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Global adaptive prescribed-time stabilization for high-order nonlinear systems

Changchun Hua[1] | **Hao Li**[1] | **Kuo Li**[2]

[1]Institute of Electrical Engineering, Yanshan University, Qinhuangdao, China

[2]Institute of Automatic Control and Complex Systems, University of Duisburg-Essen, Duisburg, Germany

Correspondence
Changchun Hua, Institute of Electrical Engineering, Yanshan University, 066004 Qinhuangdao, China.
Email: cch@ysu.edu.cn

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Abstract

In this article, the global adaptive prescribed-time stabilization problem of high-order (p-normal form) nonlinear systems with parametric uncertainties is studied. Compared with the existing results on time-optimal control, our control algorithm eliminates the residual term caused by unknown parameters for the first time. By introducing the new Lyapunov-like function and time-varying function, a prescribed-time controller with adaptive law is constructed, which ensures that the system state variables converge to the origin rather than a residual term, and the setting time can be arbitrarily specified. Moreover, with the help of the state transformations and the power integrator technique, the difficulties of system structure caused by the high power are overcome. Then, based on the differential equations, it is proved that the boundedness of all system signals. Finally, an example of underdrive system simulation is presented to verify the availabilities of the developed control algorithm.

KEYWORDS

adaptive control, high-order nonlinear systems, parametric uncertainty, prescribed-time control

1 | INTRODUCTION

In many physical systems, such as tactical missile track and autonomous vehicle rendezvous, we usually need to complete tasks without errors within the time specified by the user to fulfill stringent time constraints. As a form of time-optimal control, finite/fixed-time control has been extensively studied during the past decades due to its appealing features, which makes the system stabilization within a time and has better robustness, faster convergence rate.[1-3] However, the estimation on settling time of finite/fixed-time control method rely on some designed parameters or initial conditions.[4-8] In order to achieve system stability in accurate time, a prescribed time control method is proposed in References 9-13. Prescribed-time control can ensure that state variables converge to origin by introducing the time-varying scaling functions. It is different from the traditional time optimal control that the convergence time can be arbitrarily specified as the practical permissible range. In Reference 14, the authors provided a prescribed-time smooth state feedback controller instead of the fractional-power controller to address stabilization problem. The inverse optimality prescribed-time stabilization problem of stochastic nonlinear systems was solved in Reference 15, and the complexity of calculation was reduced by proposing a nonscaling backstepping design method. According to the time-varying scaling high gain technique and a novel temporal transformation with terminal time terms, the authors studied a class of state-dependent systems with generalized strict-feedback structure in Reference 16. However, all the above-mentioned works do not consider the influence of unknown parameters on the system.

As is well known uncertain factors such as unknown parameters exist widely in practical systems, which can cause instability or poor performance of systems. Recently, many excellent control methods for parameter uncertainty system have been reported, such as fuzzy/neural network control, dynamic gain method, adaptive control,[17-21] and so forth. Among those algorithms, adaptive control is a basic method to address system unknown parameters. An adaptive time-varying output feedback control algorithm was put forward in Reference 22 for p-normal uncertain nonlinear systems, where the nonlinear function has unknown growth rate. In Reference 23, by combining the sign function with adaptive control algorithm, the authors designed globally finite-time controller for p-normal nonlinear systems with external disturbance and unknown control coefficient. The fixed-time output tracking problem was investigated in Reference 24 for nonlinear systems with parameter uncertainty and output asymmetric time-varying constraints. However, these algorithms only ensured that state variables approach to neighborhood of origin within a finite time interval. The essential obstacle is that the uncertain parameters of systems, which may result in some residual terms in control design and analysis. In References 25 and 26, global finite-time stabilization problems were investigated, a two-phase analytical approaches was presented to deal with unknown parameters, which rendered that the system state converged to origin within a finite time, their global stability analysis was divided into two cases based on two sets related to the unknown parameters and initial conditions. Moreover, because of the existence of adaptive parameters, it is difficult to construct the whole Lyapunov function $V(t)$ so that its derivative satisfies $\dot{V} \le -aV^{\alpha} - bV^{\beta}$ to make sure that the state converges to zero, where $a, b > 0, \alpha > 1, 0 < \beta < 1$. When the above two-phase analytical method of References 25 and 26 is extended to the fixed-time control, the settling-time still relies on the initial conditions. To our best knowledge, the adaptive fixed-time algorithms can only achieve bounded stability or local stabilization. Therefore, how to provide a unified control algorithm to eliminate the residual term and assign the settling-time in advance is a valuable research issue.

High-order (p-normal) nonlinear systems have received extensive attention because it has a more general construction and potential practical value. Due to the Jacobian linearization is uncontrollable and unobservable, studying high-order nonlinear uncertain system is more challenging than the widely studied strict feedback nonlinear systems. Motivated by the aforementioned survey, this article aims to propose a global adaptive prescribed-time control algorithm for nonlinear systems with p-normal form and unknown parameter. The main contributions of this article can be summarized as follows.

1. The global adaptive prescribed-time control problem of nonlinear systems with p-normal form is investigated for the first time. Different from the results of adaptive finite/fixed-time control, the settling time is irrelevant with the initial value and the system state can converge globally to zero.
2. A new coordinate transformation combining time-varying function is proposed, which is critical for the state converges to origin within the specified time. Moreover, different from the finite/fixed-time stability theory based on fractional Lyapunov differential inequality, which generates residual term when using adaptive technology. We design a novel Lyapunov-like function to handle this trouble, the global adaptive prescribed-time stabilization problem with zero error is thoroughly solved.

Notations: In this article, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}^n$ represent the set of real number, nonnegative real number, and the n-dimensional real spaces, respectively. We denote $\lceil \eta \rfloor^{\alpha} = |\eta|^{\alpha}\mathrm{sign}(\eta)$; for any constant $\alpha \ge 1$, $\lceil \eta \rfloor^{\alpha}$ is a continuously one-order differentiable function, and $\frac{d\lceil \eta \rfloor^{\alpha}}{dt} = \alpha|\eta|^{\alpha-1}$, where $|\eta|^0 = 1$ if $\eta = 0$. $\mathbb{R}^{\ge 1}_{\mathrm{odd}} := \left\{ q \mid q = \frac{q_1}{q_2} \text{ with } q_1 \text{ and } q_2 \text{ are odd positive integers, and } q_1 \ge q_2 \right\}$; $\overline{\eta}_i = [\eta_1, \eta_2, \ldots, \eta_i]^T \in \mathbb{R}^i$.

2 | PROBLEM FORMULATION AND PREPARATION

In this article, we study the following high-order uncertain nonlinear system

$$\begin{cases} \dot{\eta}_i = g_i(t, \overline{\eta}_i)\eta_{i+1}^{p_i} + f_i(t, \theta_i, \overline{\eta}_i), \quad i = 1, 2, \ldots, n-1, \\ \dot{\eta}_n = g_n(t, \overline{\eta}_n)u^{p_n} + f_n(t, \theta_n, \overline{\eta}_n), \end{cases} \tag{1}$$

where $\eta_i \in \mathbb{R}$ and $u \in \mathbb{R}$ are state variable and control input of the system, respectively; $p_i \in \mathbb{R}^{\geq 1}_{\text{odd}}$ is the order of system; θ_i is an unknown constant; $g_i(t,\overline{\eta}_i)$: $\mathbb{R}^+ \times \mathbb{R}^i \to \mathbb{R}$ is an unknown control gain and $f_i(t,\theta_i,\overline{\eta}_i)$: $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^i \to \mathbb{R}$ is the uncertain continuous nonlinear function.

Objective: The control objective of this article is to design a prescribed time controller with adaptive law, so that the state variables converge to the origin within any time T, and all signals of the system are bounded. In order to achieve the control objective, the following assumptions and lemmas are proposed, which are critical to controller design.

