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Brief paper

Leader-following control of high-order multi-agent systems under directed graphs: Pre-specified finite time approach[☆]

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ABSTRACT

In this work we address the full state finite-time distributed consensus control problem for high-order multi-agent systems (MAS) under directed communication topology. Existing protocols for finite time consensus of MAS are normally based on the signum function or fractional power state feedback, and the finite convergence time is contingent upon the initial conditions and other design parameters. In this paper, by using regular local state feedback only, we present a distributed and smooth finite time control scheme to achieve leader–follower consensus under the communication topology containing a directed spanning tree. The proposed control consists of a finite time observer and a finite time compensator. The salient feature of the proposed method is that both the finite time intervals for observing leader states and for reaching consensus are independent of initial conditions and any other design parameters, thus can be explicitly pre-specified. Leader-following problem of MAS with both single and multiple leaders are studied.

1. Introduction

Background and motivation. Distributed consensus is known as one of the most important issues in cooperative control of MAS (Qu, 2009). The common important goal in this aspect is to achieve consensus with the least possible topological requirement and in a timely and distributive manner. In this regard, finite time distributed consensus has been an emerging popular topic in the control community in recent years as it offers numerous benefits including faster convergence rate, better disturbance rejection and robustness against uncertainties (Bhat and Bernstein, 2000) leading to fruitful results in literature, see Cao and Ren (2014), Chen, Lewis, and Xie (2011), Du, Wen, Yu, Li, and Chen (2015), Huang, Wen, Wang, and Song (2015), Hui, Haddad, and Bhat (2008), Li, Du, and Lin (2011), Li and Qu (2014), Liu, Lam, Yu, and Chen (2016), Wang and Xiao (2010), Wang, Song, and Krstic (2017), Wang, Song, Krstic, and Wen (2016a, b), Wang, Song, and Ren (2017) and Zuo (2015), to just name a few. However, among the existing results, there is no reported work on finite time consensus that is with smooth control action, further, there is no reported effort in achieving consensus with an explicitly pre-specified finite convergence time, not even for single integrator MAS, to the authors' best knowledge. As for non-networked systems, although prescribed regulation is achievable with the existing fractional approaches (Polyakov, Efimov, & Perruquetti, 2015; Polyakov & Fridman, 2014) by properly choosing the design parameters, it is nontrivial to make such choice as initial conditions and other constraints are involved, this is particularly true for high-order systems.

Note that in engineering, many systems are modeled by higher-order dynamics. For instance, a single link flexible joint manipulator is well modeled by a fourth-order nonlinear system (Khalil, 2002). In this paper we present a smooth control method for high-order MAS under directed communication constraint to achieve leader-following consensus within a finite time that is independent of initial conditions and any other design parameters, thus can be explicitly pre-specified. We achieve this by employing two different time-varying scaling functions to construct a finite-time observer and a finite time compensator, respectively, where the concept of time-varying scaling function, originated from the recent work on finite-time regulation of SISO system by Song, Wang, Holloway, and Krstic (2017), is used.

Contributions of the work. The novelty and contributions of the proposed solution can be summarized as follows: (1) A pre-specified finite time observer for each follower is constructed, and

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we have developed a lemma (Lemma 2) as a tool to analyze the stability and performance of the pre-specified finite time observer. Note that for leader–follower tracking (containment) control of MAS with high-order dynamics, observer is needed to estimate the states information of each order of the leader(s), and this has to be done within a pre-specified finite time, otherwise it would be uncertain when the control for leader-following tracking (containment) should be initiated. This is difficult to achieve with existing finite time control methods, for instance, Li et al. (2011) and Wang, Li, and Shi (2014), because the finite time T therein cannot be preset; (2) the proposed leader-following consensus protocol for the high-order MAS, consisting of a finite time observer and a finite time compensator, is able to achieve consensus within a finite time T that can be uniformity pre-specified, such T is fully independent of initial conditions and any other design parameters. Technical difficulty occurs in analyzing the boundedness of the high-order residue term θ, and we have introduced a lemma (Lemma 3) to deal with such situation; (3) the proposed finite-time control is built upon regular feedback of local system states, and the control action is continuous everywhere and smooth almost everywhere except for one single switching time instant; and (4) the proposed controller can be extended to solve the case with multiple dynamic leaders, where the containment problem involving multiple dynamic leaders arises.

Notation: $\otimes$ represents the Kronecker product; 0 is a vector/matrix with each entry being 0 with appropriate dimension; I_n represents the identity matrix of dimension n; $A > 0$ represents that A is positive definite; we call $A = [a_{ij}] \in R^{N\times N}$ a nonsingular M-matrix, if $a_{ij} < 0, i \neq j$, and all eigenvalues of A have positive real parts.

2. System dynamical model

We consider a multi-agent system consisting of N follower agent(s) and M leader agent(s) $(N, M \geq 1)$. Let $\mathcal{F} = \{1, 2, \ldots, N\}$ and $\mathcal{L} = \{N + 1, \ldots, N + M\}$ be the follower set and leader set, respectively. The dynamics of the ith ($i \in \mathcal{F}$) follower agent is of the form,

$$\begin{aligned}\dot{x}_{i,q} &= x_{i,q+1}, \quad q = 1, \ldots, n-1,\\ \dot{x}_{i,n} &= u_i,\end{aligned} \tag{1}$$

where $x_{i,q} \in R^m$ $(q = 1, \ldots, n)$ and $u_i \in R^m$ are the system state and control input, respectively. For convenience, we take $m = 1$ (the case of $m > 1$ can be established similarly). The dynamics of the ith ($i \in \mathcal{L}$) leader agent is,

$$\begin{aligned}\dot{x}_{i,q} &= x_{i,q+1}, \quad q = 1, \ldots, n-1,\\ \dot{x}_{i,n} &= 0.\end{aligned} \tag{2}$$

Suppose that the communication topology among the follower(s) and the leader(s) is described by a directed graph $\mathcal{G} = (\iota, \varepsilon)$, where $\iota = \{\iota_1, \ldots, \iota_{N+M}\}$ is the set of vertices representing $N + M$ agents and $\varepsilon \subseteq \iota \times \iota$ is the set of edges of the graph (Ren & Cao, 2010). The directed edge $\varepsilon_{ij} = (\iota_i, \iota_j)$ denotes that vertex ι_j can obtain information from ι_i. The set of in-neighbors of vertex ι_i is denoted by $\mathcal{N}_i = \{\iota_j \in \iota | (\iota_j, \iota_i) \in \varepsilon\}$. We denote by $\mathcal{A} = [a_{ij}] \in R^{(N+M)\times(N+M)}$ the weighted adjacency matrix of $\mathcal{G}$, where $\varepsilon_{ji} \in \varepsilon \Leftrightarrow a_{ij} > 0$, otherwise, $a_{ij} = 0$. In addition, $a_{ii} = 0$. $\mathcal{D} = \mathrm{diag}(\mathcal{D}_1, \ldots, \mathcal{D}_N) \in R^{(N+M)\times(N+M)}$, with $\mathcal{D}_i = \sum_{j\in\mathcal{N}_i} a_{ij}$ being the weighted in-degree of node i, denotes the in-degree matrix such that the Laplacian matrix is defined as $L = [l_{ij}] = \mathcal{D} - \mathcal{A}$.

3. Coordinated tracking control with one leader

In this section, we consider the pre-specified finite time coordinated tracking problem with 1 leader, i.e., $M = 1$. In such case, a leader is an agent without in-neighbors, a follower is an agent that has at least one in-neighbor, and the Laplacian L is represented as $L = \begin{bmatrix} L_1 & L_2 \\ 0_{1\times N} & 0 \end{bmatrix}$ with $L_1 \in R^{N\times N}$ and $L_2 \in R^{N\times 1}$.

