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Global fast finite-time partial state feedback stabilization of high-order nonlinear systems with dynamic uncertainties

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ABSTRACT

This paper is concerned with the problem of finite-time stabilization for a class of high-order nonlinear systems with zero dynamic. As a significant feature, the systems considered suffer from both the unmeasurable dynamic uncertainties and inherent nonlinearities, including high-order and low-order nonlinear growth rates. On the basis of integral Lyapunov functions equipped with sign functions and the notion of input-to-state stability, a partial state feedback stabilizer is proposed to provide a faster finite-time state convergence compared to traditional finite-time one. The novelty of this paper lies in a distinct perspective to applying the concept of fast finite-time stability developed recently in partial state feedback control design, which has been previously regarded as a rather difficult problem.

1. Introduction

Feedback linearization is one of the most popular methods in the field of nonlinear control. The key strategy of the feedback linearization approach is to transform the nonlinear system into a linear or at least partially linear one by a suitable change of coordinates together with a preliminary feedback. However, since high-order nonlinear systems have uncontrollable linearization around the origin, which makes the feedback linearization applicable, the stabilization of high-order nonlinear systems has been widely recognized as a challenging problem. Fortunately, with the aid of adding a power integrator technique initially proposed in [20], the difficulties arising from the lack of controllability of the linearization were resolved successfully, and a number of elegant asymptotic stabilization results for high-order nonlinear systems have been proposed in the past decades, such as [13,18,21,23,24,27].

On the other hand, due to the limited measurement and/or modeling techniques, system dynamics usually cannot be modeled accurately, and thereby resulting in dynamic uncertainty (i.e., zero dynamics). Inspired by small-gain theorem [10] and changing supply functions [26], a great deal of research has been devoted to stabilization of nonlinear system with dynamic uncertainty, as evidenced by the comprehensive book [10] and the references [14,15,28,34,36] to name just a few. It is necessary to point out that some extra conditions should be imposed on the systems to be investigated such as growth hypotheses. For example, under nonlinear growth rate, the study [14] solved the problem of asymptotic stabilization for a class of strict feedback cascade systems (i.e., including zero dynamics) by means of small-gain theorem and variable

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separation technique. As for high-order nonlinear systems, the work [28] was concerned with the case of uncertain control coefficients, and the paper [34] further investigated the robust regulation without a prior knowledge of the signs of control directions. Under stochastic environment, the works [9,15,36] focused on the effect of stochastic inverse dynamic for a class of high-order nonlinear systems.

Compared to asymptotic stabilization, the study of finite-time control has attracted considerable attention because of the benefits including faster convergence, high tracking precision and good robustness against various uncertainties. Based on the notion of finite-time stability together with the property of settling time function proposed in [1], the finite-time stabilization has been intensively studied for deterministic/stochastic nonlinear systems in the past two decades (see, e.g., [2–4,7,8,12,16,19,25,29,33,37,38]).

It is worth pointing out that, as shown in [17], the construction of a suitable Lyapunov function for ensuring finite-time stability is in general a difficult task due to the lack of effective nonsmooth analysis and constructive tools, especially in a complicated control plant. Besides, when the initial state is far away from the origin (trivial solution), the finite-time stabilization will deliver a slower convergence rate relative to the exponential convergence rate. Delightedly, these intrinsic obstacles were partly solved in [22] where a linear term is implanted into control design to speed up the convergent time. More recently, Sun et al. [30] proposed a fast finite-time stability theorem and provided a unified construction of Lyapunov function to implement fast finite-time stabilization for a class of high-order nonlinear systems by using adding a power integrator method. It should noted that the effect of dynamic uncertainties was not considered in [17,22,30]. As a result, an interesting question is put forward spontaneously:

Is it possible to extend the method of fast finite-time stabilization to high-order nonlinear systems with dynamic uncertainties?

To solve aforementioned question, one has to deal with three troublesome difficulties. The first difficulty is to investigate the influence of dynamic uncertainties; in this paper, a small-gain type condition is utilized to tackle dynamic uncertainties. Another intricate issue is associated with the nonlinearities which include both the high-order and low-order terms. As the third issue, a series of obstacles inevitably increase the complexity in the construction of integral Lyapunov functions combined with sign functions and the analysis of the global finite-time convergence. In consequence, the contributions of this paper are summarized as follows:

(i) This work delicately addresses the global fast finite-time stabilization problem for high-order nonlinear systems with dynamic uncertainties; thus, it provides a different insight on how to deal with finite-time stabilization for a more general class of nonlinear systems.
(ii) This paper presents a less conservative estimation of convergent time dependent on the initial value; moreover, the convergent time can be accelerated by suitably tuning design parameters. This improves the potential drawbacksin finite-time control.
(iii) With the aid of effective transformations and skillful manipulations of sign function, this paper presents a unified approach to simplifying the construction of integral Lyapunov functions combined with sign functions whereby the fast finite-time stability can be guaranteed.

2. Problem formulation and preliminaries

2.1. Preliminaries

We adopt the following notations throughout this paper. For a real vector $x=[x_1,\dots,x_n]^T\in\mathbb{R}^n$, $\bar{x}_i\triangleq[x_1,\dots,x_i]^T\in\mathbb{R}^i$, $i=1,\dots,n$, especially $\bar{x}_n=x$, and the norm $\|x\|$ of $x\in\mathbb{R}^n$ is defined by $\|x\|=\sqrt{\sum_{i=1}^n x_i^2}$. The symbol $\mathbb{R}^{\geq 1}_{odd}$ denotes the set of real numbers whose element is in the form of $\frac{q_1}{q_2}$ with q_1 and q_2 being positive odd integers and $q_1\geq q_2$. For a continuously differentiable function $V:\mathbb{R}^n\to\mathbb{R}^+$, it is positive definite if $V(x)\geq 0$ and $V(x)=0$ if and only if $x=0$; it is radially unbounded if $V(x)\to\infty$, $\|x\|\to\infty$. A function h: $[0, b)\to[0,\infty)$ is referred to as a $\mathcal{K}$ function if it is strictly increasing and $h(0)=0$. It is referred to as a $\mathcal{K}_\infty$ function if $b=\infty$ and $h(y)\to\infty$ as $y\to\infty$. The symbol $\ll$ denotes much less than. The arguments of functions are sometimes simplified, for instance, a function $f(x(t))$ can be denoted by $f(x)$, $f(\cdot)$ or f. To avoid the complex expressions of sign functions, a new definition is introduced as follows:

Definition 1. For a given positive constant a, define $\lceil y\rfloor^a\triangleq|y|^a\mathrm{sign}(y), \forall y\in\mathbb{R}$, where $\mathrm{sign}(y)=1$ if $y>0$; $\mathrm{sign}(y)=0$ if $y=0$; and $\mathrm{sign}(y)=-1$ if $y<0$.

The definition of finite-time stability proposed in [1] is given as follows.

Definition 2. Consider autonomous system

$$\dot{x}(t)=f(x(t)),\ \ f(0)=0, \tag{1}$$

where $f:U_0\to\mathbb{R}^n$ is continuous on an open neighborhood U_0 of the origin $x=0$. Suppose that $x(t)$ is defined on $[0,\infty)$. The equilibrium $x=0$ of system (1) is finite-time stable (FTS) if it is Lyapunov stable and finite-time convergent in a neighborhood $U\subseteq U_0$ of the origin. Finite-time convergence means that for any initial state $x_0\in U$, there is a function τ: $U\backslash\{0\}\to(0,\infty)$, such that every solution $x(t;x_0)$ of system (1) defined with $x(t;x_0)\in U\backslash\{0\}$ for $t\in(0,\tau(x_0))$ satisfies $\lim_{t\to\tau(x_0)}x(t;x_0)=0$ and $x(t;x_0)=0$ for any $t\geq\tau(x_0)$. When $U_0=\mathbb{R}^n$, the equilibrium $x=0$ is globally finite-time stable.

Next, we provide a useful lemma proposed in [30] which is used to characterize the feature of fast FTS and plays an important role in theoretical analysis.

Lemma 1 [30]. *Let continuously differentiable function $W:\mathbb{R}^n\to\mathbb{R}^+$ be positive definite and radially unbounded. Assume that time derivative of $W(x)$ along the solution of system (1) satisfies $\dot W(x(t))+m_1W^{\beta_1}(x(t))+m_2W^{\beta_2}(x(t))\le 0$ with $m_1,\ m_2>0$ and $\beta_1\ge 1$, $0<\beta_2<1$ being known constants. Then, there exists a finite-time $T\ge 0$, such that $x(t)=0$, $\forall t\ge T$, where*

$$T=\begin{cases}\dfrac{1}{m_2(1-\beta_2)}+\dfrac{W^{1-\beta_1}(x_0)-1}{m_1(1-\beta_1)}, & \beta_1>1,\\ \dfrac{1}{m_1(1-\beta_2)}\ln\left(1+\dfrac{m_1}{m_2}W^{1-\beta_2}(x_0)\right), & \beta_1=1.\end{cases}$$

Finally, we list some technical lemmas which will be employed in deriving the global finite-time stabilizer. The proofs of Lemmas 2–7 can be found in [13,21,30–32].

Lemma 2 [31]. *For given $r>0$ and every $x\in\mathbb{R}$, $y\in\mathbb{R}$, there holds $|x+y|^r\le c_r(|x|^r+|y|^r)$, where $c_r=2^{r-1}$ if $r\ge 1$, and $c_r=1$ if $0<r<1$. In particular, if $r\ge 1$ is an odd integer, there holds $|x-y|^r\le 2^{r-1}|x^r-y^r|$.*

Lemma 3 [21]. *For given positive real numbers m,n and a function $a(x,y)$, there holds*

$$|a(x,\ y)x^my^n|\le c(x,\ y)|x|^{m+n}+\frac{n}{m+n}\left(\frac{m}{(m+n)c(x,\ y)}\right)^{\frac{m}{n}}|a(x,\ y)|^{\frac{m+n}{n}}|y|^{m+n},$$

where $c(x,y)>0$, for any $x\in\mathbb{R}, y\in\mathbb{R}$.

Lemma 4 [30]. *Suppose $\frac{a}{b}\in\mathbb{R}^{\ge 1}_{odd}$, $b\ge 1$, then $|x^{\frac{a}{b}}-y^{\frac{a}{b}}|\le 2^{1-\frac{1}{b}}\left|\lceil x\rfloor^a-\lceil y\rfloor^a\right|^{\frac{1}{b}}$ for all $x\in\mathbb{R}, y\in\mathbb{R}$.*

Lemma 5 [13]. *For any real-valued continuous function $f(x,y)$ with $x\in\mathbb{R}^m$ and $y\in\mathbb{R}^n$, there are smooth scalar-value functions $a(x)\ge 0$, $b(y)\ge 0$, $c(x)\ge 1$, $d(y)\ge 1$ such that $|f(x,\ y)|\le a(x)+b(y)$, $|f(x,y)|\le c(x)d(y)$.*

Lemma 6 [30]. *The function $f(x)=\lceil x\rfloor^a, a\ge 1$ is continuously differentiable on $(-\infty,\ \infty)$, and its derivative satisfies $\dot f(x)=a|x|^{a-1}$ if $a>1$; $\dot f(x)=1$ if $a=1$.*

Lemma 7 [32]. *For the continuous function $f:[a,b]\to\mathbb{R}(a\le b)$, if it is monotone increasing and satisfies $f(a)=0$, then $|\int_a^b f(x)dx|\le|f(b)|\cdot|b-a|$.*

2.2. Problem formulation

This paper considers the following high-order nonlinear systems:

$$\begin{cases}\dot z\ =\ f_0(z,x_1),\\ \dot x_i=x_{i+1}^{p_i}+f_i(x,u,z),\ \ i=1,\ \dots,\ n-1,\\ \dot x_n=u^{p_n}+f_n(x,u,z),\end{cases}\tag{2}$$

where $x\in\mathbb{R}^n$ is measurable state, $z\in\mathbb{R}^m$ is unmeasured state, and $u\in\mathbb{R}$ is control input. Initial condition is $x(0)=x_0, z(0)=z_0$. For each $i=1,\dots,n$, $p_i\in\mathbb{R}^{\ge 1}_{odd}$ is named high-order of the systems, $f_i(\cdot)$ and f_0 are continuous functions. The objective is to design a continuous controller u such that the state of closed-loop systems globally converges to the origin in finite time for any initial condition $[x_0,z_0]^T\in\mathbb{R}^{n+m}$.

