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Fixed-time output stabilization and fixed-time estimation of a chain of integrators

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Summary

A solution to the problem of global fixed-time output stabilization and estimation of a chain of integrators is proposed. A nonlinear scheme comprising a state feedback controller and a dynamic observer are designed to guarantee both fixed-time estimation and fixed-time control. Robustness with respect to exogenous disturbances and measurement noises is established, and a parameter optimization algorithm is provided. The performance of the obtained control and estimation algorithms are illustrated by numeric experiments.

KEYWORDS

chain of integrators, finite-time stability, homogeneous systems, implicit Lyapunov functions

1 | INTRODUCTION

Feedback stabilization of linear and nonlinear systems is one of the central problems in control systems theory. There are many methods to design a control,[1-3] which differ in the requirements imposed on the plant model and on the guaranteed performance of the closed-loop system. Among performance criteria, the robustness with respect to external disturbances or measurement noises and the rate of convergence are the most popular. To assess robustness of nonlinear systems the input-to-state stability theory is frequently used.[4,5] The rate of convergence to the goal state or set can be asymptotic (eg, exponential) or nonasymptotic, ie, finite time or fixed time.[6-8] In the latter case, the system converges from any initial conditions of the state space in a uniform finite time (see more rigorous definitions as follows).

The present work studies the problem of output feedback design providing a fixed-time rate of convergence for a chain of integrators.[9] The chain of integrators is a very versatile and well-studied system since all linear controllable systems and many nonlinear ones can be transformed, through a linear coordinate transformation in the first case and through feedback linearization in the second, into this particular form.[10,11] Fixed-time stability, ie, globally bounded convergence rate, is a relatively recent topic,[12-14] and in the work of Basin et al,[15] a fixed-time feedback regulator for a chain of integrators is presented based on Lyapunov analysis. Here, we present a full control scheme, endowed with a state and output feedback control and with a state observer, all of which exhibit the fixed-time stability property. Instead of using a Lyapunov approach, the proposed solution is based on homogeneity theory,[3,16]; and contrarily to the work of Polyakov et al,[9] the presented control has an explicit form. Fixed-time stability of the chain of integrators has also been achieved in the works of Harmouche et al[17] and Ríos and Teel.[18] In both works, a switching in the homogeneous degree is applied to obtain fixed-time stability, and while the work of Harmouche et al[17] used a recursive controller to complete a homogeneous system, Ríos and Teel[18] used the hybrid formulation. Nonetheless, to the best of our knowledge, there is not any work that fully discloses the parameter tuning and its relationship with the settling time estimates. In contrast, in this paper, we provide a homogeneous nonrecursive fixed-time output feedback controller, and we provide a tuning algorithm, based on linear matrix inequalities (LMIs) and on the implicit Lyapunov function approach, that allows to assert fixed-time stability and to adjust the settling time. In addition, the obtained control and estimation algorithms do not have a complex computational structure usually inherited by the solutions based on the implicit Lyapunov function method.

A robustness analysis with respect to bounded external disturbances and measurement noises, based on previous results on homogeneous systems,[19] is also carried out.

The outline of this paper is as follows. Notation and preliminary results are introduced in Sections 2 and 3. The precise problem statement is given in Section 4. The proposed algorithms are presented in Section 5, and the parameter optimization is treated in Section 6. Concluding remarks and the discussion appear in Section 7.

2 | NOTATION

Throughout this paper, the following notations are used.

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where $\mathbb{R}$ is the set of real numbers.
- $|\cdot|$ denotes the absolute value in $\mathbb{R}$, $\|.\|$ denotes the Euclidean norm on $\mathbb{R}^n$, $\|x\|_{\mathcal{A}} = \inf_{\xi\in\mathcal{A}}\|x-\xi\|$ is the distance from a point $x \in \mathbb{R}^n$ to a set $\mathcal{A} \subset \mathbb{R}^n$, and $\|A\|_2$ is the induced matrix norm for $A \in \mathbb{R}^{n\times n}$.
- For a (Lebesgue) measurable function $d : \mathbb{R}_+ \to \mathbb{R}^m$, we use $||d||_{[t_0,t_1)} = \text{ess}\sup_{t\in[t_0,t_1)}\|d(t)\|$ to define the norm of $d(t)$ in the interval $[t_0, t_1)$, then $||d||_\infty = ||d||_{[0,+\infty)}$ and the set of essentially bounded measurable functions $d(t)$ with the property $||d||_\infty < +\infty$ further denoted as $\mathcal{L}_\infty$; $\mathcal{L}_D = \{d \in \mathcal{L}_\infty : ||d||_\infty \leq D\}$ for any $D > 0$.
- A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is increasing to infinity. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each fixed $t \in \mathbb{R}_+$ and $\lim_{t\to+\infty}\beta(s, t) = 0$ for each fixed $s \in \mathbb{R}_+$.
- The notation $DV(x)f(x)$ stands for the directional derivative of a continuously differentiable function V with respect to the vector field f evaluated at point x.
- A series of integers $1, 2, \ldots, n$ is denoted by $\overline{1, n}$.

3 | PRELIMINARIES

Consider the following nonlinear system:

$$\dot{x}(t) = f(x(t), d(t)),\ t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $d(t) \in \mathbb{R}^m$ is the input, $d \in \mathcal{L}_\infty$; $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ ensures forward existence of the system solutions at least locally, and $f(0, 0) = 0$. For an initial condition $x_0 \in \mathbb{R}^n$ and input $d \in \mathcal{L}_\infty$, define the corresponding solution by $X(t, x_0, d)$ for any $t \geq 0$ for which the solution exists.

Following other works,[7,8,10] let Ω be an open neighborhood of the origin in $\mathbb{R}^n$ and $D > 0$.

Definition 1. At the steady state $x = 0$, the system (1) for $d \in \mathcal{L}_D$ is said to be

1. *uniformly Lyapunov stable* if, for any $x_0 \in \Omega$ and $d \in \mathcal{L}_D$, the solution $X(t, x_0, d)$ is defined for all $t \geq 0$, and for any $\epsilon > 0$, there is $\delta > 0$ such that, for any $x_0 \in \Omega$, if $\|x_0\| \leq \delta$, then $\|X(t, x_0, d)\| \leq \epsilon$ for all $t \geq 0$;
2. *uniformly asymptotically stable* if it is uniformly Lyapunov stable, and for any $\kappa > 0$ and $\epsilon > 0$, there exists $T(\kappa, \epsilon) \geq 0$ such that, for any $x_0 \in \Omega$ and $d \in \mathcal{L}_D$, if $\|x_0\| \leq \kappa$, then $\|X(t, x_0, d)\| \leq \epsilon$ for all $t \geq T(\kappa, \epsilon)$;
3. *uniformly finite-time stable* if it is uniformly Lyapunov stable and *uniformly finite-time converging from* Ω, ie, for any $x_0 \in \Omega$ and all $d \in \mathcal{L}_D$, there exists $0 \leq T < +\infty$ such that $X(t, x_0, d) = 0$ for all $t \geq T$. The function $T_0(x_0) = \inf\{T \geq 0 : X(t, x_0, d) = 0,\ \forall t \geq T,\ \forall d \in \mathcal{L}_D\}$ is called the *uniform settling time* of the system (1);
4. *uniformly fixed-time stable* if it is uniformly finite-time stable and $\sup_{x_0\in\Omega} T_0(x_0) < +\infty$.
   The set Ω is called the *domain of stability/attraction*.

If $\Omega = \mathbb{R}^n$, then the corresponding properties are called *global* uniform Lyapunov/asymptotic/finite-time/fixed-time stability of (1) for $d \in \mathcal{L}_D$ at $x = 0$.

Stability notions can be similarly defined with respect to a set by replacing the distance to the origin in Definition 1 with the distance to an invariant set. For example, the global uniform finite-time stability with respect to a set $\mathcal{A} \subset \mathbb{R}^n$ is equivalent to the following two properties:

i. *uniform Lyapunov stability*: for any $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_D$, the solution $X(t, x_0, d)$ is defined for all $t \geq 0$, and for any $\epsilon > 0$, there is $\delta > 0$ such that, if $\|x_0\|_{\mathcal{A}} \leq \delta$, then $\|X(t, x_0, d)\|_{\mathcal{A}} \leq \epsilon$ for all $t \geq 0$;
ii. *uniform finite-time convergence*: for any $x_0 \in \mathbb{R}^n$ and all $d \in \mathcal{L}_D$, there exists $0 \leq T < +\infty$ such that $\|X(t, x_0, d)\|_{\mathcal{A}} = 0$ for all $t \geq T$.

3.1 | Input-to-state stability

More details about this theory can be found in the work of Dashkovskiy et al.[5]

Definition 2. The system (1) is called *input-to-state practically stable* (ISpS), if, for any input $d \in \mathcal{L}_\infty$ and any $x_0 \in \mathbb{R}^n$, there are some functions $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$, and $c \geq 0$ such that

$$\|X(t, x_0, d)\| \leq \beta(\|x_0\|, t) + \gamma\left(\|d\|_{[0,t)}\right) + c, \quad \forall t \geq 0.$$

The function γ is called *nonlinear asymptotic gain*. The system is called *input-to-state stable* (ISS) if $c = 0$.

Definition 3. A smooth function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called *ISpS Lyapunov function* for the system (1) if, for all $x \in \mathbb{R}^n$, $d \in \mathbb{R}^m$ and some $r \geq 0$, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, and $\theta \in \mathcal{K}$

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad DV(x)f(x, d) \leq r + \theta(\|d\|) - \alpha_3(\|x\|).$$

Such a function V is called *ISS Lyapunov function* if $r = 0$.

Note that an ISS Lyapunov function can also satisfy the following equivalent condition for some $\chi \in \mathcal{K}_\infty$:

$$\|x\| > \chi(\|d\|) \Rightarrow DV(x)f(x, d) \leq -\alpha_3(\|x\|).$$

Theorem 1. *The system (1) is ISS (ISpS) if and only if it admits an ISS (ISpS) Lyapunov function.*

3.2 | Weighted homogeneity

Following the works of Bacciotti and Rosier[3] and Zubov,[20] for strictly positive numbers r_i, $i = \overline{1, n}$ called weights and $\lambda > 0$, one can define the following:

- the *vector of weights* $r = (r_1, \ldots, r_n)^T$, $r_{\max} = \max_{1 \leq j \leq n} r_j$, and $r_{\min} = \min_{1 \leq j \leq n} r_j$;
- the *dilation matrix* function $\Lambda_r(\lambda) = \mathrm{diag}\{\lambda^{r_i}\}_{i=1}^n$; note that $\forall x \in \mathbb{R}^n$ and $\forall \lambda > 0$, we have $\Lambda_r(\lambda)x = (\lambda^{r_1} x_1, \ldots, \lambda^{r_i} x_i, \ldots, \lambda^{r_n} x_n)^T$;
- the r-homogeneous norm $\|x\|_r = (\sum_{i=1}^n |x_i|^{\frac{\rho}{r_i}})^{\frac{1}{\rho}}$ for any $x \in \mathbb{R}^n$ and $\rho \geq r_{\max}$; hence, there exist some $\underline{\sigma}, \overline{\sigma} \in \mathcal{K}_\infty$ such that

$$\underline{\sigma}(\|x\|_r) \leq \|x\| \leq \overline{\sigma}(\|x\|_r), \quad \forall x \in \mathbb{R}^n.$$

- the *unit sphere* and *a ball in the homogeneous norm* $S_r = \{x \in \mathbb{R}^n : \|x\|_r = 1\}$ and $\mathcal{B}_r(\rho) = \{x \in \mathbb{R}^n : \|x\|_r \leq \rho\}$ for $\rho \geq 0$.

