Received: 20 September 2021 | Revised: 23 February 2022 | Accepted: 13 April 2022

DOI: 10.1002/acs.3419

RESEARCH ARTICLE

WILEY

Adaptive finite-time stabilization with prescribed output convergence via an integrated scheme

Caiyun Liu | **Yungang Liu**

School of Control Science and Engineering, Shandong University, Jinan, People's Republic of China

Correspondence
Yungang Liu, School of Control Science and Engineering, Shandong University, Jinan 250061, People's Republic of China.
Email: lygfr@sdu.edu.cn

Funding information
National Natural Science Foundation of China, Grant/Award Numbers: 62033007, 61873146; Taishan Scholars Climbing Program of Shandong Province; Key and Development Plan of Shandong Province, Grant/Award Number: 2019JZZY010433

Summary

This article investigates global finite-time stabilization with prescribed output convergence for uncertain nonlinear systems. It is possible to achieve finite-time stabilization via continuous adaptive feedback for the systems, but the transient performance could not be prescribed for such feedback. Although funnel control is capable of tackling serious uncertainties and ensuring prescribed performance (such as convergence rate and overshoot), it alone cannot achieve (global) finite-time stabilization. To this end, we present an integrated controller to achieve the desirable stabilization, retaining the respective advantages of the above two schemes and circumventing their own disadvantages. Specifically, based on the funnel control scheme with suitable design parameters, the system output converges from any initial value to an arbitrarily adjustable region with the prescribed convergence rate before a pregiven time instant, while the system states keep bounded. The adaptive controller, in conjunction with a time-dependent function, ensures that the system output and states converge to zero in finite time, and particularly the system output with prescribed convergence rate evolves within the prespecified envelope. The effectiveness of the proposed scheme is illustrated by two simulation examples.

KEYWORDS

adaptive control, finite-time stabilization, funnel control, integrated control scheme, prescribed output convergence, uncertain nonlinear systems.

1 | INTRODUCTION

Nonlinear control has been arguably the most ubiquitous in the control community, whether in theory or in practice.[1-5] In 1970's and 1980's, its basic issues, such as controllability, observability, stabilizability, and robustness, have attracted extensive attention, but rarely underlining system performance. The research focus later turned to the construction/realization of various nonlinear controls to achieve stabilization, tracking, noise rejection and adaptive regulation, yet lacking the systemic treatment on system performance. In the last years, the controls ensuring distinctive performance (e.g., prescribed convergence rate, desired overshoot, finite-/fixed-time stability) have been typically explored to meet the urgent needs from practical applications (e.g., safety, efficiency and precision).

Finite-time control, owning the advantages of precision and rapidness, has been extensively investigated for classes of nonlinear systems (see e.g., References 6-9). Early works (e.g., References 6 and 7) considered finite-time stabilization for the nonlinear systems without serious uncertainties, by introducing low order terms into the controllers. Note that the uncertainty is ubiquitous in practice and adaptive feedback has powerful ability in compensating uncertainties.

Many works (e.g., References 8-10) utilized continuous adaptive feedback to achieve finite-time stabilization for nonlinear systems with serious uncertainties. Specifically,[10] investigated the switched nonlinear systems, which merely ensures the convergence to a small neighborhood of the origin. To achieve finite-time convergence (to the origin),[8] proposed the typical analysis approach for the systems with smooth nonlinearities, in which the term with serious uncertainty has higher power than its stabilizing term in Lyapunov analysis. Based on the analysis approach, the system nonlinearities are extended to be Hölder continuous in Reference 9. Whereas the adaptive strategies probably make that the change of the system states is dramatic even near the settling time. Such nature makes the system lack safety/reliability, and renders the scope of applications of adaptive finite-time stability rather limited.

Recognizing the limitation of adaptive feedback, some works have begun to take prescribed transient performance into account. For instance, work,[11] with the premise that the uncertainties belong to a known compact set, not only achieved finite-time stabilization but also guaranteed prescribed transient performance (e.g., convergence rate). For nonlinear systems with serious uncertainties, the adaptive schemes almost rule out the possibility of prescribed performance. To guarantee prescribed performance, various controls based on funnel, barrier Lyapunov function and performance-induced transformation were presented,[12-20] which force some specific system signals to evolve within the prescribed envelopes and achieve desirable performances by suitably selecting parameters. So far funnel control has realized global stabilization with prescribed convergence rate and global practical tracking with prescribed arrival time (e.g., References 14-16), however it cannot achieve finite-time stabilization. Moreover, controls based on barrier Lyapunov function and performance-induced transformation have achieved finite-time stabilization with constant output constraint and practical tracking with prescribed maximum overshoot (e.g., References 13 and 20), but they only obtained the semiglobal results, namely, design parameters therein depend on the initial data.

Both (global) finite-time stability and prescribed transient performance are important performance indices, which are needed simultaneously in many practical applications with high requirements (e.g., ship steering control, missile guidance and photovoltaic power). So far numerous continuous schemes have achieved the sole index,[8,9,11] but any one is powerless to guarantee the two indices at the same time for uncertain nonlinear systems. Although discontinuous feedback (such as switching adaptive feedback and sliding mode control; see References 21-24) can establish the two indices, the use of discontinuous strategy would cause serious chattering behavior. Within mind the current status, one should strive to strengthen continuous adaptive feedback to achieve (global) finite-time stabilization for uncertain nonlinear systems and meanwhile ensure transient performance of some specific system signals by integrating other powerful schemes.

This article focuses on global finite-time stabilization with prescribed output convergence for a class of nonlinear systems with serious uncertainties (see system (1) and Assumption 1 below). Although continuous adaptive control and funnel control act as effective schemes for finite-time stabilization and prescribed performance, respectively, either one cannot achieve the aforementioned objective alone. To retain the respective advantages of the two schemes and meanwhile circumvent their own disadvantages, we pursue an integrated controller to realize the desirable stabilization. Whereas the realization, due to the integration and high performance requirements, would encounter at least two difficulties: (i) The integration renders the controller inevitably discontinuous, and consequently how to guarantee the effectiveness of the controller (typically at the discontinuous point) and meanwhile reduce chattering behavior as possible would be a crucial difficulty in control design. (ii) Since the continuous adaptive scheme hardly guarantees the desirable transient performance, how to make the system output stay in the predefined region and have prescribed convergence rate would be another crucial difficulty. To overcome the crucial difficulties, two time-dependent functions, which characterize prescribed performance, are incorporated into the controller. First, via funnel control scheme with a suitable time-dependent function, the system output converges from any initial value to an arbitrarily adjustable region with the prescribed convergence rate before a pregiven time instant, and meanwhile the system states keep bounded. By means of another time-dependent function, the adaptive controller then ensures that the system states and the system output converge to zero in finite time, and particularly the system output with prescribed convergence rate evolves within the prespecified envelope.

We would like to outline the main contributions of the article: (i) *System in question is fairly general, compared with the relevant works (e.g., References 11 and 20).* In fact, no essential result, beyond the generality of system (1) under Assumption 1, has been found, to the authors' knowledge. (ii) *The integrated controller is discontinuous only at a pregiven time instant.* Therefore, serious chattering behavior is circumvented in contrast to switching adaptive feedback and sliding mode control (e.g., References 21-24). (iii) *Global finite-time stabilization and prescribed output convergence are realized simultaneously.* But the related schemes can merely achieve sole/degenerated performance, such as, global finite-time stabilization,[8] global stabilization with prescribed convergence rate[15] and semiglobal finite-time stabilization with constant output constraint.[20]

The remainder of this article is organized as follows. Section 2 formulates the system and the control objective. Section 3 collects some notations and preliminaries. Section 4 specifies the design procedure of the integrated controller. In Section 5, the main result of this article is presented. Two examples are provided in Section 6 to illustrate the effectiveness of the proposed scheme. Section 7 gives some concluding remarks. Appendix gathers the proof of three important relations arising in Section 4.

2 | PROBLEM FORMULATION

In this article, we pursue more sophisticated finite-time stabilization of the following uncertain nonlinear system:

$$\begin{cases} \dot{x}_i = x_{i+1} + f_i(t, x_{[i]}), \quad i = 1, \dots, n-1, \\ \dot{x}_n = u + f_n(t, x), \\ y = x_1, \end{cases} \tag{1}$$

where $x = [x_1, \dots, x_n]^{\mathrm{T}} \in \mathbf{R}^n$ is the system state with the initial condition $x(0) = x_0$ and $x_{[i]} = [x_1, \dots, x_i]^{\mathrm{T}}$; $u \in \mathbf{R}$ and $y \in \mathbf{R}$ are the control input and system output, respectively; f_i's are continuous functions satisfying $f_i(t, 0) \equiv 0$, termed the system nonlinearities. Remarkably, system (1) covers numerous practical models, such as tunnel-diode circuit,[1] ship steering model[2] and robotic manipulator,[3] and hence has been constantly studied during the past few decades (see e.g., References 4 and 5).

The control objective of this article is to achieve global finite-time stabilization with prescribed output convergence for system (1). Specifically, a controller will be constructed such that the system output enters an arbitrarily adjustable region before a pregiven time instant and thereafter converges to zero in finite time with the prescribed convergence rate, while the system states converge to zero in finite time. For later development, the objective is formulated more accurately by the following three respects:

(i) *Predefined reachability*: $|y(t)| < \varepsilon, \ \forall t \geq T_{\mathrm{p}}$, where T_{p} is pregiven and ε is an arbitrarily adjustable positive constant.
(ii) *Prescribed convergence rate*: The system output y evolves within the prespecified envelope, whose convergence rate can be arbitrarily assigned.
(iii) *Finite-time convergence*: $\lim_{t \to T(x_0)} x_i(t) = 0$ and $x_i(t) = 0, \ \forall t \geq T, \ i = 1, \dots, n$, where $T(x_0)$ is the settling time function.

To achieve the objective, we impose the following assumption on the system nonlinearities, which refers to assumption 1 in Reference 9.

Assumption 1 (9). There exist known non-negative continuous functions $\bar{f}_i(x_{[i]})$, $i = 1, \dots, n$ with $\bar{f}_i(0) = 0$ such that

$$|f_i(t, x_{[i]})| \leq \theta \bar{f}_i\left(x_{[i]}\right) \sum_{j=1}^{i} |x_j|^{\frac{r_i+\tau}{r_j}},$$

where θ is an unknown positive constant, $\tau \in (-\frac{1}{n}, 0)$ is a fractional number with even numerator and odd denominator and r_j's are positive constants satisfying $r_1 = 1$, $r_j = r_{j-1} + \tau, j = 2, \dots, n$.

Assumption 1, which is mild in comparison with References 8,11,20, indicates system (1) can weaken to be Hölder continuous and its bounding system is homogeneous of negative degree τ if neglecting growth rate functions $\bar{f}_i$'s. Actually, Assumption 1 is quite essential, since the linear or high-order growth can be converted to Assumption 1 by incorporating redundant factors into growth rate functions. In what follows, we shall strive to substantially improve system performance of adaptive finite-time control instead of further relaxing the growth condition.

3 | NOTATIONS AND PRELIMINARIES

Throughout this article, the following notations will be frequently used. We use $\mathbf{R}^{+}$ to denote the set of all non-negative real numbers and $\mathbf{R}_{\geq t_0}$ the set of all real numbers not less than t_0. Let $\mathbf{R}_{\mathrm{odd}}^{\leq 1} = \{x \in \mathbf{R} \mid 0 < x \leq 1$ and

$x = \frac{p}{q}$ withpositiveoddintegers p and q}. For $x \in \mathbf{R}^n$, we use $x_{[i]}$ to denote the column vector consisting of the first i elements of x, that is, $x_{[i]} = [x_1, \dots, x_i]^{\mathrm{T}}$. A continuous function $f : \mathbf{R}^+ \to \mathbf{R}^+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $f(0) = 0$, and belong to class $\mathcal{K}_\infty$ if it belongs to class $\mathcal{K}$ and satisfies $\lim_{x\to+\infty} f(x) = +\infty$.