The following assumptions are needed.

Assumption 1. *For each $i=1,\dots,n$, there are nonnegative continuous functions $\bar f_0(\cdot)$, $\bar f_i(\cdot)$ with $\bar f_i(0)=0$ such that*

$$|f_i(\cdot)|\le\bar f_0(\|z\|)+\sum_{j=1}^{i}|x_j|^{\frac{p_jq_j}{q_{i+1}}}\bar f_i(\bar x_i),\tag{3}$$

where $q_1,\dots,q_{n+1}$ are recursively defined by $q_1=1$, $q_{j+1}=\frac{p_jq_j}{1+(\tau-1)q_j}$ for $j=1,\dots,n$ with $0<\tau<1$.

Assumption 2. *There exists a continuous differentiable function $V_0(z)$ such that*

$$\begin{cases}\underline{\pi}(\|z\|)\le V_0(z)\le\overline{\pi}(\|z\|),\\ \dfrac{\partial V_0(z)}{\partial z}f_0(z,x_1)\le -c_0V_0^{\alpha}(z)+r_0(|x_1|),\end{cases}\tag{4}$$

where $\overline{\pi}$, $\underline{\pi}$ and r_0 are $\mathcal{K}_\infty$ functions, and c_0, $\alpha<1$ are positive constants.

In order to explain the meaning of the inequality (3), let us consider the following example:

$$\begin{cases} \dot{z} = -z^{\frac{1}{3}} + \frac{5}{6}x_1^{\frac{1}{3}}, \\ \dot{x}_1 = x_2^{\frac{5}{3}} + x_1 \sin x_1 + z^3 \triangleq x_2^{\frac{5}{3}} + f_1, \\ \dot{x}_2 = u + x_2^{\frac{4}{3}} \cos x_1 + z^3 \triangleq u + f_2. \end{cases} \tag{5}$$

According to $p_1 = \frac{5}{3}$, $p_2 = 1$ and $q_1 = 1$, we can calculate

$$q_2 = \frac{p_1 q_1}{1 + (\tau - 1)q_1} = \frac{5}{3\tau}, \quad q_3 = \frac{p_2 q_2}{1 + (\tau - 1)q_2} = \frac{5}{8\tau - 5}, \quad \frac{p_2}{q_3} = \frac{8\tau - 5}{5}. \tag{6}$$

Since $0 < \tau < 1$ and $0 < \frac{p_2}{q_3} < 1$, selecting $\tau = \frac{3}{4} \in (\frac{5}{8}, 1)$, one gets $q_2 = \frac{20}{9}$ and $q_3 = 5$ from (6), which implies $\frac{p_1 q_1}{q_2} = \frac{3}{4}$, $\frac{p_2 q_1}{q_3} = \frac{1}{5}$, $\frac{p_2 q_2}{q_3} = \frac{4}{9}$. Then, we deduce from (3) that

$$\begin{aligned} |f_1| &= |x_1 \sin x_1 + z^3| \le |x_1|^{\frac{3}{4}} |x_1|^{\frac{1}{4}} + |z|^3, \\ |f_2| &= |x_2^{\frac{4}{3}} \cos x_1 + z^3| \le |x_2|^{\frac{4}{9}} |x_2|^{\frac{8}{9}} + |z|^3 \le (|x_1|^{\frac{1}{5}} + |x_2|^{\frac{4}{9}})|x_2|^{\frac{8}{9}} + |z|^3, \end{aligned}$$

so $\bar{f}_0 = |z|^3$, $\bar{f}_1 = |x_1|^{\frac{1}{4}}$ and $\bar{f}_2 = |x_2|^{\frac{8}{9}}$. One has to know the information of zero dynamics for control design. In general, zero dynamics at least is asymptotic stable [10,26]. Specifically, Assumption 2 has been widely used in the existing papers, such as [5,6,35]. As stated in [5], the relation (4) implies that $V_0(z)$ satisfies a small-gain type condition for small signals. However, it is just applicable to strict feedback systems; that is, $p_i = 1$ in (2). This paper combines it with certain nonlinear growth rates, and addresses the possibility of finite-time stabilization for a class of high-order nonlinear systems.

Finally, we use two concluding remarks to end this subsection.

Remark 1. It should be noticed that the existing results usually used the following restriction to illustrate the range of nonlinearities:

$$|f_i| \le \sum_{j=1}^{i} |x_j|^{a_j} \bar{f}_i(\bar{x}_i), \ \ i = 1, \dots, n,$$

where $a_1, \dots, a_n$ are appropriate constants and called the powers of growth rate, and $\bar{f}_1, \dots, \bar{f}_n$ are known continuous functions. For state feedback stabilization, it is intuitive to find the smaller value of a_j to enlarge the range of nonlinearities; hence, we adopt $a_j = \frac{p_i q_j}{q_{i+1}}$ in this paper and explain that Assumption 1 enlarges a class of high-order uncertain nonlinear systems by relaxing the restriction of nonlinearities from three aspects. (i) Proposition 1 in Section 3 shows that $0 < 1 - \tau = \frac{1}{q_i} - \frac{p_i}{q_{i+1}} \le \frac{p_i}{q_i} - \frac{p_i}{q_{i+1}}$, which implies $q_{i+1} \ge q_i$. Thus, $|x_j|^{p_i} = |x_j|^{\frac{p_i q_j}{q_{i+1}}} |x_j|^{p_i - \frac{p_i q_j}{q_{i+1}}}$, and the term $|x_j|^{p_i - \frac{p_i q_j}{q_{i+1}}}$ can be included in a new continuous function $\bar{f}_i(\bar{x}_i)$; that is, Assumption 1 in this paper is weaken than those in [13,36] with $a_j = p_i$. (ii) If $p_i = 1$, the inequality $q_{i+1} \ge q_i$ again allows that Assumption 1 is weaken than those in [8,21] with $a_j = 1$. (iii) If one selects $\omega = \tau - 1$ and $r_i = \frac{1}{q_i}$, then there is $\frac{p_i q_j}{q_{i+1}} = \frac{r_i + \omega}{r_j}$ which has been used in [27,29,30,37] with $a_j = \frac{r_i + \omega}{r_j}$. Some tedious calculations show that $a_j > 0$ and can remain close to 0 which can be proved by Proposition 1 in Section 3, thus high-order and low-order nonlinear growth rates can be emerged in the nonlinearities simultaneously. The later theoretical analysis will exhibit the advantages of $a_j = \frac{p_i q_j}{q_{i+1}}$ in coping with unmeasurable states in (3).

Remark 2. The celebrated terminal sliding mode control forces trajectories to reach a sliding manifold in finite time and stay on the manifold for all future time. Except for better robustness, the prominent advantage of sliding mode control also lies in that more serious uncertainties can be included in the system to be investigated, such as completely unknown nonlinearities. Inspired by this strategy, a number of interesting results have been proposed in the past decades [10,25]. However, sliding mode control suffers from chatting because of imperfections in switching devices and delays. Chattering may excite unmodeled high-frequency dynamics, which degrades the performance of the system and may even lead to instability. As a result, this paper aims to achieve the fast finite-time partial state feedback stabilization instead of traditional sliding mode control, at the expense of known structure of nonlinearities.

3. Main results

First of all, we list useful properties of q_i which are crucial in control design.

Proposition 1. *For each $i = 1, \dots, n$, q_i satisfies*

$$q_i \ge 1, \ \ 2 - \frac{1}{q_i} + \frac{p_i}{q_{i+1}} = 1 + \tau, \ \ 0 < \frac{p_i}{q_{i+1}} < 1.$$

Proof. See Appendix A. □

Then, we introduce the following variable transformations:

$$\begin{cases} z_i = \lceil x_i \rceil^{q_i} - \lceil \alpha_{i-1}(\bar{x}_{i-1}) \rceil^{q_i}, \\ u = \alpha_n(x), \\ \alpha_i(\bar{x}_i) = -g_i^{\frac{1}{q_{i+1}}}(\bar{x}_i) \lceil z_i \rceil^{\frac{1}{q_{i+1}}}, \end{cases} \tag{7}$$

where $i = 1, \ldots, n,\ g_1(\cdot), \ldots, g_n(\cdot)$ are smooth positive functions to be specified later. For the sake of consistency, we let $g_0 = 0$ and $\alpha_0 = 0$. It follows from Lemma 6 and $q_i \geq 1$ that $z_1, \ldots, z_n$ are continuously differentiable on t. We emphasize that the sign functions are introduced in (7) to guarantee the validity of the transformations.

To solve the troublesome problem caused by sign functions, we define $W_k : \mathbb{R}^k \to \mathbb{R}$ as

$$W_k(\bar{x}_k) = \int_{\alpha_{k-1}(\bar{x}_{k-1})}^{x_k} \lceil F(s, \bar{x}_{k-1}) \rfloor^{2-\frac{1}{q_k}} \, ds, \ \ k = 2, 3, \ldots, n,$$

where $F(s, \bar{x}_{k-1}) = \lceil s \rfloor^{q_k} - \lceil \alpha_{k-1}(\bar{x}_{k-1}) \rceil^{q_k}$. Obviously, $W_k(\cdot)$ is continuously differentiable. According to Lemmas 4 and 7, it is not hard to prove that W_k satisfies

$$\frac{\partial W_k}{\partial x_k} = \lceil z_k \rfloor^{2-\frac{1}{q_k}}, \tag{8}$$

$$\frac{\partial W_k}{\partial x_i} = -\left(2 - \frac{1}{q_k}\right) \int_{\alpha_{k-1}}^{x_k} |F(s, \bar{x}_{k-1})|^{1-\frac{1}{q_k}} ds \cdot \frac{\partial}{\partial x_i} (\lceil \alpha_{k-1} \rfloor^{q_k}), \tag{9}$$

$$c_{k1} |x_k - \alpha_{k-1}|^{2q_k} \leq W_k \leq c_{k2} z_k^2, \tag{10}$$

where $c_{k1} = \frac{1}{2q_k} 2^{\frac{-2q_k^2 + 3q_k - 1}{q_k}}$, $c_{k2} = 2^{1-\frac{1}{q_k}}$, $i = 1, \ldots, k-1$. Based on (7), one has

$$u = -\left\lceil \sum_{l=1}^{n} \left(\prod_{j=l}^{n} g_j \right) \lceil x_l \rfloor^{q_l} \right\rceil^{\frac{1}{q_{n+1}}}. \tag{11}$$

Clearly, in order to achieve the detailed expression of u, we have to determine $g_1, \ldots, g_n$, which can be completed by the following proposition.

Proposition 2. *For each $k = 1, \ldots, n$, the function g_k can be calculated through the relationship $g_k = \left(\phi_k + n - k + 1 + (1 + z_k{}^2)^{(1-\tau)/2}\right)^{\frac{q_{k+1}}{p_k}}$, and a smooth positive function ϕ_k is delicately chosen to guarantee that the time derivative of V_k defined as $V_1 = \frac{1}{2} z_1^2$ for $k = 1$ and $V_k = \frac{1}{2} z_1^2 + \sum_{i=2}^{k} W_k$ for $k = 2, \ldots, n$ along the trajectories of systems (2) satisfies*

$$\begin{aligned} \dot{V}_k \leq & -\big(1 + \rho(x_1)\big) r_0(|x_1|) - (n - k + 1) \sum_{i=1}^{k} |z_i|^{1+\tau} + \sum_{i=1}^{k} \bar{f}_0^{\frac{(1+\tau)q_{i+1}}{p_i}} \\ & + k \bar{f}_0^{\frac{1+\tau}{\tau}} - \sum_{i=1}^{k} z_i^2 + \lceil z_k \rceil^{2-\frac{1}{q_k}} (x_{k+1}^{p_k} - \alpha_k^{p_k}), \end{aligned} \tag{12}$$

where $x_{n+1} = u$, and ρ is a continuously differentiable function to be specified later.

