Prescribed-time tracking with guaranteed performance for a class of self-switching systems under unknown control directions

Yujuan Wang*, Yongduan Song, *Fellow, IEEE*, Xiang Chen

***Abstract*—This paper investigates the problem of prescribed-time tracking control for a class of self-switching systems subject to non-vanishing/non-parametric uncertainties and unknown control directions. Due to the existence of the unknown inherent nonlinear dynamics and the undetectable actuation faults, the resultant control gain of the system becomes unknown and time-varying, making the control impact on the system uncertain and the prescribed-time control synthesis nontrivial. The underlying problem becomes further complex as the switching is arbitrary and unknown. To circumvent the aforementioned difficulties, the following major steps are employed. Firstly, by integrating a novel time-varying feedback gain and performance function into the control synthesis, the non-vanishing uncertainties are completely rejected and the transient performance is guaranteed; Secondly, to facilitate the stability analysis under arbitrarily switching, the concept of constraining function is introduced and incorporated into a skillfully chosen common Lyapunov function; Thirdly, to deal with the uncertain control gain, a new Nussbaum related lemma is derived. The proposed control is shown to be capable of ensuring that the tracking error not only evolves within the prescribed bound during all the operation time but also converges to zero at the rate of convergence that can be pre-assigned as fast as desired, in the presence of self-switching dynamics and unknown control directions. Both theoretical analysis and numerical simulation confirm the effectiveness of the proposed method.**

***Index Terms*—Self-switching systems; Non-vanishing uncertainties; Prescribed-time control; Unknown control directions.**

I. Introduction

Motivation. Most social, biological and physical systems may undergo sudden or gradual variation in dynamics and actuation directions, either consciously or unconsciously due to the influence of external environment and/or operational conditions. Such variations might also stem from natural self-development or various mission requirements. For instance, for a system after multiple repetitive operation, unknown changes may occur in the structure and control direction in the system due to aging or undetectable component failures, or other short circuit reasons. Such unknown changes are frequently encountered in unexpected fault systems, biological systems in self-developing or self-growing. To address the tracking control problem for such systems, in this work we explicitly consider the scenario that involves unknown self-switching (the switching time is unknown and unobtainable) dynamics and unknown actuation directions. The nature of self-switching would literally lead to quite different form of system nonlinearities within different time periods. Up to now, the territory of designating tracking control for such systems remains underexplored [1] and results capable of achieving prescribed-time tracking are rather scarce and incomplete in literature, which is the primary motivation of this work.

As tracking is the most common goal in control engineering, the method that achieves tracking in prescribed time with guaranteed transient performance is of particular interest. The prescribed-time control has entices much attention, resulting in several interesting works including [2]- [6], and so on. Considerable amount of research has been conducted on tracking control with guaranteed performance, see, for instance, [7]- [10], and so on. The work [7] provides a means to improve the transient performance of adaptive systems in terms of L_2 and L_∞ norms of the tracking error, while the transient performance of the adaptive system cannot be adjusted through changing controller design parameters. The pioneering work [8] and [9] establish a promising PPB (prescribed performance bounds) control method for addressing the transient and steady-state performance through two decay boundary functions. The work [10] investigates the prescribed performance control (PPC) problem for a family of strict-feedback nonlinear systems with actuator faults, component faults and unknown control directions. However, these methods can only guarantee the tracking error evolving within the prescribed bound rather than converging to zero. By using the concept of funnel control [11], the work [12] proposes a control method for nonlinear systems with unknown control directions and nonparametric uncertainties, which however only achieves stabilization (rather than tracking) for vanishing uncertainties in the assumed form. How to design a control scheme capable of guaranteeing prescribed performance and at the same time ensuring convergence with zero-error accuracy and in prescribed time for systems subject to non-vanishing uncertainties represents the second motivation of this work.

In dealing with unknown control directions [13], there have been some efforts based on Nussbaum function [10], [14]-

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Y. J. Wang, Y. D. Song and X. Chen are with the State Key Laboratory of Power Transmission Equipment & System Security and New Technology, School of Automation, Chongqing University, and X. Chen is also with Star Institute of Intelligent Systems (SIIS), Chongqing, 400044, China (e-mail: yjwang66@cqu.edu.cn; ydsong@cqu.edu.cn; chenx@cqu.edu.cn).

[15]. Although regular Nassbuam function based method has proven effective in addressing the issue of unknown control directions, this method is not directly applicable for dealing with the nonvanishing uncertainties and for achieving zero-error tracking at prescribed rate of convergence due to the utilization of time-varying feedback control, calling for a new Nussbaum related lemma to be established. This is the third motivation of our work.