Assumption 1. For $i = 1, \dots, n$, the sign of $g_i(t,\overline{\eta}_i)$ is unchanged, without loss of generality, supposing that it is strict positive, and there are known positive constants $\overline{g}_i$ and $\underline{g}_i$ such that $\underline{g}_i \leq |g_i(\cdot)| \leq \overline{g}_i$.

Assumption 2. For $i = 1, \dots, n$, there exists a known nonnegative C^1 function $\overline{\varphi}_i\left(t,\overline{\eta}_i\right)$ such that

$$\left|f_i(t,\theta_i,\overline{\eta}_i)\right| \leq \overline{\theta}\,\overline{\varphi}_i\left(t,\overline{\eta}_i\right)\left(|\eta_1| + |\eta_2| + \cdots + |\eta_i|\right), \tag{2}$$

where $\overline{\theta}$ is an unknown positive constant.

Remark 1. Assumption 1 is universal in the stabilization problem of nonlinear systems, which represents global controllability and can be found in Reference 26. Assumption 2 shows that the system has an unknown nonlinear growth conditions, which can be found in References 15 and 23.

Lemma 1 (27). *Let $\kappa > 0$, for any $\eta_i \in \mathbb{R}$ with $i = 1, \dots, n$, we have $\left(|\eta_1| + |\eta_2| + \cdots + |\eta_n|\right)^{\kappa} \leq a\left(|\eta_1|^{\kappa} + |\eta_2|^{\kappa} + \cdots + |\eta_n|^{\kappa}\right)$, where $a = n^{\kappa-1}$ if $\kappa \geq 1$ and $a = 1$ if $0 < \kappa < 1$.*

Lemma 2 (28). *If $a_1 > 0$ and $0 < a_2 \leq 1$, then for $\forall x, y \in \mathbb{R}$, one has $\left|\lceil x \rceil^{a_1 a_2} - \lceil y \rceil^{a_1 a_2}\right| \leq 2^{1-a_2}\left|\lceil x \rceil^{a_1} - \lceil y \rceil^{a_1}\right|^{a_2}$.*

Lemma 3 (29). *Let a_1 and a_2 be positive real numbers, there is a positive function $\lambda(x,y)$ such that the inequality $\delta|x|^{a_1}|y|^{a_2} \leq \lambda|x|^{a_1+a_2} + \frac{a_2}{a_1+a_2}\left(\frac{a_1}{(a_1+a_2)\lambda}\right)^{\frac{a_1}{a_2}}\delta^{\frac{a_1+a_2}{a_2}}|y|^{a_1+a_2}$ holds for $\forall x, y \in \mathbb{R}$, $\delta > 0$.*

3 | MAIN RESULTS

3.1 | Controller design

The adaptive prescribed-time controller of nonlinear systems (1) for any given settling time will be designed. We first introduce the following state transformations

$$x_i = L(t)^{-m_i}\eta_i, \tag{3}$$

where $m_n = 1$, $m_i p_{i-1} = m_{i-1} - 1$, and $L(t)$ is a positive continuously differentiable bounded function, it has the following characteristics

$$\lim_{t\to T} L(t) = 0, \dot{L}(t) = c(t) \leq 0, \tag{4}$$

where $c(t)$ is a bounded function. From (1) and (3), we have

$$\begin{cases} \dot{x}_i = -m_i\frac{\dot{L}(t)}{L(t)}x_i + \frac{g_i x_{i+1}^{p_i}}{L(t)} + L(t)^{-m_i}f_i\left(t,\theta_i,\overline{\eta}_i\right), \\ \dot{x}_n = -m_n\frac{\dot{L}(t)}{L(t)}x_n + \frac{g_n u^{p_n}}{L(t)} + L(t)^{-m_n}f_n\left(t,\theta_n,\overline{\eta}_n\right). \end{cases} \tag{5}$$

Remark 2. According to Figure 1, the original prescribed time stabilization problems of η_i are converted to the control problems of x_i by introducing time-varying function $L(t)$. $L(t)$ as part of the control gain, plays an important role in ensuring that all of state variables converge to origin within prescribed time due to the characteristics of $\lim_{t\to T} L(t) = 0$ and $\dot{L}(t) \leq 0$. From state transformation (5), it is easy to find that if x_i is bounded on $t \in [0,T]$, one has $\lim_{t\to T}\eta_i(t) = 0$. Moreover, parameter m_i, function $L(t)$ and variable x_i determine the convergence rate of η_i, in particular, the system performances are determined by $L(t)$. Therefore, we need to choose appropriate time-varying functions to ensure the system

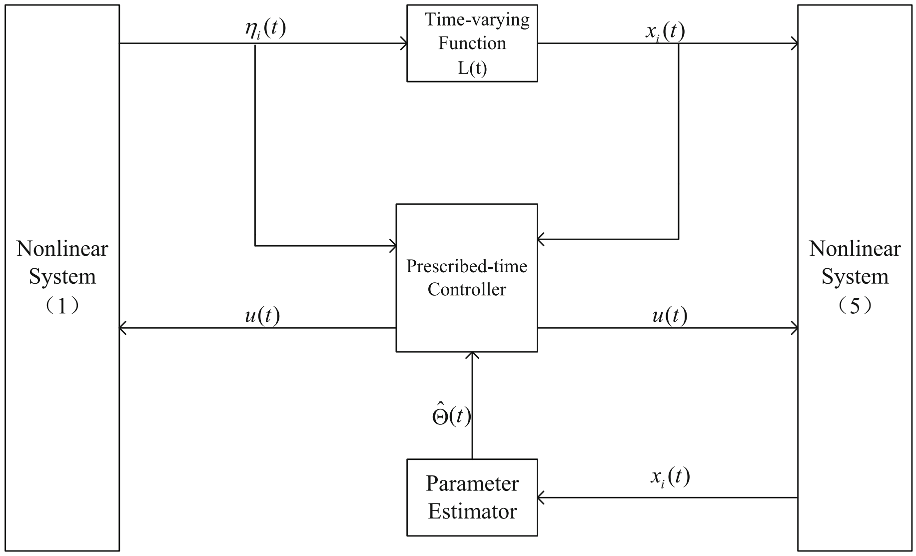

FIGURE 1 Block diagram of control strategy

transient steady-state performance. Actually, $L(t)$ can be chosen widely in this premise of its properties, such as designing function as $L(t) = (T-t)^a, \log_b(T-t+1)$, where $a \geq 1, b > 1$ are constants.

We define a positive and continuously differentiable function $W_i(t, \bar{x}_i)$ as

$$W_i(t, \bar{x}_i) = \int_{x_i^*}^{x_i} \left\lceil \lceil s \rceil^{\frac{\sigma}{r_i}} - \lceil x_i^* \rceil^{\frac{\sigma}{r_i}} \right\rceil^{\frac{2\sigma - r_i}{\sigma}} ds, \tag{6}$$

where $i = 1, 2, \ldots, n; \sigma \geq 1; r_1 = 1, r_i = p_i r_{i+1}$. It is clear that W_i is a positive and continuously differentiable function, the virtual control laws $x_1^*, x_2^*, \ldots, x_{n+1}^*$ are defined as

$$\begin{aligned} x_1^* &= 0, & \xi_1 &= \lceil x_1 \rceil^{\frac{\sigma}{r_1}} - \lceil x_1^* \rceil^{\frac{\sigma}{r_1}}, \\ x_2^* &= -\beta_1^{\frac{1}{p_1}} \lceil \xi_1 \rceil^{\frac{r_2}{\sigma}}, & \xi_2 &= \lceil x_2 \rceil^{\frac{\sigma}{r_2}} - \lceil x_2^* \rceil^{\frac{\sigma}{r_2}}, \\ &\vdots & &\vdots \\ x_{n+1}^* &= -\beta_n^{\frac{1}{p_n}} \lceil \xi_n \rceil^{\frac{r_{n+1}}{\sigma}}, & \xi_{n+1} &= \lceil u \rceil^{\frac{\sigma}{r_{n+1}}} - \lceil x_{n+1}^* \rceil^{\frac{\sigma}{r_{n+1}}}, \end{aligned} \tag{7}$$

where $\beta_i(t, \bar{x}_i, \hat{\Theta})$ is a positive C^1 function. Define an unknown positive constant in the form of

$$\Theta = \max\left\{ \bar{\theta}^2, \bar{\theta}^{\frac{4\sigma}{2\sigma - r_i}}, \bar{\theta}^{\frac{4\sigma}{r_i}} | 2 \leq i \leq n \right\}. \tag{8}$$

Let $\hat{\Theta}$ is the estimate of Θ, $\tilde{\Theta} = \Theta - \hat{\Theta}$ is the estimate error and the adaptive updating law as

$$\dot{\hat{\Theta}} = \Gamma_n(\bar{x}_n, \hat{\Theta}) \tag{9}$$

with

$$\Gamma_i = \Gamma_{i-1} + \phi_i(t, \bar{x}_i, \hat{\Theta}) \xi_i^2, \quad i = 1, 2, \ldots, n, \tag{10}$$

where $\hat{\Theta}(0) \geq 0, \Gamma_0 = 0, \phi_i(t, \bar{x}_i, \hat{\Theta})$ is a positive C^1 function. We introduce some useful propositions, the proof of which are given in the Appendix.

