, respectively; $f_k = [f_{x,k}, f_{y,k}, f_{z,k}]^T$ represents coriolis and centripetal forces acting on the body. In the simulation, the physical parameters are taken as: $M_k = \mathrm{diag}\{600 + 6(-1)^k, 800 + 8(-1)^k, 700 + 7(-1)^k\}$, $A_k = -1 + 0.1(-1)^k$, $B_k = -25 + 2.5(-1)^k$, $C_k = -10 + (-1)^k$, $D_k = -200 + 20(-1)^k$, $E_k = -0.5 + 0.05(-1)^k$, and $F_k = -1500 + 150(-1)^k$ for $k = 1, \ldots, 14$. The directed communication topology among the UAVs is shown in Fig. 1. Each edge weight is taken as 0.1.

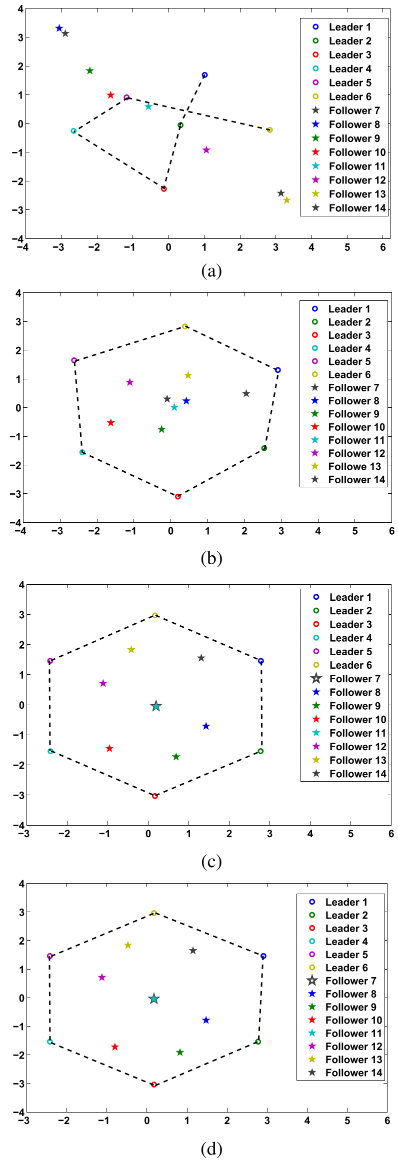
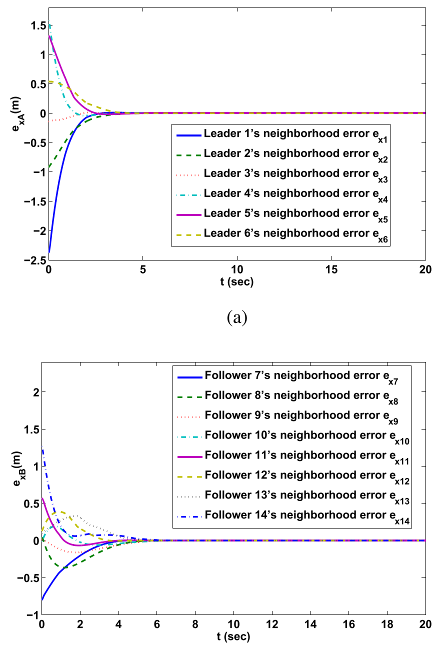


Fig. 3. Trajectory snapshots of the 14 UAVs at different times. (a) $t = 1$ s. (b) $t = 3$ s. (c) $t = 5$ s. (d) $t = 8$ s.

