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Brief paper

Practical prescribed time tracking control over infinite time interval involving mismatched uncertainties and non-vanishing disturbances ☆

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ABSTRACT

Most existing prescribed time control methods are valid for finite time intervals rather than infinite time intervals, and can only achieve regulation under vanishing or no uncertainties/disturbances. In this work, a control method that achieves pre-assignable tracking precision within the prescribed time is presented for nonlinear systems in the presence of non-vanishing disturbances and mismatched uncertainties over the infinite time interval. More specifically, by introducing a novel time-varying constraining function, the original finite-time tracking control problem is converted into one with a deferred constraint on tracking error, then by stabilizing of which, the prescribed time tracking control problem is solved with the following appealing features: 1) both the settling time and the tracking accuracy can be explicitly specified in advance regardless of initial conditions and any design parameter; 2) the control gains and the control input remain bounded and continuous everywhere; 3) the system is allowed to operate continuously even after the prescribed time; and 4) the results are global in that pre-set tracking accuracy is ensured within the predefined time for states starting from anywhere within the physically possible domain.

1. Introduction

For many time-critical application, such as auto parts assembling, spacecraft docking, and target interception, it is highly desirable that the preferred (desired) tracking precision is achieved within a short period of time. Apparently, finite-time control, fixed-time control and predefined/prescribed time control play a vital role in this regard, and thus have received much attention from control community during the past decades, see Basin, Panathula, and Shtessel (2016), Hong (2002), Hong and Jiang (2006), Huang, Lin, and Yang (2005), Huang, Wen, Wang, and Song (2016), Polyakov (2012), Polyakov, Efimov, and Perruquetti (2015), Ryan (1979), Sun, Xue, and Zhang (2015), Tian, Zuo, Yan, and Wang (2017), Utkin (1992), Zhang and Wu (2017) and Zuo (2014, 2015), for instance. It is noted that fixed-time control requires the convergence time to be bounded by a constant irrespective of the initial conditions, while predefined/prescribed time control is able to preset any practically allowable convergence time by the designer, rendering a stronger and more attractive property than fixed-time control in terms of convergence time. Thus, predefined/prescribed time control has recently gained increasing attention and can be broadly classified into fractional power of state feedback based and regular state feedback based including those recently proposed by Krishnamurthy, Khorrami, and Krstic (2020), Sánchez-Torres, Defoort, and Munoz-Vázquez (2019) and the earlier works by Becerra, Vázquez, Arechavaleta, and Delfin (2018), Jiménez-Rodríguez, Sánchez-Torres, and Loukianov (2017), Sánchez-Torres, Gómez-Gutiérrez, López, and Loukianov (2018), Song, Wang, Holloway, and Krstic (2017), Tran, Yucelen, and Sarsilmaz (2018) and Wang and Song (2018) . Based on the feedbacks with fractional power of the state variables, the predefined time stabilization controllers are combined with the sliding model control and distributed control of the first-order systems in Jiménez-Rodríguez et al. (2017) and Sánchez-Torres et al. (2018), respectively. Then, the predefined time property in these works is extended to the high-order nonholonomic systems in Sánchez-Torres et al. (2019).

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As stated in Hong (2002) , the state feedback control design with fractional powers in the above mentioned references could improve the transient behaviors and robustness properties of the system. By employing time-varying scaling functions, new prescribed finite-time control methods are developed with the regular state feedback in Becerra et al. (2018), Song et al. (2017), Tran et al. (2018) and Wang and Song (2018), but only specific classes of systems such as integrator systems and normal form systems with matched uncertainties can be handled in those literature. In Aldana-López, Gómez-Gutiérrez, Jiménez-Rodríguez, Sánchez-Torres, and Defoort (2021), a methodology for generating autonomous and non-autonomous systems with a predefined time equilibrium point is provided, which benefits for constructing new predefined time stable systems and designing predefined time controller. The recent work by Krishnamurthy et al. (2020) is the first that can explicitly deal with strict feedback like systems but requires rather restrictive conditions on the control gains and uncertainties. Besides, the control gains of these methods all grow to infinity as the system approaches its equilibrium, bringing difficulties to implementation, and these methods are invalid beyond the prescribed time interval. It is therefore of theoretical significance and practical importance to develop a more practical prescribed time control method that allows the system to operate over an infinite time interval and is able to handle strict feedback systems with nonvanishing yet mismatched uncertainties.

On the other hand, as stated in Krishnamurthy et al. (2020) and Song et al. (2017), infinite gains are inevitable to achieve zero steady-state error for the prescribed time control methods (including optimal control with a terminal constraint and sliding mode approaches with time-vary gains), while control with sufficient steady-state accuracy is acceptable for practical applications. This motivates a less ambitious but practical control objective, namely the practical finite-time control, where the steady-state error converges to a small neighborhood of origin (rather than zero) within a finite time, see for examples in Jin (2017), Li, Zhao, He, and Lu (2019), Song, Huang, and Wen (2016), Wang, Chen, Liu, and Lin (2017), Wang, Song, Krstic, and Wen (2016) and the references therein. However, in all the above-mentioned results, the size of the steady-state error literally depends on some unknown constants such that the final control precision after the settling time is unknown and uncertain, which is highly undesirable in practice. Furthermore, most of these practical finite-time control schemes are built upon fractional power of state feedback and the "adding power integration" technique is required for the high-order system, see for examples in Hong and Jiang (2006), Huang et al. (2005, 2016), Sun et al. (2015) and Zhang and Wu (2017). In this way, not only the control design and stability analyze are rather complicated, but also the control action is non-smooth, yet the settling time cannot be pre-specified since it depends on several design parameters and/or initial conditions.

In this paper, a new approach for practical prescribed time (PPT) control is presented, where both settling time and tracking precision are truly assignable in advance by the designer. Firstly, by constructing a time-varying constraining boundary, the prescribed time tracking control problem is converted into one with *deferred constraint* on tracking error (i.e., no constraint or equivalently infinite constraining boundary is imposed on tracking error initially). In order to achieve a global result, this deferred constraint imposed here is in contrast to the common bounded constraint in the references (Cao, Song, & Wen, 2019; Liu et al., 2018; Tee, Ge, & Tay, 2009). Secondly, the operation region is divided into the safe region and the collision avoidance region, and a new compound function is introduced to map the distance between the tracking error and the constraining boundary into the interval [0, 1]. In this way, the uncontrollability issue caused by the deferred constraint is avoided elegantly. Finally, two PPT control schemes are developed for the first-order nonlinear system and the high-order strict-feedback system, respectively. The originality and main contributions of this paper are summarized as follows.

(1) Compared with the predefined/prescribed time control methods in Becerra et al. (2018), Krishnamurthy et al. (2020), Song et al. (2017), Tsuji, Tanaka, Morasso, Sanguineti, and M. (2002) and Wang and Song (2018), more general high-order systems with uncertain terms and disturbances can be addressed with our proposed PPT control method. Besides, the infinite gain phenomenon is gracefully avoided such that our proposed controllers are applied for systems that operate continuously even after the predefined settling time;
(2) Different from the practical finite-time control results in Jin (2017), Li et al. (2019), Song et al. (2016) and Wang et al. (2017, 2016), where the convergence precision at the steady-state phase is unknown, here both the settling time and the final tracking precision are independent of initial conditions and any other design parameters, thus can be determined in advance as needed;
(3) The proposed method is a global solution for PPT tracking control that is able to steer the tracking error (starting from anywhere without the need for changing control structure/parameter) into the prescribed precision set within the prescribed time.