Contributions of the paper. Motivated by tactical missile guidance [2], [16], safety-critical systems and other applications in which there exists a short, finite amount of time remaining to achieve control objectives (i.e., missile interception, emergency braking control, etc, where the whole process must be accomplished within short period of time) and that time is known to within some small uncertainty, in this work we consider the prescribed-time control with guaranteed performance for a class of nonlinear systems with self-switching dynamics, non-parametric/non-vanishing uncertainties and unknown actuation directions. The main contributions of the work can be summarized as follows: 1) A time-varying scaling function is introduced to construct the state transformed variable such that prescribed performance with zero steady-state error convergence is guaranteed, in which the convergence rate can be pre-assigned as fast as desired. This distinguishes itself from most existing PPC based works that only ensure the tracking error to be within the prescribed error bound [7]-[10]; 2) the concept of constraining function is introduced and incorporated into a skillfully chosen common Lyapunov function to facilitate the stability analysis under arbitrarily switching; 3) Owing to the unknown control directions, the existing PPC approaches that are suitable for the situations where the control directions are known a priori [17]-[18] are inapplicable directly. A new Nussbaum lemma based on time-varying function is established to allow the guranteed tracking performance under the situation of unknown control directions.

Notation: Throughout this paper, $\Re$ denotes the set of real numbers; By C^{∞}, we denote the class of functions that have continuous derivatives of order ∞; $\bullet^{(j)}$ denotes the jth derivative of $\bullet$. $\begin{pmatrix} i \\ j \end{pmatrix}$ denotes the combination, which is computed as $\begin{pmatrix} i \\ j \end{pmatrix} = \frac{i!}{j!(i-j)!}$.

II. Problem Formulation

Consider the following dynamic system undergoing structural variation:

$$\begin{aligned} \dot{x}_q =& x_{q+1}, \qquad\qquad q = 1, 2, \cdots, n-1, \\ \dot{x}_n =& \sum_{k=1}^{N} \xi_k(t) g_k(x,t) u(x,t) + \sum_{m=1}^{M} \sigma_m(t) f_{dm}(x,t), \end{aligned} \tag{1}$$

where $x = [x_1 \ \dots \ x_n]^T \in \Re^n$ is the state vector; $u \in \Re$ is the control input; k and m correspond to the subscript of switching signals of $g_k(x,t)$ and $f_{dm}(x,t)$ respectively, which are picked in such a way that there are finite number of switchings in finite time and $k \in U_1 = \{1, 2, \dots, N\}$ and $m \in U_2 = \{1, 2, \dots, M\}$; ξ_k and σ_m are the switching signals defined, respectively, by $\xi_k(t) = \begin{cases} 1, & t \in [T_{k-1}, T_k) \\ 0, & \text{otherwise} \end{cases}$, $\sigma_m(t) = \begin{cases} 1, & t \in [t_{m-1}, t_m) \\ 0, & \text{otherwise} \end{cases}$, where T_k ($k \in U_1$) and t_m ($m \in U_2$) are two sequences of time instants that $g_k(x,t)$ and $f_{dm}(x,t)$ restructure their patterns respectively [20], with $T_0 = t_0$ and $T_N = t_M = t_f < \infty$ (here the step function can be replaced by ramp function to denote the gradual variation); $g_k(x,t)$ ($k \in U_1$) represents the control gain that is unknown and time-varying on $[T_{k-1}, T_k)$; $f_{dm}(x,t)$ ($m \in U_2$) denotes the lumped uncertainty on $[t_{m-1}, t_m)$, which accounts for all the other modeling uncertainties and external disturbances.

Assumption 1: Suppose that the state of system (1) does not jump at the switching time instants, i.e., the solution $x(t)$ is continuous everywhere.

Assumption 2 (global controllability): The signs of $g_k(x,t)$ ($k \in U_1$) are unknown yet uniform with respect to k, and there exist unknown continuous functions $\underline{g}_k(x)$ and $\bar{g}_k(x)$ such that for any $x \in \Re^n$ and $t \in [t_0, t_f)$, $\underline{g}_k(x) \leq g_k(x,t) \leq \bar{g}_k(x)$ with $0 < \underline{g}_k(x) < \bar{g}_k(x)$ or $0 > \bar{g}_k(x) > \underline{g}_k(x)$, implying that there exist some unknown finite constants $\underline{g}$ and $\bar{g}$ with $0 < \underline{g} < \bar{g}$ or $0 > \bar{g} > \underline{g}$ such that for all $k \in U_1$,

$$\underline{g} \leq g_k(x,t) \leq \bar{g}, \ \forall x \in \Omega_1 \subset \Re^n, \ \forall t \in [t_0, t_f) \tag{2}$$

where Ω_1 denotes a compact set.