Proposition 1. *For $i = 2, \dots, n$, there is a positive constant φ_{i1} such that*

$$\frac{\lceil \xi_{i-1} \rfloor^{\frac{2\sigma - r_{i-1}}{\sigma}} g_{i-1} \left(x_i^{p_{i-1}} - x_i^{*p_{i-1}} \right)}{L} \leq \frac{1}{3L} \xi_{i-1}^2 + \frac{\varphi_{i1}}{L} \xi_i^2. \tag{11}$$

Proposition 2. *For $i = 2, \dots, n$, there are positive C^1 functions $\phi_{i1}(t, \overline{x}_i, \hat{\Theta})$ and $\varphi_{i2}(t, \overline{x}_i, \hat{\Theta})$ such that*

$$\begin{aligned} \frac{\partial W_i}{\partial x_i} L^{-m_i} f_i - m_i \frac{\dot{L}}{L} \lceil \xi_i \rfloor^{\frac{2\sigma - r_i}{\sigma}} x_i \leq & \frac{1}{4} \sum_{k=1}^{i-1} \xi_k^2 + \frac{1}{3L} \xi_{i-1}^2 + \frac{1}{2L} \xi_i^2 \\ & + \frac{\varphi_{i2}(t, \overline{x}_i, \hat{\Theta})}{L} \xi_i^2 + L\Theta \phi_{i1}(t, \overline{x}_i, \hat{\Theta}) \xi_i^2. \end{aligned} \tag{12}$$

Proposition 3. *For $i = 2, \dots, n$, there are positive C^1 functions $k_{i1}(t, \overline{x}_i, \hat{\Theta})$, $\varphi_{i3}(t, \overline{x}_i, \hat{\Theta})$ and $\phi_{i2}(t, \overline{x}_i, \hat{\Theta})$ such that*

$$\frac{\partial W_i}{\partial t} + \sum_{k=1}^{i-1} \frac{\partial W_i}{\partial x_k} \dot{x}_k \leq \sum_{k=1}^{i-1} \left(\frac{1}{2} \xi_k^2 + \frac{1}{3L} \xi_k^2 \right) + \frac{\xi_i^2}{2L} + \left(k_{i1} + \frac{\varphi_{i3}}{L} \right) \xi_i^2 + L\Theta \phi_{i2}(t, \overline{x}_i, \hat{\Theta}) \xi_i^2, \tag{13}$$

where $\frac{\partial W_i}{\partial t}$ represents the time derivative of W_i with respect to the explicit variable t.

Proposition 4. *For $i = 2, \dots, n$, there is a positive C^1 function $k_{i2}(t, \overline{x}_i, \hat{\Theta})$ such that*

$$\sum_{k=1}^{i-1} \frac{\partial W_k}{\partial \hat{\Theta}} \phi_i \xi_i^2 + \frac{\partial W_i}{\partial \hat{\Theta}} \Gamma_i \leq \sum_{k=1}^{i-1} \frac{1}{4} \xi_k^2 + k_{i2}(t, \overline{x}_i, \hat{\Theta}) \xi_i^2. \tag{14}$$

Next, we will give specific controller design steps by inductive method.

Step 1: Choose the Lyapunov function as

$$V_1 = W_1 + V_\theta, \tag{15}$$

where $V_\theta = \frac{L(t)\tilde{\Theta}^2}{2}$. From (5), (6), and (15), the differential of V_1 satisfies

$$\begin{aligned} \dot{V}_1 \leq & \frac{\dot{L}}{L} V_\theta - L\tilde{\Theta}\dot{\hat{\Theta}} + \lceil \xi_1 \rfloor^{\frac{2\sigma - r_1}{\sigma}} \left(-m_1 \frac{\dot{L}}{L} x_1 + \frac{g_1 x_2^{p_1}}{L} + L^{-m_1} f_1(t, \theta_1, \eta_1) \right) \\ \leq & \frac{-m_1 \dot{L}}{L} \xi_1^2 + \lceil \xi_1 \rfloor^{\frac{2\sigma - r_1}{\sigma}} g_1 \frac{x_2^{p_1} - x_2^{*p_1}}{L} + \overline{\theta} \overline{\varphi}_1 (t, \eta_1) \xi_1^2 \\ & + \lceil \xi_1 \rfloor^{\frac{2\sigma - r_1}{\sigma}} \frac{g_1 x_2^{*p_1}}{L} - L\tilde{\Theta}\dot{\hat{\Theta}} + \frac{\dot{L}}{L} V_\theta \\ \leq & \frac{-m_1 \dot{L}}{L} \xi_1^2 + \lceil \xi_1 \rfloor^{\frac{2\sigma - r_1}{\sigma}} g_1 \frac{x_2^{p_1} - x_2^{*p_1}}{L} + L\Theta \phi_1 \xi_1^2 \\ & + \frac{\xi_1^2}{L} + \lceil \xi_1 \rfloor^{\frac{2\sigma - r_1}{\sigma}} \frac{g_1 x_2^{*p_1}}{L} - L\tilde{\Theta}\dot{\hat{\Theta}} + \frac{\dot{L}}{L} V_\theta, \end{aligned} \tag{16}$$

where $\phi_1(t, x_1) \geq 0.25 \overline{\varphi}_1^2$ is a positive function, then, we design the virtual controller as

$$\begin{aligned} x_2^* &= -\beta_1^{\frac{1}{p_1}} (t, \hat{\Theta}, x_1) \lceil \xi_1 \rfloor^{\frac{r_2}{\sigma}}, \\ \beta_1 &= \underline{g}_1^{-1} (-(m_1 + 2)\dot{L} + (L + 1)(n - 1) + 1 + L^2 \hat{\Theta} \phi_1). \end{aligned} \tag{17}$$

Substituting (17) into (16) yields

$$\begin{aligned} \dot{V}_1 \leq & \frac{2\dot{L}}{L} \xi_1^2 - \left(1 + L^{-1} \right) (n - 1) \xi_1^2 \\ & + \lceil \xi_1 \rfloor^{\frac{2\sigma - r_1}{\sigma}} g_1 \frac{x_2^{p_1} - x_2^{*p_1}}{L} + L\tilde{\Theta}(\Gamma_1 - \dot{\hat{\Theta}}) + \frac{\dot{L}}{L} V_\theta. \end{aligned} \tag{18}$$