Proof. See Appendix A. □

Remark 3. It is necessary to point out the difficulties in the proof of Proposition 2 as follows. (i) It is not easy to manipulate $r_0(\cdot)$ which is not differentiable but continuous, since the small-gain type conditions for small signals must be used delicately. (ii) Although the powers of growth rate $\frac{p_i q_j}{q_{i+1}}$ are successfully used to separate $\bar{f}_0$ from $x_1, \ldots, x_i$, how to dominate $\bar{f}_0$ remains unsolved. (iii) Sign functions guarantee the validity of the transformation and integral Lyapunov function, which leads to a more intricate construction of smooth functions g_i's. (iv) The introduction of $\rho(\cdot)$ creates an extra freedom in dealing with unmeasurable states and destabilized terms in control design, but it cannot dominate $\bar{f}_0$ completely because of the irrelevance of x_1 and z.

With Proposition 2 in mind, we state the main results of the paper.

Theorem 1. *Under Assumptions 1 and 2, if the following conditions are satisfied:*

$$\limsup_{s \to 0^+} \frac{r_0(s)}{s^2} < \infty, \ \ \limsup_{s \to 0^+} \frac{\bar{f}_0^2(s)}{\underline{\pi}(s)} < \infty, \tag{13}$$

then there exists a continuous state-feedback controller guaranteeing that the state of closed-loop systems composed of (2), (11) globally converges to the origin in finite time, and the convergent speed is faster than that of finite-time convergence.

Proof. For clarification, the whole proof is divided into three parts.

(a) Positive definiteness and radial unboundedness of $V(x)$**.** Select the Lyapunov function as

$$V(x,z)=V_n(x)+V_c(z),\ \ V_c(z)=\int_0^{V_0(z)}\zeta(s)ds, \tag{14}$$

and the function $\zeta:\mathbb{R}^+\to\mathbb{R}^+$ is specified as follows. Note that $\frac{1+\tau}{\tau}>2$ and $\frac{(1+\tau)q_{i+1}}{p_i}>2$ which are obtained by the identity $1+\tau=2-\frac{1}{q_i}+\frac{p_i}{q_{i+1}}$. It follows from (13) and the boundedness of $\bar{f}_0^{\frac{1-\tau}{\tau}}$ and $\bar{f}_0^{(q_{i+1}(1+\tau)-2p_i)/p_i}$ near the origin that $\limsup_{s\to0^+}\frac{\hat{k}_1(s)}{\underline{\pi}^\alpha(s)}<\infty$, where $\hat{k}_1(\|z\|)=n\bar{f}_0^{\frac{1+\tau}{\tau}}+\sum_{i=1}^n\bar{f}_0^{(1+\tau)q_{i+1}/p_i}(\|z\|)$. Now, we can define $\zeta(s)$ as

$$\zeta(s)=\begin{cases}\dfrac{2}{(1-\varepsilon)c_0}\limsup\limits_{s\to0^+}\dfrac{\hat{k}_1(s)}{\underline{\pi}^\alpha(s)}+1,\ \ s=0,\\[2ex] \dfrac{2}{(1-\varepsilon)c_0}\sup\limits_{0<s'\le s}\dfrac{\hat{k}_1(s')}{\underline{\pi}^\alpha(s')}+1,\ \ s>0,\end{cases} \tag{15}$$

where $0<\varepsilon<1$ is a given constant. It can be verified that $\zeta(s)$ is continuous, nondecreasing and positive over $[0,\infty)$.

In light of (10), one immediately arrives at

$$V(x,z)\ge\sum_{k=1}^n c_{k1}|x_k-\alpha_{k-1}|^{2q_k}+V_c(z)\triangleq U_n(x)+V_c(z), \tag{16}$$

where $c_{11}=\frac{1}{2}$. As the similar procedures in [30], one can prove that $U_n(x)$ is positive definite and radially unbounded. In addition, with $\zeta(s)>0$ in mind, it is easy to prove that $V_c(z)$ is positive definite. By using the mean value theorem in [10], $\zeta(s)>1$ and (4), one has

$$V_c(z)=\int_0^{V_0(z)}\zeta(s)ds=\zeta(\theta V_0(z))V_0(z)\ge V_0(z)\ge\underline{\pi}(\|z\|)\to\infty,\ \ \|z\|\to\infty,$$

where $0<\theta<1$. In other words, $V_c(z)$ is radially unbounded. As a result, $V(x,z)$ is positive definite and radially unbounded.

(b) Calculation on time derivative of V**.** In view of delicate constructions of $g_1,\dots,g_n$ in Proposition 2, one can deduce that

$$\dot{V}_n\le-\big(1+\rho(x_1)\big)r_0(x_1)-E_n(x)+\hat{k}_1(\|z\|), \tag{17}$$

where $E_n(x)=\sum_{i=1}^n|z_i|^{1+\tau}+\sum_{i=1}^n z_i^2$. It follows from (14) and Assumption 2 that

$$\begin{aligned}\dot{V}&\le-\big(1+\rho(x_1)\big)r_0(|x_1|)-E_n(x)+\hat{k}_1(\|z\|)\\&\quad-c_0\zeta\big(V_0(z)\big)V_0^\alpha(z)+\zeta\big(V_0(z)\big)r_0(|x_1|).\end{aligned} \tag{18}$$

Now, one can determine $\rho(x_1)$ according to the inequality

$$\rho(x_1)+1\ge\zeta\bigg(\Big(\frac{r_0(|x_1|)}{c_0\varepsilon}\Big)^{\frac{1}{\alpha}}\bigg),$$

which yields

$$\big(1+\rho(x_1)\big)r_0(|x_1|)\ge\zeta\bigg(\Big(\frac{r_0(|x_1|)}{c_0\varepsilon}\Big)^{\frac{1}{\alpha}}\bigg)r_0(|x_1|)\ge0. \tag{19}$$

In what follows, we will prove

$$\begin{aligned}&-\big(1+\rho(x_1)\big)r_0(|x_1|)-c_0\zeta\big(V_0(z)\big)V_0^\alpha(z)+\hat{k}_1(\|z\|)+\zeta\big(V_0(z)\big)r_0(|x_1|)\\&\le-\frac{(1-\varepsilon)c_0}{2}\zeta\big(V_0(z)\big)V_0^\alpha(z).\end{aligned} \tag{20}$$

To begin with, (15) shows

$$-\frac{(1-\varepsilon)c_0}{2}\zeta\big(V_0(z)\big)V_0^\alpha(z)+\hat{k}_1(\|z\|)\le0. \tag{21}$$

In addition, one has

$$-\big(1+\rho(x_1)\big)r_0(|x_1|)-c_0\varepsilon\zeta\big(V_0(z)\big)V_0^\alpha(z)+\zeta\big(V_0(z)\big)r_0(|x_1|)\le0, \tag{22}$$

which can be done by considering two cases. (i) If $V_0(z) \le \left(\frac{r_0(|x_1|)}{c_0\varepsilon}\right)^{\frac{1}{\alpha}}$. On the basis of the monotonicity of $\zeta(\cdot)$ and (19), the inequality $\big(1+\rho(x_1)\big)r_0(|x_1|) \ge \zeta(\cdot)r_0(|x_1|) \ge 0$ holds. (ii) If $V_0(z) > \left(\frac{r_0(|x_1|)}{c_0\varepsilon}\right)^{\frac{1}{\alpha}}$, which is equivalent to $c_0\varepsilon V_0^{\alpha}(z) - r_0(|x_1|) > 0$, we have $-c_0\varepsilon\zeta\big(V_0(z)\big)V_0^{\alpha}(z) + \zeta\big(V_0(z)\big)r_0(|x_1|) \le 0$. Therefore, combining (21) and (22) together gives the correctness of (20). In view of Lemma 2 and (10), it is clear that

$$\begin{aligned}
-E_n(x) &= -\sum_{i=1}^{n} |z_i^2|^{\frac{1+\tau}{2}} - \sum_{i=1}^{n} z_i^2 \\
&\le -\left(\sum_{i=1}^{n} z_i^2\right)^{\frac{1+\tau}{2}} - \sum_{i=1}^{n} z_i^2 \\
&\le -\left(\frac{1}{2}\right)^{\frac{1+\tau}{2}} V_n^{\frac{1+\tau}{2}} - \frac{1}{2}V_n.
\end{aligned} \tag{23}$$

Substituting (20) and (23) into (18) renders

$$\dot{V} \le -\left(\frac{1}{2}\right)^{\frac{1+\tau}{2}} V_n^{\frac{1+\tau}{2}} - \frac{1}{2}V_n - \frac{(1-\varepsilon)c_0}{2}\zeta\big(V_0(z)\big)V_0^{\alpha}(z) \triangleq -V^*(x,z) \le 0. \tag{24}$$

It is obvious that $V^*(x, z)$ is positive definite and radially unbounded.

(c) Global finite-time convergence. According to (24), it can be deduced from Theorem 1 in page 23 of [11] that the trivial solution $z=0, x=0$ of the closed-loop systems is globally strongly stable; in other words, each admissible solution $[z(t), x(t)]^T$ is defined on $[0, \infty)$ and $\lim_{t\to\infty} z(t) = 0$, $\lim_{t\to\infty} x(t) = 0$.

Define $U(\delta) = \{Y \in \mathbb{R}^{m+n} \,\big|\, \|Y\| \le \delta\}$ with $Y = [z, x]^T$, where the positive constant δ is specified as follows. It is apparent to obtain

$$\lim_{\|z\|\to 0} \frac{V_0^{\alpha}(z)}{\zeta\big(V_0(z)\big)V_0^{\alpha}(z)} = \lim_{\|z\|\to 0} \frac{1}{\zeta\big(V_0(z)\big)} = \frac{1}{\zeta(0)} < \infty,$$

which implies that there is a $\delta_1 > 0$ such that $V_0^{\alpha}(z) \le c_2\zeta(V_0(z))V_0^{\alpha}(z)$, $\forall\ \|z\| < \delta_1$, where c_2 is a positive constant. Moreover, the definition of $V_c(z)$ shows

$$\lim_{\|z\|\to 0} \frac{V_c(z)}{V_0(z)} = \lim_{\|z\|\to 0} \zeta\big(V_0(z)\big) < \infty.$$

Thus, there is a constant $\delta_2 > 0$ such that $V_c(z) \le c_3V_0(z)$ with c_3 being a positive constant for all $\|z\| < \delta_2$. With this in mind, the inequality $V_c^{\alpha}(z) \le c_3^{\alpha}c_2\zeta(V_0(z))V_0^{\alpha}(z)$ holds for all $\|z\| < \min\{\delta_1, \delta_2\}$; that is,

$$-\frac{(1-\varepsilon)c_0}{2}\zeta(V_0(z))V_0^{\alpha}(z) \le -\frac{(1-\varepsilon)c_0c_4}{2}V_c^{\alpha}(z),\ \ c_4 \triangleq \frac{1}{c_3^{\alpha}c_2}. \tag{25}$$

By the continuity of $V_c^{1-\alpha}(z)$ and $V_c^{1-\alpha}(0) = 0$, there exists a constant $\delta_3 > 0$ to guarantee

$$V_c^{1-\alpha}(z) < \frac{c_4c_0(1-\varepsilon)}{2},\ \ \forall\ \|z\| < \delta_3. \tag{26}$$

