Collectively Rotating Formation and Containment Deployment of Multiagent Systems: A Polar Coordinate-Based Finite Time Approach

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***Abstract*—This paper investigates the problem of achieving rotating formation and containment simultaneously via finite time control schemes for multiagent systems. It is nontrivial to maintain rotating formation where the desired formation structure is time-varying and only neighboring information is available. The underlying problem becomes even more complicated if containment is imposed yet finite time convergence is required at the same time. To tackle this problem, a polar coordinate-based approach is exploited in this paper. Finite time control protocols are established for leader agents and follower agents, respectively, such that three goals are achieved in finite time concurrently: 1) all the agents maintain a stable rotating motion around a common circular center with a common (possibly time-varying) angular velocity; 2) the leader agents form and maintain a prespecified rotating formation structure; and 3) the follower agents converge to the shifting convex hull shaped by the dynamically moving (circling) leaders. It is the polar coordinate expression that simplifies the formulation of the rotating formation-containment problem and facilitates the finite time control design process. The effectiveness of the proposed control scheme is illustrated via both formative mathematical analysis and numerical simulation.**

***Index Terms*—Finite-time control, multiagent systems (MASs), polar coordinate, rotating formation-containment.**

I. Introduction

THIS paper addresses the rotating formation-containment deployment problem for a group of dynamic agents, arising from many important applications such as formation flight of satellites around the earth, circular mobile sensor networks, spacecraft docking and so forth. By rotating formation-containment deployment, we refer in this paper the problem that all agents in the multiagent system (MAS) finally maintain a stable rotating motion around a common point, with leader agents keeping a specific rotating formation structure and meanwhile the follower agents converging to the convex hull formed by the moving leaders.

As the relative position of the desired rotating formation is time varying due to the existence of the rotating motion, the rotating formation control is challenging. The underlying problem becomes even more complex if finite time convergence is required. For only rotating formation control, there are several interesting solutions available in the literature. For instance, by applying real numbering system theory [1] invested the rotating formation problem with all agents finally circling around a common point with some special structures at a unit speed. By introducing a cyclic pursuit policy [2] studied the collective circular motion, and the results were extended in [3] by introducing a rotation matrix to an existing second-order consensus protocol. Based on the complex numbering system theory, the collective rotating formation control problem for second-order MAS was investigated in [4]. The problem of cyclic pursuits was studied in [5] and [6], and in [7] a remark on collective circular motion of heterogeneous MAS was given. There are also some works investigating containment control of MAS (see [8]–[12], to just name a few). However, those methods have not solved the finite time rotating formation or finite time rotating containment control problem.

Finite-time convergence is highly desirable for controlled systems not only because of its faster convergence rate but also due to the better disturbance rejection and robustness against uncertainties [13]. Considerable amount of research has been conducted on finite time control of MAS modeled by linear ordinary differential equations, such as [14]–[24], to just name a few. Xiao *et al.* [14] and Wang and Xiao [15] studied the finite time consensus and finite time formation control for first-order linear MAS, respectively. The terminal sliding mode approach was used to design finite time consensus algorithms in [16] for second-order leader-follower MAS. By employing a power integrator method, the work of [17] proposed a continuous finite-time consensus algorithm for second-order MAS, and the results were extended to MAS with nonlinearities in [18]. The finite-time consensus tracking and finite-time containment problem was studied in [19]. Meurer and Krstic [25] addressed the finite time transitions between desired deployment formations along predefined spatial-temporal paths. It is worth noting that the finite-time control schemes developed in the above works are inapplicable

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to the rotating formation-containment problem considered here. One is because the leaders here are not only dynamical but also the relative position of them is time varying; the other is due to the fact that one of the ultimate goals of rotating formation-containment control is to make all the agents maintain a rotating motion along a common central point continuously with a common angular velocity (possibly time-varying), while the existing finite time control methods are based on regulating the speed of the agents to zero.

Thus far, although containment control or rotating formation control of MAS has been studied extensively, each of them has been normally addressed separately and there is no reported work on collectively rotating formation and rotating containment control of MAS with particular attention to finite time convergence. This paper presents a solution to this problem. The goal is to develop distributed finite time control algorithms to achieve simultaneously rotating formation and rotating containment for a group of networked agents, each with different initial rotating speed, and within a finite time period all the followers being contained by the leaders, yet at the same time all the agents rotating around a common center with a common time-varying angular speed.

The contributions of this paper can be summarized as follows.

1) The rotating formation problem for leaders and rotating containment problem for followers are addressed simultaneously via finite time control algorithms.
2) By exploiting a polar coordinate-based approach, the formulation of the rotating formation and rotating containment problem is largely simplified, facilitating the control design and stability analysis for finite time control of the MAS.
3) By introducing a novel angular shift transformation, a continuous and simple solution is established to make all the agents in the MAS rotate with the same and *possibly time-varying* angular speed along a common circular center, while initially all the agents can *have different rotating speed*, in contrast to existing rotating control methods that require initially all the agents must *have the same rotating speed* as the final common rotating speed. Furthermore, with the proposed method, the common rotating speed can be *time-varying* rather than *constant* as required in most existing works [1]–[4], [26].

II. Preliminaries

Throughout this paper, the initial time t_0 is set as $t_0 = 0$ without loss of generality; $\lambda_{\max}(F)$ and $\lambda_{\min}(F)$ denote the maximum and minimum eigenvalue of matrix F, respectively.

Suppose that the communication among the N agents to be represented by a weighted graph $G = (P, Q)$, with P and Q denoting the set of nodes and the set of edges, respectively. A directed edge is denoted by $Q_{ij} = (p_i, p_j)$, implying that node p_j can receive the state and structure information from node p_i, and the weighted adjacency matrix $A = [a_{ij}]$, where $Q_{ji} \in Q \Leftrightarrow a_{ij} > 0$, and otherwise, $a_{ij} = 0$. Moreover, it is assumed that $a_{ii} = 0$ for all $i = 1, \ldots, N$. We denote by $D = \mathrm{diag}\{d_1, \ldots, d_N\}$ the in-degree matrix, with $d_i = \sum_{j \in N_i} a_{ij}$. The Laplacian matrix is defined as $L = D - A$. The set of neighbors of node p_i is denoted by $\mathcal{N}_i = \{p_j \in P | (p_j, p_i) \in Q\}$.

Lemma 1 [28]: If $0 < p = p_1/p_2 \le 1$, where $p_1, p_2 > 0$ are positive odd integers, then $|x^p - y^p| \le 2^{1-p}|x - y|^p$.

Lemma 2 [28]: For $x, y \in R$, if $c, d > 0$, then $|x|^c|y|^d \le c/(c+d)|x|^{c+d} + d/(c+d)|y|^{c+d}$.

Lemma 3 [29]: For $x_i \in R$, $i = 1, \ldots, n$, $0 < p \le 1$, then $(\sum_{i=1}^{n} |x_i|)^p \le \sum_{i=1}^{n} |x_i|^p \le n^{1-p}(\sum_{i=1}^{n} |x_i|)^p$.

Lemma 4 [30]: If graph G is undirected, then $X^T LX = (1/2)\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x_i - x_j)^2$. If graph G is undirected and connected, then $LX = 0_n$ or $XLX = 0$ if and only if $x_i = x_j$, for all $i, j = 1, \ldots, n$.

III. Problem Formulation

A. Agent Dynamics in Polar Coordinates

Consider a group of networked systems consisting of N agents with dynamics expressed in polar coordinate

$$\begin{aligned}\left(\dot{\rho}_k(t), \dot{\theta}_k(t)\right) &= \left(v_{\rho k}(t), v_{\theta k}(t)\right)\\ \left(\dot{v}_{\rho k}(t), \dot{v}_{\theta k}(t)\right) &= \left(u_{\rho k}(t), u_{\theta k}(t)\right), \quad k = 1, \ldots, N \end{aligned} \tag{1}$$

where (ρ_k, θ_k) denotes the position vector of the kth agent ($\rho_k \in R^+$ and $\theta_k \in [0, 2n\pi)$ with $n \in Z_+$ are the components of the polar coordinates, respectively, without loss of generality, counter clockwise is assumed), $(v_{\rho k}, v_{\theta k})$ and $(u_{\rho k}, u_{\theta k})$ denote the velocity and control input vector of the kth agent, respectively.

Remark 1: It should be stressed that with the Polar coordinate notation, θ_i being defined on R_+ $[0, 2n\pi)$ is *equivalent* to θ_i being defined on S^1 domain as in [1], which can be seen clearly from Fig. 1. However, as seen shortly, by using the polar coordinate expression, the rotating formation and containment problem can be formulated in a simplified way and can be solved gracefully with finite time approach.

B. Problem Formulation

Definition 1 [27]: For the formation-containment problem, an agent is called a leader if its neighbors are only leaders and it coordinates with its neighbors to achieve a formation, and an agent is called a follower if it has at least one neighbor in the MAS and coordinates with its neighbors to achieve containment.

Let $A = \{1, 2, \ldots, M\}$ be the set of integers for the leaders and $B = \{M+1, M+2, \ldots, N\}$ be the set of integers for the followers, respectively. Let G denote the interaction topology among all the agents, and G_A denote the interaction topology among leaders, respectively. The Laplacian matrix associated with G is denoted by $L = \begin{bmatrix} L_1 & L_2 \\ 0 & L_3 \end{bmatrix}$, with $L_1 \in R^{(N-M)\times(N-M)}$, $L_2 \in R^{(N-M)\times M}$, and $L_3 \in R^{M\times M}$ (L_3 denotes the Laplacian matrix of G_A).

Assumption 1: The communication topology G_A is undirected and connected, and the communication among different followers is bidirectional. However, the communication between a leader and a follower is unidirectional where only the leader issues the command. For each follower, there exists at least one leader that has a directed path linking to it.

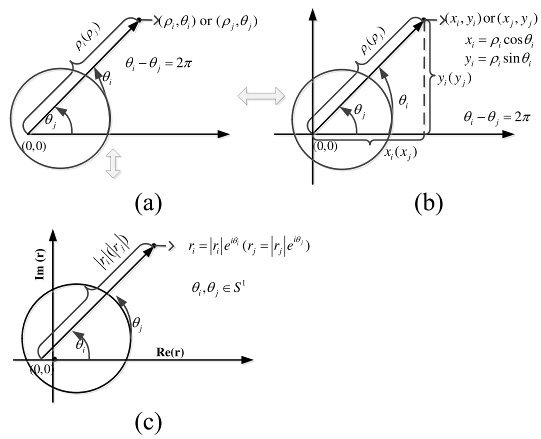

Fig. 1. Equivalence of the expression for rotating motion in the three coordinates: (a) polar, (b) Euclidean, and (c) complex coordinates.

Under Assumption 1, the following lemma can be obtained.

Lemma 5 [10]: Under Assumption 1, the Laplacian matrix L_1 is symmetrical and positive definite. In addition, each entry of $-L_1^{-1}L_2$ is non-negative, and each row of $-L_1^{-1}L_2$ has a sum equal to one.

