Distributed Average Tracking of Multiple Time-Varying Reference Signals With Bounded Derivatives

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Abstract—**We present a distributed discontinuous control algorithm for a team of agents to track the average of multiple time-varying reference signals with bounded derivatives. We use tools from nonsmooth analysis to analyze the stability of the system. For time-invariant undirected connected network topologies, we prove that the states of all agents will converge to the average of the time-varying reference signals with bounded derivatives in finite time provided that the control gain is properly chosen. The validity of this result is also established for scenarios with switching undirected connected network topologies. For time-invariant directed network topologies with a directed spanning tree, we show that all agents will still reach a consensus in finite time, but the convergent value is generally not the average of the time-varying reference signals with bounded derivatives. Simulation examples are presented to show the validity of the above results.**

Index Terms—**Distributed average tracking, multiagent system, nonsmooth system, switching topology.**

I. Introduction

In the past, researchers have been working on consensus problems with different properties of graphs, types of agent dynamics, and analysis tools (see, e.g., [1]–[6]). Here, the states of all agents usually converge to the average or the weighted average of the initial conditions of these agents, which is a constant value. In particular, finite-time consensus algorithms have attracted much attention due to its better disturbance rejection property and robustness against uncertainties. Normalized and signed versions of gradient dynamical systems were introduced in [7]. Two results on finite-time convergence and the second-order information of the Lyapunov functions were derived. The finite-time consensus problem with continuous state feedback was discussed in [8] for undirected and directed graphs. A general framework for designing semistable protocols in dynamical networks for achieving coordination tasks in finite time was developed in [9]. In addition, finite-time consensus problems for discrete-time systems were also considered, see [10], for example.

In a consensus problem, when there exists a dynamic leader (e.g., an agent that moves by itself regardless of the other agents) or a time-varying reference signal, the consensus problem becomes a coordinated tracking problem. Here, the objective is that the states of all agents track the state of the dynamic leader or the time-varying reference signal. A coordinated tracking problem was studied in [11] for a group of autonomous agents and the results were extended to second-order systems in [12]. Ref. [13] studied a distributed discrete-time coordinated tracking problem where a team of agents communicating with their local neighbors at discrete-time instants tracks a time-varying reference signal available to only a subset of the team members. In [14] and [15], coordinated tracking problems were studied where the reference velocity is available to only one agent while the other agents estimate the reference velocity with an adaptive design.

When there exist multiple reference signals, dynamic average consensus problems were studied. Here, the objective is that the states of all agents track the average of the reference signals. One motivation for this problem is from coordinated tracking for multi-camera systems, where multiple nodes equipped with cameras track objects cooperatively. In [16], a distributed algorithm was proposed to guarantee that a consensus is reached on the average of multiple reference signals with steady-state values. The result was proved by frequency-domain techniques and was applied in [17] to obtain least-squares fused estimates based on spatially distributed measurements. In [18], two dynamic average consensus algorithms were proposed, namely, a proportional (or high-pass) algorithm and a proportional-integral algorithm. These two algorithms were used in [19] to build a framework for decentralized estimation and control. An extension of the proportional-integral algorithm was proposed in [20] to study a dynamic average consensus problem. The internal model principle was employed to achieve zero steady-state error for some classes of time-varying inputs, including polynomial inputs of known orders and sinusoidal inputs with known frequencies. A consensus filter was used in [21] to study swarm dynamics where interagent forces are governed by repulsive-attractive forces.

In this note, we aim at designing distributed algorithms that enable a team of agents to track the average of multiple time-varying reference signals, referred to as a *distributed average tracking* problem hereafter.[1] The contributions of this note lie in the following facts. We propose a simple but compelling control algorithm to solve the distributed average tracking problem by using the signum function. To the best of our knowledge, the proposed algorithm is the first distributed algorithm that guarantees accurate tracking of the average of multiple arbitrary time-varying reference signals with bounded derivatives. Because of the discontinuity in the control algorithm, we exploit tools from nonsmooth analysis to investigate the stability of the closed-loop system. We have gained motivation and insights from [7] and other works on finite-time consensus. For example, the signum function is also employed in the algorithm in the current note. However, it is worth noting the differences between the current work and [7] and other works on finite-time consensus. First, in the *distributed average tracking* problem the agents not only need to reach a consensus but also need to track the average of multiple time-varying reference signals in a distributed manner. Second, the control algorithm of the current note is different from that in [7] because the signum function is used in a different way. In the current note, the signum function is applied to each neighbor, while in [7] the signum function was applied to the whole neighborhood. It is worthwhile to point out that if the signum function is applied to the whole neighborhood, then the distributed average tracking cannot be guaranteed due to the existence of multiple time-varying reference signals and the loss of the invariant property (the sum of all the time-varying signals is equal to the sum of the estimator states). Third, in the current note, nonsmooth Lyapunov functions have been used to

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[1]Here, we use the term *average tracking* rather than *dynamic consensus* to emphasize the tracking nature of the problem.

show finite-time convergence (see Theorem 1 for example), while [7] exploited smooth Lyapunov functions. Note that, in comparison with [20], the reference signals considered in this note are not assumed to have any specific form and hence more general.

Notation: Let $\mathbb{R}$ and $\mathbb{R}^+$ denote, respectively, real numbers and nonnegative real numbers. Define **1** as the column vector with all ones. Let $\mathrm{sgn}(\cdot)$ denote the signum function. Let $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ denote, respectively, the 1-norm, 2-norm, and ∞-norm of a vector. The absolute value of a scalar is denoted by $|\cdot|$. Let $\#S$ be the cardinality of the set S and let co and $\overline{\mathrm{co}}$ denote, respectively, the convex hull and convex closure. Let A^T denote the transpose of A. The Lebesgue measure of Ω is denoted by $u(\Omega)$. Let $B(x,r)$ be the open ball of radius r centered at x. The gradient of $\mathfrak{f}$ is denoted by $\nabla \mathfrak{f}$. Let $\mathcal{B}(\mathbb{R}^d)$ be the collection of subsets of $\mathbb{R}^d$. The term *almost everywhere* is abbreviated a.e. and *if and only if* is abbreviated iff. The upper Dini derivative of a continuous function $\mathfrak{v}:\mathbb{R}\to\mathbb{R}$ is denoted by $D^+\mathfrak{v}$.