2. Some definitions

We begin with recalling the basic definition of practical finite-time stability and then introducing the concept of PPT stability for nonlinear systems with the form

$$\dot{x} = f(t; x; \theta) \tag{1}$$

where $x \in \mathbb{R}^n, t \in \mathbb{R}^+, \theta \in \mathbb{R}^m$ is a free control parameter vector, and $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz in x and piecewise continuous in t for all θ.

Definition 1 (*Li et al., 2019*)**.** Let $\Theta \subset \mathrm{R}^m$ be a set of parameters. System (1) is said to be semi-global practical finite-time stable, if for all $x(0) = x_0$, there exist a scalar $\varepsilon > 0$, a settling time $T(\varepsilon, x_0) < \infty$ and $\theta \in \Theta$ such that $\|x(t, \theta)\| < \varepsilon$ for all $t \geq T(\varepsilon, x_0)$.

Definition 2. Let $\Theta \subset \mathrm{R}^m$ be a set of parameters. System (1) is called PPT stable, if for any given positive constants $\varepsilon > 0$ and $T > 0$, there exists $\theta \in \Theta$ such that

$$\|x(t, \theta)\| < \varepsilon, \quad \forall t \geq T \text{ and } \forall x(0) = x_0 \in \mathbb{R}^n \tag{2}$$

In Definition 1, the specific value of ε may not be known, and the settling time $T(\varepsilon, x_0)$ depends on the initial condition x_0 and the tracking precision ε. Some representative results in the field of semi-global practical finite-time stable have been reported, for example, see the works of Jin (2017), Li et al. (2019), Song et al. (2016) and Wang et al. (2017, 2016). Different from Definition 1, the PPT stable system in Definition 2 can preset the value of ε and T regardless of the initial condition and any other parameters, a highly desirable feature that motivates this study.

3. General way to achieve PPT tracking

To achieve PPT tracking, the key is to introduce a time-varying constraining function, with which the original PPT tracking control problem is converted into one with a deferred constraint on tracking error.

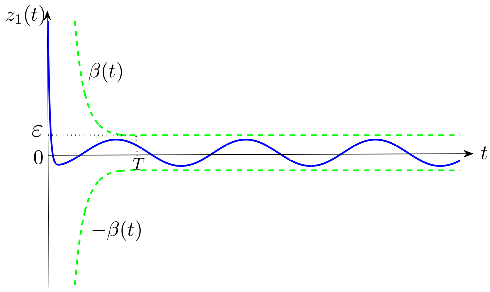

Fig. 1. Tracking error with deferred time-varying constraint.

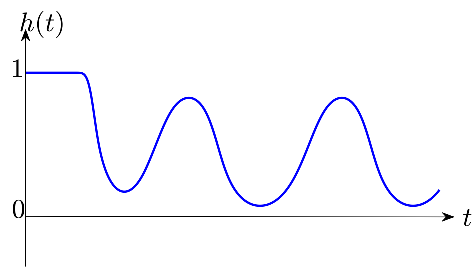

Fig. 2. The evolution of function $h(t)$.

3.1. Time-varying constraining function

Based on the predefined time T and the given precision ε, the time-varying constraining function $\beta(t)$ is constructed as

$$\beta(t)=\begin{cases}\left(\dfrac{1}{t}-\dfrac{1}{T}\right)^{2p}+\varepsilon, & 0\le t\le T\\ \varepsilon, & t>T\end{cases} \tag{3}$$

where $0<T<\infty$ and $0<\varepsilon<\infty$ represent the user-assigned settling time and steady-state tracking accuracy, n is the order of the system, p is a positive integer satisfying $2p>n+1$ so as to ensure the continuous differentiability of $\beta^{(i)}(t)(i=0,1,\ldots,n)$.

Define $z_1(t)=y(t)-y_d(t)$. Note that $\beta(t)$ monotonically decreases from infinity to ε for $t\in[0,T]$ and keeps constant ε for $t\in(T,\infty)$. Then, as shown in Fig. 1, the desired PPT tracking objective is achieved as long as the following deferred tracking error constraint is satisfied

$$-\beta(t)<z_1(t)<\beta(t). \tag{4}$$

In view of the definition of the scalar function β as given in (3), we have from (4) that

$$|z_1(t)|<\varepsilon \quad \text{for} \quad t\ge T. \tag{5}$$

In other words, the tracking error $z_1(t)$ converges to the region $\Omega=\{z_1\in\mathbb{R}:|z_1(t)|<\varepsilon\}$ within the settling time T if the constraint (4) is ensured. The main features of the constraint boundary $\beta(t)$ are that: (i) it is related to the settling time T and tracking precision ε, with which the PPT stability is linked with the problem of tracking error constraints; (ii) both parameters T and ε are at user's disposal, thus the settling time and tracking accuracy can be prescribed and preassigned as the desired value; and (iii) it is designed as infinity at the initial time, which helps to achieve a global tracking result as seen in the sequel.

3.2. Tracking error-dependent transformation

Note that most existing constraint control methods can only handle situations where the constraint boundary is bounded at all times (the constraints exist from the beginning of system operation), here the deferred constraint boundary in (4) is infinite at the initial time (which is free of constraint at the initial time), thus new constraint control methods need to be developed. Define

$$\gamma(t)=c(\beta(t)-z_1(t))(\beta(t)+z_1(t))=c(\beta^2(t)-z_1^2(t)) \tag{6}$$

with $c>0$, where $\beta(t)-z_1(t)$ and $\beta(t)+z_1(t)$ represent the distances from the tracking error to the constraint boundaries $\beta(t)$ and $-\beta(t)$, respectively. Then, based on function $\gamma(t)$, a new compound function $h(\gamma(t))$ is constructed to map the value of $\gamma(t)$ to the interval of $(0,1]$ as follows

$$h(\gamma(t))=\begin{cases}1-\left(\dfrac{\gamma(t)}{a}-1\right)^{2p}, & 0<\gamma(t)\le a\\ 1, & \gamma(t)>a\end{cases} \tag{7}$$

where $a>0$ is a user-designed safety distance. Also, since $2p>n+1$ in (7), it follows that $\lim_{\gamma(t)\to a^-}\frac{\partial^k h(\gamma(t))}{\partial\gamma(t)^k}=\lim_{\gamma(t)\to a^+}\frac{\partial^k h(\gamma(t))}{\partial\gamma(t)^k}=0$ with $k=1,\ldots,n$. Thus, the well defined function $h(\gamma(t))$ is C^n smooth including the points $\gamma(t)=a$.

About the parameter a, the functions $\gamma(t)$ and $h(\gamma(t))$, the following properties hold: (i) the operating area is divided according to the distance of the tracking error to the constraint boundary. When the distance $\gamma(t)\in(0,a]$, the system is nearby the boundary that could cause collision. Otherwise when distance $\gamma(t)\in(a,+\infty)$, the system is operating in a safe region far from the constraining boundary; (ii) when $t>T$, we have from $\beta(t)=\varepsilon$ and the definition of $\gamma(t)$ in (6) that $\gamma(t)\le c\varepsilon^2$. In this paper, we choose $a=c\varepsilon^2$, which is only one of the options but not a critical choice, guaranteeing that $\gamma(t)<a$ for $t>T$. This prevents the system from being regarded as in the safe region at all times; and (iii) note that $\gamma(0)=\infty>a$ at the initial time, thus $h(t)\equiv 1$ from the initial time instant to a certain period of time and thereafter $h(t)$ varies over $(0,1]$. One possible evolution profile of $h(t)$ is conceptually illustrated in Fig. 2.