Assumption 3: There exist unknown, continuous and positive functions such that, $|f_{dm}(x,t)| \leq \bar{f}_{dm}(x)$ ($\forall x \in \Re^n$, $\forall t \in [t_0, t_f)$), meaning that there exists an unknown finite positive constant $\bar{f}_d$ such that for all $m \in U_2$,

$$|f_{dm}(x,t)| \leq \bar{f}_d, \ \forall x \in \Omega_2 \subset \Re^n \tag{3}$$

where Ω_2 is a compact set.

Assumption 4: Suppose that the desired state trajectory is $x^* = [x_1^*, \dots, x_1^{*(n-1)}]^T \in \Re^n$. For all $t \in [t_0, t_f)$, the desired trajectory $x_1^*(t)$ and its qth ($q = 1, \dots, n-1$) order derivatives $x_1^{*(q)}$ are known and bounded, and $x_1^{*(n)}$ can be unknown but bounded by an unknown finite constant $\bar{x}_d$, that is, $|x_1^{*(n)}| \leq \bar{x}_d < \infty$.

Define the full state tracking error as $\epsilon = [\epsilon_1 \ \dots \ \epsilon_n]^T \in \Re^n$, with $\epsilon_q = x_q - x^{*(q-1)}$ being the qth ($q = 1, \dots, n$) component of the tracking error ϵ. The objective of this work is to design a control scheme for system (1) to achieve zero steady-state error tracking with prescribed performance in spite of the non-vanishing uncertainties and unknown control directions. Namely, the control goals are: 1) the prescribed tracking performance is guaranteed during the entire system operation process; 2) the tracking error ϵ is ensured to converge to zero with the rate of convergence user-assignable as fast as desired; and 3) all the internal signals in the closed-loop system are ensured to be bounded.

III. Controller Design and Stability Analysis

A. Controller Design

To develop a feasible control scheme for system (1) with order up to n ($n \geq 1$), we need to introduce the following classes of time-varying function K^+, K^{**}, $\mathcal{N}$, and $\mu(t)$ into the control input, such that the prescribed-time tracking control objective is achieved with guaranteed performance.

Definition 1: [19] The function $\nu : [t_0, t_f) \to \Re$ belongs to class K^+ if ν is non-decreasing C^∞ function with $\nu(t_0) = 1$ and $\lim_{t\to t_f^-} \nu(t) = +\infty$.

Definition 2: [19] The function $\nu : [t_0, t_f) \to \Re$ belongs to class K^{**} if $\nu \in K^+$ and for any finite $n \in Z_+$, it holds that $\frac{\nu^{(n-i-j+1)}}{\nu}\left(\frac{1}{\nu}\right)^{(i)} \in L_\infty$ for all $i = 0, 1, \ldots, n-j$ and $j = 1, \ldots, n$, and that $(1/\nu)^{(k)}\nu^\alpha \in L_\infty$ for all $k = 0, 1, \ldots, n$ and $0 < \alpha < 1$.

It can be shown that $\nu(t) = \frac{(t_f - t_0)^{n+m}}{(t_f - t)^{n+m}} \in K^{**}$ defined on $[t_0, t_f)$, with positive integers m, n [19], [2].

Definition 3: [21] A function $N(\cdot)$ is called a Nussbaum-type function $\mathcal{N}$ if $N(\cdot)$ bears the following properties: $\lim_{s\to\infty} \sup \frac{1}{s}\int_0^s N(\tau)d\tau = +\infty$, $\lim_{s\to\infty} \inf \frac{1}{s}\int_0^s N(\tau)d\tau = -\infty$.

Using a function $\varsigma(t) \in K^+$, we construct the performance function $\mu(t)$ as,

$$\mu(t) = (\mu_0 - \mu_\infty)\varsigma^{-1}(t) + \mu_\infty, \tag{4}$$

where $0 < \mu_\infty < \mu_0 < \infty$ are constants. To make the tracking rate be as fast as desired for system (1), we perform the following state transformation upon using $\nu \in K^{**}$,

$$w_1(t) = \nu(t)\epsilon_1(t), \tag{5}$$

$$w_q(t) = dw_{q-1}(t)/dt, \quad q = 2, \ldots, n+1, \tag{6}$$

and introduce a variable z as

$$z = w_n + \lambda_{n-1}w_{n-1} + \cdots + \lambda_1 w_1 = w_n + \Lambda_{n-1}^T r_1 \tag{7}$$