II. Problem Descriptions and Mathematical Preliminaries

A. Problem Descriptions

Suppose that there are n time-varying reference signals, $r_i(t)\in\mathbb{R}^m$, $i=1,\ldots,n$, satisfying the following dynamics $\dot{r}_i(t)=f_i(t)$. Here $f_i(t)\in\mathbb{R}^m$ is assumed to be measurable and bounded, i.e., $\sup_{t\in[0,\infty)}\|f_i(t)\|_\infty\le\bar{f}$ for all $i=1,\ldots,n$, where $\bar{f}$ is a positive constant. Suppose that there are n agents with $x_i(t)\in\mathbb{R}^m$ being the state of agent i, $i=1,\ldots,n$. We assume that agent i has access to $r_i(t)$. We also assume that at time $t\ge 0$ agent i can obtain information from a subset of the other agents, called its neighbors and denoted by $\mathcal{N}_i(t)$.

We use a graph $\mathcal{G}(t)\triangleq\{\mathcal{V},\mathcal{E}(t)\}$ to describe the network topology between the agents at time t, where $\mathcal{V}\triangleq\{1,\ldots,n\}$ is the node set and $\mathcal{E}(t)\triangleq\{(i,j)|i\in\mathcal{N}_j(t), j=1,\ldots,n\}$ is the edge set. The parameter t will be omitted if the graph is time invariant. A graph is undirected if $j\in\mathcal{N}_i$ implies $i\in\mathcal{N}_j$. If $i\in\mathcal{N}_j$, node i is the parent node while node j is the child node. For a digraph $\mathcal{G}=(\mathcal{V},\mathcal{E})$, a directed path is defined as a sequence of nodes $v_0,v_1,\ldots,v_k\in\mathcal{V}$ such that $(v_i,v_{i+1})\in\mathcal{E}$ for $0\le i\le k-1$. A directed tree is a directed graph, where every node has exactly one parent except for one node, called the root, and the root has directed paths to every other node. A directed spanning tree of a directed graph is a direct tree that contains all nodes of the directed graph. A directed graph has a spanning tree if there exists a directed spanning tree as a subset of the directed graph. An undirected path in an undirected graph is defined analogously. An undirected graph is connected if there is an undirected path between every pair of distinct nodes.

Our main objective is to design a distributed algorithm for agent i based on $r_i(t)$ and $x_j(t)$, $j\in\mathcal{N}_i(t)$, such that all agents will finally track the average of the n time-varying reference signals, i.e.,$\|x_i(t)-(1/n)\sum_{j=1}^n r_j(t)\|\to 0$ as $t\to\infty$. We design the following control algorithm:

$$\begin{aligned}\dot{z}_i(t)&=\alpha\sum_{j\in\mathcal{N}_i(t)}\mathrm{sgn}\left[x_j(t)-x_i(t)\right]\\ x_i(t)&=z_i(t)+r_i(t)\end{aligned}\tag{1}$$

where $z_i(t)$ is the internal state of the local filter, $\alpha>0$ is a constant, and $\mathrm{sgn}(\cdot)$ is the signum function defined component-wise, and initialize the internal states of all agents such that[2]

$$\sum_{i=1}^n z_i(0)=0.\tag{2}$$

[2]A special choice is $z_i(0)=0$ for all $i=1,\ldots,n$.

From (1), we obtain the following closed-loop system:

$$\dot{x}_i(t)=f_i(t)+\alpha\sum_{j\in\mathcal{N}_i(t)}\mathrm{sgn}\left[x_j(t)-x_i(t)\right]\tag{3}$$

with the initial conditions $\sum_{i=1}^n x_i(0)=\sum_{i=1}^n r_i(0)$. We note that each component of $x_i(t)$ is decoupled in (3). Therefore, in the following, we will only tackle the one-dimensional case, i.e., $m=1$. The same conclusions hold for any $m\ge 2$.

B. Mathematical Preliminaries

In the following, the solutions of (3) are understood in the Filippove sense. Filippov solutions are defined below.

Definition 1: [22] For a vector field $\mathfrak{f}(t,x):\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^d$, define the Filippov set-valued map $K[\mathfrak{f}](t,x)\triangleq\cap_{\delta>0}\cap_{u(N)=0}\overline{\mathrm{co}}\mathfrak{f}(t,B(x,\delta)-N)$, where $\cap_{u(N)=0}$ denotes the intersection over all sets of Lebesgue measure zeroes.

Definition 2: [23] Consider a vector differential equation

$$\dot{x}(t)=\mathfrak{f}(t,x),\tag{4}$$

where $x(t)=[x_1(t),\ldots,x_n(t)]^T$. A vector function $x(\cdot)$ is called a Filippov solution of (4) on $[t_0,t_1]$, where t_1 could be ∞, if $x(\cdot)$ is absolutely continuous and for almost all $t\in[t_0,t_1]$, $\dot{x}(t)\in K[\mathfrak{f}](t,x)$.

The next lemma establishes mild conditions under which Filippov solutions exist.

Lemma 1: [24] Given (4), let $\mathfrak{f}(t,x)$ be measurable and locally essentially bounded, that is, bounded on a bounded neighborhood of every point excluding sets of measure zero. Then, for all $x_0\in\mathbb{R}^n$, there exists a Filippov solution of (4) with the initial condition $x(0)=x_0$.

It follows from Lemma 1 that Filippov solutions for the system (3) exist because $f_i(t)$, $i=1,\ldots,n$, and $\mathrm{sgn}(\cdot)$ are measurable and bounded.

