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Brief paper

Finite-time consensus algorithm for multi-agent systems with double-integrator dynamics☆

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ABSTRACT

In this paper, we discuss the finite-time consensus problem for leaderless and leader–follower multi-agent systems with external disturbances. Based on the finite-time control technique, continuous distributed control algorithms are designed for these agents described by double integrators. Firstly, for the leaderless multi-agent systems, it is shown that the states of all agents can reach a consensus in finite time in the absence of disturbances. In the presence of disturbances, the steady-state errors of any two agents can reach a region in finite time. Secondly, for the leader–follower multi-agent systems, finite-time consensus algorithms are also designed based on distributed finite-time observers. Rigorous proof is given by using Lyapunov theory and graph theory. Finally, one example is employed to verify the efficiency of the proposed method.

1. Introduction

In recent years, the consensus problem for multi-agent systems has attracted more and more attention. This is due to the fact that the multi-agent systems have broad applications in many areas, e.g., formation control (Dong & Farrell, 2009; Fax & Murray, 2004; Ren & Atkins, 2007), attitude alignment (Dimarogonas, Tsiotras, & Kyriakopoulos, 2009), flocking (Olfati-Saber, 2006), etc. The consensus of multi-agent systems means that the states of all the agents converge to a common desired value by implementing an appropriate consensus protocol. Usually, the multi-agent systems can be classified into leaderless and leader–follower systems. Many consensus algorithms have been reported in Dimarogonas et al. (2009), Dong and Farrell (2009), Hong, Chen, and Bushnell (2008), Hong, Hu, and Gao (2008), Olfati-Saber and Murray (2004), Ren (2007, 2008) and Ren and Atkins (2007), to name just a few.

An important topic in the study of the consensus problem is the convergence rate. So far, most of the existing consensus control algorithms for multi-agent systems are asymptotic consensus algorithms, which means that the convergence rate is at best exponential with infinite settling time. In other words, the states cannot reach a consensus in finite time. Obviously, finite-time consensus algorithms are more desirable. Besides a faster convergence rate, the closed-loop systems under finite-time control usually demonstrate better disturbance rejection properties (Bhat & Bernstein, 2000; Li, Ding, & Li, 2009). Because of the above superiorities, several kinds of finite-time consensus protocols have been developed for the first-order multi-agent systems in Cortes (2006), Hui, Haddad, and Bhat (2008), Wang and Xiao (2010) and Xiao, Wang, Chen, and Gao (2009) recently.

Since in many practical individual systems, especially mechanical systems, are of second-order dynamics, it is significant and necessary to study finite-time consensus algorithms for the second-order multi-agent systems. However, it is worthy of noting that the extension of finite-time consensus algorithms from the first-order case to the second-order case is nontrivial even in the linear consensus algorithms as shown by Ren and Atkins (2007). Although the analysis of finite-time consensus for the second-order multi-agent systems is more difficult than that of first-order dynamic systems, there are still some results. For the leaderless multi-agent systems, based on homogeneous method, continuous finite time consensus algorithms were designed in Wang and Hong (2008). For the leader–follower multi-agent systems, in Khoo, Xie, and Man (2009), the terminal sliding mode technique was used to design finite time consensus algorithms. However, the distributed

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controllers in Khoo et al. (2009) are discontinuous. In practical implementations, discontinuous control may lead to chattering behavior and excite unmodeled high-frequency system dynamics.

In this paper, also for the second-order multi-agent systems, continuous finite-time consensus algorithms are addressed. Compared with Khoo et al. (2009) and Wang and Hong (2008), the difference of this paper lies in two aspects. First, based on adding a power integrator method (Qian & Lin, 2001), new continuous finite-time consensus algorithms are proposed for the second-order multi-agent systems in two cases: leaderless and leader–follower. In Wang and Hong (2008), a continuous finite-time consensus algorithm was only proposed for the leaderless multi-agent systems. Moreover, in contrast to Wang and Hong (2008), we prove finite-time convergence directly by constructing a suitable Lyapunov function and obtain the explicit expression of the finite settling time. Second, the disturbance rejection property of the closed-loop system under continuous finite-time control law is analyzed in detail. Although some results have been reported for other systems (Bhat & Bernstein, 2000; Li et al., 2009), no results for multi-agent systems are available till now.

2. Preliminaries and problem formulation

In this paper, let $P > 0$ denote a symmetric positive definite matrix P. Let $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and minimum eigenvalues of matrix P, respectively. For simplicity, let $\mathbf{1} = [1, \dots, 1]^T$ and $\mathbf{0} = [0, \dots, 0]^T$. Let $\mathrm{sig}^{\alpha}(x) = \mathrm{sign}(x)|x|^{\alpha}$, where $\alpha > 0$, $x \in R$ and $\mathrm{sign}(\cdot)$ is the standard signum function.

2.1. Graph theory

For the leaderless multi-agent systems, assume that each agent is a node and the information exchange of n agents is denoted by an undirected graph $G = \{V, E, A\}$. $V = \{v_i, i = 1, \dots, n\}$ is the set of vertices, $E \subseteq V \times V$ is the set of edges and $A = [a_{ij}] \in R^{n\times n}$ is the weighted adjacency matrix of the graph G. The node indexes belong to a finite index set $\Gamma = \{1, \dots, n\}$. If there is an edge between agent i and j, i.e., $(v_i, v_j) \in E$, then $a_{ij} = a_{ji} > 0$. If there is no edge between agent i and j, then $a_{ij} = a_{ji} = 0$. Moreover, we assume that $a_{ii} = 0$ for all $i \in \Gamma$. The set of neighbors of node v_i is denoted by $N_i = \{j : (v_i, v_j) \in E\}$. The out-degree of node v_i is defined as $\deg_{\mathrm{out}}(v_i) = d_i = \sum_{j=1}^{n} a_{ij} = \sum_{j\in N_i} a_{ij}$. Then the degree matrix of digraph G is $D = \mathrm{diag}\{d_1, \dots, d_n\}$ and the Laplacian matrix of digraph G is $L = D - A$. A path in the graph G from v_i to v_j is a sequence of distinct vertices starting with v_i and ending with v_j such that consecutive vertices are adjacent. The graph G is connected if there is a path between any two vertices.

For the leader–follower multi-agent systems, assume that the leader (labeled by 0) is represented by vertex v_0 and the followers are represented by vertices $v_1, \dots, v_n$. $\bar{G}$ is a graph with vertex set $\bar{V} = V \cup \{v_0\}$. The communication topology graph for the followers is denoted by G. The connection between the follower and the leader is directed. There are only edges from some followers to the leader, but there is no edge from the leader to any follower. The connection weight between the follower and the leader is denoted by b_i, $i \in \Gamma$. If the i-th follower is connected to the leader, then $b_i > 0$, otherwise, $b_i = 0$. Let $B = \mathrm{diag}\{b_1, \dots, b_n\}$.

