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ORIGINAL PAPER

Finite-time stability of multi-agent system in disturbed environment

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Abstract Finite-time stability problem of multi-agent system in disturbed environment is a question with practical significance. In this paper, a multi-agent system moving with obstacle avoidance is studied. The multi-agent system is expected to form a desired formation in finite time. Finite-time control law for continuous multi-agent system is proposed, which ensures that all the agents can pass the obstacles on their way, and the relative position between two agents reaches a constant value in finite time. Based on some notations and proposition given in the paper, the stability analysis is presented. Finally some simulations are presented to show the effectiveness of the method.

Keywords Multi-agent system · Finite time stability · Disturbed environment · Formation · Obstacle avoidance

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1 Introduction

In the last few years, finite-time stability problem of multi-agent system has attracted great attention. In many applications involving multi-agent systems, agents are required to agree on certain quantities of interest. In particular, it is desirable that a dynamical system converges to a stable state in finite time. More recently, researchers have considered finite-time stabilization of higher-order systems [1] as well as finite-time stabilization using output feedback [2].

Distributed decision-making for coordination of networks of dynamic agents involving information flow can be naturally captured by graph-theoretic notions. These dynamical network systems cover a very broad spectrum of applications including cooperative control of unmanned air vehicles (UAVs) [3], autonomous underwater vehicles (AUVs) [4], distributed sensor networks [5], air and ground transportation systems [6], swarms of air and space vehicle formations [7, 8], and congestion control in communication networks [9], to cite but a few examples. Hence, it is not surprising that a considerable research effort has been devoted to control of networks and control over networks in recent years [8, 10–16]. However, with the notable exception [17, 18], finite-time coordination has been addressed in the literature.

Most of those algorithms, however, do not treat obstacle avoidance in complex environments. Obstacle avoidance is fundamental for any practical mobile system, recent contributions to obstacle avoidance for multi-agent system are presented in [19–21]. In [19], the proposed treatment can not only ensure obstacle avoidance and flocking in the group but also allows for the agents to pursue additional objectives which are guaranteed to be met. General obstacles are allowed in [21], but the scheme is limited to rigid centralized leader-follower formations. The idea to map certain obstacle shapes to points and then apply a scheme using switching interaction potentials is discussed in [20].

Both finite-time stability and obstacle avoidance are important to multi-agent system. Our goal in this paper is to address finite-time stability of multi-agent system in disturbed environment. We develop a nonlinear finite-time control law using undirected graphs to accommodate for a full range of possible graph information topologies without limitations of bidirectional communication. With this control law, the multi-agent system can achieve coordination tasks in disturbed environment and in finite time. We demonstrate the effectiveness of our results.

This paper is organized as follows. The preliminary used in this paper is briefly described in the second section. The problem addressed in this paper and main results of our work are proposed in the third section. The results are verified via numerical simulations in the fourth section. And summary of the results and highlight of our key points are presented in the last section.

2 Preliminary

The distributed dynamical system studied in this paper consists of n autonomous agents, e.g. particles or robots, labeled 1 through n. All these agents share a common state space R^m, m is the dimension. $I_n = \{1, 2, \dots, n\}$, x_i denotes the state of agent $i, i \in I_n$, agent i is with the following dynamics:

$$\ddot{x}(t) = f\big(x(t)\big), \qquad x(0) = x_0, \qquad \dot{x}(0) = \dot{x}_0 \tag{1}$$

where $x(t) \in D \in R^m$.

In order to establish our main results, the following notations and definitions are needed.