Let W be a locally Lipschitz function of x, where $x=[x_1,\ldots,x_n]^T$. The generalized gradient of the function W with respect to x_i (cf. [23]) is defined by

$$(\partial W)_i\triangleq\mathrm{co}\left\{\lim_{j\to\infty}\frac{\partial W}{\partial x_j'}|x_j'\to x_i, x_j'\notin\Omega_W\cup S\right\},$$

where Ω_W is the set of points where W fails to be differentiable and S is a set of zero measure that can be arbitrarily chosen to simplify the calculation. Then the generalized gradient of W is $\partial W\triangleq[(\partial W)_1,\ldots,(\partial W)_n]^T$. The set-valued Lie derivative of W with respect to x, the trajectory of (4), is defined as $\dot{\tilde{W}}\triangleq\cap_{\xi\in\partial W}\xi^T K[\mathfrak{f}]$.

III. Stability Analysis

In this section, we analyze (1) under undirected and directed network topologies. Before moving on, we need the following lemma.

Lemma 2: For (1), if graph $\mathcal{G}(t)$ is undirected and $|x_i(t)-x_j(t)|=0$ for all $i,j=1,\ldots,n$, then $|x_i(t)-(1/n)\sum_{j=1}^n r_j(t)|=0$, for all $i=1,\ldots,n$.

Proof: It follows from (1) that $\sum_{i=1}^n x_i(t)=\sum_{i=1}^n z_i(t)+\sum_{i=1}^n r_i(t)$. Because the graph $\mathcal{G}(t)$ is undirected, it follows that:

$$\sum_{i=1}^n\dot{z}_i(t)=\alpha\sum_{i=1}^n\sum_{j\in\mathcal{N}_i(t)}\left\{\mathrm{sgn}\left[x_j(t)-x_i(t)\right]\right\}=0.\tag{5}$$

Because $\sum_{i=1}^n z_i(0)=0$ [see (2)], it follows from (5) that $\sum_{i=1}^n z_i(t)=0$ for all $t\ge 0$. Therefore, it follows that:

$$\sum_{i=1}^n x_i(t)=\sum_{i=1}^n r_i(t)\tag{6}$$

for all $t \geq 0$. If $|x_i(t) - x_j(t)| = 0$ for all $i, j = 1, \ldots, n$, it follows from (6) that $|x_i(t) - (1/n)\sum_{j=1}^n r_j(t)| = 0$ for all $i = 1, \ldots, n$. ■

We next present the first main result of this note.

Theorem 1: For (1), if $\mathcal{G}$ is time invariant, undirected, and connected, and $\alpha > \bar{f}$, then $|x_i(t) - (1/n)\sum_{j=1}^n r_j(t)| \to 0$ in finite time for all $i = 1, \ldots, n$, and the convergence time is upper bounded by $(1/2(\alpha - \bar{f}))\sum_{i=1}^n \sum_{j \in \mathcal{N}_i} |x_i(0) - x_j(0)|$.

Proof: Define $e(t)$ as the column stack vector formed by all $x_i(t) - x_j(t)$, $(i,j) \in \mathcal{E}$. Consider the Lyapunov function candidate

$$V[e(t)] = \frac{1}{2}\|e(t)\|_1 = \frac{1}{2}\sum_{i=1}^n \sum_{j \in \mathcal{N}_i} |x_i(t) - x_j(t)|. \tag{7}$$

Because $V[e(t)]$ is nonsmooth, the time derivative of $V[e(t)]$ is not defined at some time instants. The function $V[e(t)]$ is regular because the absolute value function $|x|$, $x \in \mathbb{R}$ is regular at the discontinuous point 0.

We next show that the function $V(e)$ satisfies

$$V_1(\|e\|_2) \leq V(e) \leq V_2(\|e\|_2) \tag{8}$$

for some class $\mathcal{K}$ functions $V_1(\cdot)$ and $V_2(\cdot)$. Because all norms on a finite-dimensional linear space are equivalent, it follows that there are positive constants a and b such that $a\|e\|_2 \leq \|e\|_1 \leq b\|e\|_2$. Therefore, it is obvious that (8) holds.

We can derive that

$$K\left[\sum_{j \in \mathcal{N}_i} \operatorname{sgn}(x_j - x_i)\right] = \begin{cases} \{\#\mathcal{N}_i^+ - \#\mathcal{N}_i^-\}, & \text{if } x_i \neq x_j,\ \forall j \in \mathcal{N}_i, \\ \{\#\mathcal{N}_i^+ - \#\mathcal{N}_i^-\} + [-\#\mathcal{N}_i^0, \#\mathcal{N}_i^0], & \text{otherwise.} \end{cases}$$

Here, $\mathcal{N}_i^+$ is the set of neighbors of agent i with $x_j > x_i$, $j \in \mathcal{N}_i$, $\mathcal{N}_i^-$ is the set of neighbors of agent i with $x_j < x_i$, and $\mathcal{N}_i^0$ is the set of neighbors of agent i with $x_j = x_i$. In addition, it can be shown that the generalized gradient of $V(e)$ with respect to x_i is

$$(\partial V)_i = \begin{cases} \{\#\mathcal{N}_i^- - \#\mathcal{N}_i^+\}, & \text{if } x_i \neq x_j,\ \forall j \in \mathcal{N}_i, \\ \{\#\mathcal{N}_i^- - \#\mathcal{N}_i^+\} + [-\#\mathcal{N}_i^0, \#\mathcal{N}_i^0], & \text{otherwise.} \end{cases} \tag{9}$$

It can be shown that

$$\dot{\tilde{V}} \subseteq \cap_{\xi \in \partial V} \xi^T \begin{Bmatrix} K[f_1] + \alpha K\left[\sum_{j \in \mathcal{N}_1} \operatorname{sgn}(x_j - x_1)\right] \\ \vdots \\ K[f_n] + \alpha K\left[\sum_{j \in \mathcal{N}_n} \operatorname{sgn}(x_j - x_n)\right] \end{Bmatrix}$$

where $\partial V \triangleq [(\partial V)_1, \ldots, (\partial V)_n]^T$.