Step 2: Let V_2 be defined as $V_2 = V_1 + W_2$, the derivative of V_2 satisfying

$$\begin{aligned}
\dot{V}_2 &= \dot{V}_1 + \frac{\partial W_2}{\partial x_1}\dot{x}_1 + \frac{\partial W_2}{\partial x_2}\dot{x}_2 + \frac{\partial W_2}{\partial t} + \frac{\partial W_2}{\partial \hat{\Theta}}\dot{\hat{\Theta}} \\
&\leq \frac{2\dot{L}}{L}\xi_1^2 - \left(1 + L^{-1}\right)(n-1)\xi_1^2 \\
&\quad + \lceil \xi_1 \rceil^{\frac{2\sigma - r_1}{\sigma}} g_1 \frac{x_2^{p_1} - x_2^{*p_1}}{L} + L\tilde{\Theta}(\Gamma_1 - \dot{\hat{\Theta}}) + \frac{\dot{L}}{L}V_\theta \\
&\quad + \lceil \xi_2 \rceil^{\frac{2\sigma - r_2}{\sigma}} \left(-m_2 \frac{\dot{L}}{L}x_2 + \frac{g_2 x_3^{p_2}}{L} + L^{-m_2} f_2(t, \theta_2, \overline{\eta}_2) \right) \\
&\quad + \frac{\partial W_2}{\partial x_1}\dot{x}_1 + \frac{\partial W_2}{\partial t} + \frac{\partial W_2}{\partial \hat{\Theta}}\dot{\hat{\Theta}}.
\end{aligned} \tag{19}$$

Based on Propositions 1–4, we have

$$\begin{aligned}
&\frac{\xi_1^2}{L} + \xi_1^2 + \frac{\varphi_2 \xi_2^2}{L} + k_2\xi_2^2 + L\Theta\phi_2\xi_2^2 \\
&\geq \lceil \xi_1 \rceil^{\frac{2\sigma - r_1}{\sigma}} g_1 \frac{x_2^{p_1} - x_2^{*p_1}}{L} + \frac{\partial W_2}{\partial x_1}\dot{x}_1 + \frac{\partial W_2}{\partial t} + \frac{\partial W_2}{\partial \hat{\Theta}}\Gamma_2 \\
&\quad + \lceil \xi_2 \rceil^{\frac{2\sigma - r_2}{\sigma}} \left(-m_2\frac{\dot{L}}{L}x_2 + L^{-m_2} f_2(t, \theta_2, \overline{\eta}_2) \right),
\end{aligned} \tag{20}$$

where $\varphi_2(t, \overline{x}_2, \hat{\Theta}) = \varphi_{21} + \varphi_{22} + \varphi_{23} + 1$, $k_2(t, \overline{x}_2, \hat{\Theta}) = k_{21} + k_{22}$, $\phi_2(t, \overline{x}_2, \hat{\Theta}) = \phi_{21} + \phi_{22}$ are positive functions.

Then, the virtual control law is designed as

$$\begin{aligned}
x_3^* &= -\beta_2^{\frac{1}{p_2}}(t, \hat{\Theta}, \overline{x}_2)\lceil \xi_2 \rceil^{\frac{r_3}{\sigma}}, \\
\beta_2 &= \frac{1}{\underline{g}_2}(-2\dot{L} + (L+1)(n-2) + \varphi_2 + Lk_2 + L^2\hat{\Theta}\phi_2).
\end{aligned} \tag{21}$$

Combined with (19)–(21), we have

$$\begin{aligned}
\dot{V}_2 &\leq \left(\frac{2\dot{L}}{L} - (1 + L^{-1})(n-2) \right) \sum_{k=1}^{2} \xi_2^2 + \frac{\dot{L}}{L}V_\theta \\
&\quad + \lceil \xi_2 \rceil^{\frac{2\sigma - r_2}{\sigma}} g_2 \frac{x_3^{p_2} - x_3^{*p_2}}{L} + \left(L\tilde{\Theta} - \frac{\partial W_2}{\partial \hat{\Theta}} \right)(\Gamma_2 - \dot{\hat{\Theta}}).
\end{aligned} \tag{22}$$

Inductive step *i* ($3 \leq i \leq n$)**:** Based on the above results, we have

$$\begin{aligned}
\dot{V}_{i-1} &\leq \left(\frac{2\dot{L}}{L} - (1 + L^{-1})(n-i+1) \right) \sum_{k=1}^{i-1} \xi_k^2 + \lceil \xi_{i-1} \rceil^{\frac{2\sigma - r_{i-1}}{\sigma}} g_{i-1} \frac{x_i^{p_{i-1}} - x_i^{*p_{i-1}}}{L} + \frac{\dot{L}}{L}V_\theta \\
&\quad + \left(L\tilde{\Theta} - \sum_{k=1}^{i-1} \frac{\partial W_k}{\partial \hat{\Theta}} \right)(\Gamma_{i-1} - \dot{\hat{\Theta}}).
\end{aligned} \tag{23}$$

We define the virtual controller as

$$\begin{aligned}
x_{i+1}^* &= -\beta_i(t, \hat{\Theta}, \overline{x}_i)^{\frac{1}{p_i}}\lceil \xi_i \rceil^{\frac{r_{i+1}}{\sigma}}, \\
\beta_i &= \frac{1}{\underline{g}_i}(-2\dot{L} + (L+1)(n-i) + \varphi_i + Lk_i + L^2\hat{\Theta}\phi_i),
\end{aligned} \tag{24}$$

where $\varphi_i(t,\bar{x}_i,\hat{\Theta}) = \varphi_{i1} + \varphi_{i2} + \varphi_{i3} + 1$, $k_i(t,\bar{x}_i,\hat{\Theta}) = k_{i1} + k_{i2}$, and $\phi_i(t,\bar{x}_i,\hat{\Theta}) = \phi_{i1} + \phi_{i2}$ are positive functions. Define the Lyapunov function as $V_i = V_{i-1} + W_i$, based on Propositions 1–4, we can obtain

$$\begin{aligned}\dot{V}_i &\le \dot{V}_{i-1} + \sum_{k=1}^{i-1} \frac{\partial W_i}{\partial x_k}\dot{x}_k + \frac{\partial W_i}{\partial x_i}\dot{x}_i + \frac{\partial W_i}{\partial t} + \frac{\partial W_i}{\partial \hat{\Theta}}\dot{\hat{\Theta}} \\ &\le \left(\frac{2\dot{L}}{L} - (1+L^{-1})(n-i)\right)\sum_{k=1}^{i}\xi_k^2 + \xi_i^{\frac{2\sigma-r_i}{\sigma}} g_i \frac{x_{i+1}^{p_i} - x_{i+1}^{*p_i}}{L} + \frac{\dot{L}}{L}V_\theta \\ &\quad + \left(L\tilde{\Theta} - \sum_{k=1}^{i}\frac{\partial W_k}{\partial \hat{\Theta}}\right)(\Gamma_i - \dot{\hat{\Theta}}). \end{aligned} \tag{25}$$

Step *n*: We choose the Lyapunov function $V_n = V_{n-1} + W_n$, using the inductive argument above, we know the differential of V_n satisfies

$$\begin{aligned}\dot{V}_n \le \frac{2\dot{L}}{L}\sum_{k=1}^{n}\xi_k^2 + \left(L\tilde{\Theta} - \sum_{k=1}^{n}\frac{\partial W_k}{\partial \hat{\Theta}}\right)(\Gamma_n - \dot{\hat{\Theta}}) \\ + \lceil \xi_n \rfloor^{\frac{2\sigma-r_n}{\sigma}} g_n \frac{u^{p_n} - x_{n+1}^{*p_n}}{L} + \frac{\dot{L}}{L}V_\theta. \end{aligned} \tag{26}$$

The controller with adaptive law is designed as

$$\begin{aligned} u &= -\beta_n^{\frac{1}{p_n}}(t,\hat{\Theta},\bar{x}_n)\lceil \xi_n \rfloor^{\frac{r_{n+1}}{\sigma}}, \\ \dot{\hat{\Theta}} &= \Gamma_n = \sum_{k=1}^{n}\phi_k\xi_k^2, \end{aligned} \tag{27}$$

where $\beta_n = \frac{1}{\underline{g}_n}(-2\dot{L} + \varphi_n + Lk_n + L^2\hat{\Theta}\phi_n)$, $\varphi_n(t,\bar{x}_n,\hat{\Theta}) = \varphi_{n1} + \varphi_{n2} + \varphi_{n3} + 1$, $k_n(t,\bar{x}_n,\hat{\Theta}) = k_{n1} + k_{n2}$ and $\phi_n(t,\bar{x}_n,\hat{\Theta}) = \phi_{n1} + \phi_{n2}$ are positive functions. Let $x_{n+1}^* = u$, then, since $W_i = \int_{x_i^*}^{x_i} \lceil \lceil s \rfloor^{\frac{\sigma}{r_i}} - \lceil x_i^* \rfloor^{\frac{\sigma}{r_i}} \rfloor^{\frac{2\sigma-r_i}{\sigma}} ds \le 2\xi_i^2$, $\dot{L} \le 0$, we have

$$\dot{V}_n \le \frac{2\dot{L}}{L}\sum_{k=1}^{n}\xi_k^2 + \frac{\dot{L}}{L}V_\theta \le \frac{\dot{L}}{L}V_n. \tag{28}$$

3.2 | Prescribed-time stability analysis

Now, we give the following theorem to summarize the main results of this article.