2.2. Some lemmas

Lemma 1 (Bhat & Bernstein, 2000). *Considering the system $\dot{x} = f(x), f(0) = 0, x \in R^n$, there exist a positive definite continuous function $V(x) : U \to R$, real numbers $c > 0$ and $\alpha \in (0, 1)$, and an open neighborhood $U_0 \subset U$ of the origin such that $\dot{V}(x) + c(V(x))^{\alpha} \le 0, x \in U_0 \setminus \{0\}$. Then $V(x)$ approaches 0 in finite time. In addition, the finite settling time T satisfies that $T \le \frac{V(x(0))^{1-\alpha}}{c(1-\alpha)}$.*

Lemma 2 (Qian & Lin, 2001). *If $0 < p = p_1/p_2 \le 1$, where $p_1 > 0, p_2 > 0$ are positive odd integers, then $|x^p - y^p| \le 2^{1-p}|x - y|^p$.*

Lemma 3 (Qian & Lin, 2001). *For $x \in R, y \in R$, if $c > 0, d > 0$, then $|x|^c|y|^d \le c/(c + d)|x|^{c+d} + d/(c + d)|y|^{c+d}$.*

Lemma 4 (Hardy, Littlewood, & Polya, 1952). *For $x_i \in R, i = 1, \dots, n, 0 < p \le 1$, then $\left(\sum_{i=1}^{n} |x_i|\right)^p \le \sum_{i=1}^{n} |x_i|^p \le n^{1-p}\left(\sum_{i=1}^{n} |x_i|\right)^p$.*

Lemma 5 (Olfati-Saber & Murray, 2004). *For a connected undirected graph G, the Laplacian matrix L of G has the following properties. $x^T Lx = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x_i - x_j)^2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j\in N_i} a_{ij}(x_i - x_j)^2$ for any $x = [x_1, \dots, x_n]^T \in R^n$, which implies that L is positive semi-definite. 0 is a simple eigenvalue of L and $\mathbf{1}$ is the associated eigenvector. Assume that the eigenvalues of L are denoted by $0, \lambda_2, \dots, \lambda_n$ satisfying $0 \le \lambda_2 \le \dots \le \lambda_n$. Then the second smallest eigenvalue $\lambda_2 > 0$. Furthermore, if $\mathbf{1}^T x = 0$, then $x^T Lx \ge \lambda_2 x^T x$.*

2.3. Problem formulation

For the leaderless systems, the dynamics of the i-th agent is described as

$$\dot{x}_i = v_i, \qquad \dot{v}_i = u_i + d_i(t), \quad i \in \Gamma, \tag{1}$$

where x_i, v_i are the position and velocity, u_i is the control input, called protocol, to be designed, $d_i(t)$ is the external disturbance of the i-th agent respectively.

Assumption 6. $|d_i(t)| \le l < +\infty$ for all $i \in \Gamma$.

For leader–follower systems, the dynamics of each follower is described as (1) and the dynamics of the leader is described as

$$\dot{x}_0(t) = v_0(t), \qquad \dot{v}_0(t) = u_0(t). \tag{2}$$

Assumption 7. For leader–follower multi-agent systems (1)–(2), the graph G for the followers is connected and there is at least one directed edge from a certain follower to the leader, i.e., $B \ne 0$.

Lemma 8 (Hong, Hu et al., 2008). *For leader–follower multi-agent systems (1)–(2), if Assumption 7 holds, then $L + B > 0$.*

The goal of this paper is to design finite-time consensus protocols for leaderless and leader–follower multi-agent systems such that the state consensus can be achieved in finite time in the absence disturbances. Moreover, in the presence of external disturbances, since the proposed consensus control algorithm is continuous, the disturbances cannot be suppressed completely. In this case, how about the disturbance rejection property for the closed-loop system compared with the asymptotic consensus case? We will also give an answer.

3. Main results

3.1. Finite-time consensus for the leaderless case

3.1.1. Finite-time consensus protocol design

In this subsection, we consider leaderless multi-agent systems (1) in the absence of disturbances.

Theorem 9. *Consider the leaderless multi-agent system (1) with $d_i(t) = 0$ for all $i \in \Gamma$. If the graph G is connected and u_i is designed as*

$$u_i = -k_1\left[v_i^p + k_2^p\left(\sum_{j\in N_i} a_{ij}(x_i - x_j)\right)\right]^{\frac{2}{p}-1}, \quad i \in \Gamma, \tag{3}$$

then the states of all the agents reach a consensus in finite time, where $k_2 \geq \frac{p2^{1-1/p}}{1+p} + \frac{(\beta+n\gamma)}{1+p} + k_3$, $k_1 \geq (2-1/p)2^{1-1/p} k_2^{1+p}\left(\frac{2^{1-1/p}+(\beta+n\gamma)p}{1+p} + \frac{(\beta+n\gamma)2^{1-1/p}}{k_2} + k_3\right)$, *$k_3$ is a positive constant,* $\beta = \max_{\forall i\in\Gamma}\left\{\sum_{j\in N_i} a_{ij}\right\}$, $\gamma = \max_{\forall i,j\in\Gamma}\{a_{ij}\}$, $1 < p = p_1/p_2 < 2$, *p_1, p_2 are positive odd integers.*

Proof. The proof can be divided into two steps. First, a virtual velocity is designed for each agent such that the states consensus is achieved in finite time. Then, we prove that the virtual velocity can be tracked by the real velocity in finite time under the control law (3).

Step 1. According to Lemma 5, consider the following candidate Lyapunov function

$$V_0(t) = \frac{1}{2}x^T Lx = \frac{1}{4}\sum_{i=1}^n \sum_{j\in N_i} a_{ij}(x_i - x_j)^2. \tag{4}$$

Taking the derivative of $V_0(t)$ yields $\dot V_0(t) = \dot x^T Lx = \sum_{i=1}^n\left[\sum_{j\in N_i} a_{ij}(x_i - x_j)\right] v_i$. Set $q_i = \sum_{j\in N_i} a_{ij}(x_i - x_j)$, $i \in \Gamma$ and take the velocity $v_i = -k_2 q_i^{1/p}$, $i \in \Gamma$ as the virtual input. Then $\dot V_0(t) = -k_2\sum_{i=1}^n q_i^{1+1/p}$. Using Lemma 4, we have

$$\dot V_0(t) = -k_2\sum_{i=1}^n q_i^{\frac{2(1+p)}{2p}} \leq -k_2\left(\sum_{i=1}^n q_i^2\right)^{(1+p)/(2p)}. \tag{5}$$