Let $(\dot{\tilde{V}})_i \triangleq \cap_{\xi_i \in (\partial V)_i} \xi_i (K[f_i] + \alpha K[\sum_{j \in \mathcal{N}_i} \operatorname{sgn}(x_j - x_i)])$. To show that $\dot{\tilde{V}} < 0$, we distinguish between two cases. The first case is that $x_i \neq x_j$ for all $i = 1, \ldots, n$, $j \in \mathcal{N}_i$. In this case, we have that $(\dot{\tilde{V}})_i = \cap_{\xi_i \in (\partial V)_i} \xi_i (K[f_i] + \alpha\{\#\mathcal{N}_i^+ - \#\mathcal{N}_i^-\})$. If $(\dot{\tilde{V}})_i \neq \emptyset$, suppose that $a_i \in (\dot{\tilde{V}})_i$ and $a_i^f \in K[f_i]$. From (9), we know that ξ_i has only one value which equals to $\#\mathcal{N}_i^- - \#\mathcal{N}_i^+$. Because $\sup_{t \in [0,\infty)} \|f_i(t)\|_\infty \leq \bar{f}$, according to the definition of $K[f_i]$, we know that $K[f_i] \subseteq [-\bar{f}, \bar{f}]$, which indicates that $|a_i^f| \leq \bar{f}$. Thus, one has $a_i = (\#\mathcal{N}_i^- - \#\mathcal{N}_i^+)a_i^f - \alpha(\#\mathcal{N}_i^- - \#\mathcal{N}_i^+)^2 \leq |\#\mathcal{N}_i^- - \#\mathcal{N}_i^+|(|a_i^f| - \alpha|\#\mathcal{N}_i^- - \#\mathcal{N}_i^+|)$. If $\#\mathcal{N}_i^+ - \#\mathcal{N}_i^- = 0$, then $a_i = 0$. If $\#\mathcal{N}_i^+ - \#\mathcal{N}_i^- \neq 0$, which implies that $|\#\mathcal{N}_i^- - \#\mathcal{N}_i^+|$ is an integer that is greater than or equal to one, we have $a_i \leq \bar{f} - \alpha < 0$. If $e \neq 0$, there always exists a node $p \in \{1, \ldots, n\}$ such that $\#\mathcal{N}_i^+ - \#\mathcal{N}_i^- \neq 0$, which immediately implies that $\dot{\tilde{V}} = \sum_{i=1}^n (\dot{\tilde{V}})_i \leq \bar{f} - \alpha < 0$.

The second case is that there exists $j \in \mathcal{N}_i$ such that $x_i = x_j$. If $\dot{\tilde{V}}_i \neq \emptyset$, suppose that $a_i \in \dot{\tilde{V}}_i$ and $a_i^f \in K[f_i]$. Then we know that $\forall \xi_i \in (\partial V)_i$, $a_i = \xi_i(a_i^f + \alpha v_i)$, where $v_i \in \{\#\mathcal{N}_i^+ - \#\mathcal{N}_i^-\} + [-\#\mathcal{N}_i^0, \#\mathcal{N}_i^0]$. Choose $\xi_i = -v_i \in \{\#\mathcal{N}_i^- - \#\mathcal{N}_i^+\} + [-\#\mathcal{N}_i^0, \#\mathcal{N}_i^0]$. Thus, we have that $a_i = -v_i a_i^f - \alpha v_i^2 \leq |v_i|(\bar{f} - \alpha|v_i|)$. If $v_i = 0$, then $a_i = 0$. If $v_i \neq 0$, it follows from [23, Proposition 2.2.9] that $|v_i| \geq 1$, which implies that $a_i \leq \bar{f} - \alpha < 0$. If $e \neq 0$, there always exists a node $p \in \{1, \ldots, n\}$ such that $v_p \neq 0$. Thus, we can conclude that $\dot{\tilde{V}} \leq \bar{f} - \alpha < 0$, for $e \neq 0$.

It follows that $e(t) \to 0$ as $t \to \infty$, which implies that $|x_i(t) - x_j(t)| \to 0, \forall (i,j) \in \mathcal{E}$, as $t \to \infty$. Because $\mathcal{G}$ is connected, we know that $|x_i(t) - x_j(t)| \to 0, \forall i, j = 1, \ldots, n$, as $t \to \infty$. It follows from Lemma 2 that $|x_i(t) - (1/n)\sum_{j=1}^n r_j(t)| \to 0$, $\forall i = 1, \ldots, n$, as $t \to \infty$.

From the Lebesgue's criterion for the Riemann integrability, we know that a function on a compact interval is Riemann integrable if and only if it is bounded and the set of its points of discontinuity has measure zero [25]. Write $V[e(t)]$ as $V(t)$ for simplicity. Therefore, although the time-derivative $\dot{V}(t)$ is discontinuous at some time instants, it is Riemann integrable. Then, we have that

$$V(t+h) - V(t) = \int_t^{t+h} \dot{V}(\tau)d\tau \leq (\bar{f} - \alpha)h \tag{10}$$

with $h > 0$. The upper Dini derivative of the function V is given by

$$D^+V \triangleq \limsup_{h \to 0^+} \frac{V(t+h) - V(t)}{h}. \tag{11}$$

One of the properties of limsup is that if $z_k \leq x_k$ for each $k = 1, 2, \ldots$, then $\limsup_{k \to \infty} z_k \leq \limsup_{k \to \infty} x_k$ [25]. Thus, from (10) and (11), we know that $D^+V \leq \bar{f} - \alpha$. Then it follows from the comparison lemma that $V(t) \leq V(0) - (\alpha - \bar{f})t$, which indicates that $|x_i(t) - (1/n)\sum_{i=1}^n r_i(t)| \to 0$ in finite time, and the convergence time is upper bounded by $V[e(0)]/(\alpha - \bar{f})$. ■