Theorem 1. *For the nonlinear systems (1) with Assumptions 1 and 2, by designing the controller and adaptive law (27), the states of systems can converge to zero within a arbitrarily specified time T and other signals are bounded.*

Proof. Based on the definition of $L(t)$ and $\dot{V}_n(t) \le \frac{\dot{L}(t)}{L(t)}V_n(t)$, we have

$$V_n(t) \le \frac{L(t)}{L(0)}V_n(0). \tag{29}$$

Because $V_n(t) = \sum_{i=1}^{n}\int_{x_i^*}^{x_i}\lceil \lceil s \rfloor^{\frac{\sigma}{r_i}} - \lceil x_i^* \rfloor^{\frac{\sigma}{r_i}} \rfloor^{\frac{2\sigma-r_i}{\sigma}} ds + \frac{L(t)\tilde{\Theta}^2}{2}$ and $\lim_{t\to T} L(t) = 0$, we can conclude $\lim_{t\to T}\|\bar{\xi}_n\| = 0$ and $|\tilde{\Theta}| \le \sqrt{\frac{2V_n(0)}{L(0)}}$, hence we know $\beta_i(\cdot)$ is a bounded function and $\lim_{t\to T}|x_i^*| = \lim_{t\to T}|\beta_{i-1}^{\frac{1}{p_{i-1}}}\lceil \xi_{i-1} \rfloor^{\frac{r_i}{\sigma}}| = 0$. From (3) and (7), we can get that $\lim_{t\to T}\|\bar{x}_n\| = 0$ and $\lim_{t\to T}\|\bar{\eta}_n\| = 0$. Since $\lim_{t\to T} u(t) = \lim_{t\to T}\dot{\hat{\Theta}}(t) = 0$ and $\lim_{t\to T}\|\bar{\eta}_n(t)\| = 0$, based on the solution continuation properties, $f_i(\cdot)$ vanish at the origin, when setting $u(t) = 0$ for all $t \ge T$, we can easy to verify that $\|\bar{\eta}_n(t)\| = 0$. In other words, the states of system reach to zero within the prescribed time T and will always keep in equilibrium point for $t \in [T, \infty]$. Then, based on the definition of $V_n(t)$ and (29), we can directly conclude that the state $\overline{\xi}_n$, $\overline{x}_n$ and the estimate error $\tilde{\Theta}$ are all bounded. Since Θ is a constant, we know the estimate $\hat{\Theta}$ is also bounded. Then, it can be proved that virtual controllers $x_i^*(t)$, the control law $u(t)$ and the adaptive law $\dot{\hat{\Theta}}(t)$ are bounded. ∎

Remark 3. In this article, by introducing the time-varying scaling transformations $x_i = L(t)^{-m_i}\eta_i$, the original prescribed time stabilization problems are converted to the control problems of transformed system. Then, by constructing a new coordinate transformation combining time-varying function with estimation error, we propose a new Lyapunov-like function $V_\theta = \frac{L(t)\tilde{\Theta}^2}{2}$. Essentially, $\lim_{t\to T} L(t) = 0$ can ensure $\lim_{t\to T} V_\theta = 0$, then taking $L(t)\tilde{\Theta}$ as a whole to design adaptive law and controller. In other words, our goal is that the whole Lyapunov function converges at the rate of $L(t)$, therefore, we construct function V_θ, which also converges in $L(t)$. The bound of the controller must be taken into account in prescribed-time control, we can ensure functions β_i only contain the positive power of time-varying function $L(t)$. Hence the controller u and error function $\tilde{\Theta}$ are bounded. Based on this, a new framework for adaptive prescribed-time control is established, that eliminate the residual term, and ensure that the states converge to the zero within a time that independent of the initial conditions.

Remark 4. It should be noted that due to the existence of unknown parameter $\tilde{\Theta}$, we cannot introduce $-\frac{1}{2}\tilde{\Theta}^\alpha$ to the right hand side of the derivative of the whole Lyapunov function. So, the traditional finite/fixed-time methods based on fractional Lyapunov differential inequality, cannot be directly used to stability analysis. References 23-26 made an attempt to propose a two-phase control strategy that the global finite-time stability analysis was divided into two parts because of the residual term. For example, by defining a set Δ associated with unknown parameters and initial value, Li et al.[26] get $\dot{V}_n \leq -\frac{1}{8}\overline{\mu}V_n^{\frac{2\sigma}{2\sigma-w}} - \gamma\overline{\mu}(\frac{1}{4} - \overline{V}_n)\sum_{i=1}^n \xi_i^2$, where $\overline{V}_n < \frac{1}{4}$ when state $x(t) \in \Delta$. If the initial value $x(0) \in \Delta$, the state will converge to origin with the time $T_1(x(0))$; if $x(0) \notin \Delta$, the state will reached the boundary of Δ within a finite-time $T_2(x(0))$, then from the previous discussion, the states globally converge to origin within finite time $T_1(x(0)) + T_2(x(0))$. Obviously, the setting time relies on the initial values. For the fixed-time two-phase control strategy, we only can get T_1 is independent of the initial values, the setting-time $T_1 + T_2(x(0))$ still depends on the initial value. In this article, we proposed a new control strategy that does not generate the residual sets, and the convergence time can be prespecified and irrelevant to the initial value.

4 | SIMULATION EXAMPLE

In this article, we consider the following practical model for a class of underactuated systems with weak coupling[26,30,31] to demonstrate the availabilities of our developed control algorithm.

$$\begin{aligned} \ddot{\omega} &= \frac{g}{l}\sin\omega + \frac{k_s}{b_2 l}(x - l\sin\omega)^3\cos\omega, \\ \ddot{x} &= -\frac{k}{b_1}x - \frac{k_s}{b_1}(x - l\sin\omega)^3 + \frac{\hat{u}}{b_1}, \end{aligned} \tag{30}$$

where b_1 and b_2 are the mass of smooth pedestal platform and the mass of inverted pendulum, respectively. ω and x are the angle of the inverted pendulum and the displacement of mass b_1; k_s, l, g are the spring coefficients, length of the bar and gravity coefficient, respectively. Our objective is to stabilize the mechanical system within a user-prescribed time, by defining the smooth change of coordinates $\eta_1 = \omega, \eta_2 = \dot{\eta}_1, \eta_3 = (x - l\sin\omega)\sqrt[3]{\cos\omega}, \eta_4 = \dot{\eta}_3 = u$ on $(\omega, \dot{\omega}, x, \dot{x}) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}^3$. According to,[26] the stabilization problem of the fourth-order system is also solvable if the following third-order system form[26] is solvable.

$$\begin{aligned} \dot{\eta}_1 &= \eta_2, \\ \dot{\eta}_2 &= \frac{k_s}{b_2 l}\eta_3^3 + \frac{g}{l}\sin\eta_1, \\ \dot{\eta}_3 &= u. \end{aligned} \tag{31}$$