2.1 Algebraic graph theory

In this multi-agent system, each agent can communicate with some other agents that are defined as its neighbors. We use a weighted undirected graph $g(A) = (\upsilon, \varepsilon, A)$ to represent the communication topology, where $A = [a_{ij}]$ is an $n \times n$ nonnegative symmetric matrix, $\upsilon = \{\upsilon_i : i \in I_n\}$ is the vertex set, and ε is the edge set. Vertex υ_i corresponds to agent i. An edge of $g(A)$ is denoted by (υ_i, υ_j), which is an unordered pair of vertices. $(\upsilon_i, \upsilon_j) \in \varepsilon \Leftrightarrow a_{ij} > 0 \Leftrightarrow$ agents i and j can communicate with each other, namely, they are adjacent. Moreover, we assume that $a_{ii} = 0$ for all $i \in I_n$. A is called the weight matrix and a_{ij} is the weight of edge (υ_i, υ_j). In consistence with the definition of agents' neighbors, the set of neighbors of vertex υ_i is denoted by $N_i = \{\upsilon_j : (\upsilon_j, \upsilon_i) \in \varepsilon\}$. A path in a graph from υ_i to υ_j is a sequence of distinct vertices starting with υ_i and ending with υ_j such that consecutive vertices are adjacent. A graph is connected if there is a path between any two vertices of the graph. More comprehensive discussions about graph can be found in [22].

2.2 Definitions and propositions

Definition 2.1 [23] An equilibrium point $x \in D$ of (1) is semistable if it is Lyapunov stable and there exists an open subset U of D containing x such that for all initial conditions in U, the trajectory of (1) converges to a Lyapunov stable equilibrium point, that is, $\lim_{t\to\infty} x(t) = y$, where $y \in D$ is a Lyapunov stable equilibrium point of (1) and $x \in U$. If, in addition, $U = D = R^m$, then an equilibrium point $x \in D$ of (1) is a globally semistable equilibrium. System (1) is said to be semistable if every equilibrium point of (1) is semistable. Finally, system (1) is said to be globally semistable if (1) is semistable and $U = D = R^m$.

Definition 2.2 $f^{-1}(0) \triangleq \{x \in D : \ddot{x}(t) = 0\}$. An equilibrium point $x_e \in f^{-1}(0)$ of (1) is said to be finite-time-semistable if there exist an open neighborhood $U \subseteq D$ of x_e and a function $T : U \backslash f^{-1}(0) \to (0, \infty)$, called the settling-time function, such that the following statements hold:

(i) For every $x \in U \backslash f^{-1}(0)$, for all $t \in [0, T(x)]$, $\lim_{t\to T(x)} x(t)$ exists and is contained in $U \cap f^{-1}(0)$.
(ii) x_e is semistable.

An equilibrium point $x_e \in f^{-1}(0)$ of (1) is said to be globally finite-time-semistable if it is finite-time-semistable with $U = D = R^m$. System (1) is said to be finite-time-semistable if every equilibrium point in $f^{-1}(0)$ is finite-time-semistable. Finally, system (1) is said to be globally finite-time-semistable if every equilibrium point in $f^{-1}(0)$ is globally finite-time-semistable.

In order to present Proposition 2.1 we consider a complete vector field ν on R^m such that the solutions of the differential equation $\ddot{y}(t) = \nu(y(t))$ define a continuous global flow $\tilde{\psi} : R \times R^m \to R^m$ on R^m, where $\nu^{-1}(0) \subseteq f^{-1}(0)$.

Proposition 2.1 [23] *Assume f is a homogeneous function of degree $k < 0$ with respect to ν. Furthermore, assume that there exists a weakly proper, continuous function $V : R^m \to R$ such that V is defined on R^m and satisfies $V(x) \leq 0$ for all $x \in R^m$. If every point in the largest invariant subset N of $V^{-1}(0)$ is a Lyapunov stable equilibrium point of system (1), then system (1) is finite-time-semistable.*

2.3 Some lemmas

In order to establish our main results, the following lemmas are needed.