Definition 4. A function $g : \mathbb{R}^n \to \mathbb{R}$ is *r-homogeneous with degree* $\eta \in \mathbb{R}$ if $\forall x \in \mathbb{R}^n$ and $\forall \lambda > 0$, we have

$$\lambda^{-\eta} g(\Lambda_r(\lambda)x) = g(x).$$

A vector field $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is *r-homogeneous with degree* $\nu \in \mathbb{R}$, with $\nu \geq -r_{\min}$ if $\forall x \in \mathbb{R}^n$ and $\forall \lambda > 0$, we have

$$\lambda^{-\nu} \Lambda_r^{-1}(\lambda) \phi(\Lambda_r(\lambda)x) = \phi(x),$$

which is equivalent to the ith component of ϕ being an r-homogeneous function of degree $r_i + \nu$.

The system (1) for $d = 0$ is r-homogeneous of degree ν if the vector field f is r-homogeneous of degree ν for $d = 0$.

The property of r-homogeneity can also be defined not for all $x \in \mathbb{R}^n$ but for a subset of the state space (or it can be approximately satisfied[21,22]).

An important advantage of r-homogeneous systems is that their rate of convergence can be evaluated qualitatively based on their degree of homogeneity.

Lemma 1. (See the work of Nakamura et al[23])
If (1) for $d = 0$ is r-homogeneous of degree ν and asymptotically stable at the origin, then it is

i. *globally finite-time stable at the origin if $\nu < 0$;*
ii. *globally exponentially stable at the origin if $\nu = 0$;*
iii. *globally fixed-time stable with respect to the unit ball $\mathcal{B}_r(1)$ if $\nu > 0$.*

Define $\tilde{f}(x,d) = [f(x,d)^T \ 0_m]^T \in \mathbb{R}^{n+m}$; it is an extended auxiliary vector field for the system (1), where 0_m is the zero vector of dimension m.

Theorem 2. (See the work of Bernuau et al[19])
Let the vector field $\tilde{f}$ be homogeneous with the weights $r = [r_1, \ldots, r_n]^T > 0$, $\tilde{r} = [\tilde{r}_1, \ldots, \tilde{r}_m]^T > 0$ with a degree $\nu \geq -r_{\min}$, ie, $f(\Lambda_r(\lambda)x, \Lambda_{\tilde{r}}(\lambda)d) = \lambda^\nu \Lambda_r(\lambda) f(x,d)$ for all $x \in \mathbb{R}^n$, $d \in \mathbb{R}^m$, and all $\lambda > 0$. Assume that the system (1) is globally asymptotically stable for $d = 0$; then, the system (1) is ISS.

Therefore, for the homogeneous system (1), its ISS property follows from its asymptotic stability for $d = 0$ (as for linear systems[5]).

3.3 | Implicit Lyapunov function method

Theorem 3. (See the work of Plyakov et al[14])
If there exist two continuous functions $Q_i : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, $i = \{1,2\}$ such that

C1 *$Q_i(V,e)$ are continuously differentiable on $\mathbb{R}_+ \times \mathbb{R}^n \backslash \{0\}$;*
C2 *for any $x \in \mathbb{R}^n \backslash \{0\}$, there exists $V > 0$ such that $Q_i(V,x) = 0$;*
C3 *for $\Omega = \{(V,x) \in \mathbb{R}^{n+1} : Q_i(V,x) = 0\}$, we have*

$$\lim_{\substack{x\to 0\\(V,x)\in\Omega}} V = 0, \quad \lim_{\substack{V\to 0^+\\(V,x)\in\Omega}} \|x\| = 0, \quad \lim_{\substack{\|x\|\to\infty\\(V,x)\in\Omega}} V = +\infty.$$

C4 *for all $V > 0$ and for all $x \in \mathbb{R}^n \backslash \{0\}$, the inequality $-\infty < \frac{\partial Q_i(V,x)}{\partial V} < 0$ holds;*
C5 *$Q_1(1,x) = Q_2(1,x)$ for all $x \in \mathbb{R}^n \backslash \{0\}$;*
C6 *$\frac{\partial Q_1(V,x)}{\partial x} f(t,x) \leq c_1 V^{1-\mu} \frac{\partial Q_1(V,x)}{\partial V}$ for all $t \in \mathbb{R}$, for all $V \in (0,1]$, and for all $x \in \{z \in \mathbb{R}^n \backslash \{0\} : Q_1(V,z) = 0\}$, where $c_1 > 0$ and $0 < \mu < 1$ are some constants;*
C7 *$\frac{\partial Q_2(V,x)}{\partial x} f(t,x) \leq c_2 V^{1+\nu} \frac{\partial Q_2(V,x)}{\partial V}$ for all $t \in \mathbb{R}$, for all $V \geq 1$, and for all $x \in \{z \in \mathbb{R}^n \backslash \{0\} : Q_2(V,z) = 0\}$, where $c_2 > 0$ and $\nu > 0$ are some constants.*

Then, the origin of (1) is fixed-time stable with the settling time estimate $T_0 \leq \frac{1}{c_1\mu} + \frac{1}{c_2\nu}$.

4 | PROBLEM STATEMENT

Consider a chain of integrators

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + bu(t) + d(t), \ t \geq 0, \\ y(t) &= Cx(t) + v(t), \end{aligned} \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $u(t) \in \mathbb{R}$ is the control input; $y(t) \in \mathbb{R}$ is the measured output; $d(t) \in \mathbb{R}^n$ and $v(t) \in \mathbb{R}$ are the exogenous disturbance and the measurement noise, respectively, $(d,v) \in \mathcal{L}_\infty$; and the matrices

$$A_0 = \begin{bmatrix} 0 & 1 & 0\ldots0 & 0 \\ 0 & 0 & 1\ldots0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0\ldots0 & 1 \\ 0 & 0 & 0\ldots0 & 0 \end{bmatrix}, \ b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \ C = [1 \ \ 0\ldots0],$$

are in the upper diagonal canonical form.

It is required to design a stabilizing dynamic output control $u(t)$ that ensures the ISpS property of the system (1) for any $(d,v) \in \mathcal{L}_\infty$, and that, for $d = v = 0$, provides global fixed-time stability of the closed-loop system at the origin.

5 | MAIN RESULT

The solution of the problem is divided in three steps. First, a state feedback is proposed ensuring the problem solution. Second, the equations of the observer are introduced. Third, a combined output feedback is presented and analyzed.

5.1 | State feedback

For $i = \overline{1,n}$, $x_i \in \mathbb{R}$ and $\alpha > 0$ and denote $\lceil x_i \rfloor^\alpha = |x_i|^\alpha \mathrm{sign}(x_i)$, then the control proposed in this work has the form

$$u(x) = \sum_{i=1}^{n} a_i \lceil x_i \rfloor^{\alpha_i(\nu(\|x\|))}, \tag{3}$$

$$\alpha_i(\nu) = \frac{1+n\nu}{1+(i-1)\nu},$$

$$\nu(\omega) = \begin{cases} \nu_1, & \text{if } \omega \le m, \\ \nu_2, & \text{if } \omega \ge M, \\ \frac{\nu_2-\nu_1}{M-m}\omega + \frac{M\nu_1 - m\nu_2}{M-m} & \text{otherwise,} \end{cases} \tag{4}$$

where $a = [a_1, \dots, a_n] \in \mathbb{R}^{1\times n}$ is the vector of control coefficients forming a Hurwitz polynomial, $-\infty < \nu_1 < 0 < \nu_2 < +\infty$, and $0 < m < M < +\infty$ are the tuning parameters to be defined later. Denote

$$r_i(\nu) = 1 + (i-1)\nu, \quad i = \overline{1,n}. \tag{5}$$

Then, it is straightforward to verify that, for $d = 0$, the systems (2) and (3) are $r(\nu_1)$-homogeneous of degree $\nu_1 < 0$ for $\|x\| \le m$ and $r(\nu_2)$-homogeneous of degree $\nu_2 > 0$ for $\|x\| \ge M$. Let us show that, for properly selected control parameters, the systems (2) and (3) are globally fixed-time stable at the origin.

Lemma 2. *Let $a \in \mathbb{R}^n$ form a Hurwitz polynomial; then, for any $0 < m < M < +\infty$, there exists $\tau \in (0, n^{-1})$ such that, if $\nu_1 \in (-\tau, 0)$ and $\nu_2 \in (0, \tau)$, then the systems (2) and (3) for $d = 0$ are globally fixed-time stable at the origin.*

Proof. Denote

$$A = A_0 + ba = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ a_1 & a_3 & a_3 & \cdots & a_{n-1} & a_n \end{bmatrix}.$$

Then, by the lemma conditions, there are matrices $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$ such that

$$P = P^T > 0, \; Q = Q^T > 0, \; A^T P + PA = -Q.$$

Following the work of Bhat and Bernstein,[24] consider for (2) and (3) a Lyapunov function

$$V(x) = x^T P x,$$

whose derivative admits a differential equation

$$\dot{V}(x) = DV(x)[A_0 x + bu(x)] = -x^T Q x + 2x^T P b \delta(x),$$

where $\delta(x) = \sum_{i=1}^n a_i(\lceil x_i \rfloor^{\alpha_i(\nu(\|x\|))} - x_i)$. By construction, $\nu(\|x\|) = 0$ for $\|x\| = \mu = \frac{m\nu_2 - M\nu_1}{\nu_2 - \nu_1}$. Since $\delta : \mathbb{R}^n \to \mathbb{R}$ is a continuous function and $\nu : \mathbb{R}_+ \to [\nu_1, \nu_2]$, it is possible to select the values of ν_1 and ν_2 sufficiently close to zero such that $|\delta(x)|$ stays small enough, and therefore, $\dot{V}(x) < 0$ on any compact containing the level $\|x\| = \mu$. Thus, there exists $\tau \in (0, n^{-1})$ (if $\tau \ge n^{-1}$, then $\alpha_i(\nu(\|x\|))$ may become nonpositive) such that, with $\nu_1 \in (-\tau, 0)$ and $\nu_2 \in (0,\tau)$ for all $x \in \{x \in \mathbb{R}^n : \underline{m} \le \|x\| \le \overline{M}\}$, we have $\dot{V}(x) < 0$ for any selection of $0 < \underline{m} < m < M < \overline{M} < +\infty$. Using the arguments of the work of Bhat and Bernstein[24] and Lemma 1, we can prove in this case that the system is $r(\nu_1)$-homogeneous of degree $\nu_1 < 0$ and fixed-time stable at the origin from $\mathcal{B}_{r(\nu_1)}(\rho_1)$ for $\mathcal{B}_{r(\nu_1)}(\rho_1) \subset$