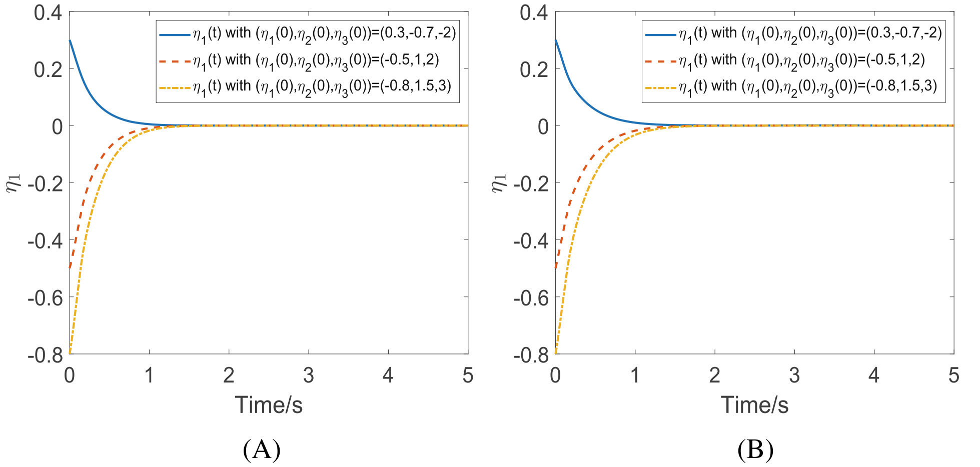

FIGURE 2 Response of η_1 under different initial values. (A) Prescribed-time $T = 3$ s and (B) prescribed-time $T = 4$ s

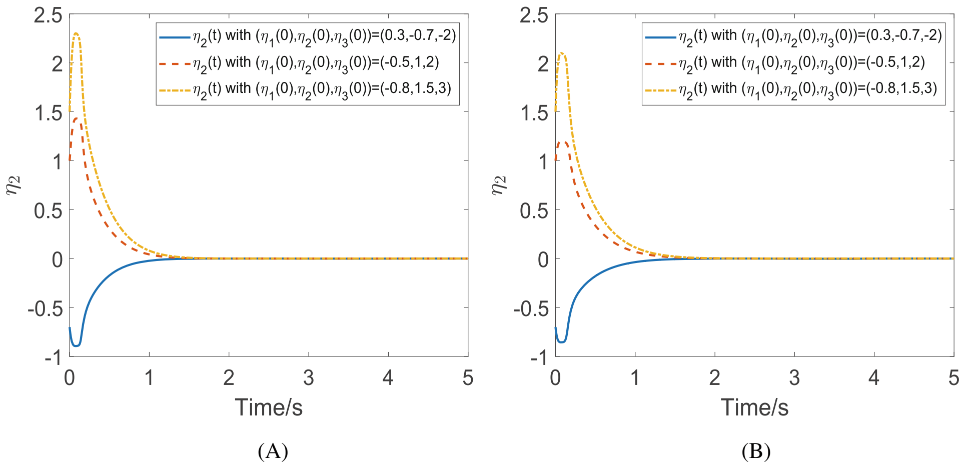

FIGURE 3 Response of η_2 under different initial values. (A) Prescribed-time $T = 3$ s and (B) prescribed-time $T = 4$ s

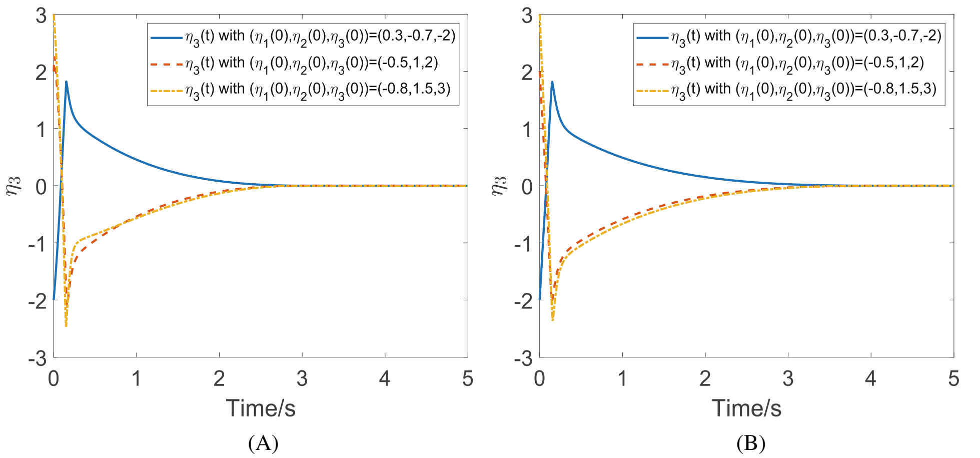

FIGURE 4 Response of η_3 under different initial values. (A) Prescribed-time $T = 3$ s and (B) prescribed-time $T = 4$ s

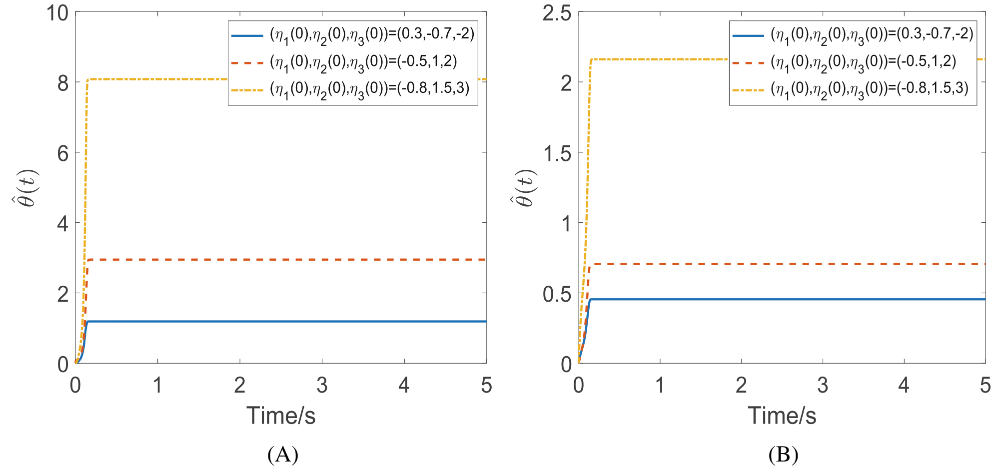

FIGURE 5 Response of $\hat{\Theta}(t)$ under different initial values. (A) Prescribed-time $T = 3$ s and (B) prescribed-time $T = 4$ s

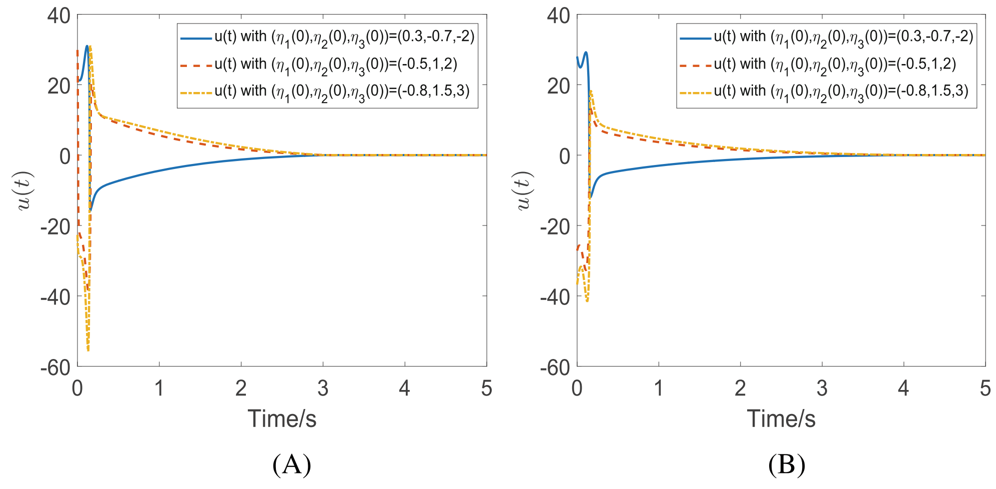

FIGURE 6 Response of $u(t)$ under different initial values. (A) Prescribed-time $T = 3$ s and (B) prescribed-time $T = 4$ s

Based on our proposed algorithm, we can take the design parameters as $m_1 = 5, m_2 = 4, m_3 = 1, r_1 = 1, r_2 = 1, r_3 = \frac{1}{3}, \sigma = 1$, $\overline{\theta} = \max\left\{1, \frac{g}{l}\right\}$. Designing time-varying function as $L(t) = T - t$. The initial values are chosen as $[\eta_1(0), \eta_2(0), \eta_3(0), \hat{\Theta}(0)]^T = [0.3, -0.7, -2, 0]^T, [-0.5, 1, 2, 0]^T$, and $[-0.8, 1.5, 3, 0]^T$, respectively. Then, by designing prescribed-time as $T = 3$ s and $T = 4$ s, respectively, we can obtain the following simulation results.