Since $Lx = [q_1, \ldots, q_n]^T$, we have $\sum_{i=1}^n q_i^2 = (Lx)^T Lx = x^T L^2 x$. Let $L^{1/2}\mathbf{1} = s = [s_1, \ldots, s_n]^T$. Then $s^T s = (L^{1/2}\mathbf{1})^T L^{1/2}\mathbf{1} = \mathbf{1}^T L\mathbf{1}$. By Lemma 5, we know $L\mathbf{1} = \mathbf{0}$. Hence, $s^T s = 0$, which implies that $s^T = \mathbf{0}^T$. It follows that $s^T x = 0$, i.e., $\mathbf{1}^T L^{1/2} x = 0$. By Lemma 5, if $\mathbf{1}^T(L^{1/2}x) = 0$, then

$$\sum_{i=1}^n q_i^2 = (L^{1/2}x)^T L(L^{1/2}x) \geq \lambda_2 x^T Lx = 2\lambda_2 V_0(t). \tag{6}$$

By (5) and (6), we get $\dot V_0(t) \leq -k_2(2\lambda_2)^{\frac{1+p}{2p}} V_0(t)^{\frac{1+p}{2p}}$. By Lemma 1, we obtain $V_0(t)$ reaches zero in finite time.

Step 2. We will show that the virtual velocity can be tracked by the real velocity in finite time under the control law (3). Take the Lyapunov function as

$$V(t) = V_0(t) + \sum_{i=1}^n V_i(t), \tag{7}$$

$$V_i(t) = \frac{1}{(2-1/p)2^{1-1/p}k_2^{1+p}}\int_{v_i^*}^{v_i}(s^p - v_i^{*p})^{2-\frac{1}{p}}\mathrm{d}s,$$
$$v_i^* = -k_2 q_i^{1/p}, \quad i \in \Gamma. \tag{8}$$

From Propositions B1 and B2 in Qian and Lin (2001), we know that $\int_{v_i^*}^{v_i}(s^p - v_i^{*p})^{2-1/p}\mathrm{d}s$, $i \in \Gamma$, are differentiable, positive definite and proper. For brevity, denote $\omega_i = v_i^p - v_i^{*p}$, $i \in \Gamma$ and $d = 1 + 1/p$. From Step 1 and Lemmas 2 and 3,

$$\begin{aligned}\dot V_0(t) &= \sum_{i=1}^n q_i v_i = \sum_{i=1}^n q_i v_i^* + \sum_{i=1}^n q_i(v_i - v_i^*)\\ &\leq -k_2\sum_{i=1}^n q_i^d + \sum_{i=1}^n 2^{1-1/p}|q_i|\,|v_i^p - v_i^{*p}|^{1/p}\\ &\leq -k_2\sum_{i=1}^n q_i^d + \sum_{i=1}^n 2^{1-1/p}\left(\frac{p|q_i|^d}{1+p} + \frac{|\omega_i|^d}{1+p}\right).\end{aligned} \tag{9}$$

Taking the derivative of $V_i(t)$ along system (1) yields

$$\begin{aligned}\dot V_i(t) &= -\frac{1}{2^{1-1/p}k_2^{1+p}}\frac{\mathrm{d}v_i^{*p}}{\mathrm{d}t}\int_{v_i^*}^{v_i}(s^p - v_i^{*p})^{1-1/p}\mathrm{d}s\\ &\quad + \frac{1}{(2-1/p)2^{1-1/p}k_2^{1+p}}\omega_i^{2-1/p}u_i, \quad i \in \Gamma.\end{aligned} \tag{10}$$

Note that $\mathrm{d}v_i^{*p}/\mathrm{d}t = -k_2^p\sum_{j\in N_i} a_{ij}(v_i - v_j) \leq k_2^p(\beta|v_i| + \gamma\sum_{m=1}^n |v_m|)$, $i \in \Gamma$, where β, γ are given in Theorem 9. Substituting this inequality into (10), and noticing that $\left|\int_{v_i^*}^{v_i}(s^p - v_i^{*p})^{1-1/p}\mathrm{d}s\right| \leq |v_i - v_i^*|\,|\omega_i|^{1-1/p}$, we have

$$\begin{aligned}\dot V_i(t) &\leq \frac{1}{2^{1-1/p}k_2}\left(\beta|v_i| + \gamma\sum_{m=1}^n |v_m|\right)|v_i - v_i^*|\,|\omega_i|^{1-\frac{1}{p}}\\ &\quad + \frac{1}{(2-1/p)2^{1-1/p}k_2^{1+p}}\omega_i^{2-1/p}u_i, \quad i \in \Gamma.\end{aligned} \tag{11}$$

Using Lemmas 2 and 3, for any $i, m \in \Gamma$, we have

$$\begin{aligned}&|v_m|\,|v_i - v_i^*|\,|\omega_i|^{1-\frac{1}{p}}\\ &\leq 2^{1-\frac{1}{p}}|v_m|\,|\omega_i|^{1/p}|\omega_i|^{1-\frac{1}{p}}\\ &\leq 2^{1-\frac{1}{p}}(|\omega_i|\,|v_m - v_m^*| + |\omega_i|\,|v_m^*|)\\ &\leq 2^{1-\frac{1}{p}}(2^{1-1/p}|\omega_i|\,|\omega_m|^{1/p} + k_2|\omega_i|\,|q_m|^{1/p})\\ &\leq 2^{1-\frac{1}{p}}\left((2^{1-\frac{1}{p}} + k_2)\frac{p|\omega_i|^d}{1+p} + \frac{2^{1-\frac{1}{p}}}{1+p}|\omega_m|^d + \frac{k_2}{1+p}|q_m|^d\right).\end{aligned}$$

Substituting this inequality into (11) leads to

$$\begin{aligned}\dot V_i(t) &\leq \frac{k_4}{k_2}|\omega_i|^d + \frac{\beta}{1+p}|q_i|^d + \frac{\gamma 2^{1-1/p}}{k_2(1+p)}\sum_{m=1}^n |\omega_m|^d\\ &\quad + \frac{\gamma}{1+p}\sum_{m=1}^n |q_m|^d + \frac{1}{(2-1/p)2^{1-1/p}k_2^{1+p}}\omega_i^{2-1/p}u_i,\end{aligned} \tag{12}$$

where $k_4 = (\beta + n\gamma)(2^{1-1/p} + k_2)\frac{p}{1+p} + \frac{\beta 2^{1-1/p}}{1+p}$. Putting together (7), (9), (12), and noticing that $|q_i|^d = |q_i|^{\frac{1+p}{p}} = |q_i|^{\frac{p_1+p_2}{p_1}} = q_i^{\frac{p_1+p_2}{p_1}} = q_i^d$, also $|\omega_i|^d = \omega_i^d$, we obtain

$$\begin{aligned}\dot V(t) &\leq -\left(k_2 - \frac{p2^{1-1/p}}{1+p} - \frac{\beta + n\gamma}{1+p}\right)\sum_{i=1}^n q_i^d\\ &\quad + \left(\frac{2^{1-1/p}}{1+p} + \frac{k_4}{k_2} + \frac{n\gamma 2^{1-1/p}}{k_2(1+p)}\right)\sum_{i=1}^n \omega_i^d\\ &\quad + \frac{1}{(2-1/p)2^{1-1/p}k_2^{1+p}}\sum_{i=1}^n \omega_i^{2-1/p}u_i.\end{aligned} \tag{13}$$

