Received: 5 March 2019 | Revised: 8 September 2019 | Accepted: 27 October 2019
DOI: 10.1002/rnc.4820

RESEARCH ARTICLE

WILEY

Prespecifiable fixed-time control for a class of uncertain nonlinear systems in strict-feedback form

Ye Cao[1,2] | **Changyun Wen**[3] | **Shilei Tan**[1,2] | **Yongduan Song**[1,2]

[1]School of Automation, Chongqing University, Chongqing, China

[2]Chongqing Key Laboratory of Intelligent Unmanned Systems, School of Automation, Chongqing University, Chongqing, China

[3]School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore

Correspondence
Yongduan Song, School of Automation, Chongqing University, Chongqing 400044, China; or Chongqing Key Laboratory of Intelligent Unmanned Systems, School of Automation, Chongqing University, Chongqing 400044, China.
Email: ydsong@cqu.edu.cn

Funding information
National Natural Science Foundation of China, Grant/Award Number: 61803053, 61833013, 61860206008 and 61773081

Summary

In this paper, we investigate the prespecifiable fixed-time control problem for a class of uncertain nonlinear systems in strict-feedback form, where the settling (convergence) time is not only bounded but also user-assignable in advance. One of the salient features of the proposed method lies in the fact that it makes it possible to achieve any practically allowable settling time by using a simple and effective control parameter selection recipe. Both fixed-time stabilization and fixed-time tracking are considered for uncertain strict-feedback systems. Firstly, by adding exponential state feedback and using fractional power integration as Lyapunov function candidate, a global stabilizing control strategy is developed. It is proved that all the system states converge to zero within prespecified fixed-time with continuous and bounded control action. Secondly, under more general uncertain nonlinearities and external disturbances, an adaptive fixed-time controller is derived such that the tracking error converges to a small neighborhood of zero within preassigned time. Theoretical results are also illustrated and supported by simulation studies.

KEYWORDS

adaptive control, backstepping, prespecifiable fixed-time, uncertain nonlinear systems

1 | INTRODUCTION

In recent years, finite-time control of nonlinear systems has received increasing attention.[1-4] Compared with asymptotically stable control, finite-time control usually has faster convergence rates, higher tracking precisions, and better disturbance rejection properties.[5-7] Many effective approaches have been developed for nonlinear systems. They can be roughly classified into four methods: time optimal control, such as bang-bang control,[8] homogeneous system finite-time stability approach,[9] sliding mode control method,[10] and finite-time Lyapunov stability approach.[11,12] Specifically, in the work of Huang et al,[11] a systematic design method is presented for achieving global finite-time stabilization of uncertain nonlinear systems dominated by a lower-triangular form. In the work of Huang et al,[12] a finite-time adaptive control scheme is proposed for a class of strict-feedback systems with parametric uncertainties based on given specifications. It should be noted that, in the abovementioned results, the settling (convergence) time is dependent on initial system conditions. In fact, the convergence time may also become unacceptably large if the magnitude of initial conditions is large. To solve this problem, the concept of fixed-time stability is proposed in the work of Polyakov.[13]

As an extension of finite-time stability, the fixed-time stability requires bounded settling time regardless of system initial states. In the work of Polyakov,[13] polynomial feedback control schemes and second-order sliding mode control schemes are presented for uncertain linear plants. In the work of Zuo,[14] fixed-time consensus tracking problem for second-order multiagent systems is investigated by using terminal sliding mode method. Based on implicit Lyapunov functions approach, fixed-time analysis for a chain of integrators is presented in the work of Polyakov et al.[15] As pointed out in the work of Sanchez-Torres et al,[16] although the settling time is bounded and adjustable, it cannot always be prespecified arbitrarily according to requirements. This is because the settling time function derived from most existing fixed-time control methods normally depends on several design parameters, such that it is difficult to find a suitable selection of these parameters to satisfy the pregiven settling time, such as those by Polyakov et al[15,17] and Zuo,[18] or the control parameters are constrained to be within certain ranges, which makes it difficult, or even impossible, to achieve the settling time at will, such as those in the results in the work of Basin et al.[19] On the other hand, however, prespecifiable settling time is an important research topic both in theory and in practice, which arises from many applications, for example, missile guidance.[20] In the work of Sanchez-Torres et al,[16] simple predefined time-based robust controllers and sliding mode controllers are presented for first-order systems. An entirely new methodology for prescribed time regulation is introduced for a class of nonlinear systems with matching conditions in the work of Song et al.[21] Meanwhile, uncertainties do exist in practical systems. Adaptive control has been proved to be an effective method to control systems with parametric uncertainties. However, to the best of our knowledge, the prespecifiable fixed-time control has never been addressed with suitable adaptive approaches for uncertain nonlinear systems in strict-feedback form. The main difficulties of addressing these issues lie in the following two respects: (1) How to design a prespecifiable fixed-time controller is unclear due to the lack of systematic method. (2) It is nontrivial to handle parameter uncertainties with fixed-time convergence and the problem becomes much more challenge for uncertain nonlinear systems in strict-feedback form.

In this paper, we aim to overcome these challenges by proposing the prespecifiable fixed-time control schemes for uncertain strict-feedback systems with unknown control gain functions. Two cases are considered. Firstly, we consider the case where system states can be factored out from the upper bounds of nonlinear functions. By adding exponential state feedback and using fractional power integration as Lyapunov function candidate, together with the analysis skills in (31)-(35), a preassigned fixed-time stabilization controller is proposed for uncertain nonlinear systems in strict-feedback form. Then, the second type of controller focuses on relaxing the conditions of system nonlinear functions to more general case, which can also be used to cope with tracking problem. An adaptive fixed-time control strategy is proposed and the tracking error converges to an arbitrarily small neighborhood of zero within preassigned time. The main contributions are summarized as follows.

1. Based on the fixed-time stability theorem in Lemma 3, where the settling time is related to only two control parameters, a recursive design algorithm is developed for the construction of a global prespecifiable fixed-time stabilizer in the first control scheme.
2. A novel adaptive parameter estimation and analysis approach is introduced in the second control scheme, such that the prespecifiable fixed-time results are extended to adaptive control.
3. The salient feature of the established results lies in the fact that both control schemes are considered for uncertain nonlinear systems in strict-feedback form and explicit algorithms are provided for adjusting the control parameters to meet the preassigned settling time in this paper.

To make our contributions more clear, more detailed discussions on our approaches and comparisons with state-of-the-art results in the area will be provided through remarks at suitable places in subsequent sections.

The remaining part of this paper is organized as follows. In Section 2, the problem is formulated and some preliminaries are given. Design and analysis of fixed-time stabilization control and fixed-time tracking control are presented for uncertain strict-feedback systems in Section 3 and Section 4, respectively. Simulation is conducted to validate the effectiveness of the proposed schemes in Section 5, and finally, the conclusions are drawn in Section 6.

2 | PRELIMINARIES

2.1 | Some useful definitions

Given the nonlinear system

$$\dot{x} = f(t, x, c), \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $c \in \mathbb{R}^b$ represents a vector of the system parameters, $f : \mathbb{R}_+ \times D \to \mathbb{R}^n$ is continuous with respect to an open neighborhood D of the origin $x = 0$ and satisfies $f(0) = 0$, and the initial condition is expressed by $x(0) = x_0$.

In the following, we review some basic definitions and then introduce the concept of prespecifiable fixed-time stability.

Definition 1 (See the work of Polyakov[13]). The origin of (1) is globally finite-time stable if it is globally asymptotically stable and any solution $x(t, x_0)$ of (1) reaches the equilibria at some finite time moment, ie, $x(t, x_0) = 0, \forall t \geq T(x_0)$, where $T : \mathbb{R}^n \rightarrow \mathbb{R}_+ \bigcup \{0\}$ is the settling time function.

For example, the origin of system $\dot{x} = -x^{1/3}, x \in \mathbb{R}$ is finite-time stable since any solution converges to the origin in finite time $T(x_0) = \frac{3}{2}x_0^{2/3}$. However, if the system initial state x_0 is not available, the settling time $T(x_0)$ cannot be estimated in advance. Actually, $T(x_0)$ is an unbound function of x_0.

Definition 2 (See the work of Polyakov[13]). The origin of (1) is globally fixed-time stable if it is globally finite-time stable and the settling time function $T(x_0)$ is bounded, ie, $\exists T_{\max} > 0$, such that $T(x_0) \leq T_{\max}, \forall x_0 \in \mathbb{R}^n$.
For example, the origin of system $\dot{x} = -(x^2+1)\mathrm{sign}(x)$ is fixed-time stable. After some calculation, we have $\arctan|x| = -t + \arctan|x_0|$ when $x \neq 0$. Thus, the convergence time $T \leq \arctan|x_0| \leq \frac{\pi}{2}$, which is independent of the initial conditions.

Definition 3. The origin of (1) is globally prespecifiable fixed-time stable if it is globally fixed-time stable and the settling time $T_{\max}$ can be arbitrarily preassigned by designing parameters c properly in advance.

Remark 1. To further illustrate the concept of prespecifiable fixed-time stable, we give the following example. For Theorem 1 in the work of Zuo,[14] the settling time cannot be preassigned by user. According to (14), the settling time is derived as $T_{\max} = T_1 + T_2 + \epsilon(\tau)$, where $T_1 = \frac{1}{\alpha_1}\frac{n_1}{m_1-n_1} + \frac{1}{\beta_1}\frac{q_1}{q_1-p_1}$, $T_2 = \frac{1}{\alpha_2}\frac{n_2}{m_2-n_2} + \frac{1}{\beta_2}\frac{q_2}{q_2-p_2}$; $\epsilon(\tau)$ is a small time margin related to the boundary width $2\tau^{p_1/(q_1-p_1)}$; m_i, n_i, p_i, and q_i are positive odd integers satisfying $m_i > n_i, p_i < q_i < 2p_i$, $\alpha_i > 0$ and $\beta_i > 0$ for $i = 1, 2$. On the one hand, the accurate value of $\epsilon(\tau)$ is unknown, although bounded. On the other hand, the settling time $T_{\max}$ depends on a number of control parameters $m_i, n_i, p_i, q_i, \alpha_i, \beta_i$, which are correlated, thus cannot be chosen independently. As a result, it is difficult to find a suitable set of those parameters to achieve the pregiven settling time $T_{\max}$.

2.2 | Some useful lemmas

The following lemmas are extended from the works of Polyakov[13] and Sanchez-Torres et al[16] to provide a Lyapunov characterization of prespecifiable fixed-time stable systems and an upper bound of the settling time function T.

Lemma 1. *Considering system (1), suppose that there exists a positive definite and radially unbounded function $V(x)$: $\mathbb{R}^n \rightarrow \mathbb{R}$ such that $V(x) = 0 \Rightarrow x = 0$ and*

$$\dot{V}(x) \leq -\frac{c}{1-\alpha}e^{\beta V(x)^{1-\alpha}}V(x)^{\alpha}, \tag{2}$$

where $c > 0, \beta > 0, 0 < \alpha < 1$ and $e^{(\bullet)}$ denotes the exponential function of $\bullet$. Then, the system is globally fixed-time stable and the settling time function T is bounded by $T \leq T_{max} = \frac{1}{c\beta}$.

Proof. Refer to the Appendix. □

Lemma 2 (See the works of Polyakov[13] and Lopez-Ramirez et al[22]). *Consider system (1). If there exists a positive definite and radially unbounded function $V(x)$: $\mathbb{R}^n \rightarrow \mathbb{R}$ such that $V(x) = 0 \Rightarrow x = 0$ and*

$$\dot{V}(x) \leq -\alpha V^p(x) - \beta V^q(x) \tag{3}$$

with $\alpha, \beta > 0, p > 1$ and $0 < q < 1$, then system (1) is globally fixed-time stable and the settling time function T is bounded as $T \leq T_{max} = \frac{1}{\alpha(p-1)} + \frac{1}{\beta(1-q)}$.