Remark 1: To reduce the errors due to incorrect initialization [Here the initialization errors are referred to z_i, that is, $\sum_{i=1}^n z_i(0) \neq 0$ [see (2)], one way is to include a "leakage" term in the state update: $\dot{z}_i(t) = -h[z_i(t)] + \alpha \sum \operatorname{sgn}[x_j(t) - x_i(t)]$ where $h[z_i(t)]$ is a linear function $cz_i(t)$ with $c > 0$.[3] The purpose of $-h[z_i(t)]$ is to drive $\sum_{i=1}^n z_i(t)$ to 0. Given $\dot{z}_i(t) = -cz_i(t) + \alpha \sum_{j \in \mathcal{N}_i(t)} \operatorname{sgn}[x_j(t) - x_i(t)]$, it follows that $\sum_{i=1}^n \dot{z}_i(t) = -c\sum_{i=1}^n z_i(t) + \alpha \sum_{i=1}^n \sum_{j \in \mathcal{N}_i(t)} \operatorname{sgn}[x_j(t) - x_i(t)]$. Because $\mathcal{G}$ is undirected, one knows that $\sum_{i=1}^n \sum_{j \in \mathcal{N}_i(t)} \operatorname{sgn}[x_j(t) - x_i(t)] = 0$. Hence, one has that $\sum_{i=1}^n \dot{z}_i(t) = -c\sum_{i=1}^n z_i(t)$, which implies that $\sum_{i=1}^n z_i(t) \to 0$ as $t \to \infty$. Of course, the tradeoff is that there will exist steady-state average tracking errors due to the leakage term.

If the communication between two agents is determined by distance, i.e., $j \in \mathcal{N}_i(t)$ if and only if $\|x_i(t) - x_j(t)\| \leq R$, where R is the communication radius of the agents, then $\mathcal{G}(t)$ is a proximity graph that could be switching. In the following, we will consider the case where $\mathcal{G}(t)$ is switching but remains connected at each time instant.

Theorem 2: For (1), if the proximity-based graph $\mathcal{G}(t)$ is switching, undirected, and connected for all $t \geq 0$, and $\alpha > \bar{f}$, then $|x_i(t) - (1/n)\sum_j r_j(t)| \to 0$ in finite time for all $i = 1, \ldots, n$.

[3]The authors would like to thank one anonymous reviewer for pointing out this remark.

Proof: Define the Lyapunov function $V(t) \triangleq (1/2)\sum_{i=1}^{n}\sum_{j\neq i}V_{ij}(x_i - x_j)$, where

$$V_{ij}(x_i - x_j) \triangleq \begin{cases} |x_i - x_j|, & \text{if } |x_i - x_j| \leq R, \\ R, & \text{otherwise.} \end{cases}$$

The proof is similar to that of Theorem 1 and is hence omitted. ■

Let Γ_c be the finite set of all undirected connected graphs with vertices $\{1,\ldots,n\}$. Let $\mathcal{I}_c \subset \mathbb{N}$ be an index set associated with the elements of Γ_c. A switching signal is a map $t \mapsto \sigma(t) : \mathbb{R}^+ \to \mathcal{I}_c$. For each time $t \in \mathbb{R}^+$, the switching signal returns the index of the graph $\mathcal{G}(t) \in \Gamma_c$.

Theorem 3: For (1), let $t \mapsto \sigma(t) : \mathbb{R}^+ \to \mathcal{I}_c$ be a switching signal. If $\alpha > (n-1)\bar{f}$, then $|x_i(t) - (1/n)\sum_j r_j(t)| \to 0$ as $t \to \infty$ for all $i = 1,\ldots,n$.

Proof: Let $\tilde{x}_i(t) \triangleq x_i(t) - (1/n)\sum_{j=1}^{n}x_j(t)$. Also let $\tilde{x}(t)$ and $x(t)$ be the column stack vectors of, respectively, all $\tilde{x}_i(t)$ and $x_i(t)$. Consider the Lyapunov function candidate

$$V(t) \triangleq \frac{1}{2}\|\tilde{x}(t)\|^2 = \frac{1}{2}\left\|x(t) - \frac{1}{n}x^T(t)\mathbf{1}\mathbf{1}\right\|^2. \tag{12}$$

Because the function $V(t)$ is continuously differentiable, we know that the generalized gradient of $V(t)$ with respect to $x_i(t)$ is a singleton. In particular, we have $(\partial V)_i = \{\partial V/\partial x_i\} = \{x_i - (1/n)x^T\mathbf{1}\}$.

In the following, we calculate the set-valued Lie derivative of the function $V(t)$. If $\dot{\tilde{V}}(t) \neq \emptyset$, suppose that $a \in \dot{\tilde{V}}(t)$ and $a_i^f \in K[f_i]$. By the definition, we know that $a = \sum_{i=1}^{n}(x_i - (1/n)x^T\mathbf{1})v_i$ with $v_i \in K[u_i]$. If $\forall i = 1,\ldots,n$, $x_j \neq x_i$ for all $j \in \mathcal{N}_i(t)$, then $K[u_i] = \{a_i^f + \alpha\sum_{j\in\mathcal{N}_i(t)}\text{sgn}(x_j - x_i)\}$. Define $\dot{\tilde{V}}_s \triangleq \{\sum_{i=1}^{n}(x_i - (1/n)x^T\mathbf{1})[a_i^f + \alpha\sum_{j\in\mathcal{N}_i(t)}\text{sgn}(x_j - x_i)]\}$. One has $\dot{\tilde{V}}_s = \sum_{i=1}^{n}(x_i - (1/n)x^T\mathbf{1})a_i^f - \alpha\sum_{i=1}^{n}(x_i - (1/n)x^T\mathbf{1})\{\sum_{j\in\mathcal{N}_i(t)}\text{sgn}[(x_i - (1/n)x^T\mathbf{1}) - (x_j - (1/n)x^T\mathbf{1})]\}$. Then, it can be obtained by direct computation that $\sum_{i=1}^{n}(x_i - (1/n)x^T\mathbf{1})a_i^f \leq (\bar{f}/n)\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}|x_i - x_j| \leq (\bar{f}/n)\sum_{i=1}^{n}\max_{i=1,\ldots,n}\{\sum_{j=1,j\neq i}^{n}|x_i - x_j|\} = \bar{f}\max_{i=1,\ldots,n}\{\sum_{j=1,j\neq i}^{n}|x_i - x_j|\}$. Since each graph is connected, we know that $\forall i \neq j$, $i,j = 1,\ldots,n$, $|x_i - x_j| \leq (1/2)\sum_{(i,j)\in\mathcal{E}(t)}|x_i - x_j|$.[4] Then, it follows that $\sum_{i=1}^{n}(x_i - (1/n)x^T\mathbf{1})a_i^f \leq (\bar{f}/2)(n-1)\sum_{(i,j)\in\mathcal{E}(t)}|x_i - x_j|$. In addition, we have $\sum_{i=1}^{n}(x_i - (1/n)x^T\mathbf{1})\sum_{j\in\mathcal{N}_i(t)}\text{sgn}[(x_i - (1/n)x^T\mathbf{1}) - (x_j - (1/n)x^T\mathbf{1})] = (1/2)\sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i(t)}[(x_i - (1/n)x^T\mathbf{1}) - (x_j - (1/n)x^T\mathbf{1})]\text{sgn}[(x_i - (1/n)x^T\mathbf{1}) - (x_j - (1/n)x^T\mathbf{1})] = (1/2)\sum_{(i,j)\in\mathcal{E}(t)}|x_i - x_j|$. Therefore, we know that $\dot{\tilde{V}}_s \leq (\bar{f}/2)(n-1)\sum_{(i,j)\in\mathcal{E}(t)}|x_i - x_j| - (\alpha/2)\sum_{(i,j)\in\mathcal{E}(t)}|x_i - x_j|$. Because $\alpha > \bar{f}(n-1)$, we know that $\dot{\tilde{V}}_s < 0$ for $\tilde{x} \neq 0$.