By the definition of k_4 and calculation, we have $\frac{2^{1-1/p}}{1+p} + \frac{k_4}{k_2} + \frac{n\gamma 2^{1-1/p}}{k_2(1+p)} = \frac{2^{1-1/p}+(\beta+n\gamma)p}{1+p} + \frac{(\beta+n\gamma)2^{1-1/p}}{k_2}$. Note that control law (3) can be rewritten as $u_i = -k_1\omega_i^{2/p-1}$, $i \in \Gamma$. Substituting this control law into (13) and from the gains condition of Theorem 9, we have

$$\dot V(t) \leq -k_3\sum_{i=1}^n q_i^d - k_3\sum_{i=1}^n \omega_i^d. \tag{14}$$

Next, from (8), by calculation and Lemma 2, we get

$$V_i(t) \leq \frac{1}{(2-1/p)2^{1-1/p}k_2^{1+p}}|v_i - v_i^*|\,|v_i^p - v_i^{*p}|^{2-\frac{1}{p}}$$

$$\leq \frac{1}{\left(2-\frac{1}{p}\right)k_2^{1+p}}|\omega_i|^{\frac{1}{p}}|\omega_i|^{2-\frac{1}{p}} = \frac{|\omega_i|^2}{\left(2-\frac{1}{p}\right)k_2^{1+p}}, \quad i \in \Gamma. \tag{15}$$

Then, combining (6) and (15), we have $V(t) = V_0(t)+\sum_{i=1}^n V_i(t) \leq c_1\left(\sum_{i=1}^n q_i^2 + \sum_{i=1}^n \omega_i^2\right)$, where $c_1 = \max\left\{\frac{1}{2\lambda_2}, \frac{1}{(2-1/p)k_2^{1+p}}\right\}$. By Lemma 4, we get

$$V^{d/2}(t) \leq c_1^{d/2}\left(\sum_{i=1}^n q_i^d + \sum_{i=1}^n \omega_i^d\right). \tag{16}$$

Let $c_2 = k_3/(2c_1^{d/2}) > 0$. Then from (14), we obtain $\dot V(t) + c_2V^{d/2}(t) \leq -\frac{k_3}{2}\sum_{i=1}^n q_i^d - \frac{k_3}{2}\sum_{i=1}^n \omega_i^d \leq 0$. By Lemma 1, there exists a time $T \leq \frac{V(0)^{1-d/2}}{c_2(1-d/2)} < +\infty$, such that $V(t) = 0, \forall t \geq T$. It means that $V_0(t) = 0, V_i(t) = 0, \forall i \in \Gamma, \forall t \geq T$. On one hand, since the graph G is connected, then $V_0(t) = 0$ implies that $x_i = x_j, \forall i,j \in \Gamma$. On the other hand, $V_i(t) = 0$ implies that $v_i(t) = v_i^*(t)$. Note that $v_i^*(t) = 0$ once $x_i = x_j, \forall i,j \in \Gamma$. Then, $V_i(t) = 0$ implies $v_i(t) = v_i^*(t) = 0, \forall i \in \Gamma$. Thus, finite-time consensus can be achieved. □

Remark 10. Observing that if we take the value of the fraction power $p = 1$, the finite-time protocol (3) will reduce to the following asymptotic linear protocol

$$u_i = -r_1v_i - r_1r_2\left[\sum_{j\in N_i} a_{ij}(x_i - x_j)\right], \quad i \in \Gamma, \tag{17}$$

where $r_2 \geq \frac{1}{2} + \frac{(\beta+n\gamma)}{2} + r_3, r_1 \geq r_2^2\left(\frac{1+(\beta+n\gamma)}{2} + \frac{(\beta+n\gamma)}{r_2} + r_3\right)$, r_3 is a positive constant.

3.1.2. Analysis of disturbance rejection performance

In this subsection, we will consider the leaderless multi-agent systems (1) in the presence of disturbances.

Theorem 11. *Consider the leaderless multi-agent systems (1). If the graph G is connected and Assumption 6 is satisfied, then under the finite-time consensus protocol (3), the steady-state errors of any two agents will converge to the region Q in finite time, where*

$$Q = \left\{|x_i - x_j| \leq c_3\left(\frac{pl2^{1/p}}{(2p-1)k_2^{1+p}k_3}\right)^{\frac{p}{2-p}}, \quad |v_i - v_j| \leq c_4\left(\frac{pl2^{1/p}}{(2p-1)k_2^{1+p}k_3}\right)^{\frac{1}{2-p}}, \forall i,j \in \Gamma\right\}, \tag{18}$$

and $c_1 = \max\{\frac{1}{2\lambda_2}, \frac{1}{(2-1/p)k_2^{1+p}}\}$, $c_3 = \sqrt{\frac{2(n-1)c_1}{\delta}}(n^{\frac{1+p}{2p-1}} + n^{\frac{3}{1+p}})^{\frac{p}{1+p}}$, $c_4 = 2[2((p-\frac{1}{2})k_2^{1+p})^{\frac{1}{2p}} + k_2(2c_5)^{\frac{1}{2p}}]c_1^{\frac{1}{2p}} \times (n^{\frac{1+p}{2p-1}} + n^{\frac{3}{1+p}})^{\frac{1}{1+p}}$, $\delta = \min\{a_{ij} : a_{ij} \neq 0, \forall i,j \in \Gamma\}$, λ_2 *is the second smallest eigenvalue of* L, $c_5 = \lambda_{\max}(L)$.

Proof. The analysis procedure is similar to that of the proof of Theorem 9, except the appearance of the disturbances. Noticing $|d_i(t)| \leq l$ and substituting the protocol (3) into $\dot V(t)$, following a similar procedure between (7) and (14) in the proof of Theorem 9, we have

$$\dot V(t) \leq -k_3\sum_{i=1}^n q_i^d - k_3\sum_{i=1}^n \omega_i^d + h\sum_{i=1}^n |\omega_i|^{2-1/p}, \tag{19}$$

where $h = \frac{l}{(2-1/p)2^{1-1/p}k_2^{1+p}}$. The following proof procedure can be divided into three steps.