Assumption 1. The topology $\mathcal{G}$ contains a directed spanning tree, where the leader acts as the root.

Lemma 1 (*Li, Wen, Duan, & Ren, 2015*). *Under Assumption 1, L_1 is a nonsingular M-matrix and is diagonally dominant, then there exists a matrix $Q = diag\{q_1, \ldots, q_N\} > 0$, in which $q_1, \ldots, q_N$ are determined by $[q_1, \ldots, q_N]^T = (L_1^T)^{-1} 1_N$, such that*

$$QL_1 + L_1^T Q > 0. \tag{3}$$

Definition 1. The pre-specified finite time full state tracking consensus of leader–follower MAS (1)–(2) with the pre-specified finite time T^* is said to be solved if, for any given initial state, it holds that

$$x_{i,q} \to x_{N+1,q} \quad \text{as} \quad t \to t_0 + T^* \tag{4}$$

$$x_{i,q} = x_{N+1,q} \quad \text{when} \quad t \geq t_0 + T^*, \tag{5}$$

for all $i \in \mathcal{F}$ and $q = 1, \ldots, n$.

In the following, we begin the controller design and stability analysis. We first introduce the local neighborhood error for the ith ($i \in \mathcal{F}$) follower as,

$$\epsilon_{i,q} = \sum_{j\in\mathcal{F}\cup\mathcal{L}} a_{ij}(x_{i,q} - x_{j,q}), \quad q = 1, \ldots, n. \tag{6}$$

We denote the state error between the ith ($i \in \mathcal{F}$) follower and the leader by

$$\delta_{i,q} = x_{i,q} - x_{N+1,q}, \quad q = 1, \ldots, n. \tag{7}$$

Denote by $\epsilon_i = [\epsilon_{i,1}, \ldots, \epsilon_{i,n}]^T \in R^n$, $x_i = [x_{i,1}, \ldots, x_{i,n}]^T \in R^n$, $x_{N+1} = [x_{N+1,1}, \ldots, x_{N+1,n}]^T \in R^n$, $\delta_i = x_i - x_{N+1}$, for $i \in \mathcal{F}$, and $E = [\epsilon_1^T, \ldots, \epsilon_N^T]^T \in R^{nN}$, $X = [x_1^T, \ldots, x_N^T]^T \in R^{nN}$, $X_{N+1} = 1_N \otimes x_{N+1} \in R^{nN}$, $\delta = X - X_{N+1}$, such that

$$E = [L_1 \otimes I_n](X - X_{N+1}) = [L_1 \otimes I_n]\delta. \tag{8}$$

To achieve finite-time control with uniformly pre-specified finite time, we need to introduce two time-varying scaling functions,

$$\varrho(t) = \frac{T_1^{1+h}}{(T_1 + t_l - t)^{1+h}}, \quad t \in [t_l, t_l + T_1), \tag{9}$$

$$\eta(t) = \begin{cases} 1, & t \in [t_0, t_1), \\ \dfrac{T^{n+h}}{(T + \tau_l - t)^{n+h}}, & t \in [\tau_l, \tau_l + T), \end{cases} \tag{10}$$

where $h > 1$ $(h \in Z_+)$ is a user chosen constant, $t_{l+1} = t_l + T_1$ $(l \in Z_+ \cup \{0\})$, and $\tau_{l+1} = \tau_l + T$ $(l \in Z_+)$ with $\tau_1 = t_0 + T_1 = t_1$. Here $T_1 > 0$ and $T > 0$ denote the pre-specified finite convergence time, respectively, both are designer-specified real number satisfying $T_1 \geq T_r$ and $T \geq T_r$, where T_r denotes the time period needed for signal processing/computing and information transmission/communication.

Properties of $\varrho(t)$: for $l \in Z_+ \cup \{0\}$ and $p > 0$, it holds that
(i) $\varrho(t)^{-p}$ is monotonically decreasing on $[t_l, t_l + T_1)$;
(ii) $\varrho(t_l)^{-p} = 1$ and $\lim_{t\to(t_l+T_1)^-} \varrho(t)^{-p} = 0$.

Properties of $\eta(t)$: for $l \in Z_+$ and $p > 0$, it holds that
(i) $\eta(t)^{-p}$ is monotonically decreasing on $[\tau_l, \tau_l + T)$;
(ii) $\eta(\tau_l)^{-p} = 1$ and $\lim_{t\to(\tau_l+T)^-} \eta(t)^{-p} = 0$.

Hereafter, we denote by $\bullet^{(q)}$ $(q = 0, \ldots, n)$ the qth derivative of $\bullet$ with $\bullet^{(0)} = \bullet$, and denote by $\bullet^k = \underbrace{\bullet \cdots \bullet}_{k}$ $(k \in N_+)$ the kth power of •. By using the scaling function $\eta(t)$ we perform the following transformation, for $i \in \mathcal{F}$, that

$$\xi_{i,1} = \eta(t)\epsilon_{i,1},$$
$$\xi_{i,q} = (\xi_{i,1})^{(q-1)}, \quad q = 2, \ldots, n, n+1, \tag{11}$$
$$r_{i,1} = \eta(t)\delta_{i,1},$$
$$r_{i,q} = (r_{i,1})^{(q-1)}, \quad q = 2, \ldots, n, n+1. \tag{12}$$

Here the case of $q = n + 1$ is also considered for the purpose of later stability analysis. Upon using the generalized Leibniz rule, we build the new variable $r_{i,q}$ ($i \in \mathcal{F}$), for $q = 1, \ldots, n, n+1$, as,

$$r_{i,q} = \sum_{j=0}^{q-1} C_{q-1}^{j}\eta^{(j)}\delta_{i,1}^{(q-1-j)} = \sum_{j=0}^{q-1} C_{q-1}^{j}\eta^{(j)}\delta_{i,q-j}, \tag{13}$$

with $C_q^j = \frac{q!}{j!(q-j)!}$ and $0! = 1$. Denote by $r = [r_1^T, \ldots, r_N^T]^T \in R^{nN}$ and $\xi = [\xi_1^T, \ldots, \xi_N^T]^T \in R^{nN}$, with $r_i = [r_{i,1}, \ldots, r_{i,n}]^T \in R^n$ and $\xi_i = [\xi_{i,1}, \ldots, \xi_{i,n}]^T \in R^n$ ($i \in \mathcal{F}$), respectively. Then it follows from (8), (11)–(12) that

$$\xi = [L_1 \otimes I_n]r. \tag{14}$$

By (11) and (13)–(14), we get the following scaled dynamics from (1)–(2) that

$$\dot{\xi} = (I_N \otimes A_0)\xi + (L_1 \otimes e_n) \cdot \eta(t) \cdot (u + \theta) \tag{15}$$

where $A_0 = \begin{bmatrix} 0_{n-1} & I_{n-1} \\ 0 & 0_{n-1}^T \end{bmatrix} \in R^n$ with $0_{n-1} = [0, \ldots, 0]^T \in R^{n-1}$, $e_n = [0_{n-1}^T, 1]^T \in R^n$, $u = [u_1, \ldots, u_N]^T \in R^N$, and $\theta = [\theta_1, \ldots, \theta_N]^T \in R^N$ with

$$\theta_i = \sum_{j=1}^{n} C_n^j \frac{\eta^{(j)}}{\eta}\delta_{i,n+1-j}, \quad i \in \mathcal{F}. \tag{16}$$

Since (A_0, e_n) is stabilizable, there exists a solution $P > 0$ to the following Riccati inequality,

$$PA_0 + A_0^T P - 2g_0 P e_n e_n^T P \le -g_0 I_n, \tag{17}$$

for any given $g_0 > 0$.