$\{x \in \mathbb{R}^n : \|x\| \leq m\}$, and it is $r(\nu_2)$-homogeneous of degree $\nu_2 > 0$ and globally fixed-time stable with respect to any ball $\mathcal{B}_{r(\nu_2)}(\rho_2)$ such that $\{x \in \mathbb{R}^n : \|x\| = M\} \subset \mathcal{B}_{r(\nu_2)}(\rho_2)$. The constants $\tau, \underline{m}, \overline{M}$ can be selected in a way that $\mathcal{B}_{r(\nu_1)}(\rho_1) \subset \{x \in \mathbb{R}^n : \underline{m} \leq \|x\| \leq m\}$ and $\mathcal{B}_{r(\nu_2)}(\rho_2) \subset \{x \in \mathbb{R}^n : M \leq \|x\| \leq \overline{M}\}$; then, (2) and (3) are globally convergent, and they are globally fixed-time stable at the origin (the time that the system spent in the set $\{x \in \mathbb{R}^n : \underline{m} \leq \|x\| \leq \overline{M}\}$ is finite). □

In order to analyze robust stability properties of the closed-loop dynamics (2) and (3), let us introduce

$$f_\nu(x, \tilde{d}) = A_0 x + b \sum_{i=1}^{n} a_i \lceil x_i + \tilde{d}_{1,i} \rfloor^{\alpha_i(\nu)} + \tilde{d}_2,$$

where $\tilde{d} = [\tilde{d}_1^T\ \tilde{d}_2^T]^T \in \mathbb{R}^{2n}$ is the new disturbance input, $\tilde{d}_1$ represents measurements noises, and $\tilde{d}_2 = d$.

Corollary 1. *Let all conditions of Lemma 2 be satisfied; then, the systems (2) and (3) are ISpS for any $\tilde{d} \in \mathcal{L}_\infty$.*

Proof. Consider the systems (2) and (3) for $\|x\| \geq M$, and $\dot{x} = f_{\nu_2}(x, \tilde{d})$ is the corresponding approximating system, which is $r(\nu_2)$-homogeneous of degree $\nu_2 > 0$ and globally fixed-time stable with respect to any ball $\mathcal{B}_{r(\nu_2)}(\rho)$ with $\rho > 0$ for $\tilde{d} = 0$. Take $\tilde{r} = \begin{bmatrix} r(\nu_2) \\ r(\nu_2) + \nu_2 \end{bmatrix}$; then, $f_{\nu_2}(\Lambda_r(\lambda)x, \Lambda_{\tilde{r}}(\lambda)\tilde{d}) = \lambda^\nu \Lambda_r(\lambda) f_{\nu_2}(x, \tilde{d})$ for all $x \in \mathbb{R}^n$, $\tilde{d} \in \mathbb{R}^{2n}$, and all $\lambda > 0$. Consequently, if all conditions of Lemma 2 are satisfied, then also all conditions of Theorem 2 are true and the system $\dot{x} = f_{\nu_2}(x, \tilde{d})$ is ISS with respect to $\tilde{d} \in \mathcal{L}_\infty$. Since $\dot{x} = f_{\nu_2}(x, \tilde{d})$ is the approximation of (2) and (3) for $\|x\| \geq M$, then (2) and (3) (the system $\dot{x} = f_{\nu(\|x\|)}(x, \tilde{d})$) are ISpS. □

Thus, the presented state control (3) solves the posed problem of robust global fixed-time stabilization for the system (2). Obviously, for any $R \in \mathbb{R}^{n \times n}, R = R^T > 0$, the systems (2) and (3) with $\nu(\|x\|_R)$, where $\|x\|_R = \sqrt{x^T R x}$, possess the same properties as (2) and (3) with $\nu(\|x\|)$.

5.2 | State observer

To explain the observer structure, let us first consider the case $d = v = 0$; then, the proposed observer takes the form (see also other works[12,13,18])

$$\begin{aligned}
\dot{z}(t) &= A_0 z(t) + bu(t) + k\left(\nu(\zeta(t)), y(t) - Cz(t)\right), \\
k_i(\nu, e) &= L_i \lceil e \rfloor^{\beta_i(\nu)}, \ \ \beta_i(\nu) = 1 + i\nu \quad i = \overline{1, n}, \\
\dot{\zeta}(t) &= -0.5\zeta(t) + p\left(\nu(\zeta(t)), y(t) - Cz(t)\right), \\
p(\nu, e) &= 4\kappa^T(\nu, e) P \kappa(\nu, e), \ \ \kappa(\nu, e) = Le - k(\nu, e),
\end{aligned} \tag{6}$$

where $z(t) \in \mathbb{R}^n$ is the state estimate; $\zeta(t) \in \mathbb{R}_+$ is an auxiliary time function; the function ν is given in (4) with $-\infty < \nu_1 < 0 < \nu_2 < +\infty$ and $0 < m < M < +\infty$, are, as previously, the tuning parameters; $L = [L_1, \ldots, L_n]^T$ is the vector of coefficients of the observer providing Hurwitz property of the matrix $A_0 - LC$; $P \in \mathbb{R}^{n \times n}$ is a matrix solution of the equations

$$P = P^T > 0, \ (A_0 - LC)^T P + P(A_0 - LC) = -P.$$

In the work of Angulo et al,[13] instead of using an auxiliary ζ-filter to compute the right-hand sides of (6) with negative and positive homogeneity degree, a time switching between two systems with positive and negative homogeneity degree is proposed. In the work of Ríos and Teel,[18] in order to switch between observers with negative and positive homogeneity degrees, a comparison of output estimation errors for these observers is used.

Lemma 3. *Let $A_0 - LC$ be a Hurwitz matrix for a given $L \in \mathbb{R}^{n \times 1}$ and assume that the solutions of (2) are defined for all $t \geq 0$; then, for any $0 < m < M < +\infty$, there exists $\tau \in (0, n^{-1})$ such that, if $\nu_1 \in (-\tau, 0)$ and $\nu_2 \in (0, \tau)$, then the systems (2) and (6) for $d = v = 0$ are globally Lyapunov stable and fixed-time convergent with respect to the set $\mathcal{A} = \{(x, z, \zeta) \in \mathbb{R}^{2n+1} : x = z, \|\zeta\| \leq M\}$ for all $(x, z) \in \mathbb{R}^{2n}$, provided that $\zeta(0) > M$ is sufficiently big.*

A more precise restriction on the value of $\zeta(0)$ is given in the proof of this lemma; it is not related with the initial conditions $x(0)$ and $z(0)$ (see (9)).

Proof. Denote $e = x - z$ as the estimation error; then,

$$\begin{aligned}\dot{e} &= A_0 e - k(v(\zeta), Ce + v) + d \\ &= (A_0 - LC)e - k(v(\zeta), Ce + v) + L(Ce + v) + d - Lv \\ &= (A_0 - LC)e + \kappa(v(\zeta), Ce + v) + d - Lv.\end{aligned}$$

Consider a Lyapunov function $V(e) = e^T Pe$; then,

$$\begin{aligned}\dot{V} &= -V + 2e^T P[\kappa(v(\zeta), Ce + v) + d - Lv] \\ &\leq -0.5V + p(v(\zeta), Ce + v) + 4(d - Lv)^T P(d - Lv).\end{aligned}$$

For an auxiliary error variable $\psi = V - \zeta$, we obtain

$$\dot{\psi} \leq -0.5\psi + 4(d - Lv)^T P(d - Lv)$$

and ψ is exponentially converging to zero (ζ is converging to V) if $d = v = 0$. In addition, if there is an instant of time $t' \geq 0$ such that $\psi(t') \geq 0$; then, $\psi(t) \geq 0$ for all $t \geq t'$.

Repeating the arguments of Lemma 2, for any $0 < m < M < +\infty$, there exists $\tau \in (0, n^{-1})$ such that, if $v_1 \in (-\tau, 0)$ and $v_2 \in (0, \tau)$, then the system

$$\dot{e} = A_0 e - k(v, Ce) \tag{7}$$

is globally asymptotically stable for any fixed value of $v \in [v_1, v_2]$. Moreover, for $v = v_1$, it is $r(v_1)$-homogeneous of degree $v_1 < 0$ and globally finite-time stable at the origin, and for $v = v_2$, the system (7) is $r(v_2)$-homogeneous of degree $v_2 > 0$ and globally fixed-time stable with respect to any ball $\mathcal{B}_{r(v_2)}(\rho)$ with $\rho > 0$. In addition,

$$\dot{V}(e) < 0, \quad \forall e \in \left\{ e \in \mathbb{R}^n : \underline{m} \leq V(e) \leq \overline{M} \right\} \tag{8}$$

for any selection of $0 < \underline{m} < m < M < \overline{M} < +\infty$ and any (possibly time varying) value of $v(\zeta(t)) \in [v_1, v_2]$.