Define

$$\Omega_1 = \left\{(x^T, v^T) : \sum_{i=1}^n q_i^d + \sum_{i=1}^n \omega_i^d < \left(n^{(1+p)/(2p-1)} + n^{3/(1+p)}\right)(2h/k_3)^{(1+p)/(2-p)}\right\},$$

$$\Omega_2 = \left\{(x^T, v^T) : V(t) < m, \quad m = c_1\left(n^{(1+p)/(2p-1)} + n^{3/(1+p)}\right)^{2/d}(2h/k_3)^{(2p)/(2-p)}\right\},$$

where $x = [x_1, \dots, x_n]^T$, $v = [v_1, \dots, v_n]^T$. First, we prove the fact that $\dot V(t) < 0$ once $(x^T, v^T) \notin \Omega_1$. However, this fact cannot guarantee the region Ω_1 is the attractive region. The reason is that it is possible the states may escape from Ω_1 after entering it, since in Ω_1, $\dot V(t) < 0$ is no longer guaranteed. However, we can find a larger region Ω_2 which contains Ω_1, i.e., $\Omega_1 \subset \Omega_2$, such that Ω_2 is the attractive region. Second, we prove that the region Ω_2 is globally attractive and can be reached in finite time, which means there exists a finite time t_1^*, such that $(x^T(t), v^T(t)) \in \Omega_2$ for $\forall t > t_1^*$. Third, we give an estimation for steady-state errors for all agents.

Step 1. Assuming $(x^T, v^T) \notin \Omega_1$, i.e., $\sum_{i=1}^n q_i^d + \sum_{i=1}^n \omega_i^d \geq \left(n^{(1+p)/(2p-1)} + n^{3/(1+p)}\right)(2h/k_3)^{(1+p)/(2-p)}$, the following analysis can be divided into two cases.

Case 1: $\sum_{i=1}^n \omega_i^d \geq n^{(1+p)/(2p-1)}(2h/k_3)^{(1+p)/(2-p)}$.

Since $1 < p < 2$ and $d = 1+1/p$, we obtain $0 < (2-1/p)/d = (2p-1)/(1+p) < 1$. Using Lemma 4, we obtain

$$\sum_{i=1}^n |\omega_i|^{2-1/p} = \sum_{i=1}^n (|\omega_i|^d)^{\frac{2-1/p}{d}} \leq n^{\frac{2-p}{1+p}}\left(\sum_{i=1}^n \omega_i^d\right)^{\frac{2p-1}{1+p}}, \tag{20}$$

which implies that $\sum_{i=1}^n \omega_i^d \geq n^{\frac{p-2}{2p-1}}\left(\sum_{i=1}^n |\omega_i|^{2-1/p}\right)^{\frac{1+p}{2p-1}}$. Then, it follows this inequality that

$$\frac{k_3}{2}\sum_{i=1}^n \omega_i^d - h\sum_{i=1}^n |\omega_i|^{2-1/p} \geq \left(\sum_{i=1}^n |\omega_i|^{2-1/p}\right) \times \left[\frac{k_3}{2}n^{(p-2)/(2p-1)}\left(\sum_{i=1}^n |\omega_i|^{2-1/p}\right)^{\frac{1+p}{2p-1}-1} - h\right]. \tag{21}$$

On the other hand, using Lemma 4 and noticing $\sum_{i=1}^n \omega_i^d \geq n^{(1+p)/(2p-1)}(2h/k_3)^{(1+p)/(2-p)}$, we obtain $\sum_{i=1}^n |\omega_i|^{2-\frac{1}{p}} \geq \left(\sum_{i=1}^n \omega_i^d\right)^{(2-1/p)/d} \geq n(2h/k_3)^{\frac{2p-1}{2-p}}$. Substituting this inequality into (21) leads to $\frac{k_3}{2}\sum_{i=1}^n \omega_i^d - h\sum_{i=1}^n |\omega_i|^{2-1/p} \geq 0$. By (19), we get $\dot V(t) < 0$.

Case 2: $\sum_{i=1}^n \omega_i^d < n^{(1+p)/(2p-1)}(2h/k_3)^{(1+p)/(2-p)}$.

If $\sum_{i=1}^n \omega_i^d < n^{(1+p)/(2p-1)}(2h/k_3)^{(1+p)/(2-p)}$, then by the definition of Ω_1, we obtain that $(x^T, v^T) \notin \Omega_1$ implies that $\sum_{i=1}^n q_i^d > n^{3/(1+p)}(2h/k_3)^{(1+p)/(2-p)}$. Meanwhile, combining (20), we can obtain $\sum_{i=1}^n |\omega_i|^{2-1/p} \leq n^{3/(1+p)}(2h/k_3)^{(2p-1)/(2-p)}$. Substituting the last two inequalities into (19), we get $\dot V(t) < -\frac{k_3}{2}\sum_{i=1}^n q_i^d - k_3\sum_{i=1}^n \omega_i^d < 0$. Hence, $\dot V(t) < 0$ once $(x^T, v^T) \notin \Omega_1$.

Step 2. In the sequel, we will prove that $\dot V(t) < 0$ once $(x^T, v^T) \notin \Omega_2$. Actually, from Step 1, we only need to prove that Ω_1 is a subset of Ω_2. The proof is given as follows. For $\forall (x^T, v^T) \in \Omega_1$, by (16), we have $V^{d/2}(t) \leq c_1^{d/2}\left(\sum_{i=1}^n q_i^d + \sum_{i=1}^n \omega_i^d\right) < c_1^{d/2}\left(n^{(1+p)/(2p-1)} + n^{3/(1+p)}\right)(2h/k_3)^{(1+p)/(2-p)}$, which leads to

$V(t) < m$. That is to say $(x^T, v^T) \in \Omega_2$, i.e., $\Omega_1 \subset \Omega_2$. Hence, once $(x^T, v^T) \notin \Omega_2$, then $\dot{V}(t) < 0$. Since Ω_2 is a level set of the Lyapunov function, then it can be concluded that the region Ω_2 is globally attractive. Therefore, there exists a finite time t_1^*, such that $(x^T(t), v^T(t)) \in \Omega_2$, i.e., $V(t) < m$ for $\forall t > t_1^*$.