Definition 2: The desired rotating formation structure for the M leader agents is specified by a vector $h = [(h_{\rho 1}, h_{\theta 1}), \dots, (h_{\rho M}, h_{\theta M})]^T = [(\bar{\rho}_1, \bar{\theta}_1), \dots, (\bar{\rho}_M, \bar{\theta}_M)]^T$, where $h_{\rho k} = \bar{\rho}_k \in R^+$ and $h_{\theta k} = \bar{\theta}_k \in [0, 2n\pi)$ ($k \in A$, $n \in Z_+$). The M leaders in system (1) are said to achieve rotating formation h in finite time if for any initial states, there exists a finite time T^* such that for all $t \geq T^*$ and all $j, k \in A$

$$\left[(\rho_k(t), \theta_k(t)) - \left(\rho_j(t), \theta_j(t)\right)\right] - \left[\left(\bar{\rho}_k, \bar{\theta}_k\right) - \left(\bar{\rho}_j, \bar{\theta}_j\right)\right] = (0, 0) \tag{2}$$

$$\dot{\rho}_k(t) = 0, \quad \dot{\theta}_k(t) = w. \tag{3}$$

Definition 3: It is said to achieve rotating containment in finite time for MAS (1) if for any initial states, there exists a finite time T^* and non-negative constants α_j ($j \in A$) satisfying $\sum_{j=1}^{M} \alpha_j = 1$ such that for all $t \geq T^*$ and $i \in B$

$$(\rho_i(t), \theta_i(t)) - \sum_{j=1}^{M} \alpha_j \left(\rho_j(t), \theta_j(t)\right) = (0, 0) \tag{4}$$

$$\dot{\rho}_i(t) = 0, \quad \dot{\theta}_i(t) = w. \tag{5}$$

Definition 4: It is said to achieve rotating formation-containment in finite time for MAS (1) if for any initial states, there exists a finite time T^* and non-negative constants α_j ($j \in A$) satisfying $\sum_{j=1}^{M} \alpha_j = 1$ such that (2)–(5) hold simultaneously for any $k \in A$ and $i \in B$.

Remark 2: It is seen that with the polar coordinates expression the rotating formation-containment problem can be formulated simply by the relations (2)–(5), where (2) and (4) imply that the rotating formation for leaders and the rotating containment for followers are achieved, respectively. In addition, the first equation in (3) [or (5)] means that each agent p_k ($k \in A \cup B$) moves in a circular orbit with a constant radius ρ_k, and the second equation in (3) [or (5)] implies that each agent p_k moves in a circle with a common angular velocity w (in this paper, w can be constant or time-varying).

Remark 3: To see the significance of polar-based coordinate method, it is worth noting that if the spatial rectangular coordinate is used, the agent dynamics are described by

$$\dot{r}_k(t) = v_k(t), \quad \dot{v}_k(t) = u_k(t) \tag{6}$$

where $r_k = x_k + i_j y_k = \rho_k e^{i_j \theta_k} \in C \approx R^2$ [1] (i_j is the imaginary unit) denotes the position of the kth agent, and the rotating formation problem is expressed as: for a given rotating formation vector $h = [h_1 \ \cdots \ h_N]^T = [\bar{\rho}_1 e^{i_j \bar{\theta}_1} \ \cdots \ \bar{\rho}_N e^{i_j \bar{\theta}_N}]^T$, if for all $j, k \in \{1, \dots, N\}$

$$\begin{aligned} \lim_{t\to\infty}\left[\left\| r_j(t) - r_k(t)\right\| - \left\| \bar{\rho}_j e^{i_j \bar{\theta}_j} - \bar{\rho}_k e^{i_j \bar{\theta}_k}\right\|\right] &= 0 \\ \lim_{t\to\infty}\left[\left(r_j(t) + w^{-1} i_j v_j(t)\right) - \left(r_k(t) + w^{-1} i_j v_k(t)\right)\right] &= 0 \end{aligned} \tag{7}$$

where $w \in R$ denotes the common (constant) angular velocity, and moreover, all the agents moves around a common circular center, that is, for all $k \in \{1, \dots, N\}$, it holds that

$$\lim_{t\to\infty}\left(\dot{r}_k(t) + w^{-1} i_j \dot{v}_k(t)\right) = 0. \tag{8}$$

That is, $\lim_{t\to\infty} \dot{v}_k(t) = i_j w v_k(t)$, from which one can see that the velocity v_k of the kth agent does not converge to zero even if the rotating formation is achieved. So if both rotating formation and rotating containment are addressed simultaneously, the formulation of the problem becomes rather complicated, not to mention the actual controller design and stability analysis, especially when finite time is imposed. However, if using the polar coordinate as proposed here, the underlying problem of collectively rotating formation and rotating containment can be simply formulated, and the finite time distributed control design involving time-varying common angular velocity can be handled neatly, as seen in next section.

IV. Main Results

The focus in this section is on deriving the distributed control schemes for the M leaders and $N - M$ followers in (1), respectively, such that the leaders achieve a prespecified rotating formation structure h in a finite time T_1 and the followers achieve rotating containment in a finite time T^* ($T^* > T_1$).

A. Finite-Time Rotating Formation Control for Leaders

1) System Transformation: As the common rotational speed for all the agents w is possibly time-varying, we introduce a new variable $\breve{\theta}_k$ ($k \in A$) as follows:

$$\breve{\theta}_k(t) = \theta_k(t) - \int_0^t w(\tau) d\tau \tag{9}$$

therefore, $\dot{\breve{\theta}}_k(t) = \dot{\theta}_k(t) - w(t) = v_{\theta k}(t) - w(t)$ such that $\dot{\breve{\theta}}_k(t) = v_{\breve{\theta} k}(t) = v_{\theta k}(t) - w(t)$, and further, $\dot{v}_{\breve{\theta} k}(t) = \dot{v}_{\theta k}(t) - \dot{w}(t) = u_{\theta k}(t) - \dot{w}(t)$, thus $\dot{v}_{\breve{\theta} k}(t) = u_{\breve{\theta} k}(t) = u_{\theta k}(t) - \dot{w}(t)$. With such new variable $\breve{\theta}_k$, the dynamical model for the kth ($k \in A$) leader in (1) then becomes

$$\begin{aligned} \left(\dot{\rho}_k(t), \dot{\breve{\theta}}_k(t) + w(t)\right) &= \left(v_{\rho k}(t), v_{\breve{\theta} k}(t) + w(t)\right) \\ \left(\dot{v}_{\rho k}(t), \dot{v}_{\breve{\theta} k}(t) + \dot{w}(t)\right) &= \left(u_{\rho k}(t), u_{\breve{\theta} k}(t) + \dot{w}(t)\right). \end{aligned} \tag{10}$$

Then the original rotating formation control problem is converted into steering the kth ($k \in A$) leader agent such that

$$\left[\left(\rho_k(t), \breve{\theta}_k(t)\right) - \left(\rho_j(t), \breve{\theta}_j(t)\right)\right] - \left[\left(\bar{\rho}_k, \bar{\theta}_k\right) - \left(\bar{\rho}_j, \bar{\theta}_j\right)\right] = 0 \tag{11}$$

$$\dot{\rho}_k(t) = 0, \quad \dot{\breve{\theta}}_k(t) = 0. \tag{12}$$

2) Controller Design: The kth ($k \in A$) neighborhood state error is introduced as

$$\begin{aligned} e_{\rho k} &= \sum_{j \in \mathcal{N}_k} a_{kj}\left(\rho_k - \rho_j - \bar{\rho}_k + \bar{\rho}_j\right) \\ e_{\breve{\theta} k} &= \sum_{j \in \mathcal{N}_k} a_{kj}\left(\breve{\theta}_k - \breve{\theta}_j - \bar{\theta}_k + \bar{\theta}_j\right). \end{aligned} \tag{13}$$

Let $E_{\rho A} = [e_{\rho 1}, \ldots, e_{\rho M}]^T$, $E_{\breve{\theta} A} = [e_{\breve{\theta} 1}, \ldots, e_{\breve{\theta} M}]^T$, $\rho_A = [\rho_1, \ldots, \rho_M]^T$, $\breve{\theta}_A = [\breve{\theta}_1, \ldots, \breve{\theta}_M]^T$, $h_\rho = (h_{\rho 1}, \ldots, h_{\rho M}]^T = [\bar{\rho}_1, \ldots, \bar{\rho}_M]^T$, $h_{\breve{\theta}} = [h_{\breve{\theta} 1}, \ldots, h_{\breve{\theta} M}]^T = [\bar{\theta}_1, \ldots, \bar{\theta}_M]^T$. It is readily seen that $E_{\rho A} = L_3(\rho_A - h_\rho)$ and $E_{\breve{\theta} A} = L_3(\breve{\theta}_A - h_{\breve{\theta}})$ from (13). Then according to Lemma 4, the rotating formation specified by (2) is achieved if and only if $E_{\rho A} = 0$ and $E_{\theta A} = 0$ simultaneously under Assumption 1.

With (13), the controller for the leaders is designed as

$$\begin{aligned} u_{\rho k} &= -c_1\left[\left(v_{\rho k}\right)^{\frac{2p+1}{2p-1}} - \left(v^*_{\rho k}\right)^{\frac{2p+1}{2p-1}}\right]^{\frac{2p-3}{2p+1}} \\ u_{\breve{\theta} k} &= -c_1\left[\left(v_{\breve{\theta} k}\right)^{\frac{2p+1}{2p-1}} - \left(v^*_{\breve{\theta} k}\right)^{\frac{2p+1}{2p-1}}\right]^{\frac{2p-3}{2p+1}} \end{aligned} \tag{14}$$

with

$$v^*_{\rho k} = -c_2\left(e_{\rho k}\right)^{\frac{2p-1}{2p+1}}, \quad v^*_{\breve{\theta} k} = -c_2\left(e_{\breve{\theta} k}\right)^{\frac{2p-1}{2p+1}} \tag{15}$$

for $k \in A$, where $p \geq 2$ is a user-chosen integer, c_1 and c_2 are design parameters chosen such that $c_2 > [(2^{1-d})/(1+d)] + [(2-d)/(1+d)]d(a+b\bar{m})$ and $c_1 > 2^{1-d}c_2^{1+1/d}[[(2-d)/(c_2(1+d))](a(2^{1-d} + 2^{1-d}d + c_2) + b\bar{m}(2^{1-d} + c_2 + 2^{1-d}d)) + [(2^{1-d}d)/(1+d)]]$, with $d = [(2p-1)/(2p+1)]$, $a = \max_{\forall k \in A}\{\sum_{j \in N_k} a_{kj}\}$, $b = \max_{\forall k, j \in A}\{a_{kj}\}$, and $\bar{m}$ being the maximum number of elements in $\mathcal{N}_k$ for all $k \in A$.

The main result in this section is summarized in the following theorem.