To carry on the control design, the transformation on the tracking error

$$\xi_1(t)=\frac{z_1(t)}{h(\gamma(t))}=\frac{z_1(t)}{h\left(c\beta^2(t)-cz_1^2(t)\right)}. \tag{8}$$

is introduced. It is interesting to note that $\xi_1(t)$ exhibits the following properties: (i) $\xi_1(t)=0$ if and only if $z_1(t)=0$; (ii) $\xi_1(t)\to\infty$ as $z_1(t)\to\beta(t)$ or $z_1(t)\to-\beta(t)$; and (iii) when $\gamma(t)>a$, the system is operating within safe region. We have from (7) that $h(\gamma(t))=1$ and therefore $\xi_1(t)=z_1(t)$. In this case, no special constraint transformation is imposed on the tracking error $z_1(t)$, which avoids the appearance of $\beta(t)$ being infinite especially at the initial time.

By using the properties of $\xi_1(t)$, the following lemma is crucial to our later technical development.

Lemma 1. *If $\xi_1(t)\in L_\infty$, then $-\beta(t)<z_1(t)<\beta(t)$ holds for all $t\in[0,\infty)$.*

Proof. Since $\beta(0)=\infty$, we have $-\beta(0)<z_1(0)<\beta(0)$ at the initial time. Now we prove $-\beta(t)<z_1(t)<\beta(t)$ for $t\in(0,\infty)$ by contradiction. Let $\varpi_1(t)=\frac{z_1(t)}{\beta(t)}$. Suppose that $z_1(t)\le-\beta(t)$ or $z_1(t)\ge\beta(t)$ for $t=t_1$, thus we have $\varpi_1(t)\ge 1$ or $\varpi_1(t)\le -1$ for $t=t_1$. Then, from the intermediate value theorem of continuous functions as in Kostrykin and Oleynik (2012), there exists a time instant $0<t_2<t_1$ such that $\varpi_1(t_2)=-1$ or $\varpi_1(t_2)=1$. As a result, $\gamma(t_2)=c\beta^2(t_2)-cz_1^2(t_2)=0$. Combining (7) and (8), we obtain that $h(\gamma(t_2))=0$ and $\xi_1(t_2)=\infty$, which leads to a contradiction for the boundedness of $\xi_1(t)$. Therefore, it is concluded that $-1<\varpi_1(t)<1$ and $-\beta(t)<z_1(t)<\beta(t)$ for all $t\in[0,\infty)$. This completes the proof. □

With the aid of Lemma 1, it is obvious that the deferred constraint on $z_1(t)$ in (4) is satisfied if $\xi_1(t)$ is ensured to be bounded for $t \in [0, \infty)$.

Remark 1. Here we pause to stress the novelty behind constructing the constraint transformation in (8). If the transformation of the form $\bar{\xi}(t) = \frac{z_1(t)}{\gamma(t)}$ rather than (8) is utilized directly, it would lead to $\dot{\bar{\xi}}(t) = \frac{1}{\gamma(t)}\dot{z}_1(t) - \frac{\dot{\gamma}(t)}{\gamma(t)^2}z_1(t)$, in which $\gamma(0) = \dot{\gamma}(0) = \infty$ (because $\beta(0)$ as defined in (3) is infinite), implying that such transformation would result in a system with vanished virtual control/actual control gains at the initial time point, thus the resultant system becomes uncontrollable initially. Similar uncontrollability issue also occur when choosing transformation function $ln(\frac{\beta+z_1}{\beta-z_1})$ as in Bechlioulis and Rovithakis (2014) . To overcome this problem, we introduce the piecewise function $h(\gamma(t))$ in (7) so that $h(\gamma(0)) = 1$ due to $\gamma(0) = \infty > a$. In this way, the transformation $\xi(t) = z_1(t)$ holds at the initial time such that the constraint boundary $\gamma(0)$ are not introduced in the derivative of $\xi(t)$ and the uncontrollability issue is avoided gracefully.

3.3. Control design and stability analysis

A general constraint transformation method is provided in the previous section. To gain better insight of the technical development, the PPT control design is started for a first-order nonlinear system, and then is extended to a high-order strict-feedback system, during which regular state feedback rather than fractional power state feedback is utilized, rendering the stability analysis and designed controller simple and low complexity.

3.3.1. Motivating example

In this subsection, a PPT tracking control law is developed for the following first-order system

$$\dot{x} = f(t, x) + u \tag{9}$$

where $x, u \in \mathbb{R}$ and function $f(t, x)$ satisfies the following assumption. Throughout this paper, for ease notation, the argument in a variable or function sometimes is dropped if no confusion is likely to occur.

Assumption 1. For lumped uncertainty $f(t, x)$, there exist an unknown constant $\delta \geq 0$ and a known continuous function $\varphi(x) \geq 0$ such that

$$|f(t, x)| \leq \delta\varphi(x). \tag{10}$$

Define tracking error $z = x - y_d$. According to the error transformation $\xi(t) = \frac{z(t)}{h(\gamma(t))} = \frac{z(t)}{h\left(c\beta^2(t) - cz^2(t)\right)}$ in (8), the original model (9) is converted into

$$\dot{\xi} = \mu\dot{z} + \nu = \mu(f + u - \dot{y}_d) + \nu \tag{11}$$

where

$$\mu = \begin{cases} \frac{1}{h} - \frac{4cp}{ah^2}(\frac{\gamma}{a} - 1)^{2p-1}z^2 > 0, & 0 < \gamma(t) \leq a \\ 1, & \gamma(t) > a, \end{cases} \tag{12}$$

$$\nu = \begin{cases} \frac{4cp}{ah^2}(\frac{\gamma}{a} - 1)^{2p-1}\beta\dot{\beta}z, & 0 < \gamma(t) \leq a \\ 0, & \gamma(t) > a, \end{cases} \tag{13}$$

with $p = 2$. Then the controller is designed as

$$u = -\frac{k}{\mu}\xi - \hat{\vartheta}\mu\varphi^2\xi + \dot{y}_d - \frac{\nu}{\mu}, \tag{14}$$

$$\dot{\hat{\vartheta}} = -\sigma\hat{\vartheta} + r\mu^2\varphi^2\xi^2, \tag{15}$$

where $k > 0, \sigma > 0$ and $r > 0$ are the positive design parameters, $\hat{\vartheta}$ is the estimation of $\vartheta = \delta^2$.

Theorem 1. *Consider the uncertain nonlinear system (9) under Assumption 1. If the adaptive controller (14)–(15) is applied, then the desired PPT tracking is achieved.*

Proof. The Lyapunov function is chosen as

$$V = \frac{1}{2}\xi^2 + \frac{1}{2r}\tilde{\vartheta}^2 \tag{16}$$

where $\tilde{\vartheta} = \hat{\vartheta} - \vartheta$. The derivative of V along (11) is

$$\begin{aligned} \dot{V} &= \xi\dot{\xi} + \frac{1}{r}\tilde{\vartheta}\dot{\hat{\vartheta}} \\ &= \mu\xi(f + u - \dot{y}_d) + \nu\xi + \frac{1}{r}\tilde{\vartheta}\dot{\hat{\vartheta}}. \end{aligned} \tag{17}$$

According to the Young's inequality as in Deng and Krstic (1997), it is easy to get that

$$\mu\xi f \leq \mu|\xi|\delta\varphi \leq \delta^2\mu^2\varphi^2\xi^2 + \frac{1}{4}. \tag{18}$$

Thus, (17) can be rewritten as

$$\dot{V} \leq \mu\xi u + \delta^2\mu^2\varphi^2\xi^2 + \nu\xi - \mu\dot{y}_d\xi + \frac{1}{4} + \frac{1}{r}\tilde{\vartheta}\dot{\hat{\vartheta}}. \tag{19}$$