In the rest of this subsection, some technical lemmas are listed for completeness of this paper.

Lemma 3 (See the work of Qian and Lin[23]). *For any real numbers $x_i, i = 1, \ldots, n$ and $0 < p \leq 1$, the following inequality holds:*

$$(|x_1| + \cdots + |x_n|)^p \leq |x_1|^p + \cdots + |x_n|^p. \tag{4}$$

If $0 < p = \frac{p_1}{p_2} \leq 1$, *where* $p_1 > 0, p_2 > 0$ *are all positive odd integers, then* $|x^p - y^p| \leq 2^{1-p}|x-y|^p$.

Lemma 4 (See the work of Qian and Lin[23]). *Let d and e be positive constants and suppose* $\gamma(x,y) > 0$ *is a real value function. Then, we have*

$$|x|^d|y|^e \leq \frac{d\gamma(x,y)|x|^{d+e}}{d+e} + \frac{e\gamma^{-d/e}(x,y)|y|^{d+e}}{d+e}. \tag{5}$$

Lemma 5 (See the work of Mitrinovic[24] and Chebyshev's inequality). *Let* $n \in \mathbb{N}$. *If* $x = (x_1, \dots, x_n)$ *and* $y = (y_1, \dots, y_n)$ *are two similarly ordered real sequences, ie,*

$$\begin{aligned} &x_1 \leq \cdots \leq x_n \text{ and } y_1 \leq \cdots \leq y_n, \\ \text{or } &x_1 \geq \cdots \geq x_n \text{ and } y_1 \geq \cdots \geq y_n, \end{aligned}$$

then the following inequality is true

$$\frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) \leq \sum_{i=1}^{n} x_i y_i. \tag{6}$$

3 | PRESPECIFIABLE FIXED-TIME STABILIZATION OF UNCERTAIN STRICT-FEEDBACK SYSTEMS

In this section, we consider the following class of uncertain nonlinear strict-feedback systems

$$\begin{aligned} \dot{x}_i &= f_i(t,\bar{x}_i) + g_i(t,\bar{x}_i)x_{i+1}, \quad i = 1, \dots, n-1 \\ \dot{x}_n &= f_n(t,\bar{x}_n) + g_n(t,\bar{x}_n)u \\ y &= x_1, \end{aligned} \tag{7}$$

where $\bar{x}_i = [x_1, x_2, \dots, x_i]^T \in \mathbb{R}^i$ and $u \in \mathbb{R}$ are the system state and input, respectively. $y \in \mathbb{R}$ is the output of the system, $f_i(t,\bar{x}_i)$ is a C^1 uncertain function with $f_i(t, 0, \dots, 0) = 0, \forall t$. The control gain $g_i(t,\bar{x}_i)$ is an unknown function that can be state and time dependent.

The control goal of this section is to design a controller u such that system (7) is prespecifiable fixed-time stable, namely, all states can converge to zero within bounded settling time regardless of system initial states and the settling time can be preassigned as required by choosing appropriate control parameters. To achieve this objective, the following assumptions are needed.

Assumption 1. For $f_i(t,\bar{x}_i), 1 \leq i \leq n$, there exist known C^1 nonnegative functions $\phi_i(\bar{x}_i)$ such that

$$|f_i(t,\bar{x}_i)| \leq (|x_1| + \cdots + |x_i|)\phi_i(\bar{x}_i). \tag{8}$$

Assumption 2. The gain functions $g_i(t,\bar{x}_i), i = 1, \dots, n$, are bounded and positive, ie, there exist known positive constants $\bar{g}_i$ and $\underline{g}_i$ such that

$$\underline{g}_i \leq g_i(\cdot) \leq \bar{g}_i. \tag{9}$$

Remark 2. It can be seen that system (7) is a lower-triangular model with time-varying gains. The same or similar version as Assumption 1 has been widely used. In the works of Qian and Lin,[25,26] practical output tracking and asymptotic stabilization, respectively, are achieved under the condition that $|f_i(t,\bar{x}_i)| \leq (|x_1|^{p_i} + \cdots + |x_i|^{p_i})\phi_i(\bar{x}_i)$, where $p_i \geq 1$ is an odd integer. Huang et al[11] and Shen and Huang[27] consider the finite-time stabilization problem for systems satisfying Assumption 1 and Pongvuthithum[28] studies global regulation of systems satisfying $|f_i(t,\bar{x}_i)| \leq (1+t^m)(|x_1|+\cdots+|x_i|)\rho_i(x_1, \dots, x_i, \theta)$ with θ being an unknown constant. In the work of Zhang and Wu,[29] fixed-time stabilization is also addressed for a class of dynamic nonholonomic systems based on Assumption 1. It should be noted that the designed controller is discontinuous and settling time cannot be preassigned within certain intervals in the work of Zhang and Wu,[29] which will be discussed further in Remark 6. Therefore, available results on prespecifiable fixed-time stabilization of system (7) satisfying Assumption 1 are still quite incomplete or unsatisfactory.

Now, we present a backstepping design procedure using induction to establish appropriate bounds for the derivatives of Lyapunov functions and, at the same time, derive virtual controls step by step. Let $d = \frac{4n}{2n+1}$ and $q_i = \frac{2n+3-2i}{2n+1}, i = 1, \dots, n-1$. For the sake of brevity, we omit the arguments of function, whenever this does not entail loss of clarity.

Step 1: We choose a Lyapunov function $V_1 = \frac{1}{2}x_1^2$. By combining (7)-(8) and using Lemma 4, the derivative of V_1 can be expressed as

$$\begin{aligned}\dot{V}_1 &\leq g_1(t,x_1)x_1x_2 + x_1^2\phi_1(x_1)\\ &\leq g_1(t,x_1)x_1x_2 + x_1^d \cdot x_1^{\frac{2}{2n+1}}\phi_1(x_1)\\ &\leq g_1(t,x_1)x_1(x_2 - x_2^*) + g_1(t,x_1)x_1x_2^* + x_1^d\tilde{\rho}_1(x_1),\end{aligned} \tag{10}$$

where $\tilde{\rho}_1(x_1) = (1+x_1^2)\phi_1(x_1)$ is a C^1 function. Then, we can design the virtual control x_2^* as

$$\begin{aligned}x_2^* &= -\frac{1}{\underline{g}_1}x_1^{\frac{2n-1}{2n+1}}\left(n-1+4n(2n+1)ce^{l(x_1^2+1)} + \tilde{\rho}_1(x_1)\right)\\ &= -\xi_1^{q_2}\beta_1(x_1),\end{aligned} \tag{11}$$

where $\xi_1 = x_1, \beta_1(x_1) = \frac{1}{\underline{g}_1}(n-1+4n(2n+1)ce^{l(x_1^2+1)} + \tilde{\rho}_1(x_1))$ is a C^1 function, c and l are positive design parameters. How to choose c and l based on the predefined settling time will be addressed later. According to Lemma 4, we get $x_1^{\frac{2}{2n+1}} \leq x_1^2 + 1$, thus

$$\dot{V}_1 \leq -\xi_1^d\left(n-1+4n(2n+1)ce^{lx_1^{2/(2n+1)}}\right) + g_1x_1\left(x_2 - x_2^*\right). \tag{12}$$

Remark 3. Different from traditional fractional power state-feedback–based methods, the virtual control in each of the constructive steps here includes an additional exponential term. For instance, in Step 1, we have introduced $4n(2n+1)ce^{l(x_1^2+1)}x_1^{\frac{2n-1}{2n+1}}$. Such a treatment, together with other design skills, renders prespecifiable fixed-time stabilization achievable as seen in the sequel.

Now, we carry out an analysis through induction. Suppose at Step $(i-1)$, where $i = 2, \dots, n$, there is a C^1 positive definite Lyapunov function V_{i-1} which satisfies

$$V_{i-1} \leq 2\left(\xi_1^2 + \cdots + \xi_{i-1}^2\right) \tag{13}$$

and a set of virtual controllers $x_1^*, x_2^*, \dots, x_i^*$ defined by

$$\begin{aligned}x_1^* &= 0, & \xi_1 &= x_1^{1/q_1} - x_1^{*1/q_1},\\ x_2^* &= -\xi_1^{q_2}\beta_1(x_1), & \xi_2 &= x_2^{1/q_2} - x_2^{*1/q_2},\\ &\vdots & &\vdots\\ x_i^* &= -\xi_{i-1}^{q_i}\beta_{i-1}(\bar{x}_{i-1}), & \xi_i &= x_i^{1/q_i} - x_i^{*1/q_i},\end{aligned}$$

with $\beta_1(\cdot) > 0, \dots, \beta_{i-1}(\cdot) > 0$ being C^1 functions, such that

$$\dot{V}_{i-1} \leq g_{i-1}\xi_{i-1}^{2-q_{i-1}}\left(x_i - x_i^*\right) - (n-i+1)\left(\sum_{k=1}^{i-1}\xi_k^d\right) - 4n(2n+1)c\left(\sum_{k=1}^{i-1}e^{l\xi_k^{2/(2n+1)}}\xi_k^d\right). \tag{14}$$

Next, we intend to derive a similar property of (14) at Step i, by proposing a suitable virtual control x_{i+1}^* with strategies including adding a power integrator technique. Consider the following Lyapunov candidate function:

$$V_i = V_{i-1} + W_i, \tag{15}$$

where $W_i = \int_{x_i^*}^{x_i} (s^{1/q_i} - x_i^{*1/q_i})^{2-q_i} ds$. Note that $W_i \geq 0$ and W_i tends to infinity as x_i increases to infinity. We first prove that condition (13) also holds at step i. With the aid of Lemma 3, we have

$$\left|x_i - x_i^*\right| \leq 2^{1-q_i}\left|x_i^{1/q_i} - x_i^{*1/q_i}\right|^{q_i} \leq 2|\xi_i|^{q_i}. \tag{16}$$

In the case that $x_i^* \leq x_i$, we have from (16) that

$$W_i \leq \int_{x_i^*}^{x_i} \left(x_i^{1/q_i} - x_i^{*1/q_i}\right)^{2-q_i} ds = \xi_i^{2-q_i}\left(x_i - x_i^*\right) \leq 2\xi_i^2. \tag{17}$$

Similar results can also be obtained for $x_i^* > x_i$, thus (13) holds at step i. Then, by taking the time derivative of (15), it follows that

$$\begin{aligned}\dot{V}_i \leq\ & g_{i-1}\xi_{i-1}^{2-q_{i-1}}\left(x_i - x_i^*\right) - (n-i+1)\left(\sum_{k=1}^{i-1}\xi_k^d\right) + g_i\xi_i^{2-q_i}x_{i+1} - 4n(2n+1)c\left(\sum_{k=1}^{i-1} e^{l\xi_k^{2/(2n+1)}}\xi_k^d\right) \\ & + \xi_i^{2-q_i} f_i(t,\bar{x}_i) - (2-q_i)\frac{d\left(x_i^{*1/q_i}\right)}{dt}\int_{x_i^*}^{x_i}\left(s^{1/q_i} - x_i^{*1/q_i}\right)^{1-q_i} ds.\end{aligned} \tag{18}$$

Now, we estimate the first term on the right-hand side of inequality (18). With the help of $q_i = q_{i-1} - \frac{2}{2n+1}$, Lemma 4 and (16), we have

$$g_{i-1}\xi_{i-1}^{2-q_{i-1}}\left(x_i - x_i^*\right) \leq 2\bar{g}_{i-1}|\xi_{i-1}|^{2-q_{i-1}}|\xi_i|^{q_i} \leq \frac{\xi_{i-1}^d}{3} + \zeta_i\xi_i^d, \tag{19}$$

where ζ_i is a positive constant. Since $\xi_i = x_i^{1/q_i} - x_i^{*1/q_i}$ and $x_i^* = -\xi_{i-1}^{q_i}\beta_{i-1}(\bar{x}_{i-1})$, we have

$$|x_i| \leq \left|\xi_i + x_i^{*1/q_i}\right|^{q_i} \leq |\xi_i|^{q_i} + |\xi_{i-1}|^{q_i}|\beta_{i-1}(\bar{x}_{i-1})|. \tag{20}$$

