Sliding mode based prescribed-time tracking control of second-order multi-agent systems

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Abstract

In this work, we investigate the prescribed-time leader-following consensus problem for second-order multi-agent systems under directed topology. A new hybrid fully distributed observer for each follower is developed to estimate the states of the leader in prescribed time. Based on the observer, a novel prescribed-time sliding mode surface that is nonsingular and the corresponding tracking controller are designed such that the full state consensus tracking is achieved within a prescribed time. The salient feature of the proposed control scheme lies in that not only accurate tracking can be achieved within the prescribed time but also the numerical implementation problem arising from the infinite time-varying gain is avoided. The effectiveness of the proposed method is confirmed by the numerical simulation.

Key words: Multi-agent systems; Directed graph; Prescribed-time control; Consensus tracking.

1 Introduction

Distributed consensus tracking of multi-agent systems (MASs) has received considerable attention during the past decades due to its broad applications in spacecraft formation flying, multi-vehicle systems, and so on [1,2]. To date, plenty of distributed tracking controllers have been developed, with some of which the asymptotic stability of the closed-loop systems can be achieved [3,4]. However, the infinite convergence time of the asymptotic stability cannot meet the specific control requirements in some real-world systems. To deal with these requirements, some developments concerning the finite-time concept are provided in [5–7]. It is noted that the settling time under the above mentioned finite-time control methods usually depends on the initial conditions of the system, making it difficult to pre-set the settling time. Recently, the concept of fixed-time stability is proposed in the literature, in which the settling time is uniformly bounded and independent of initial conditions [8–10], while the finite settling time has a complicated relationship with the system and control tunable parameters that can not be pre-specified arbitrarily.

To address this concern, a new concept called prescribed/predefined-time stability has been proposed in [11–17], in which the convergence time is irrespective of initial conditions and any design parameters, thus can be pre-assigned. Two main approaches have been proposed to achieve the prescribed-time stability. The first one is the fractional power feedback based method, where the least upper bound of the settling time is set a priori as a parameter of the control protocol [11–13]. With this control concept, the prescribed-time consensus problem has been addressed in [18–20] for MASs with integrator dynamics. Nevertheless, owing to the topology structure and the fractional power feedback technique used in the controller, the precise settling time can not be provided under the method proposed in [11–13].

The second one is the time-varying feedback based method. Different from the former method, the settling time can be provided precisely with the help of a time-varying gain function [14–17]. Based on this control concept, there have been many efforts devoted to the prescribed time control for first- [21–24], second- [25,26], high-order [27–29] linear MASs, and nonlinear MASs [30–32]. However, in the most of the aforementioned works, such as in [14–17, 22, 25, 30–32], the time-varying gain is only effective during the prescribed time interval but not applicable after that time. It is also noted that the gain would grow to be unbounded as the time approaches the prescribed time, which inevitably leads to numerical problems in the controller implementation. Moreover, it is proven in [33] that arbitrarily small measurement noise would lead to bad control performance at the prescribed time. To maintain a bounded gain, several straightforward approaches (e.g., increasing the prescribed time to slightly exceed the desired one, adding a state dead zone) are provided in [14, 15], while the main drawback of which is to

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somewhat sacrifice the precise convergence accuracy.

To solve the numerical problem arising from the infinite gain, a design method based on finite time-varying gains is proposed in [34] to achieve the prescribed-time differentiation, where it can still perform well with high control precision after the settling time. Further, the uniformly bounded time-varying gains, regardless of the initial conditions, are also exploited in [35,36] to derive the prescribed-time stabilization control algorithms. More recently, the necessary and sufficent conditions are provided in [37], enabling the systems with time-varying gains to possess the uniform stability property. However, it is still unclear how this finite time-varying gain technique could be extended to the distributed consensus control of MASs due to the local directed communication constraint.

In this paper, we aim at achieving prescribed-time consensus tracking control for perturbed second-order MASs under directed topology. Inspired by the concept of finite time-varying gain in [34, 37], we construct a distributed prescribed-time observer and prescribed-time sliding mode based tracking controller. The main contributions of this paper are as follows: 1) a new hybrid distributed observer is proposed for each follower to estimate the full states of the leader, where the observing time is ensured to be finite that can be pre-assigned freely by the designer regardless of the local directed communication topology. Moreover, the proposed observer is fully distributed by using only the local observing information, allowing the resulting control scheme to be fully distributed. In addition, a switching mechanism, in which the switched time instants are fully independent of the initial conditions of the followers, is provided such that the implementation problem is avoided effectively, distinguishing itself from the prescribed-time observer design methods given in [15,16,28–30]; 2) a new prescribed-time sliding mode surface that is nonsingular is introduced into the distributed controller design, which allows the full state consensus tracking to be achieved in prescribed time. The salient feature of the proposed control method lies in that the proposed control scheme is fully valid for the whole time interval and does not share the numerical implementation difficulties as the time approaches the prescribed time, and meanwhile, the precise consensus tracking can be achieved within the prescribed time and maintained after that time instant.

Notations: I_N represents the identity matrix of dimension N. $\mathbf{1}_N \in \mathbb{R}^N$ denotes a vector with each entry being 1. $\otimes$ denotes the Kronecker product. $\underline{\sigma}(\cdot)$ represents the minimum singular value of a matrix. $|\cdot|$ denotes the absolute value of a number. Given a vector $x = [x_1, ..., x_n]^T$ and a real number β, denote $\text{sig}^{[\beta]}(x) = [\text{sig}^{\beta}(x_1), ..., \text{sig}^{\beta}(x_n)]^T$ and $|x|^{\beta} = [|x_1|^{\beta}, ..., |x_n|^{\beta}]^T$, where $\text{sig}^{\beta}(x_i) = \text{sign}(x_i)|x_i|^{\beta} (i = 1, ..., n)$ and $\text{sign}(\cdot)$ denotes the signum function, and $\text{diag}\{x\}$ denotes a diagonal matrix with entries x_i on its diagonal.

2 Problem formulation and Preliminaries

Consider a class of second-order MASs consisting of N follower agents and one leader agent. The dynamics of the ith follower agent is described by

$$\dot{x}_i(t) = v_i(t), \ \dot{v}_i(t) = u_i(t) + w_i(t), \tag{1}$$

where $i = 1, ..., N$, $x_i(t) \in \mathbb{R}^n$, $v_i(t) \in \mathbb{R}^n$ are the system states, $u_i(t) \in \mathbb{R}^n$ is the control input, and $w_i(t) \in \mathbb{R}^n$ represents the nonvanishing uncertainty. The leader agent, labeled as $i = 0$, is described by

$$\dot{x}_0(t) = v_0(t), \ \dot{v}_0(t) = 0, \tag{2}$$

where $x_0 \in \mathbb{R}^n$ and $v_0 \in \mathbb{R}^n$ are the state vectors of the leader.