Denote by $T_M > 0$ the uniform settling time of convergence to the ball $\mathcal{B}_{r(v_2)}(\rho_M)$ of (7) for $v = v_2$, where $\rho_M > 0$ is such that $\{e \in \mathbb{R}^n : V(e) \leq M\} \subset B_{r(v_2)}(\rho_M)$. Let

$$T_M < 2\ln(\zeta(0) - M). \tag{9}$$

Then, $\zeta(t) \geq M$ for $t \in [0, t_M)$ with $t_M \geq T_M$ (t_M can also be infinite); therefore, $v(\zeta(t)) = v_2$ for $t \in [0, t_M)$ from (4), and the estimation error dynamics is $r(v_2)$-homogeneous and fixed-time stable with respect to the ball $\mathcal{B}_{r(v_2)}(\rho_M)$ on this interval of time. Since $t_M \geq T_M$, then the system (7) with $v = v(\zeta(t))$ enters in the ball $\mathcal{B}_{r(v_2)}(\rho_M)$, and there is an instant of time $t' \in [0, t_M)$ such that $\psi(t) \geq 0$ for all $t \geq t'$ (ie, $\zeta(t) \geq V(e(t))$ for all $t \geq t'$). Next, due to (8), the system (7) with $v = v(\zeta(t))$ reaches the set $\{e \in \mathbb{R}^n : V(e) \leq \overline{m}\}$ in a finite time $T_m > T_M$, where it stays for all $t \geq T_m$. By the properties of dynamics of ψ and ζ, the instant $t_M < +\infty$ and there is another time instant $t_m \geq \max\{t', T_m\}$ such that $\zeta(t) \leq m$ for all $t \geq t_m$ ($\zeta(t)$ is exponentially approaching $V(e(t))$ from above, while $V(e(t)) \leq \overline{m} < m$ for $t \geq T_m$). Consequently, $v(\zeta(t)) = v_1$ for $t \geq t_m$ and it reaches for the origin in a uniform time. Summarizing the arguments, we obtain that the system (7) with $v = v(\zeta(t))$ is globally fixed-time stable at the origin if $d = v = 0$. The variable ζ is also bounded and exponentially converging to zero. □

Corollary 2. *Let all conditions of Lemma 3 be satisfied; then, the systems (2) and (6) are ISpS with respect to the set $\mathcal{A}$ for any $(d, v) \in \mathcal{L}_\infty$.*

Proof. From the equation for ψ, we have that

$$V(t) \leq \zeta(t) + (V(0) - \zeta(0))e^{-0.5t} + 8\chi(\|v\|_\infty, \|d\|_\infty),$$

where $\sup_{t\geq 0}(d(t) - Lv(t))^T P(d(t) - Lv(t)) \leq \chi(\|v\|_\infty, \|d\|_\infty) = \|P\|_2(\|d\|_\infty^2 + \|L\|_2^2\|v\|_\infty^2)$ and $\|\cdot\|_2$ is the induced matrix norm. Consider the set $\Upsilon_0 = \{(e, \zeta) \in \mathbb{R}^{n+1} : V(e) \geq M + \max\{0, V(0) - \zeta(0)\} + 8\chi(\|v\|_\infty, \|d\|_\infty)\}$. From the aforementioned inequality, $\zeta(t) \geq M$ for $e(t) \in \Upsilon_0$. Thus, $v(t) = v_2$ if $e(t) \in \Upsilon_0$, and the estimation error dynamics takes the form

$$\dot{e} = f(e, \tilde{d}) = A_0 e - k\left(v_2, Ce + \tilde{d}_1\right) + \tilde{d}_2.$$

Denote $\tilde{r} = \begin{bmatrix} r_1(v_2) \\ r(v_2) + v_2 \end{bmatrix}$; then, $f(\Lambda_r(\lambda)e, \Lambda_{\tilde{r}}(\lambda)\tilde{d}) = \lambda^v \Lambda_r(\lambda) f(x, \tilde{d})$ for all $x \in \mathbb{R}^n$, $\tilde{d} \in \mathbb{R}^{n+1}$, and all $\lambda > 0$. Consequently, if all conditions of Lemma 3 are satisfied, the system $\dot{e} = f(e, 0)$ corresponds to (7); then, also all conditions of

Theorem 2 are true, and the system $\dot{e} = f(e, \tilde{d})$ is ISS with respect to $\tilde{d} \in \mathcal{L}_\infty$. Therefore, there exists an ISS Lyapunov function $W : \mathbb{R}^n \to \mathbb{R}_+$ (an alternative choice of W can be found in the work of Bernuau et al[19]) and some function ϱ of class $\mathcal{K}_\infty$ such that $W(e) \geq \varrho(M + \max\{0, V(0) - \zeta(0)\} + 8\chi(\|v\|_\infty, \|d\|_\infty))$ implies that $e \in \Upsilon_0$. Then, either an ISS estimate holds for W or $W(e) \leq \varrho(M + \max\{0, V(0) - \zeta(0)\} + 8\chi(\|v\|_\infty, \|d\|_\infty))$, which implies boundedness of W, and the same property for ζ. Consequently, all solutions of (6) are defined for all $t \geq 0$ ($x(t)$ is also defined for all $t \geq 0$ by conditions of Lemma 3). Then, the term $(V(0) - \zeta(0))e^{-0.5t}$ can be skipped and the aforementioned consideration can be repeated for the set $\Upsilon = \{(e, \zeta) \in \mathbb{R}^{n+1} : V(e) \geq M + 8\chi(\|v\|_\infty, \|d\|_\infty)\}$ to prove ISpS with respect to the set $\mathcal{A}$ of the systems (2) and (6). Obviously, the variable $\zeta(t)$ is also asymptotically bounded by $M + 16\chi(\|v\|_\infty, \|d\|_\infty)$ in this case. □

5.3 | Output feedback

The proposed dynamic output feedback consists in the application of the state feedback (3) with the state estimates generated by the observer (6)

$$u(z) = \sum_{i=1}^{n} a_i \lceil z_i \rfloor^{\alpha_i(\nu(\|z\|))}. \tag{10}$$

Then, the dynamics of the closed-loop system (2), (6), and (10) can be written in the coordinates x, $e = x - z$, and ζ as follows:

$$\begin{aligned} \dot{x} &= A_0 x + b \sum_{i=1}^{n} a_i \lceil x_i - e_i \rfloor^{\alpha_i(\nu(\|x-e\|))} + d, \\ \dot{e} &= A_0 e - k\left(\nu(\zeta), Ce + v\right) + d, \\ \dot{\zeta} &= -0.5\zeta + p\left(\nu(\zeta), Ce + v\right). \end{aligned} \tag{11}$$

The main result is a direct consequence of Lemmas 2 and 3 and Corollaries 1 and 2.

Theorem 4. *Let the following conditions be satisfied.*

i. $a \in \mathbb{R}^{1\times n}$ *forms a Hurwitz polynomial;*
ii. $A_0 - LC$ *is a Hurwitz matrix for given* $L \in \mathbb{R}^n$*;*
iii. $\zeta(0) > M$ *is sufficiently big.*

Then, for any $0 < m < M < +\infty$, *there exists* $\tau \in (0, n^{-1})$ *such that, for* $\nu_1 \in (-\tau, 0)$ *and* $\nu_2 \in (0, \tau)$, *the systems (2), (6), and (10) are*

1. *fixed-time converging with respect to the set* $\{(x, z, \zeta) \in \mathbb{R}^{2n+1} : x = z = 0\}$ *for* $d = v = 0$ *and for any initial conditions* $(x(0), z(0)) \in \mathbb{R}^{2n}$ *and*
2. *ISpS for any* $(d, v) \in \mathcal{L}_\infty$.

Proof. The system (11) is a cascade of the (e, ζ)- and x-dynamics. If $d = v = 0$, then (e, ζ)-subsystem is autonomous and globally fixed-time converging with respect to the set $\{(e, \zeta) \in \mathbb{R}^{n+1} : e = 0\}$ (with the uniform settling time $T_o > 0$) according to Lemma 3. During the interval $[0, T_o]$, the system (2) has bounded trajectories due to the ISpS property with respect to measurement noises (estimation errors e) established in Corollary 1, and for $t \geq T_o$, the x-subsystem is also autonomous and globally fixed-time converging at the origin by Lemma 2.

For $(d, v) \in \mathcal{L}_\infty$ the ISpS property follows the results of Corollaries 1 and 2 and the cascade structure of (11). □

In Theorem 4, the same parameters m, M, ν_1, and ν_2 have been selected for the controller (10) and for the observer (6) to keep the notation compact; however, they can be chosen differently in applications and the result of Theorem 4 stays correct.

Example 1. Let $n = 3, L = [1.5\ \ 1.01\ \ 0.25]^T, a = -2.5[1\,1\,1], m = 1, M = 5, \nu_2 = -\nu_1 = 0.1, \zeta(0) = 5M$, and

$$P = \begin{bmatrix} 0.121 & -0.13 & 0.047 \\ -0.13 & 0.261 & -0.308 \\ 0.047 & -0.308 & 0.617 \end{bmatrix}.$$

Then, all conditions of Theorem 4 are satisfied. We test first the system without any perturbations; the results are depicted in the upper plot of Figure 1 where the initial conditions of the system are $x_0 = (5, 10, 0)$ and those of the

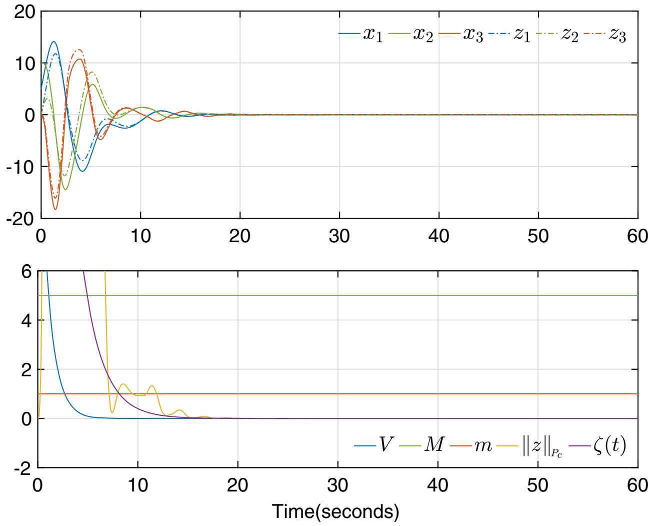

FIGURE 1 Results of simulation of (2), (6), and (10) for $n = 3$ without disturbances and with initial conditions $x_0 = (5, 10, 0)$ [Colour figure can be viewed at wileyonlinelibrary.com]

observer are $z_0 = (0, 0, 0)$. The solid color lines represent the actual state x while the dotted color lines represent the estimated state z. It can be seen how the estimated states converge rapidly to the actual states before converging both to zero. In the lower part of Figure 1, we can appreciate the elements of the control scheme; the upper and lower limits of the homogeneity degree M and n are shown as straight lines. The norm of the observed states $||z||$ is depicted in purple; while this norm is between M and m, the control's degree of homogeneity lies over the line $\frac{\nu_2-\nu_1}{M-m}||z|| + \frac{M\nu_1-m\nu_2}{M-m}$. In the case of the observer, the filter $\zeta(t)$ acts as the modulator of the observer's homogeneity degree.