According to (8), (20) and Lemma 4, we can easily find C^1 positive functions $\tilde{\gamma}_i(\bar{x}_i)$ and $\bar{\gamma}_i(\bar{x}_i)$ such that

$$\begin{aligned}|f_i(t,\bar{x}_i)| &\leq \left(|\xi_1| + \sum_{k=2}^{i}(|\xi_k|^{q_k} + |\xi_{k-1}|^{q_k}|\beta_{k-1}(\cdot)|)\right)\phi_i(\bar{x}_i) \\ &\leq (|\xi_1|^{q_i} + |\xi_2|^{q_i} + \cdots + |\xi_i|^{q_i})\tilde{\gamma}_i(\bar{x}_i)\end{aligned} \tag{21}$$

$$\leq (|\xi_1|^{q_i-2/(2n+1)} + \cdots + |\xi_i|^{q_i-2/(2n+1)})\bar{\gamma}_i(\bar{x}_i). \tag{22}$$

Thus, for the fifth term on the right hand of (18), we have from Lemma 4 that

$$\left|\xi_i^{2-q_i} f_i(t,\bar{x}_i)\right| \leq \left|\xi_i^{2-q_i}\right|\left(\sum_{k=1}^{i}|\xi_k|^{q_i-2/(2n+1)}\right)\bar{\gamma}_i(\bar{x}_i) \leq \frac{1}{3}\left(\sum_{k=1}^{i-1}|\xi_k|^d\right) + \xi_i^d\tilde{\rho}_i(\bar{x}_i), \tag{23}$$

where $\tilde{\rho}_i(\bar{x}_i) \geq 0$ is a C^1 function.

Before completing the deduction and facilitating the construction of a prespecifiable fixed-time controller, the following proposition is presented, whose technical proof is given in the Appendix.

Proposition 1. *We can construct a C^1 function $\varpi_i(\bar{x}_i) \geq 0$ such that*

$$\left|\frac{d\left(x_i^{*1/q_i}\right)}{dt}\right| \leq \left(|\xi_1|^{(2n-1)/(2n+1)} + \cdots + |\xi_i|^{(2n-1)/(2n+1)}\right)\varpi_i(\bar{x}_i). \tag{24}$$

Now, according to (24) and Lemma 4, it can be deduced that

$$\begin{aligned}\left|(2-q_i)\frac{d\left(x_i^{*1/q_i}\right)}{dt}\int_{x_i^*}^{x_i}\left(s^{1/q_i}-x_i^{*1/q_i}\right)^{1-q_i}ds\right| &\leq (2-q_i)\left|x_i-x_i^*\right||\xi_i|^{1-q_i}\left(\sum_{k=1}^{i}|\xi_k|^{(2n-1)/(2n+1)}\right)\varpi_i(\bar{x}_i)\\ &\leq 2(2-q_i)|\xi_i|\left(\sum_{k=1}^{i}|\xi_k|^{(2n-1)/(2n+1)}\right)\varpi_i(\bar{x}_i)\\ &\leq \frac{1}{3}\left(\sum_{k=1}^{i-1}|\xi_k|^d\right)+\xi_i^d\tilde{\varpi}_i(\bar{x}_i),\end{aligned} \tag{25}$$

where $\tilde{\varpi}(\cdot)>0$ is a C^1 function and the second inequality holds due to the result $|x_i-x_i^*|=|(x_i^{1/q_i})^{q_i}-(x_i^{*1/q_i})^{q_i}|\leq 2|\xi_i|^{q_i}$. Substituting (19), (23), and (25) into (18), we have

$$\begin{aligned}\dot{V}_i \leq& -(n-i)\left(\sum_{k=1}^{i-1}\xi_k^d\right)+g_i\xi_i^{2-q_i}x_{i+1}^*+g_i\xi_i^{2-q_i}\left(x_{i+1}-x_{i+1}^*\right)\\ &-4n(2n+1)c\left(\sum_{k=1}^{i-1}e^{l\xi_k^{2/(2n+1)}}\xi_k^d\right)+\xi_i^d\left(\zeta_i+\tilde{\rho}_i(\bar{x}_i)+\tilde{\varpi}_i(\bar{x}_i)\right).\end{aligned} \tag{26}$$

Clearly, the virtual control

$$x_{i+1}^*=-\xi_i^{q_{i+1}}\beta_i(\bar{x}_i), \tag{27}$$

where $\beta_i(\bar{x}_i)=-\frac{1}{\underline{g}_i}(n-i+4n(2n+1)ce^{l(\xi_i^2+1)}+\zeta_i+\tilde{\rho}_i(\bar{x}_i)+\tilde{\varpi}_i(\bar{x}_i))$ with $c>0$ and $l>0$ being positive design parameters as in (11), results in

$$\dot{V}_i \leq g_i\xi_i^{2-q_i}\left(x_{i+1}-x_{i+1}^*\right)-(n-i)\left(\sum_{k=1}^{i}\xi_k^d\right)-4n(2n+1)c\left(\sum_{k=1}^{i}e^{l\xi_k^{2/(2n+1)}}\xi_k^d\right). \tag{28}$$

Thus, a similar property of (14) is derived and this completes the inductive analysis for Step i.

Step n: By using the inductive argument above at step n, we can design the following controller:

$$u=-\frac{1}{\underline{g}_n}\xi_n^{q_{n+1}}\left(4n(2n+1)ce^{l(\xi_n^2+1)}+\zeta_n+\tilde{\rho}_n(\bar{x}_n)+\tilde{\varpi}_n(\bar{x}_n)\right)=-\xi_n^{q_{n+1}}\beta_n(\bar{x}_n), \tag{29}$$

with $\beta_n(\bar{x}_n)>0$ being a C^1 function such that

$$\dot{V}_n \leq -4n(2n+1)c\left(\sum_{k=1}^{n}e^{l\xi_k^{2/(2n+1)}}\xi_k^d\right). \tag{30}$$

Theorem 1. *Consider the uncertain nonlinear system (7). Under Assumptions 1 and 2, if the controller (29) is applied, then all the trajectories of system (7) converge to zero within a preassigned time $T=\frac{1}{cl}$, where c and l are designed positive parameters.*

Proof. For $\xi_k (k=1,\ldots,n)$, there exists a sequence $(i_1,\ldots,i_n)$ such that $0\leq \xi_{i_1}^d\leq\cdots\leq\xi_{i_n}^d$ with $d=\frac{4n}{2n+1}$. According to the fact that $e^{lx^{1/2n}}$ is a monotonically increasing function with respect to $x\geq 0$, we have $e^{l(\xi_{i_1}^d)^{\frac{1}{2n}}}\leq\cdots\leq e^{l(\xi_{i_n}^d)^{\frac{1}{2n}}}$, which is equivalent to

$$e^{l\xi_{i_1}^{2/(2n+1)}}\leq\cdots\leq e^{l\xi_{i_n}^{2/(2n+1)}}. \tag{31}$$

By using Chebyshev's inequality in Lemma 5, it is easy to get that

$$\begin{aligned}\frac{1}{n}\left(\sum_{k=1}^{n} e^{l\xi_{ik}^{2/(2n+1)}}\right)\left(\sum_{k=1}^{n}\xi_{ik}^{d}\right) &\leq \sum_{k=1}^{n} e^{l\xi_{ik}^{2/(2n+1)}}\xi_{ik}^{d}\\ \text{ie}\quad \frac{1}{n}\left(\sum_{k=1}^{n} e^{l\xi_{k}^{2/(2n+1)}}\right)\left(\sum_{k=1}^{n}\xi_{k}^{d}\right) &\leq \sum_{k=1}^{n} e^{l\xi_{k}^{2/(2n+1)}}\xi_{k}^{d}.\end{aligned} \tag{32}$$

Substituting (32) into (30) results in

$$\dot{V}_n \leq -4(2n+1)c\left(\sum_{k=1}^{n} e^{l\xi_{k}^{2/(2n+1)}}\right)\left(\sum_{k=1}^{n}\xi_{k}^{d}\right). \tag{33}$$

According to the inequality of arithmetic and geometric means $\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} a_i$, where $a = (a_1, \dots, a_n)$ is a sequence of positive numbers, we have

$$n\left(e^{l\xi_1^{2/(2n+1)}+\cdots+l\xi_n^{2/(2n+1)}}\right)^{\frac{1}{n}} \leq e^{l\xi_1^{2/(2n+1)}} + \cdots + e^{l\xi_n^{2/(2n+1)}}, \tag{34}$$

which further implies that

$$\dot{V}_n \leq -4n(2n+1)ce^{\frac{l}{n}\left(\xi_1^{2/(2n+1)}+\cdots+\xi_n^{2/(2n+1)}\right)}\left(\sum_{k=1}^{n}\xi_{k}^{d}\right). \tag{35}$$

On the other hand, from the inductive step, we know that

$$V_n \leq 2\left(\xi_1^2 + \cdots + \xi_n^2\right). \tag{36}$$

For the sake of simplicity, we define $\alpha = \frac{2n}{2n+1}$ and $1-\alpha = \frac{1}{2n+1}$. With the aid of Lemma 3, we have

$$V_n^{\alpha} \leq 2\left(\xi_1^d + \cdots + \xi_n^d\right) \quad \text{and} \quad V_n^{1-\alpha} \leq 2\left(\xi_1^{2/(2n+1)} + \cdots + \xi_n^{2/(2n+1)}\right). \tag{37}$$

Substituting (37) into (35), it follows that

$$\dot{V}_n \leq -2n(2n+1)ce^{\frac{l}{2n}V_n^{1-\alpha}}V_n^{\alpha} = -\frac{2nc}{1-\alpha}e^{\frac{l}{2n}V_n^{1-\alpha}}V_n^{\alpha}. \tag{38}$$

Therefore, from Lemma 1, it is concluded that all closed-loop system states converge to zero within the settling time $T \leq T_{\max} = \frac{1}{cl}$. The proof is completed. □

Remark 4. A prespecifiable fixed-time controller is established in (29) for a class of uncertain nonlinear systems. In practical applications, the functions $\zeta_i, \tilde{\rho}_i(\bar{x}_i), \tilde{\varpi}_i(\bar{x}_i)(i = 1, \dots, n)$ can be constructed following the proof of Theorem 1 step by step. This inductive construction idea is common in finite-time control literatures.[11,12,27,30]

Remark 5. Although exponential function is included in the controller, its value remains bounded and decreases as the states approach zero. For any preassigned settling time T, we can first set a small value of control parameter l to avoid excessive exponential terms and then simply design $c = \frac{1}{Tl}$ in (29). In this way, system (7) is stabilized within predefined time T. In practical applications, a smaller settling time T leads to faster convergence, but might cause larger control effort, especially at the initial stage. Therefore, certain trade-off between convergence time and the control effort is normally needed in practice.