Assumption 1 *The nonvanishing uncertainty $w_i(t)$ is continuous and bounded, i.e., $|w_{ik}(t)| \leq d_w$ for $k = 1, ..., n$, where d_w is a known positive constant.*

We model the communication toplogy of the N followers by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{\nu_1, ..., \nu_N\}$ is a node set representing the N followers, $\mathcal{E} \in (\mathcal{V} \times \mathcal{V})$ is an edge set representing the communication connection between the followers, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the associated adjacency matrix. An edge rooted at agent j and ending at agent i is denoted by (ν_j, ν_i), and $a_{ij} > 0$ denotes the value of the edge weight if $(\nu_j, \nu_i) \in \mathcal{E}$, otherwise $a_{ij} = 0$. It is assumed that $a_{ii} = 0$. Define the in-degree of agent i as $d_i = \sum_{j=1}^{N} a_{ij}$, the in-degree matrix as $\mathcal{D} = \text{diag}\{d_1, ..., d_N\} \in \mathbb{R}^{N\times N}$, and the Laplacian matrix as $\mathcal{L} = \mathcal{D} - \mathcal{A}$. We denote by $\mathcal{B} = \text{diag}\{b_1, ..., b_N\} \in \mathbb{R}^{N\times N}$ the leader adjacency matrix, in which $b_i > 0$ if and only if the leader is accessible by the agent i in $\mathcal{G}$ directly. For simplicity, denote $\mathcal{L} + \mathcal{B}$ by $\mathcal{H}$.

Assumption 2 *The graph $\mathcal{G}$ has a directed spanning tree, and the root agent has access to the leader.*

The control objective of this work is to solve the distributed prescribed-time tracking control problem for the networked systems (1), (2), i.e., for any given initial states, to design a distributed controller such that

$$\begin{cases} \lim_{t\to T} x_i(t) - x_0(t) = 0, \ v_i(t) - v_0(t) = 0, \\ x_i(t) \equiv x_0(t), \ v_i(t) \equiv v_0(t), \quad \forall t > T \end{cases} \tag{3}$$

for $i = 1, ..., N$, and $T > 0$ is a prior-given and user-defined time, which is independent of initial states.

To achieve the control objective, we first introduce the following time transformation function

$$t = \theta(s) \triangleq T_*(1 - e^{-s}), \tag{4}$$

where $T_* > 0$ denotes the prescribed convergence time [22, 25]. Note that $\theta(s)$ has the following properties: (1) $\theta(s)$ is strictly increasing and continuously differentiable with respect to s; (2) $\theta(s)$ converts the finite-time interval $t \in [0, T_*)$ to the infinite-time interval $s \in [0, \infty)$.

It is worth noting that here we employ a time transformation function as in (4). The intuition behind such a transformation function is that: let $\psi(t)$ denote the solution of the equation $\dot{\psi}(t) = f(t, \psi(t))$ with initial value ψ_0 at time τ. Then from [38], by using the time transformation function (4) and the definition $\phi(s) \triangleq \psi(t)$, one has $\phi'(s) = \theta'(s) f(\theta(s), \phi(s)), \phi(\theta^{-1}(\tau)) = \psi_0$, where $\phi'(s) \triangleq d\phi(s)/ds$, $\theta'(s) \triangleq d\theta(s)/ds$.

Consider the system of differential equations

$$\dot{x}(t) = f(t, x), x(0) = x_0, t \in \mathbb{R}_+ \tag{5}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is a nonlinear function. Suppose that the origin is an equilibrium point of (5).

Lemma 1 *[6] Consider system (5). If there exists a continuous radially unbounded and positive definite function $V(x(t))$ such that $\dot{V}(x(t)) \le -cV^{\alpha}(x(t))$ for some positive constants $c > 0$, $0 < \alpha < 1$, then, the origin of system (5) is finite-time stable with a finite-time settling time $T(x_0) \le \frac{V(x_0)^{1-\alpha}}{c(1-\alpha)}$.*

Lemma 2 *[8] If there exists a continuous radially unbounded and positive definite function $V(x(t))$ such that $\dot{V}(x(t)) \le -c_1 V^{r_1}(x(t)) - c_2 V^{r_2}(x(t))$ for some positive constants $c_1 > 0$, $c_2 > 0$, and $0 < r_2 < 1 < r_1$, then, the origin of system (5) is fixed-time stable with the settling time estimated by $T(x_0) \le \frac{1}{c_1(r_1-1)} + \frac{1}{c_2(1-r_2)}$.*

Lemma 3 *[39] Under Assumption 2, $\mathcal{H}$ is a nonsingular M-matrix. Let $p = [p_1, ..., p_N]^T = \mathcal{H}^{-T}\mathbf{1}_N$, $P = \mathrm{diag}\{p\}$, and $Q = P\mathcal{H} + \mathcal{H}^T P$, then P and Q are positive definite.*

Lemma 4 *[40] For $x_i \in R$, $i = 1, ..., n$, if $h > 1$, then $\sum_{i=1}^n |x_i|^h \le (\sum_{i=1}^n |x_i|)^h \le n^{h-1} \sum_{i=1}^n |x_i|^h$, and if $0 < h \le 1$, then $(\sum_{i=1}^n |x_i|)^h \le \sum_{i=1}^n |x_i|^h \le n^{1-h} (\sum_{i=1}^n |x_i|)^h$.*

3 Distributed Prescribed-time Observer Design

In this section, a new distributed observer scheme is developed for each follower to estimate the states of the leader over a directed graph in prescribed time.

Denote $\hat{x}_i$ and $\hat{v}_i$ as the estimates of the leader's states for the ith follower, $i = 1, ..., N$. Define the neighbor state estimation errors of the follower i $(i = 1, ..., N)$ as $e_{\hat{x}i} = \sum_{j=1}^N a_{ij}(\hat{x}_i - \hat{x}_j) + b_i(\hat{x}_i - x_0)$, $e_{\hat{v}i} = \sum_{j=1}^N a_{ij}(\hat{v}_i - \hat{v}_j) + b_i(\hat{v}_i - v_0)$. Then we propose the hybrid distributed prescribed-time observer as

$$\begin{cases} \dot{\hat{x}}_i = \hat{v}_i - \beta_i(e_{\hat{x}i}) \\ \dot{\hat{v}}_i = -\beta_i(e_{\hat{v}i}), \end{cases} \tag{6}$$

in which $\beta_i(\mathcal{E}_i)$ ($\mathcal{E}_i = e_{\hat{x}i}$ or $\mathcal{E}_i = e_{\hat{v}i}$) is defined as

$$\beta_i(\mathcal{E}_i) = \begin{cases} \frac{1}{T_1 - t}(k_1 \operatorname{sig}^{[\alpha]}(\mathcal{E}_i) + k_2 \operatorname{sig}^{[\gamma]}(\mathcal{E}_i)), & t \in [0, T_o] \\ k_1 \operatorname{sig}^{[\alpha]}(\mathcal{E}_i) + k_2 \operatorname{sig}^{[\gamma]}(\mathcal{E}_i), & t \in (T_o, \infty) \end{cases}$$

where $0 < \alpha < 1$, $\gamma > 1$; $k_1 > 0$ and $k_2 > 0$ are the observer gains; $0 < T_1 < T$ and $0 < T_o < T_1$ denote the user-defined convergence time and the switching time of the proposed observer, respectively. The implicit expression of T_o will be given in (19).