Consider the following nonautonomous nonlinear system:

$$\dot{x} = f(t, x), \tag{2}$$

where $x \in \mathbf{R}^n$ is the system state with the initial condition $t_0 \geq 0$ and $x(t_0) = x_0$, and $f : \mathbf{R}_{\geq t_0} \times \mathbf{R}^n \to \mathbf{R}^n$ is a continuous function with $f(t, 0) \equiv 0$.

For system (2), let's introduce the concept and Lyapunov-based condition of finite-time stability, which can be found in References 25-27. The proof of Theorem 1 is similar to that in Reference 27 and hence omitted here.

Definition 1. The origin of system (2) is

(i) stable if for each $\epsilon > 0$, there is $\delta = \delta(\epsilon, t_0) > 0$ such that for any $x_0 \in \mathcal{U} \triangleq \{\|x\| < \delta\}$, there holds $\|x(t)\| < \epsilon$, $\forall t \geq t_0 \geq 0$.
(ii) finite-time stable if it is stable and there exists a domain $\mathcal{D} \subseteq \mathbf{R}^n$ of the origin and a function $T : \mathbf{R}^+ \times \mathcal{D} \to \mathbf{R}^+$ with $T(\cdot, 0) \equiv 0$ such that, for any $(t_0, x_0) \in \mathbf{R}^+ \times \mathcal{D}$, there hold $x(t) \neq 0$, $\forall t_0 \leq t < T(t_0, x_0)$, $\lim_{t\to T(t_0,x_0)} x(t) = 0$ and $x(t) = 0$, $\forall t \geq T(t_0, x_0)$. The function $T(\cdot)$ is called the settling-time function. Furthermore, if $\mathcal{D} = \mathbf{R}^n$, then the origin is global finite-time stable.

Theorem 1. *For a domain $\mathcal{D} \subseteq \mathbf{R}^n$ of the origin, if there exists a continuously differentiable function $V : \mathbf{R}_{\geq t_0} \times \mathcal{D} \to \mathbf{R}^+$ and a continuous positive definite function $r : \mathbf{R}^+ \to \mathbf{R}^+$ such that for any $x \in \mathcal{D}$ and some $\sigma \in \mathbf{R}^+$,*

$$\begin{cases} \alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|), \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -r(V(t, x)), \\ \int_0^\sigma \frac{\mathrm{d}z}{r(z)} < +\infty, \end{cases}$$

where α_1 and α_2 belong to class $\mathcal{K}$, then the origin of system (2) is finite-time stable with the settling-time function $T : \mathbf{R}^+ \times \mathcal{D} \to \mathbf{R}^+$ satisfying

$$T(t_0, x_0) \leq \int_0^{V(t_0,x_0)} \frac{\mathrm{d}z}{r(z)}, \quad \forall t \geq t_0, \quad \forall x \in \mathcal{D}.$$

Furthermore, if $\mathcal{D} = \mathbf{R}^n$, and α_1 and α_2 belong to class $\mathcal{K}_\infty$, then the origin is globally finite-time stable. Particularly, if we choose $r(z) = cz^\alpha$, then the settling-time function satisfies

$$T(t_0, x_0) \leq \frac{1}{c(1-\alpha)} V^{1-\alpha}(t_0, x_0),$$

where $c > 0$ and $0 < \alpha < 1$ are constants.

We next introduce funnel control method which is capable of tackling unknowns and achieving prescribed performance. Roughly speaking, the funnel controller makes some specific system signal evolve within the desirable envelope of the following form:

$$\mathcal{F}_\rho \triangleq \left\{ (t, x) \in \mathbf{R}_{\geq t_0} \times \mathbf{R}^n \middle| \rho(t)\|x\| < 1 \right\},$$

where x denotes the specific system signal (e.g., the tracking error and the system state), and then prescribed performance (e.g., convergence rate and overshoot) can be achieved by choosing an appropriate function $\rho(t)$.

To understand funnel control method intuitively, we present a simple example to exhibit its powerful capability (the rigorous proof refers to Reference 16).

Consider the following system:

$$\dot{x} = u + f(x,\theta), \tag{3}$$

where unknown function $f(x,\theta)$ with $f(0,\theta) \equiv 0$ is continuously differentiable and continuous in the first argument and the second argument, respectively.

We design funnel controller as

$$u = -\frac{x}{1-(\rho(t)x)^2}, \qquad \rho(t) = \frac{t}{\varepsilon T}.$$

Note by the continuous differentiability of $f(x,\theta)$ in x and $f(0,\theta) = 0$ that $f(x,\theta) \le |x|\overline{f}(x,\theta)$ for a non-negative continuous function $\overline{f}(\cdot)$. Then, the derivative of $(\rho(t)x)^2$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((\rho(t)x)^2\right) = 2\rho(t)x\left(\frac{x}{\varepsilon T} + \rho(t)(u+f(\cdot))\right) \le \frac{2\rho(t)x^2}{\varepsilon T} - \frac{2(\rho(t)x)^2}{1-(\rho(t)x)^2} + 2(\rho(t)x)^2\overline{f}(x,\theta).$$

From the fact that on the interval of existence, there holds $|\rho(t)x| < 1$, we see that x is bounded, and in turn $\frac{2\rho(t)x^2}{\varepsilon T} + 2(\rho(t)x)^2\overline{f}(x,\theta) \le \frac{2}{\varepsilon T}|x| + 2\overline{f}(x,\theta) \le c_1$ for a positive constant c_1. Thus, on the interval of existence, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((\rho(t)x)^2\right) \le -\frac{2(\rho(t)x)^2}{1-(\rho(t)x)^2} + c_1.$$

By this, and noting that $\frac{1}{1-(\rho(t)x)^2}$ is sufficiently large when $|\rho(t)x|$ is sufficiently close to 1, we have $\sup_{t\ge 0}\rho(t)|x| < 1$ (the rigorous proof can be completed by that of theorem 2.8 in Reference 16). This, together with the definition of $\rho(t)$, implies that $|x(t)| < \varepsilon$, $\forall t \ge T$ and $\lim_{t\to\infty} x(t) = 0$.

Remarkably, the continuous adaptive controllers based on dynamic compensation are rather difficult to globally stabilize system (3) mainly due to the presence of unknown nonlinearity $f(\cdot)$, which exhibits the notable advantages of funnel control scheme in the context of prescribed performance and unknowns.

We end this section with the following lemmas which are used frequently in later development and can be found in References 9 and 28.

Lemma 1 (28). *For any positive numbers a, b, and c, there holds*

$$|x|^a|y|^b \le c|x|^{a+b} + \frac{b}{(a+b)}\left(\frac{a}{c(a+b)}\right)^{\frac{a}{b}}|y|^{a+b}.$$

Lemma 2 (9). *For a positive constant p, there hold*

$$\begin{cases} (|x_1| + \cdots + |x_n|)^p \le n^{p-1}(|x_1|^p + \cdots + |x_n|^p), & \text{if} \quad p \ge 1; \\ (|x_1| + \cdots + |x_n|)^p \le |x_1|^p + \cdots + |x_n|^p, & \text{if} \quad 0 < p \le 1; \\ |x^p - y^p| \le 2^{1-p}|x-y|^p, \quad \text{if} \quad p \in \mathbf{R}_{\text{odd}}^{\le 1}. \end{cases}$$

Lemma 3 (9). *If continuous function $f(\cdot)$ is monotone on $[a,b]$ and satisfies $f(a) = 0$, then there holds*

$$\left|\int_a^b f(x)\mathrm{d}x\right| \le |f(b)| \cdot |b-a|.$$

4 | CONTROLLER DESIGN

This section aims to integrate funnel and adaptive control schemes to achieve the aforementioned control objective for system (1). The integration makes the global feedback controller to be constructed own the following form:

$$u = \begin{cases} u_{\rm f}, & t \in [0, T_{\rm p}), \\ u_{\rm a}, & t \in [T_{\rm p}, \infty). \end{cases} \tag{4}$$

Under the controller, the convergence of system (1) naturally contains two stages. In the first stage, controller $u_{\rm f}$ guarantees that the system output converges from any initial value to an arbitrarily adjustable region before the pregiven time $T_{\rm p}$ while the system states keep bounded. Subsequently, controller $u_{\rm a}$ makes the system output and the system states converge to zero in finite time. Particularly, the convergence rate of the system output during the entire process can be arbitrarily assigned. It is worth pointing out that the controller is discontinuous only at the pregiven time $T_{\rm p}$, and hence serious chattering behavior probably arising in switching adaptive feedback and sliding mode control is circumvented. However, how to guarantee the effectiveness of controller (4) (typically at the pregiven time $T_{\rm p}$) would be a crucial difficulty. Moreover, how to design controller $u_{\rm a}$ would be another crucial difficulty, since the existing continuous schemes could hardly ensure simultaneously desirable prescribed performance and finite-time convergence (to zero) for uncertain nonlinear systems.

Remark 1. Funnel scheme, as known, not only can tackle serious uncertainties, but also can ensure prescribed performance, such as convergence rate and overshoot. However, it alone cannot achieve (global) finite-time stabilization (see e.g., References 12 and 15). If there is no concern for prescribed high performance, the adaptive scheme can be utilized to present continuous feedback to achieve (global) finite-time stabilization for uncertain nonlinear systems (see e.g., References 8 and 9). To retain the respective advantages of the two schemes and circumvent their own disadvantages, we pursue integrated controller (4) to achieve the objective of this article.

Control design proceeds with introducing the following two design functions of time:

$$\rho_1(t) = \frac{\omega_1}{T_{\rm p}} t, \quad \rho_2(t) = \frac{\varepsilon}{2} e^{-\omega_2(t-T_{\rm p})} + \frac{\varepsilon}{2}, \tag{5}$$

where ω_1 and ω_2 are design parameters satisfying $\omega_1 \geq \frac{2}{\varepsilon}$ and $\omega_2 \geq 0$.

We then design funnel controller $u_{\rm f}$ as

$$u_{\rm f} = \varphi_n(t, \eta_n), \tag{6}$$

where $\varphi_n(\cdot)$ is recursively generated by

$$\begin{cases} \varphi_1(t, \eta_1) = -\frac{\eta_1}{1-(\rho_1\eta_1)^2}, & \eta_1 = x_1, \\ \varphi_i(t, \eta_i) = -\frac{\eta_i}{1-(\rho_1\eta_i)^2}, & \eta_i = x_i - \varphi_{i-1}(\cdot), \quad i = 2, \dots, n. \end{cases} \tag{7}$$

We next design adaptive controller $u_{\rm a}$ as

$$u_{\rm a} = \alpha_n(\rho_2, x, \hat{\Theta}), \quad \dot{\hat{\Theta}} = \gamma_n(\rho_2, x, \hat{\Theta}), \quad \hat{\Theta}(T_{\rm p}) > 0, \tag{8}$$

where α_n is recursively defined by

$$\begin{cases} \begin{cases} \alpha_1(\rho_2, x_1, \hat{\Theta}) = -z_1^{r_2}\phi_1(\rho_2, x_1, \hat{\Theta}), \\ z_1 = \frac{x_1}{\sqrt{\rho_2^2 - x_1^2}}, \end{cases} \\ \begin{cases} \alpha_i(\rho_2, x_{[i]}, \hat{\Theta}) = -z_i^{r_{i+1}}\phi_i(\rho_2, x_{[i]}, \hat{\Theta}), \\ z_i = x_i^{\frac{1}{r_i}} - \alpha_{i-1}^{\frac{1}{r_i}}(\rho_2, x_{[i-1]}, \hat{\Theta}), \quad i = 2, \dots, n, \end{cases} \end{cases} \tag{9}$$

r_i's are as in Assumption 1 with $r_{n+1} = r_n + \tau$, $\gamma_n(\cdot)$ and $\phi_i(\cdot)$'s are to-be-determined design functions, continuous and smooth, respectively, and particularly $\gamma_n(\cdot) \geq 0$, $\gamma_n(\rho_2, 0, \hat{\Theta}) \equiv 0$ and $\phi_i(\cdot) > 0$.