To establish the consensus protocol, we need to first construct a distributed observer for each follower to obtain the accurate estimate of the leader's qth ($q = 1, \ldots, n$) state in finite time. We denote by $\hat{x}_{i,q}$ ($q = 1, \ldots, n$) the estimate of the leader's qth state for the ith ($i \in \mathcal{F}$) follower, then we propose the distributed finite-time observer, upon using $\varrho(t)$ in (9), as

$$\dot{\hat{x}}_{i,q} = \frac{1}{\sum_{j\in\mathcal{F}\cup\mathcal{L}} a_{ij}} \sum_{j\in\mathcal{F}\cup\mathcal{L}} a_{ij}\dot{\hat{x}}_{j,q} - \left(c + \frac{\dot{\varrho}}{\varrho}\right) \times \frac{1}{\sum_{j\in\mathcal{F}\cup\mathcal{L}} a_{ij}} \sum_{j\in\mathcal{F}\cup\mathcal{L}} a_{ij}\left(\hat{x}_{i,q} - \hat{x}_{j,q}\right) \tag{18}$$

where $\hat{x}_{N+1,q} = x_{N+1,q}$, and $c > 0$ is a user-chosen constant (as for how to obtain the information $\hat{x}_{j,q}$, see the discussion in Du et al. (2015)). Note that $\sum_{j\in\mathcal{F}\cup\mathcal{L}} a_{ij} \ne 0$ under Assumption 1, thus (18) is well defined. Based on the estimate values, we propose the finite-time consensus protocol for systems (1)–(2) as,

$$u = -\frac{1}{\eta}[I_N \otimes (e_n^T P)]\xi - \hat{\theta}, \tag{19}$$

where $\hat{\theta} = [\hat{\theta}_1, \ldots, \hat{\theta}_N]^T \in R^N$ with

$$\hat{\theta}_i = \sum_{j=1}^{n} C_n^j \frac{\eta^{(j)}}{\eta}(x_{i,n+1-j} - \hat{x}_{i,n+1-j}), \quad i \in \mathcal{F}. \tag{20}$$

Before moving on, we need the following lemmas (see Appendix for the proof).

Lemma 2. *Consider a scalar function $y(t) \ge 0$, if there exists some positive constant $\gamma > 0$ such that*

$$\dot{y}(t) = -\left(\gamma + 2\frac{\dot{\varrho}}{\varrho}\right)y(t), \quad t \in [t_l, t_l + T_1), \tag{21}$$

for $l \in Z_+ \cup \{0\}$, where ϱ is defined as in (9), then it holds that,
(i) $y(t) \to 0$ *as* $t \to (t_0 + T_1)^-$; (ii) $y(t) \equiv 0$ *when* $t \ge t_0 + T_1$.

Lemma 3. *For θ and its qth derivative $\theta^{(q)}$ ($q = 1, 2, \ldots$), there exist bounded matrices H, Γ, and Ξ_q ($q = 0, 1, 2, \ldots$) properly defined, such that*

$$\theta = \eta^{-(\frac{h}{n+h})}(I_N \otimes H)r = \Xi_0 r, \tag{22}$$
$$\theta^{(1)} = \eta^{-(\frac{h-1}{n+h})}(I_N \otimes \Gamma)r + \eta^{-(\frac{h}{n+h})} \times [I_N \otimes (HA_0) - L_1 \otimes (He_n e_n^T P)]r = \Xi_1 r, \tag{23}$$
$$\theta^{(q)} = \Xi_q r, \quad q = 2, 3, \ldots \tag{24}$$

Theorem 1. *Consider the MAS (1)–(2) with the control scheme (19)–(20). The full state leader–follower consensus is achieved within the pre-specified finite time $T^* = T_1 + T$. In particular,*

$$\|\delta(t)\| \le \eta^{-(\frac{1+h}{n+h})} Nn\|B\|\|L_1^{-1}\|\sqrt{\bar{\lambda}/\underline{\lambda}}\exp^{-\frac{g_0(t-t_0-T_1)}{2\lambda_{\max}(P)}}\|E(t_0 + T_1)\|, \tag{25}$$

for $t \in [t_0 + T_1, t_0 + T_1 + T)$, which depicts the overall convergence performance of $\delta(t)$, with $B = [b_{qk}]_{n\times n}$ being a lower triangular matrix, whose elements are,

$$b_{qk} = \eta^{-(\frac{n-1-q+k}{n+h})} C_{q-1}^{q-k} \frac{(n+h)!(-1)^{q-k}}{T^{q-k}(n+h-q+k)!}, \quad 1 \le k \le q \le n, \tag{26}$$

$\bar{\lambda} = \lambda_{\max}(Q) \cdot \lambda_{\max}(P)$, $\underline{\lambda} = \lambda_{\min}(Q) \cdot \lambda_{\min}(P)$, $g_0 = g\lambda_{\min}(Q^{-1}) > 0$ with $g = \lambda_{\min}(QL_1 + L_1^T Q)$. In addition, the consensus is maintained over $[t_0 + T_1 + T, \infty)$ in which u remains zero. Further, the resultant control input is continuous and uniformly bounded everywhere and smooth almost everywhere except for the switching time $t = t_0 + T_1$.

Proof. The proof consists of three steps.

***Step* 1.** $t \in [t_0, t_0 + T_1)$

We first prove that $\hat{x}_{i,q} \equiv x_{i,q}$ ($i \in \mathcal{F}$) for $q = 1, \ldots, n$ when $t \ge t_0 + T_1$. Let $\hat{e}_{i,q} = \sum_{j\in\mathcal{F}\cup\mathcal{L}} a_{ij}\left(\hat{x}_{i,q} - \hat{x}_{j,q}\right)$, by which (18) becomes

$$\dot{\hat{e}}_{i,q} = -\left(c + \frac{\dot{\varrho}}{\varrho}\right)\hat{e}_{i,q}. \tag{27}$$

Let $y_{i,q} = (\hat{e}_{i,q})^2$ ($i \in \mathcal{F}$, $q = 1, \ldots, n$), from (27) it follows

$$\dot{y}_{i,q} = 2\hat{e}_{i,q}\dot{\hat{e}}_{i,q} = -\left(2c + 2\frac{\dot{\varrho}}{\varrho}\right)y_{i,q}, \tag{28}$$

which yields $y_{i,q} \to 0$ as $t \to (t_0 + T_1)^-$ and $y_{i,q} \equiv 0$ when $t \ge t_0 + T_1$ for all $i \in \mathcal{F}$ and $q = 1, \ldots, n$ by Lemma 2. This implies that $\hat{e}_{i,q} \equiv 0$ when $t \ge t_0 + T_1$. Note that $\hat{e}_{i,q} = \sum_{j\in\mathcal{F}\cup\mathcal{L}} a_{ij}\left[(\hat{x}_{i,q} - x_{N+1,q}) - (\hat{x}_{j,q} - x_{N+1,q})\right]$. Then we have $[\hat{e}_{1,q}, \ldots, \hat{e}_{N,q}]^T = L_1([\hat{x}_{1,q}, \ldots, \hat{x}_{N,q}]^T - 1_N x_{N+1,q}) \equiv 0$ when $t \ge t_0 + T_1$ for all $q = 1, \ldots, n$. Thus $\hat{x}_{i,q} \equiv x_{N+1,q}$ ($i \in \mathcal{F}$, $q = 1, \ldots, n$) when $t \ge t_0 + T_1$ under Assumption 1.