Figure 2 shows the same setup with initial conditions $x_0 = 10^3(5, 10, 0)$; it can be seen that, although the initial state is significantly larger, the settling time remains within the same interval, showing the expected uniformity with respect to

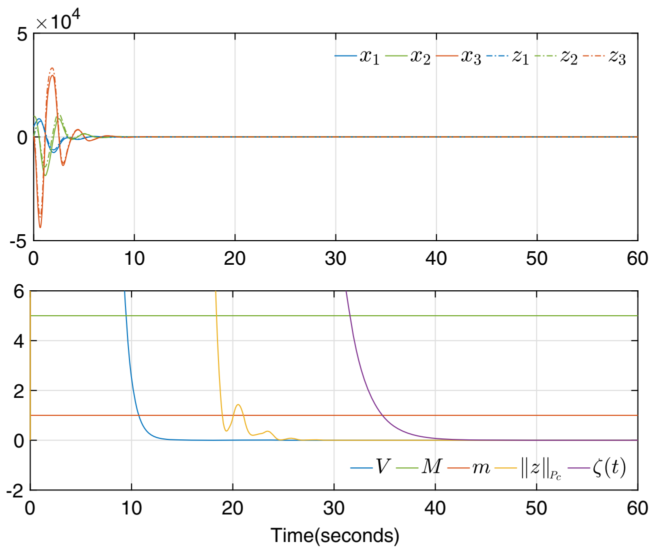

FIGURE 2 Results of simulation of (2), (6), and (10) for $n = 3$ with initial conditions $x_0 = 10^3(5, 10, 0)$ [Colour figure can be viewed at wileyonlinelibrary.com]

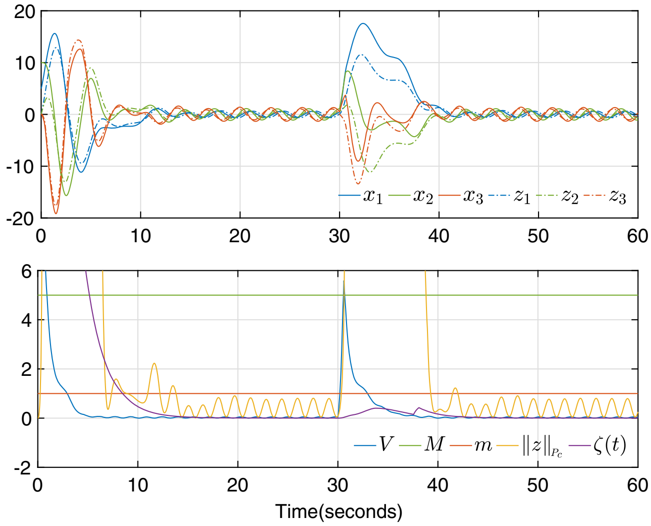

FIGURE 3 Simulation plots of (2), (6), and (10) for $n = 3$ with initial conditions $x_0 = (5, 10, 0)$ and the disturbance (12) [Colour figure can be viewed at wileyonlinelibrary.com]

the initial state. We next go back to the previous initial settings and introduce in the control scheme the disturbance

$$d(t) = \sin(2t) + \begin{cases} 10, & \text{if } t \in [30, 31] \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

The results are shown in Figure 3. We can notice that the system is robust against this disturbance and its effect in the control scheme elements are depicted in the lower plot of this figure. In particular, we can see that the disturbance modifies both $||z||$ and $\zeta(t)$ therefore changing the homogeneity degree of both the controller and the observer.

Remark 1. Since, in Theorem 4, the ISpS property with respect to d is proven; then, considering d as a function of x and assuming that the norm of such a function is less than the asymptotic gain function of the system for x sufficiently large, it is possible to prove fixed-time convergence to a zone and global boundedness of the system solutions for a nonlinear system also in the closed-loop with the proposed control (10) and the observer (6).

6 | PARAMETER TUNING

In this section, we provide effective algorithms to tune the parameters involved in the feedback controller. Based on the concept of implicit Lyapunov functions,[14] these algorithms transform the design procedure to an LMI feasibility problem, which simplifies significantly the practical applicability of the proposed method. In addition, they will provide an upper bound, even if conservative, of the settling time.

Consider the following implicit Lyapunov function candidate:

$$Q(V, x) = x^T \Lambda_{r(v)}(V^{-1}) P \Lambda_{r(v)}(V^{-1}) x - 1, \tag{13}$$

where $V \in \mathbb{R}_+, x \in \mathbb{R}^n, P \in \mathbb{R}^{n\times n}$, and $P = P^T > 0$. Although this particular function possesses many properties,* for the parametrization of the feedback controller and the state observer, it will be sufficient to remark that $Q(1, x) = 0$ implies $x^T P x = 1$ and that the implicitly defined function $V(x)$ is r-homogeneous of degree 1.

6.1 | Controller parametrization

For brevity in the notation, the following representation of systems (2) and (3) will be used

$$\dot{x} = A_0 x + ba\lceil x \rfloor^{\alpha(\nu(\|x\|_P))}, \tag{14}$$

*For a detailed explanation of the function (13) and its properties, the reader is referred to the work of Polyakov et al.[14]

where $\lceil x \rfloor^{\alpha(\nu)} = (|x_1|^{\alpha_1(\nu)}\text{sign}(x_1), \dots, |x_n|^{\alpha_n(\nu)}\text{sign}(x_n))^T$. Remark that, without loss of generality, the usual norm of the argument of ν in control (3) has been replaced with a weighted one. Let us introduce the matrix $H_{r(\nu)} = -\text{diag}(r_1(\nu), r_2(\nu), \dots, r_n(\nu))$. When ν takes a fixed value ν_j, denote, for brevity, the homogeneous weights $r_j = r(\nu_j)$, $H_{r_j} = -\text{diag}(r_{j,1}, r_{j,2}, \dots, r_{j,n})$, $\Lambda_{r_j}(\lambda) = \text{diag}(\lambda^{r_{j,1}}, \lambda^{r_{j,2}}, \dots, \lambda^{r_{j,n}})$ and $\alpha_j = \alpha(\nu_j)$. Accordingly,

$$Q_j(V, z) = z^T \Lambda_{r_j}(V^{-1}) P \Lambda_{r_j}(V^{-1}) z - 1. \tag{15}$$

Theorem 5. *Let, for some $\nu_1 \in (-\frac{1}{n}, 0)$, $\nu_2 = -\nu_1$, $\phi, \beta, \kappa, \gamma_j > 0$, $\beta < \phi$, and $\epsilon \in (0, 1)$, the system of matrix inequalities*

$$A_0 X + X A_0^T + bY + Y^T b^T + \phi X + \beta b b^T \leq 0 \tag{16a}$$

$$-\gamma_j X \leq H_{r_j} X + X H_{r_j} < 0, \tag{16b}$$

$$\xi I_n \leq X \leq \frac{1}{\kappa} I_n, \quad \begin{pmatrix} \frac{\beta^2 \xi}{\|\bar{z}_j\|^2} & Y \\ Y^T & X \end{pmatrix} \geq 0, \tag{16c}$$

$$\bar{z}_{j,i} = \begin{cases} g_{j,i}(\kappa^{-1/2}) + \kappa^{-1/2} \bar{p}_i \epsilon, & \text{if } \kappa^{-1/2} \leq \alpha_{j,i}^{1/(1-\alpha_{j,i})} \\ \max\left\{ g_{j,i}\left(\alpha_{j,i}^{1/(1-\alpha_{j,i})}\right), g_{j,i}(\kappa^{-1/2}) \right\} + \kappa^{-1/2} \bar{p}_i \epsilon, & \text{if } \kappa^{-1/2} > \alpha_{j,i}^{1/(1-\alpha_{j,i})}, \end{cases} \tag{17}$$

where $g_{j,i} : \mathbb{R} \to \mathbb{R}$, $g_{j,i}(z) = |\lceil z \rfloor^{\alpha_{j,i}} - z|$, $\bar{p}_i = (n - i + 1)\nu_2$, and $\alpha_{j,i} = (\frac{1+n\nu_j}{r_{j,1}}, \dots, \frac{1+n\nu_j}{r_{j,n}})$, be feasible for some $\xi > 0$, $X \in \mathbb{R}^{n \times n}$, $X = X^T > 0$, and $Y \in \mathbb{R}^{1 \times n}$, with $j = 1, 2$ and $i = \overline{1, n}$.

Then, the systems (2) and (3) for $d = 0$, $a = YP$ and $P = X^{-1}$ are globally fixed-time stable at the origin with

$$m = \sqrt{\lambda_{\max}\left(P^{-\frac{1}{2}} \Lambda_{r_1}(1-\epsilon) P \Lambda_{r_1}(1-\epsilon) P^{-\frac{1}{2}}\right)} < 1 < M = \sqrt{\lambda_{\min}\left(P^{-\frac{1}{2}} \Lambda_{r_2}(1+\epsilon) P \Lambda_{r_2}(1+\epsilon) P^{-\frac{1}{2}}\right)}$$

and the settling time estimate $T_{\max} \leq \frac{\gamma_1}{(\phi-\beta)|\nu_1|} + \frac{\gamma_2}{(\phi-\beta)\nu_2}$.

Before presenting the proof of this theorem, let us introduce a constructive procedure to calculate the controller's parameters. To this end, the system of LMI (16) can be solved using standard optimization tools such as MATLAB.

Constructive procedure to obtain the controller's parameters P, a, m, **and** M.

1. Set the size of the chain of integrators n.
2. Fix a negative value for ν_1. Start with values close to zero, eg, -0.001.
3. Fix positive values for ϵ and κ (a possible initial value is 0.5 for both).
4. Calculate the vectors $\bar{z}_{i,j}$ using (17).
5. Fix positive values for ϕ, β, γ_1, γ_2, and $\beta < \phi$ (possible starting values are given in the examples).
6. Verify the feasibility of the system of inequalities (16).
7. If unfeasible, reduce the value of ϵ or modify the value of κ and repeat from step 3. If feasible, the value of $|\nu_1|$ might be increased in step 2 until a desired value of $T_{\max}$ is obtained without loosing feasibility (recall that, in practice, this value might be conservative).
8. From the obtained matrices X and Y, calculate P, a, m, and M as described in Theorem 5.

Note that the parameters that influence directly the settling time are $\nu_1, \nu_2, \gamma_1, \gamma_2, \phi$, and β. The parameters ϵ and κ modify the bounds of the inequality (16c), so that its manipulation may relax the feasibility conditions of (16).

Proof of Theorem 5. **I**: The functions $Q_j(V, x)$, $j = 1, 2$ defined in (15) satisfy the conditions **C1** to **C3** of Theorem 3. Indeed, they are continuously differentiable for all $V \in \mathbb{R}_+$ and for all $x \in \mathbb{R}^n$. Since $P > 0$, then the inequalities

$$\frac{\lambda_{\min}(P)\|x\|^2}{\max\{V^{2\min r_i}, V^{2\max r_i}\}} \leq Q_j(V, x) + 1 \leq \frac{\lambda_{\max}(P)\|x\|^2}{\min\{V^{2\min r_i}, V^{2\max r_i}\}},$$

imply that, for any $x \in \mathbb{R}^n \backslash \{0\}$, there exist $V^- \in \mathbb{R}_+$ and $V^+ \in \mathbb{R}_+ : Q_j(V^-, x) < 0 < Q_j(V^+, x)$. Moreover, if $Q_j(V, x) = 0$, then, obviously, the condition **C3** of Theorem 3 holds and there exists a Lyapunov function candidate $V : \mathbb{R}^n \to \mathbb{R}_+$ implicitly defined by the identity $x^T \Lambda_{r_j}(V^{-1}) P \Lambda_{r_j}(V^{-1}) x = 1$.