Lemma 2.1 [12] *If $y_1, y_2, \ldots, y_n \geq 0$ and $0 < p \leq 1$, then*

$$\sum_{i=1}^{n} y_i^p \geq \left(\sum_{i=1}^{n} y_i \right)^p$$

Lemma 2.2 [12] *$L[A] = [l_{ij}] \in R^{m\times m}$ denotes the graph Laplacian of $g(A)$, which is defined by*

$$l_{ij} = \begin{cases} \sum_{k=1,k\neq i}^{n} a_{ik} & j = i \\ -a_{ij} & j \neq i \end{cases}$$

$L[A]$ has the following properties:

(1) *0 is an eigenvalue of $L[A]$ and 1 is the associated eigenvector, where $1 = [1, 1, \ldots, 1]^T \in R^m$.*
(2) *$x^T L[A]x = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x_j - x_i)^2$, and the semi-positive definiteness of $L[A]$ implies that all eigenvalues of $L[A]$ are real and not less than zero.*
(3) *If $g(A)$ is connected, the second smallest eigenvalue of $L[A]$, which is denoted by $\lambda_2(L_A)$ and called the algebraic connectivity of $g(A)$, is larger than zero.*

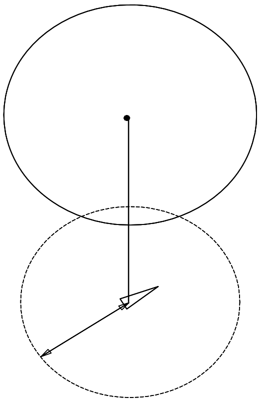


Fig. 1 Situation between agent and obstacle

(4) *The algebraic connectivity of $g(A)$ is equal to $\min_{x\neq 0, 1^T x=0} \frac{x^T L(A)x}{x^T x}$, and therefore, if $1^T x = 0$, then $x^T L(A)x \geq \lambda_2(L_A)x^T x$.*

3 Main results

Now system (1) is expressed as follows:

$$\ddot{x}_i(t) = u_i(t), \qquad x_i(0) = x_{i0}, \qquad \dot{x}_i(0) = \dot{x}_{i0} \tag{2}$$

where $x(t) \in D \in R^m$, $u_i(t)$ is the state feedback, called protocol, to be designed based on the state information received by agent i from its neighbors.

We seek a continuous distributed feedback control law involving transmission of both x_i and $\dot{x}_i$ between agents and a potential field term that is used for navigation in disturbed environment so that finite-time parallel formation is achieved: that is, the velocity $\dot{x}_i$ reaches a constant vector in finite time for all $i \in I_n$ and the relative position between two agents reaches a constant value in finite time.

$$u_i = u_{\text{iswarm}} + u_{\text{iobs}} \tag{3}$$

We restrict our study to obstacles that are connected convex regions in R^m with boundaries that are smooth manifolds. Specifically, we focus on obstacles that are spheres as shown in Fig. 1.

We present the agreement protocols that solve finite-time stability of multi-agent system in disturbed environment:

$$u_i = \sum_{j=1, j\neq i}^{n} \Phi_{ij}(\dot{x}_i, \dot{x}_j) - \sum_{j=1, j\neq i}^{n} A(i,j)\,\mathrm{sign}\big(\psi_\alpha(x_i, x_j)\big)\big|\psi_\alpha(x_i, x_j)\big|^{\frac{\alpha}{2-\alpha}} - \sum_{l=1}^{p} \nabla\sigma(x_i) \tag{4}$$

$$\Phi_{ij}(x_i, x_j) = A(i,j)\,\mathrm{sign}(x_j - x_i)|x_j - x_i|^{\alpha} \tag{5}$$

$$\psi_\alpha(x_i, x_j) = x_i - x_j - d_{ij}, \quad d_{ij} = -d_{ji} \tag{6}$$

$$\nabla\sigma(x_i) = \frac{k_\sigma}{(x_i - n_\sigma) - R_\sigma} \tag{7}$$

where d_{ij} is the desired distance between agent i and agent j, k_σ is repulsive force coefficient, n_σ is the situation that the obstacle is centered at, R_σ is the radius of the obstacle.

