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Finite-time stabilization with arbitrarily prescribed settling-time for uncertain nonlinear systems☆

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ABSTRACT

This paper considers global finite-time stabilization with arbitrarily prescribed settling-time for nonlinear systems with unknown control directions and other serious uncertainties. In view of the challenge of the problem and the scarcity of available methods, several motivating facts are offered by elaborating examples and counterexamples. The facts disclose the potential obstructions in pursuing continuous controllers and force us to resort to powerful discontinuous feedbacks. By combining the method of adding a power integrator, homogeneous approach and desingularization technique, a state-feedback controller with design parameters is first proposed. Then a switching logic is proposed to tune the parameters online, wherein two types of switching sequences are involved to suppress serious uncertainties and to guarantee arbitrarily prescribed settling-time, respectively. Also, fixed-time stabilization is considered as a special case, aiming at the trade-off between arbitrary rapidness and high control cost. Two examples are provided to demonstrate the effectiveness of the proposed strategy.

1. Introduction

Finite-time stability/stabilization has attracted much attention during the past few decades (see, e.g., [1–6]). Owing to its finite-time convergence, finite-time stability manifests the rapid transient of the system. Moreover, the system with finite-time stability owns stronger robustness than that with exponential stability, and hence is more capable of maintaining stability when the disturbance/noise goes worse. Towards the issue, early works (see, e.g., [1,7]) focused on the rigorous framework of finite-time stability and presented its Lyapunov-based conditions. Subsequently, numerous works were devoted to achieving finite-time stabilization for various classes of (uncertain) nonlinear systems (see e.g., [2–6]).

Remark that the settling-time of a finite-time stable system could heavily rely on initial conditions. This means that when the magnitude of the initial state is large, achieving the perfect result would take a long time and finite-time stability would lose its superiority of rapidness. To overcome the defect, the concept of fixed-time stability (see Definition 2 in Section 3) was thus proposed (see e.g., [8–10]), in which the settling-time function is required to be uniformly bounded with respect to initial conditions. In essence, fixed-time stability retains the local behavior of finite-time stability while augmenting the behavior of backward finite-time escape in the large. These two behaviors, as shown in [11], can be guaranteed by the so-called low-power and high-power terms of the system state, respectively, since the latter term can make the system state converge from infinity to a vicinity of the origin in a finite time, and the former term can make the system state converge from the vicinity to zero in another finite time. The two convergence processes spend the total time which is uniformly bounded with respect to initial conditions, and fixed-time stability is thus achieved. The essence and guarantee have been fully shown in the recent works on fixed-time control design (see, e.g., [12–14]).

Despite the nice uniformity for fixed-time stability/stabilization, its fixed settling time might not be small enough for some scenarios. To this end, prescribed-time stability/stabilization was proposed, in which the settling-time that can be arbitrarily prescribed in advance, and it has received special attention recently (see, e.g., [15–20]). Remark that, works, e.g., [17–19], developed a time-varying gain to achieve prescribed-time stabilization, but the controllers therein take effect as the gain grows to infinity in a prescribed finite time. Therefore, the controllers are meaningful in the finite time and are meaningless beyond the time, and apply just to peculiar scenarios, such as missile guidance and vehicle rendezvous [18]. However, many more plants are required to continue to operate after an a priori prescribed time.

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For example, in the ship steering control (see, e.g., [21]), the actual heading angle needs to be regulated to the desired angle before a prescribed time and kept at it thereafter. Hence, there emerged another type of prescribed-time stabilization/stability (see Definition 3 in Section 3), which has an arbitrarily prescribed settling-time and for which zero equilibrium is maintained from the time to infinity. Instead, by endowing fixed-time stabilization/stability with adjustable parameters, not by a time-varying gain, the Lyapunov-based approach has been proposed to classes of systems recently [15,16].

Remarkably, using continuous feedback (or smooth time-varying feedback), prescribed-time stabilization has been achieved for classes of nonlinear systems (see, e.g., [15,16,18,19]). Although therein uncertainties are involved (such as those belonging to a known set), the unknown control direction is completely excluded. As well known, the method based on Nussbaum function is the sole one to present continuous feedback for unknown control directions [22–25]. But unfortunately, the method comes along with extremely violent dynamics of compensation, and the finite-time effectiveness of the dynamic compensation is quite difficult to achieve. Therefore, finite-time stabilization for the systems with unknown control directions almost disproves any continuous feedback, let alone fixed-time stabilization and prescribed-time stabilization. Recently, noting the uncommon capability of switching adaptive feedback in compensating uncertainties [26–29], the finite-time stabilization was achieved in [28] for a class of nonlinear systems, and fixed-time consensus was achieved in [29] for leader-following multiagent systems, both in the context of unknown control directions. However, the settling-time therein cannot be prescribed in advance, since it critically relies on serious system uncertainties which have not any known bounds.

This paper is devoted to global finite-time stabilization with arbitrarily prescribed settling-time for a class of nonlinear systems with unknown control directions and other serious uncertainties (see system (1) and Assumption 1). Unknown control directions highlight the innovation and contribution of this paper, which have not been involved in the works on arbitrarily prescribed settling-time (see, e.g., [16–19]). High system powers and low-order growth co-exist while they are excluded in the relevant literature (see, e.g., [16,19,28]). Irrespective of the system generality, the high performance (arbitrarily prescribed settling-time) and the uncertainties (without any known bounds) give rise to primary challenges of the problem and appeal to deep insights, integrated strategies and capable feedback types. First of all, examples and counterexamples are elaborated to offer several motivating facts (see Section 4). They disclose the vital influence of serious uncertainties and bring the intuition and motivation for our later study on general systems (as well as relevant techniques): ***(i)*** static continuous feedback is impossible (see Fact 3 below); ***(ii)*** dynamic continuous feedback is quite hard to synthesize (see Fact 4); ***(iii)*** continuous feedback can be attempted if uncertainties are confined or objective is degenerated (see Facts 1 and 2); ***(iv)*** switching adaptive feedback is feasible if discontinuous feedbacks are admitted (see Fact 5).

With the motivating facts, especially Fact 5, we pursue a switching adaptive controller to accomplish the desired prescribed-time stabilization. The controller design entails not only switching adaptive feedback but other constructive design techniques, which renders it possible the arbitrarily prescribed settling-time in the context of multiple uncertainties. First, a state-feedback controller with design parameters is presented by the method of adding a power integrator [3], homogeneous approach [8] and desingularization technique [30]. Then, an elegant switching logic, motivated by Fact 5, is proposed to tune the parameters online (see Section 5), in which two types of switching sequences are introduced to suppress serious uncertainties and to guarantee arbitrarily prescribed settling-time, respectively. Recognize that arbitrarily prescribed settling-time could cause large control gain (hence high control cost) typically. Aiming at the trade-off between them, we take into account fixed-time stabilization as a special case and moderately reduce the largeness of control gain. Simulation experiments, in Section 6, demonstrate the proposed controller indeed ensures arbitrarily prescribed settling-time. However, the experiments in the related works, e.g., [15–19], rule out any unknown control direction. We would like to underline the advantages of this paper. On the one hand, it turns out that serious uncertainties and high performance render discontinuous feedbacks rather necessary. On the other hand, a switching adaptive controller is proposed for the breakthrough of the challenging problem, achieving system generality and performance perfection.

The remainder of the paper is organized as follows. Section 2 formulates the uncertain nonlinear systems and control objective. Section 3 gives some preliminaries. Section 4 elaborates several motivating facts for further study. Section 5 presents switching adaptive controllers with the intuition and motivation offered in Section 4. Two examples are provided in Section 6 to illustrate the proposed approach. Concluding remarks are given in Section 7. Appendix collects several proofs.

Throughout this paper, the following notations and abbreviations will be adopted.

$\mathbf{R}$, $\mathbf{R}^+$, $\mathbf{R}^n$, $\mathbf{Z}^+$	the sets of real numbers, nonnegative real numbers, real n-dimensional vectors and positive integers, respectively
$\lvert s\rvert$	the absolute value of real number s
$\{s\}^p$	$\{s\}^p = \lvert s\rvert^p\mathbf{sign}(s)$, $s \in \mathbf{R}$, $p \in \mathbf{R}^+$, where $\mathbf{sign}(\cdot)$ is the sign function
x_i, $x_{[i]}$	for a vector $x \in \mathbf{R}^n$, x_i denotes its ith element and $x_{[i]} = [x_1, \dots, x_i]^{\mathrm{T}} \in \mathbf{R}^i$, $i \in \{1, \dots, n\}$
$T_{\mathrm{s}}(\cdot)$	the settling-time function
T_{p}	an arbitrarily prescribed time
GFTS	global finite-time stability
GFTS-UBST	GFTS with uniformly bounded settling-time
GFTS-APST	GFTS with arbitrarily prescribed settling-time

2. Problem formulation

We consider the global finite-time stabilization with arbitrarily prescribed settling-time for the following uncertain nonlinear system:

$$\begin{cases} \dot{x}_i &= g_i\{x_{i+1}\}^{p_i} + f_i(x_{[i]}),\ \ i = 1, \dots, n-1, \\ \dot{x}_n &= g_n\{u\}^{p_n} + f_n(x), \end{cases} \tag{1}$$

where $p_i \geq 1$, $i = 1, \dots, n$ are called the system powers; $x = [x_1, \dots, x_n]^{\mathrm{T}} \in \mathbf{R}^n$ is the system state with the initial value $x(0) = x_0$; $u \in \mathbf{R}$ is the control input; $f_i(x_{[i]})$, $i = 1, \dots, n$, are unknown continuous functions with $f_i(0) = 0$, termed the unknown system nonlinearities; g_i, $i = 1, \dots, n$, are nonzero and unknown constants, termed the unknown control coefficients, and particularly their signs called control directions are unknown.

System (1) and its variants could cover numerous practical plants, such as ship steering autopilot [21], robot manipulator [26] and tunnel-diode circuit [31], and have been extensively investigated during the past few decades (e.g., [4,19,25,32]). Particularly, the unknown control coefficients therein can be regarded as the failure/fault of the actuator and/or physical parameters with unknown signs, such as the time constant in ship steering model (see (26)).

We impose the following mild assumption on the nonlinearities of system (1), despite the sophisticated stabilization task.

Assumption 1. For each $f_i(x_{[i]})$, $i = 1, \ldots, n$, there exists a known smooth nonnegative function $\bar{f}_i(x_{[i]})$ such that

$$|f_i(x_{[i]})| \leq \theta \bar{f}_i(x_{[i]}) \sum_{j=1}^{i} |x_j|^{\frac{r_i+\tau}{r_j}}, \tag{2}$$

where $\theta > 0$ is an unknown constant, $\tau \in \left(-\frac{1}{1+\sum_{j=1}^{n-1} p_1 p_2 \cdots p_j}, 0\right)$ and r_i is defined by $r_1 = 1$, $r_j = \frac{r_{j-1}+\tau}{p_{j-1}}$, $j = 2, \ldots, n$.