II: Since

$$\frac{\partial Q_j}{\partial V} = V^{-1} x^T \Lambda_{r_j}(V^{-1})(P H_{r_j} + H_{r_j} P) \Lambda_{r_j}(V^{-1}) x,$$

taking into account (16b), we have that $\frac{\partial Q_j}{\partial V} < 0, \forall V \in \mathbb{R}_+$ and $x \in \mathbb{R}^n \backslash \{0\}$. Therefore, the condition **C4** of Theorem 3 also holds, and therefore, $Q_j(V, z) = 0$ implicitly defines a proper positive definite Lyapunov function candidate $V : \mathbb{R}^n \to \mathbb{R}$.

III: By denoting $\frac{\partial Q_j(V,x)}{\partial x}$ as the partial derivative of Q_j along the trajectories of (14), we obtain

$$\frac{\partial Q_j(V,x)}{\partial x} = 2x^T \Lambda_{r_j}(V^{-1}) P \Lambda_{r_j}(V^{-1}) \left[A_0 x + ba\lceil x \rfloor^{\alpha(\nu)}\right].$$

By adding and subtracting inside the brackets the auxiliary term $V^{1+n\nu_j} ba\Lambda_{r_j}(V^{-1})x$ and taking into account that $\Lambda_{r_j}(V^{-1})A_0\Lambda_{r_j}^{-1}(V^{-1}) = V^{-\nu_j}A_0$ and that $\Lambda_{r_j}(V^{-1})b = V^{-1-(n-1)\nu_j}b$, we simplify the derivative as

$$\frac{\partial Q_j(V,x)}{\partial x} = \begin{pmatrix} \Lambda_{r_j}(V^{-1})x \\ d(V,x) \end{pmatrix}^T \begin{pmatrix} V^{-\nu_j}\left(PA_0 + A_0^T P + Pba + a^T b^T P\right) & Pb \\ b^T P & 0 \end{pmatrix} \begin{pmatrix} \Lambda_{r_j}(V^{-1})x \\ d(V,x) \end{pmatrix},$$

where $d(V,x) = V^{-\nu_j}(a(V^{-1-n\nu_j}\lceil x \rfloor^{\alpha(\nu)} - \Lambda_{r_j}(V^{-1})x))$. By adding and subtracting the terms $\phi V^{-\nu_j} x^T \Lambda_{r_j}(V^{-1})P\Lambda_{r_j}(V^{-1})x$ and $\frac{1}{\beta}V^{\nu_j} d^T(V,x)d(V,x)$, we obtain

$$\frac{\partial Q_j(V,x)}{\partial x} = \begin{pmatrix} \Lambda_{r_j}(V^{-1})x \\ d(V,x) \end{pmatrix}^T \Theta \begin{pmatrix} \Lambda_{r_j}(V^{-1})x \\ d(V,x) \end{pmatrix} - \phi V^{-\nu_j} x^T \Lambda_{r_j}(V^{-1})P\Lambda_{r_j}(V^{-1})x + \frac{V^{\nu_j}}{\beta} d^T(V,x)d(V,x),$$

where

$$\Theta = \begin{pmatrix} V^{-\nu_j}\left(PA_0 + A_0^T P + Pba + a^T b^T P + \phi P\right) & Pb \\ b^T P & \frac{V^{\nu_j}}{\beta} \end{pmatrix}.$$

Using the Schur complement, $\begin{pmatrix} P^{-1} & 0 \\ 0 & \eta \end{pmatrix}^T \Theta \begin{pmatrix} P^{-1} & 0 \\ 0 & \eta \end{pmatrix}$, for any $\eta \in \mathbb{R}$, it is equivalent to the left-hand side of (16a) and we arrive to

$$\frac{\partial Q_j(V,x)}{\partial x} \leq -V^{-\nu_j}\phi x^T \Lambda_{r_j}(V^{-1})P\Lambda_{r_j}(V^{-1})x + V^{\nu_j}\beta^{-1}d^T(V,x)d(V,x).$$

The term $d(V,x)$ can be rewritten as $d(V,x) = V^{-\nu_j}d_0(V,x)$, where $d_0(V,x) = a(V^{-1-n\nu_j}\lceil x \rfloor^{\alpha(\nu)} - \Lambda_{r_j}(V^{-1})x)$, and we have that

$$\frac{\partial Q_j(V,x)}{\partial x} \leq -V^{-\nu_j}\phi x^T \Lambda_{r_j}(V^{-1})P\Lambda_{r_j}(V^{-1})x + V^{-\nu_j}\beta^{-1}d_0^T(V,x)d_0(V,x).$$

If the latter term is bounded as follows $d_0^T(V(x),x)d_0(V(x),x) < \beta^2$, then we derive

$$\frac{\partial Q_j(V,x)}{\partial x} \leq -\phi V^{-\nu_j} + \beta V^{-\nu_j},$$

and using inequality (16b), we arrive to

$$\frac{\partial Q_j(V,x)}{\partial x} \leq \frac{\phi - \beta}{\gamma_j} V^{1+\nu_j} \frac{\partial Q_j}{\partial V},$$

for $(V,x) : x^T \Lambda_{r_j}(V^{-1})P\Lambda_{r_j}(V^{-1})x = 1$. Then, by defining $c_j = \frac{\phi-\beta}{\gamma_j}$, the conditions of Theorem 3 are satisfied.

IV Proof of the estimate: $d_0^T(V,x)d_0(V,x) \leq \beta^2$.

IV.a: Let us introduce the sets $\Omega_1 = \{x : m^2 \leq x^T P x \leq 1\}$ and $\Omega_2 = \{x : 1 \leq x^T P x \leq M^2\}$. Considering first the set Ω_2 and using the change of variables $y = \Lambda_{r_2}(V^{-1})x$, we obtain

$$\max_{\substack{(V,x): x^T \Lambda_{r_2}(V^{-1})P\Lambda_{r_2}(V^{-1})x=1 \\ \text{and } x^T P x \in [1,M^2]}} V = \max_{\substack{(V,y): y^T P y=1 \\ \text{and } y^T \Lambda_{r_2}(V)P\Lambda_{r_2}(V)y \in [1,M^2]}} V = 1 + \epsilon.$$

Indeed, from inequality (16b), we know that $\frac{\partial}{\partial V} y^T \Lambda_{r_2}(V)P\Lambda_{r_2}(V)y > 0$, and therefore, $\min\limits_{y: y^T P y=1} y^T \Lambda_{r_2}(V)P\Lambda_{r_2}(V)y > \min\limits_{y: y^T P y=1} y^T \Lambda_{r_2}(1+\epsilon)P\Lambda_{r_2}(1+\epsilon)y = M^2$ if $V > 1 + \epsilon$ and $1 < \|x\|_P \leq M$ implies $1 \leq V(x) \leq 1 + \epsilon$. Similarly, we show

$$\min_{\substack{(V,x): x^T \Lambda_{r_1}(V^{-1})P\Lambda_{r_1}(V^{-1})x=1 \\ \text{and } x^T P x \in [m^2,1]}} V = 1 - \epsilon.$$

Hence, we immediately conclude that $m \leq \|x\|_P^2 \leq M \Rightarrow 1 - \epsilon \leq V(x) \leq 1 + \epsilon$.

IV.b: Recall that the function ν is defined as follows:

$$\nu(\|x\|_P) = \begin{cases} \nu_1, & \text{if } \|x\|_P \le m, \\ \nu_2, & \text{if } \|x\|_P \ge M \\ \frac{\nu_2-\nu_1}{M-m}\|x\|_P + \frac{M\nu_1 - m\nu_2}{M-m}, & \text{otherwise.} \end{cases}$$

Then, we have that

$$\max_{x\in\Omega_2} d_0^T(V(x),x)d_0(V(x),x) = \max_{x\in\Omega_2} \delta_\epsilon(V(x),x)^T a^T a \delta_\epsilon(V(x),x) \le aa^T \max_{x\in\Omega_2} \delta_\epsilon(V(x),x)^T \delta_\epsilon(V(x),x),$$

where

$$\delta_\epsilon(V,x) = V^{-1-n\nu_2}\left(\lceil x \rfloor^{\alpha(\nu(\|x\|_P))} - \Lambda_{r_2}(V^{-1})x\right).$$

Using again the change of variables $y = \Lambda_{r_2}(V^{-1})x$ and considering that $x \in \Omega_2 \implies V \in [1, 1+\epsilon]$, we have that

$$aa^T \max_{x\in\Omega_2} \delta_\epsilon(V(x),x)^T \delta_\epsilon(V(x),x) \le aa^T \max_{\substack{V\in[1,1+\epsilon]\\ \|y\|_P=1}} \psi(V,y)^T\psi(V,y),$$

where $\psi(V,y) = |V^{-1-n\nu_2}\lceil \Lambda_{r_2}(V)y \rfloor^{\alpha(\nu)} - y|$, and in a component-wise expression, we have

$$\psi_i(V,y) = \left|V^{-1-n\nu_2+r_{2,i}\alpha_i(\nu)}\lceil y_i \rfloor^{\alpha_i(\nu)} - y_i\right| = V^{-p_i(\nu)}\left|\lceil y_i \rfloor^{\alpha_i(\nu)} - V^{p_i(\nu)}y_i\right|,$$

where $p_i(\nu) = 1 + n\nu_2 - r_{2,i}\alpha_i(\nu) \ge 0$. Hence,

$$\begin{aligned} \psi_i(V,y) &\le \max_{\substack{V\in[1,1+\epsilon]\\ \|y\|_P=1}} \left|\lceil y_i \rfloor^{\alpha_i(\nu)} - V^{p_i(\nu)}y_i\right| \\ &= \max_{y:y^TPx=1}\left\{\left||y_i|^{\alpha_i(\nu)} - |y_i|\right|, \left||y_i|^{\alpha_i(\nu)} - (1+\epsilon)^{p_i(\nu)}|y_i|\right|\right\} \\ &= \max_{y:y^TPx=1}\left\{\left|\lceil y_i \rfloor^{\alpha_i(\nu)} - y_i\right|, \left||y_i|^{\alpha_i(\nu)} - y_i + y_i - (1+\epsilon)^{p_i(\nu)}y_i\right|\right\} \\ &\le \max_{y:y^TPx=1}\left||y_i|^{\alpha_i(\nu)} - |y_i|\right| + |y_i|\left|1 - (1+\epsilon)^{p_i(\nu)}\right|. \end{aligned}$$

Lemma 4. *The function $g : \mathbb{R}_+^2 \to \mathbb{R}_+$, defined as $g(s,\varepsilon) = |s^\varepsilon - s|$, $s, \varepsilon \in \mathbb{R}_+$, admits the following estimate:*

$$\max_{s\in[0,\bar{s}],\varepsilon\in[0,\bar{\varepsilon}]} g(s,\varepsilon) = \begin{cases} g(\bar{s},\bar{\varepsilon}), & \text{for } \bar{s} \le \bar{\varepsilon}^{1/(1-\bar{\varepsilon})} \\ \max\left\{g\left(\bar{\varepsilon}^{1/(1-\bar{\varepsilon})},\bar{\varepsilon}\right), g(\bar{s},\bar{\varepsilon})\right\}, & \text{for } \bar{s} > \bar{\varepsilon}^{1/(1-\bar{\varepsilon})}. \end{cases} \tag{18}$$