By substituting (14)–(15) into (19), we have

$$\begin{aligned} \dot{V} &\leq -k\xi^2 - \tilde{\vartheta}\mu^2\varphi^2\xi^2 + \frac{1}{4} + \frac{1}{r}\tilde{\vartheta}\dot{\hat{\vartheta}} \\ &\leq -k\xi^2 - \frac{\sigma}{r}\tilde{\vartheta}\hat{\vartheta} + \frac{1}{4}. \end{aligned} \tag{20}$$

According to the fact that $-\tilde{\vartheta}\hat{\vartheta} = -\tilde{\vartheta}(\tilde{\vartheta} + \vartheta) \leq -\tilde{\vartheta}^2 + \frac{\tilde{\vartheta}^2}{2} + \frac{\vartheta^2}{2} \leq -\frac{\tilde{\vartheta}^2}{2} + \frac{\vartheta^2}{2}$, we obtain

$$\begin{aligned} \dot{V} &\leq -k\xi^2 - \frac{\sigma}{2r}\tilde{\vartheta}^2 + \frac{\sigma}{2r}\vartheta^2 + \frac{1}{4} \\ &\leq -\Lambda V + \Pi \end{aligned} \tag{21}$$

where $\Lambda = \min\{2k, \sigma\}$ and $\Pi = \frac{\sigma}{2r}\vartheta^2 + \frac{1}{4}$. By integrating both sides of the inequality (21), we obtain $V(t) \leq e^{-\Lambda t}V(0) + \frac{\Pi}{\Lambda}(1 - e^{-\Lambda t}) \leq V(0) + \frac{\Pi}{\Lambda}$. Therefore, $V \in L_\infty$ for any finite initial conditions, which implies that $\xi \in L_\infty$ and $\tilde{\vartheta} \in L_\infty$. Then, according to Lemma 1, we have $-\beta(t) < z(t) < \beta(t)$ for all times. Note that $\beta(t)$ as defined in (3) monotonically decreases to ϵ within finite time T, thus the tracking error $z(t)$ not only obeys the performance boundary $|z(t)| < \beta$, but also converges to the prescribed region $\Omega_1 = \{z \in \mathbb{R} : |z(t)| < \varepsilon\}$ within assignable prescribed settling time T. Besides, both the size of compact set ε and the settling time T can be determined in advance. This completes the proof. □

Remark 2. Since the constraint boundary $\beta(0)$ in (3) is infinite, we have from (6) that $\gamma(0) = \infty$. Define $t_1 = \inf\{t : \gamma(t) = a\}$. According to the continuity of $\gamma(t)$, when $t \in [0, t_1)$, it can be obtained that $\gamma(t) > a$. According to the definition of $h(t)$ in (7), $h(t) \equiv 1$ holds, then we have from (12)–(13) that $\mu = 1$, $\nu = 0$ and $\xi = z$. Based on the Lyapunov stability analysis, $\xi = z \in L_\infty$ is established, which further implies that $x \in L_\infty$ and $\varphi(x) \in L_\infty$ due to the continuity of function $\varphi(x)$. Thus, the control signal in (14)–(15) is bounded for $t \in [0, t_1)$. When $t \in [t_1, +\infty)$, the constraint boundary $\beta(t)$ as defined in (3) is bounded, the boundedness of control signal in (14)–(15) can be easily derived according to $|z(t)| < \beta$. Consequently, under any unknown initial condition, the control action generated by the proposed algorithm is continuous and bounded during the entire operational process of the system.

3.3.2. High-order strict-feedback systems

Now we develop our results to high-order strict-feedback systems:

$$\begin{aligned}\dot{x}_i &= f_i(t,\bar{x}_i)+g_i(t,\bar{x}_i)x_{i+1}+d_i(t), \quad i=1,\ldots,n-1\\ \dot{x}_n &= f_n(t,\bar{x}_n)+g_n(t,\bar{x}_n)u+d_n(t)\\ y &= x_1 \end{aligned} \tag{22}$$

where $\bar{x}_i = [x_1,x_2,\ldots,x_i]^T \in \mathbb{R}^i$ and $u \in \mathbb{R}$ are the system state and input respectively, $y \in \mathbb{R}$ is the system output, $f_i(t,\bar{x}_i)$ represents an uncertain nonlinear continuous function, $g_i(t,\bar{x}_i)$ denotes an unknown gain function that can be state and time dependent, and $d_i(t)$ is the unknown time-varying disturbances.

Assumption 2. For lumped uncertainties $f_i(t,\bar{x}_i)(i=1,\ldots,n)$, there exist unknown constants $a_i \geq 0$ and known continuously differentiable functions $\varphi_i(\bar{x}_i) \geq 0$ such that

$$|f_i(\bar{x}_i)| \leq a_i\varphi_i(\bar{x}_i). \tag{23}$$

Assumption 3. The gain functions $g_i(t,\bar{x}_i)(i=1,\ldots,n)$ are positive and there exist unknown constants $0<\underline{g}_i<\bar{g}_i<\infty$ such that

$$0<\underline{g}_i \leq g_i(\cdot) \leq \bar{g}_i. \tag{24}$$

Assumption 4. The unknown time-varying disturbances $d_i(t)(i=1,\ldots,n)$ are bounded, i.e., $|d_i(t)|<D_i$ with D_i as an unknown constant.

Assumption 5. The desired trajectory y_d and its $i(i=1,\ldots,n)$-order derivatives are known, piecewise continuous and bounded.

Remark 3. As mentioned in Song et al. (2016), $\varphi_i(\bar{x}_i)$ in Assumption 2 is easy to be obtained only with some crude model information in practice. For example, consider the uncertain function $L_i(\bar{x}_i) = \rho_1 cos(\rho_2\bar{x}_i)+\bar{x}_i e^{-|\rho_3\bar{x}_i|}$, where ρ_1, ρ_2 and ρ_3 are unknown constants. Clearly, $L_i(\bar{x}_i)$ cannot be linearly parameterized as in Jin (2017), and neither state $\bar{x}_i$ nor unknown parameters ρ_1, ρ_2 and ρ_3 can be factored out from $L_i(\bar{x}_i)$. However, it is effortless to obtain function $\varphi_i(\bar{x}_i)=1+|\bar{x}_i|$, such that $|L_i(\bar{x}_i)| \leq a\varphi(\bar{x}_i)$ with $a=\max\{\rho_1,1\}$. Besides, both $\underline{g}_i$ and $\bar{g}_i$ in Assumption 3 are unknown and are not used in building the corresponding controller.

Remark 4. It is worth comparing the difference between our proposed method and the one in Bechlioulis and Rovithakis (2014). In Bechlioulis and Rovithakis (2014), although Assumptions 1–2 on the lumped uncertainties $f_i(\bar{x}_i)$ are not required, the continuous differentiability of $f_i(\bar{x}_i)$ is needed, based on which the boundedness of $f_i(\bar{x}_i)$ is then established such that $|f_i(\bar{x}_i)| \leq \bar{F}_i$ with $\bar{F}_i$ being an unknown positive constant. It is worth noting that in establishing such bounded relationship, the fact that $\beta(t)$ is bounded is used, i.e., if $-\beta(t)<x_1(t)-y_d(t)<\beta(t)$ holds for $t\in[0,\tau_{max})$, then the boundedness of $x_1(t)$ for $t\in[0,\tau_{max})$ can be deduced with the bounded boundary $\beta(t)$, which, together with the continuity, allows the boundedness of $f(t,x_1)$ to be established. Whereas in our paper, the constraint boundary $\beta(t)$ in (3) is infinite initially, which is purposely designed this way to cope with any unknown initial condition (one of the contributions of the paper). Consequently, the boundedness of x_1 and $f(t,x_1)$ cannot be directly established with this setting, making the method for control design and stability analysis in Bechlioulis and Rovithakis (2014) inapplicable.