Remark 6. The main differences compared to existing relevant references are worth noting:

(1) Although various control methods have been developed for fixed-time stabilization problem, most of them are considered for linear systems such as those in the works of Polyakov et al[13,15] and one-dimensional systems such as the results in the works of Sanchez-Torres et al,[16] Basin et al,[19] and Defoort et al.[31] When systems are

in strict-feedback form, the problem becomes difficult and complicated in the construction of Lyapunov functions and stability analysis, as evidenced from the recursive design and analysis approaches toward Theorem 1 involving techniques such as induction. (2) In the works of Tian et al[32] and Zhang and Wu,[29] fixed-time control schemes are developed for double integrator systems by using bilimit homogeneous technique and nonholonomic systems with switching approach, respectively. It should be noted that the value of partial settling time T_1' in the work of Tian et al[32] is unknown, although it is independent of the initial condition. In addition, the settling time in the work of Zhang and Wu[29] is $T = \frac{2}{1-\sigma} + \frac{2^d n^{d-1}}{l(d-1)}$, where $\sigma = \frac{2n}{2n+1}, d = \frac{\gamma+\rho}{2}, l > 0$ and $\gamma + \rho > 2$. According to the first term in T, we have $T > 2(2n+1)$, which implies that the settling time cannot be preassigned within the interval $(0, 2(2n+1)]$. While in this paper, the settling time can be prespecified **as needed**, even in the presence of uncertain nonlinearities. (3) Compared with finite-time stabilization results in the works of Huang et al,[11] Huang et al,[12] Shen and Huang,[27] and Sun et al,[30] where an inductive construction process is also used in control design, a new fixed-time stabilization controller in our paper is derived as presented in Theorem 1, in which the settling time is user-assignable in advance by designing the control parameters as discussed in Remark 5.

4 | PRESPECIFIABLE FIXED-TIME TRACKING CONTROL OF UNCERTAIN STRICT-FEEDBACK SYSTEMS WITH EXTERNAL DISTURBANCES

In the study of prespecifiable fixed-time stabilization in Section 3, it is assumed that the system states can be factored out from the upper bounds of nonlinear functions $f_i(\bar{x}_i)$. In this section, we relax this condition to more general case, under which the prespecifiable fixed-time tracking control problem (more general yet more challenging than stabilization) is addressed. The following class of uncertain nonlinear strict-feedback systems with external disturbances is considered:

$$\begin{aligned}
\dot{x}_i &= f_i(t, \bar{x}_i) + g_i(t, \bar{x}_i)x_{i+1} + d_i(t), \quad i = 1, \dots, n-1 \\
\dot{x}_n &= f_n(t, \bar{x}_n) + g_n(t, \bar{x}_n)u + d_n(t) \\
y &= x_1,
\end{aligned} \tag{39}$$

where $d_i(t), i = 1, \dots, n$ denote the unknown time-varying disturbances and the notions of other symbols are the same as (7).

For systems with nonparametric uncertainties and external disturbances, it is nontrivial and too costly to develop continuous control to achieve zero steady-state error. On the other hand, convergence with sufficient steady-state accuracy is acceptable for most practical applications. Therefore, the control goal of this section is to design a prespecifiable fixed-time tracking controller u such that the tracking error $z_1 = y - y_d$ converges to an arbitrarily small neighborhood of zero with prespecifiable settling time, and meanwhile, all the internal signals of the closed-loop systems are bounded. The following assumptions are needed to achieve this goal.

Assumption 3. For lumped uncertainties $f_i(t, \bar{x}_i), 1 \le i \le n$, there exist unknown constants $a_i \ge 0$ and known C^1 functions $\varphi_i(\bar{x}_i) \ge 0$ such that

$$|f_i(\bar{x}_i)| \le a_i \varphi_i(\bar{x}_i). \tag{40}$$

Assumption 4. The gain functions $g_i(t, \bar{x}_i), i = 1, \dots, n$ are positive and there exist unknown constants $0 < \underline{g}_i < \bar{g}_i < \infty$ such that

$$0 < \underline{g}_i \le g_i(\cdot) \le \bar{g}_i. \tag{41}$$

Assumption 5. The unknown time-varying disturbances $d_i(t) (i = 1, \dots, n)$ are bounded, ie, $|d_i(t)| < D_i, \forall t \ge 0$ with D_i as an unknown constant.

Assumption 6. The desired trajectory y_d and its derivatives up to the nth order are known and bounded.

Remark 7. In the work of Huang et al,[12] finite-time stabilization is achieved for systems on assumption that $|f_i(\bar{x}_i)| \le a_i(|x_1| + \cdots + |x_i|)\varphi_i(\bar{x}_i)$. In this case, the states x_i are assumed to be factored out from the upper bound of $f_i(\bar{x}_i)$. Whereas here in Assumption 3, this condition is not required. On the other hand, Assumption 3 is also more general than linearly parameterizable nonlinearity $f_i(\bar{x}_i) = \theta_i^T \varpi_i(\bar{x}_i)$ as assumed in the work of Jin,[33] where $\theta_i \in \mathbb{R}^m$ is an unknown vector. Indeed, from $f_i(\bar{x}_i) = \theta_i^T \varpi_i(\bar{x}_i)$, it is easy to obtain that $|f_i(\bar{x}_i)| \leq \|\theta_i^T\|\|\varpi_i(\bar{x}_i)\| \leq a_i\varphi_i(\bar{x}_i)$ with $a_i = \|\theta_i^T\|$ and $\varphi_i(\bar{x}_i) = \|\varpi_i(\bar{x}_i)\|$ and then Assumption 3 naturally holds. Besides, Assumption 3 is similar to the core function method in the work of Song et al,[34] and covers a wider class of unknown nonlinear functions. Consider the uncertain function $L(x) = \rho_1 \cos(\rho_2 x) + xe^{-|\rho_3 x|}$, where ρ_1, ρ_2, and ρ_3 are unknown constants. Clearly, neither state x nor unknown parameters ρ_1, ρ_2, and ρ_3 can be factored out from $L(x)$. However, it is effortless to obtain function $\varphi(x) = 1 + |x|$, such that $|L(x)| \leq a\varphi(x)$ with $a = \max\{\rho_1, 1\}$. In fact $\varphi_i(\bar{x}_i)$ can be easily derived with only crude model information, which can also be seen from Example 2 in simulation section. To our best knowledge, there is no result reported in the literature on the fixed-time tracking problem for system (39) satisfying (40). Different from Assumption 2, here, $\bar{g}_i$ in Assumption 4 can be unknown rather than known, which is not included in the corresponding controller although they are used for stability analysis.

Here, we present an adaptive backstepping control design procedure. As it is difficult to establish the fixed-time convergence result by using the directly regular adaptive design procedure, a novel adaptive parameter estimation and analysis approach is introduced to handle system uncertainties. Some standard design procedures, similar to the work of Krstic et al,[35] are omitted here just for simplicity. First, the following change of coordinates are introduced:

$$\begin{aligned} z_1 &= y - y_d, \\ z_i &= x_i - x_i^*, i = 2, \dots, n. \end{aligned} \tag{42}$$

Step 1: The derivative of the tracking error z_1 is given as

$$\dot{z}_1 = f_1 + g_1\left(z_2 + x_2^*\right) + d_1 - \dot{y}_d. \tag{43}$$

Based on Assumption 3 and Assumption 5, the time derivative of $\frac{1}{2}z_1^2$ along (43) is

$$z_1\dot{z}_1 \leq |z_1|a_1\varphi_1 + |z_1|D_1 + g_1z_1z_2 - z_1\dot{y}_d + g_1z_1x_2^* \leq |z_1|b_1\phi_1 + g_1z_1z_2 - z_1\dot{y}_d + g_1z_1x_2^*, \tag{44}$$

where $b_1 = \max\{a_1, D_1\}$ and $\phi_1 = \varphi_1 + 1$. Note that, for any variable $\rho \in \mathbb{R}$ and constant $\varepsilon > 0$, the following relationship holds: $0 \leq |\rho| - \frac{\rho^2}{\sqrt{\rho^2+\varepsilon^2}} < \varepsilon$, where $|\rho|$ denotes the absolute value of ρ. Thus,

$$|z_1|b_1\phi_1 \leq b_1\frac{z_1^2\phi_1^2}{\sqrt{z_1^2\phi_1^2 + \varepsilon^2}} + b_1\varepsilon.$$

By defining $\theta = [b_1, \bar{g}_1, \dots, b_{n-1}, \bar{g}_{n-1}, b_n]^T \in \mathbb{R}^{2n-1}$ with $b_i = \max\{a_i, D_i\}(i = 1, \dots, n)$ and $\eta_1 = [\frac{z_1\phi_1^2}{\sqrt{z_1^2\phi_1^2+\varepsilon^2}}, 0, \dots, 0]^T \in \mathbb{R}^{2n-1}$, it follows from (44) that

$$z_1\dot{z}_1 \leq z_1\theta^T\eta_1 + g_1z_1z_2 - z_1\dot{y}_d + g_1z_1x_2^* + b_1\varepsilon. \tag{45}$$

Now, we introduce an unknown parameter θ_0 to describe the bounding effect of all the unknown terms in (45), instead of addressing these unknown terms themselves. More specifically,

$$z_1\theta^T\eta_1 \leq \theta_0\frac{z_1^2\|\eta_1\|^2}{\sqrt{z_1^2\|\eta_1\|^2 + \varepsilon^2}} + \theta_0\varepsilon, \tag{46}$$

where θ_0 is an unknown constant such that $\|\theta\| \leq \theta_0$.

Remark 8. Here, we pause to stress that instead of estimating unknown constants a_i, g_i, and D_i directly, a new unknown vector θ is introduced to avoid overparametrization.[36] Then, we estimate only the upper bound of θ and analyze the adaptive law at the final step. This one parameter estimation approach is beneficial to prespecifiable fixed-time Lyapunov stability analysis as seen shortly.

The Lyapunov function at this step is designed as

$$V_1 = \frac{1}{2}z_1^2 + \frac{1}{2\sigma_0}\tilde{\theta}_0^2, \tag{47}$$

where $\tilde{\theta}_0 = \hat{\theta}_0 - \theta_0$ with $\hat{\theta}_0$ being the estimate of θ_0, and σ_0 is a positive design parameter. Hence, the derivative of V_1 along (45) is

$$\begin{aligned}\dot{V}_1 \leq & -k\left(\frac{1}{2}z_1^2\right)^{\frac{3}{4}} - k(n+1)\left(\frac{1}{2}z_1^2\right)^2 + g_1 z_1 z_2 + g_1 z_1 x_2^* + z_1\bar{x}_2^* - \tilde{\theta}_0\frac{z_1^2\|\eta_1\|^2}{\sqrt{z_1^2\|\eta_1\|^2+\varepsilon^2}} + \frac{1}{\sigma_0}\tilde{\theta}_0\dot{\hat{\theta}}_0 + \varepsilon(\theta_0+b_1)\\ \leq & -k\left(\frac{1}{2}z_1^2\right)^{\frac{3}{4}} - k(n+1)\left(\frac{1}{2}z_1^2\right)^2 + g_1 z_1 z_2 + g_1 z_1 x_2^* + z_1\bar{x}_2^* + \frac{1}{\sigma_0}\tilde{\theta}_0(\dot{\hat{\theta}}_0 - \varsigma_1) - \frac{\sigma_1}{\sigma_0}\tilde{\theta}_0\hat{\theta}_0 - \frac{\sigma_2}{\sigma_0^2}\tilde{\theta}_0\hat{\theta}_0^3 + \varepsilon(\theta_0+b_1),\end{aligned} \tag{48}$$

where k, σ_1, σ_2 are all positive design parameters, and

$$\bar{x}_2^* = -\dot{y}_d + \left(\frac{1}{2}\right)^{\frac{3}{4}} kS(z_1) + \left(\frac{1}{2}\right)^2 k(n+1)z_1^3 + \hat{\theta}_0\frac{z_1\|\eta_1\|^2}{\sqrt{z_1^2\|\eta_1\|^2+\varepsilon^2}} \tag{49}$$

$$\varsigma_1 = \sigma_0\frac{z_1^2\|\eta_1\|^2}{\sqrt{z_1^2\|\eta_1\|^2+\varepsilon^2}} - \sigma_1\hat{\theta}_0 - \frac{\sigma_2}{\sigma_0}\hat{\theta}_0^3, \tag{50}$$

with $S(z_1) = \begin{cases} z_1^{\frac{1}{2}}, & z_1 \geq 0 \\ -(-z_1)^{\frac{1}{2}}, & z_1 < 0. \end{cases}$

Then, we can design the virtual control x_2^* as

$$x_2^* = -\frac{1}{\underline{g}_1}\frac{z_1\bar{x}_2^{*2}}{\sqrt{z_1^2\bar{x}_2^{*2}+\varepsilon^2}}. \tag{51}$$

It should be noted that both $\bar{x}_2^*$ are combined with z_1 in (51) rendering $z_1 S(z_1)$ in the designed virtual controller x_2^* also continuously differentiable. By using the fact that

$$g_1 z_1 x_2^* \leq -\frac{z_1^2\bar{x}_2^{*2}}{\sqrt{z_1^2\bar{x}_2^{*2}+\varepsilon^2}} \leq -\left|z_1\bar{x}_2^*\right| + \varepsilon \tag{52}$$

and substituting (52) into (48), we obtain

$$\dot{V}_1 \leq -k\left(\frac{1}{2}z_1^2\right)^{\frac{3}{4}} - k(n+1)\left(\frac{1}{2}z_1^2\right)^2 - \frac{\sigma_1}{\sigma_0}\tilde{\theta}_0\hat{\theta}_0 - \frac{\sigma_2}{\sigma_0^2}\tilde{\theta}_0\hat{\theta}_0^3 + g_1 z_1 z_2 + \frac{1}{\sigma_0}\tilde{\theta}_0(\dot{\hat{\theta}}_0 - \varsigma_1) + C_1, \tag{53}$$

where $C_1 = \varepsilon(\theta_0 + b_1 + 1)$.