In the following, we will show that $\dot{\tilde{V}}_s = \dot{\tilde{V}}$ for almost every $t \in \mathbb{R}^+$ at the discontinuous points, that is, the points at which there exists $i \in \{1,\ldots,n\}$, $j \in \mathcal{N}_i$ such that $x_j = x_i$, except for the point $x_1 = x_2 = \cdots = x_n$. Let P denote the set of all discontinuous points. Then it suffices to show that for all $p \in P \setminus \{x | x_1 = x_2 = \cdots = x_n\}$, the system cannot stay at p for a time interval whose length is greater than 0. Define $M \triangleq \{k = 1,\ldots,n, |x_k = \max_j x_j\}$. Because each graph is connected, there must be a node $k \in M$ such that node k has a neighbor in $\{1,\ldots,n\} \setminus M$. Because $a_k^f + \alpha\sum_{j\in\mathcal{N}_k(t)}\text{sgn}(x_j - x_k) \leq \bar{f} - \alpha < 0$, we know that the system cannot stay at p during a time interval. Therefore, we have $\dot{\tilde{V}} =^{a.e.} \dot{\tilde{V}}_s < 0$ for all $t \in \mathbb{R}^+$, which indicates that $\|\tilde{x}(t)\| \to 0$ as $t \to \infty$. This implies that $|x_i(t) - x_j(t)| \to 0$ as $t \to \infty$ for all $i,j = 1,\ldots,n$. Therefore, according to Lemma 2, we know that $|x_i(t) - (1/n)\sum_{i=1}^{n}r_i(t)| \to 0$ as $t \to \infty$ for all $i = 1,2,\ldots,n$. ■

[4]Here note that because $\mathcal{G}$ is undirected, $(i_1,j_1) \in \mathcal{E}$ implies $(j_1,i_1) \in \mathcal{E}$. That is, in the summation sign, both (i_1,j_1) and (j_1,i_1) should be counted.

In the following, we will show that if $\mathcal{G}(t)$ is directed, a consensus can still be reached, but the convergent value is generally not $(1/n)\sum_{i=1}^{n}r_i(t)$, but some value depending on $(1/n)\sum_{i=1}^{n}r_i(t)$ and the unidirectional edges. That is, bidirectional edges have no contributions to the final convergent value. Define $U(t)$ to be the set of unidirectional edges in the graph $\mathcal{G}(t)$, and define $k[\mathcal{G}(t)] \triangleq \sum_{(i,j)\in U(t)}\text{sgn}[x_j(t) - x_i(t)]$.

Theorem 4: For (1), if $\mathcal{G}(t)$ is directed and has a directed spanning tree at each time instant, and $\alpha > (n-1)\bar{f}$, then $|x_i(t) - x_j(t)| \to 0$ as $t \to \infty$ for all $i,j = 1,\ldots,n$. In particular, if $k[\mathcal{G}(t)] = 0$ for all $t \geq 0$, then $|x_i(t) - (1/n)\sum_j r_j(t)| \to 0$ as $t \to \infty$ for all $i = 1,\ldots,n$.

Proof: Consider the same Lyapunov function candidate as defined by (12). We can show that $\sum_{i=1}^{n}x_i(t) = \sum_{i=1}^{n}r_i(t) + \alpha\int_0^t\sum_{(i,j)\in U(t)}\text{sgn}[x_j(\tau) - x_i(\tau)]d\tau$.

Therefore, by using similar arguments to those in the proof of Theorem 3, we can obtain that $|x_i(t) - (1/n)\sum_{j=1}^{n}x_j(t)| \to 0$ as $t \to \infty$, which implies that $|x_i(t) - x_j(t)| \to 0$ as $t \to \infty$ for all $i,j = 1,\ldots,n$. Note that $(1/n)\sum_{i=1}^{n}x_i(t) = (1/n)\sum_{i=1}^{n}r_i(t) + (\alpha/n)\int_0^t\sum_{(i,j)\in U(t)}\text{sgn}[x_j(\tau) - x_i(\tau)]d\tau$. In particular, if $k[\mathcal{G}(t)] = 0$ for all $t \geq 0$, we know that $|x_i(t) - (1/n)\sum_{j=1}^{n}r_j(t)| \to 0$ as $t \to \infty$. ■

Remark 2: When all $f_i \equiv 0$, $i = 1,\ldots,n$, we can obtain from Theorem 4 the consensus result under a directed switching network topology.