Differentiating (8) with respect to time, from the definition of tracking error $z_1=y-y_d$ and the system (22), we have

$$\dot{\xi}_1=\mu_1\dot{z}_1+\nu_1=\mu_1(f_1+g_1x_2+d_1-\dot{y}_d)+\nu_1 \tag{25}$$

where

$$\mu_1=\begin{cases}\dfrac{1}{h}-\dfrac{4cp}{ah^2}(\dfrac{\gamma}{a}-1)^{2p-1}z_1^2>0, & 0<\gamma(t)\leq a\\ 1, & \gamma(t)>a.\end{cases} \tag{26}$$

$$\nu_1=\begin{cases}\dfrac{4cp}{ah^2}(\dfrac{\gamma}{a}-1)^{2p-1}\beta\dot{\beta}z_1, & 0<\gamma(t)\leq a\\ 0, & \gamma(t)>a.\end{cases} \tag{27}$$

It can be easily verified that μ_1 is strictly positive and computable. Also μ_1 is C^n smooth because $\lim_{\gamma\to a^-}\mu_1=\lim_{\gamma\to a^+}\mu_1=1$, $\lim_{\gamma\to a^-}\mu_1^{(j)}=\lim_{\gamma\to a^+}\mu_1^{(j)}=0$ and $\lim_{\gamma\to a^-}\nu_1^{(k)}=\lim_{\gamma\to a^+}\nu_1^{(k)}=0$ with $j=1\cdots,n$ and $k=0,1\cdots,n$. With the help of (25), system (22) is transformed to the following form

$$\begin{aligned}\dot{\xi}_1&=\mu_1(f_1+g_1x_2+d_1-\dot{y}_d)+\nu_1,\\ \dot{x}_i&=f_i+g_ix_{i+1}+d_i, \quad i=2,\ldots,n-1\\ \dot{x}_n&=f_n+g_nu+d_n,\end{aligned} \tag{28}$$

with both μ_1 and ν_1 being available for control design, whose stability is sufficient to ensure the deferred tracking error constraint given by (4). Therefore, we only need to focus on designing controller u to stabilize the transformed system (28).

Here we present an adaptive backstepping control design to stabilize system (28). Although standard procedure exists in Krstic, Kanellakopoulos, and Kokotovic (1992), some new techniques need to be adopted in the first two steps, thus these two steps are elaborated in details. First, we introduce the following change of coordinates

$$\xi_i=x_i-\alpha_{i-1}, i=2,\ldots,n \tag{29}$$

where α_{i-1} is a virtual controller to be designed at the step $i-1$ and the controller u is derived at the final step.

Step 1*:* From (28) and (29), it follows that

$$\dot{\xi}_1=\mu_1(f_1+g_1\xi_2+g_1\alpha_1+d_1-\dot{y}_d)+\nu_1. \tag{30}$$

Then, the time derivative of $\frac{1}{2}\xi_1^2$ along (30) is

$$\begin{aligned}\xi_1\dot{\xi}_1=&\mu_1(\xi_1f_1+g_1\xi_1\xi_2+g_1\xi_1\alpha_1\\ &+\xi_1d_1-\xi_1\dot{y}_d)+\xi_1\nu_1.\end{aligned} \tag{31}$$

With the help of Assumptions 1–3 and Young's inequality as in Deng and Krstic (1997), it is easy to obtain that

$$\begin{aligned}\xi_1\nu_1&\leq\underline{g}_1\nu_1^2\xi_1^2+\frac{1}{4\underline{g}_1},\\ \mu_1\xi_1f_1&\leq\mu_1|\xi_1|a_1\varphi_1\leq\underline{g}_1a_1^2\mu_1^2\varphi_1^2\xi_1^2+\frac{1}{4\underline{g}_1},\\ \mu_1\xi_1d_1&\leq\mu_1|\xi_1|D_1\leq\underline{g}_1\mu_1^2\xi_1^2+\frac{D_1^2}{4\underline{g}_1},\\ -\mu_1\xi_1\dot{y}_d&\leq\mu_1|\xi_1||\dot{y}_d|\leq\underline{g}_1\mu_1^2\dot{y}_d^2\xi_1^2+\frac{1}{4\underline{g}_1},\\ \mu_1g_1\xi_1\xi_2&\leq\mu_1\bar{g}_1|\xi_1||\xi_2|\leq\underline{g}_2\mu_1^2\xi_1^2\xi_2^2+\frac{\bar{g}_1^2}{4\underline{g}_2}.\end{aligned} \tag{32}$$

By adding both sides of above inequalities, we get

$$\begin{aligned}\xi_1\dot{\xi}_1\leq&g_1\mu_1\xi_1\alpha_1+\underline{g}_1\mu_1^2\Phi_1\xi_1^2+\underline{g}_1\mu_1^2\vartheta_1\Psi_1\xi_1^2\\ &+\underline{g}_2\mu_1^2\xi_1^2\xi_2^2+\frac{3+D_1^2}{4\underline{g}_1}+\frac{\bar{g}_1^2}{4\underline{g}_2},\end{aligned} \tag{33}$$

where $\vartheta_1=a_1^2$, $\Phi_1=1+\dot{y}_d^2+\frac{\nu_1^2}{\mu_1^2}$ and $\Psi_1=\varphi_1^2$. Hence the virtual control is constructed as

$$\begin{aligned}\alpha_1&=-\frac{k_1}{\mu_1}\xi_1-\mu_1\hat{\vartheta}_1\Psi_1\xi_1-\mu_1\Phi_1\xi_1,\\ \dot{\hat{\vartheta}}_1&=-\sigma_1\hat{\vartheta}_1+r_1\Psi_1\mu_1^2\xi_1^2,\end{aligned} \tag{34}$$

where $k_1 > 0, \sigma_1 > 0$ and $r_1 > 0$ are the positive design constants, $\hat{\vartheta}_1$ is the estimation of ϑ_1. Note that $\xi_1\alpha_1 \leq 0$, thus

$$g_1\mu_1\xi_1\alpha_1 \leq -\underline{g}_1 k_1\xi_1^2 - \underline{g}_1\mu_1^2\hat{\vartheta}_1\Psi_1\xi_1^2 - \underline{g}_1\mu_1^2\Phi_1\xi_1^2. \tag{35}$$

Now we define a Lyapunov function candidate

$$V_1 = \frac{1}{2}\xi_1^2 + \frac{\underline{g}_1}{2r_1}\tilde{\vartheta}_1^2 \tag{36}$$

where $\tilde{\vartheta}_1 = \hat{\vartheta}_1 - \vartheta_1$. Differentiating (36) and combining (33)–(35), we have

$$\dot{V}_1 \leq -\underline{g}_1 k_1\xi_1^2 - \frac{\underline{g}_1\sigma_1}{r_1}\tilde{\vartheta}_1\hat{\vartheta}_1 + \underline{g}_2\mu_1^2\xi_1^2\xi_2^2 + \frac{3+D_1^2}{4\underline{g}_1} + \frac{\bar{g}_1^2}{4\underline{g}_2}. \tag{37}$$

According to the fact that $-\tilde{\vartheta}_1\hat{\vartheta}_1 = -\tilde{\vartheta}_1(\tilde{\vartheta}_1 + \vartheta_1) \leq -\tilde{\vartheta}_1^2 + \frac{\tilde{\vartheta}_1^2}{2} + \frac{\vartheta_1^2}{2} \leq -\frac{\tilde{\vartheta}_1^2}{2} + \frac{\vartheta_1^2}{2}$, we obtain

$$\dot{V}_1 \leq -\underline{g}_1 k_1\xi_1^2 - \frac{\underline{g}_1\sigma_1}{2r_1}\tilde{\vartheta}_1^2 + \underline{g}_2\mu_1^2\xi_1^2\xi_2^2 + \chi_1 \tag{38}$$

where $\chi_1 = \frac{\underline{g}_1\sigma_1}{2r_1}\vartheta_1^2 + \frac{1}{\underline{g}_1} + \frac{\bar{g}_1^2}{4\underline{g}_2}$ and $\underline{g}_2\mu_1^2\xi_1^2\xi_2^2$ will be handled in the next step.