Define the global state estimation errors of the follower i $(i = 1, ..., N)$ as $\delta_{\hat{x}i} = \hat{x}_i - x_0$, $\delta_{\hat{v}i} = \hat{v}_i - v_0$. Let $\delta_{\hat{x}} = [\delta_{\hat{x}1}^T, ..., \delta_{\hat{x}N}^T]^T$, $\delta_{\hat{v}} = [\delta_{\hat{v}1}^T, ..., \delta_{\hat{v}N}^T]^T$, and $e_{\hat{x}} = [e_{\hat{x}1}^T, ..., e_{\hat{x}N}^T]^T$, $e_{\hat{v}} = [e_{\hat{v}1}^T, ..., e_{\hat{v}N}^T]^T$, then $e_{\hat{x}} = (\mathcal{H} \otimes I_n)\delta_{\hat{x}}$, $e_{\hat{v}} = (\mathcal{H} \otimes I_n)\delta_{\hat{v}}$. By using (6) and (2), a straightforward derivation of $e_{\hat{x}}$, $e_{\hat{v}}$ yields

$$\begin{cases} \dot{e}_{\hat{x}} = e_{\hat{v}} - (\mathcal{H} \otimes I_n)\beta(e_{\hat{x}}) \\ \dot{e}_{\hat{v}} = -(\mathcal{H} \otimes I_n)\beta(e_{\hat{v}}) \end{cases} \tag{7}$$

with $\beta(e_{\hat{x}}) = [\beta_1^T(e_{\hat{x}1}), ..., \beta_N^T(e_{\hat{x}N})]^T$, $\beta(e_{\hat{v}}) = [\beta_1^T(e_{\hat{v}1}), ..., \beta_N^T(e_{\hat{v}N})]^T$. The design of the prescribed-time observer is to make $\delta_{\hat{x}}$, $\delta_{\hat{v}}$ converge into zero in a pre-given time T_1, and the estimation errors remain zero for $t \in [T_1, \infty)$. We now present the following theorem.

Theorem 1 *Consider the networked systems given by (1), (2) under Assumption 2. By using the distributed prescribed-time observer given by (6), if $k_1 > 0, k_2 > 0$, then the accurate estimates of the states of the leader are obtained for each follower within the finite time $T_o < T_1$, and are also remained over $t \in (T_o, \infty)$.*

Proof: The proof consists of two steps.

Step 1. $t \in [0, T_o]$. Consider the time transformation function

$$t = \theta_1(s) \triangleq T_1(1 - e^{-s}), \tag{8}$$

and let $(e_{\hat{x}}(t), e_{\hat{v}}(t))$ be a solution of the dynamical system (7). For $\phi_{e\hat{x}}(s) \triangleq e_{\hat{x}}(t)$ and $\phi_{e\hat{v}}(s) \triangleq e_{\hat{v}}(t)$, by using $t = \theta_1(s)$, the error dynamics (7) can be rewritten as

$$\begin{cases} \phi'_{e\hat{x}} = T_1 e^{-s}\phi_{e\hat{v}} - (\mathcal{H} \otimes I_n)(k_1 \operatorname{sig}^{[\alpha]}(\phi_{e\hat{x}}) + k_2 \operatorname{sig}^{[\gamma]}(\phi_{e\hat{x}})) \\ \phi'_{e\hat{v}} = -(\mathcal{H} \otimes I_n)(k_1 \operatorname{sig}^{[\alpha]}(\phi_{e\hat{v}}) + k_2 \operatorname{sig}^{[\gamma]}(\phi_{e\hat{v}})). \end{cases} \tag{9}$$

Construct the following Lyapunov function

$$V_{o1}(s) = \sum_{i=1}^N \sum_{k=1}^n \left(\frac{p_i k_1}{1+\alpha}|\phi_{e\hat{v}ik}(s)|^{1+\alpha} + \frac{p_i k_2}{1+\gamma}|\phi_{e\hat{v}ik}(s)|^{1+\gamma}\right), \tag{10}$$

where p_i is defined in Lemma 3. Differentiating $V_{o1}(s)$ with respect to s along (9), and considering Lemma 3, one has

$$
\begin{aligned}
V_{o1}'(s) &= -(k_1 \operatorname{sig}^{[\alpha]}(\phi_{e\hat{v}}) + k_2 \operatorname{sig}^{[\gamma]}(\phi_{e\hat{v}}))^T (P\mathcal{H} \otimes I_n) \\
&\quad (k_1 \operatorname{sig}^{[\alpha]}(\phi_{e\hat{v}}) + k_2 \operatorname{sig}^{[\gamma]}(\phi_{e\hat{v}})) \\
&\leq -\frac{k_1^2 \underline{\sigma}(Q)}{2} \sum_{i=1}^{N}\sum_{k=1}^{n} |\phi_{e\hat{v}ik}|^{2\alpha} - \frac{k_2^2 \underline{\sigma}(Q)}{2} \sum_{i=1}^{N}\sum_{k=1}^{n} |\phi_{e\hat{v}ik}|^{2\gamma} \\
&\quad - k_1 k_2 \underline{\sigma}(Q) \sum_{i=1}^{N}\sum_{k=1}^{n} |\phi_{e\hat{v}ik}|^{\alpha+\gamma} \\
&\leq -\frac{\xi_1 \underline{\sigma}(Q)}{2} (\Xi^{2\alpha} + \Xi^{2\gamma} + \Xi^{\alpha+\gamma}), \qquad (11)
\end{aligned}
$$

where $\xi_1 = \min\{k_1^2 (nN)^{-1}, k_2^2 (nN)^{1-2\gamma}, 2k_1 k_2 (nN)^{1-\alpha-\gamma}\}$, $\Xi = \sum_{i=1}^{N}\sum_{k=1}^{n} |\phi_{e\hat{v}ik}|$, and Lemma 4 is used to obtain the second inequality.

From (10), we obtain that

$$
\begin{aligned}
V_{o1}^{\frac{2\alpha}{1+\alpha}}(s) &\leq (\chi_1 k_1 \Xi^{1+\alpha} + \chi_2 k_2 \Xi^{1+\gamma})^{\frac{2\alpha}{1+\alpha}} \\
&\leq (\chi_1 k_1)^{\frac{2\alpha}{1+\alpha}} \Xi^{2\alpha} + (\chi_2 k_2)^{\frac{2\alpha}{1+\alpha}} \Xi^{\frac{2\alpha(1+\gamma)}{1+\alpha}}, \qquad (12)
\end{aligned}
$$

and

$$
V_{o1}^{\frac{2\gamma}{1+\gamma}}(s) \leq 2^{\frac{\gamma-1}{\gamma+1}} ((\chi_1 k_1)^{\frac{2\gamma}{1+\gamma}} \Xi^{\frac{2\gamma(1+\alpha)}{1+\gamma}} + (\chi_2 k_2)^{\frac{2\gamma}{1+\gamma}} \Xi^{2\gamma}), \qquad (13)
$$

where $\chi_1 = \frac{p_M}{1+\alpha}$, $\chi_2 = \frac{p_M}{1+\gamma}$, $p_M = \max\{p_1, ..., p_N\}$, and Lemma 4 is used. In light of the Young's inequality, we further have

$$
\Xi^{\frac{2\alpha(1+\gamma)}{1+\alpha}} = \Xi^{\frac{2\alpha(\alpha+\gamma)}{1+\alpha}} \Xi^{\frac{2\alpha(1-\alpha)}{1+\alpha}} \leq \frac{2\alpha}{1+\alpha} \Xi^{\alpha+\gamma} + \frac{1-\alpha}{1+\alpha} \Xi^{2\alpha}, \qquad (14)
$$

and

$$
\Xi^{\frac{2\gamma(1+\alpha)}{1+\gamma}} = \Xi^{\frac{2(\alpha+\gamma)}{1+\gamma}} \Xi^{\frac{2\alpha(\gamma-1)}{1+\gamma}} \leq \frac{2}{1+\gamma} \Xi^{\alpha+\gamma} + \frac{\gamma-1}{1+\gamma} \Xi^{2\alpha}. \qquad (15)
$$