Theorem 1: Consider the group of networked systems (1) satisfying Assumption 1. For the M leaders with the dynamics as described in (1) or equivalently in (10), if the distributed control law (14) with (15) is applied, then the M leaders are ensured to achieve the prespecified rotating formation structure and rotate at a common angular speed $w(t)$ [i.e., both (2) and (3) are satisfied] simultaneously in a finite time T_1. In addition, the finite time T_1 is specified by

$$T_1 \leq \frac{2p+1}{c} V_A(t_0)^{1/(2p+1)} \tag{16}$$

where $V_A(t_0)$ is decided by (17), (26), and (37) which is known, and $c \leq \min\{(2^\alpha k_3/\lambda^\alpha_{\max}), c_2^{[8p^2/(4p^2-1)]}k_4\}$ with $\alpha = [2p/(2p+1)]$ and k_3, k_4 being decided by (35) and (39) which is explicitly computable.

Proof: The proof consists of five steps.

Step 1: Define a distributed Lyapunov function candidate.

Denote the eigenvalues of L_3 by $0, \lambda_2, \ldots, \lambda_M$, where $\lambda_k > 0$ for $k = 2, \ldots, M$, which is guaranteed by Lemma 4. Thus there exists an orthogonal matrix U_M such that $L_3 = U_M^T \mathrm{diag}\{0, \lambda_2, \ldots, \lambda_M\} U_M = U_M^T \Lambda U_M$, with $\Lambda = \mathrm{diag}\{0, \lambda_2, \ldots, \lambda_M\}$. Define $\bar{\Lambda} = \mathrm{diag}\{a, \lambda_2, \ldots, \lambda_n\}$ ($0 < a \leq \lambda_2$ is a constant) and $Q = U_M^T \bar{\Lambda}^{-1} U_M$. Note that matrix Q is symmetrical and positive definite, which motivates us to construct the following Lyapunov function:

$$V_1(t) = \frac{1}{2}E^T_{\rho A} Q E_{\rho A} + \frac{1}{2}E^T_{\breve{\theta} A} Q E_{\breve{\theta} A}. \tag{17}$$

Note that

$$\begin{aligned} &E^T_{\rho A} Q E_{\rho A} \\ &= \left(\rho_A - h_\rho\right)^T L_3^T\left(U_M^T \bar{\Lambda}^{-1} U_M\right) L_3\left(\rho_A - h_\rho\right) \\ &= \left(\rho_A - h_\rho\right)^T\left(U_M^T \Lambda U_M\right)^T\left(U_M^T \bar{\Lambda}^{-1} U_M\right)\left(U_M^T \Lambda U_M\right)\left(\rho_A - h_\rho\right) \\ &= \left(\rho_A - h_\rho\right)^T U_M^T \Lambda U_M\left(\rho_A - h_\rho\right) \\ &= \left(\rho_A - h_\rho\right)^T L_3\left(\rho_A - h_\rho\right) \end{aligned} \tag{18}$$

and similarly

$$E^T_{\breve{\theta} A} Q E_{\breve{\theta} A} = \left(\breve{\theta}_A - h_{\breve{\theta}}\right)^T L_3\left(\breve{\theta}_A - h_{\breve{\theta}}\right). \tag{19}$$

According to Lemma 4, it holds that

$$\begin{aligned} V_1(t) &= \frac{1}{2}\left(\rho_A - h_\rho\right)^T L_3\left(\rho_A - h_\rho\right) \\ &\quad + \frac{1}{2}\left(\breve{\theta}_A - h_{\breve{\theta}}\right)^T L_3\left(\breve{\theta}_A - h_{\breve{\theta}}\right) \\ &= \frac{1}{4}\sum_{k=1}^M \sum_{j \in N_k} a_{kj}\left(\rho_k - \bar{\rho}_k - \rho_j + \bar{\rho}_j\right)^2 \\ &\quad + \frac{1}{4}\sum_{k=1}^M \sum_{j \in N_k} a_{kj}\left(\breve{\theta}_k - \bar{\theta}_k - \breve{\theta}_j + \bar{\theta}_j\right)^2. \end{aligned} \tag{20}$$

Thus the derivative of $V_1(t)$ can be computed as

$$\begin{aligned} \dot{V}_1(t) &= \dot{\rho}_A^T L_3\left(\rho_A - h_\rho\right) + \dot{\breve{\theta}}_A^T L_3\left(\breve{\theta}_A - h_{\breve{\theta}}\right) \\ &= \sum_{k=1}^M e_{\rho k}\dot{\rho}_k + \sum_{k=1}^M e_{\breve{\theta} k}\dot{\breve{\theta}}_k = \sum_{k=1}^M e_{\rho k} v_{\rho k} + \sum_{k=1}^M e_{\breve{\theta} k} v_{\breve{\theta} k} \end{aligned} \tag{21}$$

in which $v_{\rho A} = [v_{\rho 1}, \ldots, v_{\rho M}]^T$ and $v_{\breve{\theta} A} = [v_{\breve{\theta} 1}, \ldots, v_{\breve{\theta} M}]^T$. Let $v^*_{\rho k}$ and $v^*_{\breve{\theta} k}$ [given in (15)] be the virtual control of $v_{\rho k}$ and $v_{\breve{\theta} k}$, respectively. Then it holds

$$\begin{aligned} \dot{V}_1(t) &= \sum_{k=1}^M e_{\rho k} v^*_{\rho k} + \sum_{k=1}^M e_{\rho k}\left(v_{\rho k} - v^*_{\rho k}\right) \\ &\quad + \sum_{k=1}^M e_{\breve{\theta} k} v^*_{\breve{\theta} k} + \sum_{k=1}^M e_{\breve{\theta} k}\left(v_{\breve{\theta} k} - v^*_{\breve{\theta} k}\right) \\ &= -c_2\sum_{k=1}^M e_{\rho k}^{1+d} + \sum_{k=1}^M e_{\rho k}\left(v_{\rho k} - v^*_{\rho k}\right) \\ &\quad - c_2\sum_{k=1}^M e_{\breve{\theta} k}^{1+d} + \sum_{k=1}^M e_{\breve{\theta} k}\left(v_{\breve{\theta} k} - v^*_{\breve{\theta} k}\right). \end{aligned} \tag{22}$$

Let $\delta_{\rho k} = (v_{\rho k})^{(1/d)} - (v^*_{\rho k})^{(1/d)}$ and $\delta_{\breve{\theta} k} = (v_{\breve{\theta} k})^{(1/d)} - (v^*_{\breve{\theta} k})^{(1/d)}$. By using Lemmas 2 and 3, it is readily seen that

$$\begin{aligned}\sum_{k=1}^{M} e_{\rho k}\left(v_{\rho k} - v^*_{\rho k}\right) &\le \sum_{k=1}^{M}\left|e_{\rho k}\right| \cdot 2^{1-d}\left|\left(v_{\rho k}\right)^{\frac{1}{d}} - \left(v^*_{\rho k}\right)^{\frac{1}{d}}\right|^{d} \\ &= \sum_{k=1}^{M} 2^{1-d}\left|e_{\rho k}\right|\left|\delta_{\rho k}\right|^{d} \le \frac{2^{1-d}}{1+d}\sum_{k=1}^{M}\left[\left(e_{\rho k}\right)^{1+d} + d\left(\delta_{\rho k}\right)^{1+d}\right]\end{aligned} \tag{23}$$

and similarly

$$\sum_{k=1}^{M} e_{\breve{\theta} k}\left(v_{\breve{\theta} k} - v^*_{\breve{\theta} k}\right) \le \frac{2^{1-d}}{1+d}\sum_{k=1}^{M}\left[\left(e_{\breve{\theta} k}\right)^{1+d} + d\left(\delta_{\breve{\theta} k}\right)^{1+d}\right]. \tag{24}$$

By combining (22)–(24), we have

$$\begin{aligned}\dot{V}_1(t) \le &-c_2\sum_{k=1}^{M} e_{\rho k}^{1+d} + \frac{2^{1-d}}{1+d}\sum_{k=1}^{M}\left[\left(e_{\rho k}\right)^{1+d} + d\left(\delta_{\rho k}\right)^{1+d}\right] \\ &- c_2\sum_{k=1}^{M} e_{\breve{\theta} k}^{1+d} + \frac{2^{1-d}}{1+d}\sum_{k=1}^{M}\left[\left(e_{\breve{\theta} k}\right)^{1+d} + d\left(\delta_{\breve{\theta} k}\right)^{1+d}\right].\end{aligned} \tag{25}$$

Step 2: Define another Lyapunov function as

$$V_2(t) = V_{\rho 2}(t) + V_{\breve{\theta} 2}(t) \tag{26}$$

with $V_{\bullet 2}(t) = \sum_{k=1}^{M} [1/(2^{1-d}c_2^{1+1/d})] \int_{v^*_{\bullet k}}^{v_{\bullet k}} (s^{(1/d)} - (v^*_{\bullet k})^{(1/d)})^{2-d} ds$. It has been known that $V_2(t)$ such defined is positive semidefinite and C^1 [31]. Taking the derivative of $V_{\rho 2}(t)$ yields

$$\begin{aligned}\dot{V}_{\rho 2}(t) = \sum_{k=1}^{M}\frac{1}{2^{1-d}c_2^{1+1/d}}\Bigg[&\delta_{\rho k}^{2-d}\dot{v}_{\rho k} - (2-d) \\ &\times \int_{v^*_{\rho k}}^{v_{\rho k}}\left(s^{\frac{1}{d}} - \left(v^*_{\rho k}\right)^{\frac{1}{d}}\right)^{1-d} ds \frac{d\left(v^*_{\rho k}\right)^{\frac{1}{d}}}{dt}\Bigg].\end{aligned} \tag{27}$$

According to Lemma 1, we obtain

$$\begin{aligned}&\int_{v^*_{\rho k}}^{v_{\rho k}}\left(s^{\frac{1}{d}} - \left(v^*_{\rho k}\right)^{\frac{1}{d}}\right)^{1-d} ds \\ &\quad\le \left|v_{\rho k} - v^*_{\rho k}\right|\left|v_{\rho k}^{\frac{1}{d}} - \left(v^*_{\rho k}\right)^{\frac{1}{d}}\right|^{1-d} \\ &\quad= \left|\left(v_{\rho k}^{\frac{1}{d}}\right)^{d} - \left(v_{\rho k}^{*\frac{1}{d}}\right)^{d}\right|\left|\delta_{\rho k}\right|^{1-d} \\ &\quad\le 2^{1-d}\left|v_{\rho k}^{\frac{1}{d}} - \left(v^*_{\rho k}\right)^{\frac{1}{d}}\right|^{d}\left|\delta_{\rho k}\right|^{1-d} = 2^{1-d}\left|\delta_{\rho k}\right|.\end{aligned} \tag{28}$$

From the definition of $v^*_{\rho k}$, it is readily seen that

$$\frac{d\left(v^*_{\rho k}\right)^{\frac{1}{d}}}{dt} = -c_2^{\frac{1}{d}}\dot{e}_{\rho k} = -c_2^{\frac{1}{d}}\sum_{j\in N_k} a_{kj}\left(v_{\rho k} - v_{\rho j}\right). \tag{29}$$