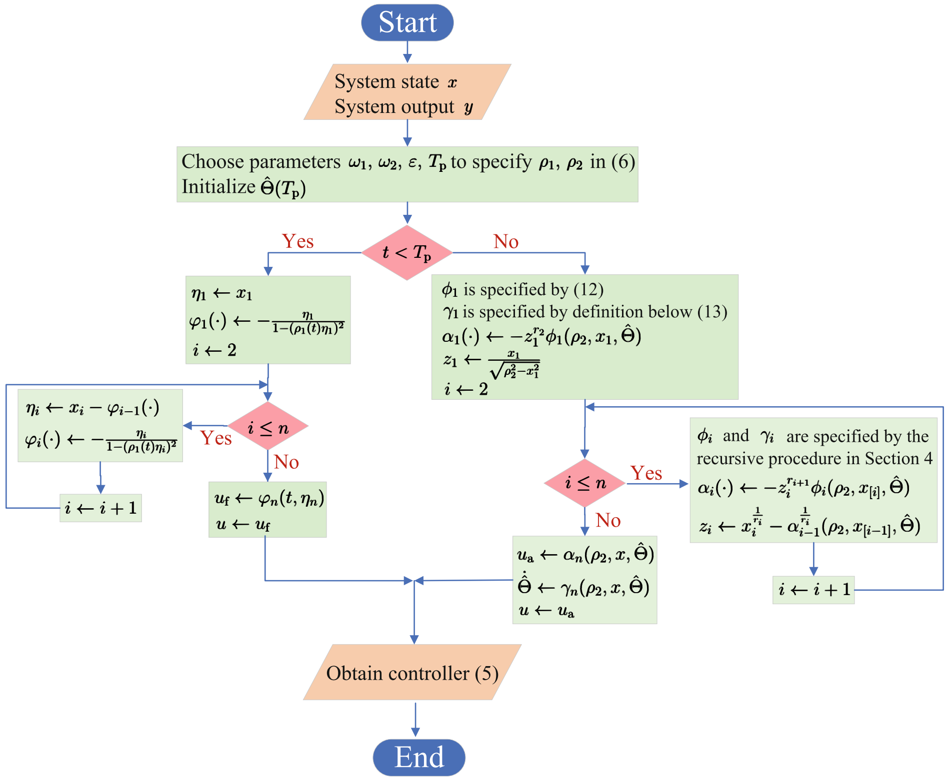

FIGURE 1 Flow diagram of the specification of the integrated controller (4)

Proposition 1. *For system* (1) *under Assumption 1, the integrated controller* (4) *with* (5)–(9) *guarantees that, for any initial value $x_0 \in \mathbf{R}^n$, all system signals are bounded on $[0, T_{\rm p} + t_f)$, where t_f is certain small positive constant.*

Proof. Similar to the proof of theorem 1 in Reference 15 (replacing $\mathcal{N}(r_i)$ and k_i used in Reference 15 with $-r_i$ and 1), we can prove signals x_i's, η_i's, φ_i's and $u_{\rm f}$ are bounded on $[0, T_{\rm p})$ and especially $|x_1(t)| < \frac{1}{\rho_1(t)}$ on $[0, T_{\rm p})$. Then, by continuity of $\rho_1(t)$ and $x_i(t)$'s, the system states x_i's are also bounded on closed interval $[0, T_{\rm p}]$ and particularly $|x_1(T_{\rm p})| \le \frac{1}{\rho_1(T_{\rm p})}$. With the specified $\omega_1 \ge \frac{2}{\varepsilon}$ in mind, we have $\frac{1}{\rho_1(T_{\rm p})} \le \frac{\varepsilon}{2} < \varepsilon = \rho_2(T_{\rm p})$. Thus, it can be derived that $|x_1(T_{\rm p})| < \rho_2(T_{\rm p})$, which implies that $z_1(T_{\rm p})$ is bounded.

With the bounded system signals $z_1(T_{\rm p})$, $x(T_{\rm p})$, $\hat{\Theta}(T_{\rm p})$ and $u_{\rm a}(T_{\rm p})$, which act as the initial values starting $T_{\rm p}$, by the continuity of its vector field, the resulting closed-loop system has bounded solutions on certain small time period starting from $T_{\rm p}$, representing the period by $[T_{\rm p}, T_{\rm p} + t_f)$. ∎

To clearly show how to implement the proposed scheme, we provide flow diagram as shown in Figure 1.

Remark 2. Typically, the continuous adaptive scheme is hard to achieve prescribed output convergence for uncertain nonlinear systems. To eliminate the disadvantage, we introduce the delicate transformation $z_1 = x_1\left(\rho_2^2 - x_1^2\right)^{-\frac{1}{2}}$, by means of which, the realization of the prescribed output convergence is converted into the proof of the boundedness and finite-time convergence of z_1. This treatment largely facilitates the control design and performance analysis later on. Recently, some relevant works (see e.g., References 13 and 17) adopted the intermediate state similar to z_1 in (9), where the initial value $x_1(t_0)$ instead of $x_1(T_{\rm p})$ was directly utilized to ensure the rationality of z_1. In this sense, design parameters depend on the initial data, namely, the existing strategies merely achieve semiglobal stabilization. Whereas our new scheme depicted above makes system state $x_1(t)$ suitably small in a pregiven time, and hence is capable of realizing global finite-time stabilization (see Proposition 1 for details).

Remark 3. Let us see why we choose time-dependent functions as in (5) and how to choose parameters in these functions to ensure feasibility of the proposed scheme and the prescribed performance. From (7) and (9), we see that the proposed scheme builds on conditions $|\rho_1(t)x_1(t)| < 1$ for $t < T_{\rm p}$ and $|x_1(t)| < \rho_2(t)$ for $t \ge T_{\rm p}$ which are used to ensure prescribed performance. The conditions, as well as predefined reachability, entail $|\rho_1(0)x_1(0)| < 1$ for any $x_1(0)$, $|x_1(T_p)| < \rho_2(T_p)$ and $\rho_2(t) \le \rho_2(T_p) \le \varepsilon$, for $t \ge T_p$. As a result, it is required that $\rho_1(0) = 0$, $\frac{1}{\rho_1(T_p)} < \rho_2(T_p) \le \varepsilon$ and $\dot{\rho}_2(t) \le 0$ for $t \ge T_p$. With this, we choose functions ρ_i's as in (5) with $\omega_1 \ge \frac{2}{\varepsilon}$ and $\omega_2 \ge 0$. By the functions and (7) and (9), the prescribed performance is transformed into the boundedness of φ_1 and z_1, which implies $|y(t)| < \frac{T_p}{\omega_1 t}$, $\forall t \in [0, T_p)$ and $|y(t)| < \frac{\varepsilon}{2}e^{-\omega_2(t-T_p)} + \frac{\varepsilon}{2}$, $\forall t \in [T_p, \infty)$. Explicitly, ω_1 and ω_2 characterize the convergence rate of the system output and parameter ε represents the reachability level. As such, the prescribed output convergence can be established by selecting appropriate parameters in ρ_i's. Particularly, the selection is sketched as follows. Suppose that T_p and ε are given and the desirable convergence rate on $[0, T_p)$ is $\frac{1}{t}$. Then, we choose $\omega_1 = \max\left\{\frac{2}{\varepsilon}, T_p\right\}$, which means $\frac{1}{\rho_1(T_p)} \le \frac{\varepsilon}{2} < \varepsilon = \rho_2(T_p)$ and $|y(t)| \le \frac{1}{\rho_1(t)} \le \frac{1}{t}$ on $[0, T_p)$. This, together with $|y(t)| < \rho_2(t)$ and the nonincreasing property of $\rho_2(t)$ on $[T_p, \infty)$, implies the predefined reachability of the system output is realized. Moreover, the prescribed convergence rate of the system output on $[T_p, \infty)$ can be ensured by suitably choosing ω_2.

For later development, we introduce functions

$$\begin{cases} W_1(\rho_2, x_1) = \frac{1}{2}z_1^2, \\ W_i(\rho_2, x_{[i]}, \hat{\Theta}) = \int_{\alpha_{i-1}}^{x_i}\left(s^{\frac{1}{r_i}} - \alpha_{i-1}^{\frac{1}{r_i}}\right)^{2-r_i} \mathrm{d}s, \qquad i = 2, \ldots, n, \end{cases} \tag{10}$$

whose basic properties are characterized by the following proposition (its proof is similar to that of proposition B.1 in Reference 28, and hence omitted here).

Proposition 2. *Functions W_i's ($i = 2, \ldots, n$) are continuously differentiable, and satisfy*

$$2^{\frac{(r_i-1)(2-r_i)}{r_i}-1} r_i |x_i - \alpha_{i-1}|^{\frac{2}{r_i}} \le W_i(\cdot) \le 2^{1-r_i} z_i^2,$$

and

$$\begin{cases} \dfrac{\partial W_i}{\partial x_i} = z_i^{2-r_i}, \\ \frac{\partial W_i}{\partial \chi_j} = -(2-r_i)\int_{\alpha_{i-1}}^{x_i}\left(s^{\frac{1}{r_i}} - \alpha_{i-1}^{\frac{1}{r_i}}\right)^{1-r_i} \mathrm{d}s \frac{\partial \alpha_{i-1}^{\frac{1}{r_i}}}{\partial \chi_j}, \end{cases}$$

where χ_j denotes the jth element of $\chi = [x_1, \ldots, x_{i-1}, \hat{\Theta}, \rho_2]^{\mathrm{T}} \in \mathbf{R}^{i+1}$.

Specification of design functions $\boldsymbol{\phi_i}$'s and $\boldsymbol{\gamma_n}$. In what follows, we develop a Lyapunov-based recursive procedure to specify design functions ϕ_i's and γ_n involved in u_a of (8) to complete the control design, and to obtain the Lyapunov function candidate for the convenience of the subsequent analysis.

Step 1. Let $V_1 = W_1 + \frac{1}{2}\tilde{\Theta}^2$ for this step, where $\tilde{\Theta} = \Theta - \hat{\Theta}$ with $\hat{\Theta}$ being the estimation of $\Theta = \max\left\{\theta, \theta^{\frac{2+\tau}{2-r_n}}, \theta^{2+\tau}\right\}$. After some elaborated calculations (see Appendix for details), we obtain

$$\dot{V}_1 \le z_1 h(\rho_2, x_1)(x_2 - \alpha_1) + z_1 h(\rho_2, x_1)\alpha_1 + z_1^{2+\tau} h(\rho_2, x_1)\theta \bar{f}_1(x_1)(\rho_2^2 - x_1^2)^{\frac{r_2}{2}} - \frac{\dot{\rho}_2}{\rho_2} h(\rho_2, x_1)(\rho_2^2 - x_1^2)^{\frac{1}{2}} z_1^2 - \tilde{\Theta}\dot{\hat{\Theta}}, \tag{11}$$

where $h(\cdot)$ is a known smooth positive function which is characterized in the detailed proof of (11) in Appendix.