Assumption 1 indicates that system (1) allows inherent nonlinearities and serious uncertainties, reflected by $\bar{f}_i(x_{[i]})$ and θ, respectively. Specifically, the growth of the nonlinearities is of low order (i.e., in (2), each power $\frac{r_i+\tau}{r_j} < 1$), unlike the related works (e.g., [16,18,28]) where the growths are linear or of high order. Actually, both high-order and linear growths can be transformed into the low-order growths by adjusting the growth rate functions (i.e., $\bar{f}_i(\cdot)$ in (2)). Thus Assumption 1 covers those in the aforementioned works.

The uncertainty θ is completely excluded in the works with the same stabilization task (e.g., [15,16]). In Section 4, it is shown by examples and counterexamples that this uncertainty poses an inherent obstruction in pursuing the convergence (to zero) in an arbitrarily prescribed time.

Beyond the general nonlinearities, unknown control directions and high system powers (i.e., $p_i \geq 1$, $i = 1, \ldots, n$) are allowed in system (1), while they are ruled out in the existing works on prescribed-time stabilization, e.g., [15–19].

The multiple ingredients undermine the relevant control schemes (e.g., [16,18,19]) and appeal to deep insights, integrated strategies and powerful controllers to achieve the wanted stabilization. In view of the essentiality and complexity of the problem, we will not immediately carry out the control design and analysis, while making adequate preparations in subsequent Sections 3 and 4.

Specifically, Section 3 provides related concepts and conditions of finite-time stability, and collects several useful inequalities. Section 4 presents several motivating facts, by addressing examples and counterexamples with respect to system (1), to disclose the vital influence of system uncertainties and the feasibility and potential techniques of the wanted stabilization.

With the motivating facts, we resort to switching adaptive feedback to complete the control design of system (1) in Section 5, extending the prescribed-time stabilization under which the system still keeps defined from the finite time to infinity.

3. Preliminaries

Let us introduce the concepts of *global finite-time stability* (GFTS), *global finite-time stability with uniformly bounded settling-time* (GFTS-UBST, i.e., global fixed-time stability) and *global finite-time stability with arbitrarily prescribed settling-time* (GFTS-APST), all with respect to the following nonlinear system:

$$\dot{x} = f(x), \tag{3}$$

where $x \in \mathbf{R}^n$ is the state with the initial condition $x(0) = x_0$, and $f : \mathbf{R}^n \to \mathbf{R}^n$ is continuous and satisfies $f(0) = 0$.

Assume that system (3) has a unique solution in forward time for any initial condition x_0.

Definition 1 (*[1]*)**.** System (3) is globally finite-time stable if

- ***(i)*** it is stable, i.e., for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that, for any $x_0 \in \mathcal{U}_\delta \triangleq \{x \mid \|x\| < \delta\}$, there holds $\|x(t)\| < \varepsilon$ for all $t \geq 0$;
- ***(ii)*** it is globally finite-time convergent, i.e., there exists a function $T_s : \mathbf{R}^n \to \mathbf{R}^+$ with $T_s(0) = 0$ (called the settling-time function), such that for any initial condition x_0, $x(t) \neq 0$ $\forall t \in [0, T_s(x_0))$, $\lim_{t\to T_s(x_0)} x(t) = 0$ and $x(t) = 0$ $\forall t \in [T_s(x_0), +\infty)$.

Definition 2 (*[10]*)**.** System (3) is globally finite-time stable with uniformly bounded settling-time (i.e., globally fixed-time stable) if it is globally finite-time stable and the associated settling-time function $T_s(x_0)$ is globally bounded on $\mathbf{R}^n$, i.e., $\sup_{x_0\in\mathbf{R}^n} T_s(x_0) < +\infty$.

If system (3) is oriented from that with a feedback controller in the loop, and the controller is with certain adjustable parameters, then the following concept of GFTS-APST is meaningful.

Definition 3 (*[16]*)**.** System (3) with adjustable parameters is globally finite-time stable with arbitrarily prescribed settling-time if it is globally finite-time stable and for an arbitrarily prescribed time $T_p > 0$, its settling-time function satisfies $\sup_{x_0\in\mathbf{R}^n} T_s(x_0) \leq T_p$ by tuning the associated parameters.

The three concepts of finite-time stability differ as the settling-time is gradually strengthened. The third one, with its orientation of controller, can be referred to as prescribed-time stabilization [15,16], for which the system and control are meaningful for all time, not just in the specified time. It clearly differs from the prescribed-time stabilization in [17–19], where the system and control are confined in the prescribed time and are meaningless beyond the time.

Subsequently, let us characterize the conditions of GFTS-UBST and GFTS-APST from a Lyapunov perspective. While the following theorem involves the same condition as [11], it provides a tighter and less conservative estimate on settling-time. As such, we merely show the estimate in the theorem.

Theorem 1. *For system (3), suppose there exists a continuously differentiable, positive definite and radially unbounded function $V(x)$ satisfying*

$$\dot{V}(x) \leq -c_1 V^\alpha(x) - c_2 V^\beta(x), \tag{4}$$

with constants $c_1 > 0$, $c_2 > 0$, $0 < \alpha < 1$ and $\beta > 1$. Then the system can achieve GFTS-UBST and the settling-time function $T_s(x_0)$ satisfies

$$\sup_{x_0\in\mathbf{R}^n} T_s(x_0) \leq \frac{\pi}{(\beta-\alpha)c_1^{\frac{\beta-1}{\beta-\alpha}} c_2^{\frac{1-\alpha}{\beta-\alpha}} \sin\left(\frac{(1-\alpha)\pi}{\beta-\alpha}\right)}. \tag{5}$$

Furthermore, if constants c_1 and c_2 are arbitrarily adjustable, then the system can achieve GFTS-APST.

Proof. Using Corollary 2 in [11], we know by (4) that the system has GFTS-UBST. The associated settling-time function, by Theorem 4 in [11], satisfies

$$\sup_{x_0\in\mathbf{R}^n} T_s(x_0) \leq \int_0^\infty \frac{1}{c_1 V^\alpha + c_2 V^\beta} \mathrm{d}V. \tag{6}$$

We next show the right-hand side of (6) is equal to that of (5). Make substitution $V = \left(\frac{c_1}{c_2} z\right)^{\frac{1}{\beta-\alpha}}$; thereby $\mathrm{d}V = \left(\frac{c_1}{c_2}\right)^{\frac{1}{\beta-\alpha}} \frac{1}{\beta-\alpha} z^{\frac{1}{\beta-\alpha}-1} \mathrm{d}z$. By the substitution, we have

$$\begin{aligned}
&\int_0^\infty \frac{1}{c_1 V^\alpha + c_2 V^\beta} \mathrm{d}V \\
&= \int_0^\infty \frac{\left(\frac{c_1}{c_2}\right)^{\frac{1}{\beta-\alpha}} \frac{1}{\beta-\alpha} z^{\frac{1}{\beta-\alpha}-1}}{c_1 \left(\frac{c_1}{c_2}\right)^{\frac{\alpha}{\beta-\alpha}} z^{\frac{\alpha}{\beta-\alpha}} + c_2 \left(\frac{c_1}{c_2}\right)^{\frac{\beta}{\beta-\alpha}} z^{\frac{\beta}{\beta-\alpha}}} \mathrm{d}z \\
&= \frac{1}{(\beta-\alpha)c_1^{\frac{\beta-1}{\beta-\alpha}} c_2^{\frac{1-\alpha}{\beta-\alpha}}} \int_0^\infty \frac{z^{\frac{1-\alpha}{\beta-\alpha}-1}}{1+z} \mathrm{d}z.
\end{aligned} \tag{7}$$

Note that $\int_0^{\infty}\frac{x^{p-1}}{1+x}\mathrm{d}x=\frac{\pi}{\sin(p\pi)}$ when $0<p<1$ (see Theorem 1.2.1, page 9 of [33]). Applying this and $0<\frac{1-\alpha}{\beta-\alpha}<1$ to (7) immediately arrives at the right-hand side of (5).

It is apparent from (5) that if constants c_1 and c_2 are adjustable, then for any prescribed $T_{\mathrm{p}}>0$, there are c_1 and c_2 such that $\sup_{x_0\in\mathbf{R}^n}T_{\mathrm{s}}(x_0)\le T_{\mathrm{p}}$. □

Let us collect several useful inequalities in Lemmas 1–3.

Lemma 1 ([4]). *For positive constants p_1 and p_2 and a scalar function $\gamma(x,y)>0$, where $x\in\mathbf{R}$ and $y\in\mathbf{R}$, there holds*

$$|x|^{p_1}|y|^{p_2}\le\frac{p_1\gamma(x,y)}{p_1+p_2}|x|^{p_1+p_2}+\frac{p_2}{(p_1+p_2)\gamma^{\frac{p_1}{p_2}}(x,y)}|y|^{p_1+p_2}.$$

Lemma 2 ([32]). *Let $z_i\in\mathbf{R}$, $i=1,\ldots,n$, $x\in\mathbf{R}$ and $y\in\mathbf{R}$. Then, for constant $p\in(0,\ 1]$, there hold*

$$\begin{cases}(|z_1|+\cdots+|z_n|)^p\ \le\ |z_1|^p+\cdots+|z_n|^p,\\ (|z_1|+\cdots+|z_n|)^{\frac{1}{p}}\ \le\ n^{\frac{1}{p}-1}\big(|z_1|^{\frac{1}{p}}+\cdots+|z_n|^{\frac{1}{p}}\big),\\ |\{x\}^p-\{y\}^p|\ \le\ 2^{1-p}|x-y|^p.\end{cases}$$

Lemma 3 ([34]). *Let $x\in\mathbf{R}$. Then, for constants p, $\underline{p}$ and $\bar{p}$ satisfying $0<\underline{p}<p<\bar{p}$, there holds*

$$|x|^p\ \le\ |x|^{\underline{p}}+|x|^{\bar{p}}.$$

4. Motivating facts

As above-introduced, this paper aims at global finite-time stabilization with arbitrarily prescribed settling-time for uncertain nonlinear systems. But in terms of the system uncertainties without any known bounds, the problem has not been investigated and cannot be settled readily. In view of its challenge and the scarcity of available methods, in this section we look into examples and counterexamples to bring intuition and motivation for our later elaborate study.

Specifically, for the typical problem below, we provide several facts to indicate the vital influence of system uncertainties on GFTS-APST. Fact 1 states that if uncertainties are reduced to a known range, then a static continuous controller can be pursued. Fact 2 shows that if the control objective degenerates into a semi-global one, it is possible to design a dynamic-high-gain controller. Without any reduction/degeneration, Fact 3 precludes the possibility of any static continuous controller; Fact 4 rules out a large class of dynamic-high-gain continuous controllers which share common features with those for adaptive and fixed-time stabilizations. However, Fact 5 shows that a switching adaptive controller is possible if discontinuous feedbacks are admitted.