Step i ($i = 2, \ldots, n-1$): The time derivative of $z_i = x_i - x_i^*$ is

$$\dot{z}_i = f_i + g_i z_{i+1} + g_i x_{i+1}^* + d_i - \dot{x}_i^*, \tag{54}$$

where $\dot{x}_i^* = \frac{\partial x_i^*}{\partial\hat{\theta}_0}\dot{\hat{\theta}}_0 + \sum_{j=1}^{i-1}\frac{\partial x_i^*}{\partial x_j}(g_j x_{j+1} + f_j + d_j) + \sum_{j=1}^{i}\frac{\partial x_i^*}{\partial y_d^{(j-1)}}y_d^{(j)}$. Thus, the time derivative of $\frac{1}{2}z_i^2$ along (54) is

$$z_i\dot{z}_i \leq |z_i|(a_i\varphi_i + D_i) + g_i z_i z_{i+1} + g_i z_i x_{i+1}^* - z_i\frac{\partial x_i^*}{\partial\hat{\theta}_0}\dot{\hat{\theta}}_0 - z_i\sum_{j=1}^{i-1}\frac{\partial x_i^*}{\partial x_j}g_j x_{j+1} + |z_i|\sum_{j=1}^{i-1}\left|\frac{\partial x_i^*}{\partial x_j}\right|(a_j\varphi_j + D_j) - z_i\sum_{j=1}^{i}\frac{\partial x_i^*}{\partial y_d^{(j-1)}}y_d^{(j)}. \tag{55}$$

For $g_{i-1}z_{i-1}z_i$ in step $i-1$ and $-z_i\sum_{j=1}^{i-1}\frac{\partial x_i^*}{\partial x_j}g_jx_{j+1}$ in (55), we have

$$g_{i-1}z_{i-1}z_i - z_i\sum_{j=1}^{i-1}\frac{\partial x_i^*}{\partial x_j}g_jx_{j+1} \le \bar{g}_{i-1}\frac{z_{i-1}^2z_i^2}{\sqrt{z_{i-1}^2z_i^2+\varepsilon^2}} + \sum_{j=1}^{i-1}\bar{g}_j\frac{z_i^2\left(\frac{\partial x_i^*}{\partial x_j}\right)^2x_{j+1}^2}{\sqrt{z_i^2\left(\frac{\partial x_i^*}{\partial x_j}\right)^2x_{j+1}^2+\varepsilon^2}} + \varepsilon\left(2\bar{g}_{i-1}+\sum_{j=1}^{i-2}\bar{g}_j\right). \tag{56}$$

For $|z_i|(a_i\varphi_i+D_i)$ and $|z_i|\sum_{j=1}^{i-1}|\frac{\partial x_i^*}{\partial x_j}|(a_j\varphi_j+D_j)$ in (55), we have

$$\begin{aligned}|z_i|(a_i\varphi_i+D_i) + |z_i|\sum_{j=1}^{i-1}\left|\frac{\partial x_i^*}{\partial x_j}\right|\left(a_j\varphi_j+D_j\right) &\le |z_i|b_i\phi_i + |z_i|\sum_{j=1}^{i-1}\left|\frac{\partial x_i^*}{\partial x_j}\right|b_j\phi_j \\ &\le b_i\frac{z_i^2\phi_i^2}{\sqrt{z_i^2\phi_i^2+\varepsilon^2}} + \sum_{j=1}^{i-1}b_i\frac{z_i^2\left(\frac{\partial x_i^*}{\partial x_j}\right)^2\phi_j^2}{\sqrt{z_i^2\left(\frac{\partial x_i^*}{\partial x_j}\right)^2\phi_j^2+\varepsilon^2}} + \varepsilon\left(\sum_{j=1}^{i}b_j\right),\end{aligned} \tag{57}$$

where $\phi_j = \varphi_j + 1(j=1,\ \dots, i)$. By designing

$$\eta_i = \left[\frac{z_i\left(\frac{\partial x_i^*}{\partial x_1}\right)^2\phi_1^2}{\sqrt{z_i^2\left(\frac{\partial x_i^*}{\partial x_1}\right)^2\phi_1^2+\varepsilon^2}}, \frac{z_i\left(\frac{\partial x_i^*}{\partial x_1}\right)^2x_2^2}{\sqrt{z_i^2\left(\frac{\partial x_i^*}{\partial x_1}\right)^2x_2^2+\varepsilon^2}}, \dots, \frac{z_i\left(\frac{\partial x_i^*}{\partial x_{i-1}}\right)^2\phi_{i-1}^2}{\sqrt{z_i^2\left(\frac{\partial x_i^*}{\partial x_{i-1}}\right)^2\phi_{i-1}^2+\varepsilon^2}},\right.$$
$$\left.\frac{z_i\left(\frac{\partial x_i^*}{\partial x_{i-1}}\right)^2x_i^2}{\sqrt{z_i^2\left(\frac{\partial x_i^*}{\partial x_{i-1}}\right)^2x_i^2+\varepsilon^2}} + \frac{z_{i-1}^2z_i}{\sqrt{z_{i-1}^2z_i^2+\varepsilon^2}}, \quad \frac{z_i\phi_i^2}{\sqrt{z_i^2\phi_i^2+\varepsilon^2}}, 0, \dots, 0\right]^T \in \mathbb{R}^{2n-1},$$

it follows that

$$g_{i-1}z_{i-1}z_i + z_i\dot{z}_i \le z_i\theta^T\eta_i + g_iz_iz_{i+1} + g_iz_ix_{i+1}^* - z_i\frac{\partial x_i^*}{\partial\hat{\theta}_0}\dot{\hat{\theta}}_0 - z_i\sum_{j=1}^{i}\frac{\partial x_i^*}{\partial y_d^{(j-1)}}y_d^{(j)} + \varepsilon\left(\sum_{j=1}^{i-2}\bar{g}_j + 2\bar{g}_{i-1} + \sum_{j=1}^{i}b_j\right). \tag{58}$$

Therefore, by using the similar procedure as in step 1 and defining $V_i = V_{i-1} + \frac{1}{2}z_i^2$, it can be obtained that

$$\dot{V}_i \le -k\sum_{j=1}^{i}\left(\frac{1}{2}z_j^2\right)^{\frac{3}{4}} - k(n+1)\sum_{j=1}^{i}\left(\frac{1}{2}z_j^2\right)^2 - \frac{\sigma_1}{\sigma_0}\tilde{\theta}_0\hat{\theta}_0 - \frac{\sigma_2}{\sigma_0^2}\tilde{\theta}_0\hat{\theta}_0^3 + \frac{1}{\sigma_0}\tilde{\theta}_0(\dot{\hat{\theta}}_0 - \varsigma_i) + g_iz_iz_{i+1} + \sum_{j=2}^{i}z_j\frac{\partial x_j^*}{\partial\hat{\theta}_0}(\varsigma_i - \dot{\hat{\theta}}_0) + C_i, \tag{59}$$

where $C_i = C_{i-1} + \varepsilon(\sum_{j=1}^{i-2}\bar{g}_j + 2\bar{g}_{i-1} + \sum_{j=1}^{i}b_j + \theta_0 + 1)$, $\varsigma_i = \varsigma_{i-1} + \sigma_0\frac{z_i^2\|\eta_i\|^2}{\sqrt{z_i^2\|\eta_i\|^2+\varepsilon^2}}$.

The virtual control x_{i+1}^* is designed as

$$x_{i+1}^* = -\frac{1}{\underline{g}_i}\frac{z_i\bar{x}_{i+1}^{*2}}{\sqrt{z_i^2\bar{x}_{i+1}^{*2}+\varepsilon^2}}, \tag{60}$$

$$\bar{x}_{i+1}^* = \left(\frac{1}{2}\right)^{\frac{3}{4}}kS(z_i) + \left(\frac{1}{2}\right)^2k(n+1)z_i^3 + \hat{\theta}_0\frac{z_i\|\eta_i\|^2}{\sqrt{z_i^2\|\eta_i\|^2+\varepsilon^2}} - \sum_{j=1}^{i}\frac{\partial x_i^*}{\partial y_d^{(j-1)}}y_d^{(j)} - \frac{\partial x_i^*}{\partial\hat{\theta}_0}\varsigma_i - \sum_{j=2}^{i-1}z_j\frac{\partial x_j^*}{\partial\hat{\theta}_0}\frac{\sigma_0z_i\|\eta_i\|^2}{\sqrt{z_i^2\|\eta_i\|^2+\varepsilon^2}} \tag{61}$$

with $S(z_i) = \begin{cases} z_i^{\frac{1}{2}}, & z_i \ge 0 \\ -(-z_i)^{\frac{1}{2}}, & z_i < 0. \end{cases}$

Step n: From (59)-(61) for $i = n$, we obtain

$$\dot{V}_n \leq -k\sum_{j=1}^{n}\left(\frac{1}{2}z_j^2\right)^{\frac{3}{4}} - k(n+1)\sum_{j=1}^{n}\left(\frac{1}{2}z_j^2\right)^2 - \frac{\sigma_1}{\sigma_0}\tilde{\theta}_0\hat{\theta}_0 - \frac{\sigma_2}{\sigma_0^2}\tilde{\theta}_0\hat{\theta}_0^3 + \frac{1}{\sigma_0}\tilde{\theta}_0(\dot{\hat{\theta}}_0 - \varsigma_n) + \sum_{j=2}^{n} z_j\frac{\partial x_j^*}{\partial\hat{\theta}_0}(\varsigma_n - \dot{\hat{\theta}}_0) + C_n, \tag{62}$$

where $C_n = C_{n-1} + \varepsilon(\sum_{j=1}^{n-2}\bar{g}_j + 2\bar{g}_{n-1} + \sum_{j=1}^{n} b_j + \theta_0 + 1)$.
Now, we can design the adaptive control law as follows:

$$\begin{aligned} u &= -\frac{1}{\underline{g}_n}\frac{z_n\bar{x}_{n+1}^{*2}}{\sqrt{z_n^2\bar{x}_{n+1}^{*2} + \varepsilon^2}}, \\ \dot{\hat{\theta}}_0 &= \varsigma_n. \end{aligned} \tag{63}$$

The above backstepping design leads to the following theorem.