Then, substituting (14) and (15) into (12) and (13), respectively, gives

$$
V_{o1}^{\frac{2\alpha}{1+\alpha}}(s) + V_{o1}^{\frac{2\gamma}{1+\gamma}}(s) \leq \xi_2 (\Xi^{2\alpha} + \Xi^{\alpha+\gamma} + \Xi^{2\gamma}), \qquad (16)
$$

where $\xi_2 = \max\{(\chi_1 k_1)^{\frac{2\alpha}{1+\alpha}} + \frac{1-\alpha}{1+\alpha} (\chi_2 k_2)^{\frac{2\alpha}{1+\alpha}} + 2^{\frac{\gamma-1}{1+\gamma}} \frac{\gamma-1}{1+\gamma} (\chi_1 k_1)^{\frac{2\gamma}{1+\gamma}}, \frac{2\alpha}{1+\alpha} (\chi_2 k_2)^{\frac{2\alpha}{1+\alpha}} + 2^{\frac{\gamma-1}{1+\gamma}} \frac{2}{1+\gamma} (\chi_1 k_1)^{\frac{2\gamma}{1+\gamma}}, 2^{\frac{\gamma-1}{1+\gamma}} (\chi_2 k_2)^{\frac{2\gamma}{1+\gamma}}\}$.

Thus, it comes from (11) and (16) that

$$
V_{o1}'(s) \leq -\frac{\xi_1 \underline{\sigma}(Q)}{2\xi_2} (V_{o1}^{\frac{2\alpha}{1+\alpha}}(s) + V_{o1}^{\frac{2\gamma}{1+\gamma}}(s)). \qquad (17)
$$

From Lemma 2, it can be obtained from (17) that $\phi_{e\hat{v}}(s)$ is fixed-time stable at the origin with a settling time bounded by $s_{o1} = \frac{2\xi_2}{\xi_1 \underline{\sigma}(Q)} (\frac{1+\alpha}{1-\alpha} + \frac{1+\gamma}{\gamma-1})$, which further implies that $\phi_{e\hat{v}}(s)$ converges to zero as $s > s_{o1}$.

By inspection, after the convergence of $\phi_{e\hat{v}}(s)$, the dynamics of $\phi_{e\hat{x}}(s)$ in (9) reduce to $\phi_{e\hat{x}}' = -k_1(\mathcal{H} \otimes I_n) \operatorname{sig}^{[\alpha]}(\phi_{e\hat{x}}) - k_2(\mathcal{H} \otimes I_n) \operatorname{sig}^{[\gamma]}(\phi_{e\hat{x}})$. Construct another Lyapunov function as

$$
V_{o2}(s) = \sum_{i=1}^{N}\sum_{k=1}^{n} (\frac{p_i k_1}{1+\alpha} |\phi_{e\hat{x}ik}(s)|^{1+\alpha} + \frac{p_i k_2}{1+\gamma} |\phi_{e\hat{x}ik}(s)|^{1+\gamma}).
$$

By following the similar analysis as in (10)-(17), the derivative of $V_{o2}(s)$ with respect to s is calculated as

$$
V_{o2}'(s) \leq -\frac{\xi_1 \underline{\sigma}(Q)}{2\xi_2} (V_{o2}^{\frac{2\alpha}{1+\alpha}}(s) + V_{o2}^{\frac{2\gamma}{1+\gamma}}(s)). \qquad (18)
$$

By Lemma 2, we have that $\phi_{e\hat{x}}(s)$ converges to zero in a fixed time $2s_{o1}$, i.e., $\phi_{e\hat{x}}(s) = 0$ when $s > 2s_{o1} \triangleq s_o$.

Therefore, one has that $\lim_{s \to s_o} \phi_{e\hat{x}}(s) = 0$ and $\lim_{s \to s_o} \phi_{e\hat{v}}(s) = 0$. According to the time transformation function $t = \theta_1(s)$ with the fact $t \to T_1(1 - e^{-s_o})$ as $s \to s_o$, and $\phi_{e\hat{x}}(s) \triangleq e_{\hat{x}}(t)$, $\phi_{e\hat{v}}(s) \triangleq e_{\hat{v}}(t)$, we have that in the original time variable t, $e_{\hat{x}}(t) = 0$, $e_{\hat{v}}(t) = 0$ as

$$
t \to T_1(1 - e^{-s_o}) \triangleq T_o < T_1. \qquad (19)
$$

Recalling that $e_{\hat{x}} = (\mathcal{H} \otimes I_n)\delta_{\hat{x}}$, $e_{\hat{v}} = (\mathcal{H} \otimes I_n)\delta_{\hat{v}}$, and $\mathcal{H}$ is a nonsingular matrix under Assumption 2, we further have that the accurate estimations of the states of the leader can be obtained by the followers within a finite time $T_o < T_1$, i.e., $\delta_{\hat{x}}(t) = 0$, $\delta_{\hat{v}}(t) = 0$ as $t \to T_o$.

Step 2. $t \in (T_o, \infty)$. The remaining task is to prove that $\delta_{\hat{x}}$, $\delta_{\hat{v}}$ remain zero for $t \in (T_o, \infty)$. Note that the utilized distributed observer is switched to the case with $\beta_i(\mathcal{E}_i) = k_1 \operatorname{sig}^{[\alpha]}(\mathcal{E}_i) + k_2 \operatorname{sig}^{[\gamma]}(\mathcal{E}_i)$ after the time instant T_o. Then construct the Lyapunov functions on $t \in (T_o, \infty)$ as

$$
V_{o3}(t) = \sum_{i=1}^{N}\sum_{k=1}^{n} \frac{p_i k_1}{1+\alpha} |e_{\hat{v}ik}(t)|^{1+\alpha} + \sum_{i=1}^{N}\sum_{k=1}^{n} \frac{p_i k_2}{1+\gamma} |e_{\hat{v}ik}(t)|^{1+\gamma},
$$

and

$$
V_{o4}(t) = \sum_{i=1}^{N}\sum_{k=1}^{n} \frac{p_i k_1}{1+\alpha} |e_{\hat{x}ik}(t)|^{1+\alpha} + \sum_{i=1}^{N}\sum_{k=1}^{n} \frac{p_i k_2}{1+\gamma} |e_{\hat{x}ik}(t)|^{1+\gamma}.
$$

By following the similar procedure as in (10)-(18) with $k_1 > 0$, $k_2 > 0$, we readily obtain that $\dot{V}_{o3}(t) \leq -\frac{\xi_1 \underline{\sigma}(Q)}{2\xi_2} (V_{o3}^{\frac{2\alpha}{1+\alpha}}(t) + V_{o3}^{\frac{2\gamma}{1+\gamma}}(t))$ and $\dot{V}_{o4}(t) \leq -\frac{\xi_1 \underline{\sigma}(Q)}{2\xi_2} (V_{o4}^{\frac{2\alpha}{1+\alpha}}(t) + V_{o4}^{\frac{2\gamma}{1+\gamma}}(t))$ for $t \in (T_o, \infty)$, which, together with $e_{\hat{x}}(T_o) = 0$, $e_{\hat{v}}(T_o) = 0$, implies that $V_{o3}(t) \equiv 0$ and $V_{o4}(t) \equiv 0$ on $t \in (T_o, \infty)$. Thus, $\delta_{\hat{x}}(t) \equiv 0$ and $\delta_{\hat{v}}(t) \equiv 0$ for all $t \in (T_o, \infty)$, which means that the accurate estimates are maintained over $(T_o, \infty)(T_o < T_1)$. The proof is completed.