Similarly, the continuity of $V(Y)$ and $V(0) = 0$ imply that there exists a constant $\delta_4 > 0$ such that $V(Y) < 1$ for all $\|Y\| < \delta_4$. Now, one can conclude that $\delta < \min\{\delta_1, \ldots, \delta_4\}$. Using (25) and (26) in (24), we have

$$\begin{aligned}
\dot{V} &\le -\frac{(1-\varepsilon)c_0c_4}{2}V_c^{\alpha}(z) - \left(\frac{1}{2}\right)^{\frac{1+\tau}{2}} V_n^{\frac{1+\tau}{2}} - \frac{1}{2}V_n \\
&= -\frac{(1-\varepsilon)c_0c_4}{4}V_c^{\alpha}(z) - \left(\frac{1}{2}\right)^{\frac{1+\tau}{2}} V_n^{\frac{1+\tau}{2}} - \frac{1}{2}(V_n + V_c) \\
&\quad + \frac{1}{2}V_c^{\alpha}\left(V_c^{1-\alpha} - \frac{(1-\varepsilon)c_0c_4}{2}\right) \\
&\le -m_1V^{\beta_1} - m_2V_c^{\alpha} - m_2V_n^{\bar{\alpha}},
\end{aligned} \tag{27}$$

where $\beta_1 = 1, \bar{\alpha} = \frac{1+\tau}{2} < 1, m_1 = \frac{1}{2}, m_2 = \min\{\frac{(1-\varepsilon)c_0c_4}{4}, \frac{1}{2^{\bar{\alpha}}}\}$. Note that $V < 1$ means $V_c < 1$ and $V_n < 1$. It follows from Lemma 2 that

$$V_c^{\alpha} + V_n^{\bar{\alpha}} \ge V_c^{\beta_2} + V_n^{\beta_2} \ge (V_c + V_n)^{\beta_2} = V^{\beta_2},$$

where $\beta_2 = \max\{\alpha, \bar{\alpha}\}$. As a result, (27) can be rewritten as $\dot{V} + m_1V^{\beta_1} + m_2V^{\beta_2} \le 0$.

Up to now, we can prove the global finite-time convergence of $Y(t)$. (i) If $Y(0) \in U(\delta)$. It can be deduced from Lemma 1 that $Y(t)$ converges to zero within a finite time $T = \frac{1}{m_1(1-\beta_2)}\ln\left(1 + \frac{m_1}{m_2}V^{1-\beta_2}(Y(0))\right)$. (ii) If $Y(0) \notin U(\delta)$. Let $\bar{T}$ be the first time that $Y(t)$ intersects the boundary of $U(\delta)$, then $\|Y(t)\| > \delta$ for $t \in [0, \bar{T})$, and $\|Y(t)\| \le \delta$ for $t \in [\bar{T}, \infty)$ because of $\dot{V} \le 0$. Of course, $\bar{T}$ must be finite, this is proved as follows: Lemma 4.3 in [10] implies that there exists a $\mathcal{K}_\infty$ function $\pi(\cdot)$ such that $\pi(\|Y\|) \le V^*(Y)$, it follows from (24) that

$$V(Y(0)) \geq V(Y(0)) - V(Y(t)) \geq \int_0^t V^*(Y(s))ds$$
$$\geq \int_0^t \pi(\|Y(s)\|)ds \geq \int_0^t \pi(\delta)ds = t\pi(\delta),$$

this promises $0 \le t \le \frac{V(Y(0))}{\pi(\delta)} < \infty$. Thus one can choose $\bar{T} = \frac{V(Y(0))}{\pi(\delta)}$. The conclusion follows from the case of (i) with the convergent time $T = \frac{V(Y(0))}{\pi(\delta)} + \frac{1}{m_1(1-\beta_2)} \ln\left(1 + \frac{m_1}{m_2} V^{1-\beta_2}(Y(\bar{T}))\right)$.

Finally, we make some comparisons on the convergence. If $Y(0) \in U(\delta)$, Lemma 1 shows that there exists a finite time $T(m_2, \beta_2) = \frac{1}{m_1(1-\beta_2)} \ln\left(1 + \frac{m_1}{m_2} V^{1-\beta_2}(Y(0))\right)$, such that $Y(t) = 0$ for all $t \ge T$. On the other hand, the finite time controller in [1,5,8] provides that the convergent time is $T_1(m_2, \beta_2) = \frac{V^{1-\beta_2}(Y(0))}{m_2(1-\beta_2)}$. It is easy to see that

$$T(m_2, \beta_2) = \frac{2}{1-\beta_2} \ln\left(1 + \frac{V^{1-\beta_2}(Y(0))}{2m_2}\right)$$
$$\ll \frac{2}{1-\beta_2} \frac{V^{1-\beta_2}(Y(0))}{2m_2} = T_1(m_2, \beta_2), \ \ 0 < m_2 \ll 1.$$

If $Y(0) \notin U(\delta)$, there exists a new convergent time $T(m_2, \beta_2) = \frac{V(Y(0))}{\pi(\delta)} + \frac{2}{1-\beta_2} \ln\left(1 + \frac{V^{1-\beta_2}(Y(\bar{T}))}{2m_2}\right)$. In what follows, one can prove $T \ll T_1$ by the delicate choice of m_2. First, we choose a small m_2 to ensure $m_2 \ll \frac{\pi(\delta)}{4(1-\beta_2)V^{\beta_2}(Y(0))}$ with defined δ and β_2, which means $\frac{V(Y(0))}{\pi(\delta)} \ll \frac{1}{4}T_1$. Then, one further restricts $m_2 \ll \frac{1}{2}V^{1-\beta_2}(Y(0))$, and deduces from the decreasing property of $V(Y)$ that

$$\frac{2}{1-\beta_2} \ln\left(1 + \frac{V^{1-\beta_2}(Y(\bar{T}))}{2m_2}\right) \le \frac{2}{1-\beta_2} \ln\left(1 + \frac{V^{1-\beta_2}(Y(0))}{2m_2}\right) \ll \frac{1}{2}T_1,$$

where the last inequality uses the fact that $\ln(1+x) \ll \frac{x}{2}$ for all $x \gg 1$. Finally, the selection of $0 < m_2 \ll \min\{\frac{\pi(\delta)}{4(1-\beta_2)V^{\beta_2}(Y(0))}, \frac{1}{2}V^{1-\beta_2}(Y(0))\}$ renders $T \ll \frac{3T_1}{4}$, and this completes the proof. □

Remark 4. We emphasize the contributions of Theorem 1 from four aspects. (i) This paper successfully extends the method of fast finite-time stabilization in [30] to high-order nonlinear systems with dynamic uncertainties. (ii) Due to the relaxed restrictions on dynamic uncertainties, we introduce a new positive definite function $V_c(z)$ instead of directly using $V_0(z)$ provided in Assumption 2; however, it is really difficult to specify it through small-gain type conditions for small signals. (iii) Derivations of time derivative of V is done by extending the idea of changing supply functions [26] to high-order uncertain nonlinear systems, where the main technique is the introduction of the new function $\rho(x_1)$. (iv) The construction of the compact set $U(\delta)$ is more convenient than those in [5,6,29], which confines the norm of $[z, x]^T$ to a small neighborhood instead of the range determined by the corresponding Lyapunov functions. Moreover, the parameters β_2 and m_2 are introduced to accelerate the convergence speed without costing large control effort.

4. Simulation example

As an application of the design method, we consider the numerical example:

$$\begin{cases} \dot{z} = -z^{\frac{1}{3}} + \frac{5}{6}x_1^{\frac{1}{3}}, \\ \dot{x}_1 = x_2^{\frac{5}{3}} + x_1 \sin x_1 + z^3 \triangleq x_2^{\frac{5}{3}} + f_1, \\ \dot{x}_2 = u + x_2^{\frac{4}{3}} \cos x_1 + z^3 \triangleq u + f_2. \end{cases} \tag{28}$$

Apparently, f_1 and f_2 satisfy Assumption 1; moreover, we have the following

$$|f_1| \le |x_1| + |z|^3 \le |x_1|^{\frac{3}{4}} \cdot \bar{f}_1 + \bar{f}_0(\|z\|),$$
$$|f_2| \le |x_2|^{\frac{4}{3}} + |z|^3 \le (|x_1|^{\frac{1}{5}} + |x_2|^{\frac{4}{9}}) \cdot \bar{f}_2 + \bar{f}_0(\|z\|),$$

where $\bar{f}_1 = (1+x_1^2)^{\frac{1}{8}}$, $\bar{f}_2 = (1+x_2^2)^{\frac{4}{9}}$, and $\bar{f}_0(\|z\|) = |z|^3$. Besides, by choosing $V_0 = \frac{1}{2}z^4$ and using Lemma 3, one has

$$\dot{V}_0 = 2z^3\dot{z} = 2z^3(-z^{\frac{1}{3}} + \frac{5}{6}x_1^{\frac{1}{3}}) \le -2z^{\frac{10}{3}} + \frac{5}{3}|z|^3 \cdot |x_1^{\frac{1}{3}}| \le -\frac{1}{2}|z|^{\frac{10}{3}} + \frac{1}{6}|x_1|^{\frac{10}{3}},$$

which implies that Assumption 2 holds with

$\alpha = \frac{5}{6}, c_0 = 1, r_0(x_1) = \frac{1}{6}|x_1|^{\frac{10}{3}}$.

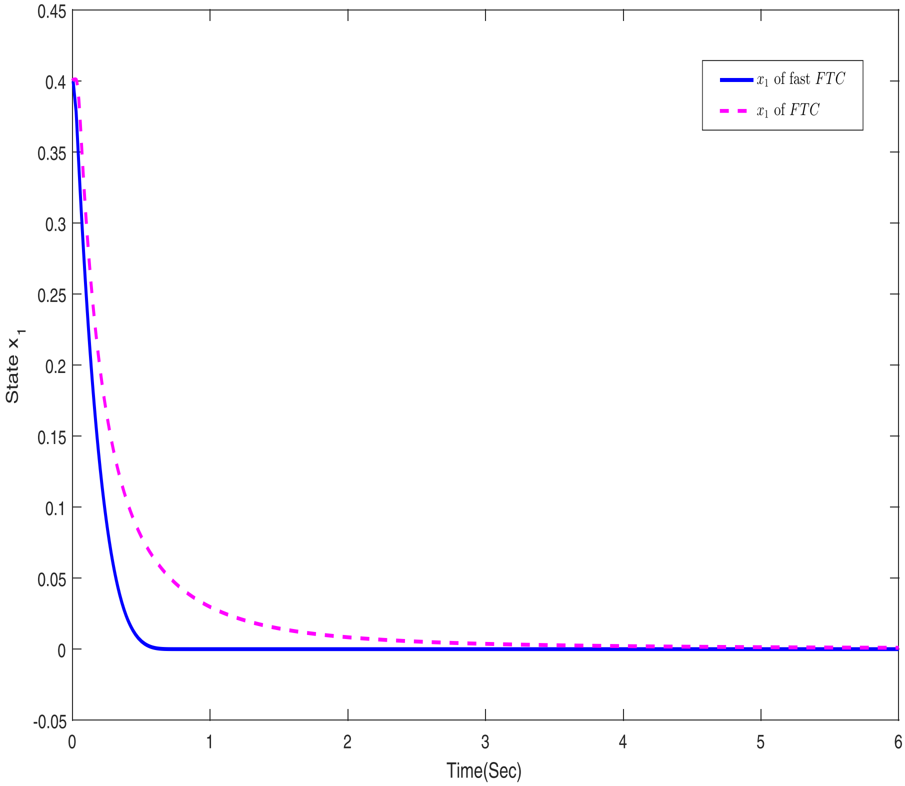

Fig. 1. The trajectories of the state x_1.