Theorem 3.1 *Consider the dynamical system given by (2). Assume that $A = A^T$. Then finite-time parallel formation with obstacles in environment is achieved under the distributed feedback control law given by (4).*

Proof For the distributed control law (4), let $z_{ij} \triangleq \psi_\alpha(x_i, x_j)$, $i \in I_n$, $i \neq j$, then

$$\dot{z}_{ij}(t) \triangleq \dot{x}_i(t) - \dot{x}_j(t), \qquad z_{ij}(0) = z_{ij0}, \quad t \geq 0 \tag{8}$$

$$\ddot{x}_i = \sum_{j=1, j\neq i}^{n} \Phi_{ij}(\dot{x}_i, \dot{x}_j) - \sum_{j=1, j\neq i}^{n} A(i,j)\,\mathrm{sign}\big(z_{ij}(t)\big)\big|z_{ij}(t)\big|^{\frac{\alpha}{2-\alpha}} - \sum_{l=1}^{p} \nabla\sigma(x_i) \tag{9}$$

$$\dot{x}_i(0) = \dot{x}_{i0}$$

The system given by (8) and (9) is homogeneous of degree $k = \frac{\alpha-1}{\alpha} < 0$ with dilation $(\lambda^{\frac{1}{\alpha}}\dot{x}_{ri}, \lambda^{\frac{2-\alpha}{\alpha}} z_{rij}, \lambda^{\frac{\alpha}{\alpha-2}}\nabla\sigma_i)$.

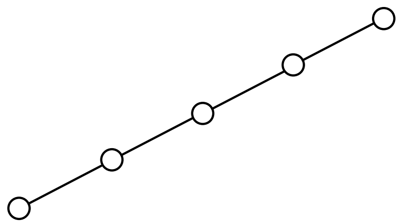


Fig. 2 Desired formation

Now consider the Lyapunov function

$$V(z, \dot{x}) = \frac{1}{2}\sum_{i=1}^{n} \dot{x}_i^2 + \frac{2-\alpha}{4}\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} A(i,j)\big|z_{ij}(t)\big|^{\frac{2}{2-\alpha}} + \sum_{i=1}^{n}\sum_{l=1}^{p}\sigma(x_i) \tag{10}$$

The derivative of V along the trajectories is

$$\begin{aligned}\dot{V}(z, \dot{x}) &= \sum_{i=1}^{n} \dot{x}_i \sum_{j=1, j\neq i}^{n} \Phi_{ij}(\dot{x}_i, \dot{x}_j) \\ &\quad - \sum_{i=1}^{n} \dot{x}_i \sum_{j=1, j\neq i}^{n} A(i,j)\,\mathrm{sign}\big(\psi_\alpha(x_i, x_j)\big) \\ &\quad \times \big|\psi_\alpha(x_i, x_j)\big|^{\frac{\alpha}{2-\alpha}} \\ &\quad - \sum_{i=1}^{n} \dot{x}_i \sum_{l=1}^{p} \nabla\sigma(x_i) \\ &\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} A(i,j)\,\mathrm{sign}\big(\psi_\alpha(x_i, x_j)\big) \\ &\quad \times \big|\psi_\alpha(x_i, x_j)\big|^{\frac{\alpha}{2-\alpha}}(\dot{x}_i - \dot{x}_j) \\ &\quad + \sum_{i=1}^{n}\sum_{l=1}^{p} \nabla\sigma(x_i)\dot{x}_i \\ &= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} (\dot{x}_i - \dot{x}_j)\Phi_{ij}(\dot{x}_i, \dot{x}_j)\end{aligned}$$

By (5),

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} (\dot{x}_i - \dot{x}_j)\Phi_{ij}(\dot{x}_i, \dot{x}_j) \leq 0$$