with $r_1 = [w_1 \ \ldots \ w_{n-1}]^T \in \Re^{n-1}$, $\Lambda_{n-1} = [\lambda_1 \ \ldots \ \lambda_{n-1}]^T \in \Re^{n-1}$, where Λ_{n-1} is an appropriately chosen coefficient vector so that the polynomial $s^{n-1} + \lambda_{n-1}s^{n-2} + \cdots + \lambda_1$ is Hurwitz. From the definition of z and r_1, it is easy to see that

$$\dot{r}_1 = Ar_1 + e_{n-1}z \tag{8}$$

with $A = \begin{bmatrix} 0 & I_{n-2} \\ -\lambda_1 & -\lambda_2 \ \ldots \ -\lambda_{n-1} \end{bmatrix}$ and $e_{n-1} = [0_{n-2}^T \ 1]^T$. According to Lemma 2.1 in [22], we then establish the relation between the norm of r_1 and supremum norm of z. Define $|z|_{[t_0,t]} = \sup_{\tau\in[t_0,t]} |z(\tau)|$, then there exist some finite positive constants c_0 and λ_0 such that

$$\|r_1(t)\| \le c_0 e^{-\lambda_0(t-t_0)}\|r_1(t_0)\| + \frac{c_0}{\lambda_0}|z|_{[t_0,t]}. \tag{9}$$

The derivative of the new state z is

$$\dot{z} = \dot{w}_n + \Lambda_{n-1}^T \dot{r}_1 = \dot{w}_n + \Lambda_{n-1}^T Dw, \tag{10}$$

with $D = \left[0_{(n-1)\times 1}, \ I_{n-1}\right]$ and $w = [w_1 \ \ldots \ w_n]^T \in \Re^n$. By noting that, for $q = 0, 1, \cdots, n$,

$$w_{q+1} = (\nu\epsilon_1)^{(q)} = \sum_{k=0}^{q} \begin{pmatrix} q \\ k \end{pmatrix} \nu^{(k)}\epsilon_{q+1-k}, \tag{11}$$

from which we then get that,

$$\begin{aligned} \dot{w}_n =& w_{n+1} = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \nu^{(k)}\epsilon_{n+1-k} \\ =& \nu\dot{\epsilon}_n + \sum_{k=1}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \nu^{(k)}\nu^{(k)}\epsilon_{n+1-k}. \end{aligned} \tag{12}$$

Introducing $\eta = \nu^{-1}$ and (12) into (10) yields

$$\begin{aligned} \dot{z} =& \nu\dot{\epsilon}_n + \sum_{k=1}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \nu^{(k)}\epsilon_{n+1-k} + \Lambda_{n-1}^T Dw \\ =& \nu\Bigg(\sum_{k=1}^{N} \xi_k g_k u + \sum_{m=1}^{M} \sigma_m f_{dm} - x^{*(n)} + l_0(\eta)w \\ & + \eta\Lambda_{n-1}^T Dw\Bigg) \end{aligned} \tag{13}$$

where the fact that $\eta\sum_{k=1}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \nu^{(k)}\epsilon_{n+1-k} = l_0(\eta)w$ has been used, with $l_0(\eta) = [l_{0,1} \ l_{0,2} \ \ldots \ l_{0,n}]$, and for $j = 1, 2, \ldots, n$, $l_{0,j}(\eta) = \sum_{i=0}^{n-j} \begin{pmatrix} n \\ n-i-j+1 \end{pmatrix} \begin{pmatrix} i+j-1 \\ i \end{pmatrix} \frac{\nu^{(n-i-j+1)}}{\nu}\eta^{(i)}$. Further, it is worth mentioning that $l_0(\eta)$ is finite, which is ensured by the fact that $\nu \in K^{**}$. This can be proven by using the method similar to Lemma 9 in [19].

The following lemma establishes the relation between the original error ϵ and the transformed error w. The boundedness of matrix $Q(\eta)$ is guaranteed by $\nu(t) \in K^{**}$.