Remark 1 *Compared with those infinite time-varying gain based prescribed-time observers provided in [15, 16,* *28–30], here in this work a time-varying gain term together with a fractional-order relative state based fixed-time term is introduced into the observer design such that the precise observing is achieved at the finite time T_o. On examining the definition of s_o defined below (18), it is seen that the value of s_o is uniformly bounded and independent of the initial conditions of all the followers, which, by (19), further implies that the switching time T_o is strictly less than the prescribed time T_1 and thus the time-varying gain in (6) can be ensured to be uniformly bounded even if $t \to T_1$. This is deemed favorable in practical implementation, especially under the condition of measurement noise.*

Remark 2 *It is worth noting that the leader's states can be observed in prescribed time under the proposed observe scheme in a fully distributed manner, distinguishing itself from those in [28–30]. Particularly, both the observing states of the neighbors and their derivatives are needed in the observer design in [28], which would inevitably lead to the communication loop problem, while in [29] and [30], the smallest real part of the nonzero eigenvalues of the Laplacian matrix is needed to be known for each follower, which is global information. In this work, all of the above mentioned information is not needed, allowing the observer to be fully distributed. In addition, the proposed observe scheme can be easily extended to high-order cases under directed topology upon using the cascaded structure and the time transformation method.*

4 Prescribed-time Tracking Controller Design

This section presents the observer-based sliding mode controller for each follower to achieve full state consensus tracking in prescribed time under directed graph. Define the tracking errors $\tilde{x}_i = x_i - \hat{x}_i$, $\tilde{v}_i = v_i - \hat{v}_i$, then we present the prescribed-time nonsingular sliding mode surface r_i for follower i as

$$r_i = \begin{cases} \kappa (T-t)^h \,\mathrm{sig}^{[h]}(\tilde{v}_i) + \tilde{x}_i, & t \in [0, T_c] \quad (20a) \\ \kappa\, \mathrm{sig}^{[h]}(\tilde{v}_i) + \tilde{x}_i, & t \in (T_c, \infty) \quad (20b) \end{cases}$$

where $i = 1, ..., N$, $1 < h < 2$, $\kappa > 0$ is a design constant, and $T_o < T_c < T$ is the switching time instant that will be given in (36).

Based on the distributed observer (6) and the nonsingular sliding mode surface (20a), (20b), we design the prescribed-time consensus tracking controller for each follower i as

$$u_i = \begin{cases} \dfrac{1}{T-t}\tilde{v}_i - \dfrac{\mathrm{sig}^{[2-h]}(\tilde{v}_i)}{\kappa h (T-t)^h} - \dfrac{c_1(r_i + \mathrm{sig}^{[\rho]}(r_i) + \mathrm{sig}^{[\varsigma]}(r_i))}{(T-t)^{1+h}} \\ \qquad - c_2 \,\mathrm{sign}(r_i), \quad t \in [0, T_c] \quad (21a) \\ -\dfrac{\mathrm{sig}^{[2-h]}(\tilde{v}_i)}{\kappa h} - c_1(r_i + \mathrm{sig}^{[\rho]}(r_i) + \mathrm{sig}^{[\varsigma]}(r_i)) \\ \qquad - c_2\,\mathrm{sign}(r_i), \quad t \in (T_c, \infty) \quad (21b) \end{cases}$$

where $0 < \rho < 1$, $\varsigma > 1$, and c_1, c_2 are the control gains. The main result of this work is now presented.

Theorem 2 *Consider the networked systems given by (1), (2) under Assumptions 1-2. Using the controller (21a), (21b) together with the observer (6), if $c_1 > 0$, $c_2 > d_w$, $\kappa > 0$, the distributed prescribed-time tracking control problem is solved in that for any given initial states and $i = 1, ..., N$*
1) when $t \in [0, T_o]$, the tracking errors $\tilde{x}_i(t)$ and $\tilde{v}_i(t)$ are bounded;
2) when $t \in (T_o, \infty)$, the tracking errors $\tilde{x}_i(t)$, $\tilde{v}_i(t)$ converge to zero in the finite time $T_c < T$, and then remain zero for all $t > T_c$;
3) the control input $u_i(t)$ remains uniformly bounded for $t \in [0, \infty)$.

Proof: 1) For $t \in [0, T_o]$, by using (1) and (6), the derivative of (20a) is

$$\begin{aligned} \dot{r}_i = \tilde{v}_i &+ \frac{1}{T_1 - t}(k_1 \mathrm{sig}^{[\alpha]}(e_{\hat{x}i}) + k_2 \mathrm{sig}^{[\gamma]}(e_{\hat{x}i})) \\ &- \kappa h (T-t)^{h-1}\mathrm{sig}^{[h]}(\tilde{v}_i) + \kappa h (T-t)^h \mathrm{diag}(\eta_i)[u_i + w_i \\ &+ \frac{1}{T_1 - t}(k_1 \mathrm{sig}^{[\alpha]}(e_{\hat{v}i}) + k_2 \mathrm{sig}^{[\gamma]}(e_{\hat{v}i}))], \quad (22) \end{aligned}$$

where $i = 1, ..., N$, $\eta_i = [\eta_{i1}, ..., \eta_{in}]^T = |\tilde{v}_i|^{h-1}$. Substituting the control input (21a) into (22) gives

$$\begin{aligned} \dot{r}_i = &\frac{1}{T_1 - t}(k_1 \mathrm{sig}^{[\alpha]}(e_{\hat{x}i}) + k_2 \mathrm{sig}^{[\gamma]}(e_{\hat{x}i})) - \frac{c_1 \kappa h}{T-t}\mathrm{diag}(\eta_i) \\ &\times [r_i + \mathrm{sig}^{[\rho]}(r_i) + \mathrm{sig}^{[\varsigma]}(r_i)] + \kappa h (T-t)^h \mathrm{diag}(\eta_i)[-c_2 \mathrm{sign}(r_i) \\ &+ w_i + \frac{1}{T_1 - t}(k_1 \mathrm{sig}^{[\alpha]}(e_{\hat{v}i}) + k_2 \mathrm{sig}^{[\gamma]}(e_{\hat{v}i}))]. \quad (23) \end{aligned}$$

By utilizing the time transformation function (8), we can rewrite (23) as

$$\begin{aligned} \phi'_{ri} = &\, k_1 \mathrm{sig}^{[\alpha]}(\phi_{e\hat{x}i}) + k_2 \mathrm{sig}^{[\gamma]}(\phi_{e\hat{x}i}) \\ &- \frac{c_1 \kappa h T_1 e^{-s}}{\Gamma}\mathrm{diag}(\phi_{\eta i})[\phi_{ri} + \mathrm{sig}^{[\rho]}(\phi_{ri}) + \mathrm{sig}^{[\varsigma]}(\phi_{ri})] \\ &+ \kappa h T_1 e^{-s}\Gamma^h \mathrm{diag}(\phi_{\eta i})[-c_2 \mathrm{sign}(\phi_{ri}) + \phi_{wi}] \\ &+ \kappa h \Gamma^h \mathrm{diag}(\phi_{\eta i})[k_1 \mathrm{sig}^{[\alpha]}(\phi_{e\hat{v}i}) + k_2 \mathrm{sig}^{[\gamma]}(\phi_{e\hat{v}i})] \quad (24) \end{aligned}$$

for $s \in [0, s_o]$, in which s_o is given below (18), $\Gamma = T - T_1 + T_1 e^{-s}$, $\phi_{ri}(s) \triangleq r_i(t)$, $\phi_{e\hat{x}i}(s) \triangleq e_{\hat{x}i}(\theta_1(s))$, $\phi_{e\hat{v}i}(s) \triangleq e_{\hat{v}i}(\theta_1(s))$, $\phi_{\eta i}(s) \triangleq \eta_i(\theta_1(s))$, and $\phi_{wi}(s) \triangleq w_i(\theta_1(s))$.