Step 2: The time derivative of $\xi_2 = x_2 - \alpha_1$ is

$$\dot{\xi}_2 = \dot{x}_2 - \dot{\alpha}_1 = f_2 + g_2\xi_3 + g_2\alpha_2 + d_2 - \dot{\alpha}_1 \tag{39}$$

where $\dot{\alpha}_1 = \frac{\partial\alpha_1}{\partial x_1}f_1 + \frac{\partial\alpha_1}{\partial x_1}g_1x_2 + \frac{\partial\alpha_1}{\partial x_1}d_1 + \frac{\partial\alpha_1}{\partial y_d}\dot{y}_d + \frac{\partial\alpha_1}{\partial\dot{y}_d}y_d^{(2)} + \frac{\partial\alpha_1}{\partial\beta}\dot{\beta} + \frac{\partial\alpha_1}{\partial\dot{\beta}}\beta^{(2)} + \frac{\partial\alpha_1}{\partial\hat{\vartheta}_1}\dot{\hat{\vartheta}}_1$. The virtual controller α_2 and the adaption law are designed as

$$\begin{aligned}\alpha_2 &= -k_2\xi_2 - \hat{\vartheta}_2\Psi_2\xi_2 - \Phi_2\xi_2 - \mu_1^2\xi_1^2\xi_2,\\ \dot{\hat{\vartheta}}_2 &= -\sigma_2\hat{\vartheta}_2 + r_2\Psi_2\xi_2^2,\end{aligned} \tag{40}$$

where $k_2 > 0$, $\sigma_2 > 0$ and $r_2 > 0$ are the positive design parameters, $\hat{\vartheta}_2$ is the estimation of $\vartheta_2 = \max\{a_1^2, a_2^2\}$, $\Psi_2 = \varphi_2^2 + \left(\frac{\partial\alpha_1}{\partial x_1}\varphi_1\right)^2$ and $\Phi_2 = \left(\frac{\partial\alpha_1}{\partial x_1}x_2\right)^2 + \left(\frac{\partial\alpha_1}{\partial x_1}\right)^2 + \sum_{j=0}^{1}\left(\frac{\partial\alpha_1}{\partial y_d^{(j)}}y_d^{(j+1)}\right)^2 + \sum_{j=0}^{1}\left(\frac{\partial\alpha_1}{\partial\beta^{(j)}}\beta^{(j+1)}\right)^2 + \left(\frac{\partial\alpha_1}{\partial\hat{\vartheta}_1}\dot{\hat{\vartheta}}_1\right)^2 + 1$.

Define a positive definite Lyapunov function

$$V_2 = \frac{1}{2}\xi_2^2 + \frac{\underline{g}_2}{2r_2}\tilde{\vartheta}_2^2 \tag{41}$$

where $\tilde{\vartheta}_2 = \hat{\vartheta}_2 - \vartheta_2$. It follows that

$$\dot{V}_2 = \xi_2 f_2 + g_2\xi_2\xi_3 + g_2\xi_2\alpha_2 + \xi_2 d_2 - \xi_2\dot{\alpha}_1 + \frac{\underline{g}_2}{r_2}\tilde{\vartheta}_2\dot{\hat{\vartheta}}_2. \tag{42}$$

By using Young's inequality and following the analysis similar to (32), we have

$$\xi_2 f_2 + \xi_2 d_2 - \xi_2\dot{\alpha}_1 + g_2\xi_2\xi_3 \leq \underline{g}_2\vartheta_2\Psi_2\xi_2^2 + \underline{g}_2\Phi_2\xi_2^2 + \underline{g}_3\xi_2^2\xi_3^2 + \frac{7}{4\underline{g}_2} + \frac{\bar{g}_1^2}{4\underline{g}_2} + \frac{1}{4\underline{g}_2}\sum_{j=1}^{2}D_j^2 + \frac{1}{4\underline{g}_3}\bar{g}_2^2. \tag{43}$$

Since $\xi_2\alpha_2 \leq 0$, thus we obtain

$$g_2\xi_2\alpha_2 \leq -\underline{g}_2 k_2\xi_2^2 - \underline{g}_2\hat{\vartheta}_2\Psi_2\xi_2^2 - \underline{g}_2\Phi_2\xi_2^2 - \underline{g}_2\mu_1^2\xi_1^2\xi_2^2. \tag{44}$$

Substituting (40) and (43)–(44) into (42), it follows that

$$\dot{V}_2 \leq -\underline{g}_2 k_2\xi_2^2 - \frac{\underline{g}_2\sigma_2}{2r_2}\tilde{\vartheta}_2^2 - \underline{g}_2\mu_1^2\xi_1^2\xi_2^2 + \underline{g}_3\xi_2^2\xi_3^2 + \chi_2 \tag{45}$$

where $\chi_2 = \frac{7}{4\underline{g}_2} + \frac{1}{4\underline{g}_2}\bar{g}_1^2 + \frac{1}{4\underline{g}_2}\sum_{j=1}^{2}D_j^2 + \frac{1}{4\underline{g}_3}\bar{g}_2^2 + \frac{\underline{g}_2\sigma_2}{2r_2}\vartheta_2^2$.

Table 1
Adaptive backstepping controller.

Virtual Control Laws: $(i = 3, \ldots, n)$	
$\alpha_i = -k_i\xi_i - \hat{\vartheta}_i\Psi_i\xi_i - \Phi_i\xi_i - \xi_{i-1}^2\xi_i$	(T1.1)
$\dot{\hat{\vartheta}}_i = -\sigma_i\hat{\vartheta}_i + r_i\Psi_i\xi_i^2$	(T1.2)
with $k_i > 0$, $\sigma_i > 0$ and $r_i > 0$ are the positive design parameters, $\hat{\vartheta}_i$ is the estimation of $\vartheta_i = \max_{1\leq j\leq i}\{a_j^2\}$, $\Psi_i = \varphi_i^2 + \sum_{j=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_j\right)^2$ and $\Phi_i = \sum_{j=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial x_j}x_{j+1}\right)^2 + \sum_{j=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial x_j}\right)^2 + \sum_{j=0}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial y_d^{(j)}}y_d^{(j+1)}\right)^2 + \sum_{j=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial\hat{\vartheta}_j}\dot{\hat{\vartheta}}_j\right)^2 + \sum_{j=0}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial\beta^{(j)}}\beta^{(j+1)}\right)^2 + 1.$	
Final Control Laws:	
$u = \alpha_n$	(T1.3)