Lemma 1: [19] The scaling transformation $\epsilon(t) \mapsto w(t)$ is given by, $w = P(\nu)\epsilon$, where the matrix $P(\nu)$ is a lower triangular matrix having elements $\{p_{ij}\}$ given by $p_{ij}(\nu) = \begin{pmatrix} i-1 \\ i-j \end{pmatrix} \nu^{(i-j)}$ $(1 \le j \le i \le n)$. The inverse transformation $w(t) \mapsto \epsilon(t)$ is given by, $\epsilon = \eta^\alpha Q(\eta)w$, where $\eta = 1/\nu$, $0 < \alpha < 1$, and $Q(\eta)$ is a lower triangular matrix having elements $\{q_{ij}\}$ given by $q_{ij}(\eta) = \begin{pmatrix} i-1 \\ i-j \end{pmatrix} \eta^{(i-j)}\nu^\alpha$ $(1 \le j \le i \le n)$. Furthermore, $\bar{q} = \sup_{\eta\in(0,1]} \|Q(\eta)\|$ is finite ensured by $\nu \in K^{**}$.

Inspired by the work of [8], [10] and [17], we construct the following constraining function $H(t)$ as

$$H(t) = \frac{\text{arctanh}\left(z(t)/\mu(t)\right)}{1 - \left(z(t)/\mu(t)\right)^2}. \tag{14}$$

It is worth mentioning that the role of the arctanh transformation in (14) is similar to the logarithm one in [8], [17] and the $\tan$ one in [10], and neither one property is necessarily preferable over the other.

To cope with time-varying control gain with unknown control direction, we employ the following Nussbaum-type function $N(\cdot) \in \mathcal{N}$,

$$N(\zeta) = e^{\zeta^2}\cos\left(\frac{\pi}{2}\zeta\right). \tag{15}$$

Now by using $H(t)$ and $N(\zeta)$, we build the controller as

$$u(t) = \lambda N(\zeta)H(t) \tag{16}$$

where λ is a finite positive constant, and

$$\dot{\zeta} = \begin{cases} 0, & \text{if} \quad |z(t)| < \beta(t), \\ H^2(t), & \text{if} \quad |z(t)| \geq \beta(t). \end{cases} \tag{17}$$

with

$$\beta(t) =(\mu_0 - \mu_\infty)\varsigma^{-1}(t) + \beta_\infty, \tag{18}$$

where $0 < \beta_\infty < \mu_\infty$ is a finite constant. The proposed control scheme is summarized into the following control diagram.

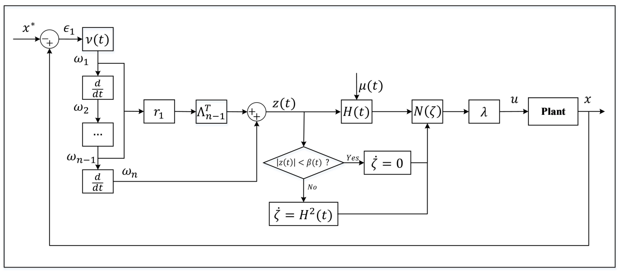


Fig. 1. The control diagram.

Remark 1: It is worth noting that in this work we employ a scaling of initial errors by a time-varying function $\nu(t)$ that belongs to K^{**} defined as in Definition 2. The intuition behind such a function is the formulation of those conditions that should be satisfied for $\nu(t) \in K^{**}$, which guarantees the boundedness of the transformed variable w and thus avoids the singularity issues in the controller. As shown shortly, it is such scaling functions $\nu(t)$ defined in Definition 2 and $\mu(t)$ given in (4) that together allow for achieving the zero-error convergence with prescribed decay rate and ensuring the prescribed performance bound.

Remark 2: It is worth mentioning that both $\nu(t)$ and $\mu(t)$ have a variety of choices (forms) and are at the designer's disposal. With the proposed control method, one can achieve prescribed time tracking with various performance bounds, by simply specifying $\nu(t), \mu_0, \mu_\infty$ and $\varsigma(t)$ properly, which is deemed a useful and favorable feature of design variety and flexibility associated with the proposed method.

B. Stability Analysis

The following lemma establishes a crucial result that extends the regular Nussbaum related lemma [14] and allows the challenging case involving time-varying function $\nu(t)$ to be addressed.

Lemma 2: Let $\zeta(t)$ and $V(t)$ be two smooth functions defined on $[t_0, t_f)$ with $V(\cdot) \geq 0$, $\forall t \in [t_0, t_f)$, and $N(\zeta)$ be an even smooth Nussbaum-type function and $\nu(t)$ be a smooth time-varying scaling function satisfying $\nu(t) \in K^+$. If for $\forall t \in [t_0, t_f)$, it holds that

$$V(t) \leq C_0 + \int_{t_0}^{t} e^{-\int_\tau^t \nu(s)ds} \left(g(x,\tau)N(\zeta) + 1\right) \dot{\zeta}\nu(\tau)d\tau \tag{19}$$

where C_0 denotes a positive finite constant, $g(t)$ is a variable satisfying $g(t) \in [\underline{l}, \bar{l}]$ with $0 < \underline{l} < \bar{l}$ or $0 > \bar{l} > \underline{l}$ ($\underline{l}$ and $\bar{l}$ denote finite constants), then $\zeta(t)$, $N(\zeta)$ and $V(t)$ remain bounded on $[t_0, t_f)$.