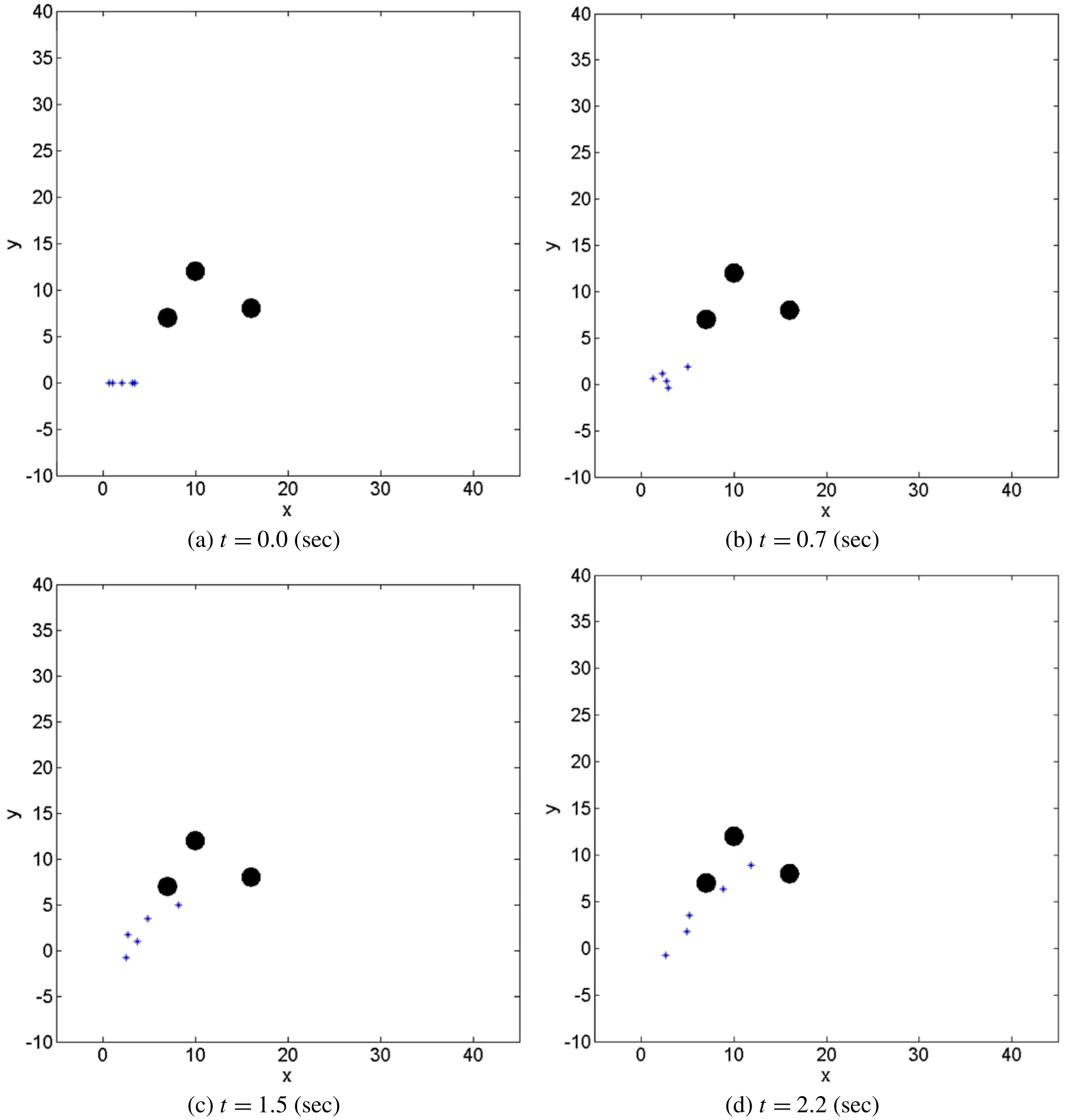


Fig. 3 Movements of agents in disturbed environment