Step 3. Since the graph G is connected, then for any nodes v_i, v_j, there exists a path $v_i = v_{i_1}, v_{i_2}, \ldots, v_{i_s} = v_j$, connecting v_i to v_j. Then $\sqrt{\delta}|x_i - x_j| \leq \sqrt{a_{i_1 i_2}}|x_{i_1} - x_{i_2}| + \cdots + \sqrt{a_{i_{s-1} i_s}}|x_{i_{s-1}} - x_{i_s}|$, where $\delta = \min\{a_{ij} : a_{ij} \neq 0, \ \forall i, j \in \Gamma\}$. By Lemma 4, we have

$$\begin{aligned}
&\sqrt{a_{i_1 i_2}}|x_{i_1} - x_{i_2}| + \cdots + \sqrt{a_{i_{s-1} i_s}}|x_{i_{s-1}} - x_{i_s}| \\
&\quad \leq (s-1)^{\frac{1}{2}} [a_{i_1 i_2}(x_{i_1} - x_{i_2})^2 + \cdots + a_{i_{s-1} i_s}(x_{i_{s-1}} - x_{i_s})^2]^{\frac{1}{2}} \\
&\quad \leq (s-1)^{\frac{1}{2}} \left[\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x_i - x_j)^2/2\right]^{\frac{1}{2}} = \sqrt{(s-1)2V_0(t)}.
\end{aligned}$$

Hence, by the definition of c_3, when $t \geq t_1^*$, we get

$$\begin{aligned}
|x_i - x_j| &\leq \sqrt{2(s-1)V_0(t)}/\sqrt{\delta} \leq \sqrt{2(n-1)V(t)}/\sqrt{\delta} \\
&\leq \sqrt{2(n-1)m}/\sqrt{\delta} = c_3(2h/k_3)^{p/(2-p)}. \qquad (22)
\end{aligned}$$

Next, let us consider $V_i(t)$. By Lemma 2, we have $|s^p - v_i^{*p}| \geq \left(2^{\frac{1}{p}-1}|s - v_i^*|\right)^p$. If $v_i \geq v_i^*$, by the previous inequality and the definition of V_i, we have

$$V_i(t) \geq \frac{2^{1-2p}}{(2p-1)k_2^{1+p}}(v_i - v_i^*)^{2p}, \quad i \in \Gamma. \qquad (23)$$

In the case of $v_i < v_i^*$, the proof of (23) is similar.

Since $V_i(t) \leq V(t) \leq m, \forall t \geq t_1^*$, by (23), we obtain $|v_i(t) - v_i^*(t)| \leq 2\left((p-1/2)k_2^{1+p}\right)^{1/(2p)} m^{1/(2p)}$ when $t \geq t_1^*$. By (6), we get $\sum_{i=1}^n q_i^2 = \left(L^{1/2}x\right)^T L\left(L^{1/2}x\right) \leq c_5 x^T Lx = 2c_5 V_0(t) \leq 2c_5 V(t)$, where $c_5 = \lambda_{\max}(L)$. Then, when $t \geq t_1^*$, we have $|v_i^*(t)| = k_2|q_i^{1/p}| \leq k_2\left(\sum_{i=1}^n q_i^2\right)^{1/(2p)} \leq k_2(2c_5)^{1/(2p)} m^{\frac{1}{2p}}$. Hence, for any $i \in \Gamma$, when $t \geq t_1^*$, $|v_i(t)| \leq |v_i^*(t)| + |v_i(t) - v_i^*(t)| \leq \frac{c_4}{2}(2h/k_3)^{1/2-p}$. Then, for any $i, j \in \Gamma$, when $t \geq t_1^*$,

$$|v_i(t) - v_j(t)| \leq |v_i(t)| + |v_j(t)| \leq c_4 (2h/k_3)^{1/2-p}. \qquad (24)$$

Substituting $h = \frac{l}{(2-1/p)2^{1-1/p}k_2^{1+p}}$ into (22) and (24) yields the final expression for the region Q as (18). $\square$

Remark 12. If $p = 1$, then it can be proven that the steady-state errors of any two agents will converge to the region M in finite time under the linear protocol (17) by a similar proof like that of Theorem 11, where

$$M = \left\{|x_i - x_j| \leq c_7\left(\frac{2l}{r_2^2 r_3}\right), \ |v_i - v_j| \leq c_8\left(\frac{2l}{r_2^2 r_3}\right), \forall i, j \in \Gamma\right\},$$

and

$$c_6 = \max\left\{\frac{1}{2\lambda_2}, \frac{1}{r_2^2}\right\}, \qquad c_7 = \sqrt{2(n-1)c_6(n^2 + n^{3/2})/\delta},$$

$$c_8 = 2r_2\left(1 + \sqrt{c_5}\right)\sqrt{2c_6(n^2 + n^{3/2})},$$

The following remark will show that the finite-time consensus protocol (3) has a better disturbance rejection property than the linear consensus protocol (17).

Remark 13. In Theorem 11 and Remark 12, under two kinds of consensus protocols, the estimations for steady-state errors for all agents are given respectively in the presence of disturbances.

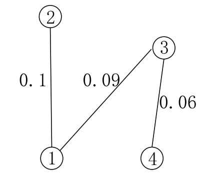


Fig. 1. The information exchange among agents.

One may argue that through adjusting control gains k_2, r_2 or/and k_3, r_3 to be large enough, both convergence regions, i.e., Q and M, can be rendered to be as small as desired, where Q, M are defined in Theorem 11 and Remark 12. It may mean that the finite-time consensus protocol does not have any prominent advantage over the linear consensus protocol on disturbance rejection. Nevertheless, due to control saturation constraint, we cannot select k_2, r_2 and k_3, r_3 to be sufficient large. In this case, for the proposed method using finite-time consensus protocol here, an additional parameter, i.e., the fractional power p, can be adjusted to enhance the disturbance rejection performance without increasing k_2 and k_3 to be sufficient large. For example, supposing $0 < \frac{pl2^{1/p}}{(2p-1)k_2^{1+p}k_3} < 1$ is satisfied, i.e., selecting k_2 such that $k_2 > \left(\frac{pl2^{1/p}}{(2p-1)k_3}\right)^{\frac{1}{1+p}}$, we can select p to approximate to 2 such that the power expression $1/(2-p)$ in Q is bigger enough than 1, which can make $\left(\frac{2^{1/p}pl}{(2p-1)k_3k_2^{1+p}}\right)^{1/(2-p)} \ll \frac{2l}{r_3 r_2^2}$.

3.2. *Finite-time consensus for the leader–follower case*

In this subsection, we consider the finite-time consensus tracking problem for the leader–follower multi-agent systems (1)–(2). We first propose a distributed finite-time observer for each follower to obtain the accurate estimates of the leader's velocity in finite time. Then, a finite-time consensus tracking algorithm is designed such that the leader can be tracked in finite time.