According to Proposition 2, one can calculate that $\phi_1 = 0.5 + \bar{f}_1 + (1+\rho(x_1))r_2$, $\phi_2 = 2.05 + \bar{f}_1(1+g_1^{\frac{1}{5}}) + 0.77\big(\bar{f}_1(1+g_1^{\frac{1}{5}})\big)^{\frac{35}{31}} + 1.46g_1(\bar{f}_1 + g_1^{\frac{3}{4}}) + \big(1.34(\bar{f}_1 + g_1^{\frac{3}{4}})^{\frac{7}{4}} + 1\big)g_1^{\frac{7}{4}}$, where $r_2 = \frac{1}{6}(\sqrt{1+x_1^2})^{\frac{19}{12}}$, $\rho(x_1) + 1 \geq \zeta\big(0.13x_1^4\big)$, and $\zeta(s) = \frac{2}{1-\varepsilon}\limsup_{s\to 0+}\frac{\hat{k}_1(s)}{\underline{\pi}^{\alpha}(s)} + 1$ if $s = 0$, and $\zeta(s) = \frac{2}{1-\varepsilon}\sup_{0<s'\leq s}\frac{\hat{k}_1(s')}{\underline{\pi}^{\alpha}(s')} + 1$ if $s > 0$, $\hat{k}_1(s) = 3s^7 + s^{\frac{105}{4}}$, $\underline{\pi}(s) = s^4$. Through complex calculations, the actual adaptive control can be constructed as $u(t) = -\big(g_1g_2x_1 + g_2\lceil x_2\rceil^{\frac{20}{9}}\big)^{\frac{1}{5}}$, where $g_1 = \big(\phi_1 + 2 + (1+z_1^2)^{\frac{1}{8}}\big)^{\frac{4}{3}}$, $g_2(\bar{x}_2) = \big(1 + \phi_2 + (1 + (x_2^{\frac{20}{9}} + g_1x_1)^2)^{\frac{1}{8}}\big)^5$. However, if we adopt the method in [1,5,6,29], then the traditional finite-time controller is given by $\tilde{u}(t) = -\Big(\tilde{g}_1\tilde{g}_2x_1 + \tilde{g}_2\lceil x_2\rceil^{\frac{20}{9}}\Big)^{\frac{1}{5}}$, where $\tilde{g}_1 = \big(\phi_1 + 2\big)^{\frac{4}{3}}$, $\tilde{g}_2 = \big(\phi_2 + 1\big)^5$.

In simulation, the initial conditions are chosen as $x_1(0) = 0.4$, $x_2(0) = -0.25$, $z(0) = 0.3$, and $\varepsilon = 0.9$. Figs. 1–4 show that the convergent time of fast finite-time control in this paper is less than existing results, such as [1,5,6,29]. Specifically, x_1 tends to 0 at 0.5 s in the fast finite-time controller, and tends to 0 at 6 s in the traditional finite-time controller; apparently, there is 5.5-s disparities between two methods. Similarly, x_2 goes to 0 at 0.8 s in the fast finite-time controller versus 4.5 s in the traditional finite-time controller, and z converges to 0 at 1 s in the fast finite-time controller versus 5 s in the traditional finite-time controller. There are no dramatic changes of control magnitudes between the fast finite-time controller and the traditional finite-time controller, but the former even has faster convergent speed.

The presented scheme is also applicable to the single-link robotic manipulator system in [10], which is described by

$$ml\ddot{\theta}(t) = u(t) - mg\sin(\theta(t)) - fl\dot{\theta}(t), \tag{29}$$

where the robotic manipulator m revolves around the pivot under the action of force, l be the distance between the centroid of robotic manipulator and the pivot, f is the unknown viscous friction coefficient, and $mg\sin(\theta(t))$ is a component of the gravity along the tangent direction of the moving direction of the moving direction. With the definitions of $x_1(t) = ml\theta(t)$, $x_2(t) = ml\dot{\theta}(t)$, the equation of motion (29) can be rewritten as

$$\dot{x}_1 = x_2,\ \ \dot{x}_2 = u - mg\sin\left(\frac{x_1}{ml}\right) - \frac{f}{m}x_2. \tag{30}$$

It is obviously that there is no zero dynamic and we choose $q_1 = 1$ and $q_2 = \frac{5}{4}$. It is easy to verify that $f_1 = 0$ and $f_2 = -mg\sin(\frac{x_1}{ml}) - \frac{f}{m}x_2$ satisfying Assumption 1 with $\bar{f}_1 = 0$, $\bar{f}_2 = (\frac{g}{l} + \frac{f}{m})\sqrt{x_1^2 + x_2^2}$. One knows that $p_1 = p_2 = 1$, $\tau = \frac{4}{5}$, $q_3 = \frac{5}{3}$. Choose $m = \frac{6}{5}$Kg, $l = \frac{20}{3}$m, $f = \frac{5}{2}$Pas. Followed by the design procedure of (11), some tedious calculations lead

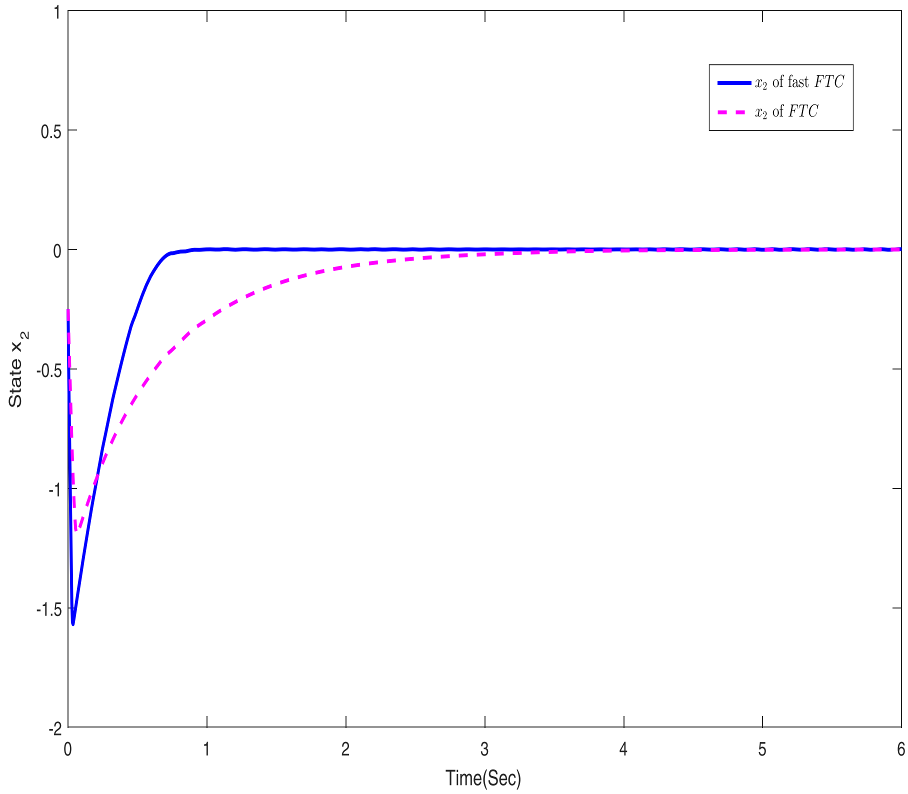

Fig. 2. The trajectories of the state x_2.

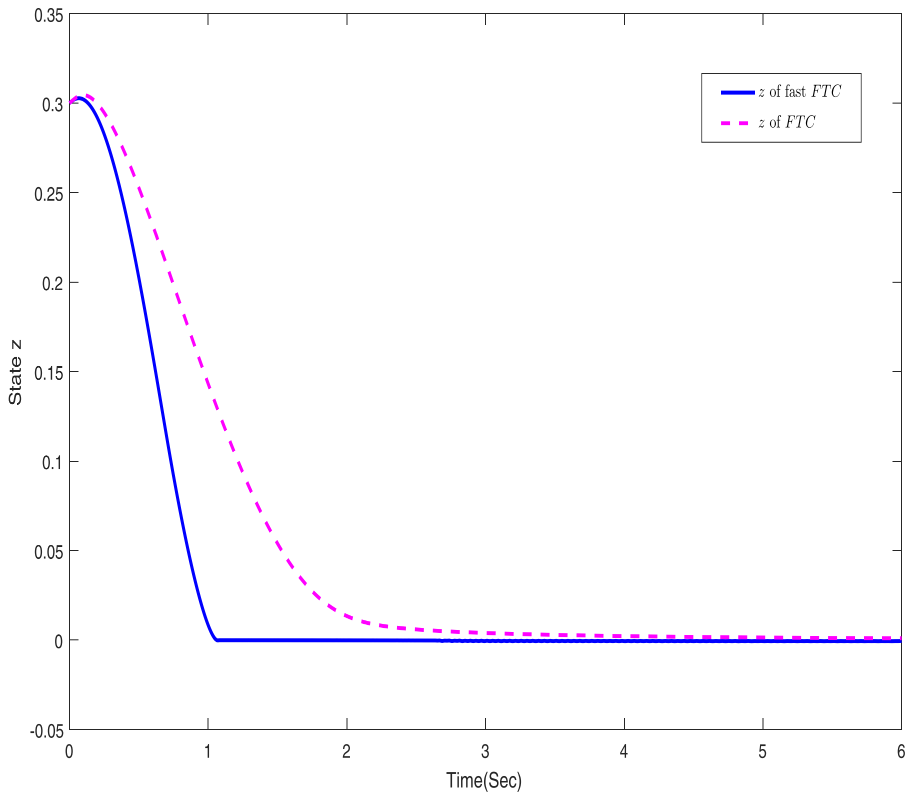

Fig. 3. The trajectories of control z.

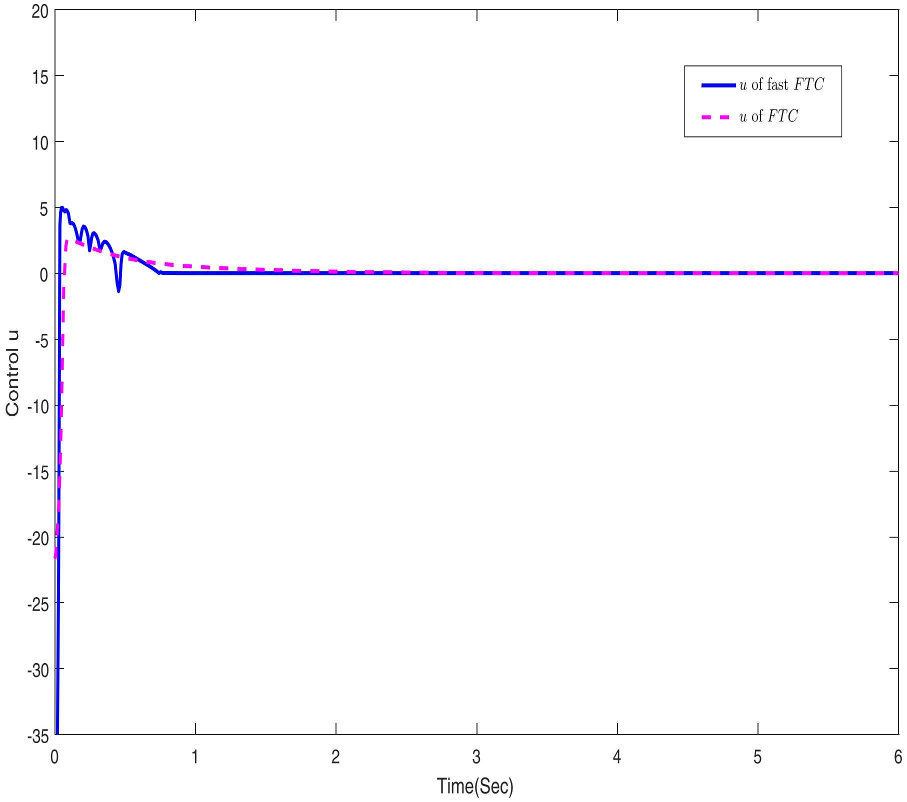

Fig. 4. The trajectories of control u.