Then we construct the Lyapunov function

$$V_{cik}(s) = |\phi_{rik}(s)| \quad (25)$$

for $i = 1, ..., N$, $k = 1, ..., n$. Note that the derivative of $V_{cik}(s)$ can be calculated in the Filippov sense [41], and thus $V'_{cik}(s) = \mathrm{sign}(\phi_{rik}(s))\phi'_{rik}(s)$. By using (24), we have

$$\begin{aligned} V'_{cik} \le &-C_0 \phi_{\eta ik}(|\phi_{rik}| + |\phi_{rik}|^{1+\rho} + |\phi_{rik}|^{1+\varsigma}) + k_1 |\phi_{e\hat{x}ik}|^\alpha \\ &+ k_2 |\phi_{e\hat{x}ik}|^\gamma + \kappa h T_1 e^{-s}\Gamma^h \phi_{\eta ik}(-c_2 + \phi_{wik}\mathrm{sign}(\phi_{rik})) \\ &+ \kappa h T^h \phi_{\eta ik}(k_1 |\phi_{e\hat{v}ik}|^\alpha + k_2 |\phi_{e\hat{v}ik}|^\gamma), \quad (26) \end{aligned}$$

where $C_0 = \frac{c_1 \kappa h T_1 e^{-s_o}}{T}$, and we have used the facts $\Gamma < T$ and $e^{-s_o} < e^{-s} < 1$ for $s \in [0, s_o]$ to get the inequality. Note that $\phi_{e\hat{x}ik}$ and $\phi_{e\hat{v}ik}$ are bounded during the time interval $[0, T_o]$ as shown in the proof of Step 1 of Theorem 1. Then there exist some bounded positive constants C_1, C_2 such that

$$\begin{aligned} V'_{cik} \leq & - C_0 \phi_{\eta ik} |\phi_{rik}| + C_1 \phi_{\eta ik} + C_2 \\ = & - C_0 \phi_{\eta ik} V_{cik} + C_1 \phi_{\eta ik} + C_2, \end{aligned} \tag{27}$$

where the fact $c_2 > d_w$ and Assumption 1 are applied. Let $\mu_{ik}(s) = e^{C_0 \int_0^s \phi_{\eta ik}(\tau) d\tau}$, and multiplying $\mu_{ik}(s)$ on both hands of (27) gives $\frac{d}{ds}(\mu_{ik}(s) V_{cik}(s)) \leq C_1 \mu_{ik}(s) \phi_{\eta ik}(s) + C_2 \mu_{ik}(s)$. Integrating it over $[0, s]$ with $s \in [0, s_o]$, we have

$$\begin{aligned} V_{cik}(s) \leq & V_{cik}(0) + \int_0^s C_1 e^{-C_0 \int_{\tau_1}^s \phi_{\eta ik}(\tau) d\tau} \phi_{\eta ik}(\tau_1) d\tau_1 \\ & + \int_0^s C_2 e^{-C_0 \int_{\tau_1}^s \phi_{\eta ik}(\tau) d\tau} d\tau_1 \\ \leq & V_{cik}(0) + \int_0^s C_1 e^{-C_0 \int_{\tau_1}^s \phi_{\eta ik}(\tau) d\tau} d\left(\int_0^{\tau_1} \phi_{\eta ik}(\tau) d\tau\right) \\ & + \int_0^s C_2 e^{-C_0 \int_{\tau_1}^s \phi_{\eta ik}(\tau) d\tau} d\tau_1 \\ \leq & V_{cik}(0) + \frac{C_1}{C_0} e^{-C_0 \int_0^s \phi_{\eta ik}(\tau) d\tau} (e^{C_0 \int_0^s \phi_{\eta ik}(\tau) d\tau} - 1) \\ & + \int_0^s C_2 e^{-C_0 \int_{\tau_1}^s \phi_{\eta ik}(\tau) d\tau} d\tau_1 \\ \leq & V_{cik}(0) + \frac{C_1}{C_0} + C_2 s_o \end{aligned} \tag{28}$$

for each $i = 1, ..., N$ and $k = 1, ..., n$, which indicates that $\phi_{rik}(s)$ is bounded over $s \in [0, s_o]$, and in turn guarantees the boundedness of $r_{ik}(t)$ over $t \in [0, T_o]$. Thus, recalling (20a), it follows from (28) that there always exists a bounded constant d_r such that $r_{ik} = \kappa (T - t)^h \operatorname{sig}^h(\tilde{v}_{ik}) + \tilde{x}_{ik} \triangleq d_r$, and then $\tilde{v}_{ik} = -\frac{1}{\kappa^{\frac{1}{h}}(T-t)} \operatorname{sig}^{\frac{1}{h}}(\tilde{x}_{ik} - d_r)$. Noting that $\tilde{v}_{ik} = \dot{\tilde{x}}_{ik} - \beta_{ik}(e_{\hat{x}ik})$ by using (1) and (6), and letting $z_{ik} = \tilde{x}_{ik} - d_r$, $V_{zik}(t) = |z_{ik}(t)|$, one has $\dot{V}_{zik}(t) < -\frac{1}{\kappa^{\frac{1}{h}} T} V_{zik}^{\frac{1}{h}}(t) + |\beta_{ik}(e_{\hat{x}ik})|$, from which, together with the boundedness of $\beta_{ik}(e_{\hat{x}ik})$ as proven in Step 1 of the proof of Theorem 1, it can be obtained that z_{ik} is bounded, and hence $\tilde{x}_i(t)$, $\tilde{v}_i(t)$ are also bounded over the time interval $[0, T_o]$.

2) For $t \in (T_o, \infty)$, note that $e_{\hat{x}}(t)$ and $e_{\hat{v}}(t)$ remain zero from Theorem 1, and then for $t \in (T_o, T_c]$, it comes from (23) that

$$\begin{aligned} \dot{r}_i = & - \frac{c_1 \kappa h}{T - t} \operatorname{diag}(\eta_i)(r_i + \operatorname{sig}^{[\rho]}(r_i) + \operatorname{sig}^{[\varsigma]}(r_i)) \\ & + \kappa h (T - t)^h \operatorname{diag}(\eta_i)(-c_2 \operatorname{sign}(r_i) + w_i). \end{aligned} \tag{29}$$