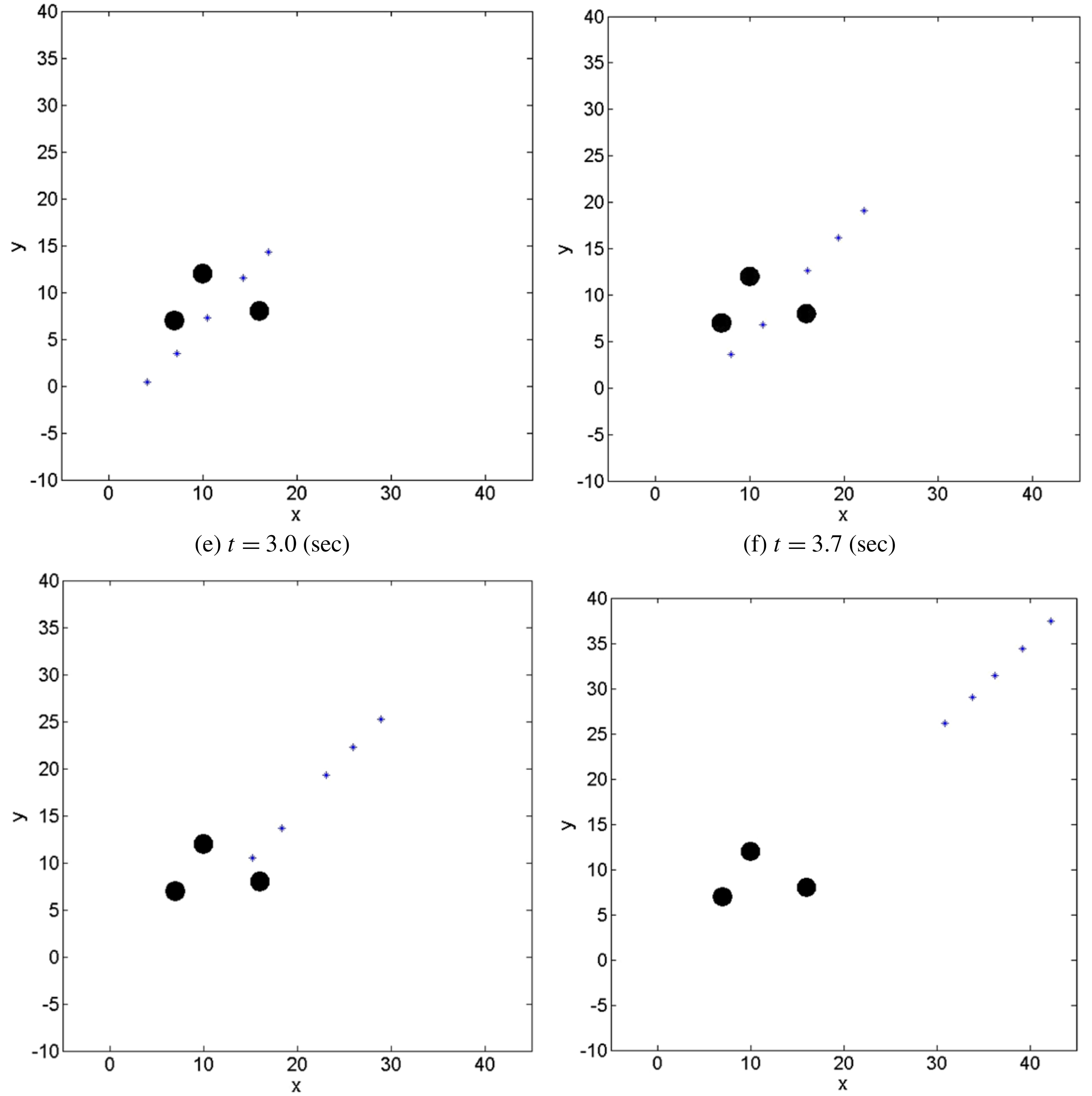
that is, $\dot{V}(z, \dot{x}) \leq 0$.