It is straightforward to derive that

$$\left|\frac{d\left(v^*_{\rho k}\right)^{\frac{1}{d}}}{dt}\right| \le c_2^{\frac{1}{d}}\left(a\left|v_{\rho k}\right| + b\sum_{j\in N_k}\left|v_{\rho j}\right|\right). \tag{30}$$

Note that the control law $u_{\rho k}$ given in (14) can be equivalently represented as $u_{\rho k} = -c_1\delta_{\rho k}^{2d-1}$ $(k \in A)$. Using such $u_{\rho k}$ and substituting (28) and (30) into (27) yield

$$\begin{aligned}\dot{V}_{\rho 2}(t) \le \sum_{k=1}^{M}\Bigg[&\frac{-c_1}{2^{1-d}c_2^{1+1/d}}\delta_{\rho k}^{1+d} \\ &+ \frac{1}{c_2}(2-d)|\delta_{\rho k}|\left(a|v_{\rho k}| + b\sum_{j\in N_k}|v_{\rho j}|\right)\Bigg].\end{aligned} \tag{31}$$

According to Lemmas 2 and 3, it holds that

$$\begin{aligned}\left|\delta_{\rho k}\right|\left|v_{\rho j}\right| &\le \left|\delta_{\rho k}\right|\left|v_{\rho j} - v^*_{\rho j}\right| + \left|\delta_{\rho k}\right|\left|v^*_{\rho j}\right| \\ &\le 2^{1-d}\left|\delta_{\rho k}\right|\left|\delta_{\rho j}\right|^{d} + c_2\left|\delta_{\rho k}\right|\left|e_{\rho j}\right|^{d} \\ &\le 2^{1-d}\frac{\left|\delta_{\rho k}\right|^{1+d}}{1+d} + 2^{1-d}\frac{d\left|\delta_{\rho j}\right|^{1+d}}{1+d} \\ &\quad+ \frac{c_2\left|\delta_{\rho k}\right|^{1+d}}{1+d} + \frac{c_2 d\left|e_{\rho j}\right|^{1+d}}{1+d}\end{aligned} \tag{32}$$

for all $k, j \in A$, which yields

$$\begin{aligned}&\sum_{k=1}^{M}\left|\delta_{\rho k}\right|(a\left|v_{\rho k}\right| + b\sum_{j\in N_k}\left|v_{\rho j}\right| \\ &\le \sum_{k=1}^{M}\Bigg[a\left(2^{1-d} + \frac{c_2}{1+d}\right)\left|\delta_{\rho k}\right|^{1+d} + \frac{ac_2 d}{1+d}\left|e_{\rho k}\right|^{1+d} \\ &\qquad+ \frac{b\left(2^{1-d} + c_2\right)}{1+d}\sum_{j\in N_k}\left|\delta_{\rho k}\right|^{1+d} + \frac{b2^{1-d}d}{1+d}\sum_{j\in N_k}\left|\delta_{\rho j}\right|^{1+d} \\ &\qquad+ \frac{bc_2 d}{1+d}\sum_{j\in N_k}\left|e_{\rho j}\right|^{1+d}\Bigg] \\ &\le \sum_{k=1}^{M}\frac{1}{1+d}\Big[a\left(2^{1-d} + 2^{1-d}d + c_2\right) \\ &\qquad+ b\bar{m}\left(2^{1-d} + c_2 + 2^{1-d}d\right)\Big]\left|\delta_{\rho k}\right|^{1+d} \\ &\quad+ \sum_{k=1}^{M}\frac{c_2 d}{1+d}(a + b\bar{m})\left|e_{\rho k}\right|^{1+d}.\end{aligned} \tag{33}$$

By substituting (33) into (31), one gets

$$\dot{V}_{\rho 2}(t) \le -k_1\sum_{k=1}^{M}\delta_{\rho k}^{1+d} - k_2\sum_{k=1}^{M} e_{\rho k}^{1+d} \tag{34}$$

where

$$k_1 = \frac{c_1}{2^{1-d}c_2^{1+1/d}} - \frac{2-d}{c_2(1+d)} \times \left[a\left(2^{1-d}+2^{1-d}d+c_2\right)+b\bar{m}\left(2^{1-d}+c_2+2^{1-d}d\right)\right]$$
$$k_2 = \frac{d-2}{1+d}d(a+b\bar{m}). \tag{35}$$

Similarly, it is readily derived that

$$\dot{V}_{\breve{\theta}2}(t) \le -k_1\sum_{k=1}^{M}\delta_{\breve{\theta}k}^{1+d} - k_2\sum_{k=1}^{M}e_{\breve{\theta}k}^{1+d} \tag{36}$$

with k_1 and k_2 given as in (35).

Step 3: Examine the Lyapunov function candidate

$$V_A(t) = V_1(t) + V_2(t) \tag{37}$$

where $V_1(t)$ and $V_2(t)$ are given in (17) and (26), respectively. Recalling (25), (34), and (36), one gets

$$\dot{V}_A(t) \le -k_3\sum_{k=1}^{M}\left(\delta_{\rho k}^{1+d}+\delta_{\breve{\theta}k}^{1+d}\right) - k_4\sum_{k=1}^{M}\left(e_{\rho k}^{1+d}+e_{\breve{\theta}k}^{1+d}\right) \tag{38}$$

where

$$k_3 = k_1 - \frac{2^{1-d}d}{1+d}, \quad k_4 = c_2 - \frac{2^{1-d}}{1+d} + k_2 \tag{39}$$

in which k_3 and k_4 can be ensured to be positive with c_1 and c_2 being chosen as in the controller design part below (14) and (15).

Step 4: Notice that $Q = U_M^T\Lambda U_M$ appeared in (17) is symmetrical and positive definite. Denote by $\lambda_{\max}$ the maximum eigenvalue of Q, then we have

$$\begin{aligned} V_1(t) &\le \frac{1}{2}\lambda_{\max}E_{\rho A}^T E_{\rho A} + \frac{1}{2}\lambda_{\max}E_{\breve{\theta}A}^T E_{\breve{\theta}A} \\ &= \frac{1}{2}\lambda_{\max}\sum_{k=1}^{M}e_{\rho k}^2 + \frac{1}{2}\lambda_{\max}\sum_{k=1}^{M}e_{\breve{\theta}k}^2. \end{aligned} \tag{40}$$

By the similar way as in the proof of (28), one gets

$$\int_{v_{\rho k}^*}^{v_{\rho k}}\left(s^{\frac{1}{d}} - \left(v_{\rho k}^*\right)^{\frac{1}{d}}\right)^{2-d} ds \le 2^{1-d}|\delta_{\rho k}|^2 \tag{41}$$

which implies that

$$V_{\rho 2}(t) \le \frac{1}{c_2^{1+1/d}}\sum_{k=1}^{M}|\delta_{\rho k}|^2. \tag{42}$$

And then

$$V_{\breve{\theta}2}(t) \le \frac{1}{c_2^{1+1/d}}\sum_{k=1}^{M}|\delta_{\breve{\theta}k}|^2. \tag{43}$$

Recalling (37), (40), (42), and (43), it is deduced that

$$\begin{aligned} V_A(t) \le{}& \frac{1}{2}\lambda_{\max}\sum_{k=1}^{M}e_{\rho k}^2 + \frac{1}{2}\lambda_{\max}\sum_{k=1}^{M}e_{\breve{\theta}k}^2 \\ &+ \frac{1}{c_2^{1+1/d}}\sum_{k=1}^{M}\delta_{\rho k}^2 + \frac{1}{c_2^{1+1/d}}\sum_{k=1}^{M}\delta_{\breve{\theta}k}^2. \end{aligned} \tag{44}$$

Step 5: From Lemma 2, one has

$$\begin{aligned} V_A^\alpha(t) \le{}& \frac{\lambda_{\max}^\alpha}{2^\alpha}\sum_{k=1}^{M}e_{\rho k}^{\frac{4p}{2p+1}} + \frac{\lambda_{\max}^\alpha}{2^\alpha}\sum_{k=1}^{M}e_{\breve{\theta}k}^{\frac{4p}{2p+1}} \\ &+ \frac{1}{c_2^{\frac{8p^2}{(4p^2-1)}}}\sum_{k=1}^{M}\left|\delta_{\rho k}\right|^{\frac{4p}{2p+1}} + \frac{1}{c_2^{\frac{8p^2}{(4p^2-1)}}}\sum_{k=1}^{M}\left|\delta_{\breve{\theta}k}\right|^{\frac{4p}{2p+1}}. \end{aligned} \tag{45}$$

With $c \le \min\{(2^\alpha k_3/\lambda_{\max}^\alpha), c_2^{(8p^2/(4p^2-1))}k_4\}$, one then has

$$\begin{aligned} cV_A^\alpha(t) \le{}& k_3\sum_{k=1}^{M}e_{\rho k}^{\frac{4p}{2p+1}} + k_3\sum_{k=1}^{M}e_{\breve{\theta}k}^{\frac{4p}{2p+1}} \\ &+ k_4\sum_{k=1}^{M}\left|\delta_{\rho k}\right|^{\frac{4p}{2p+1}} + k_4\sum_{k=1}^{M}\left|\delta_{\breve{\theta}k}\right|^{\frac{4p}{2p+1}} \end{aligned}$$

which, together with (38), implies that

$$\dot{V}_A(t) + cV_A^\alpha(t) \le 0. \tag{46}$$

Note that $0 < \alpha < 1$, according to Theorem 4.2 in [13] it is concluded that $V_A(t)$ converges to zero in a finite time T_1, satisfying

$$T_1 \le \frac{V_A(t_0)^{1-\alpha}}{c(1-\alpha)} \tag{47}$$

where $V_A(t_0)$ is known, and c is computable. Thus the leaders achieve finite time stability with time T_1 satisfying (16).