We then define

$$\phi_1(\cdot) = \hat{\Theta}\tilde{f}_1(x_1)(\rho_2^2 - x_1^2)^{\frac{r_2}{2}} + \frac{c_1 + n - 1}{h(\rho_2, x_1)} - \frac{\dot{\rho}_2}{\rho_2}(\rho_2^2 - x_1^2)^{\frac{1}{2}}(1 + z_1^2)^{\frac{1-r_2}{2}}, \tag{12}$$

where c_1 is a positive design parameter and $\tilde{f}_1(\cdot) \ge \bar{f}_1(\cdot)$ is a known smooth non-negative function. By this and the definition of α_1 in (9), and noting $\hat{\Theta} + \tilde{\Theta} = \Theta \ge \theta$, we have

$$\dot{V}_1 \le -(c_1 + n - 1)z_1^{2+\tau} + z_1 h(\rho_2, x_1)(x_2 - \alpha_1) - \tilde{\Theta}\left(\dot{\hat{\Theta}} - \gamma_1(\rho_2, x_1)\right), \tag{13}$$

where $\gamma_1(\rho_2, x_1) = h(\rho_2, x_1)\bar{f}_1(x_1)(\rho_2^2 - x_1^2)^{\frac{r_2}{2}} z_1^{2+\tau} =: \beta_1^{\gamma}(\rho_2, x_1) z_1^{2+\tau}$.

Step 2. Let $V_2 = V_1 + W_2$ for this step. It can be deduced that the time derivative of V_2 along system (1) satisfies (see Appendix for details)

$$\dot{V}_2 \leq -(c_1+n-2)\sum_{i=1}^{2} z_i^{2+\tau} + z_2^{2-r_2}(x_3-\alpha_2) + z_2^{2-r_2}\alpha_2 - \left(\tilde{\Theta} - \frac{\partial W_2}{\partial \hat{\Theta}}\right)\left(\dot{\hat{\Theta}} - \gamma_1(\cdot) - \beta_2^{\gamma}(\rho_2, x_{[2]}, \hat{\Theta})z_2^{2+\tau}\right)$$
$$+ \beta_2^{\phi}(\rho_2, x_{[2]}, \hat{\Theta})z_2^{2+\tau} + (c_1+n-2)z_2^{2+\tau}, \tag{14}$$

where $\beta_2^{\phi}(\cdot)$ is a known smooth non-negative function and $\beta_2^{\gamma}(\cdot)$ is a known continuous non-negative function with $\beta_2^{\gamma}(\rho_2, 0, \hat{\Theta}) \equiv 0$, which are characterized in the detailed proof of (14) in Appendix.

Choose

$$\phi_2(\rho_2, x_{[2]}, \hat{\Theta}) = c_1 + n - 2 + \beta_2^{\phi}(\rho_2, x_{[2]}, \hat{\Theta}).$$

Then, (14) is changed into the following form:

$$\dot{V}_2 \leq -(c_1+n-2)\sum_{i=1}^{2} z_i^{2+\tau} + z_2^{2-r_2}(x_3-\alpha_2) - \left(\tilde{\Theta} - \frac{\partial W_2}{\partial \hat{\Theta}}\right)\left(\dot{\hat{\Theta}} - \gamma_2(\rho_2, x_{[2]}, \hat{\Theta})\right),$$

where $\gamma_2(\cdot) = \gamma_1(\cdot) + \beta_2^{\gamma}(\rho_2, x_{[2]}, \hat{\Theta})z_2^{2+\tau}$.

Recursive design step *i* (*i* = 3, …, *n*). Suppose that the first $i-1$ steps have been achieved, that is, smooth non-negative functions $\phi_j(\rho_2, x_{[j]}, \hat{\Theta})$ and continuous non-negative functions $\gamma_j(\rho_2, x_{[j]}, \hat{\Theta}) = \beta_j^{\gamma}(\rho_2, x_{[j]}, \hat{\Theta})z_j^{2+\tau}$ with $\beta_j^{\gamma}(\rho_2, 0, \hat{\Theta}) \equiv 0$, where $j = 1, \ldots, i-1$, have been found such that

$$\dot{V}_{i-1} \leq -(c_1+n-i+1)\sum_{j=1}^{i-1} z_j^{2+\tau} + z_{i-1}^{2-r_{i-1}}(x_i-\alpha_{i-1}) - \left(\tilde{\Theta} - \sum_{j=2}^{i-1}\frac{\partial W_j}{\partial \hat{\Theta}}\right)\left(\dot{\hat{\Theta}} - \gamma_{i-1}(\rho_2, x_{[i-1]}, \hat{\Theta})\right). \tag{15}$$

Let $V_i = V_{i-1} + W_i$. By (15), the time derivative of V_i can be computed as (see Appendix for details)

$$\dot{V}_i \leq -(c_1+n-i)\sum_{j=1}^{i} z_j^{2+\tau} + z_i^{2-r_i}(x_{i+1}-\alpha_i) + z_i^{2-r_i}\alpha_i + \left(\beta_i^{\phi}(\rho_2, x_{[i]}, \hat{\Theta}) + c_1 + n - i\right)z_i^{2+\tau}$$
$$-\left(\tilde{\Theta} - \sum_{j=2}^{i}\frac{\partial W_j}{\partial \hat{\Theta}}\right)\left(\dot{\hat{\Theta}} - \gamma_{i-1}(\cdot) - \beta_i^{\gamma}(\rho_2, x_{[i]}, \hat{\Theta})z_i^{2+\tau}\right), \tag{16}$$

where $\beta_i^{\phi}(\cdot)$ is a known smooth non-negative function and $\beta_i^{\gamma}(\cdot)$ is a known continuous non-negative function with $\beta_i^{\gamma}(\rho_2, 0, \hat{\Theta}) \equiv 0$, which are characterized in the detailed proof of (16) in Appendix.

Then, by defining

$$\phi_i(\rho_2, x_{[i]}, \hat{\Theta}) = c_1 + n - i + \beta_i^{\phi}(\rho_2, x_{[i]}, \hat{\Theta}),$$

we obtain

$$\dot{V}_i \leq -(c_1+n-i)\sum_{j=1}^{i} z_j^{2+\tau} + z_i^{2-r_i}(x_{i+1}-\alpha_i) - \left(\tilde{\Theta} - \sum_{j=2}^{i}\frac{\partial W_j}{\partial \hat{\Theta}}\right)\left(\dot{\hat{\Theta}} - \gamma_i(\rho_2, x_{[i]}, \hat{\Theta})\right),$$

where $\gamma_i(\cdot) = \gamma_{i-1}(\cdot) + \beta_i^{\gamma}(\rho_2, x_{[i]}, \hat{\Theta})z_i^{2+\tau}$.

So far, we have completed the design procedure of desirable controller (8), by which and noting $x_{n+1} = \alpha_n$ and $\dot{\hat{\Theta}} = \gamma_n$, we readily achieve the following important proposition.

Proposition 3. *Let $V_n = \sum_{i=1}^{n} W_i(\cdot) + \frac{1}{2}\tilde{\Theta}^2$. Then, functions $\phi_i(\cdot)$'s and $\gamma_n(\cdot)$ specified above guarantee that the time derivative of V_n along system (1) satisfies*

$$\dot{V}_n \leq -c_1 \sum_{j=1}^{n} z_j^{2+\tau}.$$

5 | MAIN RESULT

We are now ready to present the main result of this article, which shows finite-time convergence of system state x and the prescribed convergence of system output y.

Theorem 2. *Consider system (1) under Assumption 1. The global feedback controller (4) with (5)–(9) guarantees that for any initial condition, there hold*

(i) *All the signals of the resulting closed-loop system are bounded on $[0, \infty)$, and particularly system state x converges to zero in finite time.*
(ii) *The system output with prescribed convergence rate enters the arbitrarily adjustable region $\Omega \triangleq \{y \in \mathbf{R} \| y| < \varepsilon\}$ before the pregiven time T_p and thereafter converges to zero in finite time, where $\varepsilon > 0$ represents the reachability level.*

Proof. We first prove the boundedness of all the signals and the prescribed performance of the system output. From Proposition 1, we see that all system signals are bounded on $[0, T_\mathrm{p}]$, and hence we only prove their boundedness after T_p. From Proposition 3, we deduce $V_n(t) \leq V_n(T_\mathrm{p}) < \infty, \forall t \geq T_\mathrm{p}$, which, together with (10) and Proposition 2, implies that z_1, x_i, $i = 2, \ldots, n$ and $\tilde{\Theta}$ are bounded on $[T_\mathrm{p}, \infty)$. Noting $\hat{\Theta} = \Theta + \tilde{\Theta}$ and $z_1 = x_1(\rho_2^2 - x_1^2)^{-\frac{1}{2}}$,we have that x_1 and $\hat{\Theta}$ are also bounded on $[T_\mathrm{p}, \infty)$. Thus, all the signals are bounded on $[0, \infty)$.

From Proposition 1, there holds $|y(t)| < \frac{1}{\rho_1(t)} = \frac{T_\mathrm{p}}{\omega_1 t}$ on $[0, T_\mathrm{p})$. By this, and noting $\omega_1 \geq \frac{2}{\varepsilon}$, we have $|y(t)| < \varepsilon$ on $\left[\frac{T_\mathrm{p}}{2}, T_\mathrm{p}\right)$. Thus, the system output enters Ω before the pregiven time T_p. Moreover, the prescribed convergence rate of the system output on $[0, T_\mathrm{p})$ can be achieved by choosing an appropriate ω_1.

Recalling the facts that $z_1(t)$ is bounded and $\rho_2(t)$ is nonincreasing on $[T_\mathrm{p}, \infty)$, we conclude from (9) that $|y(t)| < \rho_2(t) \leq \varepsilon, \forall t \geq T_\mathrm{p}$, which implies the system output stays in Ω after the pregiven time T_p. Furthermore, by selecting suitable ω_2 in ρ_2, the prescribed convergence rate of the system output on $[T_\mathrm{p}, \infty)$ can be guaranteed.

From the two stages of the system output convergence depicted above, predefined reachability and prescribed convergence rate of the system output are achieved.

We next prove finite-time convergence of system state x and system output y.

Due to the fact that all signals of system (1) are bounded, we can deduce from (9) that $\dot{z}_i$, $i = 1, \ldots, n$ are bounded. Moreover, it follows from Proposition 3 that $z_i \in \mathcal{L}_p$ with $p = 2 + \tau > 1$. Therefore, we obtain $\lim_{t\to\infty} z_i(t) = 0$ by Barbălat Lemma. This, in conjunction with (9), implies $\lim_{t\to\infty} x_i(t) = 0$, $i = 1, \ldots, n$.

Since T_p is finite, we merely prove finite-time convergence on the time interval $[T_\mathrm{p}, \infty)$, and particularly we later omit the time interval for simplicity if there is no confusion.

From Lemma 2 and Proposition 2, there holds

$$\sum_{j=1}^{n} z_j^{2+\tau} \geq \sum_{j=1}^{n} 2^{\frac{(2+\tau)(r_j-1)}{2}} W_j^{\frac{2+\tau}{2}} \geq 2^{\frac{(2+\tau)(r_n-1)}{2}} V^{\frac{2+\tau}{2}}. \tag{17}$$

Let $V(\rho_2, x, \hat{\Theta}) = \sum_{i=1}^{n} W_i(\rho_2, x_{[i]}, \hat{\Theta})$. Then, by (17) and Proposition 3, the derivative of V along system (1) satisfies

$$\dot{V} = \dot{V}_n + \tilde{\Theta}\dot{\hat{\Theta}} \leq -\frac{c_1}{2} 2^{\frac{(2+\tau)(r_n-1)}{2}} V^{\frac{2+\tau}{2}} - \sum_{i=1}^{n} \left(\frac{c_1}{2} - \tilde{\Theta}\beta_i^{\gamma}(\rho_2, x_{[i]}, \hat{\Theta})\right) z_i^{2+\tau}. \tag{18}$$

Note that $\lim_{t\to\infty} x_i(t) = 0$ and $\beta_i^{\gamma}(\rho_2, 0, \hat{\Theta}) \equiv 0$. Then, by the continuity of $\beta_i^{\gamma}(\rho_2, x_{[i]}, \hat{\Theta})$ in $x_{[i]}$ and the boundedness of $\tilde{\Theta}(t)$ on $[T_\mathrm{p}, \infty)$, we have $\lim_{t\to\infty} \tilde{\Theta}(t)\beta_i^{\gamma}(\rho_2(t), x_{[i]}(t), \hat{\Theta}(t)) = 0$, which implies that there exists a finite time $T_1 > T_\mathrm{p}$ such that

$\tilde{\Theta}\beta_i^{\gamma}(\rho_2, x_{[i]}, \hat{\Theta}) \leq \frac{c_1}{2}$ for $t \geq T_1$. Thus, it follows from (18) that

$$\dot{V}(t) \leq -\frac{c_1}{2} 2^{\frac{(2+\tau)(r_n-1)}{2}} V^{\frac{2+\tau}{2}}(t), \quad t \geq T_1,$$

where we denote $V(\rho_2(t), x(t), \hat{\Theta}(t))$ by $V(t)$ for simplicity. This, together with Theorem 1, means that $x(t)$ converges to zero before finite time $T \leq \frac{2^{2+\frac{(2+\tau)(1-r_n)}{2}}}{c_1|\tau|} V^{\frac{|\tau|}{2}}(T_1) + T_1$.