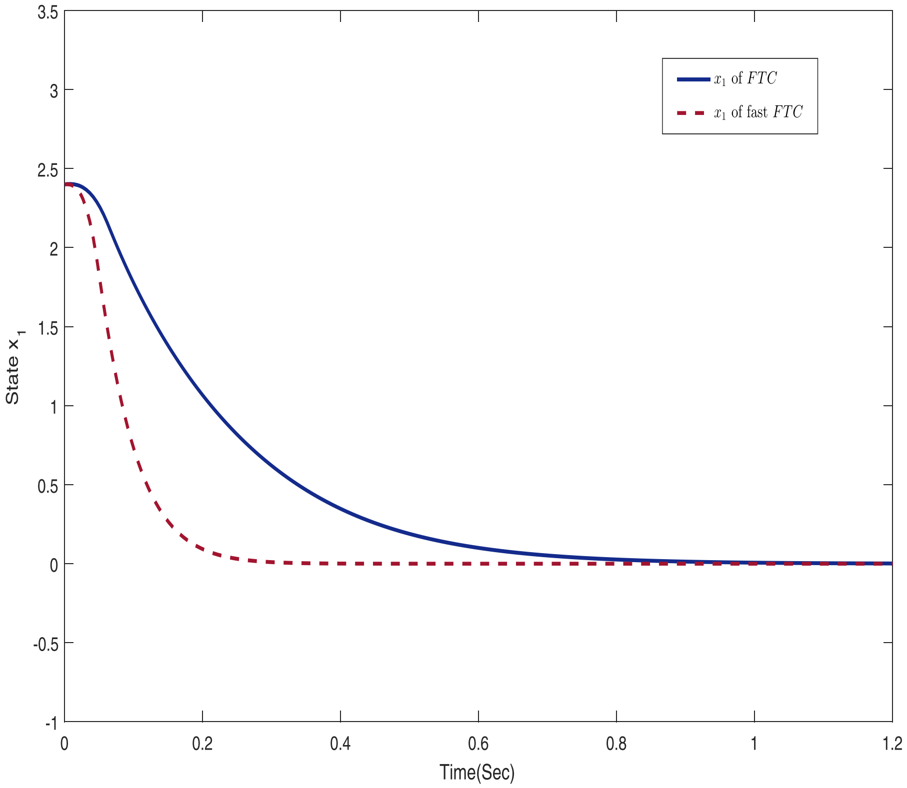

Fig. 5. The trajectories of the state x_1.

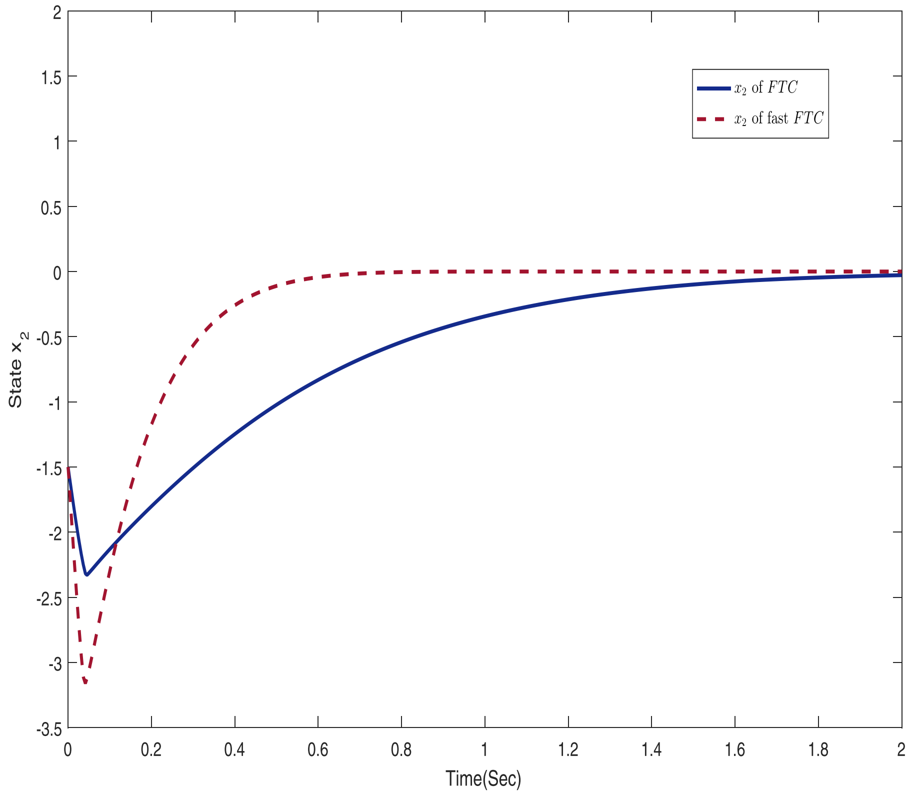

Fig. 6. The trajectories of the state x_2.

to $g_1 = \left(2.5 + (1 + x_1^2)^{\frac{1}{10}}\right)^{\frac{3}{4}}$ and $g_2 = \left(\phi_2 + 1 + (\sqrt{1 + z_1^2})^{\frac{1}{5}}\right)^{\frac{5}{4}}, \phi_2 = 1.15 + 3.6\sqrt{x_1^2 + x_2^2} + (1 + g_1^{\frac{3}{5}}) + 0.67(3.6\sqrt{x_1^2 + x_2^2} + (1 + g_1^{\frac{3}{5}}))^{\frac{3}{2}} + g_1^{\frac{4}{5}} + 1.8g_1^{\frac{9}{5}} + 0.9g_1^{\frac{81}{25}}$. In simulation, choose initial values as $x_1(0) = 2.4$, $x_2(0) = -1.5$. Figs. 5 and 6 show the effectiveness of the constructed controller.

5. Conclusions

In this paper, we propose a systematic design method to achieve global finite-time stabilization for a family of uncertain nonlinear systems. Motivated by the improved finite-time stability theorem in [30] and the adding a power integrator method in [20], an iterative algorithm is developed to make system state variables converge to zero in finite time. However, the problem of the stabilization is unsolved for high-order nonlinear systems with parameter and dynamic uncertainties. Another interesting question is to develop a robust fast finite-time control scheme by investigating the effects of unknown time-delay and unclear external disturbance.

Acknowledgments

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Appendix A. Proofs of inequalities and propositions

Proof of inequality (10). We provide the detailed deduction of (10) as follows.

(i) The proof of the right-hand side of (10). According to Lemma 4, one has

$$\begin{aligned} W_k &\le \left| \lceil x_k \rceil^{q_k} - \lceil \alpha_{k-1}(\bar{x}_{k-1}) \rceil^{q_k} \right|^{2-\frac{1}{q_k}} \cdot |x_k - \alpha_{k-1}| \\ &\le |z_k|^{2-\frac{1}{q_k}} \cdot 2^{1-\frac{1}{q_k}} \cdot |z_k|^{\frac{1}{q_k}} \end{aligned}$$

$$= c_{k2} z_k^2,$$

where $c_{k2} = 2^{1-\frac{1}{q_k}}$.

(ii) The proof of the left-hand side of (10). This can be proved by discussing two cases. If $x_k \geq \alpha_{k-1}$, it is easy to see from Lemma 4 that

$$\begin{aligned} W_k &= \int_{\alpha_{k-1}(\bar{x}_{k-1})}^{x_k} \left(\lceil s \rceil^{q_k} - \lceil \alpha_{k-1} \rceil^{q_k}\right)^{2-\frac{1}{q_k}} \mathrm{d}s \\ &= \int_{\alpha_{k-1}(\bar{x}_{k-1})}^{x_k} \left(\left(\lceil s \rceil^{q_k} - \lceil \alpha_{k-1} \rceil^{q_k}\right)^{\frac{1}{q_k}}\right)^{q_k \cdot (2-\frac{1}{q_k})} \mathrm{d}s \\ &\geq \int_{\alpha_{k-1}(\bar{x}_{k-1})}^{x_k} 2^{\frac{-2q_k^2+3q_k-1}{q_k}} (s-\alpha_{k-1})^{2q_k-1} \mathrm{d}s \\ &= \frac{1}{2q_k} 2^{\frac{-2q_k^2+3q_k-1}{q_k}} (x_k - \alpha_{k-1})^{2q_k} \\ &\triangleq c_{k1} (x_k - \alpha_{k-1})^{2q_k}, \end{aligned}$$

where the inequality is obtained by letting $a = q_k, b = q_k$ in Lemma 4. The proof for the case of $x_k < \alpha_{k-1}$ is similar to that of $x_k \geq \alpha_{k-1}$, and is omitted here. □

Proof of Proposition 1. $q_i \geq 1$ is proved in an inductive way. The case of $i = 1$ is obvious. Suppose that $q_i \geq 1$. In view of the definition of q_i, $0 < \tau < 1$ and $p_i \geq 1$, we conclude

$$p_i q_i - 1 - (\tau - 1) q_i = (p_i - \tau + 1) q_i - 1 > p_i q_i - 1 \geq 0.$$

Hence, $p_i q_i \geq 1 + (\tau - 1) q_i$, which is equivalent to $q_{i+1} = \frac{p_i q_i}{1+(\tau-1)q_i} \geq 1$. To prove the second inequality, we can rewrite q_{i+1} as

$$\frac{p_i}{q_{i+1}} = \frac{1 + (\tau - 1) q_i}{q_i} = \frac{1}{q_i} + \tau - 1.$$

In consequence, the relation $2 - \frac{1}{q_i} + \frac{p_i}{q_{i+1}} = 1 + \tau$ holds. The last inequality directly follows from $\frac{p_i}{q_{i+1}} = \frac{1}{q_i} + \tau - 1 < \frac{1}{q_i} \leq 1$. This completes the proof. □

Proof of Proposition 2. To determine g_1, we define $V_1 = \frac{z_1^2}{2}$ whose time derivative along the trajectories of (7) is

$$\dot{V}_1 = z_1 (x_2^{p_1} + f_1) = z_1 (x_2^{p_1} - \alpha_1^{p_1}) + z_1 \alpha_1^{p_1} + z_1 f_1. \tag{A.31}$$

With Proposition 1 and Assumption 1 in mind, one deduces from Lemma 3 that

$$\begin{aligned} z_1 f_1 &\leq |z_1| \bar{f}_0 + |z_1| \cdot |x_1|^{\frac{p_1}{q_2}} \bar{f}_1 \\ &\leq |z_1|^{1+\tau} \tilde{\phi}_1 (x_1) + 2 \bar{f}_0^{\frac{1+\tau}{\tau}} \\ &= |z_1|^{1+\tau} \tilde{\phi}_1 (x_1) + \bar{f}_0^{\frac{1+\tau}{\tau}} + \bar{f}_0^{\frac{(1+\tau)q_2}{p_1}}, \end{aligned} \tag{A.32}$$

where $\tilde{\phi}_1 = \bar{f}_1 + \left(\frac{\tau}{2}\right)^\tau \cdot \frac{1}{(1+\tau)^{1+\tau}}$ is continuous. Then, Lemma 5 shows that there is a smooth and positive $\bar{\phi}_1$ such that $\bar{\phi}_1(x_1) \geq \tilde{\phi}_1(x_1)$. Let $r_0(|x_1|) = |z_1|^{1+\tau} r_1(|z_1|), r_1(|z_1|) = |z_1|^{1-\tau} \frac{r_0(|z_1|)}{z_1^2}$. According to (13), it follows that

$$r_1(0) = \limsup_{|z_1| \to 0^+} |z_1|^{1-\tau} \cdot \frac{r_0(|z_1|)}{z_1^2} = \lim_{|z_1| \to 0} |z_1|^{1-\tau} \cdot \limsup_{|z_1| \to 0^+} \frac{r_0(|z_1|)}{z_1^2} = 0,$$

which implies $r_1(s)$ is continuous over $[0, \infty)$. This together with Lemma 5 shows that there is a smooth positive $r_2(x_1)$ such that $r_0(x_1) = r_1(|z_1|)|z_1|^{1+\tau} \leq r_2(x_1)|z_1|^{1+\tau}$. Now, choose $g_1(\bar{x}_1) = \left(\phi_1 + n + (1+{z_1}^2)^{(1-\tau)/2}\right)^{\frac{q_2}{p_1}}$ with $\phi_1 = \bar{\phi}_1 + (1 + \rho(x_1)) r_2$ and ρ being a continuously differentiable function to be specified later. Then, by using (A.32) in (A.31), one infers from $z_1 \alpha_1^{p_1} = -g_1^{\frac{p_1}{q_2}} |z_1|^{1+\tau}$ that

$$\begin{aligned} \dot{V}_1 &\leq -\big(1 + \rho(x_1)\big) r_0(|x_1|) - n|z_1|^{1+\tau} + z_1 (x_2^{p_1} - \alpha_1^{p_1}) - z_1^2 \\ &\quad + |z_1|^{1+\tau} \left(\phi_1 + n + (1+{z_1}^2)^{(1-\tau)/2} - g_1^{\frac{p_1}{q_2}}\right) + \bar{f}_0^{\frac{1+\tau}{\tau}} + \bar{f}_0^{\frac{(1+\tau)q_2}{p_1}} \\ &\leq -\big(1 + \rho(x_1)\big) r_0(|x_1|) - n|z_1|^{1+\tau} + \bar{f}_0^{\frac{1+\tau}{\tau}} + \bar{f}_0^{\frac{(1+\tau)q_2}{p_1}} \\ &\quad + z_1 (x_2^{p_1} - \alpha_1^{p_1}) - z_1^2. \end{aligned} \tag{A.33}$$