Proof. The function g is depicted in Figure 4. It is easy to show that function g_ε attains a local maximum at $s_{\max_1} = \varepsilon^{1/(1-\varepsilon)}$ within the interval $s \in [0,1]$; therefore, if $\bar{s} \le s_{\max_1}$, then $\max_{s\in[0,\bar{s}]} g(s,\varepsilon) = g(\bar{s},\varepsilon)$, and if $\bar{s} > s_{\max_1}$, then $\max_{s\in[0,\bar{s}]} g(s,\varepsilon) = \max\{g(\varepsilon^{1/(1-\varepsilon)},\varepsilon), g(\bar{s},\varepsilon)\}$. Taking into account that $\frac{\partial(s^\varepsilon - s)}{\partial\epsilon} = s^\varepsilon \ln s$, we complete the proof. □

Since $y^TPy \implies |y_i| \le \lambda_{\min}^{-1/2}(P) = \kappa^{-1/2}$, then, using Lemma 4, the term $\left||y_i|^{\alpha_i(\nu)} - |y_i|\right|$ can be bounded as follows:

$$\left|\lceil y_i \rfloor^{\alpha_i(\nu)} - y_i\right| \le \tilde{z}_{2,i} = \begin{cases} g_{\alpha_{2,i}}(\kappa^{-1/2}), & \text{for } \kappa^{-1/2} \le \alpha_{2,i}^{1/(1-\alpha_{2,i})} \\ \max\left\{g_{2,i}\left(\alpha_{2,i}^{1/(1-\alpha_{2,i})}\right), g_{2,i}(\kappa^{-1/2})\right\}, & \text{for } \kappa^{-1/2} > \alpha_{2,i}^{1/(1-\alpha_{2,i})}. \end{cases}$$

Then, for $V \in [1, 1+\epsilon]$ and $y^TPy = 1$, one has

$$\psi_i(V,y) \le \tilde{z}_{2,i} + \kappa^{-1/2}\left((1+\epsilon)^{p_i(\nu)} - 1\right) \le \tilde{z}_{2,i} + \kappa^{-1/2}\left((1+\epsilon)^{\bar{p}_i} - 1\right),$$

where $\nu = \nu(\|x\|_P) \in [0, \nu_2]$, $p_i(\nu) = 1 + n\nu_2 - (1+(i-1)\nu_2)\frac{1+n\nu}{1+(i-1)\nu}$, and $\bar{p}_i = p_i(0) = (n-i+1)\nu_2$. Here, the fact that, for any x in the interval $1 \le \|x\|_P \le M$, $p_i(\nu(\|x\|_P)) \le \bar{p}_i$ has been used.

By applying the mean value theorem with $h : \mathbb{R} \to \mathbb{R}$, $h(\theta) = \theta^{\bar{p}_i}$, we obtain that $h(1+\epsilon) - h(1) = h'(\theta^*)\epsilon$, where $\theta^* \in [1, 1+\epsilon]$. Noting that $0 < \bar{p}_i < 1$, we have that $(1+\epsilon)^{\bar{p}_i} - 1 \le \bar{p}_i\epsilon$, and we arrive to the following estimate:

$$\psi_i(V,y) \le \tilde{z}_{2,i} + \kappa^{-1/2}\bar{p}_i\epsilon,$$

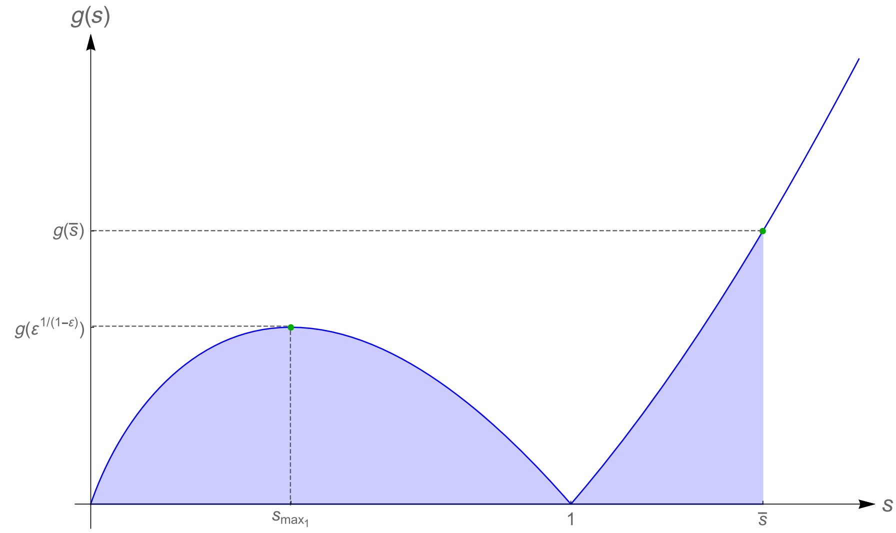

FIGURE 4 Plot of function $g(s,\varepsilon) = |\lceil s\rfloor^{\varepsilon} - s|$, with fixed $\varepsilon \in \mathbb{R}_+$. Remark that, if $\bar{s} > 1$, the maximum value of $g(s)$ in the interval $s \le \bar{s}$ is the maximum between $g(\varepsilon^{1/(1-\varepsilon)}, \varepsilon)$ and $g(\bar{s}, \varepsilon)$ [Colour figure can be viewed at wileyonlinelibrary.com]

for $V \in [1, 1+\epsilon]$ and $\|y\|_P = 1$. Therefore, it has been proven that

$$\max_{x\in\Omega_2} d_0^T(V(x),x)d_0(V(x),x) \le a^T a\|\bar{z}_2\|^2,$$

where $\bar{z}_2 = (\bar{z}_{2,1}, \dots, \bar{z}_{2,n})^T \in \mathbb{R}_+^n$ and $\bar{z}_{2,i} = \tilde{z}_{2,i} + \kappa^{-1/2}\bar{p}_i\epsilon$.

Accordingly, proceeding in the same fashion for the set Ω_1, we obtain

$$\max_{x\in\Omega_1\cup\Omega_2} d_0^T(V(x),x)d_0(V(x),x) \le a^T a\|\bar{z}_2\|^2.$$

Using the Schur complement, inequality (16c) becomes $aP^{-1}a^T \le \frac{\xi\beta^2}{\|\bar{z}_j\|^2}$, and since $aa^T\xi \le aP^{-1}a^T$, we have that $aa^T \le \frac{\beta^2}{\|\bar{z}_j\|^2}$ and we conclude that

$$d_0(V(x),x)^T d_0(V(x),x) \le \beta^2$$

if $x \in \Omega_1 \cup \Omega_2$.

IV.c Boundedness of $d_0^T(V(x),x)d_0(V(x),x)$ **in** $\mathbb{R}^n\backslash(\Omega_1\cup\Omega_2)$: If $x \in \mathbb{R}^n\backslash\{x : \|x\|_P \le M\}$, then $\nu(\|x\|_P) = \nu_2$ and

$$d_0(V(x),x) = aV^{-1-n\nu_2}\left(\lceil x\rfloor^{\alpha(\nu_2)} - \Lambda_{r_2}(V^{-1})x\right) = a\left(\lceil y\rfloor^{\alpha(\nu_2)} - y\right),$$

where $y = \Lambda_{r_2}(V^{-1}(x))x$ and $y^T P y = 1$ (since $Q(V(x),x) = 0$). Hence, $\psi_i(V,y) \le \tilde{z}_{2,i}$ and the required estimate $d_0(V,x)^T d_0(V,x) \le \beta^2$ is straightforward. Similar considerations can be provided for $x : \|x\|_P \le m$. □

Finally let us remark that, following a similar procedure, along with the ideas presented in the work of Lopez-Ramirez et al,[25] analogous algorithms can be developed for the observer.

Example 2. (Controller parametrization)
We start by choosing $n = 3$, $\nu_1 = -0.02$, $\nu_2 = -\nu_1$, $\epsilon = \kappa = 0.5$, and calculating $\bar{z}_j$; following Theorem 5, we obtain $\bar{z}_1 = (0.0715, 0.0662, 0.0605)$ and $\bar{z}_2 = (0.0721, 0.0689, 0.0661)$. We now choose the parameters $\phi = 0.5$, $\beta = 0.3$, $\gamma_1 = \gamma_2 = 3$, and solve the set of LMIs (16a) to (16c) to obtain

$$P_c = \begin{pmatrix} 2.3367 & 1.8430 & 1.7795 \\ 1.8430 & 4.8778 & 2.1295 \\ 1.7795 & 2.1295 & 4.2590 \end{pmatrix}, \quad a = (-0.5952, -1.7576, -1.3889).$$

With this parameter choice, the maximum settling time of the controller is $T_{\max} = 1500$ seconds.

Figure 5 depicts the substitution of these values in the systems (2) and (6) with initial conditions $x(0) = (5, 10, 0)$. It is possible to see a fast convergence to the real states before converging also rapidly to the origin. In Figure 6, the initial conditions have been changed to $x(0) = 10^3(5, 10, 0)$; it is possible to see that the settling time is not significantly

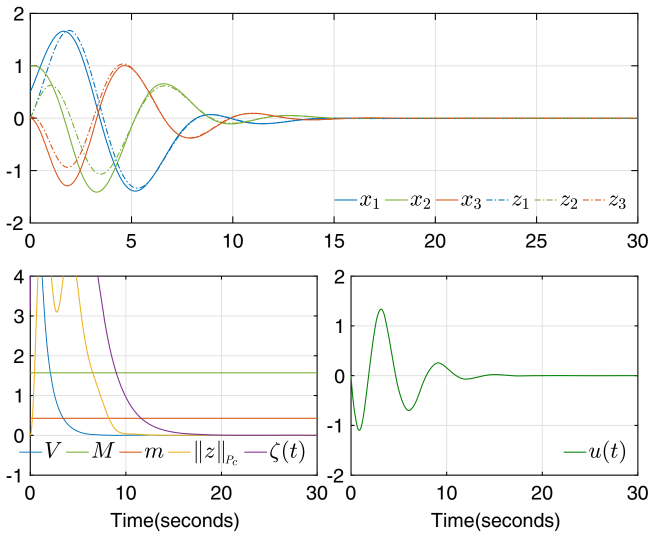

FIGURE 5 Results of simulation of (2), (6), and (10) for $n = 3$ without disturbances and with initial conditions $x_0 = (0.5, 1, 0)$ [Colour figure can be viewed at wileyonlinelibrary.com]