According to the above arguments, it can be deduced that for any initial condition, system state x and system output y converge to zero in finite time.

This completes the proof. ■

Remark 4. The controls ensuring distinctive performance (e.g., prescribed convergence rate, desired overshoot, finite-time stability) are of much significance in practical applications with high requirements (e.g., ship steering control, missile guidance and photovoltaic power). Let us see this point in detail by an example. In a photovoltaic power system, the speed of maximum power point tracking should be fast enough to ensure efficiency,[15] which requires a adequately large convergence rate. In this article, the proposed scheme, via funnel control and adaptive feedback, simultaneously guarantees global finite-time stability, predefined reachability and prescribed convergence rate. Moreover, serious chattering behavior probably arising in switching adaptive feedback and sliding mode control is circumvented. The guarantee and circumvention highlight the main contribution of this article.

6 | SIMULATION EXAMPLES

In this section, two examples are provided to illustrate the effectiveness and merit of the proposed scheme.

We first consider the following robotic manipulator system:

$$J\ddot{q} + B\dot{q} + MgL\sin q = u, \tag{19}$$

where the relevant definitions can be found in Reference 3, and are omitted here for brevity.

Define $x_1 = Jq, x_2 = J\dot{q}$. Then (19) can be changed into the following form:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = u - \frac{B}{J}x_2 - MgL\sin\left(\frac{x_1}{J}\right), \\ y = x_1, \end{cases} \tag{20}$$

which satisfies Assumption 1 with $\tau = -\frac{2}{5}, \overline{f}_1 = 0, \overline{f}_2 = |x_1|^{\frac{4}{5}} + |x_2|^{\frac{2}{3}}$ and $\theta = \left|\frac{B}{J}\right| + \left|\frac{MgL}{J}\right|$, where M and L are assumed to be unknown in view of the limitation of sensor.

The control objective is to achieve global finite-time stabilization for system (20) with initial condition $[x_1(0), x_2(0)]^{\mathrm{T}} = [4, 1]^{\mathrm{T}}$, and to guarantee that the system output converges to $\{y \in \mathbf{R} \| y| < 1\}$ (i.e., $\varepsilon = 1$ in Theorem 2) before 3 s and thereafter remains inside with prescribed convergence rate satisfying $\omega_1 = 2$ and $\omega_2 = 1$.

Following the controller design in Section 4, we obtain the design functions in the following form:

$$\begin{cases} \phi_1 = \frac{1}{h(\cdot)}\left(2 - \frac{\dot{\rho}_2}{\rho_2}h(\cdot)(1+z_1^2)^{\frac{1}{5}}\sqrt{\rho_2^2 - x_1^2}\right), \\ \phi_2 = 1 + \beta_{21}(\cdot) + \beta_{22}(\cdot) + \beta_{23}(\cdot), \\ \dot{\hat{\Theta}} = \beta_{24}(\cdot)z_2^{\frac{8}{5}}, \quad \hat{\Theta}(T_{\mathrm{p}}) = 2, \quad T_{\mathrm{p}} = 3, \end{cases}$$

where $\beta_{21} = 2.5h^{\frac{8}{3}}$, $\beta_{22} = 0.8\left(\left(1+x_1^2\right)\frac{2}{5} + \left(1+x_2^2\right)^{\frac{1}{3}}\right)\left(\left(\rho_2^2 - x_1^2\right)^{\frac{4}{35}} + 1 + \phi_1^{\frac{8}{21}}\right)$, $\beta_{23} = 1.8\left|\frac{\partial\alpha_1^{\frac{5}{3}}}{\partial x_1}\right| z_2^{\frac{8}{5}} + 2.6\left(\frac{\partial\alpha_1^{\frac{5}{3}}}{\partial x_1}\phi_1\right)^{\frac{8}{5}} + 2.9\left((1+z_1^2)^{\frac{1}{5}}\frac{\partial\phi_1}{\partial\rho_2}\dot{\rho}_2\right)^{\frac{8}{3}} z_2^{\frac{8}{5}}$ and $\beta_{24} = 0.8\left(|x_1|^{\frac{4}{5}} + |x_2|^{\frac{2}{3}}\right)\left((\rho_2^2 - x_1^2)^{\frac{4}{35}} + 1 + \phi_1^{\frac{8}{21}}\right)\hat{\Theta}$.

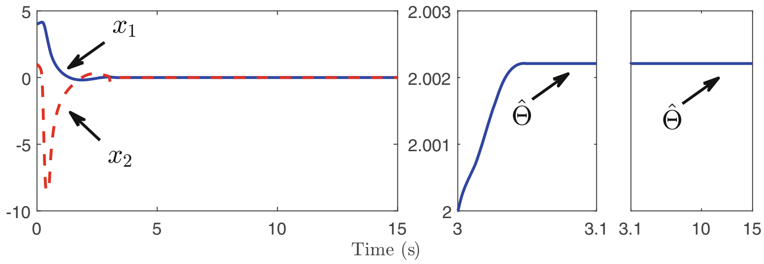

FIGURE 2 Evolution of signals x_1, x_2, and $\hat{\Theta}$ for system (20)

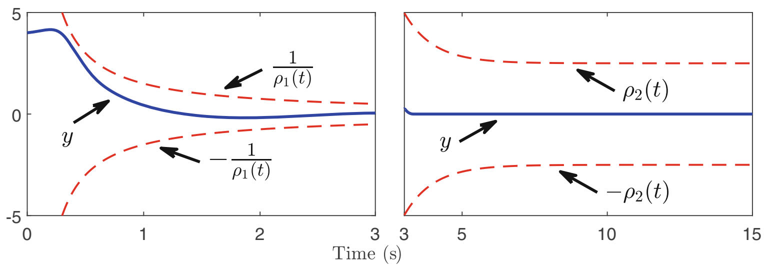

FIGURE 3 Evolution of system output y for system (20)

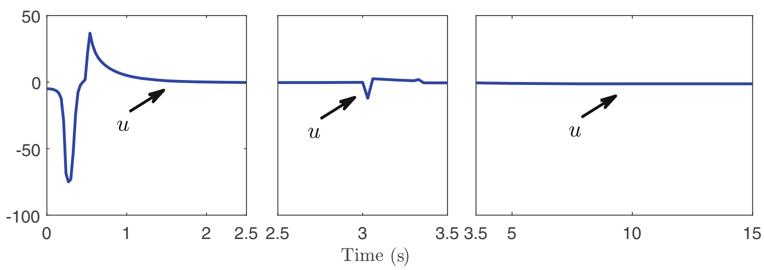

FIGURE 4 Evolution of control input u for system (20)

Suppose $g = 9.8, M = B = 0.5, J = L = 1$. By Matlab, we get Figures 2–4. Specifically, Figures 2 and 4 exhibit all system signals are bounded, and particularly system states x_1 and x_2 converge to zero in finite time. Figure 3 shows that $|y(t)| < \frac{1}{\rho_1(t)}$ for $t < 3$ and $|y(t)| < \rho_2(t) \leq 1$ for $t \geq 3$, that is, the prescribed convergence rate and predefined reachability of system output y are realized.

We next consider the following 2-dimensional system typically with inherent system nonlinearities:

$$\begin{cases} \dot{x}_1 = x_2 + \theta_1 \sin(x_1)|x_1|^{\frac{19}{21}} + x_1 \cos(x_1), \\ \dot{x}_2 = u + \theta_2 x_1^2 + \sin(x_1) x_2^{\frac{17}{19}}, \\ y = x_1, \end{cases} \tag{21}$$

where θ_1 and θ_2 are unknown constants. Obviously, system (21) satisfies Assumption 1 with $\tau = -\frac{2}{21}, \overline{f}_1 = |x_1| + |x_1|^{\frac{2}{21}}$, $\overline{f}_2 = |x_1| + |x_1|^{\frac{25}{21}}$ and $\theta = \max\{1 + |\theta_1|, 1 + |\theta_2|\}$.

The control objective is to devise a controller such that for initial condition $[x_1(0), x_2(0)]^{\mathrm{T}} = [1, -5]^{\mathrm{T}}$, system states x_1 and x_2 converge to zero in finite time, and particularly the system output converges to $\{y \in \mathbf{R} \| y| < 1\}$ (i.e., $\varepsilon = 1$ in Theorem 2) before 2 s and thereafter remains inside with prescribed convergence rate satisfying $\omega_1 = 2$ and $\omega_2 = 1$.

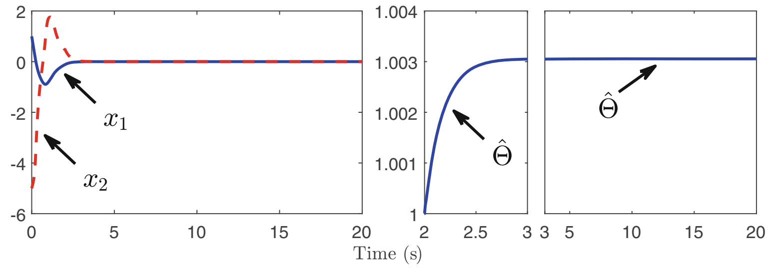

FIGURE 5 Evolution of signals x_1, x_2 and $\hat{\Theta}$ for system (21)

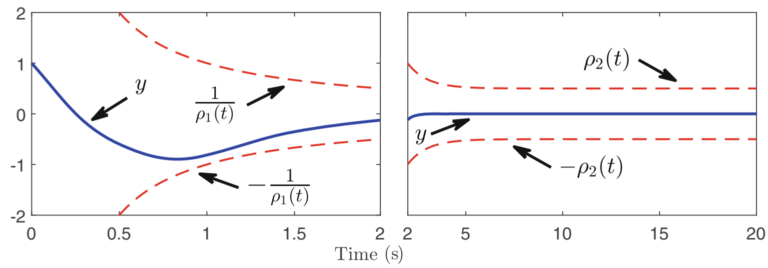

FIGURE 6 Evolution of system output y for system (21)

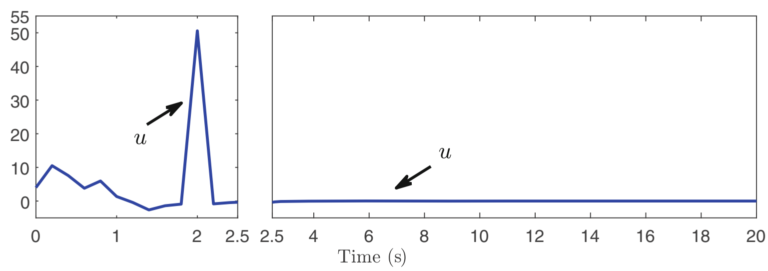

FIGURE 7 Evolution of control input u for system (21)

By the controller design in Section 4, design functions are specified as follows:

$$\begin{cases} \phi_1 = \frac{2(\rho_2^2 - x_1^2)^{\frac{3}{2}}}{\rho_2^2} + \hat{\Theta}(1 + x_1^2)^{\frac{1}{21}}(\rho_2^2 - x_1^2)^{\frac{19}{42}} - \frac{\dot{\rho}_2}{\rho_2}(\rho_2^2 - x_1^2)^{\frac{1}{2}}(1 + z_1^2)^{\frac{1}{21}}, \\ \phi_2 = 1 + \beta_2^{\phi}(x_{[2]}, \hat{\Theta}), \\ \dot{\hat{\Theta}} = \rho_2^2 \overline{f}_1(x_1)(\rho_2^2 - x_1^2)^{\frac{3}{2} - \frac{r_2}{2}} z_1^{2+\tau} + \beta_2^{\gamma}(x_{[2]}, \hat{\Theta}) z_2^{2+\tau}, \quad \hat{\Theta}(T_{\mathrm{p}}) = 1, \quad T_{\mathrm{p}} = 2, \end{cases}$$

where $\beta_2^{\phi}(\cdot)$ and $\beta_2^{\gamma}(\cdot)$, specified readily by the Lyapunov-based recursive procedure in Section 4, are with cumbersome expressions and are omitted here.