Corollary 1: For (1), let agent ℓ be a leader whose state satisfies $\dot{x}_\ell(t) = f_\ell(t)$ and who does not have a neighbor. Suppose that $\sup_{t\in[0,\infty)}\|f_\ell(t)\|_\infty$ is bounded. In addition, let $f_i \equiv 0$, $i \neq \ell$. If $\mathcal{G}(t)$ is directed and agent ℓ has directed paths to all other agents at each time instant, and $\alpha > (n-1)\sup_{t\in[0,\infty)}\|f_\ell(t)\|_\infty$, then $|x_i(t) - x_\ell(t)| \to 0$ as $t \to \infty$ for all $i \neq \ell$.

Proof: The corollary is an immediate consequence of Theorem 4. ■

IV. Initialization Errors, Time Delays, and Discrete-Time Implementation

In this section, the robustness of (1) to initialization errors and time delays are considered. A discrete-time version of (1) is proposed and is shown to have steady-state errors. For simplicity, time-invariant and undirected graphs are assumed.

A. Robustness to Initialization Errors

Suppose that (1) is not correctly initialized, that is, $\sum_{i=1}^{n}z_i(0) \neq 0$ or equivalently $\sum_{i=1}^{n}x_i(0) \neq \sum_{i=1}^{n}r_i(0)$. One would expect the algorithm to exhibit steady-state errors. Let $r_j'(t)$ be the signal with the initial condition $x_j(0)$ and the derivative $f_j(t)$. From Theorem 1, it follows that $|x_i(t) - (1/n)\sum_{j=1}^{n}r_j'(t)| \to 0$ in finite time. Because $\dot{r}_i'(t) = \dot{r}_i(t) = f_i(t)$ for $i = 1,\ldots,n$, one has $r_i'(t) - r_i(t) = x_i(0) - r_i(0)$ for $t \geq 0$. Then it is straightforward to obtain that $\limsup_{t\to\infty}|x_i(t) - (1/n)\sum_{j=1}^{n}r_j(t)| = \limsup_{t\to\infty}|(1/n)\sum_{j=1}^{n}r_j'(t) - (1/n)\sum_{j=1}^{n}r_j(t)| = (1/n)|\sum_{j=1}^{n}x_j(0) - \sum_{j=1}^{n}r_j(0)|$.

B. Robustness to Time Delays

In the *distributed average tracking* problem, if time delays appear, the system inevitably suffers from steady-state errors no matter which control algorithm is used. For simplicity, a homogeneous time delay $\tau \geq 0$ is assumed. Then (3) becomes

$$\dot{x}_i(t) = f_i(t) + \alpha\sum_{j\in\mathcal{N}_i}\text{sgn}\left[x_j(t-\tau) - x_i(t-\tau)\right]. \tag{13}$$

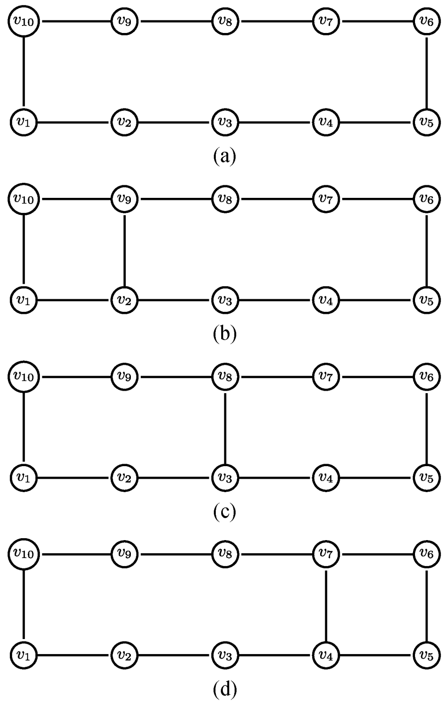

Fig. 1. Graphs for undirected switching topologies. (a) $\mathcal{G}_a$. (b) $\mathcal{G}_b$. (c) $\mathcal{G}_c$. (d) $\mathcal{G}_d$.

For notational convenience, define that $(\dot{\tilde{V}}(t))_i \triangleq \cap_{\xi_i \in (\partial V(t))_i} \xi_i (K[f_i(t)] + \alpha K[\sum_{j \in \mathcal{N}_i} \mathrm{sgn}(x_j(t-\tau) - x_i(t-\tau))])$ and $\dot{\tilde{V}}(t) \triangleq \sum_{i=1}^n (\dot{\tilde{V}}(t))_i$. During the time interval $[t-\tau, t]$, if $\#\mathcal{N}_i^-(t) = \#\mathcal{N}_i^-(t-\tau)$ and $\#\mathcal{N}_i^+(t) = \#\mathcal{N}_i^+(t-\tau)$, then by using similar arguments to those in Theorem 1, one has $\dot{\tilde{V}}(t) < 0$. By using this fact, it follows that if the tracking error is large (for example, $|x_i(t) - x_j(t)| > \beta$ for some constant $\beta \gg 0$ and for all $i \in \{1, \ldots, n\}$, and $j \in \mathcal{N}_i$) and the time delay τ is sufficiently small, one has $\dot{\tilde{V}}(t) < 0$, which implies that the tracking error is upper bounded. However, when the tracking error is small or τ is large, one might have $\#\mathcal{N}_i^-(t) \neq \#\mathcal{N}_i^-(t-\tau)$ or $\#\mathcal{N}_i^+(t) \neq \#\mathcal{N}_i^+(t-\tau)$. Then $\dot{\tilde{V}}(t) < 0$ is not guaranteed. In this case, the magnitude of the difference of $\dot{\tilde{V}}(t)$ (therefore the upper bound of the tracking error) is related to α and τ: the larger α and τ, the larger the difference of $\dot{\tilde{V}}(t)$ (and of the upper bound). However, from the definitions of $\mathcal{N}_i^-$ and $\mathcal{N}_i^+$, one knows that to reach a consensus $\#\mathcal{N}_i^-$ and $\#\mathcal{N}_i^+$ must be reduced to 0 (therefore must be varying), which indicates that there are steady-state errors.