Suppose that one can find a continuously differential function $V_{k-1}(\cdot)$ and positive smooth functions $g_1,\dots,g_{k-1}$ such that

$$\begin{aligned}\dot V_{k-1} \le& -\big(1+\rho(x_1)\big)r_0(|x_1|) - (n-k+2)\sum_{i=1}^{k-1}|z_i|^{1+\tau} + \sum_{i=1}^{k-1}\bar f_0^{\frac{(1+\tau)q_{i+1}}{p_i}}\\ &+(k-1)\bar f_0^{\frac{1+\tau}{\tau}} - \sum_{i=1}^{k-1} z_i^2 + \lceil z_{k-1}\rfloor^{2-\frac{1}{q_{k-1}}}(x_k^{p_{k-1}} - \alpha_{k-1}^{p_{k-1}}).\end{aligned}\tag{A.34}$$

Obviously, (A.34) reduces to (A.33) when $k=2$. In the following, one needs to find g_k such that (A.34) holds for k. To see this point, consider $V_k = V_{k-1} + W_k$. It is not hard to see that

$$\begin{aligned}\dot V_k \le& -\big(1+\rho(x_1)\big)r_0(|x_1|) - (n-k+2)\sum_{i=1}^{k-1}|z_i|^{1+\tau} + \sum_{i=1}^{k-1}\bar f_0^{\frac{(1+\tau)q_{i+1}}{p_i}}\\ &+(k-1)\bar f_0^{\frac{1+\tau}{\tau}} - \sum_{i=1}^{k-1} z_i^2 + \lceil z_k\rfloor^{2-\frac{1}{q_k}}\alpha_k^{p_k} + \lceil z_k\rfloor^{2-\frac{1}{q_k}}(x_{k+1}^{p_k} - \alpha_k^{p_k})\\ &+\lceil z_k\rfloor^{2-\frac{1}{q_k}} f_k + \lceil z_{k-1}\rfloor^{2-\frac{1}{q_{k-1}}}(x_k^{p_{k-1}} - \alpha_{k-1}^{p_{k-1}}) + \sum_{i=1}^{k-1}\frac{\partial W_k}{\partial x_i}\dot x_i.\end{aligned}\tag{A.35}$$

Next, we estimate last three terms on the right hand side of (A.35). Firstly, one can prove the following inequality:

$$\Big|\frac{\partial}{\partial x_i}\big(\lceil\alpha_{k-1}\rfloor^{q_k}\big)f_i\Big| \le \beta_{ki}(\bar x_{k-1})(\bar f_0 + \sum_{j=1}^{k-1}|z_j|^{\tau}),\tag{A.36}$$

where β_{ki}, $k=2,\dots,n$, $i=1,\dots,k-1$ are smooth and positive functions. The proof of (A.36) is provided as follows. when $k=2$, it follows from Assumption 1 and (7) that

$$\Big|\frac{\partial\lceil\alpha_1\rfloor^{q_2}}{\partial x_1}f_1\Big| \le \Big(g_1 + \Big|\frac{\partial g_1}{\partial z_1}\Big|\cdot|z_1|\Big)\cdot(\bar f_0 + |z_1|^{\frac{p_1}{q_2}}\bar f_1) \le \beta_{21}(x_1)(\bar f_0 + |z_1|^{\tau}),$$

where $\beta_{21}(x_1)$ is a smooth and positive function, and satisfies $\beta_{21} \ge (g_1 + |\frac{\partial g_1}{\partial z_1}|\cdot|z_1|)(1+\bar f_1)$. The existence of β_{21} can be derived from Lemma 5. Assume (A.36) holds for $k=m-1$. When $k=m$, for $i=1,\dots,m-2$, it can be observed that

$$\begin{aligned}&\Big|\frac{\partial\lceil\alpha_{m-1}\rfloor^{q_m}}{\partial x_i}f_i\Big|\\ \le& \Big|\frac{\partial\lceil\alpha_{m-2}\rfloor^{q_{m-1}}}{\partial x_i}g_{m-1}f_i\Big| + \Big|\frac{\partial g_{m-1}}{\partial x_i}z_{m-1}f_i\Big|\\ \le& g_{m-1}\beta_{m-1,i}\Big(\bar f_0 + \sum_{j=1}^{m-2}|z_j|^{\tau}\Big) + |z_{m-1}|\cdot\Big|\frac{\partial g_{m-1}}{\partial x_i}\Big|\cdot(\bar f_0 + \sum_{j=1}^{i}|x_j|^{\frac{p_iq_j}{q_{i+1}}}\bar f_i)\\ =& \Big(g_{m-1}\beta_{m-1,i} + |z_{m-1}|\cdot\Big|\frac{\partial g_{m-1}}{\partial x_i}\Big|\Big)\bar f_0 + g_{m-1}\beta_{m-1,i}\sum_{j=1}^{m-2}|z_j|^{\tau}\\ &+|z_{m-1}|^{1-\tau}\Big|\frac{\partial g_{m-1}}{\partial x_i}\Big|\sum_{j=1}^{i}|x_j|^{\frac{p_iq_j}{q_{i+1}}}\bar f_i|z_{m-1}|^{\tau}\\ \le& \Big(g_{m-1}\beta_{m-1,i} + |z_{m-1}|^{1-\tau}\Big|\frac{\partial g_{m-1}}{\partial x_i}\Big|\sum_{j=1}^{i}|x_j|^{\frac{p_iq_j}{q_{i+1}}}\bar f_i\Big)\sum_{j=1}^{m-1}|z_j|^{\tau}\\ &+\Big(g_{m-1}\beta_{m-1,i} + |z_{m-1}|\cdot\Big|\frac{\partial g_{m-1}}{\partial x_i}\Big|\Big)\bar f_0\\ \le& \beta_{mi}(\bar x_{m-1})(\bar f_0 + \sum_{j=1}^{m-1}|z_j|^{\tau}),\end{aligned}\tag{A.37}$$

where the smooth and positive function β_{mi} satisfies

$$\beta_{mi} \ge g_{m-1}\beta_{m-1,i} + |z_{m-1}|^{1-\tau}\Big|\frac{\partial g_{m-1}}{\partial x_i}\Big|\sum_{j=1}^{i}|x_j|^{\frac{p_iq_j}{q_{i+1}}}\bar f_i + |z_{m-1}|\cdot\Big|\frac{\partial g_{m-1}}{\partial x_i}\Big|,$$

which can be obtained from Lemma 5. For $i = m-1$, using Lemma 2 and (7) renders

$$\begin{aligned} \left|\lceil x_{m-1}\rfloor^{q_{m-1}-1}\right| &\le \left|z_{m-1}\right|^{1-\frac{1}{q_{m-1}}} + g_{m-2}^{1-\frac{1}{q_{m-1}}}\left|z_{m-2}\right|^{1-\frac{1}{q_{m-1}}} \\ &\le (1+g_{m-2}^{1-\frac{1}{q_{m-1}}})\cdot\left(\left|z_{m-1}\right|^{1-\frac{1}{q_{m-1}}} + \left|z_{m-2}\right|^{1-\frac{1}{q_{m-1}}}\right). \end{aligned} \tag{A.38}$$

Meanwhile, there holds

$$\begin{aligned} \sum_{j=1}^{m-1}|x_j|^{\frac{p_{m-1}q_j}{q_m}} &= \sum_{j=1}^{m-1}\left|z_j + \lceil\alpha_{j-1}\rfloor^{q_j}\right|^{\frac{p_{m-1}}{q_m}} \\ &\le \sum_{j=1}^{m-1}\left(|z_j|^{\frac{p_{m-1}}{q_m}} + |g_{j-1}z_{j-1}|^{\frac{p_{m-1}}{q_m}}\right) \\ &\le \sum_{j=1}^{m-1}\left(1+g_{j-1}^{\frac{p_{m-1}}{q_m}}\right)\left(|z_j|^{\frac{p_{m-1}}{q_m}} + |z_{j-1}|^{\frac{p_{m-1}}{q_m}}\right). \end{aligned} \tag{A.39}$$

With Proposition 1 and the definition of smooth and positive function $\tilde{\beta}_{m,m-1} = \left(1+g_{m-1}^{1-\frac{1}{q_{m-1}}}\right)\sum_{j=1}^{m-1}\left(1+g_{j-1}^{\frac{p_{m-1}}{q_m}}\right)$ in hand, using (A.38), (A.39) and Lemma 3, one can get

$$\begin{aligned} &\left|\lceil x_{m-1}\rfloor^{q_{m-1}-1}\right|\sum_{j=1}^{m-1}|x_j|^{\frac{p_{m-1}q_j}{q_m}} \\ &\le \tilde{\beta}_{m,m-1}(\bar{x}_{m-1})\left(|z_{m-1}|^{1-\frac{1}{q_{m-1}}} + |z_{m-2}|^{1-\frac{1}{q_{m-1}}}\right) \\ &\quad\cdot\sum_{j=1}^{m-1}\left(|z_j|^{\frac{p_{m-1}}{q_m}} + |z_{j-1}|^{\frac{p_{m-1}}{q_m}}\right) \\ &\le \bar{\beta}_{m,m-1}(\bar{x}_{m-1})\sum_{j=1}^{m-1}|z_j|^{\tau}, \end{aligned} \tag{A.40}$$

where a smooth and positive function $\bar{\beta}_{m,m-1}$ and a positive constant c_m are defined as follows:

$$\begin{aligned} \bar{\beta}_{m,m-1} &= (2m-3)(c_m+1)\tilde{\beta}_{m,m-1}, \\ c_m &= \frac{p_{m-1}}{(1+\tau)q_m}\cdot\left(\frac{q_{m-1}-1}{(1+\tau)q_{m-1}}\right)^{\frac{(q_{m-1}-1)q_m}{q_{m-1}p_{m-1}}}. \end{aligned}$$