The simulation results are showed in Figures 2–6. According to Figures 2–4, it is obvious to get that the state variable converges to zero within $T = 3$ s and $T = 4$ s, where the setting time T of the system can be arbitrarily specified by user, and independent of any design parameters and initial values. From Figures 5 and 6, we can know the estimated values $\hat{\Theta}$ and controller $u(t)$ are bounded and $\lim_{t\to T} u(t) = 0$ due to $\lim_{t\to T} \|\overline{\eta}_n\| = 0$. According to the simulation results, it can be concluded that our proposed adaptive prescribed-time control algorithm is effective. Moreover, from the part figures A and B, we can find that the control gain actually depends on given convergence time, larger control gain will lead to faster convergence. Theoretically, the convergence time is any prescribed time, since it does not rely on the initial values. However, if the prescribed time T decreases, the control gain will become large. In practical application, we need to balance the relationship between the convergence time and the control gain to find an optimal situation.

5 | CONCLUSION

This article has proposed a novel adaptive prescribed-time algorithm to handle the stabilization problem for a class of uncertainty systems with p-normal form. By utilizing the time-varying scaling transformation, an adaptive controller has been structured to ensure that system state variables converge to the origin within the specified time and all the signals are bounded. Different from the existing results, our control algorithm does not generate residual sets, hence avoiding classified discussion caused by residual sets. Finally, the availabilities of the developed algorithm are demonstrated by a underactuated simulation example. Further research will focus on the global adaptive prescribed-time tracking problem without residual sets.

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CONFLICT OF INTEREST

The authors declare that they have no conflict of interest to this work.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

ORCID

Changchun Hua https://orcid.org/0000-0001-6311-2112
Hao Li https://orcid.org/0000-0001-8164-3830
Kuo Li https://orcid.org/0000-0002-9049-5558

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APPENDIX

Proof of Proposition 1. Based on Lemmas 2 and 3 and $p_i = \frac{r_i}{r_{i+1}}$, we have

$$\begin{aligned}
&L^{-1}|\xi_{i-1}|^{\frac{2\sigma-r_{i-1}}{\sigma}} g_{i-1}\left|x_i^{p_{i-1}} - x_i^{*p_{i-1}}\right| \\
&\leq \overline{g}_{i-1}L^{-1}|\xi_{i-1}|^{\frac{2\sigma-r_{i-1}}{\sigma}}\left|\lceil x_i\rceil^{\frac{\sigma}{r_i}\frac{r_{i-1}}{\sigma}} - \lceil x_i^*\rceil^{\frac{\sigma}{r_i}\frac{r_{i-1}}{\sigma}}\right| \\
&\leq \overline{g}_{i-1}2^{1-\frac{r_{i-1}}{\sigma}}L^{-1}|\xi_{i-1}|^{\frac{2\sigma-r_{i-1}}{\sigma}}|\xi_i|^{\frac{r_{i-1}}{\sigma}} \\
&\leq \frac{1}{3L}{\xi_{i-1}}^2 + \frac{\varphi_{i1}}{L}{\xi_i}^2,
\end{aligned} \tag{A1}$$

where φ_{i1} is a positive designed real number. ■

Proof of Proposition 2. We know $m_{i-1} - m_i \geq 1$, since $L < L(0)$, there is a known function $\tilde{\varphi}_{i1}\left(t, \overline{x}_i\right)$ such that $\tilde{\varphi}_{i1}\left(t, \overline{x}_i\right) \geq \overline{\varphi}_i = \overline{\varphi}_i\left(t, L^{m_1}x_1, \ldots, L^{m_i}x_i\right)$, then

$$\frac{\partial W_i}{\partial x_i}L^{-m_i}f_i\left(t, \theta_i, \overline{x}_i\right) \leq \overline{\theta}\overline{\varphi}_i\left(t, \overline{\eta}_i\right)|\xi_i|^{2-\frac{r_i}{\sigma}}\left(L^{m_1-m_i}|x_1| + \cdots + |x_i|\right)$$

$$
\begin{aligned}
&\leq \max\left\{L^{m_1-m_i}, L^{m_2-m_i}, \dots, L^0\right\}\bar{\theta}\tilde{\varphi}_i\left(t,\bar{x}_i\right) \\
&\quad \times |\xi_i|^{2-\frac{r_i}{\sigma}}(|\xi_1|^{\frac{r_1}{\sigma}} + \cdots + \beta_{i-1}^{\frac{1}{p_{i-1}}}|\xi_{i-1}|^{\frac{r_i}{\sigma}} + |\xi_i|^{\frac{r_i}{\sigma}}) \\
&\leq \frac{1}{4}\sum_{k=1}^{i-1}\xi_k^2 + \frac{\xi_i^2}{2L} + L\Theta\phi_{i1}(t,\bar{x}_i,\hat{\Theta})\xi_i^2,
\end{aligned} \tag{A2}
$$

we know $\phi_{i1}(t,\bar{x}_i,\hat{\Theta})$ is a known positive C^1 function.

According to (7) and Lemma 3, we can obtain

$$
\begin{aligned}
-m_i\frac{\dot{L}}{L}\lceil\xi_i\rfloor^{\frac{2\sigma-r_i}{\sigma}}x_i &\leq -m_i\frac{\dot{L}}{L}|\xi_i|^{\frac{2\sigma-r_i}{\sigma}}(|\xi_i|^{\frac{r_i}{\sigma}} + \beta_{i-1}^{\frac{1}{p_{i-1}}}|\xi_{i-1}|^{\frac{r_i}{\sigma}}) \\
&\leq \frac{1}{3L}|\xi_{i-1}|^2 + \frac{\varphi_{i2}(t,\bar{x}_{i-1},\hat{\Theta})}{L}|\xi_i|^2,
\end{aligned} \tag{A3}
$$

where $\varphi_{i2}(t,\bar{x}_{i-1},\hat{\Theta})$ is a C^1 function. Combining (A2) and (A3), we can get Proposition 2. ∎