Considering another time transformation function

$$t = \theta_2(s) \triangleq T(1 - e^{-s}), \tag{30}$$

we can rewrite (29) as

$$\begin{aligned} \phi'_{ri} = & - c_1 \kappa h \operatorname{diag}(\phi_{\eta i})(\phi_{ri} + \operatorname{sig}^{[\rho]}(\phi_{ri}) + \operatorname{sig}^{[\varsigma]}(\phi_{ri})) \\ & + (T e^{-s})^{1+h} \kappa h \operatorname{diag}(\phi_{\eta i})(-c_2 \operatorname{sign}(\phi_{ri}) + \phi_{wi}) \end{aligned} \tag{31}$$

for $s \in (\theta_2^-(T_o), \theta_2^-(T_c)]$. Let $V_c(s) = \sum_{i=1}^N \sum_{k=1}^n V_{cik}(s)$, in which $V_{cik}(s)$ is defined in (25). Then differentiating $V_c(s)$ along (31) and using Assumption 1 yield

$$\begin{aligned} V'_c(s) \leq & - \sum_{i=1}^N \sum_{k=1}^n c_1 \kappa h \phi_{\eta ik} (|\phi_{rik}(s)|^\rho + |\phi_{rik}(s)|^\varsigma) \\ \leq & -\xi_3 (Nn)^{1-\varsigma} (V_c^\rho(s) + V_c^\varsigma(s)), \end{aligned} \tag{32}$$

where $\xi_3 = \min\{c_1 \kappa h \phi_{\eta ik}\}$ for all $i = 1, ..., N$, $k = 1, ..., n$, and we have used the relation $c_2 > d_w$ and Lemma 4 to get the inequalities. According to Lemma 2, ϕ_{ri} will converge to zero in a fixed time bounded by $s_{c1} = \frac{(Nn)^{\varsigma-1}}{\xi_3}(\frac{1}{1-\rho} + \frac{1}{\varsigma - 1})$ in the case of any $\phi_{\eta ik} \neq 0$. By the time transformation function $\theta_2(s)$, one further obtains that in the original time variable t, the sliding mode surface $r_i(t)$ can be reached when

$$t \to T(1 - e^{-(\theta_2^-(T_o) + s_{c1})}) \triangleq T_{c1} < T. \tag{33}$$

For the case of $\phi_{\eta ik} = 0$, we show that $\phi_{\eta ik} = 0$ is not an attractor in the reaching phase, and would not affect the reachability of the sliding mode surface. Note that $\phi_{\eta i}(s) \triangleq |\tilde{v}_i(t)|^{h-1} = 0$, and then calculate the derivative of $\tilde{v}_i(t)$ by applying the control input (21a) as

$$\dot{\tilde{v}}_i = -\frac{c_1 (r_i + \operatorname{sig}^{[\rho]}(r_i) + \operatorname{sig}^{[\varsigma]}(r_i))}{(T - t)^{1+h}} - c_2 \operatorname{sign}(r_i) + w_i. \tag{34}$$

It is clearly seen that $\dot{\tilde{v}}_i \neq 0$ in the case of $\tilde{v}_i = 0$ and $r_i \neq 0$. Thus we conclude that $\tilde{v}_i(t) = 0$ (i.e., $\phi_{\eta i}(s) = 0$) is not an attractor and the fixed-time reachability of sliding mode surface $r_i(t)$ is still ensured.

Next, for $t \in [T_{c1}, T_c]$, note that the sliding mode surface $r_i(t)$ is reached, i.e., $r_i(t) = 0$, and it then follows from (20a) that $\dot{\tilde{x}}_i = -\frac{\xi_4}{T-t} \operatorname{sig}^{[\frac{1}{h}]}(\tilde{x}_i)$ with $\xi_4 = \kappa^{-\frac{1}{h}}$. Let $\phi_{xi}(s) \triangleq \tilde{x}_i(t)$, and by using (30), one has that $\phi'_{\tilde{x}i}(s) = -\xi_4 \operatorname{sig}^{[\frac{1}{h}]}(\phi_{\tilde{x}i}(s))$ for $s \in [\theta_2^-(T_{c1}), \theta_2^-(T_c)]$. Construct the Lyapunov function $V_{c1}(s) = \sum_{i=1}^N \sum_{k=1}^n |\phi_{\tilde{x}ik}(s)|$, whose derivative gives

$$\dot{V}_{c1}(s) = -\xi_4 \sum_{i=1}^N \sum_{k=1}^n |\phi_{\tilde{x}ik}(s)|^{\frac{1}{h}} \leq -\xi_4 V_{c1}^{\frac{1}{h}}(s), \tag{35}$$

which indicates that $\phi_{\tilde{x}ik}$ for $i = 1, ..., N$, $k = 1, ..., n$ will converge to zero in a finite time $s_{c2} = \frac{h V_{c1}^{\frac{h-1}{h}}(\theta_2^-(T_{c1}))}{\xi_4(h-1)}$ by Lemma 1. Thus in the original time variable t, when

$$t \to T(1 - e^{-(\theta_2^-(T_{c1}) + s_{c2})}) \triangleq T_c < T, \tag{36}$$

one has that $\tilde{x}_i(t) = 0$ and $\tilde{v}_i(t) = 0$.

Note that after the time instant $t = T_c$, the sliding mode surface $r_i(t)$ and the control input $u_i(t)$ are switched from (20a) to (20b) and from (21a) to (21b), respectively. Thus in what follows, we show that the switched controller (21b) keeps the tracking errors $\tilde{x}_i(t)$ and $\tilde{v}_i(t)$ in the origin for all $t > T_c$.

By using (21b), the derivative of (20b) is $\dot{r}_i = -\kappa h \eta_i (c_1(r_i + \mathrm{sig}^{[\rho]}(r_i) + \mathrm{sig}^{[\varsigma]}(r_i)) + c_2 \,\mathrm{sign}(r_i) - w_i)$. Construct the Lyapunov function $V_{c2}(t) = \frac{1}{2}\sum_{i=1}^{N} r_i^T(t) r_i(t)$, whose derivative gives

$$\dot{V}_{c2}(t) \leq -\sum_{i=1}^{N} c_1 \kappa h \eta_i r_i^T(t) r_i(t) \leq 0, \qquad (37)$$

where $c_2 > d_w$ is used. Then integrating it over (T_c, ∞) yields $V_{c2}(t) \leq V_{c2}(T_c) = 0$, which implies that $r_i(t)$ remains zero for $t \in (T_c, \infty)$. Then, from (20b), one finally obtains that $\tilde{x}_i(t) \equiv 0$, $\tilde{v}_i(t) \equiv 0$ on (T_c, ∞).

3) For $t \in [0, T_c]$, it comes from (28) and (32) that all the internal signals $r_i(t)$, $\tilde{x}_i(t)$ and $\tilde{v}_i(t)$ are bounded. Then from the expression of (21a), we obtain that $u_i(t)$ is also bounded over $t \in [0, T_c]$. On the other hand, for $t \in (T_c, \infty)$, it can be also concluded from (37) that all the internal signals are bounded, and hence (21b) is also bounded. Therefore, based on the above analysis, we conclude that the control input $u_i(t)$ remains uniformly bounded for $t \in [0, \infty)$. The proof is completed.

Remark 3 *It is worth noting that the switching time T_c given in (36) depends on the value of $\theta_2^-(T_{c1})$, which further depends on the switching time T_o and the control design parameters given in (33). By recalling that T_o is uniformly bounded explained in Remark 1, it is seen that the value of T_c is also uniformly bounded, which further implies that the relation $T_c < T$ is strictly satisfied under all initial conditions. This feature is essential to guarantee the time-varying gain in the observer-based controller to be uniformly bounded, and thus the numerical problems can be effectively avoided in the controller implementation.*

Remark 4 *It is worth mentioning that a nonsingular sliding mode surface (20a) on $[0, T_c]$ is proposed, which allows that the corresponding consensus tracking controller (21a) is also nonsingular. It is inspired by yet different from that proposed in [42]. Here the proposed sliding mode based controller does not suffer from the singularity problem by introducing a proper selection of the fractional power h with $1 < h < 2$, and meanwhile allows the states to converge to the equilibrium point within a prescribed time, regardless of the second-order dynamics and local directed communication.*

Remark 5 *The proposed observer-based control scheme, gaining its inspiration from [34], is independent of and differs from [34] in several aspects. First, in this paper, the networked multiple systems with limited communication connection is considered, whereas [33] is on prescribed-time differetiator design for single system. Second, only the relative observing information of neighboring agents is utilized in this paper and there is no need for global information of the common origin, thus allowing the observer and controller to be fully distributed, while the direct state feedback is used in the observer as in [34]. Third, for the controller design, a prescribed-time nonsingular sliding mode surface and a sliding mode based tracking controller are further proposed. As far as we know, there has scarcely been any work to address such a prescribed-time sliding mode controller design by using the concept of finite varying gains.*

5 Simulation results

In this section, a simulation example is provided to validate the effectiveness of the theoretical results. We consider a MASs with four followers indexed by 1 to 4 and one leader indexed by 0. The directed communication graph is depicted in Fig. 1.