The distributed finite-time observer is proposed as

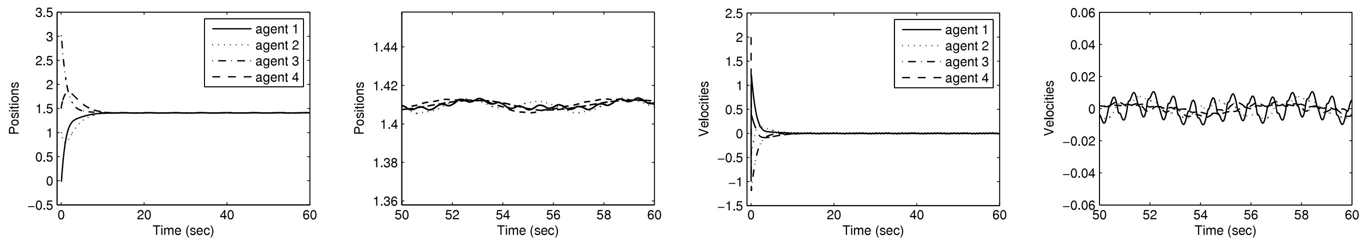
$$\begin{aligned}
\dot{\widehat{v}}_i &= \frac{1}{\sum\limits_{j\in N_i} a_{ij} + b_i}\left(\sum_{j\in N_i} a_{ij}\dot{\widehat{v}}_j + b_i\dot{v}_0\right) - \frac{\eta}{\sum\limits_{j\in N_i} a_{ij} + b_i} \\
&\quad \times \operatorname{sig}^{\alpha}\left[\sum_{j\in N_i} a_{ij}(\widehat{v}_i - \widehat{v}_j) + b_i(\widehat{v}_i - v_0)\right], \quad i \in \Gamma, \qquad (25)
\end{aligned}$$

where $\widehat{v}_i$ is the estimate of the leader's velocity for the i-th follower, $0 < \alpha < 1$ and $\eta > 0$. Then, based on the estimate values, we propose the following finite-time consensus protocol for systems (1)–(2) as follows

$$\begin{aligned}
u_i &= \dot{\widehat{v}}_i - k_1\left[(v_i - \widehat{v}_i)^p\right. \\
&\quad \left. + k_2^p\left(\sum_{j\in N_i} a_{ij}(x_i - x_j) + b_i(x_i - x_0)\right)\right]^{\frac{2}{p}-1}, \quad i \in \Gamma \qquad (26)
\end{aligned}$$

where k_1, k_2, and p are the same as that of Theorem 9.

Theorem 14. *Consider the leader–follower multi-agent systems (1)–(2) which satisfy Assumptions 6 and 7. Under the finite-time consensus protocol (25)–(26), if $d_i(t) = 0$, each follower can track the*



(a) Agents positions. (b) Agents positions in $t \in [50, 60]$. (c) Agents velocities. (d) Agents velocities in $t \in [50, 60]$.

Fig. 2. Responses under the finite-time protocol (3).

dynamic leader in finite time, if $d_i(t) \neq 0$, the states of i-th follower can reach the region Ψ_i in finite time, where

$$\Psi_i = \left\{ (x_i, v_i) : |x_i(t) - x_0(t)| \le c_{10} \left(\frac{pl2^{1/p}}{(2p-1)k_2^{1+p}k_3} \right)^{\frac{p}{2-p}}, \right.$$

$$\left. |v_i(t) - v_0(t)| \le c_{11} \left(\frac{pl2^{1/p}}{(2p-1)k_2^{1+p}k_3} \right)^{1/(2-p)} \right\}, \quad i \in \Gamma,$$

and $c_9 = \max\left\{ \frac{1}{2c}, \frac{1}{(2-1/p)k_2^{1+p}} \right\}$, $c = \lambda_{\min}(L+B)$, $c_{10} = \sqrt{\frac{2c_9}{c}} \left(n^{\frac{1+p}{2p-1}} + n^{\frac{3}{1+p}} \right)^{\frac{p}{1+p}}$, $c_{11} = \left[2\left(\left(p - \frac{1}{2}\right) k_2^{1+p} \right)^{\frac{1}{2p}} + k_2 \left(2c_{12}\right)^{\frac{1}{2p}} \right] c_9^{\frac{1}{2p}} \left(n^{\frac{1+p}{2p-1}} + n^{\frac{3}{1+p}} \right)^{\frac{1}{1+p}}$, $c_{12} = \lambda_{\max}(L+B)$.

Proof. We first prove that $\widehat{v}_i \to v_0(t)$, $\forall i \in \Gamma$ in finite time. Since the graph G for the followers is connected due to Assumption 7, then $\sum_{j\in N_i} a_{ij} + b_i \neq 0$, $\forall i \in \Gamma$. Hence, (25) is well defined. By (25), we have

$$\sum_{j\in N_i} a_{ij}(\dot{\widehat{v}}_i - \dot{\widehat{v}}_j) + b_i(\dot{\widehat{v}}_i - \dot{v}_0)$$

$$= -\eta \operatorname{sig}^{\alpha}\left[\sum_{j\in N_i} a_{ij}(\widehat{v}_i - \widehat{v}_j) + b_i(\widehat{v}_i - v_0) \right], \quad \forall i \in \Gamma. \tag{27}$$

Define $e_i = \sum_{j\in N_i} a_{ij}(\widehat{v}_i - \widehat{v}_j) + b_i(\widehat{v}_i - v_0)$, $i \in \Gamma$, then system (27) can be written as $\dot{e}_i = -\eta \operatorname{sig}^{\alpha}(e_i)$, $\forall i \in \Gamma$. Using Lyapunov function $V_i = \frac{1}{2}e_i^2$, it follows from Lemma 1 that $V_i(t)$ reaches zero in finite time, i.e., there exists a finite time $T_i^* < +\infty$, such that $V_i(t) = 0$, $\forall t \ge T_i^*$, $\forall i \in \Gamma$. Define $T^* = \max\{T_1^*, \dots, T_n^*\}$. Hence $e_i = \sum_{j\in N_i} a_{ij}\left((\widehat{v}_i - v_0) - (\widehat{v}_j - v_0) \right) + b_i(\widehat{v}_i - v_0) = 0$, $\forall t \ge T^*$, $\forall i \in \Gamma$. According to the definitions of matrices L and B, we obtain $[e_1, e_2, \dots, e_n]^T = (L+B)[\widehat{v}_1 - v_0, \widehat{v}_2 - v_0, \dots, \widehat{v}_n - v_0]^T = \mathbf{0}$, when $t \ge T^*$. By Lemma 8, we get $\widehat{v}_i - v_0 = 0$, $\forall t \ge T^*$, $\forall i \in \Gamma$. That is to say $\widehat{v}_i = v_0(t)$, $\dot{\widehat{v}}_i = \dot{v}_0(t) = u_0(t)$, $\forall i \in \Gamma$, when $t \ge T^*$.