Typical Problem. *Whether or not a continuous controller exists to achieve GFTS-APST of the following typical scalar uncertain system:*

$$\dot{x}\ =\ u+\theta|x|,\qquad x(0)=x_0,\tag{8}$$

where θ is an unknown constant (assumed to be positive).

We first provide two positive facts, on the typical problem, to show that if system uncertainties are confined and the control objective is degenerated, then there indeed exist static and dynamic continuous controllers to achieve global and semi-global finite-time stabilization with arbitrarily prescribed settling-time, respectively.

Positive Fact 1. *For system (8), assume $\theta\le\theta_0$ for a known constant $\theta_0>0$. Then for an arbitrarily prescribed time $T_{\mathrm{p}}>0$, the following static continuous controller* can *guarantee its GFTS-APST:*

$$u=-\theta_0x-c\big(x^{\frac{1}{3}}+x^3\big),\quad c\ge\frac{3\pi}{2^{\frac{3}{2}}T_{\mathrm{p}}}.$$

Proof. By Theorem 1 with $V=\frac{x^2}{2}$, we can readily verify GFTS-APST of the controller. □

Recognize that it is expected for GFTS-APST that the control gain is equipped with the mechanism of finite-time escape. We thus achieve the following positive fact.

Positive Fact 2. *For system (8), the following dynamic-high-gain controller* can *guarantee its* semi-global finite-time stability with arbitrarily prescribed settling-time:

$$\begin{cases}u=-2k^2x^{\frac{1}{3}}-2k^2x^3,\\ \dot{k}=k^2x^2,\quad k(0)=k_0(x_0,T_{\mathrm{p}})>0,\end{cases}\tag{9}$$

where k_0 is to-be-determined as required.

Proof. See Appendix A.1. □

We next provide two negative facts for the above typical problem: it is impossible to pursue a static continuous controller or a large class of dynamic-high-gain continuous controllers.

Negative Fact 3. *For system (8), any static continuous feedback controller, independent of unknown θ,* cannot *guarantee its GFTS-APST.*

Proof. Suppose by contradiction that the following continuous controller can guarantee GFTS-APST of system (8):

$$u=\varphi(x,T_{\mathrm{p}}).$$

This means that under the controller, $x(t)\equiv0\ \forall t\ge T_{\mathrm{p}}$, no matter what x_0 or θ. Particularly, we know by continuity that for any initial value $x_0>0$ and any $\theta>0$, there exists a time instant $0<t_1<T_{\mathrm{p}}$ such that $\dot{x}(t_1)=\varphi(x(t_1),T_{\mathrm{p}})+\theta|x(t_1)|<0$ and $x(t)>0\ \forall t\le t_1$.

Fixing a positive x_0 and noting the arbitrariness of unknown θ and the independence of $\varphi(\cdot)$ in θ, there is the case that $\theta>\frac{|\varphi(x_0,T_{\mathrm{p}})|}{x_0}$, which implies $\dot{x}(0^+)>0$. Then, by $\dot{x}(t_1)<0$, we learn that there exists another time instant $0<t_2<t_1$ such that $\dot{x}(t_2)=0$ and in turn $x(t)\equiv x(t_2)>0\ \forall t\ge t_2$. This directly contradicts the supposition. □

In view of Facts 2 and 3, we conjecture that dynamic continuous controllers are still impossible for GFST-APST. Since dynamic continuous controllers are too vast to be delineated, we prove the impossibility for a large class of such controllers; see Fact 4 below. We choose this class, not others, because: ***(i)*** they share low-power and high-power terms as in fixed-time stabilization [11]; ***(ii)*** they are of dynamic high gain and own sufficiently strong capability for uncertain nonlinear systems. The negative Fact 4 shows that it is quite hard to pursue a dynamic continuous controller if it does exist for the above typical problem.

We let $\omega(\cdot,T_{\mathrm{p}})$ be a continuous, positive and monotonously increasing function.

We also let $\psi(\cdot,\cdot,T_{\mathrm{p}})$ be a continuous vector-valued function with nonnegative entries such that

$$\dot{\kappa}=\psi(v,\kappa,T_{\mathrm{p}}),$$

is *zero-input globally bounded*,[1] and its solution is continuously dependent on v, T_{p} and initial value.

Negative Fact 4. *For system (8), the following dynamic-high-gain controller* cannot *guarantee its GFTS-APST:*

$$\begin{cases}u=-\omega(k,T_{\mathrm{p}})\big(\{x\}^{\alpha}+\{x\}^{\beta}\big),\\ \dot{k}=\psi(x,k,T_{\mathrm{p}}),\quad k(0)=k_0(T_{\mathrm{p}})>0,\end{cases}\tag{10}$$

where $0<\alpha<1$ and $\beta>1$.

[1] A dynamical system is said to be *zero-input globally bounded* if once its input vanishes, it is globally bounded. Here, for κ-dynamics, with $v=0$, for any initial value $\kappa(0)$, there is $\epsilon>0$ such that $\|\kappa(t)\|<\epsilon$ for $t\ge0$.

Proof. See Appendix A.2. □

With the negative facts on continuous controllers, we turn to pursue a switching adaptive controller, noting that discontinuous feedback has remarkable advantages over continuous feedback in compensating serious uncertainties.

We let sequences $\{m_k, k \in \mathbf{Z}^+\}$ and $\{h_k, k \in \mathbf{Z}^+\}$ satisfy

$$\begin{cases} m_k = \frac{2^k}{k\delta}, \\ 0 < |h_k| < |h_{k+1}|, \\ \lim_{k\to+\infty} |h_k| = +\infty, \\ \mathbf{sign}(h_k) = -\mathbf{sign}(h_{k+1}), \end{cases} \tag{11}$$

where $\delta > 0$ is a design parameter to be determined later.

We also let $\chi(t, a, c)$ be a nonnegative solution of integral equation $\int_a^\chi \frac{1}{1+s^2} ds = -ct$ with $a > 0$ and $c > 0$, i.e.,

$$\chi(t, a, c) = \begin{cases} \tan\big(-ct + \arctan(a)\big), & t \le \frac{1}{c}\arctan(a), \\ 0, & t > \frac{1}{c}\arctan(a). \end{cases} \tag{12}$$

Positive Fact 5. *For the following variant of system (8) (by admitting an unknown control coefficient):*

$$\dot{x} = gu + \theta|x|, \qquad x(0) = x_0, \tag{13}$$

a suitable switching logic on parameters $\{m_k\}$ and $\{h_k\}$ can be found such that the following set of controllers can guarantee its GFTS-APST:

$$u_k = -h_k(m_k + |h_k|)\big(x^{\frac{3}{5}} + x^{\frac{7}{5}}\big), \quad k \in \mathbf{Z}^+, \tag{14}$$

where g is a nonzero unknown constant whose unknown sign incurs unknown control direction to the system.

Proof. We need to search for a switching logic to render controllers (14) feasible.

Define the switching times $\{t_k, k \in \mathbf{Z}^+\}$ as

$$t_k = \inf\Big\{t > t_{k-1} \Big| V^{\frac{1}{5}}(t) > \chi\big(t - t_{k-1}, V^{\frac{1}{5}}(t_{k-1}^+) + \varepsilon, c_k\big)\Big\}, \tag{15}$$

where $\varepsilon > 0$ is a prescribed constant, $c_k = 2^{\frac{4}{5}} \cdot \frac{m_k}{5}$ and $V = \frac{x^2}{2}$.

The initialization of the switching logic is set as follows:

(***i***) Assign an arbitrary constant $\varepsilon > 0$;

(***ii***) Select h_k satisfying (11) and $\delta \le 2^{\frac{4}{5}} \cdot \frac{T_p}{5\pi}$;

(***iii***) Set $t_0 = 0$, $k = 1$ and in turn specify m_1 and h_1.

The following switching logic ($k \in \mathbf{Z}^+$) is implemented: At first, controller (14) with the parameters m_1 and h_1 works on system (13) until the first switching time t_1 is detected. Just after t_1, parameters m_1 and h_1 are changed into m_2 and h_2, while controller (14) is parameterized by m_2 and h_2. Moreover, the controller (14) with the parameters m_2 and h_2 acts until the second switching time t_2 is detected. This process will continue until the switching stops. Remarkably, controller u_k works on switching interval $(t_{k-1}, t_k]$.

Appendix A.3 provides the rest proof: only finite switchings happen for controller (14) under the given switching logic; there establishes GFTS-APST of the resulting closed-loop system. □

The above facts disclose the feasibility and potential techniques of the above typical problem. With these, our later study is devoted to achieving GFTS-APST for a class of general uncertain nonlinear systems (i.e., system (1)), with the aid of switching adaptive feedback (as in Fact 5).

5. Switching adaptive controller design

This section focuses on global finite-time stabilization with arbitrarily prescribed settling-time for system (1). First, a state-feedback controller with design parameters is recursively designed by integrating constructive design methods. Then a switching logic is proposed to tune the parameters online, wherein two types of switching sequences (respectively reflected by $\{h_{[i],k}, k \in \mathbf{Z}^+\}$ and $\{m_k, k \in \mathbf{Z}^+\}$ in (20)) are involved to suppress serious uncertainties and to guarantee arbitrarily prescribed settling-time, respectively. Likewise, fixed-time stabilization is considered as a special case, by choosing sequence $\{m_k, k \in \mathbf{Z}^+\}$ as a fixed positive constant c, aiming at the trade-off between arbitrary rapidness and high control cost.

Inspired by Fact 5 above, we design the following state-feedback controller (parameterized by m and $h_{[n]}$):

$$u = x_{n+1}^* = -\Big\{h_n\phi_n(x, m, h_{[n-1]})\{z_n\}^{r_n+\tau}\Big\}^{\frac{1}{p_n}}, \tag{16}$$

where z_n is recursively defined by

$$\begin{cases} z_1 = x_1, \qquad x_1^* = 0, \\ z_i = \{x_i\}^{\frac{1}{r_i}} - \{x_i^*\}^{\frac{1}{r_i}}, \\ x_i^* = -\Big\{h_{i-1}\phi_{i-1}(x_{[i-1]}, m, h_{[i-2]})\{z_{i-1}\}^{r_{i-1}+\tau}\Big\}^{\frac{1}{p_{i-1}}}, \\ \quad i = 2, \dots, n. \end{cases} \tag{17}$$

In (16) and (17), $h_0 = 0$, for each i, $i = 1, \dots, n$, r_i and τ are the same as in (2), m and h_i are design parameters whose tuning laws will be determined later, and positive function $\phi_i(\cdot)$ is continuously differentiable in the first argument and continuous in the rest two ones.