In the sequel, we prove that the error δ is bounded by u proposed in (19). Note that for $t \in [t_0, t_0 + T_1)$, $\eta(t) = 1$ and $\eta(t)^{(j)} = 0$ for $j = 1, \ldots, n$, then $\theta = \hat{\theta} = 0$, (15) can be expressed as

$$\dot{\xi} = (I_N \otimes A_0)\xi + (L_1 \otimes e_n)u, \quad t \in [t_0, t_0 + T_1) \tag{29}$$

and the control law in (19) reduces to

$$u = -[I_N \otimes (e_n^T P)]\xi, \quad t \in [t_0, t_0 + T_1). \tag{30}$$

We then insert (30) into (29) to get

$$\dot{\xi} = (I_N \otimes A_0)\xi - [L_1 \otimes (e_n e_n^T P)]\xi. \tag{31}$$

Choose the Lyapunov function on $[t_0, t_0 + T_1)$ as

$$V = \xi^T (Q \otimes P)\xi. \tag{32}$$

The time derivative of V along (31) is

$$\begin{aligned}\dot{V} &= 2\xi^T(Q \otimes P)\big[(I_N \otimes A_0)\xi - \big(L_1 \otimes (e_n e_n^T P)\big)\xi\big] \\ &= 2\xi^T\big[Q \otimes (PA_0)\big]\xi - 2\xi^T\big[(QL_1) \otimes (Pe_n e_n^T P)\big]\xi \\ &= \xi^T\big[Q \otimes (PA_0 + A_0^T P)\big]\xi \\ &\quad - \xi^T\big[(QL_1 + L_1^T Q) \otimes (Pe_n e_n^T P)\big]\xi \\ &\le \xi^T\big[Q \otimes (PA_0 + A_0^T P)\big]\xi - g\xi^T\big[I_N \otimes (Pe_n e_n^T P)\big]\xi,\end{aligned} \tag{33}$$

where $g = \lambda_{\min}(QL_1 + L_1^T Q)$, which is positive according to Lemma 1. Let $\tilde{\xi} = (\sqrt{Q} \otimes I_n)\xi$. Then (33) becomes

$$\begin{aligned}\dot{V} &\le \tilde{\xi}^T\big[I_N \otimes (PA_0 + A_0^T P)\big]\tilde{\xi} - g\tilde{\xi}^T\big[Q^{-1} \otimes (Pe_n e_n^T P)\big]\tilde{\xi} \\ &\le \tilde{\xi}^T\big[I_N \otimes (PA_0 + A_0^T P - g_0 Pe_n e_n^T P)\big]\tilde{\xi},\end{aligned} \tag{34}$$

where $g_0 = g\lambda_{\min}(Q^{-1}) > 0$ due to $Q^{-1} > 0$. Recalling (17), we then arrive at

$$\dot{V} \le -g_0\tilde{\xi}^T(I_N \otimes I_n)\tilde{\xi} = -g_0\xi^T(Q \otimes I_n)\xi \le -\frac{g_0}{\lambda_{\max}(P)}V. \tag{35}$$

Further, we solve the differential inequality (35) to obtain

$$V(t) \le \exp^{-\frac{g_0}{\lambda_{\max}(P)}(t-t_0)}V(t_0), \tag{36}$$

which yields

$$\|\xi(t)\| \le \sqrt{\bar{\lambda}/\underline{\lambda}}\exp^{-\frac{g_0}{2\lambda_{\max}(P)}(t-t_0)}\|\xi(t_0)\|. \tag{37}$$

Note that $\xi(t) = E(t)$ on $[t_0, t_0 + T_1)$, from (8) and (37) we get

$$\begin{aligned}\|\delta(t)\| &= \|(L_1^{-1} \otimes I_n)\, E(t)\| \\ &\le \sqrt{\bar{\lambda}/\underline{\lambda}}\exp^{-\frac{g_0}{2\lambda_{\max}(P)}(t-t_0)} n\|L_1^{-1}\|\|E(t_0)\|,\end{aligned} \tag{38}$$

which guarantees the boundedness of δ on $[t_0, t_0 + T_1)$.

We now prove that the control input u is uniformly bounded on $[t_0, t_0 + T_1)$. By examining u given in (30), we then arrive at

$$\|u\| = \left\|[I_N \otimes (e_n^T P)]\xi\right\| \le N\left\|e_n^T P\right\|\|\xi\| \in L_\infty, \tag{39}$$

on $[t_0, t_0 + T_1)$ following from (37), which ensures the uniform boundedness of u on $[t_0, t_0 + T_1)$. Note that from (31) it is straightforward to show, by differentiating $\dot{\xi}$ step by step, that, there exist some bounded matrix Σ_q $(q = 1, 2, \ldots)$ such that

$$\xi^{(q)} = \Sigma_q\xi, \quad q = 1, 2, \ldots, \tag{40}$$

which, together with (30), yields

$$u^{(q)} = \big[I_N \otimes (e_n^T P)\big]\xi^{(q)} = \big[I_N \otimes (e_n^T P)\big]\Sigma_q\xi, \quad q = 1, 2, \ldots \tag{41}$$

From (30) it is clear that u is smooth up to ∞ order (C^∞) w.r.t. the state on $[t_0, t_0 + T_1)$ because u is composed of the regular state. From (30) and (41), it is clear that u is smooth up to ∞ order (C^∞) w.r.t. t on $[t_0, t_0 + T_1)$ because all the derivatives of u w.r.t. t exist and are continuous.

Step 2. $t \in [t_0 + T_1, t_0 + T_1 + T)$

We prove that the leader–follower consensus is achieved in the finite time $T^* = T_1 + T$. Further, we claim that the control input u is smooth and uniformly bounded on $[t_0 + T_1, t_0 + T_1 + T)$.

From the proof in step 1 we see that $\hat{x}_{i,q} \equiv x_{N+1,q}$ when $t \ge t_0 + T_1$, which means that $\hat{\theta} \equiv \theta$ when $t \ge t_0 + T_1$. Then we insert the control law (19) into (15) to obtain

$$\dot{\xi} = (I_N \otimes A_0)\xi - [L_1 \otimes (e_n e_n^T P)]\xi, \tag{42}$$

which, consequently, equals (31). Thus we follow the same procedure as in the proof of (33)–(35) by taking the same Lyapunov function as in (32) on $[t_0 + T_1, t_0 + T_1 + T)$, and arrive at

$$\dot{V} \le -\frac{g_0}{\lambda_{\max}(P)}V, \quad t \in [t_0 + T_1, t_0 + T_1 + T) \tag{43}$$

which ensures the exponential stability of ξ on $[t_0 + T_1, t_0 + T_1 + T)$. Similarly, we solve the differential inequality (43) to get

$$V(t) \le \exp^{-\frac{g_0}{\lambda_{\max}(P)}(t-t_0-T_1)}V(t_0 + T_1). \tag{44}$$

By (14), it then follows from (44) that

$$\|r(t)\| \le n\|L_1^{-1}\|\sqrt{\bar{\lambda}/\underline{\lambda}}\exp^{-\frac{g_0(t-t_0-T_1)}{2\lambda_{\max}(P)}}\|\xi(t_0 + T_1)\|, \tag{45}$$

ensuring the boundedness of both ξ and r on $[t_0 + T_1, t_0 + T_1 + T)$.