Hence, combining (7), (A.37) and (A.40) yields

$$\begin{aligned} \left|\frac{\partial\lceil\alpha_{m-1}\rfloor^{q_m}}{\partial x_{m-1}}f_{m-1}\right| &\le \left(\left|\frac{\partial g_{m-1}}{\partial x_{m-1}}z_{m-1}\right| + q_{m-1}\left|\lceil x_{m-1}\rfloor^{q_{m-1}-1}\right|g_{m-1}\right)\cdot|f_{m-1}| \\ &= \left|\frac{\partial g_{m-1}}{\partial x_{m-1}}z_{m-1}\right|\cdot\left(\bar{f}_0 + \sum_{j=1}^{m-1}|x_j|^{\frac{p_{m-1}q_j}{q_m}}\bar{f}_{m-1}\right) + g_{m-1} \\ &\quad\cdot q_{m-1}\left|\lceil x_{m-1}\rfloor^{q_{m-1}-1}\right|\cdot\left(\bar{f}_0 + \sum_{j=1}^{m-1}|x_j|^{\frac{p_{m-1}q_j}{q_m}}\bar{f}_{m-1}\right) \\ &\le \left(\left|\frac{\partial g_{m-1}}{\partial x_{m-1}}\right|\cdot|z_{m-1}| + g_{m-1}q_{m-1}|x_{m-1}|^{q_{m-1}-1}\right)\bar{f}_0 \\ &\quad+\left|\frac{\partial g_{m-1}}{\partial x_{m-1}}\right|\cdot|z_{m-1}|^{1-\tau}\sum_{j=1}^{m-1}|x_j|^{\frac{p_{m-1}q_j}{q_m}}\bar{f}_{m-1}\cdot|z_{m-1}|^{\tau} \\ &\quad+g_{m-1}q_{m-1}\bar{f}_{m-1}\cdot\bar{\beta}_{m,m-1}\sum_{j=1}^{m-1}|z_j|^{\tau} \end{aligned}$$

$$\le \hat{\beta}_{m,m-1}(\bar{x}_{m-1})\left(\bar{f}_0 + \sum_{j=1}^{m-1}|z_j|^{\tau}\right), \tag{A.41}$$

where the continuous function $\hat{\beta}_{m,m-1}$ is defined by

$$\hat{\beta}_{m,m-1} = \left|\frac{\partial g_{m-1}}{\partial x_{m-1}}\right| \cdot \left(|z_{m-1}| + |z_{m-1|}^{1-\tau}\sum_{j=1}^{m-1}|x_j|^{\frac{p_{m-1}q_j}{q_m}}\bar{f}_{m-1}\right) + g_{m-1}q_{m-1}\left(|x_{m-1|}^{q_{m-1}-1} + \bar{f}_{m-1}\bar{\beta}_{m,m-1}\right).$$

Then, Lemma 5 illustrates that there is a smooth and positive function $\beta_{m,m-1}$ such that $\beta_{m,m-1} \ge \hat{\beta}_{m,m-1}$, so the proof of (A.36) is completed. Moreover, the deductions in [30] show that there exist smooth and positive functions ρ_{ki}, $i = 1, \ldots, k-1$, such that

$$\left|\frac{\partial}{\partial x_i}\left(\lceil\alpha_{k-1}\rceil^{q_k}\right)x_{i+1}^{p_i}\right| \le \rho_{ki}(\bar{x}_k)\sum_{j=1}^{k}|z_j|^{\tau}, \ \ k = 2, \ldots, n. \tag{A.42}$$

In addition, using Lemma 4, one has

$$-\left(2 - \frac{1}{q_k}\right)\int_{\alpha_{k-1}}^{x_k}\left|\lceil s\rceil^{q_k} - \lceil\alpha_{k-1}(\bar{x}_{k-1})\rceil^{q_k}\right|^{1-\frac{1}{q_k}}ds \le \tilde{c}_k|z_k|, \tag{A.43}$$

where $\tilde{c}_k = (2 - \frac{1}{q_k}) \cdot 2^{1-\frac{1}{q_k}}$. Consequently, based on (A.36), (A.42) and (A.43), applying Lemma 3 and Proposition 1 shows that

$$\begin{aligned}
\sum_{i=1}^{k-1}\frac{\partial W_k}{\partial x_i}\dot{x}_i &= -\left(2 - \frac{1}{q_k}\right)\int_{\alpha_{k-1}}^{x_k}\left|\lceil s\rceil^{q_k} - \lceil\alpha_{k-1}(\bar{x}_{k-1})\rceil^{q_k}\right|^{1-\frac{1}{q_k}}ds \\
&\quad \cdot \sum_{i=1}^{k-1}\frac{\partial}{\partial x_i}(\lceil\alpha_{k-1}\rceil^{q_k})(x_{i+1}^{p_i} + f_i) \\
&\le m_k(\bar{x}_k)|z_k|\left(\bar{f}_0 + \sum_{j=1}^{k}|z_j|^{\tau}\right) \\
&\le \hat{\phi}_{k_1}(\bar{x}_k)|z_k|^{1+\tau} + |\bar{f}_0|^{\frac{1+\tau}{\tau}} + \frac{1}{2}\sum_{i=1}^{k-2}|z_i|^{1+\tau} + \frac{1}{3}|z_{k-1}|^{1+\tau},
\end{aligned} \tag{A.44}$$

where $m_k = \tilde{c}_k\sum_{i=1}^{k-1}(\rho_{ki} + \beta_{ki})$, and the smooth and positive function $\hat{\phi}_{k_1}$ is defined by

$$\hat{\phi}_{k_1} = m_k^{1+\tau} \cdot \frac{\tau^{\tau}}{(1+\tau)^{1+\tau}} + \frac{m_k^{1+\tau}}{(1+\tau)^{1+\tau}} \cdot \left((3\tau)^{\tau} + (k-2)(2\tau)^{\tau}\right) + m_k.$$

Secondly, it can be deduced from Lemmas 3 and 4 that

$$\begin{aligned}
&\lceil z_{k-1}\rceil^{2-\frac{1}{q_{k-1}}}(x_k^{p_{k-1}} - \alpha_{k-1}^{p_{k-1}}) \le |z_{k-1}|^{2-\frac{1}{q_{k-1}}}|x_k^{p_{k-1}} - \alpha_{k-1}^{p_{k-1}}| \\
&\le 2^{1-\frac{p_{k-1}}{q_k}}|z_{k-1}|^{2-\frac{1}{q_{k-1}}}|z_k|^{\frac{p_{k-1}}{q_k}} \le \frac{1}{3}|z_{k-1}|^{1+\tau} + \hat{\phi}_{k_2}|z_k|^{1+\tau},
\end{aligned} \tag{A.45}$$

where a constant $\hat{\phi}_{k_2}$ is defined by

$$\hat{\phi}_{k_2} = \frac{p_{k-1}}{(1+\tau)q_k} \cdot \left(\frac{3(2q_{k-1}-1)}{(1+\tau)q_{k-1}}\right)^{\frac{q_k(2q_{k-1}-1)}{p_{k-1}q_{k-1}}} \cdot 2^{\frac{(1+\tau)(q_k-p_{k-1})q_{k-1}}{q_k(2q_{k-1}-1)}} > 0.$$

Thirdly, according to $|x_i|^{\frac{q_i p_k}{q_{k+1}}} \le |z_i|^{\frac{p_k}{q_{k+1}}} + |g_{i-1}|^{\frac{p_k}{q_{k+1}}}|z_{i-1}|^{\frac{p_k}{q_{k+1}}}$ for $i = 1, \ldots, k$, it can be deduced from Lemma 3, Proposition 1 and Assumption 1 that

$$
\begin{aligned}
\lceil z_k \rceil^{2-\frac{1}{q_k}} f_k \le\ & |z_k|^{2-\frac{1}{q_k}} \bar{f}_0 + |z_k|^{2-\frac{1}{q_k}} \sum_{i=1}^{k} |x_i|^{\frac{p_k q_i}{q_{k+1}}} \bar{f}_k \\
\le\ & |z_k|^{2-\frac{1}{q_k}} \bar{f}_0 + |z_k|^{2-\frac{1}{q_k}} \sum_{i=1}^{k} \bar{g}_{k-1} |z_i|^{\frac{p_k}{q_{k+1}}} \bar{f}_k \\
\le\ & c_{k_3} |z_k|^{1+\tau} + |\bar{f}_0|^{\frac{1+\tau}{\tau-1+\frac{1}{q_k}}} + \tilde{\phi}_{k_3}(\bar{x}_k)|z_k|^{1+\tau} \\
& + \frac{1}{2} \sum_{i=1}^{k-2} |z_i|^{1+\tau} + \frac{1}{3} |z_{k-1}|^{1+\tau} \\
\le\ & \hat{\phi}_{k_3}(\bar{x}_k)|z_k|^{1+\tau} + |\bar{f}_0|^{\frac{(1+\tau)q_{k+1}}{p_k}} + \frac{1}{3} |z_{k-1}|^{1+\tau} \\
& + \frac{1}{2} \sum_{i=1}^{k-2} |z_i|^{1+\tau},
\end{aligned} \tag{A.46}
$$

where a smooth and positive function $\bar{g}_{k-1}$, a positive constant c_{k_3} and a continuous and nonnegative function $\tilde{\phi}_{k_3}$ are defined as follows:

$$
\begin{aligned}
\bar{g}_{k-1} &= 1 + \sum_{i=1}^{k-1} g_i^{\frac{p_k}{q_{k+1}}}, \\
c_{k_3} &= \frac{2q_k - 1}{(1+\tau)q_k} \cdot \left(\frac{(\tau-1)q_k + 1}{(1+\tau)q_k} \right)^{\frac{(\tau-1)q_k+1}{2q_k-1}}, \\
\tilde{\phi}_{k_3} &= \bar{g}_{k-1}\bar{f}_k + \left(\bar{g}_{k-1}\bar{f}_k\right)^{\frac{(1+\tau)q_k}{2q_k-1}} \cdot \left((k-2)2^{\frac{(\tau-1)q_k+1}{2q_k-1}} + 3^{\frac{(\tau-1)q_k+1}{2q_k-1}} \right) \\
&\quad \cdot \frac{2q_k - 1}{(1+\tau)q_k} \cdot \left(\frac{(\tau-1)q_k + 1}{(1+\tau)q_k} \right)^{\frac{(\tau-1)q_k+1}{2q_k-1}},
\end{aligned}
$$

and the smooth positive function $\hat{\phi}_{k_3}$ satisfies $\hat{\phi}_{k_3} \ge c_{k_3} + \tilde{\phi}_{k_3}$ which can be obtained by Lemma 5.

Up to now, let $\phi_k(\bar{x}_k) = \sum_{i=1}^{3} \hat{\phi}_{k_i}$ which is a smooth and positive function. Substituting (A.44)–(A.46) into (A.35) yields

$$
\begin{aligned}
\dot{V}_k \le\ & -\big(1+\rho(x_1)\big) r_0(|x_1|) - (n-k+1) \sum_{i=1}^{k-1} |z_i|^{1+\tau} \\
& + \sum_{i=1}^{k} \bar{f}_0^{\frac{(1+\tau)q_{i+1}}{p_i}} + k\bar{f}_0^{\frac{1+\tau}{\tau}} - \sum_{i=1}^{k-1} z_i^2 + \phi_k |z_k|^{1+\tau} \\
& + \lceil z_k \rceil^{2-\frac{1}{q_k}} \alpha_k^{p_k} + \lceil z_k \rceil^{2-\frac{1}{q_k}} (x_{k+1}^{p_k} - \alpha_k^{p_k}).
\end{aligned} \tag{A.47}
$$

If one chooses the smooth function g_k as

$$
g_k(\bar{x}_k) = \left(\phi_k + n - k + 1 + (1+z_k^2)^{(1-\tau)/2}\right)^{\frac{q_{k+1}}{p_k}}, \quad k \ge 2,
$$

then one has

$$
\begin{aligned}
\dot{V}_k \le\ & -\big(1+\rho(x_1)\big) r_0(|x_1|) - (n-k+1) \sum_{i=1}^{k} |z_i|^{1+\tau} + \sum_{i=1}^{k} \bar{f}_0^{\frac{(1+\tau)q_{i+1}}{p_i}} \\
& + k\bar{f}_0^{\frac{1+\tau}{\tau}} - \sum_{i=1}^{k} z_i^2 + \lceil z_k \rceil^{2-\frac{1}{q_k}} (x_{k+1}^{p_k} - \alpha_k^{p_k}).
\end{aligned} \tag{A.48}
$$

This completes the proof. □

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