It remains to determine design functions $\phi_i(\cdot)$, $i = 1, \dots, n$ to complete the control design. The functions will be specified in the following proposition (also in its proof), by the method of adding a power integrator [3], homogeneous approach [8] and desingularization technique [30].

Proposition 1. *For system (1) under Assumption 1, positive functions $\phi_i(\cdot)$, $i = 1, \dots, n$ can be found such that function $V_n(x, m_1, h_{[n-1]}) = \sum_{i=1}^n \int_{x_i^*}^{x_i} \Big\{\{s\}^{\frac{1}{r_i}} - \{x_i^*\}^{\frac{1}{r_i}}\Big\}^{2-r_i} ds$ satisfies*

$$\begin{aligned} \dot{V}_n \le\ & -2^{\frac{(r_n-1)(2+\tau)}{2}} m V_n^{\frac{2+\tau}{2}} - 2^{\frac{(r_n-1)(2-\tau)}{2}} n^{\frac{\tau}{2}} m V_n^{\frac{2-\tau}{2}} \\ & - \sum_{i=1}^n (h_i g_i - 1)\phi_i(\cdot)|z_i|^{2+\tau} \\ & + \sum_{i=1}^n \big(F_i(x_{[i]}, h_{[i-1]}, \theta, g_{[i-1]}) \\ & - F_i(x_{[i]}, h_{[i-1]}, |h_1|, h_{[i-1]})|z_i|^{2+\tau}, \end{aligned} \tag{18}$$

where $g_0 = 0$, and functions $F_i(\cdot)$, $i = 1, \dots, n$ are nonnegative and continuous (furthermore continuously differentiable in the first argument), and strictly increasing in the third argument (on $\mathbf{R}^+$) and in the amplitude of each entry of the last argument.

Proof. See Appendix A.4. □

To compensate the uncertainties in a prescribed time, we would like to tune parameters m and $h_{[n]}$ in (16) and (17) online in a switching manner. To this end, we derive the following set of controllers from (16):

$$\begin{aligned} u_k &= u(x, m_k, h_{[n],k}) \\ &= -\Big\{h_{n,k}\phi_n(x, m_k, h_{[n-1],k})\{z_n\}^{r_n+\tau}\Big\}^{\frac{1}{p_n}}, \quad k \in \mathbf{Z}^+, \end{aligned} \tag{19}$$

where $\{m_k, k \in \mathbf{Z}^+\}$ is a switching sequence and $h_{[n],k} = [h_{1,k}, \dots, h_{n,k}]^{\mathrm{T}}$ with $\{h_{i,k}, k \in \mathbf{Z}^+\}$, $i = 1, \dots, n$ being switching sequences. The $n+1$ sequences are defined by

$$\begin{cases} m_k \;=\; \frac{2^k}{k\delta}, \\ 0 \;<\; |h_{i,k}| \;<\; |h_{i,k+1}|, \\ \lim_{k\to+\infty}|h_{i,k}| \;=\; +\infty, \\ \begin{cases} \mathbf{sign}(h_{i,k_1}) = 1,\;\; k_1 = 1,\ldots,2^{n-i}, \\ \mathbf{sign}(h_{i,k_2}) = -1,\;\; k_2 = 1+2^{n-i},\ldots,2^{n-i+1}, \\ \mathbf{sign}(h_{i,k+2^{n-i+1}}) = \mathbf{sign}(h_{i,k}), \end{cases} \\ \quad i = 1,\ldots,n, \end{cases} \tag{20}$$

where $\delta > 0$ is a design parameter to be determined later.

Next, we design the switching logic to tune parameters m_k and $h_{[n],k}$ in controllers (19).

The initialization of the switching logic is set as follows:

(i) Assign an arbitrary number $\varepsilon > 0$;

(ii) Choose$\{h_{i,k}\}$, $i = 1, \ldots, n$ satisfying (20), and $\delta \le 2^{\frac{(r_n-1)(2-\tau)}{2}-1}\cdot n^{\frac{\tau}{2}}\cdot\frac{|\tau|T_p}{\pi}$ for arbitrarily prescribed $T_p > 0$.

(iii) Set $t_0 = 0$ and $k = 1$. Accordingly, $h_{i,1}$, $i = 1,\ldots,n$ and m_1 are determined.

The following switching logic ($k \in \mathbf{Z}^+$) is implemented: At initial time t_0, controller (19) with parameters $h_{[n],1}$ and m_1 is applied to system (1) and it works until t_1 is detected. Just after t_1, parameters m_1 and $h_{[n],1}$ are switched to parameters m_2 and $h_{[n],2}$, respectively. Accordingly, the controller (19) is parameterized by m_2 and $h_{[n],2}$, which works until t_2 is detected. This process will continue until the switching stops. It is worth pointing out that controller (19) with parameters $h_{[n],k}$ and m_k works on switching time interval $(t_{k-1}, t_k]$, where switching time sequence $\{t_k, k \in \mathbf{Z}^+\}$ satisfies

$$t_k = \inf\Big\{t > t_{k-1}\,\Big|\,V_{n,k}^{\frac{|\tau|}{2}}(t) > \chi\left(t - t_{k-1}, V_{n,k}^{\frac{|\tau|}{2}}(t_{k-1}^+) + \varepsilon, \bar{c}_k\right)\Big\}, \tag{21}$$

where $\chi(\cdot)$ is defined by (12), $\bar{c}_k = 2^{\frac{(r_n-1)(2-\tau)}{2}-1}n^{\frac{\tau}{2}}|\tau|m_k$ and $V_{n,k}(t) = V_n(x(t), m_k, h_{[n],k})$.

Theorem 2. *Consider system (1) under Assumption 1. The switching adaptive controller derived from (19) guarantees that for any initial value x_0, signals $x(t)$ and $u(t)$ are bounded on $[0,\infty)$, and particularly, state $x(t)$ converges to zero in an arbitrarily prescribed time T_p and maintains zero for all $t \ge T_p$.*

Proof. From (16), it follows that the right-hand side of system (1) is continuous on each switching time interval $(t_{k-1}, t_k]$. By this and the existence theorem, and concatenating solutions between successive switching intervals, we know that the system state $x(t)$ can be defined on the maximal existence interval $[0, t_f)$ with $0 < t_f \le \infty$.

We next prove by contradiction that only finite switchings happen. Suppose that there are infinite switchings. Then, there exists a sufficiently large k^* such that (for $t \in (t_{k^*-1}, t_{k^*}]$)

$$\dot{V}_{n,k^*}(t) \;\le\; -2^{\frac{(r_n-1)(2-\tau)}{2}}n^{\frac{\tau}{2}}m_{k^*}\left(V_{n,k^*}^{\frac{2+\tau}{2}}(t) + V_{n,k^*}^{\frac{2-\tau}{2}}(t)\right), \tag{22}$$

where t_{k^*} denotes the k^*-th switching time.

Let us verify the correctness of (22). The nonnegative functions $F_i(x_{[i]}, h_{[i-1],k}, \theta, g_{[i-1]})$, $i = 1,\ldots,n$ are strictly increasing and continuous in θ and $|g_{i-1}|$, $i = 1,\ldots,n$, which ensures that when $|h_{1,k}| \ge \max\{\theta, |g_1|\}$ and $|h_{i,k}| \ge |g_i|$, $i = 2,\ldots,n-1$, there hold $F_i(x_{[i]}, h_{[i-1],k}, \theta, g_{[i-1]}) - F_i(x_{[i]}, h_{[i-1],k}, |h_{1,k}|, h_{[i-1],k}) \le 0$, $i = 1,\ldots,n$. Moreover, there exists k such that $\mathbf{sign}(g_{[n]}) = [\mathbf{sign}(g_1),\ldots,\mathbf{sign}(g_n)]^{\mathrm{T}} = [\mathbf{sign}(h_{1,k}),\ldots,\mathbf{sign}(h_{n,k})]^{\mathrm{T}} = \mathbf{sign}(h_{[n],k})$ in every 2^n period, since $\{\mathbf{sign}(h_{i,k})\}$, $i = 1,\ldots,n$ are periodic with period 2^{n-i+1}. Thus, there always exists k^* such that ($i = 1,\ldots,n$)

$$\begin{cases} 1 - g_i h_{i,k^*} \le 0, \\ F_i(x_{[i]}, h_{[i-1],k^*}, \theta, g_{[i-1]}) \le \\ \qquad\qquad F_i(x_{[i]}, h_{[i-1],k^*}, |h_{1,k^*}|, h_{[i-1],k^*}), \end{cases} \tag{23}$$

which, together with (18), implies that (22) holds.

Solving (22) and noting the definition of $\chi(\cdot)$ in (12), we have (for $t \in (t_{k^*-1}, t_{k^*}]$)

$$\begin{aligned} V_{n,k^*}^{\frac{|\tau|}{2}}(t) &\le \chi\left(t - t_{k^*-1}, V_{n,k^*}^{\frac{|\tau|}{2}}(t_{k^*-1}^+), \bar{c}_{k^*}\right) \\ &\le \chi\left(t - t_{k^*-1}, V_{n,k^*}^{\frac{|\tau|}{2}}(t_{k^*-1}^+) + \varepsilon, \bar{c}_{k^*}\right), \end{aligned}$$

which implies that t_{k^*} is not a switching time. This contradicts the supposition, and thus only finite switchings happen.

Subsequently, we prove that $t_f = \infty$ and state and control are bounded on $[0,\infty)$. Let $t_{\bar{k}}$ denote the last switching time. Then, we obtain from (21) that

$$V_{n,\bar{k}}^{\frac{|\tau|}{2}}(t) \le \chi\left(t - t_{\bar{k}}, V_{n,\bar{k}}^{\frac{|\tau|}{2}}(t_{\bar{k}}^+) + \varepsilon, \bar{c}_{\bar{k}+1}\right) < \infty,\;\; t \in (t_{\bar{k}}, t_f).$$

With this, noting the definition of $V_{n,k}(\cdot)$ and (19), we have signals $x(t)$ and $u(t)$ are bounded on $(t_{\bar{k}}, t_f)$, which implies that all signals of the closed-loop system are bounded on $[0, t_f)$. Thus, $t_f = \infty$.