Proof: The proof is given in Appendix A. ■

Lemma 3: Define $V(t) = \frac{1}{2}\text{arctanh}^2\left(\frac{z(t)}{\mu(t)}\right)$. If $|z(t)| < \mu(t)$ and $V(t) \in L_\infty$ hold, then there exists a positive constant $\underline{\mu}_c \leq \mu_\infty$ such that $|z(t)| \leq \mu(t) - \underline{\mu}_c$.

Proof: Since $V(t) \in L_\infty$, we let $C = \sup_{t\in[t_0,t_f)} \left|\text{arctanh}\left(\frac{z(t)}{\mu(t)}\right)\right|$, which is finite. Note that arctanh$(\cdot)$ is a monotonic increasing function, it follows that $C \geq \text{arctanh}\left|\frac{z(t)}{\mu(t)}\right|$, and thus

$$\left|\frac{z(t)}{\mu(t)}\right| \leq \tanh(C). \tag{20}$$

Let $c = 1 - \tanh(C)$, from (20) it follows that

$$|z(t)| \leq \mu(t)\tanh(C) = \mu(t)(1 - c). \tag{21}$$

Note that $\mu(t) = (\mu_0 - \mu_\infty)\varsigma^{-1}(t) + \mu_\infty \geq \mu_\infty$, we then arrive at

$$|z(t)| \leq \mu(t) - \mu(t)c \leq \mu(t) - \mu_\infty c = \mu(t) - \underline{\mu}_c \tag{22}$$

where $\underline{\mu}_c = \mu_\infty c \leq \mu_\infty$ $(0 < c = 1 - \tanh(C) \leq 1)$. ■

Now we are ready to state the following result on tracking control of system (1). In what follows, we prove that the filtered variable $z(t)$ is ensured to be confined within the performance bound $\mu(t)$. Therefore the tracking error $\epsilon(t)$ is naturally guaranteed to evolve within a performance bound related to $\mu(t)$, which can be well shaped by adjusting $\lambda_1, \ldots, \lambda_{n-1}$ in Λ_{n-1} and μ_0, μ_∞ in $\mu(t)$ properly.

Theorem 1: Consider system (1) under Assumptions 1-4. If $|z(t_0)| < \mu(t_0)$ holds, then the proposed control law (16) guarantees that,

i) $|z(t)| < \mu(t)$ for all $t \in [t_0, t_f)$;

ii) the tracking error $\epsilon(t)$ converges to zero with a prescribed rate that can be user-specified as fast as desired. More specifically,

$$\|\epsilon(t)\| \leq \nu^{-\alpha}\bar{q}\|B^{-1}\|\left[c_0 e^{-\lambda_0(t-t_0)}\|r_1(t_0)\| + \frac{c_0}{\lambda_0}\mu_0 + \mu(t)\right], \tag{23}$$

for all $t \in [t_0, t_f)$, where $0 < \alpha < 1$ is given as in Lemma 1 and $B = \begin{bmatrix} I_{n-1} & 0_{n-1} \\ \lambda_1 \quad \ldots & \lambda_{n-1} \ 1 \end{bmatrix}$, implying that $\epsilon(t) \to 0$ as $t \to t_f^-$ with the convergence rate $\nu^{-\alpha}$ that can be pre-assigned as fast as desired;

iii) all the states and control input signal in the closed-loop system remain continuous and bounded.

Proof: Introducing the control input $u = \lambda N(\zeta)H(t)$ into the transformed system model (13) yields

$$\begin{aligned}\dot{z} =& \nu\Bigg(\sum_{k=1}^{N} \xi_k g_k \lambda N(\zeta)H(t) + \sum_{m=1}^{M}\sigma_m f_{dm} - x^{*(n)} \\ &+ l_0(\eta)w + \eta\Lambda_{n-1}^T Dw\Bigg).\end{aligned} \tag{24}$$

Step 1: We prove that $z(t)$ is guaranteed in a prescribed performance $|z(t)| < \mu(t)$ for all $t \in [t_0, t_f)$.