Theorem 2. *Consider the uncertain nonlinear system (39). Under Assumptions 3 to 6, if the adaptive controller (63) is applied, then the following objectives are achieved.*

$\mathcal{O}_1$) *The closed-loop system is stable and all the signals are uniformly bounded.*
$\mathcal{O}_2$) *The output tracking error z_1 converges to a set which can be designed as an arbitrarily small neighborhood of zero, within a predefined time $T = \frac{4}{k} + \frac{1}{(1-\lambda)k}$, where $0 < \lambda < 1$ and k are positive designed parameters.*

Proof. For the term $-\frac{\sigma_1}{\sigma_0}\tilde{\theta}_0\hat{\theta}_0$ in (62), it can be obtained $-\frac{\sigma_1}{\sigma_0}\tilde{\theta}_0\hat{\theta}_0 \leq -\frac{\sigma_1}{2\sigma_0}\tilde{\theta}_0^2 + \frac{\sigma_1}{2\sigma_0}{\theta_0}^2$. Note that

$$-\frac{\sigma_1}{2\sigma_0}\tilde{\theta}_0^2 = -\frac{\sigma_1}{4\sigma_0}\tilde{\theta}_0^2 + \frac{1}{2\sqrt{2\sigma_0}}\frac{k^2}{\sigma_1}|\tilde{\theta}_0| - k\left(\frac{1}{2\sigma_0}\tilde{\theta}_0^2\right)^{\frac{3}{4}} - \frac{1}{4\sigma_0}\left(\sqrt{\sigma_1}|\tilde{\theta}_0| - (2\sigma_0)^{\frac{1}{4}}\frac{k}{\sqrt{\sigma_1}}\sqrt{|\tilde{\theta}_0|}\right)^2 \tag{64}$$

and

$$\frac{1}{2\sqrt{2\sigma_0}}\frac{k^2}{\sigma_1}|\tilde{\theta}_0| \leq \frac{\sigma_1}{8\sigma_0}|\tilde{\theta}_0|^2 + \frac{k^4}{4\sigma_1^3}, \tag{65}$$

we further have

$$-\frac{\sigma_1}{\sigma_0}\tilde{\theta}_0\hat{\theta}_0 \leq -k\left(\frac{1}{2\sigma_0}\tilde{\theta}_0^2\right)^{\frac{3}{4}} + \frac{\sigma_1}{2\sigma_0}{\theta_0}^2 + \frac{k^4}{4\sigma_1^3}. \tag{66}$$

For the term $-\frac{\sigma_2}{\sigma_0^2}\tilde{\theta}_0\hat{\theta}_0^3$ in (62), it follows that $-\frac{\sigma_2}{\sigma_0^2}\tilde{\theta}_0\hat{\theta}_0^3 = -\frac{\sigma_2}{\sigma_0^2}\tilde{\theta}_0^4 - \frac{3\sigma_2}{\sigma_0^2}\tilde{\theta}_0^3\theta_0 - \frac{3\sigma_2}{\sigma_0^2}\tilde{\theta}_0^2\theta_0^2 - \frac{\sigma_2}{\sigma_0^2}\tilde{\theta}_0\theta_0^3$. From Lemma 4, we have

$$\begin{aligned} -\frac{3\sigma_2}{\sigma_0^2}\tilde{\theta}_0^3\theta_0 &\leq \frac{3\sigma_2}{\sigma_0^2}\frac{3}{4\tau^{\frac{4}{3}}}\left|\tilde{\theta}_0^3\right|^{\frac{4}{3}} + \frac{3\sigma_2}{\sigma_0^2}\frac{\tau^4}{4}\theta_0^4 \\ -\frac{\sigma_2}{\sigma_0^2}\tilde{\theta}_0\theta_0^3 &\leq \frac{3\sigma_2}{\sigma_0^2}\tilde{\theta}_0^2\theta_0^2 + \frac{\sigma_2}{12\sigma_0^2}\theta_0^4. \end{aligned} \tag{67}$$

Thus, by choosing $\tau^{\frac{4}{3}} = 3$, it is easy to get that

$$-\frac{\sigma_2}{\sigma_0^2}\tilde{\theta}_0\hat{\theta}_0^3 \leq -\sigma_2\left(\frac{1}{2\sigma_0}\tilde{\theta}_0^2\right)^2 + \frac{61\sigma_2}{3\sigma_0^2}\theta_0^4. \tag{68}$$

By substituting (63), (66), and (68) into (62), we can obtain that

$$\dot{V}_n \leq -k\sum_{j=1}^{n}\left(\frac{1}{2}z_j^2\right)^{\frac{3}{4}} - k(n+1)\sum_{j=1}^{n}\left(\frac{1}{2}z_j^2\right)^2 - k\left(\frac{\tilde{\theta}_0^2}{2\sigma_0}\right)^{\frac{3}{4}} - \sigma_2\left(\frac{\tilde{\theta}_0^2}{2\sigma_0}\right)^2 + \bar{C}, \tag{69}$$

where

$$\bar{C} = C_n + \frac{\sigma_1}{2\sigma_0}{\theta_0}^2 + \frac{k^4}{4\sigma_1^3} + \frac{61\sigma_2}{3\sigma_0^2}\theta_0^4. \tag{70}$$

Now, with the aid of Lemma 3 and Cauthy-Schwarz inequality $\left(\sum_{i=1}^{n} x_i\right)^2 \leq n \sum_{i=1}^{n} x_i^2$, we have

$$\dot{V}_n \leq -kV_n^{\frac{3}{4}} - kV_n^2 + \bar{C} \tag{71}$$

by choosing $\sigma_2 = k(n+1)$. Then, it can be concluded from (71) that V_n enters the set $\Omega_1 = \{V_n | V_n^2 < \frac{\bar{C}}{k}\}$ as time goes by. Once V_n is outside the set Ω_1, we have $\dot{V}_n < 0$. Therefore, $V_n \in L_\infty$, which indicates that $z_i \in L_\infty$ and $\tilde{\theta}_0 \in L_\infty$. Based on this result, we can conclude that all the internal signals are bounded.

On the other hand, when $0 < \lambda < 1$ and $V_n^2 \geq \frac{\bar{C}}{\lambda k}$, we have $\bar{C} \leq \lambda k V_n^2$. From (71), it is easy to get that

$$\dot{V}_n \leq -kV_n^{\frac{3}{4}} - (1-\lambda)kV_n^2. \tag{72}$$

With the aid of Lemma 2, the output tracking error z_1 converges to the set $\Omega_2 = \{z_1 || z_1| < \sqrt{\frac{\bar{C}}{\lambda k}}\}$ in predefined time, with the guaranteed convergence time estimated as

$$T \leq T_{\max} = \frac{4}{k} + \frac{1}{(1-\lambda)k}. \tag{73}$$

The proof is completed. □

Remark 9. Based on the result in Theorem 2, we provide an explicit parameter adjustment scheme so that the settling time $T_{\max}$ can meet the preset value. One simple way is to choose λ properly to fix the maximum magnitude of the steady-state error first, then determine the control gain by setting $k = \frac{4(1-\lambda)+1}{T_{max}(1-\lambda)}$ for given $T_{\max}$ and λ based on (73). Note that the settling time $T_{\max}$ only related to the parameters k and λ, thus we call them time parameters in this paper. After that, one can still make the residual set smaller by just decreasing ε, increasing σ_0, σ_1, and $\frac{2\sigma_0}{\sigma_1}$, which means that $2\sigma_0$ should be increased by a value larger than that of σ_1. One the other hand, reducing the size of the residual set or settling time might demand increasing control gain, thus certain trade-off should be made in balancing tracking precision/speed and control action.

Remark 10. Some characteristics of the proposed controller are summarized as follows:

(1) Since this section studies the tracking problem of systems with nonparametric uncertainties and external disturbances, it is acceptable for practical applications that the tracking error converges to an adjustable compact set (rather than zero) within finite time. Several other works also addressed such types of finite-time control issue, for example, the works of Li et al,[37] Wang et al,[38] and Jin.[33] By comparing with these results, it is seen that the settling time therein is dependent on the initial conditions of the system and may become unacceptably large if the magnitude of the initial condition is large, whereas, here in this paper, our control scheme is able to achieve tracking with adjustable size of accuracy within any practically allowable presettling time regardless of the system initial states, which is a significant advancement from the existing results. (2) Since it is nontrivial to synthesize the adaptive compensation satisfying the relation (3), few fixed-time results with adaptive approach are available except the results in the work of Zheng et al,[39] Huang and Jia,[40] and Zhang et al,[41] where the adaptive fixed-time control schemes are all developed for one- or two-dimensional systems by using terminal sliding mode technique. Whereas in this paper, our proposed adaptive control scheme is able to handle high-dimensional uncertain nonlinear systems with unknown control gains. (3) As pointed out in Remark 1 in the work of Zheng et al,[39] the results in the works of Zheng et al,[39] Huang and Jia,[40] and Zhang et al[41] are all based on the assumption that the estimation errors of adaptive parameters are bounded. However, such a condition cannot be ensured before the closed-loop system stability is established. The results in Theorem 2 of this paper do not need such a condition. (4) The proposed adaptive prespecifiable fixed-time controller is continuous and the control parameter selection recipe is presented in Remark 9 such that the settling time can be preassigned as required.

5 | SIMULATION RESULTS

In this section, we give two examples on stabilization and tracking, respectively, to illustrate the proposed prespecifiable fixed-time control schemes and verify their effectiveness with comparisons.

5.1 | Example 1: stabilization case

In this part, we consider the system model as those in the work of Meng et al,[42] ie,

$$\begin{aligned}
\dot{x}_1 &= c_1 e^{x_1^2} + x_2 \\
\dot{x}_2 &= \frac{1 - e^{x_1 x_2}}{1 + e^{x_1 x_2}} + x_3 \\
\dot{x}_3 &= c_2 e^{x_1^2 x_3^2} + u,
\end{aligned} \tag{74}$$

where $c_1 = c_2 = 0.2$. With some calculation, we obtain $\phi_1 = \frac{c_1 e^{x_1^2}(x_1^2+1)}{2x_1^2}$, $\phi_2 = \frac{x_2^2+1}{2x_2^2}$, and $\phi_3 = \frac{c_2 e^{x_2^2 x_3^2}(x_3^2+1)}{2x_3^2}$. Then, the controller is designed according to Theorem 1 and the convergence time $T \leq T_{\max} = \frac{5}{cl}$.

In simulation, the initial conditions are set as $x_0 = [1, -3, 3]^T$. To verify that the convergence time is adjustable with the proposed method, we choose two sets of parameters $c = 1, l = 0.5$ and $c = 2, l = 0.5$ in order to meet the preassigned settling times of 10 and 5 seconds, respectively. The state trajectories and control signals are shown in Figure 1 and Figure 2, respectively. It can be observed that the convergence time with our proposed control scheme indeed matches the preassigned 10 and 5 seconds, achieved by designing the corresponding control parameters in advance. For further illustrations, we now predefine the convergence time to 10 seconds for different initial conditions $x_0 = [2, -5, 8]^T$ and $x_0 = [2, -8, 30]^T$. It is shown in Figure 3 that all states converge to zero before 10 seconds by choosing control parameters $c = 2$ and $l = 0.25$. The control signals presented in Figure 4 show that higher control effort is required to stabilize the system with larger magnitude of initial conditions in this case. On the other hand, due to the fractional power of the system states in (29), the control signal is continuous but nonsmooth.