In the following, we establish the relation from $r \to \delta$ to obtain the full state leader–follower consensus result. Note that $\delta_{i,1} = \eta^{-1}r_{i,1}$, $\delta_{i,q} = \delta_{i,1}^{(q-1)}$ $(q = 1, \ldots, n)$ from (1) and (12)–(2). In addition, for $j = 0, \ldots, n, \ldots, n + h$, it holds that,

$$(\eta^{-1})^{(j)} = \eta^{-(\frac{1+h}{n+h})}\frac{(n+h)!(-1)^j}{T^j(n+h-j)!}\eta^{-(\frac{n-1-j}{n+h})}, \tag{46}$$

and $(\eta^{-1})^{(n+h+l)} = 0$ $(l = 1, 2, \ldots)$. By again using the generalized Leibniz rule, we have, for $q = 1, \ldots, n$, that

$$\begin{aligned}\delta_{i,q} &= \sum_{j=0}^{q-1} C_{q-1}^j(\eta^{-1})^{(j)}r_{i,1}^{(q-1-j)} = \sum_{j=0}^{q-1} C_{q-1}^j(\eta^{-1})^{(j)}r_{i,q-j} \\ &= \eta^{-(\frac{1+h}{n+h})}\sum_{j=0}^{q-1} C_{q-1}^j\frac{(n+h)!(-1)^j}{T^j(n+h-j)!}\eta^{-(\frac{n-1-j}{n+h})}r_{i,q-j}.\end{aligned} \tag{47}$$

By substituting $k = q - j$ with $j = 0, \ldots, q - 1$, we arrive at

$$\begin{aligned}\delta_{i,q} &= \eta^{-(\frac{1+h}{n+h})}\sum_{k=1}^{q} C_{q-1}^{q-k}\frac{(n+h)!(-1)^{q-k}}{T^{q-k}(n+h-q+k)!} \\ &\quad \times \eta^{-(\frac{n-1-q+k}{n+h})}r_{i,k}.\end{aligned} \tag{48}$$

It then follows from (48), by inspection, that

$$\delta = \eta^{-(\frac{1+h}{n+h})}(I_N \otimes B)r, \tag{49}$$

where $B = [b_{qk}]_{n\times n}$ with its elements given in (26). Note that $\|I_N \otimes B\|$ is a continuous function of η^{-p} $(p > 0)$, which is bounded by $\eta^{-p} \in (0, 1]$, thus $\|I_N \otimes B\|$ is finite. Recalling (45), (49), and $\xi(t_0 + T_1) = E(t_0 + T_1)$, we then arrive at (25). By the properties of η, where $\eta^{-(\frac{1+h}{n+h})}$ is monotonically decreasing and converges to zero as $t \to t_0 + T_1 + T$, we see that the overall convergence performance of $\delta(t)$ in terms of precision and convergence velocity is clearly depicted by (25). In addition, we see from (25) that

$$\delta \to 0 \ \text{ as } \ t \to (t_0 + T_1 + T)^-, \tag{50}$$

which means that the full state leader–follower consensus of (1)–(2) is achieved within the finite time $T^* = T_1 + T$.

We still need to prove that the control input u is smooth and uniformly bounded on $[t_0 + T_1, t_0 + T_1 + T)$. By examining each term in u on $[t_0 + T_1, t_0 + T_1 + T)$, it is clear, by (14), (45), and the properties of η, that

$$\left\|\frac{1}{\eta}[I_N \otimes (e_n^T P)]\xi\right\| \le \frac{1}{\eta}\|[L_1 \otimes (e_n^T P)]\|\|r\| \in L_\infty, \tag{51}$$

and by (22), (45), and the properties of η, that

$$\begin{aligned}\|\hat{\theta}\| &= \|\theta\| = \left\|\eta^{-(\frac{h}{n+h})}(I_N \otimes H)r\right\| \\ &\le \eta^{-(\frac{h}{n+h})}N\|H\|\|r\| \in L_\infty.\end{aligned} \tag{52}$$

Both of (51) and (52) imply that

$$\|u\| \le \left\| \frac{1}{\eta}[I_N \otimes (e_n^T P)]\xi \right\| + \|\hat{\theta}\| \in L_\infty \tag{53}$$

on $[t_0 + T_1, t_0 + T_1 + T)$. By examining $u^{(q)}$ $(q = 1, 2, \ldots)$ on $[t_0 + T_1, t_0 + T_1 + T)$, we have

$$u^{(q)} = -\underbrace{\eta^{-1}[I_N \otimes (e_n^T P)]\xi^{(q)}}_{u_1^{(q)}} - \underbrace{(\eta^{-1})^{(q)}[I_N \otimes (e_n^T P)]\xi}_{u_2^{(q)}} - \underbrace{\theta^{(q)}}_{u_3^{(q)}}. \tag{54}$$

From (40), (14), (45) and (24), we see that

$$u_1^{(q)} = \eta^{-1}\big[I_N \otimes (e_n^T P)\big]\Sigma_q(L_1 \otimes I_n)r, \tag{55}$$

$$u_2^{(q)} = (\eta^{-1})^{(q)} L_1 \otimes (e_n^T P)r, \tag{56}$$

$$u_3^{(q)} = \Xi_q r. \tag{57}$$

By the analysis similar to that in Step 1, we then conclude that u is smooth to order ∞ (C^∞) w.r.t. both the state and t on $[t_0 + T_1, t_0 + T_1 + T)$.

***Step* 3.** $t \in [t_0 + T_1 + T, \infty)$

We prove that the consensus is maintained over $[t_0 + T_1 + T, \infty)$ and the control input u remains zero for $t \in [t_0 + T_1 + T, \infty)$.

Firstly, we choose the same Lyapunov function as in (32) on $[t_0 + T_1 + T, t_0 + T_1 + 2T)$. By noting that

$$\delta(\tau_2) = \delta(t_0 + T_1 + T) = \lim_{t \to (t_0 + T_1 + T)^-} \delta(t) = 0, \tag{58}$$

and $r(\tau_2) = \delta(\tau_2) = 0$, we then obtain

$$V(\tau_2) = \xi^T (Q \otimes P)\xi \Big|_{\tau_2} = r^T\big[(L_1^T Q L_1) \otimes P\big] r\Big|_{\tau_2} = 0. \tag{59}$$

On the other hand, by following the same procedure as in (42)–(43) with the control law (19), we readily obtain $\dot{V}(t) \le 0$, which, together with (59), implies

$$0 \le V(t) \le V(t_0 + T_1 + T) = V(\tau_2) = 0 \tag{60}$$

for $t \in [\tau_2, \tau_2 + T)$, with $\tau_2 = t_0 + T_1 + T$. That is, $V(t) \equiv 0$ on $[\tau_2, \tau_2 + T)$. Thus $\xi(t) \equiv 0$ and then $r \equiv 0$ and $\delta \equiv 0$ on $[\tau_2, \tau_2 + T)$. From the definition of u in (19) for $t \in [\tau_2, \tau_2 + T)$ and (22), we deduce that $u \equiv 0$ on $[\tau_2, \tau_2 + T)$. Similarly, we can obtain that $\delta \equiv 0$ and $u \equiv 0$ on $[\tau_l, \tau_l + T)$ for $l = 3, 4, \ldots$. Thus $\delta(t) \equiv 0$ and $u \equiv 0$ for all $t \ge \tau_2 = t_0 + T_1 + T$, which means that the consensus is maintained over $[t_0 + T_1 + T, \infty)$, within which the control input u remains zero with the control law (19).