By seeking a contradiction, suppose that there exists a finite time $t_0 < t_s \leq t_f$ such that $|z(t_s)| \geq \mu(t_s)$. We now prove that $z(t)$ is continuous for all $t \in [t_0, t_f)$. The claimed property of z follows from (7) and (11), with the continuousness of x and x^* following from Assumption 1 and Assumption 4, and the continuousness of $\nu(t)$ and its up to n-th order derivatives established in Definition 2. Since $z(t)$ is continuous and $|z(t_0)| < \mu(t_0)$, we have

$$|z(t)| < \mu(t), \quad \forall t < t_s, \tag{25}$$

and there exists a sequence τ_k which approaches t_s from left side with $\lim_{k\to\infty} \tau_k = t_s$ such that

$$\lim_{k\to\infty} |z(\tau_k)| = \mu(t_s). \tag{26}$$

Notice that the following discussions and analysis from (27) to (39) are based on the time interval $[t_0, t_s)$.

We first prove that $g_k(x,t) \in L_\infty$ and $f_{dm}(x,t) \in L_\infty$ for all $k \in U_1$ and $m \in U_2$. To this end, we introduce a variable $\bar{w} = \left[r_1^T, z\right]^T$ to derive the boundedness of $\epsilon(t)$. Note that

$$\bar{w} = [r_1^T, z]^T = \begin{bmatrix} I_{n-1} & 0_{n-1} \\ \lambda_1 \quad \dots \quad \lambda_{n-1} & 1 \end{bmatrix} w = Bw, \tag{27}$$

which then implies $w = B^{-1}\bar{w}$. We see from Lemma 1 that, $\epsilon = \eta^\alpha Q(\eta) w$, which then implies $\epsilon = \eta^\alpha Q(\eta) B^{-1}\bar{w}$. By noting that $0 < \eta < 1$ and $Q(\eta)$ is bounded according to Lemma 1, the boundedness of ϵ can thus be derived from the boundedness of $\bar{w}$. We now analyze the boundedness of $\bar{w}$. By combining (27), (9) and (25), we arrive at

$$\begin{aligned}\|\bar{w}\| \leq& \|r_1(t)\| + |z(t)| \\ \leq& c_0 e^{-\lambda_0(t-t_0)}\|r_1(t_0)\| + \frac{c_0}{\lambda_0}|z|_{[t_0,t]} + |z(t)| \\ \leq& c_0 e^{-\lambda_0(t-t_0)}\|r_1(t_0)\| + \frac{c_0}{\lambda_0}\mu_0 + \mu(t) \in L_\infty, \end{aligned} \tag{28}$$

which then implies that $\epsilon(t) = \eta^\alpha Q(\eta) B^{-1}\bar{w} \in L_\infty$. Since x^* is bounded according to Assumption 4, we then derive that $x(t) = \epsilon(t) + x^* \in L_\infty$, which then implies that $g_k(x,t) \in L_\infty$ and $f_{dm}(x,t) \in L_\infty$ for all $k \in U_1$ and $m \in U_2$ from Assumptions 2-3.

To move on, we need to consider two cases.

Case i: When $|z(t)| < \beta(t)$. It follows from the definition of $\mu(t)$ in (4) and $\beta(t)$ in (18) that

$$|z(t)| < \beta(t) = \mu(t) - (\mu_\infty - \beta_\infty) < \mu(t). \tag{29}$$

In addition, it is not hard to see that $\zeta(t) \in L_\infty$ from (17), which further implies that $N(\zeta) \in L_\infty$ from the definition of $N(\zeta)$ given in (15).

Case ii: When $|z(t)| \geq \beta(t)$. We choose

$$V(t) = \frac{1}{2}\text{arctanh}^2\left(\frac{z(t)}{\mu(t)}\right), \quad \forall t \in [t_0, t_s). \tag{30}$$

Differentiating (30) yields

$$\begin{aligned}\dot{V}(t) =& \text{arctanh}\left(\frac{z(t)}{\mu(t)}\right) \cdot \frac{1}{1 - (z(t)/\mu(t))^2} \cdot \frac{\dot{z}\mu(t) - z\dot{\mu}(t)}{\mu^2(t)} \\ =& \frac{H(t)}{\mu(t)}\left(\dot{z} - \frac{\dot{\mu}(t)}{\mu(t)}z\right) \\ =& \frac{H(t)}{\mu(t)}\Bigg[\nu\Bigg(\sum_{k=1}^N \xi_k g_k \lambda N(\zeta) H(t) + \sum_{m=1}^M \sigma_m f_{dm} \\ & - x^{*(n)} + l_0(\eta) w + \eta\Lambda_{n-1}^T D w\Bigg) - \frac{\dot{\mu}(t)}{\mu(t)} z\Bigg] \\ =& \nu g_z(x,t) N(\zeta) H^2(t) + \nu F_z(t) H(t) \end{aligned} \tag{31}$$