C. *Discrete-Time Implementation*

In this subsection, a discrete-time version of (1) is proposed. For $i = 1, \ldots, n$, the dynamics of the reference signals are described by $r_i(k+1) = r_i(k) + Tf_i(k)$, where T is the sampling period and k is the discrete time index, and the system is governed by $x_i(k+1) = x_i(k) + Tf_i(k) + \alpha T \sum_{j \in \mathcal{N}_i} \mathrm{sgn}[x_j(k) - x_i(k)]$ with the initial condition $\sum_{i=1}^n x_i(0) = \sum_{i=1}^n r_i(0)$. It is straightforward to obtain that $\sum_{i=1}^n x_i(k+1) = \sum_{i=1}^n x_i(k) + T\sum_{i=1}^n f_i(k)$. It follows that $|\sum_{i=1}^n x_i(k+1) - \sum_{i=1}^n r_i(k+1)| = |\sum_{i=1}^n x_i(k) - \sum_{i=1}^n r_i(k)| = |\sum_{i=1}^n x_i(0) - \sum_{i=1}^n r_i(0)| = 0$. Define $V(k) = \max_i[x_i(k) - (1/n)\sum_j x_j(k)]$. It follows that $V(k) \geq 0$ and $V(k) = 0$ if and only if $x_i(k) = (1/n)\sum_{j=1}^n x_j(k) = (1/n)\sum_{j=1}^n r_j(k)$ for all $i = 1, \ldots, n$. By using a similar analysis to that in Section IV-B, one can show that there are steady-state errors in the system which are related to α and T.

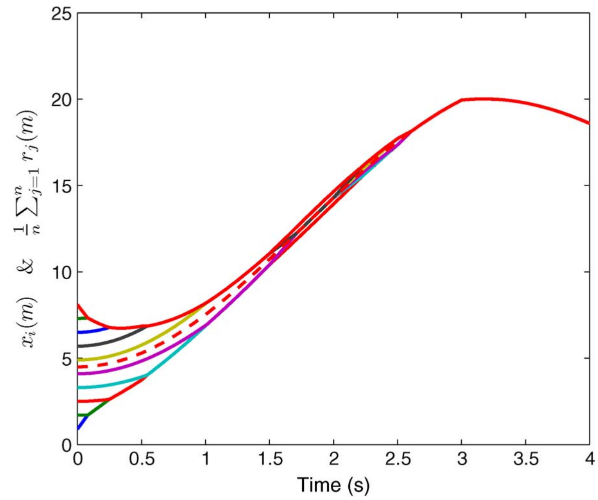

Fig. 2. State trajectories of the agents under discontinuous f_i and a switching, undirected, and connected topology. The solid lines are the agents' states. The dashed line is the average of all reference signals.

V. Simulation Examples

We show an example to verify Theorem 3. In the example, we use the four connected graphs shown in Fig. 1. The initial graph is $\mathcal{G}_a$. Then, after every 0.5 s, the network topology switches to the next one and the procedure repeats. The initial states of the agents are generated in the range [0,2] randomly. Agent i has a reference signal $r_i(t)$ satisfying that $\dot{r}_i = i \times \sin(t) + (t \bmod 3)$. The control gain α is chosen as 5. The state trajectories of the agents are shown in Fig. 2. We can see that the agents track the average of all reference signals eventually.

VI. Conclusion

We have presented in this note a simple but appealing distributed control algorithm for a team of agents to solve the average tracking problem, which could find applications in various fields including mobile sensor networks, synchronization of oscillators, distributed estimation, decision making, or optimization. In the algorithm, each agent has a reference signal or measurement whose derivative is assumed to be bounded, and updates its states based on the information received from its neighbors and its reference signal. Our analysis started with undirected connected graphs. In this part, we have shown that the average tracking problem can be solved if the control gain is properly chosen. Then the result was extended to switching graphs. We have also considered the case of directed graphs.

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A Minimum Phase Output in the Exact Tracking Problem for the Nonminimum Phase Underactuated Surface Ship

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***Abstract*—This note considers the output tracking problem for a nonminimum-phase underactuated surface ship, where the actuation is provided by a stern propeller and rudder. As tracking output, we choose the position of a point placed on the longitudinal symmetry axis, at a positive distance from the position of the ship's geometric center. We show that the exact tracking problem for a smooth reference trajectory always admits a solution for this particular output and that the state space of the zero dynamics features an invariant region that lies in an arbitrarily small neighborhood of zero, provided that the reference trajectory is such that its speed does not vary too much with respect to an assigned reference value and its second derivative is sufficiently small. We present an output tracking controller which guarantees local asymptotic stability of the origin of the tracking error dynamics. Finally, we provide bounds on the zero dynamics' state by means of a positive invariant region of a discontinuous comparison system. We do not take explicitly into account environmental disturbances but present a simulation in presence of a nonrotational current.**

***Index Terms*—Nonminimum phase systems, output tracking, ship control, stability of NL systems.**

I. Introduction

The trajectory tracking problem for the underactuated surface ship has generated great interest in the last couple of decades (see the recent book [1] with the references therein and papers [2]–[13]). This paper considers a common actuator configuration, consisting of a stern propeller and rudder. When the output is chosen at the position of the geometric center, the system is nonminimum-phase, i.e., it has unstable internal dynamics (see the discussion in [2]). Even if this actuator configuration is of great practical interest, it has not received much attention in literature.

References [3], [4], [9]–[13] consider the tracking problem for this type of underactuated ship. In particular, [3] describes how to use backstepping to develop a controller to perform trajectory tracking. It extends earlier backstepping designs (developed in [5]) explaining how to select outputs when generalized forces act on the vessel. In [4], the controller uses a state estimator to handle uncertainties on the state measurement. In these two papers, the control law is aimed at stabilizing the ship to a known bounded reference full-state trajectory. Finding this reference trajectory is not trivial since the system is nonminimum phase: in general, it requires noncausal dynamic inversion.

A different approach consists in redefining the tracking output for removing the nonminimum-phase behavior. This method was first applied in [14] for the dynamic positioning of a ship in changing environmental conditions. With respect to the output tracking problem, it was first introduced by Lindegaard [9] with the notion of *virtual reference point*. This idea is developed in [10] and [11], which present a control law that guarantees asymptotic zero position and speed error in presence of a constant unknown disturbance. They also provide a proof of

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