Step i($i = 3, \ldots, n$): Based on the results of the previous two steps and with the consideration of the function $V_i = \frac{1}{2}\xi_i^2 + \frac{\underline{g}_i}{2r_i}\tilde{\vartheta}_i^2$, where $\tilde{\vartheta}_i = \hat{\vartheta}_i - \vartheta_i$, the final control and adaptive laws can be recursively obtained by following the standard backstepping procedure, as summarized in Table 1. Then we have

$$\dot{V}_m \leq -\underline{g}_m k_m\xi_m^2 - \frac{\underline{g}_m\sigma_m}{2r_m}\tilde{\vartheta}_m^2 - \underline{g}_m\xi_{m-1}^2\xi_m^2 + \underline{g}_{m+1}\xi_m^2\xi_{m+1}^2 + \chi_m \tag{46}$$

for $m = 2, \ldots, n-1$ and

$$\dot{V}_n \leq -\underline{g}_n k_n\xi_n^2 - \frac{\underline{g}_n\sigma_n}{2r_n}\tilde{\vartheta}_n^2 - \underline{g}_n\xi_{n-1}^2\xi_n^2 + \chi_n \tag{47}$$

where $\chi_m = \frac{(4m-1)}{4\underline{g}_m} + \frac{1}{4\underline{g}_m}\sum_{j=1}^{m-1}\bar{g}_j^2 + \frac{1}{4\underline{g}_m}\sum_{j=1}^{m}D_j^2 + \frac{\bar{g}_m^2}{4\underline{g}_{m+1}} + \frac{\underline{g}_m\sigma_m}{2r_m}\vartheta_m^2$ and $\chi_n = \frac{(4n-1)}{4\underline{g}_n} + \frac{1}{4\underline{g}_n}\sum_{j=1}^{n-1}\bar{g}_j^2 + \frac{1}{4\underline{g}_n}\sum_{j=1}^{n}D_j^2 + \frac{\underline{g}_n\sigma_n}{2r_n}\vartheta_n^2$.
Now we are at the position to state the following theorem.

Theorem 2. *Consider the uncertain nonlinear system (22). Under Assumptions 2–5, if the adaptive controller (T1.3) is applied, then the following objectives are achieved.*
(i) The closed-loop system is stable.
(ii) The tracking error converges to the prescribed region $\Omega_2 = \{z_1 \in \mathbb{R} : |z_1(t)| < \varepsilon\}$ within prescribed time T, where ε and T are both user-assignable.
(iii) All the internal signals are bounded.

Proof. Consider the following Lyapunov function $V = V_1 + V_2 + \cdots + V_n$. By combining (38), (45), (46) and (47), it follows that

$$\dot{V} \leq -\sum_{i=1}^{n}\underline{g}_i k_i\xi_i^2 - \sum_{i=1}^{n}\frac{\underline{g}_i\sigma_i}{2r_i}\tilde{\vartheta}_i^2 + \sum_{i=1}^{n}\chi_i \leq -\gamma V + \chi \tag{48}$$

where $\gamma = \min_{1\leq i\leq n}\{2\underline{g}_i k_i, \sigma_i\}$ and $\chi = \sum_{i=1}^{n}\chi_i$.
(*i*) By integrating both sides of the inequality (48), we obtain $V(t) \leq e^{-\gamma t}V(0) + \frac{\chi}{\gamma}(1 - e^{-\gamma t}) \leq V(0) + \frac{\chi}{\gamma}$. Therefore, $V \in L_\infty$ for any finite initial conditions, which implies that $\xi_i \in L_\infty$ and $\tilde{\vartheta}_i \in L_\infty$. Thus, the closed-loop system is stable and the signals ξ_i and $\tilde{\vartheta}_i$ are ultimately uniformly bounded.
(*ii*) Since ξ_1 is bounded, we have from Lemma 1 that $-\beta(t) < z_1(t) < \beta(t)$. By recalling the definition of $\beta(t)$ in (3), it can be concluded that $|z_1(t)| < \varepsilon$ for $t > T$. Thus, the tracking error converges to the prescribed region $\Omega_2 = \{z_1 \in \mathbb{R} : |z_1(t)| < \varepsilon\}$ within prescribed time T. Besides, by choosing corresponding parameters ε and T for constraining function $\beta(t)$ in advance, the tracking precision and settling time can be prescribed as desired values.

(*iii*) According to the fact that $\beta(0) = \infty$, we have $\gamma(0) = \infty > a$ for all finite initial conditions. Define $T_1 = \inf\{t : \gamma(t) = a\}$. Thus, when $t \in [0, T_1)$, we have $\gamma(t) > a$ and $h(t) \equiv 1$, which further implies that $\mu_1 = 1$ and $\nu_1 = 0$. Note that ξ_1, Ψ_1 and Φ_1 are bounded, then it follows from (34) that $\alpha_1 \in L_\infty$ for $t \in [0, T_1)$. On the other hand, when $t \in [T_1, \infty)$, we have $0 < \beta(t) \leq \left(\frac{1}{T_1} - \frac{1}{T}\right)^{2p} + \varepsilon$ and $|\dot{\beta}(t)| \leq \frac{2p}{T_1^2}\left(\frac{1}{T_1} - \frac{1}{T}\right)^{2p-1}$. With the help of $\gamma(t) = c\beta^2(t) - cz_1^2(t)$ and $z_1^2(t) \leq \beta^2(t)$ as stated in objective *ii*, we have $0 < \gamma(t) \leq c\beta^2(t)$ and $0 < h(t) \leq 1$, which implies that μ_1 and ν_1 are bounded. Since all the signals in α_1 are bounded, the boundedness of α_1 is easily ensured for $t \in [T_1, \infty)$. Thus, the virtual control α_1 is bounded for $t \in [0, \infty)$. By using the analysis similar to the above, the boundedness of $\alpha_i (i = 2, \ldots, n-1)$ and u can also be established. This completes the proof. □

Remark 5. Our proposed PPT control has several attractive features: (1) tracking for more general nonlinear systems; (2) avoiding high gain; (3) allowing for the system to operate in infinite time interval rather than a finite time period; (4) achieving acceptable precision/accuracy ε (pre-specifiable) within prescribed time T, a favorable feature in practice. Whereas in current prescribed time control related works (Becerra et al., 2018; Krishnamurthy et al., 2020; Song et al., 2017; Tsuji et al., 2002; Wang & Song, 2018), zero error is achieved only for regulation (rather than tracking) and is at the cost of high gain, and the resultant control is operational only within $t \in [0, T)$ rather than $t \in [0, \infty)$, which substantially limits its applicability. On the other hand, due to the existence of inevitable measurement noises and calculation errors, zero error cannot literally be fully realized and certain tracking error is acceptable in practical applications. After all, extremely high control gain and limited operational periods are undesirable in some applications.

4. Simulation results

To further illustrate our proposed method and verify its effectiveness, we apply it to the following robotic manipulator system as in Xing, Wen, Liu, Su, and Cai (2017), i.e.,

$$J\ddot{q}(t) + D\dot{q}(t) + MgL\sin(q(t)) + d(t) = u \tag{49}$$

where q and $\dot{q}$ represent the angle and angular velocity of the rigid link, respectively. J is the rotation inertia of the servo motor, D is the damping coefficient, L denotes the length from the axis of joint to the mass center, M is the mass of the link and g represents the gravitational acceleration. Similar to the work of Xing et al. (2017), for simulation the physical parameters are chosen as $J = 1, MgL = 10$ and $D = 2$, while they are all unknown for controller design. The control objective is to achieve the desired PPT tracking for the angle $x_1 = q(t)$ and the desired trajectory $q_d(t) = 0.5\sin(t)$.