Let $\theta_1 = 0.5$ and $\theta_2 = 2$. Then, by Matlab, we obtain Figures 5–7. Concretely, Figures 5 and 7 show that system states x_1 and x_2, gain $\hat{\Theta}$ and control input u are bounded, and particularly system states x_1 and x_2 converge to zero in finite time. Moreover, the prescribed output convergence can be seen from Figure 6.

Remarkably, the devised controllers for systems (20) and (21) have advantages in ensuring system performance, compared with continuous and discontinuous schemes. On the one hand, the devised controllers are discontinuous only at the pregiven time instant, and thus circumvent serious chattering behavior probably arising in discontinuous schemes.[23,24] On the other hand, the related continuous schemes (e.g., References 13 and 15) cannot achieve the above objectives of

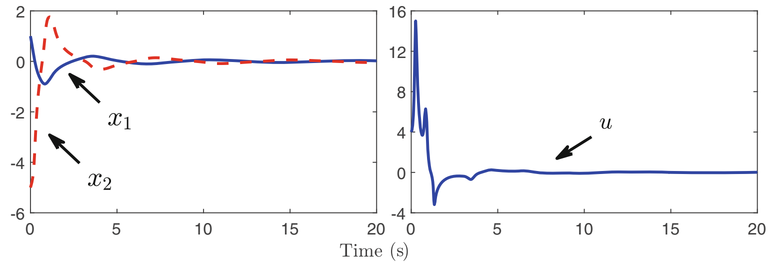

FIGURE 8 Evolution of system states x_1 and x_2 and control input u for system (21) under funnel control

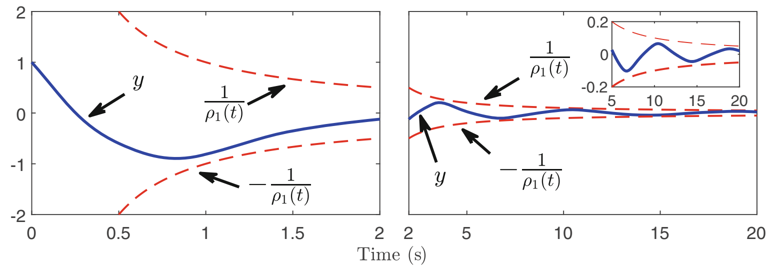

FIGURE 9 Evolution of system output y for system (21) under funnel control

systems (20) and (21). Concretely,[15] can realize global stabilization and the prescribed performance of the system output for systems (20) and (21), however finite-time convergence is excluded. With the help of performance-induced transformation,[13] can make the system output converge to a prescribed neighborhood of the origin, but which cannot ensure finite-time convergence (to zero). While the scheme based on the barrier Lyapunov function in Reference 20 can realize semiglobal finite-time stabilization with constant output constraint, it is incapable of global objectives of systems (20) and (21).

To explicitly show the advantage of the proposed scheme in terms of system performance, we finally adopt funnel control scheme in Reference 15 to act on system (21). By controller u_f in (6) with $\omega_1 = 2$ and $\varepsilon = 1$, we obtain Figures 8 and 9. From Figure 9, it can be seen that funnel control scheme ensures prescribed convergence rate and predefined reachability of the system output. But it, as shown in Figures 8 and 9, cannot make the system states and system output converge to zero in 20 s, while our scheme can (see Figures 5 and 6 above).

7 CONCLUDING REMARKS

In this article, global finite-time stabilization with prescribed output convergence has been achieved for a class of uncertain nonlinear systems via integrating funnel control and adaptive control schemes. Although the two schemes are powerful to achieve prescribed performance and finite-time stabilization, respectively, neither of the schemes can be used alone to achieve the objective of this article, which is the main reason why we pursue an integrated controller. Specifically, by utilizing funnel control scheme and incorporating a delicate time-dependent function into adaptive controller, the system output evolves within two prespecified envelopes, which guarantees that the system output with prescribed convergence rate converges from any initial value to an arbitrarily adjustable region before a pregiven time instant and thereafter stays inside. Moreover, the adaptive controller ensures that the system states and the system output converge to zero in finite time. Particularly, the controller is discontinuous only at the pregiven time instant, and hence the serious chattering behavior probably arising in switching adaptive feedback and sliding mode control is circumvented. Whereas this article only realizes the prescribed performance of the system output, one may explore whether it is possible to integrate more powerful schemes to realize prescribed performance of other system signals. Irrespective of prescribed performance, merely finite-time stabilization can be established in this article, while prescribed-time stabilization has been achieved recently (see e.g., References 29 and 30). Hence, how to achieve prescribed-time stabilization on the premise of prescribed performance deserves further investigation.

ACKNOWLEDGMENT

This work was supported by the National Natural Science Foundation of China (Grant Nos. 62033007 and 61873146), by the Taishan Scholars Climbing Program of Shandong Province, and by the Key and Development Plan of Shandong Province (Grant No. 2019JZZY010433).

ORCID

Yungang Liu https://orcid.org/0000-0002-4753-9578

REFERENCES

1. Khalil HK. *Nonlinear Systems*. 3rd ed. Prentice Hall; 2002.
2. Fossen TI. *Guidance and Control of Ocean Vehicles*. John Wiley; 1994.
3. Cruz-Zavala E, Sanchez T, Nuño E, Moreno JA. Lyapunov-based finite-time control of robot manipulators. *Int J Robust Nonlinear Control*. 2021;31(8):3090-3114.
4. Krstić M, Kanellakopoulos I, Kokotović PV. *Nonlinear and Adaptive Control Design*. John Wiley; 1995.
5. Jiang ZP, Mareels I, Hill DJ, Huang J. A unifying framework for global regulation via nonlinear output feedback: from ISS to iISS. *IEEE Trans Automat Contr*. 2004;49(4):549-562.
6. Huang XQ, Lin W, Yang B. Global finite-time stabilization of a class of uncertain nonlinear systems. *Automatica*. 2005;41(5):881-888.
7. Bhat SP, Bernstein DS. Continuous finite-time stabilization of the translational and rotational double integrators. *IEEE Trans Automat Contr*. 1998;43(5):678-682.
8. Hong YG, Wang JK, Cheng DZ. Adaptive finite-time control of nonlinear systems with parametric uncertainty. *IEEE Trans Automat Contr*. 2006;51(5):858-862.
9. Sun ZY, Xue LR, Zhang KM. A new approach to finite-time adaptive stabilization of high-order uncertain nonlinear system. *Automatica*. 2015;58:60-66.
10. Li S, Ahn CK, Xiang ZR. Command-filter-based adaptive fuzzy finite-time control for switched nonlinear systems using state-dependent switching method. *IEEE Trans Fuzzy Syst*. 2020;29(4):833-845.
11. Huang JS, Wen CY, Wang W, Song YD. Design of adaptive finite-time controllers for nonlinear uncertain systems based on given transient specifications. *Automatica*. 2016;69:395-404.
12. Ilchmann A, Ryan EP, Sangwin CJ. Tracking with prescribed transient behaviour. *ESAIM Control Optim Cal Var*. 2002;7:471-493.
13. Bechlioulis CP, Rovithakis GA. Adaptive control with guaranteed transient and steady state tracking error bounds for strict feedback systems. *Automatica*. 2009;45(2):532-538.
14. Berger T, Le HH, Reis T. Funnel control for nonlinear systems with known strict relative degree. *Automatica*. 2018;87:345-357.
15. Li FZ, Liu YG. Control design with prescribed performance for nonlinear systems with unknown control directions and non-parametric uncertainties. *IEEE Trans Automat Contr*. 2018;63(10):3573-3580.
16. Li FZ, Liu YG. Prescribed-performance control design for pure-feedback nonlinear systems with most severe uncertainties. *SIAM J Control Optim*. 2018;56(10):517-537.
17. Liu CG, Wang HQ, Liu XP, Zhou YC. Adaptive finite-time fuzzy funnel control for nonaffine nonlinear systems. *IEEE Trans Syst Man Cybern Syst*. 2021;51(5):2894-2903.
18. Wu ZW, Zhang TP. Adaptive finite-time tracking control for parameterized nonlinear systems with full state constraints. *Int J Adapt Control Signal Process*. 2021;35(9):1768-1788.
19. Tang L, Yang MY, Sun J. Adaptive fuzzy constraint control for switched nonlinear systems in nonstrict feedback form. *Int J Adapt Control Signal Process*. 2021;35(8):1594-1611.
20. Chen CC, Sun ZY. A unified approach to finite-time stabilization of high-order nonlinear systems with an asymmetric output constraint. *Automatica*. 2020;111:108581.
21. Miller DE, Davison EJ. An adaptive controller which provides an arbitrarily good transient and steady-state response. *IEEE Trans Automat Contr*. 1991;36(1):68-81.
22. Ding SH, Levant A, Li SH. Simple homogeneous sliding-mode controller. *Automatica*. 2016;67:22-32.
23. Wu J, Li J, Zong GD, Chen WS. Global finite-time adaptive stabilization of nonlinearly parametrized systems with multiple unknown control directions. *IEEE Trans Syst Man Cybern Syst*. 2017;47(7):1405-1414.
24. Liu YJ, Chen H. Adaptive sliding mode control for uncertain active suspension systems with prescribed performance. *IEEE Trans Syst Man Cybern Syst*. 2021;51(10):6414-6422.
25. Lopez-Ramirez F, Efimov D, Polyakov A, Perruquetti W. Conditions for fixed-time stability and stabilization of continuous autonomous systems. *Syst Control Lett*. 2019;129:26-35.
26. Bhat SP, Bernstein DS. Finite-time stability of continuous autonomous systems. *SIAM J Control Optim*. 2000;38(3):751-766.
27. Moulay E, Perruquetti W. Finite time stability conditions for non-autonomous continuous systems. *Int J Control*. 2008;81(5):797-803.

28. Qian CJ, Lin W. A continuous feedback approach to global strong stabilization of nonlinear systems. *IEEE Trans Automat Contr*. 2001;46(7):1061-1079.
29. Krishnamurthy P, Khorrami F, Krstić M. Adaptive output-feedback stabilization in prescribed time for nonlinear systems with unknown parameters coupled with unmeasured states. *Int J Adapt Control Signal Process*. 2021;35(2):184-202.
30. Liu CY, Liu YG. Finite-time stabilization with arbitrarily prescribed settling-time for uncertain nonlinear systems. *Syst Control Lett*. 2022;159:105088.