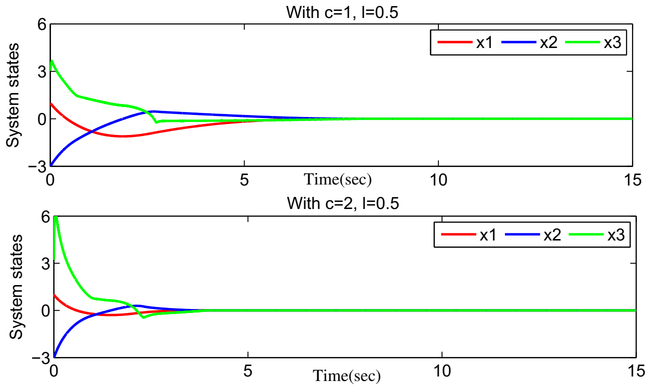

FIGURE 1 System states with different time parameters [Colour figure can be viewed at wileyonlinelibrary.com]

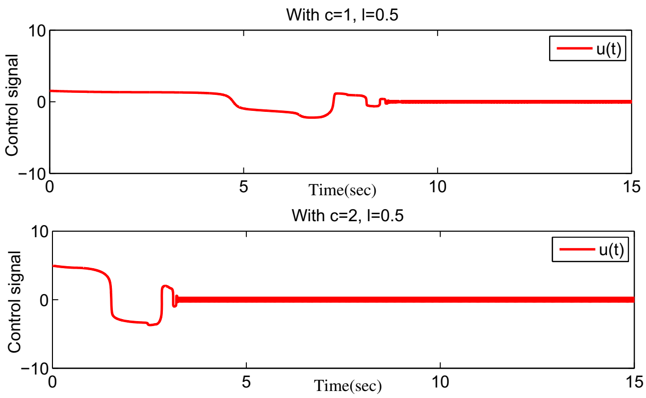

FIGURE 2 Control signal with different time parameters [Colour figure can be viewed at wileyonlinelibrary.com]

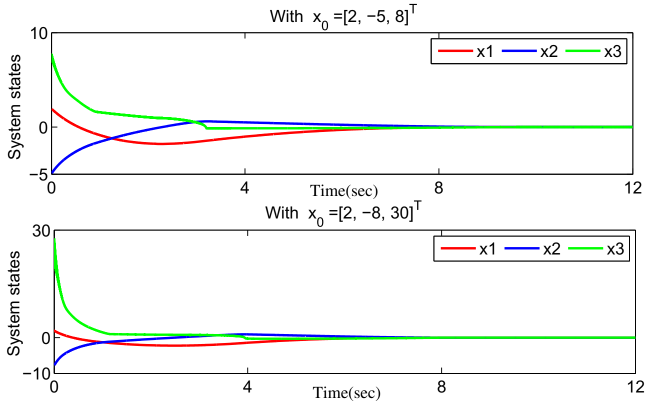

FIGURE 3 System states with different initial conditions [Colour figure can be viewed at wileyonlinelibrary.com]

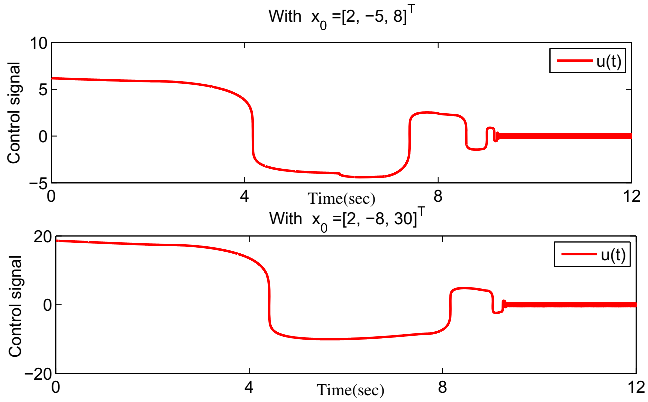

FIGURE 4 Control signal with different initial conditions [Colour figure can be viewed at wileyonlinelibrary.com]

5.2 | Example 2: tracking case

In this part, we consider the following robotic manipulator system as the work of Xing et al,[43] ie,

$$J\ddot{q}(t) + D\dot{q} + MgL\sin(q(t)) + d(t) = u, \tag{75}$$

where q and $\dot{q}$ are the angle and angular velocity of the rigid link, respectively. J denotes the rotation inertia of the servo motor, B is the damping coefficient, L is the length from the axis of joint to the mass center, M is the mass of the link, and g is the gravitational acceleration. Similar to the work of Xing et al,[43] for simulation, the physical parameters are given as $J = 1, MgL = 10, D = 2$ and the disturbance is chosen as $d(t) = \sin(10t) + 0.4$, while they are all unknown for controller design. Let $x_1 = q, x_2 = \dot{q}, \theta_1 = -\frac{D}{J}, \theta_2 = -\frac{MgL}{J}$ and $g_2 = \frac{1}{J}$, then (75) can be transformed into

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \theta_1 x_2 + \theta_2 \sin(x_1) + g_2 u + d. \end{aligned} \tag{76}$$

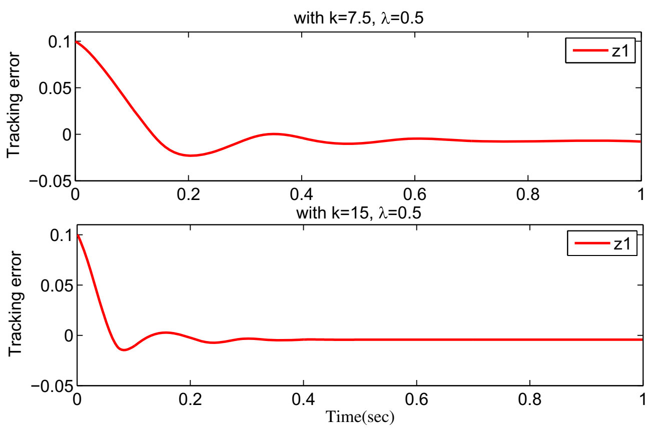

FIGURE 5 Tracking error with different time parameters [Colour figure can be viewed at wileyonlinelibrary.com]

Since $|\theta_1 x_2 + \theta_2 sin(x_1)| \leq a(|x_2| + |sin(x_1)|)$ where a is an unknown constant, we can simply choose $\varphi(\cdot) = |x_2| + |sin(x_1)|$. The control objective is for the angle $x_1 = q$ to track the desired trajectory $y_d = 0.3\sin(t)$ and the tracking error z_1 converges to an arbitrarily small neighborhood of zero within preassigned time. For the sake of simulation, we first set $x_1(0) = 0.1, x_2(0) = 0, a(0) = 0, \sigma_0 = 0.1, \sigma_1 = 1, \varepsilon = 0.01$, $\lambda = 0.5$, and $\underline{g}_2 = 0.1$. According to (73), we choose $k = 7.5$ and $k = 15$ to meet the preassigned settling times of 0.8 and 0.4 seconds, respectively. As shown in Figure 5, tracking error converges to a steady-state compact set with these two parameters within 0.8 and 0.4 seconds, respectively. The control signals are presented in Figure 6that show even though a smaller settling time can be achieved by increasing the control gain k, the magnitude of control signal will be larger, especially at the initial stage. In order to further verify that the size of the compact set can be reduced by adjusting the control parameters, as stated in Remark 9, we select two sets of parameters $\sigma_0 = 0.2, \sigma_1 = 1.2, \epsilon = 0.005$ and $\sigma_0 = 0.3, \sigma_1 = 1.5, \epsilon = 0.002$, respectively, with same values of parameters $k = 15$ and $\lambda = 0.5$. It is shown in Figure 7 that the steady-state residual set is smaller with the settling time unaltered

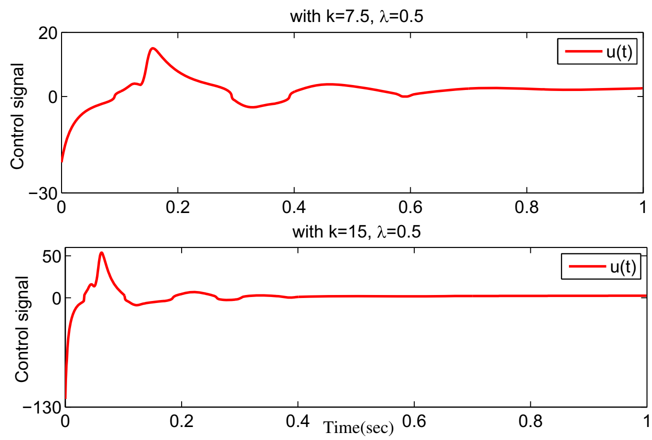

FIGURE 6 Control signal with different time parameters [Colour figure can be viewed at wileyonlinelibrary.com]

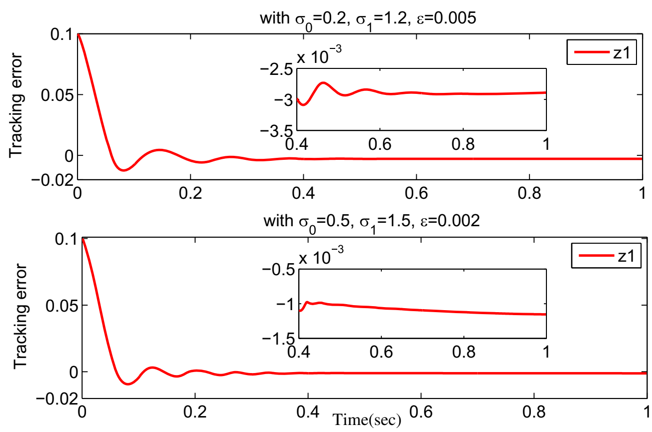

FIGURE 7 Tracking error with different control parameters [Colour figure can be viewed at wileyonlinelibrary.com]

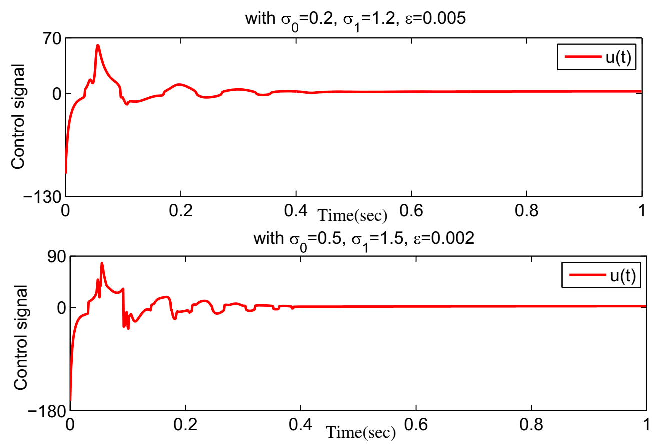

FIGURE 8 Control signal with different control parameters [Colour figure can be viewed at wileyonlinelibrary.com]

by decreasing ε and increasing σ_0, σ_1, and $\frac{2\sigma_0}{\sigma_1}$. As observed from Figure 8, higher control effort is required to achieve a smaller magnitude of steady-state tracking error within preassigned settling time. Therefore, there is a balance between tracking speed/precision and control action.

6 | CONCLUSION

We have developed two control schemes in this paper for a class of uncertain nonlinear systems in strict-feedback form, in order to achieve prespecifiable fixed-time control with fixed-time Lyapunov stability characterization and backstepping approach. The settling time of the proposed methods can be determined by designing the corresponding control parameters in advance, which is an appealing feature that is nontrivial to establish even for linear or single integrator systems. On the other hand, certain constraint on the preassigned settling time should be imposed in consideration of the fact that the control is normally implemented digitally (in discrete time) and that there is certain limit on sampling rate and signal processing/transformation. Therefore, in practice, the preassigned settling time must be not shorter than the combination of the sampling period and the time required for signal processing and transformation. More about this issue is worth further studying as an interesting future research topic.

ACKNOWLEDGEMENT

This work was supported by the National Natural Science Foundation of China under Grants 61803053, 61833013, 61860206008, and 61773081.