On one hand, $V_A(t) = 0$ for all $t \ge T_1$, which implies that $V_1(t) = 0$ and $V_2(t) = 0$ for all $t \ge T_1$. Since the communication graph G_A is connected, from Lemma 4, $V_1(t) = 0$ implies that $(\rho_k - \rho_j) - (\bar{\rho}_k - \bar{\rho}_j) = 0$ and $(\breve{\theta}_k - \breve{\theta}_j) - (\bar{\theta}_k - \bar{\theta}_j) = 0$, i.e., $(\theta_k - \theta_j) - (\bar{\theta}_k - \bar{\theta}_j) = 0$, $\forall k, j \in A$, which means that the prespecified rotating formation h for the leaders is achieved. On the other hand, $V_2(t) = 0$ implies that $\dot{\rho}_k = v_{\rho k} = v_{\rho k}^* = 0$, $\forall k \in A$, which means that all the leaders in MAS (1) move in a circular orbit with a constant radius ρ_k. In addition, we also have $\dot{\theta}_k - w(t) = \dot{\breve{\theta}}_k = v_{\breve{\theta}k} = v_{\breve{\theta}k}^* = 0$, $\forall k \in A$, which means that all the leaders in MAS (1) rotate with a common (time-varying) angular velocity $w(t)$. In summary, all the leaders achieve rotating formation in finite time. ■

B. Finite-Time Rotating Containment Control for Followers

We first use the angular displacement transformation $\breve{\theta}_i(t) = \theta_i(t) - \int_0^t w(\tau)d\tau$ $(i \in B)$ to convert the original agent dynamics for followers in (1) into the following form:

$$\begin{aligned} \left(\dot{\rho}_i(t), \dot{\breve{\theta}}_i(t) + w(t)\right) &= \left(v_{\rho i}(t), v_{\breve{\theta}i}(t) + w(t)\right) \\ \left(\dot{v}_{\rho i}(t), \dot{v}_{\breve{\theta}i}(t) + \dot{w}(t)\right)(t) &= \left(u_{\rho i}(t), u_{\breve{\theta}i}(t) + \dot{w}(t)\right) \end{aligned} \tag{48}$$

for all $i \in B$ with $v_{\breve{\theta}i}(t) = v_{\theta i}(t) - w(t)$ and $u_{\breve{\theta}i}(t) = u_{\theta i}(t) - \dot{w}(t)$. With $\breve{\theta}_i$ $(i \in B)$, the containment objective for followers is equivalently expressed as, for all $t \ge T^*$

$$\rho_i(t) - \sum_{j=1}^{M}\alpha_j\rho_j(t) = 0, \quad \breve{\theta}_i(t) - \sum_{j=1}^{M}\alpha_j\breve{\theta}_j(t) = 0 \tag{49}$$

$$\dot{\rho}_i(t) = 0, \quad \dot{\breve{\theta}}_i(t) = 0 \tag{50}$$

for all $i \in B$, where $\alpha_j \geq 0$ $(j \in A)$ satisfy $\sum_{j=1}^{M} \alpha_j = 1$.

The ith $(i \in B)$ neighborhood error is introduced as

$$e_{\rho i} = \sum_{j \in \mathcal{N}_i} a_{ij}(\rho_i - \rho_j), \quad e_{\breve{\theta} i} = \sum_{j \in \mathcal{N}_i} a_{ij}\left(\breve{\theta}_i - \breve{\theta}_j\right). \tag{51}$$

Let $E_{\rho B} = [e_{\rho M+1}, \ldots, e_{\rho N}]^T$, $E_{\breve{\theta} B} = [e_{\breve{\theta} M+1}, \ldots, e_{\breve{\theta} N}]^T$, $\rho_B = [\rho_{M+1}, \ldots, \rho_N]^T$, and $\breve{\theta}_B = [\breve{\theta}_{M+1}, \cdots, \breve{\theta}_N]^T$. Then

$$\begin{aligned} E_{\rho B} &= L_1 \rho_B + L_2 \rho_A = L_1\left[\rho_B - \left(-L_1^{-1} L_2 \rho_A\right)\right] \\ E_{\breve{\theta} B} &= L_1 \breve{\theta}_B + L_2 \breve{\theta}_A = L_1\left[\breve{\theta}_B - \left(-L_1^{-1} L_2 \breve{\theta}_A\right)\right]. \end{aligned} \tag{52}$$

Note that each entry of $-L_1^{-1}L_2$ is non-negative and each row sum of $-L_1^{-1}L_2$ is equal to one, and moreover, L_1 is symmetric and positive definite from Lemma 5. Thus $E_{\rho B} = 0$ (or $E_{\breve{\theta} B} = 0$) leads to $\rho_B - (-L_1^{-1}L_2\rho_A) = 0$ (or $\breve{\theta}_B - (-L_1^{-1}L_2\breve{\theta}_A) = 0$), i.e., $\rho_i - \sum_{k=1}^{M} l_{ik}\rho_k = 0$ (or $\breve{\theta}_i - \sum_{k=1}^{M} l_{ik}\breve{\theta}_k = 0$), where l_{ik} $(i = M+1, \ldots, N,\ k = 1, \ldots, M)$ denotes the (i,k)th element of matrix $-L_1^{-1}L_2$ and is non-negative and satisfies $\sum_{k=1}^{M} l_{ik} = 1$. This implies that the finite time containment objective for followers can be achieved by steering $E_{\rho B} = 0$ and $E_{\breve{\theta} B} = 0$ in a finite time.

The containment control scheme for the ith $(i \in B)$ follower is designed as

$$\begin{aligned} u_{\rho i} &= -c_3\left[\left(v_{\rho i}\right)^{\frac{2q+1}{2q-1}} - \left(v_{\rho i}^*\right)^{\frac{2q+1}{2q-1}}\right]^{\frac{2q-3}{2q+1}} \\ u_{\breve{\theta} i} &= -c_3\left[\left(v_{\breve{\theta} i}\right)^{\frac{2q+1}{2q-1}} - \left(v_{\breve{\theta} i}^*\right)^{\frac{2q+1}{2q-1}}\right]^{\frac{2q-3}{2q+1}} \end{aligned} \tag{53}$$

with

$$v_{\rho i}^* = -c_4\left(e_{\rho i}\right)^{\frac{2q-1}{2q+1}}, \quad v_{\breve{\theta} i}^* = -c_4\left(e_{\breve{\theta} i}\right)^{\frac{2q-1}{2q+1}} \tag{54}$$

where $q \geq 2$ is a integer, c_3 and c_4 are design parameters chosen such that $c_4 > [(2^{1-s})/(1+s)] + [(2-s)/(1+s)]s(\tilde{a} + \tilde{b}\tilde{m}) > 0$ and $c_3 > 2^{1-s}c_4^{1+1/s}[[(2-s)/(c_4(1+s))](\tilde{a}(2^{1-s} + 2^{1-s}s + c_4) + \tilde{b}\tilde{m}(2^{1-s} + c_4 + 2^{1-s}s)) + [(2^{1-s}s)/(1+s)]] > 0$, with $s = [(2q-1)/(2q+1)]$, $\tilde{a} = \max_{\forall i \in B}\{\sum_{j \in \tilde{\mathcal{N}_i}} a_{ij}\}$, $\tilde{b} = \max_{\forall i, j \in B}\{a_{ij}\}$, and $\tilde{m}$ being the maximum number of the elements in $\mathcal{N}_i/A$ for all $i \in B$.

Theorem 2: Consider the dynamical model for the followers given in (1) or equivalently (48) under Assumption 1. If the followers are controlled by (53) and (54), then the states of the followers converge to the convex hull shaped by those of the leaders, and meanwhile, all the followers maintain a rotating motion as described by (5), in a finite time T^* as determined by

$$T^* = T_1 + T_2 \tag{55}$$

with T_1 given as in (16) and T_2 satisfying

$$T_2 \leq \frac{2q+1}{\kappa} V_B(t_0)^{1/(2q+1)} \tag{56}$$

in which $V_B(t_0)$ is decided by (57), (71), and (72), which is known, and $\kappa \leq \min\{(2^\sigma k_7/\bar{\lambda}_{\max}^\sigma),\ c_4^{(8q^2/(4q^2-1))}k_8\}$ with $\sigma = [2q/(2q+1)] \in (0,1)$ and k_3, k_4 being decided by (74) and (76), which is explicitly computable.

Proof: To steer the followers to enter the convex hull formed by the leaders, which will take time T_1 to achieve, we have to wait for at least T_1 to initiate the containment control law. We first prove that the states of the followers remain bounded around that of the leaders for $t \in [0, T_1)$, and also that the rotating containment is achieved in a finite time T^* $(T^* > T_1)$. Thus, we need to consider two cases: 1) $t \in [0, T_1)$ and 2) $t \geq T_1$.

Case 1 [$t \in [0, T_1)$]: Construct the first part of the Lyapunov function candidate

$$V_3(t) = \frac{1}{2} E_{\rho B}^T L_1^{-1} E_{\rho B} + \frac{1}{2} E_{\breve{\theta} B}^T L_1^{-1} E_{\breve{\theta} B} \tag{57}$$

which is positive definite guaranteed by Lemma 5. Let $v_{\rho B} = [v_{\rho(M+1)}, \ldots, v_{\rho N}]^T$ and $v_{\breve{\theta} B} = [v_{\breve{\theta}(M+1)}, \ldots, v_{\breve{\theta} N}]^T$. Then the time derivative of $V_3(t)$ is computed as

$$\begin{aligned} \dot{V}_3(t) &= E_{\rho B}^T\left[v_{\rho B} - \left(-L_1^{-1}L_2 v_{\rho A}\right)\right] \\ &\quad + E_{\breve{\theta} B}^T\left[v_{\breve{\theta} B} - \left(-L_1^{-1}L_2 v_{\breve{\theta} A}\right)\right] \\ &= E_{\rho B}^T v_{\rho B} + E_{\rho B}^T L_1^{-1} L_2 v_{\rho A} + E_{\breve{\theta} B}^T v_{\breve{\theta} B} + E_{\breve{\theta} B}^T L_1^{-1} L_2 v_{\breve{\theta} A} \\ &\leq \underbrace{E_{\rho B}^T v_{\rho B} + E_{\breve{\theta} B}^T v_{\breve{\theta} B}}_{\Delta_1} + \underbrace{\frac{1}{2} E_{\rho B}^T E_{\rho B} + \frac{1}{2} E_{\breve{\theta} B}^T E_{\breve{\theta} B}}_{\Delta_2} \\ &\quad + \underbrace{\frac{1}{2}\left(L_1^{-1}L_2 v_{\rho A}\right)^T\left(L_1^{-1}L_2 v_{\rho A}\right) + \frac{1}{2}\left(L_1^{-1}L_2 v_{\breve{\theta} A}\right)^T\left(L_1^{-1}L_2 v_{\breve{\theta} A}\right)}_{\Delta_3}. \end{aligned} \tag{58}$$

In the following, we first examine Δ_3 to derive that there exists some bounded positive constant l such that:

$$\begin{aligned} &\frac{1}{2}\left(L_1^{-1}L_2 v_{\rho A}\right)^T\left(L_1^{-1}L_2 v_{\rho A}\right) \\ &\qquad + \frac{1}{2}\left(L_1^{-1}L_2 v_{\breve{\theta} A}\right)^T\left(L_1^{-1}L_2 v_{\breve{\theta} A}\right) < l. \end{aligned} \tag{59}$$

According to Lemma 5, one has

$$\begin{aligned} &\left(L_1^{-1}L_2 v_{\rho A}\right)^T\left(L_1^{-1}L_2 v_{\rho A}\right) \\ &\quad = \left(-L_1^{-1}L_2 v_{\rho A}\right)^T\left(-L_1^{-1}L_2 v_{\rho A}\right) = \sum_{i=M+1}^{N}\left(\sum_{k=1}^{M} l_{ik} v_{\rho k}\right)^2 \\ &\quad \leq \sum_{i=M+1}^{N}\left(\sum_{k=1}^{M} v_{\rho k}\right)^2 = (N-M)\left(\sum_{k=1}^{M} v_{\rho k}\right)^2 \\ &\left(L_1^{-1}L_2 v_{\breve{\theta} A}\right)^T\left(L_1^{-1}L_2 v_{\breve{\theta} A}\right) \leq (N-M)\left(\sum_{k=1}^{M} v_{\breve{\theta} k}\right)^2. \end{aligned} \tag{60}$$