How to cite this article: Liu C, Liu Y. Adaptive finite-time stabilization with prescribed output convergence via an integrated scheme. *Int J Adapt Control Signal Process*. 2022;36(7):1759-1777. doi: 10.1002/acs.3419

APPENDIX. PROOFS OF IMPORTANT RELATIONS

In this section, we provide the detailed proofs of three important relations arising in Section 4. It is worth emphasizing that these proofs are indispensable, since they determine some important functions involved in the control design, such as β_i^{ϕ}'s.

Proof of (11). From (9), there holds

$$\dot{z}_1 = \frac{\rho_2^2}{(\rho_2^2 - x_1^2)^{\frac{3}{2}}}\left(\dot{x}_1 - \frac{\dot{\rho}_2}{\rho_2}x_1\right) = \frac{\rho_2^2}{(\rho_2^2 - x_1^2)^{\frac{3}{2}}}\left(x_2 + f_1 - \frac{\dot{\rho}_2}{\rho_2}x_1\right) =: h(\rho_2, x_1)\left(x_2 + f_1 - \frac{\dot{\rho}_2}{\rho_2}x_1\right).$$

Then, by this, (9) and Assumption 1, we have

$$\begin{aligned}\dot{V}_1 &= z_1 h(\rho_2, x_1)(x_2 - \alpha_1) + z_1 h(\rho_2, x_1)\alpha_1 + z_1 h(\rho_2, x_1)\left(f_1(t, x_1) - \frac{\dot{\rho}_2}{\rho_2}x_1\right) - \tilde{\Theta}\dot{\hat{\Theta}} \\ &\leq z_1 h(\rho_2, x_1)(x_2 - \alpha_1) + z_1 h(\rho_2, x_1)\alpha_1 + z_1^{2+\tau} h(\cdot)\theta \overline{f}_1(x_1)(\rho_2^2 - x_1^2)^{\frac{r_1+\tau}{2}} - \frac{\dot{\rho}_2}{\rho_2} h(\cdot)(\rho_2^2 - x_1^2)^{\frac{1}{2}} z_1^2 - \tilde{\Theta}\dot{\hat{\Theta}}.\end{aligned}$$

This implies (11) holds. ∎

Proof of (14). From (13) and Proposition 2, it follows that

$$\begin{aligned}\dot{V}_2 &= \dot{V}_1 + \frac{\partial W_2}{\partial \hat{\Theta}}\dot{\hat{\Theta}} + \sum_{i=1}^{2}\frac{\partial W_2}{\partial x_i}\dot{x}_i + \frac{\partial W_2}{\partial \rho_2}\dot{\rho}_2 \\ &\leq -(c_1 + n - 1)z_1^{2+\tau} + z_1 h(\rho_2, x_1)(x_2 - \alpha_1) - \tilde{\Theta}\left(\dot{\hat{\Theta}} - \gamma_1(\cdot)\right) + \frac{\partial W_2}{\partial \hat{\Theta}}\dot{\hat{\Theta}} + z_2^{2-r_2}(x_3 - \alpha_2) + z_2^{2-r_2}\alpha_2 \\ &\quad + z_2^{2-r_2} f_2(t, x_{[2]}) - (2 - r_2)\int_{\alpha_1}^{x_2}\left(s^{\frac{1}{r_2}} - \alpha_1^{\frac{1}{r_2}}\right)^{1-r_2}\mathrm{d}s\frac{\partial \alpha_1^{\frac{1}{r_2}}}{\partial x_1}\dot{x}_1 - (2 - r_2)\int_{\alpha_1}^{x_2}\left(s^{\frac{1}{r_2}} - \alpha_1^{\frac{1}{r_2}}\right)^{1-r_2}\mathrm{d}s\frac{\partial \alpha_1^{\frac{1}{r_2}}}{\partial \rho_2}\dot{\rho}_2.\end{aligned} \tag{A1}$$

We next estimate the second term and last three terms on the right-hand side of (A1) (denoted by ①–④, respectively). After some tedious calculations (the details are given later for readability), we have

$$\begin{cases} ① \leq \frac{1}{4}z_1^{2+\tau} + \beta_{21}(\rho_2, x_1)z_2^{2+\tau}, \\ ② \leq \frac{1}{4}z_1^{2+\tau} + \tilde{\Theta}\beta_{22}(\rho_2, x_{[2]}, \hat{\Theta})z_2^{2+\tau} + \beta_{23}(\rho_2, x_{[2]}, \hat{\Theta})z_2^{2+\tau}, \\ ③ + ④ \leq \frac{1}{4}z_1^{2+\tau} + \tilde{\Theta}\beta_{24}(\rho_2, x_{[2]}, \hat{\Theta})z_2^{2+\tau} + \beta_{25}(\rho_2, x_{[2]}, \hat{\Theta})z_2^{2+\tau}, \end{cases} \tag{A2}$$

where $\beta_{2j}(\cdot), j = 1, 3, 5$ are known smooth non-negative functions and $\beta_{2j}(\cdot), j = 2, 4$ are known continuous non-negative functions with $\beta_{2j}(\rho_2, 0, \hat{\Theta}) \equiv 0$.

Define $\beta_2^{\gamma}(\rho_2, x_{[2]}, \hat{\Theta}) = \beta_{22}(\cdot) + \beta_{24}(\cdot)$. Then, from (A1) and (A2), it is deduced that

$$\dot{V}_2 \le -\left(c_1 + n - \frac{7}{4}\right) z_1^{2+\tau} + z_2^{2-r_2}(x_3 - \alpha_2) + z_2^{2-r_2}\alpha_2 - \left(\tilde{\Theta} - \frac{\partial W_2}{\partial \hat{\Theta}}\right)\left(\dot{\hat{\Theta}} - \left(\gamma_1(\cdot) + \beta_2^{\gamma}(\cdot) z_2^{2+\tau}\right)\right)$$
$$+ \frac{\partial W_2}{\partial \hat{\Theta}}\left(\gamma_1(\cdot) + \beta_2^{\gamma}(\cdot) z_2^{2+\tau}\right) + \left(\beta_{21}(\cdot) + \beta_{23}(\cdot) + \beta_{25}(\cdot)\right) z_2^{2+\tau}. \tag{A3}$$

From Lemmas 2 and 3, there holds

$$\int_{\alpha_1}^{x_2}\left(s^{\frac{1}{r_2}} - \alpha_1^{\frac{1}{r_2}}\right)^{1-r_2} \mathrm{d}s \le |z_2|^{1-r_2}|x_2 - \alpha_1| \le 2^{1-r_2}|z_2|. \tag{A4}$$

Note that $\gamma_1(\cdot) = \beta_1^{\gamma}(\cdot) z_1^{2+\tau}$. Then using (A4) and Lemma 1 yields

$$\begin{aligned}\frac{\partial W_2}{\partial \hat{\Theta}}\left(\gamma_1(\cdot) + \beta_2^{\gamma}(\cdot) z_2^{2+\tau}\right) &= -(2 - r_2)\int_{\alpha_1}^{x_2}\left(s^{\frac{1}{r_2}} - \alpha_1^{\frac{1}{r_2}}\right)^{1-r_2} \mathrm{d}s \frac{\partial \alpha_1^{\frac{1}{r_2}}}{\partial \hat{\Theta}} \sum_{j=1}^{2} \beta_j^{\gamma}(\cdot) z_j^{2+\tau} \\ &\le (2 - r_2)2^{1-r_2}|z_2| \cdot \left|\frac{\partial \alpha_1^{\frac{1}{r_2}}}{\partial \hat{\Theta}}\right| \left(\beta_1^{\gamma}(\cdot) z_1^{2+\tau} + \beta_2^{\gamma}(\cdot) z_2^{2+\tau}\right) \\ &\le \frac{1}{4} z_1^{2+\tau} + \beta_{26}(\rho_2, x_{[2]}, \hat{\Theta}) z_2^{2+\tau},\end{aligned}$$

where $\beta_{26}(\cdot)$ is a known smooth non-negative function. Substituting this into (A3) results in

$$\dot{V}_2 \le -(c_1 + n - 2) z_1^{2+\tau} + z_2^{2-r_2}(x_3 - \alpha_2) + z_2^{2-r_2}\alpha_2 - \left(\tilde{\Theta} - \frac{\partial W_2}{\partial \hat{\Theta}}\right)\left(\dot{\hat{\Theta}} - \left(\gamma_1(\cdot) + \beta_2^{\gamma}(\cdot) z_2^{2+\tau}\right)\right) + \beta_2^{\phi}(\rho_2, x_{[2]}, \hat{\Theta}) z_2^{2+\tau},$$

where $\beta_2^{\phi}(\cdot) = \beta_{21}(\cdot) + \beta_{23}(\cdot) + \beta_{25}(\cdot) + \beta_{26}(\cdot)$. This implies (14) holds.

Let's verify the correctness of (A2). By Lemmas 1 and 2, and noting $2 + \tau = 1 + r_2$, we get

$$① \le 2^{1-r_2} h(\cdot)|z_1| \cdot |z_2|^{r_2} \le \frac{1}{4} z_1^{2+\tau} + \beta_{21}(\rho_2, x_1) z_2^{2+\tau},$$

where $\beta_{21}(\cdot) = \frac{r_2}{2+\tau}\left(\frac{4}{2+\tau}\right)^{\frac{1}{r_2}}\left(2^{1-r_2}h(\cdot)\right)^{\frac{2+\tau}{r_2}}$.

In virtue of (9) and Lemma 2, it is not difficult to obtain that

$$|x_1|^{\frac{r_2+\tau}{r_1}} = |z_1|^{r_2+\tau}(\rho_2^2 - x_1^2)^{\frac{r_2+\tau}{2}}, \qquad |x_2|^{\frac{r_2+\tau}{r_2}} = |z_2 + \alpha_1^{\frac{1}{r_2}}|^{r_2+\tau} \le |z_2|^{r_2+\tau} + |z_1|^{r_2+\tau}\phi_1^{\frac{r_2+\tau}{r_2}}.$$

From this and Assumption 1, it follows that

$$\begin{aligned}② &\le \theta |z_2|^{2-r_2}\bar{f}_2(x_{[2]}) \sum_{j=1}^{2} |x_j|^{\frac{r_2+\tau}{r_j}} \\ &\le \theta |z_2|^{2-r_2}\bar{f}_2(x_{[2]})|z_1|^{r_2+\tau}(\rho_2^2 - x_1^2)^{\frac{r_2+\tau}{2}} + \theta |z_2|^{2-r_2}\bar{f}_2(x_{[2]})\left(|z_2|^{r_2+\tau} + |z_1|^{r_2+\tau}\phi_1^{\frac{r_2+\tau}{r_2}}\right).\end{aligned}$$

Then, by this and Lemma 1, and noting $\Theta \ge \theta^{\frac{2+\tau}{2-r_2}}$ and $\Theta = \hat{\Theta} + \tilde{\Theta}$, we obtain

$$\begin{aligned}② &\le \frac{1}{4} z_1^{2+\tau} + \Theta \bar{f}_2(\cdot) z_2^{2+\tau} + \frac{(2 - r_2)(4(r_2 + \tau))^{\frac{r_2+\tau}{2-r_2}}}{(2 + \tau)^{\frac{2+\tau}{2-r_2}}}\Theta \cdot \bar{f}_2^{\frac{2+\tau}{2-r_2}}(\cdot)\left((\rho_2^2 - x_1^2)^{\frac{(r_2+\tau)(2+\tau)}{4-2r_2}} + \phi_1^{\frac{(r_2+\tau)(2+\tau)}{2r_2 - r_2^2}}\right) z_2^{2+\tau} \\ &\le \frac{1}{4} z_1^{2+\tau} + \tilde{\Theta}\beta_{22}(\cdot) z_2^{2+\tau} + \beta_{23}(\cdot) z_2^{2+\tau}.\end{aligned}$$