Finally, let us prove that the closed-loop system is globally finite-time stable with arbitrarily prescribed settling-time. From (21), it follows that

$$V_{n,\bar{k}+1}^{\frac{|\tau|}{2}}(t) \;\le\; \chi\left(t - t_{\bar{k}}, V_{n,\bar{k}+1}^{\frac{|\tau|}{2}}(t_{\bar{k}}^+) + \varepsilon, \bar{c}_{\bar{k}+1}\right), \quad t > t_{\bar{k}},$$

which, together with the definition of $\chi(\cdot)$ in (12), implies $V_{n,\bar{k}+1}^{\frac{|\tau|}{2}}(t) \le 0$ for $t \ge t_{\bar{k}} + \frac{1}{\bar{c}_{\bar{k}+1}}\arctan\left(V^{\frac{|\tau|}{2}}(t_{\bar{k}}^+) + \varepsilon\right) =: \tilde{t}$. By this and the definition of $V_n(\cdot)$ and noting $V_{n,\bar{k}+1}(t) = V_n(x(t), m_{\bar{k}+1}, h_{[n],\bar{k}+1}) \ge 0$, we have $x(t) = 0$, $\forall t \ge \tilde{t}$, and thus the settling-time function satisfies $T_s(x_0) \le \tilde{t} < t_{\bar{k}} + \frac{\pi}{2\bar{c}_{\bar{k}+1}}$ by $\arctan(\cdot) \le \frac{\pi}{2}$. In addition, by (21) and the definition of $\chi(\cdot)$ in (12), we obtain $t_k - t_{k-1} < \frac{\pi}{2\bar{c}_k}$, $k = 1,2,\ldots,\bar{k}$. Therefore, from the definitions of $\bar{c}_k$ and δ and the fact $\sum_{k=1}^{\infty}\frac{k}{2^k} = 2$, there holds

$$T_s(x_0) < \sum_{k=1}^{\bar{k}+1}\frac{\pi}{2\bar{c}_k} < \frac{\pi\delta}{2^{\frac{(r_n-1)(2-\tau)}{2}-1}n^{\frac{\tau}{2}}|\tau|} \le T_p, \tag{24}$$

which shows that state $x(t)$ converges to zero before T_p.

This completes the proof. □

From the above proof, especially (24), one can see that a upper bound of settling-time function $T_s(x_0)$, instead of settling-time function itself, is arbitrarily prescribed. In this way, the settling-time function is achieved arbitrarily prescribed in the sense of arbitrary smallness. This substitute is because that for the settling-time function (unlike its upper bound in (24)), we have no access to its explicit expression on design parameters and hence it cannot be adjusted by the parameters.

Note that prescribed-time T_p, in theory, is arbitrary and its magnitude has no restriction. But in practice, we typically pursue an arbitrarily small prescribed settling-time to fulfill the demand of rapidness. Indeed, T_p in this paper can be arbitrarily small by choosing an enough small δ (see (24)), but which would render the control gain containing $\frac{2^k}{k\delta}$ sufficiently large.

No doubt arbitrarily prescribed settling-time simultaneously concerns rapidness (transient speed) and precision (steady-state behavior), and typically the former would confront high control cost (e.g., large control gain). If the rapidness is not pursued overly, we can achieve fixed-time stabilization for system (1) and get rid of the largeness of control gain by the following controller (derived from (19)):

$$u_k \;=\; u(x, c, h_{[n],k}),\;\; k \in \mathbf{Z}^+, \tag{25}$$

where $c > 0$ is a fixed constant, not a switching sequence as in (19); and $h_{[n],k}$ is the same as in (19), which is tuned by the switching logic above with $m_k \equiv c$.

Theorem 3. *Consider system (1) under Assumption 1. The switching adaptive controller derived from (25) can ensure that for any initial value x_0, signals $x(t)$ and $u(t)$ are bounded on $[0, \infty)$, and particularly, state $x(t)$ converges to zero before a fixed time (independent of the initial condition) and maintains zero after that.*

Proof. It suffices to prove the fixed-time convergence, as following the proof of Theorem 2, one can readily obtain the global boundedness of signals $x(t)$ and $u(t)$ on $[0, \infty)$.

Owing to the increasing property of $|h_{i,k}|$ with respect to k (see (20)), we have a finite integer $k' > 0$ which is the smallest one such that $|h_{1,k}| \geq \max\{\theta, |g_1|\}$, $|h_{i,k}| \geq |g_i|$, $i = 2, \ldots, n-1$ and $1 - |g_i h_{i,k}| \leq 0$, $i = 2, \ldots, n$ hold for any $k \geq k'$. Moreover, from the 2^n periodicity of $\mathbf{sign}(h_{[n],k}) = [\mathbf{sign}(h_{1,k}), \ldots, \mathbf{sign}(h_{n,k})]^{\mathrm{T}}$, we know that there exists an infinite subsequence $\{k_j | \, k_j \geq k'\}$ of $\{k\}$ such that $\mathbf{sign}(g_{[n]}) = [\mathbf{sign}(g_1), \ldots, \mathbf{sign}(g_n)]^{\mathrm{T}} = \mathbf{sign}(h_{[n],k_j})$. Choose $k^* = \min_{j \in \mathbf{Z}^+}\{k_j\}$ and recall the increasing properties of functions $F_i(x_{[i]}, h_{[i-1],k}, \theta, g_{[i-1]})$, $i = 1, \ldots, n$ in θ and $|g_{i-1}|$ (see Proposition 1). We have that (23) holds, regardless of the system states. Thus, k^* is independent of the initial condition, while depending on uncertainties θ and g_i, $i = 1, \ldots, n$.

From the proof of Theorem 2, it follows that the switching times are not larger than k^*. As such, letting $t_{\bar{k}}$ denote the last switching time, we have $\bar{k} \leq k^*$. Note that $\bar{c}_k$ in (21) is changed into $\bar{c}_k = 2^{\frac{(rn-1)(2-\tau)}{2}-1} n^{\frac{\tau}{2}} |\tau| c$. Then, similar to (24), we get

$$T_s(x_0) < \sum_{k=1}^{\bar{k}+1} \frac{\pi}{2\bar{c}_k} \leq \sum_{k=1}^{k^*+1} \frac{\pi}{2\bar{c}_k} = \frac{(k^*+1)\pi}{2^{\frac{(rn-1)(2-\tau)}{2}} n^{\frac{\tau}{2}} |\tau| c}.$$

This implies that the settling-time function $T_s(x_0)$ is bounded by a finite constant independent of the initial condition, and thus fixed-time convergence is achieved. □

Remark 1. The control schemes involved in Theorems 2 and 3 can also be applied to the more general systems by a minor modification. For example, the unknown control coefficients are allowed to be functions satisfying $a_i \leq |g_i(t, x_{[i]})| \leq b_i \bar{g}_i(x_{[i]})$, $i = 1, \ldots, n$ with unknown positive constants a_i and b_i and known smooth positive functions $\bar{g}_i(\cdot)$, $i = 1, \ldots, n$.

6. Simulation examples

In this section, we give two examples to illustrate the effectiveness of the control scheme, in which unknown control directions underline the benefits of the proposed strategy over the related results.

We first study the ship steering model described by

$$T_1 \dddot{\eta} + \dot{\eta} + \sigma \dot{\eta}^3 = K\rho, \tag{26}$$

where the relevant definitions can be found in [21,28]. Note by [21] that T_1 is positive when a ship is straight-line stable, otherwise it is negative. Moreover, all parameters vary with operation conditions, such as the speed of the ship and the water dept. We thus assume that T_1, σ and K are unknown, and particularly the sign of T_1 is also unknown.

The control objective is to find a controller such that the actual heading angle η is regulated to the desired angle, denoted by d, before 25 s and thereafter remains it. To this end, we introduce transformations $x_1 = \eta - d$, $x_2 = \dot{\eta}$ and $u = \rho$, and in turn obtain

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \frac{K}{T_1}u - \frac{1}{T_1}x_2 - \frac{\sigma}{T_1}x_2^3. \end{cases} \tag{27}$$

As such, the objective is transformed into that x_1 and x_2 converge to zero before 25 s and remain zero thereafter. Note that the settling-time function $T_s(x_0)$ satisfies (24). Then, to achieve the objective, design parameter δ needs to satisfy $\frac{\pi\delta}{2^{\frac{(rn-1)(2-\tau)}{2}-1} n^{\frac{\tau}{2}} |\tau|} \leq 25$, i.e., $\delta \leq 0.592$.

Obviously, system (27) satisfies Assumption 1 with $\tau = -\frac{2}{11}$, $\theta = \max\{\frac{1}{T_1}, \frac{\sigma}{T_1}\}$, $\bar{f}_1 = 0$ and $\bar{f}_2(\cdot) = (1+x_2^2)^{\frac{1}{9}} + x_2^{\frac{20}{9}}$. Following the control design in Section 5, we devise controller $u_k = -h_k\big(m_k + m_k(1+z_2^2)^{\frac{2}{11}} + F(\cdot)\big)z_2^{\frac{7}{11}}$ with $F(\cdot) = 0.68\big(0.36 m_k(1+x_1^2)^{\frac{2}{11}} + |h_k|\bar{f}_2(\cdot) + 1.2\phi_1^{\frac{11}{9}}(\cdot) + 0.18\phi_1^{\frac{20}{11}}(\cdot) + \bar{f}_2(\cdot)\phi_1^{\frac{7}{9}}(\cdot)\big)^{\frac{20}{13}} |h_k|^{\frac{40}{13}}$ and $\phi_1(\cdot) = m_k\big(1+x_1^2\big)^{\frac{2}{11}} + m_k + 1$, and choose $\varepsilon = 8$, $\delta = 0.59$, $m_k = \frac{2^k}{k\delta}$ and $|h_{1,k}| = 0.25k^2$.

Letting $K = \sigma = T_1 = 1$ and $x(0) = [-0.1, 1]^{\mathrm{T}}$, we get Fig. 1. Apparently, system states x_1 and x_2 converge to zero before 25 s, which shows the effectiveness of the proposed scheme. From Fig. 1, we see that the settling-time is about 2 s, which is far less than the prescribed time 25 s. This is because that as stated in Section 5, we actually prescribe the upper bound of the settling-time function, rather than the settling-time itself.

Next, we consider the following uncertain system:

$$\begin{cases} \dot{x}_1 = g_1 x_2 + \theta_1 x_1^{\frac{9}{11}}, \\ \dot{x}_2 = g_2 u^{\frac{9}{7}} + \theta_2 x_1, \end{cases} \tag{28}$$

where θ_1 and θ_2 are unknown positive constants, g_1 and g_2 are unknown nonzero constants with unknown signs. It can be seen that system (28) satisfies Assumption 1 with $\tau = -\frac{2}{11}$, $\bar{f}_1 = 1$, $\bar{f}_2(\cdot) = (1+x_1^2)^{\frac{2}{11}}$ and $\theta = \max\{|\theta_1|, |\theta_2|\}$.

The control objective is to find a switching adaptive controller such that the states x_1 and x_2 converge to zero before 10 s. From (24), we see that $T_s(x_0) < 10$ can be guaranteed by $\frac{\pi\delta}{2^{\frac{(rn-1)(2-\tau)}{2}-1} n^{\frac{\tau}{2}} |\tau|} \leq 10$, i.e., $\delta \leq 0.2368$.