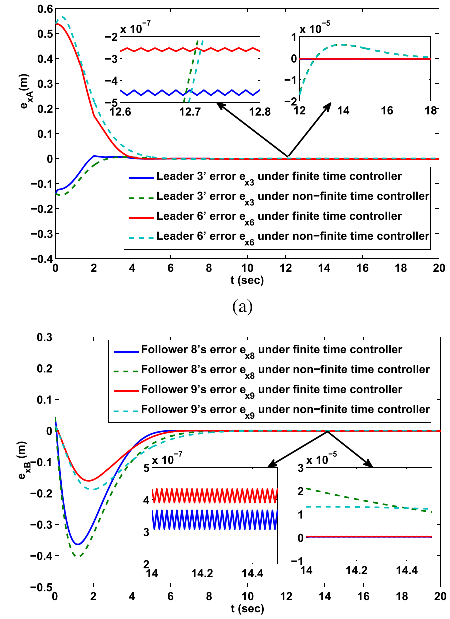
The simulation objective is described as follows. First, the six leaders are required to achieve the prespecified parallel hexagon formation in finite time using the proposed control law given in (7) and (8), where the formation is defined as $\varpi_k = [3\sin(k\pi/3),\ 3\cos(k\pi/3),\ -3\sin(k\pi/3)]^T$ for $k = 1, \ldots, 6$. Second, the eight followers are required to converge to the convex hull shaped by the formation of



(a)

(b)

Fig. 4. Neighborhood errors of the 14 UAVs under the finite-time control scheme. (a) Leaders' neighborhood errors in x-direction. (b) Followers' neighborhood errors in x-direction.



(a)

(b)

Fig. 5. Neighborhood error comparison under the finite-time and nonfinite-time control schemes. (a) Leaders' neighborhood error comparison in x-direction. (b) Followers' neighborhood error comparison in x-direction.

the six leaders in finite time by applying the control law (41) and (42) to the eight followers. The initial condition of the 14 UAVs are taken as $r_k(0) \in [-5, 5]$ for $k \in \mathscr{A}$ and $r_k(0) \in [-6, 6]$ for $k \in \mathscr{B}$, and $v_k(0) = (0, 0, 0)$ ($k = 1, \ldots, 14$). The control parameters are taken as $s = 6$, $c_1 = 20\,000$, $c_2 = 5$, $c_3 = 20\,000$, and $c_4 = 5$. In addition, the initial values of the estimates are chosen as $\hat{\beta}_k = 0_{3\times 3}$ for $k \in \mathscr{A}$ and $k \in \mathscr{B}$, respectively. The trajectories of all the leaders and followers during the transient process from the initial position to the final position are shown in Fig. 2. The formation and containment process are shown in Fig. 3(a)–(d) by the trajectories snapshots of the 14 agents at different instants, where the state trajectories of the leaders and the followers are denoted by the circle and pentagram, respectively, and the convex hull formed by the states of leaders is marked by solid lines. It is seen from Fig. 3(a)–(d) that the leaders achieve the desired parallel hexagon formation and the followers converge to the convex hull shaped by the leaders in finite time. The error convergence results under the proposed finite-time control scheme are represented in Fig. 4.

To show the effectiveness of our proposed scheme, we make a comparison on convergence results between the proposed finite-time control scheme given in Theorems 1 and 2 and the typical nonfinite-time-based control scheme corresponding to the case of $h = 1$. The two control schemes are applied to the same group of UAVs modeled by (74) with the same design parameters mentioned above. The convergence comparison results are shown in Fig. 5. It is observed from the results that the proposed finite-time controller leads to faster convergence rate and higher accuracy as compared with the nonfinite-time controller.

Remark 3: It should be mentioned that the control parameter c_1 (c_3) is set relatively larger compared with c_2 (c_4) in the simulation. This is because $\max\{m_{x,k}, m_{y,k}, m_{z,k}\} = 808$, leading to the equivalent control gain $1/808$, thus in order to have enough power to steer the system, the designed control parameter c_1 (c_3) has to be chosen large enough to counteract the small control gain $1/808$.

V. Conclusion

This paper explicitly addressed finite-time formation–containment control of MAS with unknown control gains and nonlinear dynamics, under one-way direction communication interactions. It is shown that under the proposed control scheme, not only can the leaders form a specific formation in finite time, but also the followers enter into the convex hull spanned by the leaders in finite time. Not considered in this paper is the collision avoidance among the agents during the operation, which obviously further complicates the underlying problem significantly. For agents with nonlinear dynamics performing containment and formation simultaneously under finite-time requirements, collision avoidance is still a challenging open problem, thus worthy of further studying in the future.

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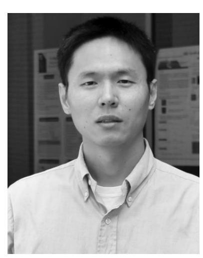

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