We now show the boundedness of $(\sum_{k=1}^{M} v_{\rho k})^2$ and $(\sum_{k=1}^{M} v_{\breve{\theta} k})^2$. Based on (46), we have $\dot{V}_A(t) \leq -cV_A^\alpha(t) \leq 0$, and therefore $V_A(t) \leq V_A(t_0)$. Let $c_a = cV_A(t_0)^{\alpha-1}$, and then $\dot{V}_A(t) + c_a V_A(t) \leq 0$, which further yields

$$V_A(t) \leq V_A(t_0) e^{-c_a t}. \tag{61}$$

In view of Lemma 3, it is readily seen that

$$\sum_{i=1}^{M}\left|v_{\rho i}^*\right|+\sum_{i=1}^{M}\left|v_{\breve{\theta} i}^*\right|=c_2\sum_{i=1}^{M}\left|\left(e_{\rho i}^2\right)^{\frac{d}{2}}\right|+c_2\sum_{i=1}^{M}\left|\left(e_{\breve{\theta} i}^2\right)^{\frac{d}{2}}\right| \leq c_2(2M)^{1-\frac{d}{2}}\left(\sum_{i=1}^{M}e_{\rho i}^2+\sum_{i=1}^{M}e_{\breve{\theta} i}^2\right)^{\frac{d}{2}}. \quad (62)$$

From the definition of $V_1(t)$ in (17), it follows that $\sum_{i=1}^{M}e_{\rho i}^2+\sum_{i=1}^{M}e_{\breve{\theta} i}^2 = E_{\rho A}^T E_{\rho A}+E_{\breve{\theta} A}^T E_{\breve{\theta} A} \leq 2\lambda_{\min}(Q)^{-1}V_1 \leq 2\lambda_{\min}(Q)^{-1}V_A \leq 2\lambda_{\min}(Q)^{-1}V_A(t_0)e^{-c_a t}$, which, together with (62), yields:

$$\sum_{i=1}^{M}\left|v_{\rho i}^*\right|+\sum_{i=1}^{M}\left|v_{\breve{\theta} i}^*\right| \leq \mu_1 e^{-k_{\mu 1}t} \quad (63)$$

where $\mu_1 = c_2(2M)^{1-(d/2)}(2\lambda_{\min}(Q)^{-1}V_A(t_0))^{(d/2)}$ and $k_{\mu 1} = (d/2)c_a$.

According to [18] we know that

$$\int_{v_{\rho k}^*}^{v_{\rho k}}\left(s^{\frac{1}{d}}-\left(v_{\rho k}^*\right)^{\frac{1}{d}}\right)^{2-d}ds \geq \frac{2p-1}{2^{\frac{(1-d)(2-d)}{d}}(4p+2)}\left|v_{\rho k}-v_{\rho k}^*\right|^{\frac{4p+2}{2p-1}} \quad (64)$$

which implies, from the definition of $V_2(t)$, that

$$V_2(t) \geq \frac{1}{2^{1-d}c_2^{1+1/d}}\frac{2p-1}{2^{\frac{(1-d)(2-d)}{d}}(4p+2)} \times\sum_{k=1}^{M}\left(\left|v_{\rho k}-v_{\rho k}^*\right|^{\frac{4p+2}{2p-1}}+\left|v_{\breve{\theta} k}-v_{\breve{\theta} k}^*\right|^{\frac{4p+2}{2p-1}}\right). \quad (65)$$

This, together with (61), leads to

$$\sum_{k=1}^{M}\left(|v_{\rho k}-v_{\rho k}^*|^{\frac{4p+2}{2p-1}}+|v_{\breve{\theta} k}-v_{\breve{\theta} k}^*|^{\frac{4p+2}{2p-1}}\right) \leq k_0 V_A(t_0)e^{-c_a t} \quad (66)$$

with $k_0 = [(2^{[(1-d)(2-d)/d]}(4p+2)\cdot 2^{1-d}c_2^{1+1/d})/(2p-1)]$. By using Lemma 3 twice, it then follows from (66) that:

$$\begin{aligned}\sum_{i=1}^{M}\left|v_{\rho i}-v_{\rho i}^*\right|+\sum_{i=1}^{M}\left|v_{\breve{\theta} i}-v_{\breve{\theta} i}^*\right| &= \sum_{i=1}^{M}\left(\left|v_{\rho i}-v_{\rho i}^*\right|^{\frac{4p+2}{2p-1}}\right)^{\frac{2p-1}{4p+2}}+\sum_{i=1}^{M}\left(\left|v_{\breve{\theta} i}-v_{\breve{\theta} i}^*\right|^{\frac{4p+2}{2p-1}}\right)^{\frac{2p-1}{4p+2}} \\ &\leq (2M)^{1-\frac{2p-1}{4p+2}}\left(\sum_{i=1}^{M}\left(\left|v_{\rho i}-v_{\rho i}^*\right|^{\frac{4p+2}{2p-1}}+\sum_{i=1}^{M}\left|v_{\breve{\theta} i}-v_{\breve{\theta} i}^*\right|^{\frac{4p+2}{2p-1}}\right)\right)^{\frac{2p-1}{4p+2}} \leq \mu_2 e^{-k_{\mu 2}t}\end{aligned} \quad (67)$$

where $\mu_2 = (2M)^{1-[(2p-1)/(4p+2)]}(k_0 V_A(t_0))^{[(2p-1)/(4p+2)]}$ and $k_{\mu 2} = [(2p-1)/(4p+2)]c_a$. By combining (63) and (67), we obtain

$$\begin{aligned}&\left(\sum_{k=1}^{M}v_{\rho k}\right)^2+\left(\sum_{k=1}^{M}v_{\breve{\theta} k}\right)^2 \\ &\leq \left(\sum_{k=1}^{M}\left|v_{\rho k}\right|\right)^2+\left(\sum_{k=1}^{M}\left|v_{\breve{\theta} k}\right|\right)^2 \leq \left(\sum_{k=1}^{M}\left|v_{\rho k}\right|+\sum_{k=1}^{M}\left|v_{\breve{\theta} k}\right|\right)^2 \\ &\leq \left(\sum_{k=1}^{M}\left|v_{\rho k}-v_{\rho k}^*\right|+\sum_{k=1}^{M}\left|v_{\rho k}^*\right|+\sum_{k=1}^{M}\left|v_{\breve{\theta} k}-v_{\breve{\theta} k}^*\right|+\sum_{k=1}^{M}\left|v_{\breve{\theta} k}^*\right|\right)^2 \\ &\leq \left(\mu_1 e^{-k_{\mu 1}t}+\mu_2 e^{-k_{\mu 2}t}\right)^2 \leq (\mu_1+\mu_2)^2.\end{aligned} \quad (68)$$

By inserting (68) into (60), one then concludes that (59) holds with $l = (1/2)(N-M)(\mu_1+\mu_2)^2$. To continue, we examine Δ_1 in (58). Let $s = [(2q-1)/(2q+1)]$. Then we have

$$\begin{aligned}E_{\rho B}^T v_{\rho B}+E_{\breve{\theta} B}^T v_{\breve{\theta} B} &= \sum_{i=M+1}^{N}e_{\rho i}v_{\rho i}+\sum_{i=M+1}^{N}e_{\breve{\theta} i}v_{\breve{\theta} i} \\ &= -c_4\sum_{i=M+1}^{N}e_{\rho i}^{1+s}+\sum_{i=M+1}^{N}e_{\rho i}\left(v_{\rho i}-v_{\rho i}^*\right) \\ &\quad -c_4\sum_{i=M+1}^{N}e_{\breve{\theta} i}^{1+s}+\sum_{i=M+1}^{N}e_{\breve{\theta} i}\left(v_{\breve{\theta} i}-v_{\breve{\theta} i}^*\right).\end{aligned} \quad (69)$$

Let $\delta_{\rho i} = (v_{\rho i})^{(1/s)}-(v_{\rho i}^*)^{(1/s)}$ and $\delta_{\breve{\theta} i} = (v_{\breve{\theta} i})^{(1/s)}-(v_{\breve{\theta} i}^*)^{(1/s)}$ ($i \in B$). By following the similar procedure as in deriving (23)–(25), we obtain:

$$\begin{aligned}&E_{\rho B}^T v_{\rho B}+E_{\breve{\theta} B}^T v_{\breve{\theta} B} \\ &\leq -c_4\sum_{i=M+1}^{N}e_{\rho i}^{1+s}+\frac{2^{1-s}}{1+s}\sum_{i=M+1}^{N}\left[\left(e_{\rho i}\right)^{1+s}+s\left(\delta_{\rho i}\right)^{1+s}\right] \\ &\quad -c_4\sum_{i=M+1}^{N}e_{\breve{\theta} i}^{1+s}+\frac{2^{1-s}}{1+s}\sum_{i=M+1}^{N}\left[\left(e_{\breve{\theta} i}\right)^{1+s}+s\left(\delta_{\breve{\theta} i}\right)^{1+s}\right].\end{aligned} \quad (70)$$

Choose the Lyapunov function candidate as

$$V_B(t) = V_3(t)+V_4(t) \quad (71)$$

with V_3 being given in (57) and

$$V_4(t) = \sum_{i=M+1}^{N}\frac{1}{2^{1-s}c_4^{1+1/s}}\left[\int_{v_{\rho i}^*}^{v_{\rho i}}\left(\varsigma^{\frac{1}{s}}-\left(v_{\rho i}^*\right)^{\frac{1}{s}}\right)^{2-s}d\varsigma +\int_{v_{\breve{\theta} i}^*}^{v_{\breve{\theta} i}}\left(\varsigma^{\frac{1}{s}}-\left(v_{\breve{\theta} i}^*\right)^{\frac{1}{s}}\right)^{2-s}d\varsigma\right]. \quad (72)$$

By following the similar lines as in deriving (27)–(36) and using $u_{\rho i}$ and $u_{\breve{\theta} i}$ ($i \in B$) given in (53), we get:

$$\dot{V}_4(t) \leq -k_5\sum_{i=M+1}^{N}\left(\delta_{\rho i}^{1+s}+\delta_{\breve{\theta} i}^{1+s}\right)-k_6\sum_{i=M+1}^{N}\left(e_{\rho i}^{1+s}+e_{\breve{\theta} i}^{1+s}\right) \quad (73)$$

with

$$\begin{aligned} k_5 &= \frac{c_3}{2^{1-s}c_4^{1+1/s}} - \frac{2-s}{c_4(1+s)} \\ &\quad \times \left[\tilde{a}\left(2^{1-s}+2^{1-s}s+c_4\right)+\tilde{b}\tilde{m}\left(2^{1-s}+c_4+2^{1-s}s\right)\right] \\ k_6 &= \frac{s-2}{1+s}s\left(\tilde{a}+\tilde{b}\tilde{m}\right). \end{aligned} \tag{74}$$

Combining (70) and (73) yields that

$$\begin{aligned} &E_{\rho B}^T v_{\rho B} + E_{\breve{\theta} B}^T v_{\breve{\theta} B} + \dot{V}_4(t) \\ &\leq -k_7 \sum_{i=M+1}^{N}\left(\delta_{\rho i}^{1+s}+\delta_{\breve{\theta} i}^{1+s}\right) - k_8 \sum_{i=M+1}^{N}\left(e_{\rho i}^{1+s}+e_{\breve{\theta} i}^{1+s}\right) \end{aligned} \tag{75}$$

in which

$$k_7 = k_5 - \frac{2^{1-s}s}{1+s}, \quad k_8 = c_4 - \frac{2^{1-s}}{1+s} + k_6 \tag{76}$$

in which k_7 and k_8 can be ensured to be positive with c_3 and c_4 being chosen as in the controller design part below (53) and (54).