From the above analysis, it is clear that u is smooth (C^∞) except at $t_0 + T_1$ and $t_0 + T_1 + T$. So we just need to examine the continuity of $u^{(1)}$ at $t_0 + T_1$ and $t_0 + T_1 + T$, respectively. From (30) and (31), we have

$$u^{(1)} = \big(I_N \otimes (e_n^T P A_0)\big)\xi - \big(L_1 \otimes (e_n^T P e_n e_n^T P)\big)\xi, \tag{61}$$

on $[t_0, t_0 + T_1)$, and from (42), (14) and (23), we see that

$$\begin{aligned} u_1^{(1)} &= -\eta^{-1}\big[I_N \otimes (e_n^T P)\big] \times \big[(I_N \otimes A_0)\xi - \big(L_1 \otimes (e_n e_n^T P)\big)\xi\big] \\ &= \eta^{-1}\big[L_1 \otimes (e_n^T P A_0)\big]r - \big[L_1^2 \otimes (e_n^T P e_n e_n^T P)\big]r \end{aligned} \tag{62}$$

$$u_2^{(1)} = \frac{n+h}{T}\eta^{-(1-\frac{1}{n+h})} L_1 \otimes (e_n^T P)r, \tag{63}$$

$$u_3^{(1)} = \eta^{-(\frac{h-1}{n+h})}\big(I_N \otimes \Gamma\big)r + \eta^{-(\frac{h}{n+h})}\big[I_N \otimes (HA_0) - L_1 \otimes (He_n e_n^T P)\big]r, \tag{64}$$

on $[t_0 + T_1, t_0 + T_1 + T)$. Then by (61)–(64) and the properties of η, we arrive at,

$$\lim_{t \to (t_0 + T_1)^-} u^{(1)} = \lim_{t \to (t_0 + T_1)^+} u_1^{(1)} \ne \lim_{t \to (t_0 + T_1)^+} u^{(1)}, \tag{65}$$

implying that u is not smooth (C^1) w.r.t. t at $t = t_0 + T_1$ and also that u is not smooth w.r.t. the state, the later can be proved by contradiction: if u is smooth w.r.t. the state, then it is smooth w.r.t. t because the state is smooth w.r.t. t. This is a contradiction. Nevertheless, it is worth mentioning that u is continuous everywhere w.r.t. the state and t, including at $t = t_0 + T_1$ due to the continuity of $\eta(t)$ at $t = t_0 + T_1$. Now let us examine the continuity and smoothness of u at $t = t_0 + T_1 + T$. From (62)–(64), we have, $\lim_{t \to (t_0 + T_1 + T)^-} u^{(1)} = 0 = \lim_{t \to (t_0 + T_1 + T)^+} u^{(1)}$, which means that $u^{(1)}$ is continuous w.r.t. t at $t = t_0 + T_1 + T$ and thus $u(t)$ is smooth to order 1 w.r.t. t at $t = t_0 + T_1 + T$. On the other hand, u is at least continuous w.r.t. the state at $t = t_0 + T_1 + T$. Therefore, it is concluded that u is: 1) continuous everywhere; 2) smooth w.r.t. t almost everywhere except for $t = t_0 + T_1$ and 3) smooth w.r.t. the state almost everywhere except for $t = t_0 + T_1$ and $t = t_0 + T_1 + T$. ■

4. Containment control with multiple leaders

We further consider the case with multiple leaders. The finite time containment control problem arises in the presence of multiple leaders, which refers to the process that the followers coordinate with their neighbors to enter the convex hull formed by the states of leaders in finite time (Meng, Ren, & You, 2010).

Suppose that there are M ($M > 1$) leaders and N followers in the directed graph $\mathcal{G}$. In such case the Laplacian L is represented as $\begin{bmatrix} L_1 & L_2 \\ 0_{M\times N} & 0_{M\times M} \end{bmatrix}$, with $L_1 \in R^{N\times N}$ and $L_2 \in R^{N\times M}$.

Assumption 2. Suppose that for each follower, there exists at least one leader that has a directed path to it.

Lemma 4. *There exists a positive diagonal matrix $Q_{\mathcal{F}} = diag\{\tilde{q}_1, \ldots, \tilde{q}_N\} \in R^{N\times N}$ such that, $Q_{\mathcal{F}} L_1 + L_1^T Q_{\mathcal{F}} > 0$, in which $\tilde{q}_1, \ldots, \tilde{q}_N$ are decided by $[\tilde{q}_1, \ldots, \tilde{q}_N]^T = (L_1^T)^{-1} 1_N$. Moreover, each entry of $-L_1 L_2$ is nonnegative, and each row of $-L_1 L_2$ has a sum equal to one.*

Proof. The second assertion is well known, see Lemma 4 in Meng et al. (2010). In the following we show the first assertion. According to Meng et al. (2010) we know that L_1 is a nonsingular M-matrix under Assumption 2. By Theorem 4.25 in Qu (2009) we deduce that L_1^{-1} exists and is nonnegative. Therefore it satisfies the same condition as that in Lemma 4 in Li et al. (2015), then following the same line as in the proof of Lemma 4 in Li et al. (2015) we arrive at the first assertion. ■

Definition 2 (*Wang & Song, 2017*)**.** The pre-specified finite time full state containment of MAS (1)–(2) ($M > 1$) with the pre-specified finite time T^* is said to be solved if, for any given initial state, there exist nonnegative constants β_j ($j \in \mathcal{L}$) satisfying $\sum_{j=1}^M \beta_j = 1$ such that, for all $i \in \mathcal{F}$ and $q = 1, \ldots, n$, $\lim_{t \to (t_0 + T^*)^-} x_{i,q} = \sum_{j=1}^M \beta_j x_{j,q}$ and $x_{i,q} - \sum_{j=1}^M \beta_j x_{j,q} = 0_m$ when $t \ge t_0 + T^*$.

Let $E_{\mathcal{F}} = [e_1^T, \ldots, e_N^T]^T$, $E_{\mathcal{L}} = [e_{N+1}^T, \ldots, e_{N+M}^T]^T$, $x_{\mathcal{F}} = [x_1^T, \ldots, x_N^T]^T$, and $x_{\mathcal{L}} = [x_{N+1}^T, \ldots, x_{N+M}^T]^T$. Then it holds

$$\begin{aligned} E_{\mathcal{F}} &= (L_1 \otimes I_n)x_{\mathcal{F}} + (L_2 \otimes I_n)x_{\mathcal{L}} \\ &= (L_1 \otimes I_n)[x_{\mathcal{F}} - ((-L_1^{-1}L_2) \otimes I_n)x_{\mathcal{L}}]. \end{aligned} \tag{66}$$

Let $x^* = [x_1^{*T}, \ldots, x_N^{*T}] = -((-L_1^{-1}L_2) \otimes I_n)x_{\mathcal{L}} \in R^{nN}$. According to Lemma 4, $x_{\mathcal{F}} \to x_e$ means that $x_{i,q}$ ($i \in \mathcal{F}$) converge to the convex hull $Co\{x_{j,q}, j \in \mathcal{L}\}$ for all $q = 1, \ldots, n$. Then the full state finite time containment objective in this paper is to design distributed controller such that $x_{\mathcal{F}} \to x_e$, i.e., $E_{\mathcal{F}} \to 0$, in the uniformly pre-specified finite time T^*.

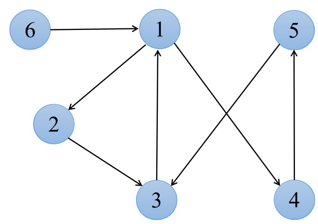


Fig. 1. The topology.

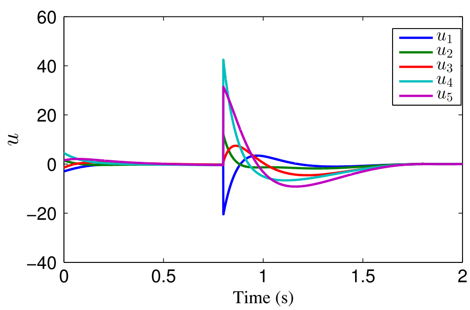


Fig. 2. The proposed u.

Firstly, we design the same distributed observer as in (18) with $M > 1$. It is seen that $[\hat{e}_{1,q}, \ldots, \hat{e}_{N,q}]^T = L_1([\hat{x}_{1,q}, \ldots, \hat{x}_{N,q}]^T - [x^*_{1,q}, \ldots, x^*_{N,q}]^T) \equiv 0$ when $t \geq t_0 + T_1$ for all $q = 1, \ldots, n$, with which we then conclude that $\hat{x}_{i,q} \equiv x^*_{i,q}$ for all $i \in \mathcal{F}$ and $q = 1, \ldots, n$ when $t \geq t_0 + T_1$ under Assumption 2. Then we design the same controller as in (19)–(20) with $M > 1$, and obtain the finite-time containment result in the following theorem.