We next verify the correctness of the last inequality of (A2). From (9), Assumption 1 and Lemma 2, there holds

$$\dot{x}_1 = x_2 + f_1(t, x_1) = \left(z_2 + \alpha_1^{\frac{1}{r_2}}\right)^{r_2} + f_1(t, x_1) \leq |z_2|^{r_2} + |z_1|^{r_2}\phi_1(\cdot) + \theta\overline{f}_1(x_1)(\rho_2^2 - x_1^2)^{\frac{r_2}{2}}|z_1|^{r_2}.$$

By this, (A4) and Lemma 1, and noting $1 + r_2 = 2 + \tau$, we have

$$\begin{aligned}
③ &\leq (2 - r_2)2^{1-r_2}|z_2| \cdot \left|\frac{\partial \alpha_1^{\frac{1}{r_2}}}{\partial x_1}\right| \cdot \left(|z_2|^{r_2} + |z_1|^{r_2}\phi_1(\cdot) + \theta\overline{f}_1(\cdot)(\rho_2^2 - x_1^2)^{\frac{r_2}{2}}|z_1|^{r_2}\right) \\
&\leq (2 - r_2)2^{1-r_2}\left|\frac{\partial \alpha_1^{\frac{1}{r_2}}}{\partial x_1}\right| z_2^{2+\tau} + \frac{1}{8}z_1^{2+\tau} + c_{21}\left(\frac{\partial \alpha_1^{\frac{1}{r_2}}}{\partial x_1}\right)^{2+\tau} \phi_1^{2+\tau}(\cdot)z_2^{2+\tau} + c_{21}\left(\frac{\partial \alpha_1^{\frac{1}{r_2}}}{\partial x_1}\right)^{2+\tau} \left(\theta(\rho_2^2 - x_1^2)^{\frac{r_2}{2}}\overline{f}_1(\cdot)\right)^{2+\tau} z_2^{2+\tau},
\end{aligned} \tag{A5}$$

where $c_{21} = \frac{1}{2+\tau}\left(\frac{16r_2}{2+\tau}\right)^{r_2}((2 - r_2)2^{1-r_2})^{2+\tau}$.

Noting $\alpha_1^{\frac{1}{r_2}} = -x_1(\rho_2^2 - x_1^2)^{-\frac{1}{2}}\phi_1^{\frac{1}{r_2}} =: -x_1\overline{\phi}_1$, we have $\frac{\partial \alpha_1^{\frac{1}{r_2}}}{\partial \rho_2} = -x_1\frac{\partial \overline{\phi}_1}{\partial \rho_2}$. This, together with (A4) and Lemma 1, implies

$$④ \leq (2 - r_2)2^{1-r_2}\left|\frac{\partial \overline{\phi}_1}{\partial \rho_2}\dot{\rho}_2\right|(\rho_2^2 - x_1^2)^{\frac{1}{2}}|z_1 z_2| \leq \frac{1}{8}z_1^{2+\tau} + c_{22}(\rho_2, x_{[2]})z_2^{2+\tau}, \tag{A6}$$

where $c_{22}(\cdot) = \frac{r_2}{2+\tau}\left(\frac{8}{2+\tau}\right)^{\frac{1}{r_2}}((2 - r_2)2^{1-r_2})^{\frac{2+\tau}{1+\tau}}(\rho_2^2 - x_1^2)^{\frac{2+\tau}{2(1+\tau)}}\left((1 + z_2^2)^{\frac{-\tau}{2}}\frac{\partial \overline{\phi}_1}{\partial \rho_2}\dot{\rho}_2\right)^{\frac{2+\tau}{1+\tau}}$.

Thus, by (A5) and (A6), and noting $\dot{\rho}_2(t) = -\omega_2\rho_2(t) + \frac{\omega_2\varepsilon}{2}$ and $\theta^{2+\tau} \leq \Theta$, we obtain the last inequality of (A2).

This completes the proof of (14). ■

Proof of (16). Using (15) and Proposition 2 yields

$$\begin{aligned}
\dot{V}_i &= \dot{V}_{i-1} + \frac{\partial W_i}{\partial \hat{\Theta}}\dot{\hat{\Theta}} + \sum_{j=1}^{i}\frac{\partial W_i}{\partial x_j}\dot{x}_j + \frac{\partial W_i}{\partial \rho_2}\dot{\rho}_2 \\
&\leq -(c_1 + n - i + 1)\sum_{j=1}^{i-1} z_j^{2+\tau} + z_{i-1}^{2-r_{i-1}}(x_i - \alpha_{i-1}) - \left(\tilde{\Theta} - \sum_{j=2}^{i-1}\frac{\partial W_j}{\partial \hat{\Theta}}\right)\left(\dot{\hat{\Theta}} - \gamma_{i-1}(\cdot)\right) + \frac{\partial W_i}{\partial \hat{\Theta}}\dot{\hat{\Theta}} + z_i^{2-r_i}(x_{i+1} - \alpha_i) \\
&\quad + z_i^{2-r_i}\alpha_i + z_i^{2-r_i}f_i(t, x_{[i]}) - (2 - r_i)\int_{\alpha_{i-1}}^{x_i}\left(s^{\frac{1}{r_i}} - \alpha_{i-1}^{\frac{1}{r_i}}\right)^{1-r_i}\mathrm{d}s\sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}^{\frac{1}{r_i}}}{\partial x_j}\dot{x}_j - (2 - r_i)\int_{\alpha_{i-1}}^{x_i}\left(s^{\frac{1}{r_i}} - \alpha_{i-1}^{\frac{1}{r_i}}\right)^{1-r_i}\mathrm{d}s\frac{\partial \alpha_{i-1}^{\frac{1}{r_i}}}{\partial \rho_2}\dot{\rho}_2.
\end{aligned} \tag{A7}$$

Similar to the verification of (A2), the second term and last three terms on the right-hand side of (A7) (denoted by ⑤–⑧, respectively) are estimated as

$$\begin{cases}
⑤ \leq \frac{1}{4}z_{i-1}^{2+\tau} + \beta_{i1}z_i^{2+\tau}, \\
⑥ \leq \frac{1}{3}\sum_{j=1}^{i-2} z_j^{2+\tau} + \frac{1}{4}z_{i-1}^{2+\tau} + \tilde{\Theta}\beta_{i2}(\rho_2, x_{[i]}, \hat{\Theta})z_i^{2+\tau} + \beta_{i3}(\rho_2, x_{[i]}, \hat{\Theta})z_i^{2+\tau}, \\
⑦ + ⑧ \leq \frac{1}{3}\sum_{j=1}^{i-2} z_j^{2+\tau} + \frac{1}{4}z_{i-1}^{2+\tau} + \tilde{\Theta}\beta_{i4}(\rho_2, x_{[i]}, \hat{\Theta})z_i^{2+\tau} + \beta_{i5}(\rho_2, x_{[i]}, \hat{\Theta})z_i^{2+\tau},
\end{cases} \tag{A8}$$

where β_{i1} is a positive constant, $\beta_{ij}(\cdot), j = 2, 4$ are known continuous non-negative functions with $\beta_{ij}(\rho_2, 0, \hat{\Theta}) \equiv 0$ and $\beta_{ij}(\cdot)$, $j = 3, 5$ are known smooth non-negative functions.

Define $\beta_i^{\gamma}(\rho_2, x_{[i]}, \hat{\Theta}) = \beta_{i2}(\rho_2, x_{[i]}, \hat{\Theta}) + \beta_{i4}(\rho_2, x_{[i]}, \hat{\Theta})$. Then, substituting (A8) into (A7) yields

$$\dot{V}_i \leq -\left(c_1 + n - i + \frac{1}{4}\right)z_{i-1}^{2+\tau} - \left(c_1 + n - i + \frac{1}{3}\right)\sum_{j=1}^{i-2} z_j^{2+\tau} + z_i^{2-r_i}(x_{i+1} - \alpha_i) + z_i^{2-r_i}\alpha_i$$

$$
-\left(\tilde{\Theta}-\sum_{j=2}^{i}\frac{\partial W_j}{\partial\hat{\Theta}}\right)\left(\dot{\hat{\Theta}}-(\gamma_{i-1}(\cdot)+\beta_i^{\gamma}(\cdot)z_i^{2+\tau})\right)+\frac{\partial W_i}{\partial\hat{\Theta}}(\gamma_{i-1}(\cdot)+\beta_i^{\gamma}z_i^{2+\tau})
$$
$$
+\sum_{j=2}^{i-1}\frac{\partial W_j}{\partial\hat{\Theta}}\beta_i^{\gamma}z_i^{2+\tau}+(\beta_{i1}(\cdot)+\beta_{i3}(\cdot)+\beta_{i5}(\cdot))\,z_i^{2+\tau}. \tag{A9}
$$

From (9) and Lemmas 2 and 3, it follows that

$$
\frac{\partial W_i}{\partial\hat{\Theta}}=-(2-r_i)\int_{\alpha_{i-1}}^{x_i}\left(s^{\frac{1}{r_i}}-\alpha_{i-1}^{\frac{1}{r_i}}\right)^{1-r_i}\mathrm{d}s\cdot\frac{\partial\alpha_{i-1}^{\frac{1}{r_i}}}{\partial\hat{\Theta}}\le(2-r_i)\left|\frac{\partial\alpha_{i-1}^{\frac{1}{r_i}}}{\partial\hat{\Theta}}\right|\cdot|z_i|^{1-r_i}|x_i-\alpha_{i-1}|\le(2-r_i)2^{1-r_i}\left|\frac{\partial\alpha_{i-1}^{\frac{1}{r_i}}}{\partial\hat{\Theta}}\right|\cdot|z_i|,
$$

which, together with the definition of $\gamma_{i-1}(\cdot)$, implies

$$
\begin{aligned}
\frac{\partial W_i}{\partial\hat{\Theta}}(\gamma_{i-1}(\cdot)+\beta_i^{\gamma}(\cdot)z_i^{2+\tau})+\sum_{j=2}^{i-1}\frac{\partial W_j}{\partial\hat{\Theta}}\beta_i^{\gamma}(\cdot)z_i^{2+\tau}&=\frac{\partial W_i}{\partial\hat{\Theta}}\sum_{j=1}^{i}\beta_j^{\gamma}(\cdot)z_j^{2+\tau}+\sum_{j=2}^{i-1}\frac{\partial W_j}{\partial\hat{\Theta}}\beta_i^{\gamma}(\cdot)z_i^{2+\tau}\\
&\le\frac{1}{3}\sum_{j=1}^{i-2}z_j^{2+\tau}+\frac{1}{4}z_{i-1}^{2+\tau}+\beta_{i6}(\rho_2,x_{[i]},\hat{\Theta})z_i^{2+\tau},
\end{aligned} \tag{A10}
$$

where $\beta_{i6}(\cdot)$ is a known smooth non-negative function.

Substituting (A10) into (A9) yields

$$
\dot{V}_i\le-(c_1+n-i)\sum_{j=1}^{i-1}z_j^{2+\tau}+z_i^{2-r_i}(x_{i+1}-\alpha_i)+z_i^{2-r_i}\alpha_i-\left(\tilde{\Theta}-\sum_{j=2}^{i}\frac{\partial W_j}{\partial\hat{\Theta}}\right)\left(\dot{\hat{\Theta}}-(\gamma_{i-1}+\beta_i^{\gamma}z_i^{2+\tau})\right)+\beta_i^{\phi}(\cdot)z_i^{2+\tau},
$$

where $\beta_i^{\phi}(\cdot)=\beta_{i1}(\cdot)+\beta_{i3}(\cdot)+\beta_{i5}(\cdot)+\beta_{i6}(\cdot)$. This implies (16) holds. ■