By the control design in Section 5, we choose controller $u_k = -h_{2,k}^{\frac{11}{9}} z_2^{\frac{49}{99}}\Big(0.3 m_k z_2^{\frac{4}{11}} + 1.14|h_{1,k}|^{\frac{20}{9}} + 0.8|h_{1,k}|^{\frac{20}{9}} z_2^{\frac{80}{99}} + m_k + 1.31|h_{1,k}|F_1(\cdot)\Big)^{\frac{7}{9}}$ with $F_1(\cdot) = \frac{4}{9} m_k |h_{1,k}|^{\frac{11}{9}} F_2^{\frac{2}{9}}(\cdot) x_1^{\frac{4}{11}} + |h_{1,k} F_2(\cdot)|^{\frac{11}{9}}$ and $F_2(\cdot) = 1 + 2m_k + m_k(1+x_1^2)^{\frac{2}{11}} + |h_{1,k}|$, the switching sequences $m_k = \frac{2^k}{k\delta}$ and $|h_{i,k}| = 0.1k^2 + 0.1$ and parameters $\varepsilon = 8$ and $\delta = 0.23$.

Let $g_1 = 3$, $g_2 = -2$, $\theta_1 = \theta_2 = 0.5$ and $x(0) = [-0.2, 1]^{\mathrm{T}}$. Simulation results are shown in Fig. 2. We find from the figure that system states x_1 and x_2 indeed converge to zero before 10 s, which exhibits the effectiveness of the proposed controller. Moreover, the figure shows that the settling-time is about 1 s, which indicates that the estimate of its upper bound is valid but conservative.

7. Concluding remarks

In this paper, global finite-time stabilization with arbitrarily prescribed settling-time has been addressed for nonlinear systems with unknown control directions and other serious uncertainties. Through examples and counterexamples, several motivating facts have been offered to illustrate that if system uncertainties are without any known bounds, static (resp. dynamic) continuous feedback is impossible (resp. at least rather hard) to achieve the desired stabilization. In view of this, a switching adaptive controller has been constructed to guarantee that the system state converges to zero in an arbitrarily prescribed settling-time. Also, fixed-time stabilization has been achieved as a special case, aiming at the trade-off between arbitrary rapidness

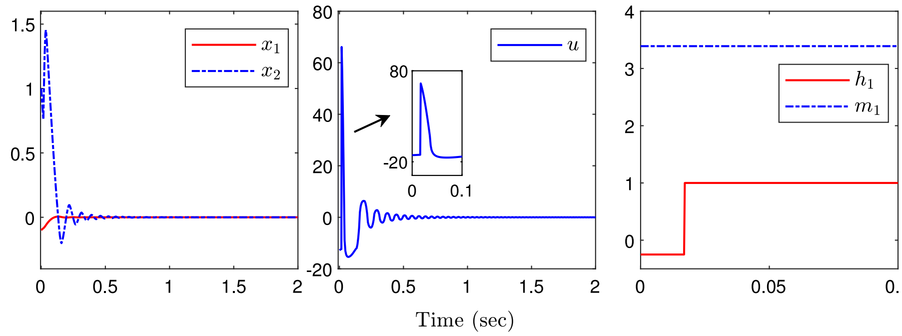

Fig. 1. The curves of states, input and parameters.

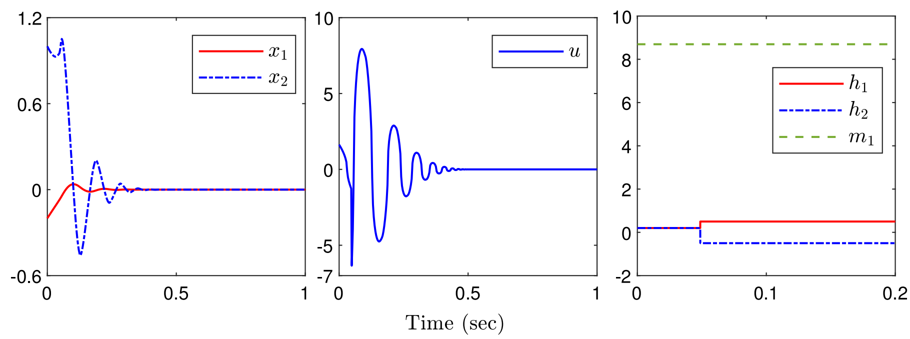

Fig. 2. The curves of states, input and parameters.

and high control cost. Whereas the expense of the proposed controllers might be high in some scenarios, especially when the system dimension is large, since the switching sequences involved, which are specified in advance, make the controllers be a worst case type. Therefore, it deserves an attempt whether or not online switching sequences are feasible, and (if so) the efficiency of the switching can be significantly improved to decrease the expense.

CRediT authorship contribution statement

Caiyun Liu: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Writing – original draft, Writing – review & editing. **Yungang Liu:** Conceptualization, Methodology, Writing – review & editing, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix. Several proofs

This section gives the proofs of Facts 2 and 4 and Proposition 1, as well as the rest proof of Fact 5.

A.1. Proof of Fact 2

Substituting (9) into (8) yields the closed-loop system:

$$\begin{cases}\dot{x} = -2k^2x^{\frac{1}{3}} - 2k^2x^3 + \theta|x|,\\ \dot{k} = k^2x^2.\end{cases} \tag{29}$$

We next show the following three properties of the closed-loop states k and x, and in turn complete the proof.

(i) *Boundedness of $k(t)$ on $[0,\infty)$.* Suppose by contradiction that $k(t)$ is unbounded on the interval of solution existence. This, together with $\dot{k}\ge 0$, implies that there exists a finite time $t_1>0$ such that $k^2(t)>\theta$ for $t\ge t_1$.

Let $V_1=\frac{x^2}{2}+k$. Its derivative along system (29) satisfies

$$\dot{V}_1 \le -2k^2x^{\frac{4}{3}} - 2k^2x^4 + \theta x^2 + k^2x^2. \tag{30}$$

From Lemma 3, it follows that $x^2\le x^{\frac{4}{3}}+x^4$. Then, by (30) and noting $k^2(t)>\theta\ \forall t\ge t_1$, we have $\dot{V}_1(t)\le 0\ \forall t\ge t_1$, and in turn $k(t)\le V_1(t)\le V_1(t_1)<\infty\ \forall t\ge t_1$. This contradicts the supposition, and thus $k(t)$ is bounded on $[0,\infty)$.

(ii) *Boundedness and convergence (to zero) of system state $x(t)$.* By $\dot{k}=k^2x^2$, we get $k(t)-k(0)=\int_0^t k^2(\tau)x^2(\tau)\mathrm{d}\tau$. This, together with the boundedness of $k(t)$, implies that $k(t)x(t)$ and in turn $x(t)$ are square integrable.

Consider $V_2=\frac{x^2}{2}$. Then its derivative along system (29) satisfies $\dot{V}_2=-2k^2x^{\frac{4}{3}}-2k^2x^4+\theta x|x|\le\theta x^2$, which means $V_2(t)\le V_2(0)+\int_0^t\theta x^2(\tau)\mathrm{d}\tau$ for $t\ge 0$. With this and the square integrability of $x(t)$, we directly have the boundedness of $V_2(t)$ and in turn $x(t)$ on $[0,\infty)$. The boundedness of x and k implies $\dot{x}(t)$ is bounded, and then $x(t)$ is uniformly continuous; thus, using Barbălat Lemma (see Lemma 8.2, page 323 of [31] or Lemma 4, page 52 of [35]) yields $\lim_{t\to\infty}x(t)=0$.

(iii) *Semi-global finite-time convergence with arbitrarily prescribed settling-time of system state $x(t)$.* It suffices to consider the case of $x_0>0$. Let us first see this. For the case of $x_0\le 0$, there would always hold $\theta x(t)|x(t)|\le 0$ and $\dot{V}_2\le -2k^2x^{\frac{4}{3}}-2k^2x^4\le -2k_0^2x^{\frac{4}{3}}-2k_0^2x^4$. From the latter and choosing suitable k_0 satisfying $2^{\frac{7}{2}}k_0^2\ge\frac{3\pi}{T_p}$, we can readily establish the wanted convergence using Theorem 1.

The rest proof is confined to $x_0>0$. Define

$$\Omega\triangleq\left\{x\in\mathbf{R}\,\middle|\,|x|\le h(x_0)\right\},\quad h(x_0)=\frac{x_0^{\frac{3}{2}}}{2^{\frac{3}{2}}\left(x_0^{\frac{1}{3}}+x_0^3\right)^{\frac{3}{2}}}. \tag{31}$$

Noting $x_0^{\frac{1}{2}} < \left(x_0^{\frac{1}{3}} + x_0^3\right)^{\frac{3}{2}}$, we have $h(x_0) < x_0$.

We then prove that there exists a time instant $t_1 \le \frac{1}{k_0 h^2(x_0)}$ such that $x(t_1) \in \Omega$. Suppose by contradiction that there is no such time instant t_1, i.e., $x(t) > h(x_0)$ for $t \le \frac{1}{k_0 h^2(x_0)} =: \tau_0$. Then, we get $\int_0^{\tau_0} x^2(t)\mathrm{d}t > \int_0^{\tau_0} h^2(x_0)\mathrm{d}t = \frac{1}{k_0}$, and in turn $1 - k_0\int_0^{\tau_0} x^2(t)\mathrm{d}t < 0$. From this and $\dot{k} = k^2x^2$, it follows that $0 < \frac{k_0}{k(\tau_0)} = 1 - k_0\int_0^{\tau_0} x^2(t)\mathrm{d}t < 0$, a contradiction. So there is such time instant t_1.

Owing to $x_0 > h(x_0) \ge x(t_1)$ and continuity of solution, there exists a time instant $t_2 \in (0, t_1]$ such that $x(t_2) = h(x_0)$, and a time instant $t_3 \in [0, t_2)$ such that $x(t_3) = x_0$ and $\dot{x}(t_3) \le 0$. Then, by (29), we have $-2k^2(t_3)\left(x_0^{\frac{1}{3}} + x_0^3\right) + \theta x_0 \le 0$, from which, together with $h(x_0)$ in (31) and $\dot{k} \ge 0$, it follows that $k^2(t_3) \ge \theta h^{\frac{2}{3}}(x_0)$ and $k^2(t) \ge k^2(t_3)$ for $t \ge t_3$.

Observe from $\dot{V}_2 = -k^2x^{\frac{4}{3}} - 2k^2x^4 - (k^2 - \theta x^{\frac{2}{3}})x^{\frac{4}{3}}$ and $k^2(t) \ge k^2(t_3) \ge \theta h^{\frac{2}{3}}(x_0)$ $\forall t \ge t_3$ that Ω is an invariant set and after t_2 (at which $x(t_2) = h(x_0)$ and $\dot{V}_2(t_2) < 0$), state $x(t)$ enters Ω and maintains there.