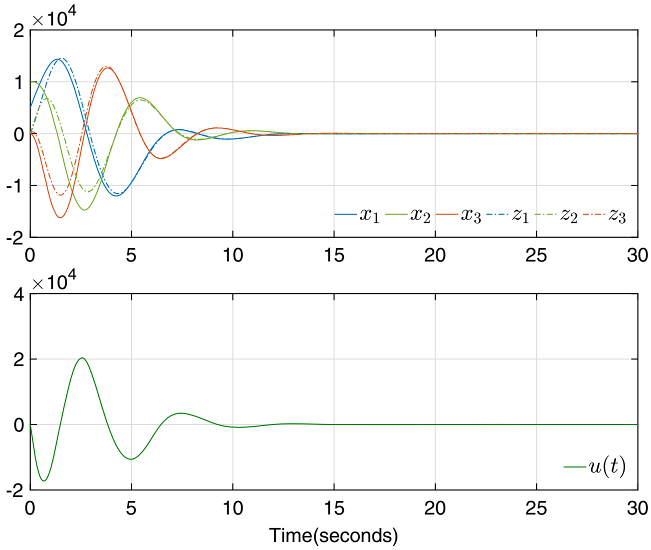

FIGURE 6 Results of simulation of (2), (6), and (10) for $n = 3$ with initial conditions $x_0 = 10^3(0.5, 1, 0)$ [Colour figure can be viewed at wileyonlinelibrary.com]

modified, and that, in both cases, the systems reach the equilibrium long before the settling time estimates. Finally, in Figure 7, the initial conditions were reset to $x(0) = (5, 10, 0)$ and the disturbance term

$$d(t) = \sin(2t) + \begin{cases} 10, & \text{if } t \in [30, 31] \\ 0, & \text{otherwise} \end{cases} \tag{19}$$

was added.

We can see that the convergence to zero is preserved and with a much better performance than in Example 1.

Example 3. (Comparative example)
This last example is meant to compare the fixed-time controllers (2) and (3), referred here as the *nonrecursive* controllers, with the finite-time ones and fixed-time ones described in the work of Harmouche et al,[17] referred accordingly as the *recursive* ones. The parameter choice for the tuning algorithm 5 is as follows: $\nu_1 = -1/400$, $\nu_2 = -\nu_1$, $\epsilon = 0.1$, $\phi = 8$, $\beta = 7$, $\kappa = 0.5$, and $\gamma_1 = \gamma_2 = 3$. Moreover, the obtained parameter values are $a = -(384, 136.08, 16.94)$,

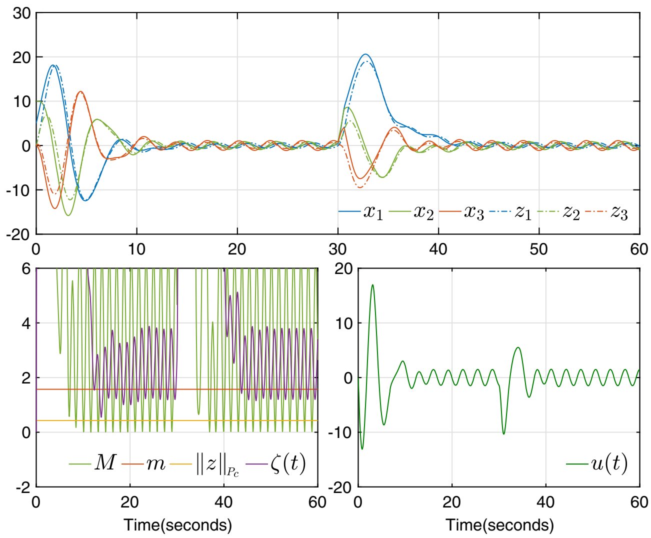

FIGURE 7 Results of simulation of (2), (6), and (10) for $n = 3$ with initial conditions $x_0 = (5, 10, 0)$ and the disturbance (19) [Colour figure can be viewed at wileyonlinelibrary.com]

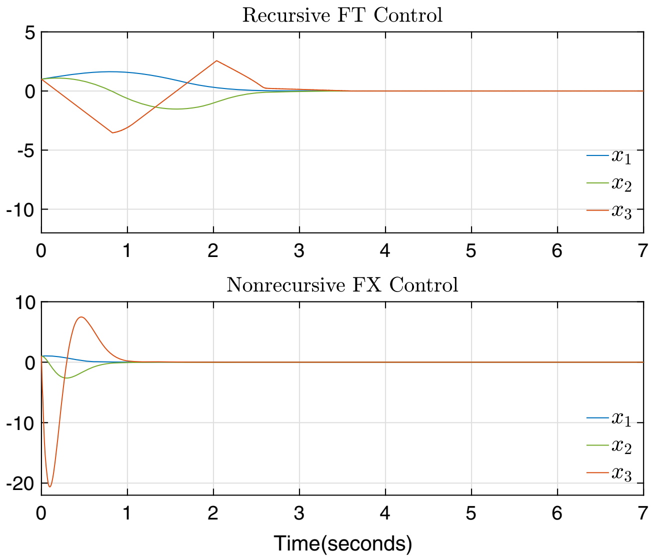

FIGURE 8 Comparison results between the recursive finite-time controller and the nonrecursive fixed-time one for $n = 3$, without disturbances and with initial conditions $x_0 = (1, 1, 1)$ [Colour figure can be viewed at wileyonlinelibrary.com]

$P_c = 10^3 \begin{pmatrix} 2.3195 & 0.5083 & 0.0359 \\ 0.5083 & 0.1455 & 0.0113 \\ 0.0359 & 0.0113 & 0.0014 \end{pmatrix}$, $\bar{z}_1 = (0.0047, 0.0056, 0.0064)$, and $\bar{z}_2 = (0.0047, 0.0056, 0.0065)$. The corresponding parameters for both the finite-time controllers and the fixed-time controllers in the work of Harmouche et al[17] are the same as the ones used in the example section of the cited article. Figure 8 shows the unperturbed states of the chain of integrators with initial conditions $x_0 = (1, 1, 1)$. Clearly, the fixed-time (FX) controller outperforms the finite-time (FT) one. In Figure 9 it is possible to see how in the finite-time case, the variation of initial conditions affects the settling-time while the fixed-one remains within two seconds. Finally, Figure 10 shows an improvement of over two seconds in the settling time with respect to the recursive fixed-time controller, illustrating that, with the tuning algorithm, the settling time can be adjusted. As shown in the interior plots, fixed-time stability is assumed whenever all the states enter a strip of magnitude 10^{-4} (depicted between black lines in Figure 10) and remain there for the rest of the simulation.

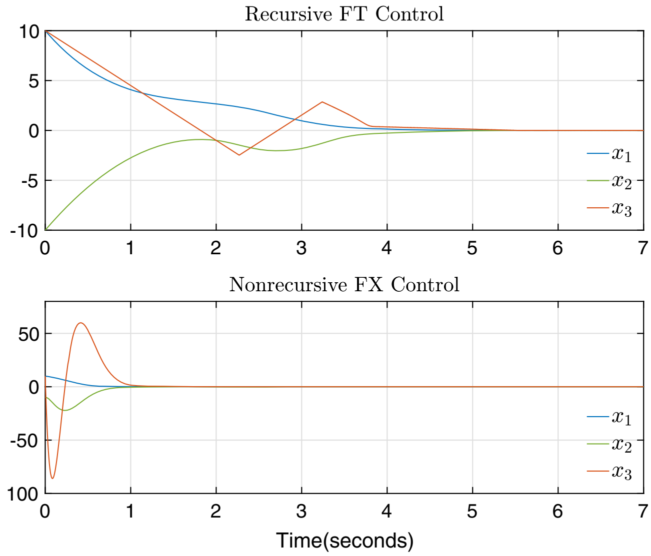

FIGURE 9 Comparison results between the recursive finite-time observer and the nonrecursive fixed-time one for $n = 3$, without disturbances and with initial conditions $x_0 = (10, 10, 10)$ [Colour figure can be viewed at wileyonlinelibrary.com]

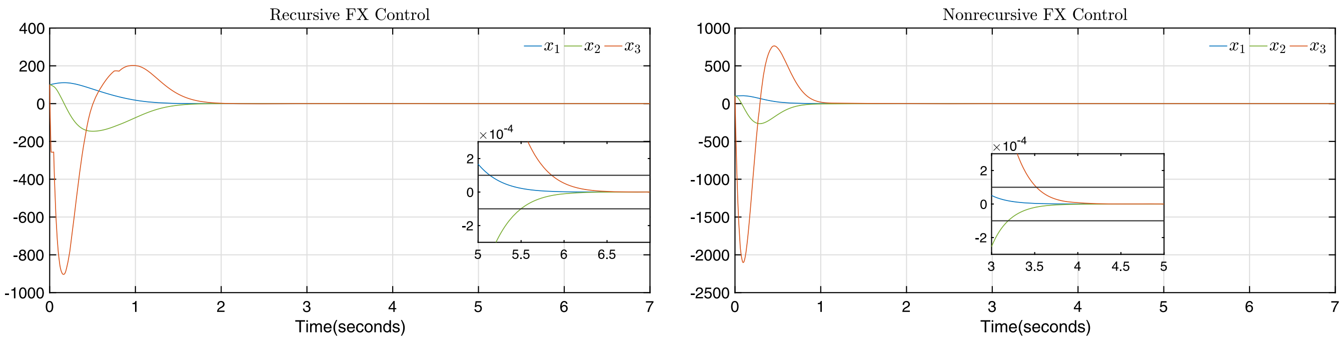

FIGURE 10 Comparison results between the recursive finite-time observer and the nonrecursive fixed-time one for $n = 3$, without disturbances and with initial conditions $x_0 = (10, 10, 10)$ [Colour figure can be viewed at wileyonlinelibrary.com]

7 | CONCLUSION

A state feedback control has been constructed for a chain of integrators, which ensures global convergence of all trajectories to the origin with an upper bound of the settling time that is independent of the initial conditions (fixed-time stability). An observer has been proposed, which provides a global estimation of the plant state (global differentiation) with a fixed-time convergence rate. Both control and estimation algorithms are robust with respect to disturbances and noises. It has been shown that the combination of these algorithms results in a global fixed-time output stabilization control law. Effective tools to optimize the scheme's parameters, allowing the maximum settling time to be estimated have been presented and the efficacy of this scheme has been demonstrated in simulations. In the perturbed case, practical fixed-time stabilization is obtained; this is fixed-time stabilization to a ball containing the origin whose radius depends on the size of the perturbation.

As with many control algorithms that focus on *fast* convergence rates, the control signal magnitude might grow significantly to cope with the time constraints imposed. In practice, these constraints can only be granted locally due to boundedness of the admissible control magnitude. However, in contrast to other control algorithms, the fixed-time controllers do not need to be retuned if the admissible control magnitude (and, consequently, the domain of fixed-time convergence) is increased. A related issue deals with the methods used for simulation, which have to be adapted to treat highly nonlinear systems.[26]

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