Theorem 2. *Consider a group of agents with N followers being described by (1) and M leaders being described by (2) under Assumption 2. The control scheme (19)–(20) with $M > 1$ solves the pre-specified finite-time full state containment problem with the pre-specified finite time $T^* = T_1 + T$. Furthermore, the containment will be maintained over $[t_0 + T_1 + T, \infty)$ in which u remains zero, and the resultant control input u is uniformly bounded and smooth almost everywhere except for $t = t_0 + T_1$.*

Proof. By choosing the Lyapunov function as

$$V = \xi^T(Q_{\mathcal{F}} \otimes P)\xi, \tag{67}$$

the result can be readily derived by following the same procedure as in the proof of Theorem 1. ■

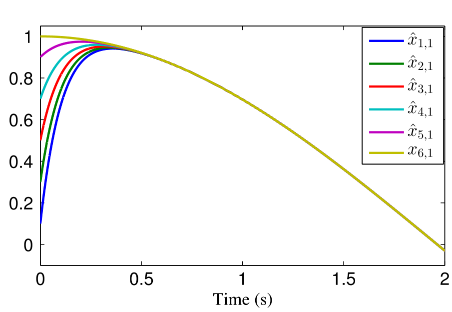
Before closing this section, it is worth mentioning that, although observer based finite time control approach has been used by Li et al. (2011) and Wang et al. (2014), their designs are segmented with the switching points depending on the convergence time of the observer, and such observing convergence time directly depends on the initial conditions and a number of other design parameters, making the switching moment hard to determine. Whereas the observer proposed in this work is based on a time-varying observer gain, which allows the observing time not only to be finite but also uniform in initial conditions, thus can be pre-assigned. This leads to a completely different yet improved method for building the distributed control upon observer.

5. Numerical simulation

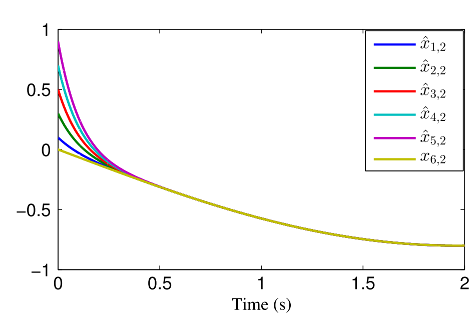
To test the effectiveness of the proposed finite-time leader-following consensus protocol, we conduct the simulation on a group of networked multiple systems consisting of 1 leader and 5 followers, whose dynamics are descried by (1)–(2) with $n = 2$. The communication topology among the 6 agents is shown as in Fig. 1, satisfying Assumption 1, with the weight being 1. The initial conditions for the proposed observer design in (18) are chosen as: $t_0 = 0$, $\hat{x}_1 = [0.1, 0.1]^T$, $\hat{x}_2 = [0.3, 0.3]^T$, $\hat{x}_3 = [0.5, 0.5]^T$, $\hat{x}_4 = [0.7, 0.7]^T$, $\hat{x}_5 = [0.9, 0.9]^T$, and the design parameters for the finite-time observer are: $T_1 = 0.8s$, $h = 2$, and $c = 6$. The initial states of the 5 followers are set as: $X(t_0) = [1, 0, 0.5, 0, 0, 0, -0.5, 0, -1, 0]^T$ with $t_0 = 0$. A straightforward calculation shows that $g = \lambda_{\min}(QL_1 + L_1^TQ) = 1.7838$ with $Q = \mathrm{diag}\{q_1, \ldots, q_5\}$ and $[q_1, \ldots, q_5]^T = L_1^{-1}1_5$, and then $g_0 = g\lambda_{\min}(Q^{-1}) = 0.2g = 0.3568$ with the topology given in Fig. 1. Solving the Riccati inequality (17) by using Matlab gives a solution $P = \begin{bmatrix} 2.9921 & 2.9921 \\ 2.9921 & 6.4842 \end{bmatrix}$. The other design parameters in the proposed control scheme (19)–(20) are chosen as: $T = 1s$ and $h = 2$.

The simulation results are presented in Figs. 2–4. Fig. 2 is the control input signal, Fig. 3 shows the estimate value of the leader's state, $\hat{x}_{i,q}$, for $i = 1, \ldots, 5$ and $q = 1, 2$, obtained by the distributed observer established in (18), and Fig. 4 is the system response under the proposed control (19)–(20). Fig. 2 verifies that the control input signals are uniformly bounded and continuous everywhere and smooth over the whole time interval except for $t = t_0 + T_1$. From Fig. 3 we see that the observed value of each follower converges to the true state value of the leader within the pre-specified finite time T_1, which makes the leader's state available for each follower after T_1 and then allows us to implement the controller (19)–(20) for the period of $t \in [T_0 + T_1, T_0 + T_1 + T)$ successfully. Fig. 4 shows that the qth ($q = 1, 2$) global errors among the followers and the leader, i.e., $\delta_{i,1}$ and $\delta_{i,2}$ ($i = 1, \ldots, 5$), converge to zero within the pre-specified finite time $T_1 + T$.

Furthermore, we have tested the consensus property for all the followers under different given finite times (with common initial conditions) and different initial conditions (with common

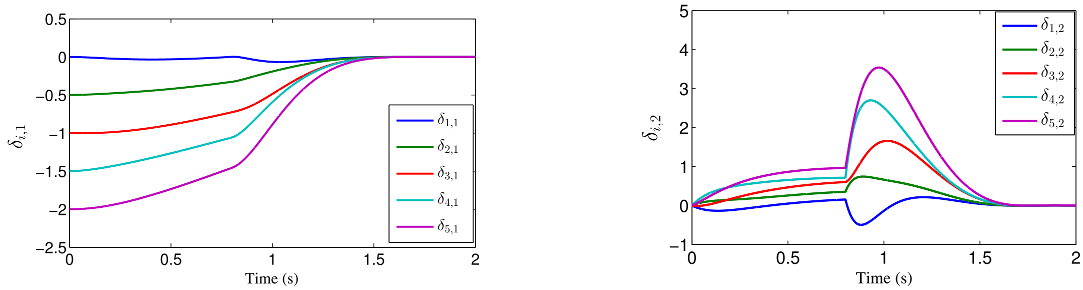


(a) $\hat{x}_{i,1}$ and $x_{6,1}$.



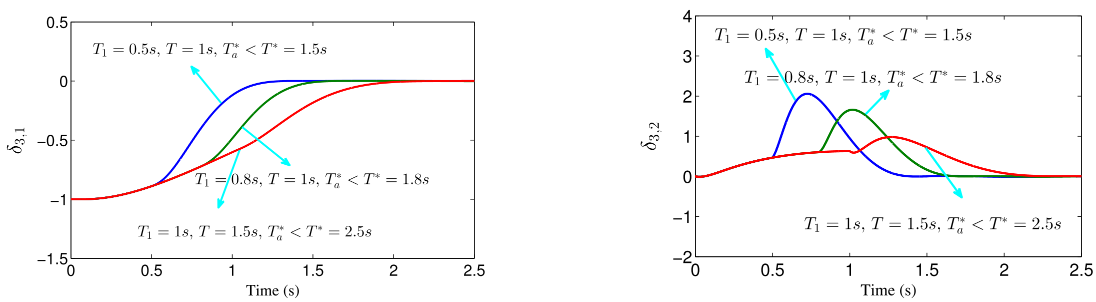
(b) $\hat{x}_{i,2}$ and $x_{6,2}$.

Fig. 3. Observing performance of the distributed observer (18).