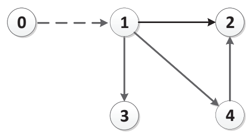

Fig. 1. Communication graph topology.

For the four followers, the nonvanishing uncertainty is set as $w_i(t) = 0.1\sin(t)$ $(i = 1, ..., 4)$, and the initial states are set as $x_1(0) = -0.5$, $v_1(0) = 0$; $x_2(0) = -1$, $v_2(0) = 0.5$; $x_3(0) = 0.5$, $v_3(0) = 0$; $x_4(0) = 2$, $v_4(0) = 1.5$. For the leader, the initial states are set as $x_0(0) = 1$, $v_0(0) = 2$. Choose the prescribed finite-time convergence constants as $T_1 = 2.5s$ and $T = 5s$. Then in light of Theorems 1 and 2, the observer and controller parameters are selected as $\alpha = 0.8$, $\gamma = 1.2$, $k_1 = k_2 = 1$, and $\rho = 0.8$, $\varsigma = 1.2$, $h = 1.5$, $c_1 = 8$, $c_2 = 0.15$, $\kappa = 0.3$, respectively.

The simulation results are shown in Figs. 2-6. Figs. 2 and 3 illustrate the estimation errors of the distributed prescribed-time observer. It is clearly seen that the observer can estimate the states of the leader with high accuracy slightly before the user-defined time $T_1 = 2.5s$, and remain the accurate estimation after $t > 2.5s$. From Figs. 4 and 5, it is seen that the full state tracking errors converge into zero slightly faster than the prescribed finite time $T = 5s$, which implies that the full-state consensus tracking is achieved within the prescribed time. The control input signals are shown in Fig. 6, which verifies that the control inputs are uniformly bounded on the entire time interval, especially before and after the prescribed time T_1 and T. It is also observed that the chattering phenomenon happens in the control inputs due to the sign function terms involved in (21a) and (21b).

To better verify the effectiveness of the proposed method, we enlarge the initial states of the followers (without retuning the other parameters) for comparision. Change the initial states of the followers as $x_1(0) = -5$, $v_1(0) = 0$; $x_2(0) = -10$, $v_2(0) = 5$; $x_3(0) = 5$, $v_3(0) = 0$; $x_4(0) = 20$, $v_4(0) = 15$. The results are revealed in Figs. 7 and 8, showing that the consensus tracking is also achieved within the prescribed time $T = 5s$, and the almost same convergence time are obtained as in Figs. 4 and 5. This implies that the enlarged initial states do not affect the convergence performance under the proposed control scheme.

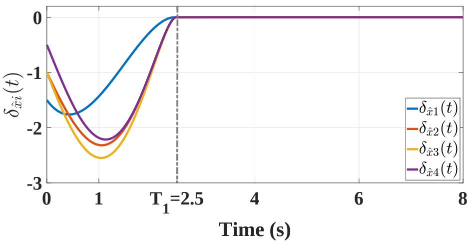

Fig. 2. State estimation errors $\delta_{\hat{x}i}(t)$ of the distributed observer (6).

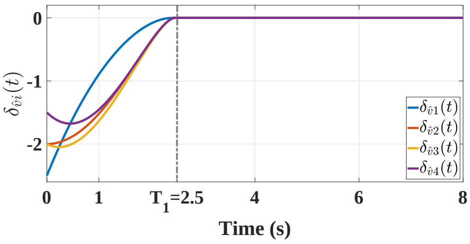

Fig. 3. State estimation errors $\delta_{\hat{v}i}(t)$ of the distributed observer (6).

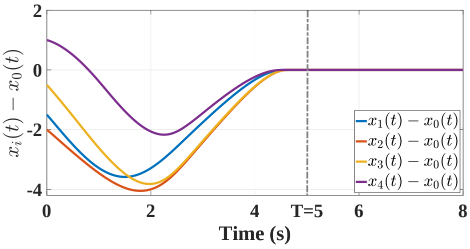

Fig. 4. Consensus tracking errors $x_i(t) - x_0(t)$ of the 4 followers.

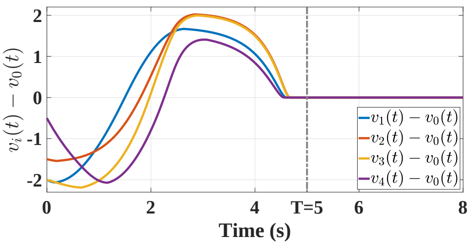

Fig. 5. Consensus tracking errors $v_i(t) - v_0(t)$ of the 4 followers.

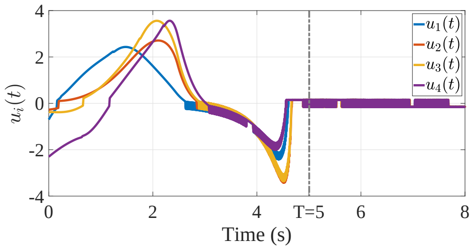

Fig. 6. Control inputs $u_i(t)$ of the 4 followers.

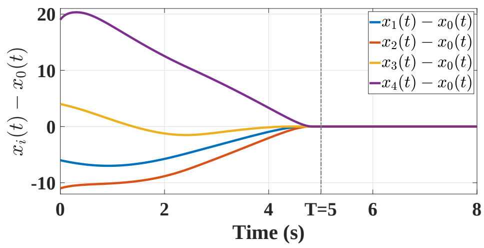

Fig. 7. Consensus tracking errors $x_i(t) - x_0(t)$ of the 4 followers under the enlarged initial scenario.

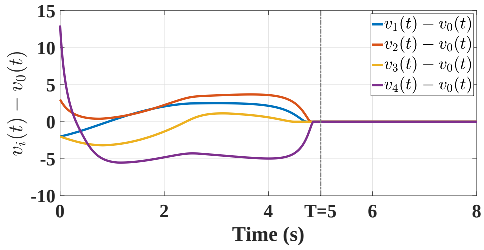

Fig. 8. Consensus tracking errors $v_i(t) - v_0(t)$ of the 4 followers under the enlarged initial scenario.

6 Conclusions

In this paper, we have addressed the prescribed-time leader-following consensus problem for second-order MASs under directed topology. A distributed prescribed-time observer-based control protocol depending only on the local observing information is designed, which is fully distributed. In addition, the proposed protocol does not suffer from the implementation problem arising from the infinite gains as the time approaches the prescribed time instant. It is worth mentioning that our analysis focuses on second-order integrator dynamics. One of the research directions for future work is to generalize the finite time-varying gains to the high-order nonlinear MASs in strict-feedback form.

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