At last, we estimate Δ_2 in (58). Note that $E_{\rho B}^T L_1^{-1} E_{\rho B} \geq \lambda_{\min}(L_1^{-1}) E_{\rho B}^T E_{\rho B}$ and $E_{\breve{\theta} B}^T L_1^{-1} E_{\breve{\theta} B} \geq \lambda_{\min}(L_1^{-1}) E_{\breve{\theta} B}^T E_{\breve{\theta} B}$. Denote $\bar{\lambda}_{\min} = \lambda_{\min}(L_1^{-1})$, then it holds that

$$\frac{1}{2}E_{\rho B}^T E_{\rho B} + \frac{1}{2}E_{\breve{\theta} B}^T E_{\breve{\theta} B} \leq \bar{\lambda}_{\min}^{-1} V_3(t) \leq \bar{\lambda}_{\min}^{-1} V_B(t). \tag{77}$$

Recalling (58), (59), (71), (75), and (77), it is readily seen that

$$\dot{V}_B(t) \leq \bar{\lambda}_{\min}^{-1} V_B(t) + l. \tag{78}$$

Solving (78) respect to t for $t \in [0, T_1)$ yields

$$\begin{aligned} V_B(t) &\leq \left(V_B(0)+\bar{\lambda}_{\min} l\right)e^{\bar{\lambda}_{\min}^{-1} t} - \bar{\lambda}_{\min} l \\ &\leq \left(V_B(0)+\bar{\lambda}_{\min} l\right)e^{\bar{\lambda}_{\min}^{-1} T_1} - \bar{\lambda}_{\min} l \end{aligned} \tag{79}$$

which means that $V_B(t)$ is bounded during $t \in [0, T_1)$. Noting that

$$\left|e_{\rho i}\right| \leq \sqrt{2\bar{\lambda}_{\min}^{-1} V_B(t)}, \quad \left|e_{\breve{\theta} i}\right| \leq \sqrt{2\bar{\lambda}_{\min}^{-1} V_B(t)} \tag{80}$$

then it is deduced from the definition of the neighborhood error in (52), together with (79) and (80), that the states of the ith follower are bounded around the states of the leaders during $[0, T_1)$ for all $i \in B$.

Case 2 ($t \geq T_1$): It is worth noting that the leaders form the desired rotating formation after T_1, i.e., for all $j, k \in A$, $[(\rho_k, \theta_k) - (\rho_j, \theta_j)] - [(\bar{\rho}_k, \bar{\theta}_k) - (\bar{\rho}_j, \bar{\theta}_j)] = (0, 0)$, and meanwhile, $\dot{\rho}_k(t) = 0$ and $\dot{\theta}_k(t) = w$. It thus follows that, when $t \geq T_1$, $v_{\rho A} = 0$ and $v_{\breve{\theta} A} = 0$. Hence, the derivative of $V_3(t)$ becomes

$$\begin{aligned} \dot{V}_3(t) &= E_{\rho B}^T v_{\rho B} + E_{\breve{\theta} B}^T v_{\breve{\theta} B} \\ &= \sum_{i=M+1}^{N} e_{\rho i} v_{\rho i} + \sum_{i=M+1}^{N} e_{\breve{\theta} i} v_{\breve{\theta} i}. \end{aligned} \tag{81}$$

By following the same lines as in deriving (23)–(25), we arrives at:

$$\begin{aligned} \dot{V}_3(t) &\leq -c_4 \sum_{i=M+1}^{N} e_{\rho i}^{1+s} + \frac{2^{1-s}}{1+s} \\ &\quad \times \sum_{i=M+1}^{N}\left[\left(e_{\rho i}\right)^{1+s} + s\left(\delta_{\rho i}\right)^{1+s}\right] - c_4 \sum_{i=M+1}^{N} e_{\breve{\theta} i}^{1+s} \\ &\quad + \frac{2^{1-s}}{1+s}\sum_{i=M+1}^{N}\left[\left(e_{\breve{\theta} i}\right)^{1+s} + s\left(\delta_{\breve{\theta} i}\right)^{1+s}\right]. \end{aligned} \tag{82}$$

Using the procedure similar to that as in (71)–(76), we obtain

$$\dot{V}_B(t) \leq -k_7 \sum_{i=M+1}^{N}\left(\delta_{\rho i}^{1+s}+\delta_{\breve{\theta} i}^{1+s}\right) - k_8 \sum_{i=M+1}^{N}\left(e_{\rho i}^{1+s}+e_{\breve{\theta} i}^{1+s}\right) \tag{83}$$

with k_7 and k_8 being given as in (76).

Notice that L_1^{-1} is symmetric and positive definite, which is guaranteed by Lemma 5, then we denote by $\bar{\lambda}_{\max}$ the maximum eigenvalue of L_1^{-1}. Thus

$$V_3(t) \leq \frac{1}{2}\bar{\lambda}_{\max}\sum_{i=M+1}^{N} e_{\rho i}^2 + \frac{1}{2}\bar{\lambda}_{\max}\sum_{i=M+1}^{N} e_{\breve{\theta} i}^2. \tag{84}$$

By following the similar way as in the proof of (28), it is derived, for $i \in B$, that:

$$\int_{v_{\rho i}^*}^{v_{\rho i}}\left(\varsigma^{\frac{1}{s}} - \left(v_{\rho i}^*\right)^{\frac{1}{s}}\right)^{2-s} d\varsigma \leq 2^{1-s}\left|\delta_{\rho i}\right|^2. \tag{85}$$

Thus

$$V_4(t) \leq \frac{1}{c_4^{1+1/s}}\sum_{i=M+1}^{N}\left|\delta_{\rho i}\right|^2 + \frac{1}{c_4^{1+1/s}}\sum_{i=M+1}^{N}\left|\delta_{\breve{\theta} i}\right|^2. \tag{86}$$

It then follows from (84) and (86) that:

$$\begin{aligned} V_B(t) &\leq \frac{1}{2}\bar{\lambda}_{\max}\sum_{i=M+1}^{N} e_{\rho i}^2 + \frac{1}{2}\bar{\lambda}_{\max}\sum_{i=M+1}^{N} e_{\breve{\theta} i}^2 \\ &\quad + \frac{1}{c_4^{1+1/s}}\sum_{i=M+1}^{N}\delta_{\rho i}^2 + \frac{1}{c_4^{1+1/s}}\sum_{i=M+1}^{N}\delta_{\breve{\theta} i}^2. \end{aligned} \tag{87}$$

With $\sigma = [2q/(2q+1)] \in (0,1)$ and $\kappa \leq \min\{(2^{\sigma}k_7/\bar{\lambda}_{\max}^{\sigma}),\ c_4^{(8q^2/(4q^2-1))}k_8\}$, it then holds

$$\dot{V}_B(t) + \kappa V_B^{\sigma}(t) \leq 0. \tag{88}$$

By Theorem 4.2 in [13], $V_B(t)$ converges to zero after T_1 in a finite time T_2, which satisfies

$$T_2 \leq \frac{V_B(t_0)^{1-\sigma}}{\kappa(1-\sigma)} \tag{89}$$

where $V_B(t_0)$ is known, and κ is computable. Therefore, the followers achieve finite time stability with time T^* ($T^* = T_1 + T_2$) as given in (55).

Furthermore, $V_B(t) = 0$ for all $t \geq T^*$, and then $V_3(t) = 0$ and $V_4(t) = 0$ for all $t \geq T^*$. From the above analysis, $V_3(t) = 0$ implies that $\rho_i - \sum_{j=1}^{M} l_{ij}\rho_j = 0$ and

$\breve{\theta}_i - \sum_{j=1}^{M} l_{ij}\breve{\theta}_j = 0$, and then $\theta_i - \sum_{j=1}^{M} l_{ij}\theta_j = 0$, where l_{ij} ($i = M+1, \ldots, N, j = 1, \ldots, M$) is the entry of $-L_1^{-1}L_2$ (l_{ij} is non-negative and satisfies $\sum_{j=1}^{M} l_{ij} = 1$), which means that the states of the followers converge to the convex hull formed by the states of the leaders. On the other hand, by following the same line as in the proof of Theorem 1, $V_4(t) = 0$ implies that all the followers in MAS (1) move in a circular orbit with a constant radius ρ_i ($i \in B$) and a common time-varying angular velocity $w(t)$. In summary, all the followers achieve rotating containment in finite time. ■

C. Finite-Time Rotating Formation-Containment Control for All the Agents

With the strategies as stated in Theorems 1 and 2, we finally arrive at the following rotating formation-containment result.

Theorem 3: Consider the MAS described by (1) under Assumption 1. Let the leaders be controlled by (14) and (15) and the followers be controlled by (53) and (54). Then the objective of rotating formation-containment in the sense of Definition 4 is achieved in a finite time T^* given in (55). More specially, the leaders achieve the prespecified formation in a finite time T_1 given in (16) and the states of the followers enter the convex hull shaped by those of the leaders in the finite time T^* ($T^* = T_1 + T_2$). In addition, after the finite time T^*, all the agents maintain the rotating motion surrounding a common circular center with the same time-varying angular velocity $w(t)$.

Proof: It is the combination of the two results established in Theorems 1 and 2, thus the proof is omitted. ■

Remark 4: It is worth mentioning that the expressions for T_1 in (16) and T_2 in (56) are sufficient conditions, which do not represent the precise convergence times and only mean that the convergence times T_1 and T_2 are finite and depend on the initial conditions and design parameters. So we can choose proper design parameters to make the finite convergence time $T^* = T_1 + T_2$ small. By examining T_1 in (16), we see that T_1 mainly depends on two classes of design parameters: one is p, another one is in the definition of c. Thus we can set p smaller and c larger to make T_1 smaller, where increasing c_1 and c_2 lead to an increasing c upon examining the definition of c. The similar conclusion also holds for T_2. However, certain tradeoff between convergence time and control effort need to be made in practice.