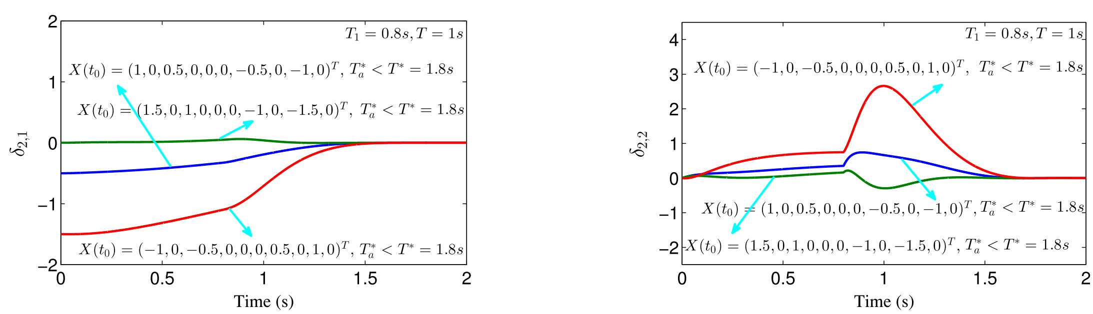
(a) $\delta_{i,1}$. (b) $\delta_{i,2}$.

Fig. 4. Consensus error of the 5 followers.



(a) $\delta_{3,1}$. (b) $\delta_{3,2}$.

Fig. 5. Consensus of the 3rd follower under three different given finite times for T_1 and T (T_a^* denotes the time at which the consensus is achieved).



(a) $\delta_{2,1}$. (b) $\delta_{2,2}$.

Fig. 6. Consensus of the 2nd follower under different initial conditions.

pre-specified finite time T_1 and T). Fig. 5 is the selected 3rd follower consensus process, showing that the consensus is reached within three different pre-specified finite times for T_1 and T. Fig. 6 confirms that the consensus is reached within the uniformly pre-specified finite time for the 2nd followers (other followers all reach the consensus within the pre-specified finite time but not shown here) although different initial conditions are involved.

All of those results verify our theoretical prediction and validate the effectiveness of the proposed control scheme. That is, the finite convergence time is independent of any initial conditions and any other design parameters, thus can be uniformly pre-specified.

6. Conclusions

We proposed in this work a new approach for finite-time leader-following consensus control of high-order MAS under directed communication topology. The salient feature of this method is that for any communication graph containing a directed spanning tree with the leader as the root, leader-following consensus is achieved within a pre-specified finite time, regardless of any initial conditions or any other design parameter. Furthermore, the resultant control action is continuous everywhere and smooth almost everywhere, except for $t = t_0 + T_1$. One research direction for future study is to extend the proposed finite-time consensus control method to more general case of high-order MAS with nonlinear and uncertain dynamics.

Appendix A. Proof of Lemma 2

Proof. (i) Multiplying ϱ^2 on both hands of (21) gives

$$\varrho^2\dot{y}(t) \le -\gamma\varrho^2 y(t) - 2\varrho\dot{\varrho}y(t) \tag{A.1}$$

which yields

$$\frac{d(\varrho^2 y(t))}{dt} = \varrho^2\dot{y}(t) + 2\varrho\dot{\varrho}y(t) \le -\gamma(\varrho^2 y(t)). \tag{A.2}$$

Note that $\varrho(t_0) = 1$, then we solve (A.2) and obtain

$$\varrho(t)^2 y(t) \le \exp^{-\gamma(t-t_0)}\varrho(t_0)^2 y(t_0) = \exp^{-\gamma(t-t_0)} y(t_0). \tag{A.3}$$

It thus follows from (A.3) that, $y(t) \le \varrho^{-2}\exp^{-\gamma(t-t_0)} y(t_0)$, by which we then arrive at $y(t) \to 0$ as $t \to (t_0+T_1)^-$ by the property of $\varrho(t)$.

(ii) From the above analysis, we see that

$$y(t_1) = y(t_0+T_1) = \lim_{t\to(t_0+T_1)^-} y(t) = 0. \tag{A.4}$$

For $t \in [t_1, t_1+T_1)$ with $t_1 = t_0+T_1$, we see from (21) that, $\dot{y} \le 0$. This, together with (A.4), yields $0 \le y(t) \le y(t_1) = 0$ for $t \in [t_1, t_1+T_1)$, implying that $y(t) \equiv 0$ on $[t_1, t_1+T)$. Similarly, $y(t) \equiv 0$ on $[t_l, t_l+T_1)$ for $l = 2, 3, \ldots$. Therefore, $y(t) \equiv 0$ when $t \ge t_0 + T_1$. ■

Appendix B. Proof of Lemma 3

Proof. Note that, $\eta^{(q)} = \frac{(n+h+q-1)!}{T^q(n+h-1)!}\eta^{1+\frac{q}{n+h}}$, $(q = 0, \ldots, n)$, by inserting which and (48) into (16), we obtain,

$$\theta_i = \eta^{-(\frac{h}{n+h})}\sum_{j=1}^{n} C_n^j \frac{(n+h+j-1)!}{T^j(n+h-1)!}\sum_{k=1}^{n+1-j} C_{n-j}^{n+1-j-k} \times \frac{(n+h)!(-1)^{n+1-j-k}}{T^{n+1-j-k}(h-1+j+k)!}\eta^{-(\frac{k-1}{n+h})} r_{i,k}, \tag{B.1}$$

which then yields (22) by inspection, where $H = [H_{ik}]_{n\times n}$ with $H_{ik} = \bar{H}_{ik}\eta^{-(\frac{k-1}{n+h})}$ and

$$\bar{H}_{ik} = \sum_{j=1}^{n} C_n^j C_{n-j}^{n+1-j-k}\frac{(n+h+j-1)!}{T^j(n+h-1)!} \times \frac{(n+h)!(-1)^{n+1-j-k}}{T^{n+1-j-k}(h-1+j+k)!}, \tag{B.2}$$

which is bounded following from the properties of η. In addition, we get the derivative of θ from (22), (42) and (14) as,

$$\begin{aligned}\dot{\theta} &= \left[\eta^{-\frac{h}{n+h}}(I_N\otimes H)\right]^{(1)} r + \eta^{-\frac{h}{n+h}}(I_N\otimes H)\dot{r} \\ &= \left[\left(I_N\otimes\left(\eta^{-\frac{h}{n+h}}H\right)\right)\right]^{(1)} r + \eta^{-\frac{h}{n+h}}\left[I_N\otimes(HA_0)\right]r \\ &\quad - \eta^{-\frac{h}{n+h}}\left[L_1\otimes(He_ne_n^TP)\right]r.\end{aligned} \tag{B.3}$$

Note that

$$\begin{aligned}\eta^{-(\frac{h-1}{n+h})}\Gamma_{ik} &= \left(\eta^{-(\frac{h}{n+h})}H_{ik}\right)^{(1)} = \left(\eta^{-(\frac{h}{n+h})}\bar{H}_{ik}\eta^{-(\frac{k-1}{n+h})}\right)^{(1)} \\ &= \eta^{-(\frac{h-1}{n+h})}\left(-\frac{h+k-1}{T}\right)\bar{H}_{ik}\eta^{-(\frac{k-1}{n+h})}\end{aligned} \tag{B.4}$$

which, combining (B.3), yields (23) with $\Gamma = [\Gamma_{ik}]_{n\times n}$, where $\Gamma_{ik} = -\frac{h+k-1}{T}\bar{H}_{ik}\eta^{-(\frac{k-1}{n+h})}$. From (46), it is clear that $(\eta^{-1})^j$ $(j = 0, 1, 2, \ldots)$ are bounded by 1, which, together with (22) and (23), yields (24). ■

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