In simulation, the first-order Euler method is used to discretizing the continuous system model and the control parameters are chosen as $a = 0.4, c = 100, k_1 = 4, k_2 = 18, \sigma_2 = 1, r_2 = 0.001$. Now, we verify that the desired tracking precision $|z_1(t)| = |q(t) - q_d(t)| < 0.02$ (rad) can be achieved within the prescribed time 2 s for different initial conditions and external disturbances. Firstly, we fix the parameters in constraining function $\beta(t)$ as $T = 2, \varepsilon = 0.02$, and then test with different initial conditions $q(0) = 10$ (rad), $q(0) = 5$ (rad), $q(0) = -5$ (rad), $q(0) = -10$ (rad). The tracking errors and control signals are shown in Figs. 3 and 4. Secondly, we fix the initial condition as $q(0) = 1$ (rad), and then choose different disturbances such as $d(t) = 10sin(t)$, $d(t) = 5cos(t)$, $d(t) = 15sin(t)cos(t)$ and $d(t) = 8arctan(t)$, respectively. The simulation results are shown in Figs. 5 and 6. It can be

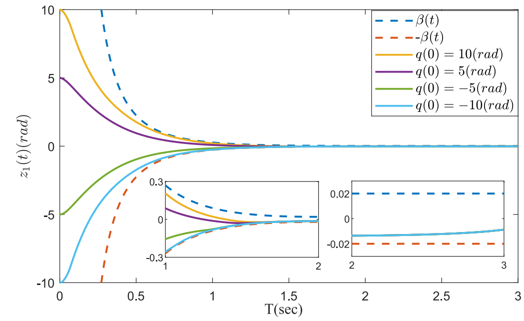

Fig. 3. Tracking error with different initial conditions.

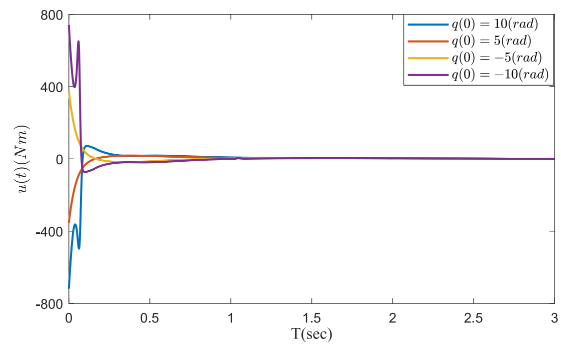

Fig. 4. Control signal with different initial conditions.

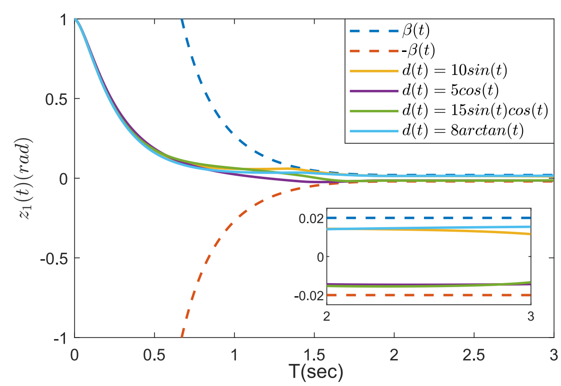

Fig. 5. Tracking error with different disturbances.

observed that for different initial conditions and disturbances, our control scheme works uniformly and the given tracking precision $|z(t)| < 0.02$ (rad) are reached when $T > 2$ s. Note that we purposely choose a relatively large range of initial conditions to test the global tracking capability of the proposed control scheme, which results in large initial control action in Fig. 4. When the initial state $q(0)$ reduces to a smaller value (for robotic manipulator it is normally within $-\pi(rad)$ to $\pi(rad)$), our control method does not demand excessively large initial control effort.

To further illustrate that different convergence times and tracking precisions can be prescribed by choosing corresponding performance parameters, we set $T = 1, \varepsilon = 0.01$ and $T = 0.5, \varepsilon = 0.01$, respectively. The tracking error and corresponding control signals (actions) under both performance parameters are depicted in Figs. 7 and 8. We can see that the tracking error converges to the predefined compact set $\Omega_1 = \{z \parallel z_1(t)| < 0.01$ (rad)$\}$ within 1 s and 0.5 s, respectively, which confirms the theoretical results. Similar results would be followed if we change ε when parameter T is fixed. Besides, as observed from

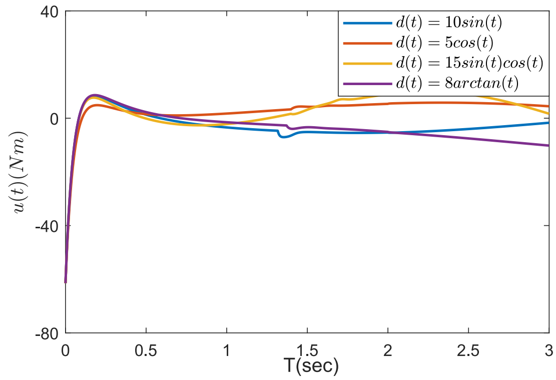

Fig. 6. Control signal with different disturbances.

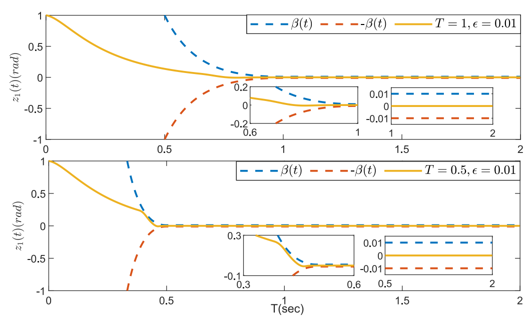

Fig. 7. Tracking error with different performance parameters.

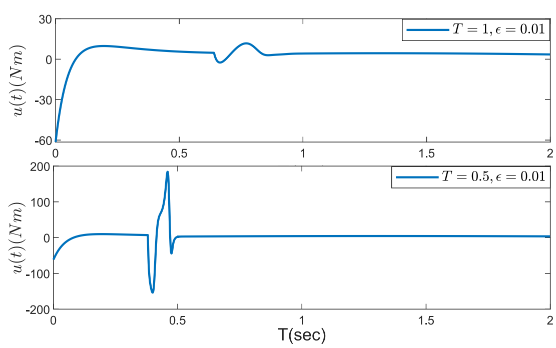

Fig. 8. Control signal with different performance parameters.

Fig. 8, the control signals fluctuate in nearly 0.7 s and 0.4 s. To further explore the reason for this fluctuation phenomenon, we examine Figs. 7 and 8 at the same time. It is shown that when the tracking error $z_1(t)$ approaches the constraint boundary $\beta(t)$, our proposed schemes provide a relatively large control action that derives $z_1(t)$ back from the boundary curve. In this way, $z_1(t)$ is ensured within the constraint boundary $\beta(t)$ and the desired tracking precision is achieved. On the other hand, we can see from Fig. 8 that larger control action is required to achieve a faster settling time. These results show that there is a balance between tracking speed/precision and control action.

5. Conclusion

In this paper, a general method for PPT tracking is presented for both the first-order and high-order uncertain systems. It is proven that, convergence time and convergence accuracy can be preset for all initial value conditions, and the control signal is bounded. An interesting topic for further study is to integrate the infinity gain avoidance methods similar to those in Gómez-Gutiérrez (2020) and Moulay, Léchappé, Bernuau, and Plestan (2021) to achieve zero-error prescribed time tracking, instead of the practical tracking with preset accuracy in this paper.

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