Furthermore, note that for $\forall t \ge t_2$

$$\dot{V}_2(t) \le -k^2x^{\frac{4}{3}}(t) - 2k^2x^4(t) \le -k_0^2 2^{\frac{2}{3}} V_2^{\frac{2}{3}}(t) - 8k_0^2 V^2(t).$$

Then by Theorem 1, we have $x(t) = 0$ for $t \ge T_s$, where $T_s \le t_2 + 2^{\frac{5}{4}} \cdot \frac{\pi}{3k_0^2} \le \frac{1}{k_0h^2(x_0)} + 2^{\frac{5}{4}} \cdot \frac{\pi}{3k_0^2}$. Thus, choosing an appropriate k_0 such that $\frac{1}{k_0h^2(x_0)} + 2^{\frac{5}{4}} \cdot \frac{\pi}{3k_0^2} \le T_p$, we readily establish the wanted convergence. This completes the proof. □

A.2. Proof of Fact 4

Suppose by contradiction that controller (10) can guarantee GFTS-APST for system (8). This means that from any initial value, system state $x(t)$ can be steered to the origin $x = 0$ in the prescribed time T_p, and hence to gain a contradiction, we merely find an initial value $x_0 > 0$, from which, state $x(t)$ does not have the convergence to zero.

Know that the function $\omega(k, T_p)$, in k, is continuous and monotonously increasing, and $k(0)$ allows for T_p. Then for given $T_p > 0$, $k(0) > 0$ and $M_1 \ge 1$, there exists $M_2 \ge 1$ such that

$$\frac{\omega(\bar{k}, T_p)}{\omega(k(0), T_p)} \le M_1, \quad \bar{k} = M_2 k(0). \tag{32}$$

Know from premise that k is zero-input globally bounded, and continuously depends on x, T_p and initial value $k(0)$. Hence, for the T_p, $k(0)$ and M_2, there exists $\delta_1 > 0$ such that if $|x(t)| < \delta_1$ on $[0, T_p]$, then for $i = 1, \ldots, m$,

$$\frac{k_i(t)}{k_i(0)} \le M_2, \quad t \in [0, T_p]. \tag{33}$$

Choose $0 < x_0 < \delta_1$ and $0 < \varepsilon < 1$ to guarantee $\frac{x_0^{\alpha-1} + x_0^{\beta-1}}{c} > M_1$, where $c = (\varepsilon x_0)^{\alpha-1} + (\varepsilon x_0)^{\beta-1}$. Then, from (32), it follows that for $0 < x_0 < \delta_1$,

$$\omega(\bar{k}, T_p) < \omega(k(0), T_p)\frac{x_0^{\alpha-1} + x_0^{\beta-1}}{c}.$$

This, together with the arbitrariness of unknown θ, implies the following relation ($0 < x_0 < \delta_1$):

$$\omega(\bar{k}, T_p) < \frac{\theta}{c} < \omega(k(0), T_p)\frac{x_0^{\alpha-1} + x_0^{\beta-1}}{c}. \tag{34}$$

With the above critical choices, we next show that the x_0 lying in $(0, \delta_1)$ is a wanted initial value which could lead to a contradiction.

Substituting (10) into (8) yields the closed-loop system:

$$\begin{cases} \dot{x} = -\omega(k, T_p)(\{x\}^\alpha + \{x\}^\beta) + \theta|x|, \\ \dot{k} = \psi(x, k, T_p), \quad k(0) > 0. \end{cases} \tag{35}$$

By (34), we have $\dot{x}(0^+) \le 0$, which, together with the increasing property of $k_i(t)$ and $\omega(k, T_p)$ with respect to k_i, implies that $0 \le x(t) \le x_0 < \delta_1$ for $t \in [0, T_p]$. Then, it follows from (33) that $k_i(t) \le \bar{k}_i$ for $t \in [0, T_p]$. Thus, by (34) and the increasing property of $\omega(k, T_p)$, we deduce

$$\omega(k(t), T_p) \le \omega(\bar{k}, T_p) < \frac{\theta}{c}, \quad \forall t \in [0, T_p].$$

This implies that the decaying rate of system state $x(t)$ on $[0, T_p]$ is slower than the following system:

$$\dot{z} = -\frac{\theta}{c}\left(\{z\}^\alpha + \{z\}^\beta - c|z|\right), \quad z(0) = x_0 > 0. \tag{36}$$

Noting $c = (\varepsilon x_0)^{\alpha-1} + (\varepsilon x_0)^{\beta-1}$, we have that $\varepsilon x_0 > 0$ is a equilibrium of system (36), and furthermore state $z(t)$ decays to εx_0 rather than 0.

Thus, for system (35) with $0 < x_0 < \delta_1$, the system state $x(t)$ cannot converge to zero in prescribed time T_p, which contradicts the supposition. This completes the proof. □

A.3. The rest proof of Fact 5

We first prove by contradiction that only finite switchings happen for controller (14) under the given switching logic. Suppose that there are infinite switchings. Then there exists a sufficiently large k^* such that $gh_{k^*} - 1 \ge 0$ and $|h_{k^*}| - \theta \ge 0$.

Substituting (14) into (13) makes the time derivative of $V(x) = \frac{x^2}{2}$ satisfy

$$\dot{V} \le -2^{\frac{4}{5}} m_k\left(V^{\frac{4}{5}} + V^{\frac{6}{5}}\right) + F(x, m_k, h_k, \theta, g), \tag{37}$$

where $F = -\left((gh_k - 1)(m_k + |h_k|) + (|h_k| - |\theta|)\right)\left(x^{\frac{8}{5}} + x^{\frac{12}{5}}\right)$.

Applying relations $gh_{k^*} - 1 \ge 0$ and $|h_{k^*}| - |\theta| \ge 0$ to (37) yields $\dot{V}(t) \le -2^{\frac{4}{5}} m_{k^*}\left(V^{\frac{4}{5}}(t) + V^{\frac{6}{5}}(t)\right)$ $\forall t \in (t_{k^*-1}, t_{k^*}]$. Solving this, and by the definition of $\chi(\cdot)$, we have $V^{\frac{1}{5}}(t) \le \chi\left(t - t_{k^*-1}, V^{\frac{1}{5}}(t_{k^*-1}^+), c_{k^*}\right) \le \chi\left(t - t_{k^*-1}, V^{\frac{1}{5}}(t_{k^*-1}^+) + \varepsilon, c_{k^*}\right)$ $\forall t \in (t_{k^*-1}, t_{k^*}]$, which implies that t_{k^*} is not the switching time and the switching will not happen after t_{k^*-1}. Thus, the finiteness of switchings is immediately showed.

We next prove the boundedness of the closed-loop system, and in turn establish GFTS-APST of the system. Assume that $t_{\bar{k}}$ is the last switching time. Then, by (15), we obtain $V^{\frac{1}{5}}(t) \le \chi\left(t - t_{\bar{k}}, V^{\frac{1}{5}}(t_{\bar{k}}^+) + \varepsilon, c_{\bar{k}+1}\right) < \infty$ $\forall t \in [t_{\bar{k}}, \infty)$, from which, the boundedness of $x(t)$ and $u(t)$ on $[0, +\infty)$ is proven.

From (15), it follows that $V^{\frac{1}{5}}(t) \le \chi\left(t - t_{\bar{k}}, V^{\frac{1}{5}}(t_{\bar{k}}^+) + \varepsilon, c_{\bar{k}+1}\right)$ for $t_{\bar{k}} < t \le t_{\bar{k}} + \frac{1}{c_{\bar{k}+1}}\arctan\left(V^{\frac{1}{5}}(t_{\bar{k}}^+) + \varepsilon\right) =: \bar{t}$. By the definition of $V(\cdot)$ and $\chi(\cdot)$, we get $x(t) = 0$ $t \ge \bar{t}$, which, together with $\arctan(\cdot) < \frac{\pi}{2}$, implies $T_s(x_0) \le \bar{t} < t_{\bar{k}} + \frac{\pi}{2c_{\bar{k}+1}}$. Moreover, it follows from (15) that $t_k - t_{k-1} < \frac{\pi}{2c_k}$, $k = 1, \ldots, \bar{k}$ and in turn $t_{\bar{k}} < \sum_{k=1}^{\bar{k}} \frac{\pi}{2c_k}$. Thus, by $\sum_{k=1}^{\infty} \frac{k}{2^k} = 2$, $c_k = 2^{\frac{4}{5}} \cdot \frac{m_k}{5}$ and $\delta \le 2^{\frac{4}{5}} \cdot \frac{T_p}{5\pi}$, we have

$$T_s(x_0) < \sum_{k=1}^{\bar{k}+1} \frac{\pi}{2c_k} = \sum_{k=1}^{\bar{k}+1} \frac{5k\pi\delta}{2^{\frac{9}{5}+k}} < \frac{5\pi\delta}{2^{\frac{4}{5}}} \le T_p.$$

This completes the proof. □

A.4. Proof of Proposition 1

The following recursive design process specifies design functions $\phi_i, i=1,\ldots,n$ in (16) and (17), and meanwhile verifies the correctness of (18).

For brevity of proof, we introduce notations ($i=1,\ldots,n$):

$$\begin{cases}\Gamma_i(\cdot)=\text{terms of the last three lines of (18) with } n=i,\\ \Xi_i(\cdot)=\sum_{j=1}^{i}\big((m+n-i)|z_j|^{2+\tau}+m|z_j|^{2-\tau}\big).\end{cases} \tag{38}$$

Step 1. We first construct continuously differentiable function $V_1=\frac{1}{2}z_1^2$. From Assumption 1, it follows that

$$\begin{aligned}\dot V_1 \le{}& z_1g_1\big(\{x_2\}^{p_1}-\{x_2^*\}^{p_1}\big)+z_1g_1\{x_2^*\}^{p_1}\\ &+\theta\bar f_1(\cdot)|z_1|^{2+\tau}.\end{aligned} \tag{39}$$

Choose the positive function

$$\phi_1(x_1,m,h_1)\ =\ |h_1|\bar f_1+2m+n-1+m\sqrt{1+z_1^2}.$$

Then (39) is changed into

$$\dot V_1\le-\Xi_1(\cdot)+\Gamma_1(\cdot)+z_1g_1\big(\{x_2\}^{p_1}-\{x_2^*\}^{p_1}\big),$$

where $F_1(\cdot)$ in $\Gamma_1(\cdot)$ satisfies $F_1(x_1,\theta)=\theta\bar f_1(x_1)$.