ORCID

Ye Cao https://orcid.org/0000-0002-0279-6175
Yongduan Song https://orcid.org/0000-0002-2167-1861

REFERENCES

1. Du H, Li S, Lin X. Finite-time formation control of multiagent systems via dynamic output feedback. *Int J Robust Nonlinear Control*. 2013;23(14):1609-1628.
2. Zhang X, Feng G, Sun Y. Finite-time stabilization by state feedback control for a class of time-varying nonlinear systems. *Automatica*. 2012;48(3):499-504.
3. Haddad WM, L'Afflitto A. Finite-time stabilization and optimal feedback control. *IEEE Trans Autom Control*. 2016;61(4):1069-1074.
4. Wen J, Nguang SK, Shi P, Peng L. Finite-time stabilization of Markovian jump delay systems-a switching control approach. *Int J Robust Nonlinear Control*. 2017;27(2):298-318.
5. Bhat SP, Bernstein DS. Finite-time stability of continuous autonomous systems. *SIAM J Control Optim*. 2000;38(3):751-766.
6. Liu Y, Ma Y, Wang Y. Reliable sliding mode finite-time control for discrete-time singular Markovian jump systems with sensor fault and randomly occurring nonlinearities. *Int J Robust Nonlinear Control*. 2018;28(2):381-402.
7. Chen X, Zhang X. Output-feedback control strategies of lower-triangular nonlinear nonholonomic systems in any prescribed finite time. *Int J Robust Nonlinear Control*. 2019;29(4):904-918. http://dx.doi.org/10.1002/rnc.4413
8. Ryan EP. Singular optimal controls for second-order saturating systems. *Int J Control*. 1979;30(4):549-564.
9. Hong Y. Finite-time stabilization and stabilizability of a class of controllable systems. *Syst Control Lett*. 2002;46(4):231-236.
10. Utkin VI. *Sliding Modes in Control Optimization*. Berlin, Germany: Springer-Verlag; 1992.
11. Huang X, Lin W, Yang B. Global finite-time stabilization of a class of uncertain nonlinear systems. *Automatica*. 2005;41(5):881-888.
12. Huang J, Wen C, Wang W, Song Y. Design of adaptive finite-time controllers for nonlinear uncertain systems based on given transient specifications. *Automatica*. 2016;69:395-404.
13. Polyakov A. Nonlinear feedback design for fixed-time stabilization of linear control systems. *IEEE Trans Autom Control*. 2012;57(8):2106-2110.
14. Zuo Z. Nonsingular fixed-time consensus tracking for second-order multi-agent networks. *Automatica*. 2015;54:305-309.
15. Polyakov A, Efimov D, Perruquetti W. Finite-time and fixed-time stabilization: implicit Lyapunov function approach. *Automatica*. 2015;51:332-340.
16. Sánchez-Torres JD, Sanchez EN, Loukianov AG. Predefined-time stability of dynamical systems with sliding modes. In: Proceedings of the American Control Conference; 2015; Chicago, IL.
17. Polyakov A, Efimov D, Perruquetti W. Finite-time stabilization using implicit Lyapunov function technique. *IFAC Proc Vol*. 2013;46(23):140-145. Part of Special Issue: 9th IFAC Symposium on Nonlinear Control Systems.
18. Zuo Z. Non-singular fixed-time terminal sliding mode control of non-linear systems. *IET Control Theory Appl*. 2014;9(4):545-552.

19. Basin M, Panathula CB, Shtessel Y. Adaptive uniform finite-/fixed-time convergent second-order sliding-mode control. *Int J Control*. 2016;89(9):1777-1787.
20. Zarchan P. *Tactical and Strategic Missile Guidance*. Reston, VA: American Institute of Aeronautics and Astronautics (AIAA); 2007.
21. Song Y, Wang Y, Holloway J, Krstic M. Time-varying feedback for regulation of normal-form nonlinear systems in prescribed finite time. *Automatica*. 2017;83:243-251.
22. Lopez-Ramirez F, Efimov D, Polyakov A, Perruquetti W. On necessary and sufficient conditions for fixed-time stability of continuous autonomous systems. Paper presented at: 2018 European Control Conference (ECC); 2018; Limassol, Cyprus. https://doi.org/10.23919/ECC.2018.8550571
23. Qian C, Lin W. Non-Lipschitz continuous stabilizers for nonlinear systems with uncontrollable unstable linearization. *Syst Control Lett*. 2001;42(3):185-200.
24. Mitrinović DS. Čebyšev's and related inequalities. *Analytic Inequalities*. 1st ed. Berlin, Germany: Springer-Verlag; 1970:36. https://doi.org/10.1007/978-3-642-99970-3
25. Qian C, Lin W. Practical output tracking of nonlinear systems with uncontrollable unstable linearization. *IEEE Trans Autom Control*. 2002;47(1):21-36.
26. Lin W, Qian C. Adding one power integrator: a tool for global stabilization of high-order lower-triangular systems. *Syst Control Lett*. 2000;39(5):339-351.
27. Shen Y, Huang Y. Global finite-time stabilisation for a class of nonlinear systems. *Int J Syst Sci*. 2012;43(1):73-78.
28. Pongvuthithum R. A time-varying feedback approach for triangular systems with nonlinear parameterization. *SIAM J Control Optim*. 2009;48(3):1660-1674.
29. Zhang Z, Wu Y. Fixed-time regulation control of uncertain nonholonomic systems and its applications. *Int J Control*. 2017;90(7):1327-1344.
30. Sun Z, Yun M, Li T. A new approach to fast global finite-time stabilization of high-order nonlinear system. *Automatica*. 2017;81:455-463.
31. Defoort M, Polyakov A, Demesure G, Djemai M, Veluvolu K. Leader-follower fixed-time consensus for multi-agent systems with unknown non-linear inherent dynamics. *IET Control Theory Appl*. 2015;9(14):2165-2170.
32. Tian B, Zuo Z, Yan X, Wang H. A fixed-time output feedback control scheme for double integrator systems. *Automatica*. 2017;80:17-24.
33. Jin X. Adaptive finite-time fault-tolerant tracking control for a class of MIMO nonliner systems with output constraints. *Int J Robust Nonlinear Control*. 2017;27(5):722-741.
34. Song Y, Huang X, Wen C. Tracking control for a class of unknown nonsquare MIMO nonaffine systems: A deep-rooted information based robust adaptive approach. *IEEE Trans Autom Control*. 2016;61(10):3227-3233.
35. Krstic M, Kanellakopoulos I, Kokotovic PV. Adaptive nonlinear control without overparametrization. *Syst Control Lett*. 1992;19(3):177-185.
36. Chen C, Wen C, Liu Z, Xie K, Zhang Y, Chen CLP. Adaptive asymptotic control of multivariable systems based on a one-parameter estimation approach. *Automatica*. 2017;83:124-132.
37. Li H, Zhao S, He W, Lu R. Adaptive finite-time tracking control of full state constrained nonlinear systems with dead-zone. *Automatica*. 2019;100:99-107.
38. Wang Y, Song Y, Krstic M, Wen C. Fault-tolerant finite time consensus for multiple uncertain nonlinear mechanical systems under single-way directed communication interactions and actuation failures. *Automatica*. 2016;63:374-383.
39. Zheng Z, Feroskhan M, Sun L. Adaptive fixed-time trajectory tracking control of a stratospheric airship. *ISA Transactions*. 2018;76:134-144.
40. Huang Y, Jia Y. Robust adaptive fixed-time tracking control of 6-DOF spacecraft fly-around mission for noncooperative target. *Int J Robust Nonlinear Control*. 2018;28(6):2598-2618.
41. Zhang Y, Tang S, Guo J. Adaptive terminal angle constraint interception against maneuvering targets with fast fixed-time convergence. *Int J Robust Nonlinear Control*. 2018;28(8):2996-3014.
42. Meng W, Yang Q, Jagannathan S, Sun Y. Distributed control of high-order nonlinear input constrained multiagent systems using a backstepping-free method. *IEEE Trans Cybern*. 2019;49(11):3923-3933. https://doi.org/10.1109/TCYB.2018.2853623
43. Xing L, Wen C, Liu Z, Su H, Cai J. Event-triggered adaptive control for a class of uncertain nonlinear systems. *IEEE Trans Autom Control*. 2017;62(4):2071-2076.

How to cite this article: Cao Y, Wen C, Tan S, Song Y. Prespecifiable fixed-time control for a class of uncertain nonlinear systems in strict-feedback form. *Int J Robust Nonlinear Control*. 2020;30:1203–1222. https://doi.org/10.1002/rnc.4820

APPENDIX

Proof of Lemma 1. By solving differential inequality (2), it follows that

$$V(x)^{1-\alpha} \leq -\frac{1}{\beta} \ln \left(\beta ct + e^{-\beta V(x_0)^{1-\alpha}} \right). \tag{A1}$$

Note that $V(x) \le 0$ if $\beta ct + e^{-\beta V(x_0)^{1-\alpha}} = 1$. As $V(x)$ is a positive definite function, we have $V(x) \equiv 0$ after the settling time function T, where

$$T \le \frac{1 - e^{-\beta V(x_0)^{1-\alpha}}}{c\beta} \le \frac{1}{c\beta}, \quad \forall x_0. \tag{A2}$$

The proof is completed.

Proof of Proposition 1. We can use an inductive argument to show that the result of Proposition 1 holds. To this end, we make the assumption that for $k = 1, \dots, i-2$, there exist C^1 functions $\mu_{i-1,k}(\cdot)$ such that

$$\left| \frac{\partial \left(x_{i-1}^{*1/q_{i-1}} \right)}{\partial x_k} \right| \le \left(|\xi_1|^{1-q_k} + \cdots + |\xi_{i-2}|^{1-q_k} \right) \mu_{i-1,k}(\bar{x}_{i-2}). \tag{A3}$$

Our goal is to show that there exist C^1 functions $\mu_{i,k}(\cdot), k = 1, \dots, i-1$ such that

$$\left| \frac{\partial (x_i^{*1/q_i})}{\partial x_k} \right| \le \left(|\xi_1|^{1-q_k} + \cdots + |\xi_{i-1}|^{1-q_k} \right) \mu_{i,k}(\bar{x}_{i-1}). \tag{A4}$$

By using the same proving method as Proposition 4.1 in the work of Huang et al,[11] we can easily get (A4), and therefore, its proof is omitted here just for simplicity. With the aid of (20) and (21), we have

$$\begin{aligned} \dot{x}_k &\le \bar{g}_k \left(|\xi_{k+1}|^{q_{k+1}} + |\xi_k|^{q_{k+1}} |\beta_k(\bar{x}_k)| \right) + \left(\sum_{j=1}^{k} |\xi_j|^{q_k} \right) \bar{\gamma}_k(\bar{x}_k) \\ &\le \left(\sum_{j=1}^{k+1} |\xi_j|^{q_{k+1}} \right) \tilde{\mu}_k(\bar{x}_{k+1}), \end{aligned} \tag{A5}$$

where $\tilde{\mu}_k(\bar{x}_{k+1}) \ge 0$ is a C^1 function. Combining (A4) and (A5) yields that for $k = 1, \dots, i-1$,

$$\begin{aligned} \left| \frac{\partial \left(x_i^{*1/q_i} \right)}{\partial x_k} \dot{x}_k \right| &\le \left(\sum_{j=1}^{i-1} |\xi_j|^{1-q_k} \right) \mu_{i,k}(\bar{x}_{i-1}) \left(\sum_{j=1}^{k+1} |\xi_j|^{q_{k+1}} \right) \tilde{\mu}_k(\bar{x}_{k+1}) \\ &\le \sum_{j=1}^{i} |\xi_j|^{(2n-1)/(2n+1)} \hat{\mu}_k(\bar{x}_i), \end{aligned}$$

where $\hat{\mu}_k(\bar{x}_i) \ge 0$ are C^1 functions. Thus,

$$\left| \frac{d \left(x_i^{*1/q_i} \right)}{dt} \right| = \left| \sum_{k=1}^{i-1} \frac{\partial \left(x_i^{*1/q_i} \right)}{\partial x_k} \dot{x}_k \right| \le \left(\sum_{j=1}^{i} |\xi_j|^{(2n-1)/(2n+1)} \right) \varpi_i(\bar{x}_i),$$

where $\varpi_i(\bar{x}_i) \ge 0$ is a C^1 function. The proof is completed. □