Let

$$\begin{aligned}\Re &\triangleq \left\{(z, \dot{x}) : \dot{V}(z, \dot{x}) = 0\right\} \\ &= \left\{(z, \dot{x}) : \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\dot{x}_i - \dot{x}_j)\Phi_{ij}(\dot{x}_i, \dot{x}_j) = 0, \right. \\ &\qquad \left. i = 1, 2, \ldots, n-1 \right\}\end{aligned}$$

$$\Re \triangleq \left\{(z, \dot{x}) : \dot{x}_1 = \cdots = \dot{x}_n\right\}$$

Since $\dot{x}_1 = \cdots = \dot{x}_n$, it follows that $\dot{z}_{ij}(t) = 0$, $i \in I_n$, $i \neq j$.



(e) $t = 3.0$ (sec)

(f) $t = 3.7$ (sec)

(g) $t = 4.5$ (sec)

(h) $t = 6.0$ (sec)

Fig. 3 (*Continued*)

Let M denote the largest invariant set contained in $\Re$.

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{n}\dot{x}_i^2$$

$$= \dot{V} - \frac{2-\alpha}{4}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} A(i,j)\frac{d}{dt}\left|z_{ij}(t)\right|^{\frac{2}{2-\alpha}}$$

$$-\sum_{i=1}^{n}\sum_{l=1}^{p}\frac{d}{dt}\sigma(x_i) = 0$$

which implies that $\dot{x}_1 = \cdots = \dot{x}_n = c$.

Finally, for each $i \in I_n$, $z_{ij} = -z_{ji}$ and $\dot{z}_{ij} = 0$, it follows that $z_{ij} = 0$.

To show Lyapunov stability of $\dot{x}(t) = ce$ and $z(t) = 0$, consider the shifted Lyapunov function candidate:

$$\tilde{V}(z,\dot{x}) = \frac{1}{2}\sum_{i=1}^{N}(\dot{x}_i - c)^2 + \frac{2-\alpha}{4}\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} A(i,j)\left|z_{ij}(t)\right|^{\frac{2}{2-\alpha}} + \sum_{i=1}^{N}\sum_{l=1}^{p}\sigma(x_i)$$

The proof follows using identical arguments as above. Then the derivative of $\tilde{V}$ along the trajectories is $\dot{\tilde{V}} \leq 0$, which implies that every point in M is a Lyapunov stable equilibrium point of system (2). Hence, it follows from Proposition 2.1 that system (2) is finite-time-semistable. That is to say, finite-time parallel formation may be achieved. □

4 Simulation results

In this section the simulation example is presented. The system consists of five agents, for the protocol α is 0.6 and the initial state and initial speed is a random set.

The communication topologies of the system are time-varying. From observations of nature swarms we can see that an agent within a swarm system tends to pay more attention to nearby agents, the further away, the less attention it is given. From this observation we choose for any agent i, it always associates with the nearest two agents.

The desired formation of this group illustrated in Fig. 2.

The matrix of obstacles is

$$M_s = \begin{pmatrix} 16 & 10 & 7 \\ 8 & 12 & 7 \\ 1 & 1 & 1 \end{pmatrix}$$

The movement process of agents under control law (4) is shown in Fig. 3. The estimated settling time is 6.0 seconds. From Fig. 3 we can see that all the agents can pass the obstacles on their way, the five agents can form the desired formation in a short period and move together with the stable formation. The relative position between two agents reaches a constant value in finite time.

5 Conclusion

This paper discussed finite-time stability of multi-agent system in disturbed environment. Finite-time control law to make multi-agent system reach the parallel formation in finite time with obstacles avoidance was established along with the stability analysis, and finally, the simulation confirmed our theoretical results.

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