Recursive design step $i+1$ ($i=1,\ldots,n-1$). Suppose that steps $1,\ldots,i$ have been completed, namely, design functions $\phi_j(\cdot)$, $j=1,\ldots,i$ have been found and functions $F_j(\cdot)$ in $\Gamma_j(\cdot)$, $j=1,\ldots,i$ have been specified to guarantee that $V_i=\sum_{j=1}^{i}\int_{x_j^*}^{x_j}\big\{\{s\}^{\frac{1}{r_j}}-\{x_j^*\}^{\frac{1}{r_j}}\big\}^{2-r_j}ds$ satisfies

$$\dot V_i\le-\Xi_i(\cdot)+\Gamma_i(\cdot)+\{z_i\}^{2-r_i}g_i\big(\{x_{i+1}\}^{p_i}-\{x_{i+1}^*\}^{p_i}\big). \tag{40}$$

Next, we would like to specify function $\phi_{i+1}(\cdot)$ to guarantee that $V_{i+1}=V_i+\int_{x_{i+1}^*}^{x_{i+1}}\big\{\{s\}^{\frac{1}{r_{i+1}}}-\{x_{i+1}^*\}^{\frac{1}{r_{i+1}}}\big\}^{2-r_{i+1}}ds=:V_i+W_{i+1}$ satisfies (40). Particularly, when $i+1=n$, we let $x_{n+1}=u$.

From (1) and (40), it follows that

$$\begin{aligned}\dot V_{i+1}\le{}&-\Xi_i(\cdot)+\Gamma_i(\cdot)+\{z_i\}^{2-r_i}g_i\big(\{x_{i+1}\}^{p_i}-\{x_{i+1}^*\}^{p_i}\big)\\ &+\{z_{i+1}\}^{2-r_{i+1}}g_{i+1}\{x_{i+2}\}^{p_{i+1}}+\{z_{i+1}\}^{2-r_{i+1}}f_{i+1}\\ &+\sum_{j=1}^{i}\frac{\partial W_{i+1}}{\partial x_j}\dot x_j.\end{aligned} \tag{41}$$

We need to estimate the third term and the last two terms on the right-hand side of (41).

From Lemma 1 and (17), it follows that

$$\begin{aligned}&\{z_i\}^{2-r_i}g_i\big(\{x_{i+1}\}^{p_i}-\{x_{i+1}^*\}^{p_i}\big)\\ &\quad\le 2^{1-r_{i+1}p_i}|g_i|\cdot|z_i|^{2-r_i}|z_{i+1}|^{r_i+\tau}\\ &\quad\le\frac{1}{3}z_i^{2+\tau}+F_{i+1,1}(g_i)|z_{i+1}|^{2+\tau},\end{aligned} \tag{42}$$

where $F_{i+1,1}(\cdot)=\frac{r_{i+1}p_i}{2+\tau}\big(\frac{3(2-r_i)}{2+\tau}\big)^{\frac{2-r_i}{r_{i+1}p_i}}2^{\frac{(1-r_{i+1}p_i)(2+\tau)}{r_{i+1}p_i}}|g_i|^{\frac{2+\tau}{r_{i+1}p_i}}$.

By (17), Assumption 1 and Lemmas 1 and 2, we obtain

$$\begin{aligned}\{z_{i+1}\}^{2-r_{i+1}}f_{i+1}\le{}&\theta\bar f_{i+1}(x_{[i]})|z_{i+1}|^{2-r_{i+1}}\sum_{j=1}^{i+1}\big(|z_j|^{r_{i+1}+\tau}\\ &\cdot(|h_{j-1}|\phi_{j-1})^{\frac{r_{i+1}+\tau}{r_jp_{j-1}}}|z_{j-1}|^{r_{i+1}+\tau}\big)\\ \le{}&\frac{1}{3}|z_i|^{2+\tau}+\frac{1}{2}\sum_{j=1}^{i-1}|z_j|^{2+\tau}\\ &+F_{i+1,2}(x_{[i+1]},h_{[i]},\theta)|z_{i+1}|^{2+\tau},\end{aligned} \tag{43}$$

where function $F_{i+1,2}(\cdot)$ is nonnegative and continuous (furthermore continuously differentiable in the first argument), and strictly increasing in the last argument (on $\mathbf{R}^+$).

After some tedious calculations, we have

$$\begin{aligned}\sum_{j=1}^{i}\frac{\partial W_{i+1}}{\partial x_j}\dot x_j\le{}& F_{i+1,3}(x_{[i]},h_{[i]},\theta,g_{[i]})|z_{i+1}|^{2+\tau}\\ &+\frac{1}{3}|z_i|^{2+\tau}+\frac{1}{2}\sum_{j=1}^{i-1}|z_j|^{2+\tau},\end{aligned} \tag{44}$$

where function $F_{i+1,3}(\cdot)$ is nonnegative and continuous (furthermore continuously differentiable in the first argument), and strictly increasing in the third argument (on $\mathbf{R}^+$) and in the amplitude of each entry of the last argument.

Let us verify the correctness of (44). By (17), we obtain

$$\{x_{i+1}^*\}^{\frac{1}{r_{i+1}}}\ =\ -\sum_{l=1}^{i}\prod_{m=l}^{i}\{h_m\}^{\frac{1}{r_{m+1}p_m}}\phi_m^{\frac{1}{r_{m+1}p_m}}(\cdot)\{x_l\}^{\frac{1}{r_l}}.$$

From this, there holds

$$\begin{aligned}\frac{\partial\big(\{x_{i+1}^*\}^{\frac{1}{r_{i+1}}}\big)}{\partial x_j}={}&-\prod_{m=j}^{i}\{h_m\}^{\frac{1}{r_{m+1}p_m}}\phi_m^{\frac{1}{r_{m+1}p_m}}\frac{1}{r_j}|x_j|^{\frac{1}{r_j}-1}\\ &-\sum_{l=1}^{i}\frac{\partial\Big(\prod_{m=l}^{i}\{h_m\}^{\frac{1}{r_{m+1}p_m}}\phi_m^{\frac{1}{r_{m+1}p_m}}\Big)}{\partial x_j}\{x_l\}^{\frac{1}{r_l}}.\end{aligned} \tag{45}$$

Moreover, from Lemma 2 and (17), it follows that

$$\begin{cases}|x_j|^{\frac{1}{r_j}-1}\le|z_j|^{1-r_j}+|h_{j-1}|^{\frac{1-r_j}{r_jp_{j-1}}}\phi_{j-1}^{\frac{1-r_j}{r_jp_{j-1}}}|z_{j-1}|^{1-r_j},\\ |x_l|^{\frac{1}{r_l}}\le|z_l|+|h_{l-1}|^{\frac{1}{r_lp_{l-1}}}\phi_{l-1}^{\frac{1}{r_lp_{l-1}}}|z_{l-1}|.\end{cases} \tag{46}$$

Similar to the proof of (43), it can be deduced that

$$\begin{aligned}|\dot x_j|\le{}&|g_j|\cdot|z_{j+1}|^{r_{j+1}p_j}+\phi_j|g_jh_j|\cdot|z_j|^{r_{j+1}p_j}\\ &+\theta\bar f_j\sum_{l=1}^{j}\Big(|z_l|^{r_j+\tau}+(|h_{l-1}|\phi_{l-1})^{\frac{r_j+\tau}{r_lp_{l-1}}}|z_{l-1}|^{r_j+\tau}\Big).\end{aligned} \tag{47}$$

In addition, it follows from Lemma 2 that

$$\begin{aligned}&\int_{x_{i+1}^*}^{x_{i+1}}\Big|\{s\}^{\frac{1}{r_{i+1}}}-\{x_{i+1}^*\}^{\frac{1}{r_{i+1}}}\Big|^{1-r_{i+1}}ds\\ &\le|z_{i+1}|^{1-r_{i+1}}|x_{i+1}-x_{i+1}^*|\le2^{1-r_{i+1}}|z_{i+1}|.\end{aligned} \tag{48}$$

By (45)–(48) and Lemma 1, and noting $\frac{\partial W_{i+1}}{\partial x_j}=-(2-r_{i+1})\int_{x_{i+1}^*}^{x_{i+1}}|\{s\}^{\frac{1}{r_{i+1}}}-\{x_{i+1}^*\}^{\frac{1}{r_{i+1}}}|^{1-r_{i+1}}ds\frac{\partial\{x_{i+1}^*\}^{\frac{1}{r_{i+1}}}}{\partial x_j}$, we get (44).

Let $F_{i+1}(\cdot)$ in $\Gamma_{i+1}(\cdot)$ satisfy $F_{i+1}(\cdot)=F_{i+1,1}(\cdot)+F_{i+1,2}(\cdot)+F_{i+1,3}(\cdot)$. Substitute (42)–(44) into (41). Then, by (17) and (38), choosing

$$\phi_{i+1}(\cdot)=2m+n-i-1+m\sqrt{1+z_{i+1}^2}+F_{i+1}(\cdot), \tag{49}$$

we have

$$\begin{aligned}\dot V_{i+1}\le{}&-\Xi_{i+1}(\cdot)+\Gamma_{i+1}(\cdot)\\ &+\{z_{i+1}\}^{2-r_{i+1}}g_{i+1}\big(\{x_{i+2}\}^{p_{i+1}}-\{x_{i+2}^*\}^{p_{i+1}}\big),\end{aligned} \tag{50}$$

which implies that the recursive step is completed.

Note by (38) and $i=1,\ldots,n-1$ that (49) and (50) still hold when $i+1=n$. Then, by (17) and (49), we can specify controller (16), and furthermore deduce from $x_{n+1}=u=x_{n+1}^*$ and (50) that

$$\dot V_n\le-\Xi_n(\cdot)+\Gamma_n(\cdot). \tag{51}$$

By the definition of V_n and Lemma 2, we get

$$\begin{cases} V_n^{\frac{2+\tau}{2}} \le \left(2^{1-r_n}\sum_{i=1}^{n} z_i^2\right)^{\frac{2+\tau}{2}} \le 2^{\frac{(1-r_n)(2+\tau)}{2}} \sum_{i=1}^{n} |z_i|^{2+\tau}, \\ V_n^{\frac{2-\tau}{2}} \le \left(2^{1-r_n}\sum_{i=1}^{n} z_i^2\right)^{\frac{2-\tau}{2}} \le 2^{\frac{(1-r_n)(2-\tau)}{2}} n^{\frac{-\tau}{2}} \sum_{i=1}^{n} |z_i|^{2-\tau}, \end{cases}$$

which implies

$$\begin{aligned} & -\sum_{j=1}^{n}\left(|z_j|^{2+\tau}+|z_j|^{2-\tau}\right) \\ & \le -2^{\frac{(r_n-1)(2+\tau)}{2}} V_n^{\frac{2+\tau}{2}} - 2^{\frac{(r_n-1)(2-\tau)}{2}} n^{\frac{\tau}{2}} V_n^{\frac{2-\tau}{2}}. \end{aligned} \tag{52}$$

By the definitions of $\Xi_n(\cdot)$ and $\Gamma_n(\cdot)$ in (38), substituting (52) into (51) yields (18). This completes the proof. □

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