Proof of Proposition 3. According to (7), there exists a nonnegative function $\overline{\varphi}(t,\hat{\Theta},\bar{x}_{i-1})$ such that

$$
\begin{aligned}
\frac{\partial\left(-x_i^*\right)^{\frac{\sigma}{r_i}}}{\partial t} &= \frac{\partial(-\beta_{i-1}^{\frac{\sigma}{r_{i-1}}}\lceil\xi_{i-1}\rfloor)}{\partial t} \\
&\leq |\xi_{i-1}|\left|\frac{\partial(\beta_{i-1}^{\frac{\sigma}{r_{i-1}}})}{\partial t}\right| + \beta_{i-1}^{\frac{\sigma}{r_{i-1}}}|\xi_{i-2}|\left|\frac{\partial(\beta^{i-2}_{\frac{\sigma}{r_{i-2}}})}{\partial t}\right| + \cdots + \beta_{i-1}^{\frac{\sigma}{r_{i-1}}}\beta_{i-2}^{\frac{\sigma}{r_{i-2}}}\dots\beta_2^{\frac{\sigma}{r_2}}|\xi_1|\left|\frac{\partial(\beta^1_{\frac{\sigma}{r_1}})}{\partial t}\right| \\
&\leq \overline{\varphi}(t,\hat{\Theta},\bar{x}_{i-1})\sum_{k=1}^{i-1}|\xi_k|
\end{aligned} \tag{A4}
$$

by a simple calculation, we have

$$
\begin{aligned}
\frac{\partial W_i}{\partial t} &\leq \left(\frac{2\sigma-r_i}{\sigma}\right)\left|\frac{\partial\left(-x_i^*\right)^{\frac{\sigma}{r_i}}}{\partial t}\right|\int_{x_i^*}^{x_i}\left(s^{\frac{\sigma}{r_i}} - x_i^{*\frac{\sigma}{r_i}}\right)^{1-\frac{r_i}{\sigma}} \\
&\leq \left(\frac{2\sigma-r_i}{\sigma}\right)\overline{\varphi}(t,\hat{\Theta},\bar{x}_{i-1})\sum_{k}^{i-1}|\xi_k|\,2^{1-\frac{r_i}{\sigma}}|\xi_i| \\
&\leq \frac{1}{4}\sum_{k=1}^{i-1}\xi_k^2 + k_{i1}\xi_i^2,
\end{aligned} \tag{A5}
$$

where $k_{i1}(t,\bar{x}_i,\hat{\Theta})$ is a known positive C^1 function. Then, based on mathematical induction, there exists a C^1 function $\lambda(t,\bar{x}_{i-1},\hat{\Theta})$ such that

$$
\left|\frac{\partial(-x_i^*)^{\frac{\sigma}{r_i}}}{\partial x_j}\right| \leq \left(\sum_{l=1}^{i-1}|\xi_l|^{1-\frac{r_j}{\sigma}}\right)\lambda(t,\bar{x}_{i-1},\hat{\Theta}). \tag{A6}
$$

Then according to (5) and (A2), we have

$$
\begin{aligned}
\frac{\partial\left(-x_i^*\right)^{\frac{\sigma}{r_i}}}{\partial x_j}\dot{x}_j &\leq \left(\sum_{l=1}^{i-1}|\xi_l|^{1-\frac{r_j}{\sigma}}\right)\tilde{\lambda}(t,\bar{x}_{i-1},\hat{\Theta}) \\
&\quad \times\left(\frac{\beta_{j-1}^{\frac{1}{p_{j-1}}}|\xi_{j-1}|^{\frac{r_j}{\sigma}} + \left(\beta_j^{\frac{1}{p_j}}+1\right)|\xi_j|^{\frac{r_j}{\sigma}}}{L} + \frac{|\xi_{j+1}|^{\frac{r_j}{\sigma}}}{L} + L^{-m_j}f_j\left(t,\theta_j,\bar{\eta}_j\right)\right)
\end{aligned}
$$

$$\leq \left(L^{-1}\sum_{l=1}^{i-1}\overline{\lambda}_l\left(t,\overline{x}_{i-1},\hat{\Theta}\right)|\xi_l|\right) + \overline{\theta}^{\frac{\sigma}{r_j}}\left(\sum_{l=1}^{i-1}\overline{\varphi}_l\left(t,\overline{x}_{i-1},\hat{\Theta}\right)|\xi_l|\right). \tag{A7}$$

Further, in view of (A6), there are positive C^1 functions $\varphi_{i3}(t,\overline{x}_i,\hat{\Theta})$, $\phi_{i2}(t,\overline{x}_i,\hat{\Theta})$ such that

$$\begin{aligned}\sum_{k=1}^{i-1}\frac{\partial W_i}{\partial x_k}\dot{x}_k &\leq \left(2-\frac{r_i}{\sigma}\right)\sum_{k=1}^{i-1}\left|\frac{\partial\left(x_i^*\right)^{\frac{\sigma}{r_i}}}{\partial x_k}\dot{x}_k\right|\int_{x_i^*}^{x_i}\left(s^{\frac{\sigma}{r_i}}-x_i^{*\frac{\sigma}{r_i}}\right)^{1-\frac{r_i}{\sigma}}ds \\ &\leq \left(\frac{2\sigma-r_i}{\sigma}\right)2^{1-\frac{r_i}{\sigma}}|\xi_i|\sum_{k=1}^{i-1}\left|\frac{\partial\left(-x_i^*\right)^{\frac{\sigma}{r_i}}}{\partial x_k}\dot{x}_k\right| \\ &\leq \sum_{k=1}^{i-1}\left(\frac{1}{4}\xi_k{}^2+\frac{1}{3L}\xi_k{}^2\right)+\frac{\varphi_{i3}(t,\overline{x}_i,\hat{\Theta})}{L}\xi_i{}^2+\frac{1}{2L}\xi_i{}^2+L\Theta\phi_{i2}(t,\overline{x}_i,\hat{\Theta})\xi_i^2. \end{aligned}\tag{A8}$$

Combining (A5) and (A8), we can get (13). ■

Proof of Proposition 4. Based on the proof of (12), we can find a function $\gamma_i(t,\hat{\Theta},\overline{x}_{i-1})$ yields

$$\begin{aligned}\frac{\partial\left(-x_i^*\right)^{\frac{\sigma}{r_i}}}{\partial\hat{\Theta}} &= \frac{\partial(-\beta_{i-1}^{\frac{\sigma}{r_{i-1}}}\xi_{i-1})}{\partial\hat{\Theta}} \\ &\leq |\xi_{i-1}|\left|\frac{\partial(\beta^{i-1}_{\frac{\sigma}{r_{i-1}}})}{\partial\hat{\Theta}}\right|+\beta_{i-1}^{\frac{\sigma}{r_{i-1}}}|\xi_{i-2}|\left|\frac{\partial(\beta^{i-2}_{\frac{\sigma}{r_{i-2}}})}{\partial\hat{\Theta}}\right| \\ &\quad+\cdots+\beta_{i-1}^{\frac{\sigma}{r_{i-1}}}\beta_{i-2}^{\frac{\sigma}{r_{i-2}}}\ldots\beta_2^{\frac{\sigma}{r_2}}|\xi_1|\left|\frac{\partial(\beta^1_{\frac{\sigma}{r_1}})}{\partial\hat{\Theta}}\right| \\ &= \gamma_i(t,\hat{\Theta},\overline{x}_{i-1}). \end{aligned}\tag{A9}$$

According to the expression of β_i, we have $\gamma_i(t,\hat{\Theta},\overline{x}_{i-1})$ is a C^1 function, then we have

$$\begin{aligned}\sum_{k=1}^{i-1}\frac{\partial W_k}{\partial\hat{\Theta}}\phi_i\xi_i^2+\frac{\partial W_i}{\partial\hat{\Theta}}\Gamma_i &\leq \sum_{k=1}^{i-1}\frac{\partial W_k}{\partial\hat{\Theta}}\phi_i\xi_i^2 \\ &\quad+\left(\frac{2\sigma-r_i}{\sigma}\right)\left|\frac{\partial\left(-x_i^*\right)^{\frac{\sigma}{r_i}}}{\partial\hat{\Theta}}\right|\int_{x_i^*}^{x_i}\left(s^{\frac{\sigma}{r_i}}-x_i^{*\frac{\sigma}{r_i}}\right)^{1-\frac{r_i}{\sigma}}ds\Gamma_i \\ &\leq \tilde{\gamma}2^{1-\frac{r_i}{\sigma}}|\xi_i|\sum_{k=1}^{i}\phi_k\xi_k^2+\sum_{k=1}^{i-1}\frac{\partial W_k}{\partial\hat{\Theta}}\phi_i\xi_i^2 \\ &\leq \sum_{k=1}^{i-1}\frac{1}{4}\xi_k^2+k_{i2}(t,\overline{x}_i,\hat{\Theta})\xi_i^2, \end{aligned}\tag{A10}$$

where $k_{i2}(t,\overline{x}_i,\hat{\Theta})$ is a known positive C^1 function. ■