Let $\bar{x}_i(t) = x_i(t) - x_0(t)$, $\bar{v}_i(t) = v_i(t) - v_0(t)$, $\bar{u}_i(t) = u_i(t) - u_0(t)$, $i = 0, 1, \dots, n$. By (1)–(2), the dynamics of the leader and the followers can be rewritten as

$$\dot{\bar{x}}_0(t) = 0, \qquad \dot{\bar{v}}_0(t) = 0, \qquad \bar{x}_0(0) = 0, \tag{28}$$

$$\dot{\bar{x}}_i(t) = \bar{v}_i(t), \qquad \dot{\bar{v}}_i(t) = \bar{u}_i(t), \quad i \in \Gamma. \tag{29}$$

From the previous results, when $t \ge T^*$, the control law (26) reduces to

$$\bar{u}_i = -k_1 \left[\bar{v}_i^{\,p} + k_2^p \left(\sum_{j\in N_i} a_{ij}(\bar{x}_i - \bar{x}_j) + b_i(\bar{x}_i - \bar{x}_0) \right) \right]^{\frac{2}{p}-1}, \quad i \in \Gamma. \tag{30}$$

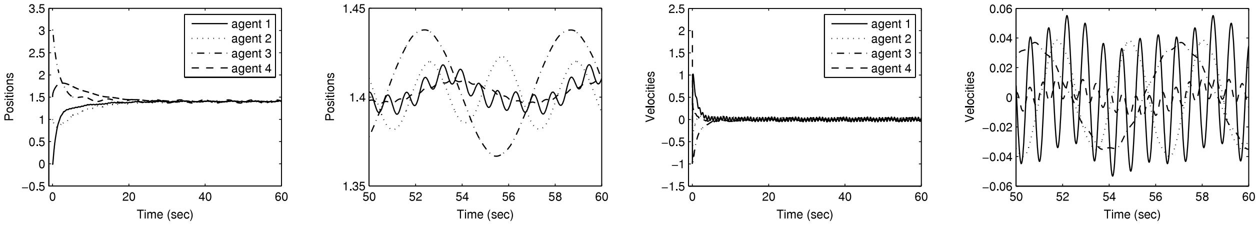
Substituting L with $L+B$, λ_2 with c and x with $\bar{x}$ into the proof procedure of Theorems 9 and 11, we can prove that under the control law (30), the consensus for systems (28)–(29) can be achieved in finite time. □

Remark 15. In Hong, Chen et al. (2008), Hong, Hu et al. (2008) and Ren (2007, 2008), the authors have studied the asymptotic consensus problem for the leader–follower systems. Here, to solve the finite-time consensus problem, distributed finite-time observers are proposed to estimate the dynamic leader's velocity. Note that an observer's control input depends on both its neighbors' states and their derivatives as in Ren (2007, 2008). The derivatives can be calculated by numerical differentiation. Moreover, note that the finite-time observer is designed for each follower to estimate the leader's velocity. Actually, for the follower with $b_i > 0$, the observer is unnecessary since in this case the follower can obtain the states v_0 and $\dot{v}_0$ directly from the leader.

Remark 16. Note that we do not prove that the control laws (25)–(26) can guarantee the boundedness of states in the interval $[0, T^*]$ since the nonlinear items are too complex. To this end, we can design the following additional control laws in the interval $[0, T^*]$: $u_i = -[\sum_{j\in N_i} a_{ij}(x_i - x_j) + b_i x_i]$, $i \in \Gamma$. By using Lyapunov function $V(t) = (1/2)x^T(L+B)x + (1/2)v^T v$, we have $\dot{V}(t) = x^T(L+B)v + v^T(u+d)$, where $u = [u_1, \dots, u_n]^T$, $d = [d_1, \dots, d_n]^T$. Substituting $u = -(L+B)x$ into $\dot{V}(t)$ and noticing $|d_i| \le l$, $\forall i \in \Gamma$, we have $\dot{V}(t) = v^T d \le (1/2)v^T v + (1/2)d^T d \le V(t) + (n/2)l^2$. By calculation, $V(t) \le \left(V(0) + \frac{n}{2}l^2\right) e^t - \frac{n}{2}l^2 \le \left(V(0) + \frac{n}{2}l^2\right) e^{T^*} - \frac{n}{2}l^2$, when $t \in [0, T^*]$. On the other hand, $|x_i(t)| \le \sqrt{x^T x} \le \sqrt{2V(t)/\lambda_{\min}(L+B)}$ and $|v_i(t)| \le \sqrt{v^T v} \le \sqrt{2V(t)}$, $\forall i \in \Gamma$. Hence, the states of i-th follower are bounded in the interval $[0, T^*]$, $\forall i \in \Gamma$. Moreover, it should be pointed out that the switching time T^* depends on the control parameters η, α and observers initial values $e_i(0)$, $i \in \Gamma$ from the proof of Theorem 14.

4. Numerical examples and simulations

Consider the leaderless multi-agent systems (1) with 4 agents. The information exchange among agents is shown in Fig. 1. The initial conditions are selected as follows: $x(0) = [0, 1, 3, 1.5]^T$, $v(0) = [-1, 0, 1, 2]^T$. The external disturbances are given as: $d_1(t) = 1.1\sin(8t-1)$, $d_2(t) = -\cos(2t)$, $d_3(t) = 0.8\sin(t) + 0.7\cos(t)$, $d_4(t) = 0.2\cos(11t-4)$, which implies l can be selected as $l = 1.1$. To have a fair comparison of the disturbance rejection performance between the finite-time protocol (3) and the linear protocol (17), the control signals are limited not to exceed 55. The control gains of protocol (3) are selected $k_3 = 1.3$, $p = 9/7$, $k_2 = 2.7$, $k_1 = 34$. The gains of linear protocol (17) are chosen as $r_3 = 1.5$, $r_2 = 3$, $r_1 = 26$. The response curves of the closed-loop system are shown in Figs. 2 and 3, respectively. It is easy to see that the finite-time protocol (3) has a faster convergence speed and a better disturbance rejection property than the linear protocol (17) from Fig. 2(a)–(d) and Fig. 3(a)–(d).



(a) Agents positions. (b) Agents positions in $t \in [50, 60]$. (c) Agents velocities. (d) Agents velocities in $t \in [50, 60]$.

Fig. 3. Responses under the linear protocol (17).

5. Conclusion

Continuous finite-time consensus algorithms have been first proposed for second-order leaderless multi-agent systems. It has been shown that the states consensus can be achieved in finite time in the absence of disturbances. In the presence of disturbances, the steady-state errors of any two agents can reach a region in finite time. For the leader–follower multi-agent systems, we have also presented a continuous finite-time consensus algorithm.

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