V. Numerical Simulation

To verify the effectiveness of the proposed finite-time control scheme, numerical simulation on a group of nine agents consisting of four leaders and five followers is conducted. The communication topology is shown in Fig. 2, which satisfies Assumption 1. Each edge weight was taken as 1. The simulation is conducted under the condition that the initial angular velocities of all agents are different and the common angular velocity $w(t)$ is time-varying.

The simulation objective is to verify that if the leaders can achieve the prespecified rotating formation and if the followers can achieve containment both in finite time by using the proposed control schemes given in Theorems 1 and 2. According to the established theory, this objective can be realized by applying the proposed control laws given in (14) and (15) to the four leaders and the control law (53) and (54) to the five followers. In the simulation, the control parameter p is chosen as $p = 11$, c_1 and c_2 are taken, respectively, as $c_1 = 49$ and $c_2 = 3$ such that $k_3 > 0$ and $k_4 > 0$, where $k_3 = k_1 - [(2^{1-d}d)/(1+d)] = 0.3920 > 0$ and $k_4 = c_2 - [(2^{1-d})/(1+d)] + k_2 = 0.3697 > 0$, with $k_1 = [c_1/(2^{1-d}c_2^{1+1/d})] - [(2-d)/c_2(1+d)][a(2^{1-d} + 2^{1-d}d + c_2) + bm_k(2^{1-d} + c_2 + 2^{1-d}d)] = 0.8047$ and $k_2 = [(d-2)/(1+d)]d(a+b\bar{m}) = -2.0751$, $d = (2p-1)/(2p+1)$, $a = \max_{\forall i\in A}\{\sum_{j\in\mathcal{N}_k} a_{kj}\} = 2$, $b = \max_{\forall k,j\in A}\{a_{kj}\} = 1$, and $\bar{m} = 2$; the control parameter q is chosen as $q = 11$, c_3 and c_4 are taken, respectively, as $c_3 = 50$ and $c_4 = 3$ such that $k_7 > 0$ and $k_8 > 0$, where $k_7 = k_5 - [2^{1-s}s/(1+s)] = 0.3920 > 0$ and $k_8 = c_4 - [2^{1-d}/(1+d)] + k_6 = 0.3697 > 0$, with $k_5 = [c_3/(2^{1-d}c_4^{1+1/d})] - [(2-d)/c_4(1+d)][\tilde{a}(2^{1-s} +$

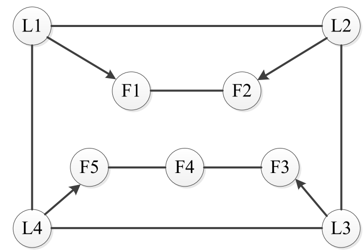

Fig. 2. Communication topology between four leaders and five followers.

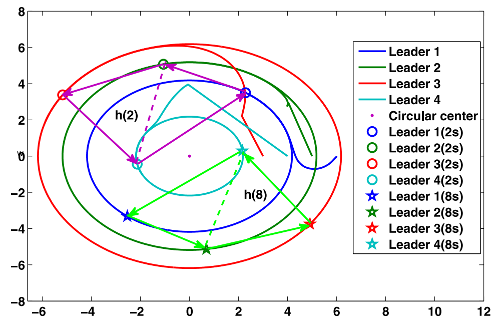

Fig. 3. Position trajectories of the four leaders with the desired rotating formation structure at different times.

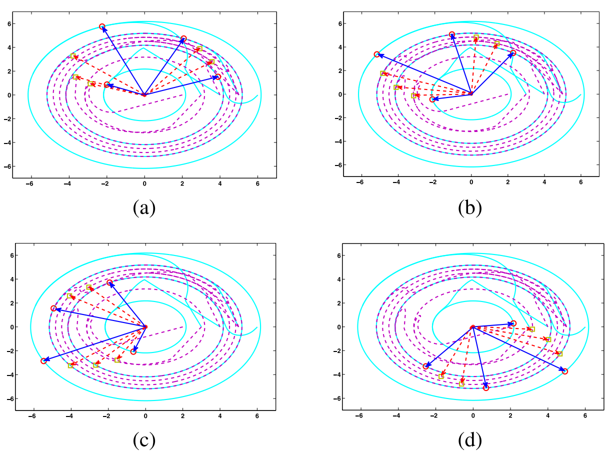

Fig. 4. Position trajectories of the nine agents with the containment structure at different times. (a) $t = 1$ s. (b) $t = 2$ s. (c) $t = 4$ s. (d) $t = 8$ s.

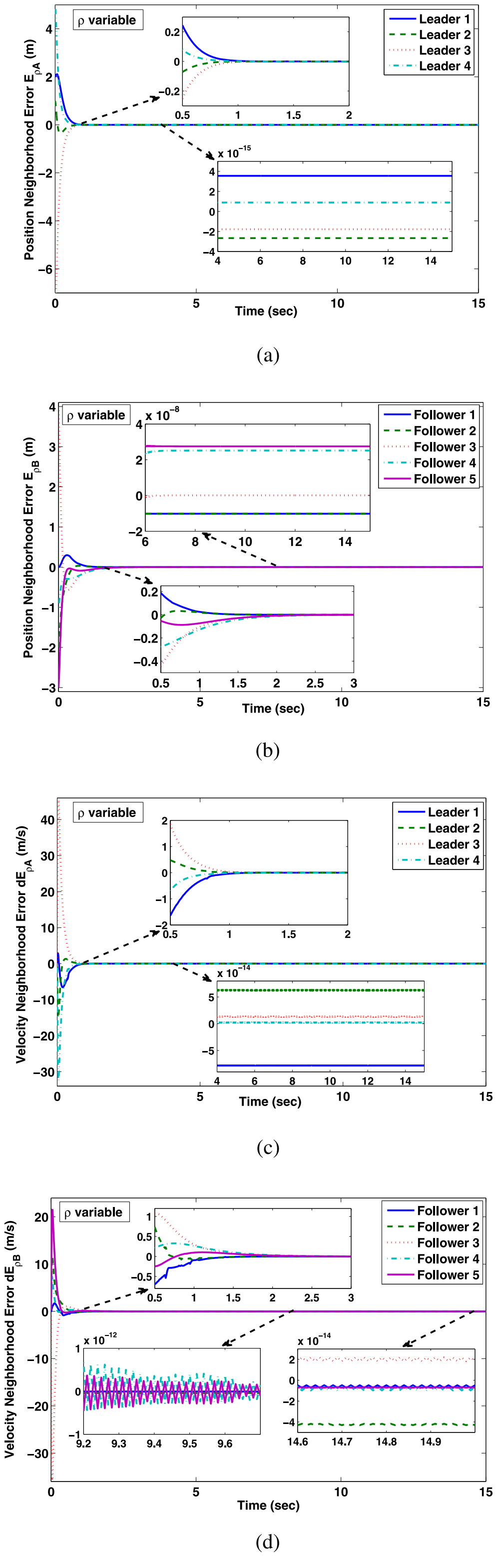

Fig. 5. Position and velocity neighborhood errors of the four leaders and five followers (expressed in ρ). (a) $E_{\rho A}$. (b) $E_{\rho B}$. (c) $\dot{E}_{\rho A}$. (d) $\dot{E}_{\rho B}$.

$2^{1-s}s + c_4) + \tilde{b}\tilde{m}(2^{1-s} + c_4 + 2^{1-s}s)] = 0.8989$ and $k_6 = [(s-2)(1+s)]s(\tilde{a} + \tilde{b}\tilde{m}) = -2.0751$, $s = (2q-1)/(2q+1)$, $\tilde{a} = \max_{\forall i \in B}\{\sum_{j \in \tilde{\mathcal{N}}_i} a_{ij}\} = 2$, $\tilde{b} = \max_{\forall i,j \in B}\{a_{ij}\} = 1$, and $\tilde{m} = 2$.

The desired rotating formation vector for the leaders is specified as $(h_\rho, h_\theta) = [(3,0), (4, \pi/4), (5, \pi/2, (1, 3\pi/4)]$. The initial states of all the agents are $\rho_A(0) = [6,5,3,4]^T$; $\rho_B(0) = [5,4,5,3,2]^T$; $\theta_A(0) = [0, \pi/5, \pi/4, \pi/2]$; $\theta_B(0) = [\pi/6, \pi/5, \pi/3, 5\pi/3, 5\pi/4]^T$. The initial velocities of all the agents are different from each other which are taken as $v_{\rho A}(0) = [0.3, 0, 0.5, 0.4]^T$; $v_{\rho B}(0) = [0.3, 0.4, 0, 0.45, 0.55]^T$; $v_{\theta A}(0) = [0.5, 0.6, 0.7, 0.8]$; $v_{\theta B}(0) = [0.6, 0.5, 0.7, 0.8, 0.6]^T$. In addition, the common rotating angular velocity $w(t)$ is taken as $w(t) = 0.5 + 0.5\exp(-t)$, which is time-varying.

The simulation results are shown in Figs. 3–5, where it is seen from Fig. 3 that the four leaders achieve the desired rotating formation structure in a finite time (within 2 s) and maintain the time-varying formation structure when rotating. The position trajectories of the nine agents (including the four leaders and five followers) with the containment structure at different times are shown in Fig. 4, where it can be seen the states of the five followers, including ρ_i and θ_i ($i = 1, \ldots, 5$), enter into the convex hull formed by that of the moving leaders in a finite time [within 2 s as shown in Fig. 4(b)]. The position neighborhood errors for the four leaders and five followers are depicted in Fig. 5(a) and (b), respectively. The velocity neighborhood errors for the four leaders and five followers are presented in Fig. 5(c) and (d), respectively. It is observed from the results that high precision rotating formation-containment is realized by the proposed control schemes given in Theorems 1 and 2 in finite time.

VI. Conclusion

This paper explicitly addresses finite-time rotating formation-containment control for MAS and it is shown that under the proposed control schemes not only the states of the leaders achieve the prespecified rotating formation but also the states of the followers converge to the convex hull shaped by those of the leaders in a finite time. Furthermore, all the agents in the system, including leaders and followers, maintain the rotating motion collectively. It is worth noting that the time-varying angular velocity rotation formation investigated here includes the constant common angular velocity w as a special case.

An interesting topic for further research is to extend the results to rotating formation and containment of MAS with nonidentical uncertain dynamics and unpredictable actuation faults [32] under the directed topology [33]. Another topic worthy of investigation is the combination of the proposed method with partial differential equation (PDE)-based method (a new continuum-based viewpoint) as investigated in [34], to enable the rotation formation and containment deployment in 3-D space [35] or to achieve PDE-based finite time convergence [25].

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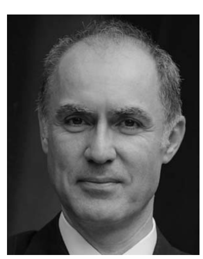

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