with $g_z(x,t) = \lambda \sum_{k=1}^N \xi_k g_k / \mu(t)$ and $F_z(t) = \left(\sum_{m=1}^M \sigma_m f_{dm} - x^{*(n)} + l_0(\eta) w + \eta \Lambda_{n-1}^T D w\right)/\mu(t) - (z\dot{\mu}) / \left(\nu\mu^2\right)$. In the following, we prove that there exist unknown finite constants $\underline{b}$, $\bar{b}$ and $\bar{F}_f$ such that

$$\underline{b} \leq g_z(x,t) \leq \bar{b}, \quad |F_z(t)| \leq \bar{F}_f, \tag{32}$$

with $0 < \underline{b} < \bar{b}$ or $0 > \bar{b} > \underline{b}$. Note that $g_k(x,t) \in L_\infty$ for all $k \in U_1$ and the signs of $g_k(x,t) \in L_\infty$ ($k \in U_1$) are uniform w. r. t. k from Assumption 2, which, together with the fact that $\mu_0^{-1} \leq \mu^{-1} \leq \mu_\infty^{-1}$, implies the claimed property of $g_z(x,t)$ in (32). Note that $f_{dm} \in L_\infty$ for all $m \in U_2$, $x^{*(n)}$ is bounded from Assumption 4, $l_0(\eta)$ is bounded guaranteed by $\nu \in K^{**}$, $w = B^{-1}\bar{w}$ is bounded, and $\frac{\dot{\mu}}{\mu^2}$ is bounded by the definition of $\mu(t)$, we then derive that $F_z(t)$ is bounded.

Notice that

$$F_z(t)H(t) \leq \frac{1}{2}F_z^2(t) + \frac{1}{2}H^2(t) \leq \bar{F}_z + \frac{1}{2}H^2(t), \tag{33}$$

with $\bar{F}_z = \frac{1}{2}\bar{F}_f^2$. Substituting (33) into (31) yields

$$\begin{aligned}\dot{V}(t) \leq& \nu g_z(t) N(\zeta) H^2(t) + \nu \bar{F}_z + \frac{1}{2}\nu H^2(t) \\ =& -\frac{1}{2}\nu H^2(t) + \nu\bar{F}_z + \nu\left[g_z(t)N(\zeta) + 1\right] H^2(t) \\ \leq& -\nu V(t) + \nu \bar{F}_z + \nu\left[g_z(t)N(\zeta) + 1\right]\dot{\zeta}. \end{aligned} \tag{34}$$

Solving the differential inequality (34) leads to

$$\begin{aligned}V(t) \leq& e^{-\int_{t_0}^t \nu(\tau)d\tau} V(t_0) + \int_{t_0}^t e^{-\int_\tau^t \nu(s)ds}\bar{F}_z \nu(\tau) d\tau \\ & + \int_{t_0}^t e^{-\int_\tau^t \nu(s)ds}\left[g_z(t)N(\zeta) + 1\right]\dot{\zeta}\nu(\tau)d\tau. \end{aligned} \tag{35}$$

Note that

$$\begin{aligned}&\int_{t_0}^t e^{-\int_\tau^t \nu(s)ds}\bar{F}_z\nu(\tau)d\tau \\ =& \bar{F}_z e^{-\int_{t_0}^t \nu(s)ds}\int_{t_0}^t e^{\int_{t_0}^\tau \nu(s)ds} d\int_{t_0}^\tau \nu(s)ds \\ =& \bar{F}_z e^{-\int_{t_0}^t \nu(s)ds} \cdot e^{\int_{t_0}^\tau \nu(s)ds}\Big|_{t_0}^t \leq \bar{F}_z. \end{aligned} \tag{36}$$

By inserting (36) into (35), we arrive at

$$\begin{aligned}V(t) \leq& V(t_0) + \bar{F}_z \\ & + \int_{t_0}^t e^{-\int_\tau^t \nu(s)ds}\left[g_z(t)N(\zeta) + 1\right]\dot{\zeta}\nu(\tau)d\tau. \end{aligned} \tag{37}$$

It then follows from Lemma 2 that $\zeta \in L_\infty$, $N(\zeta) \in L_\infty$ and $V(t) \in L_\infty$ on $[t_0, t_s)$. According to Lemma 3, there exists a finite positive constant $0 < \underline{\mu}_c < \mu_\infty$ such that

$$|z(t)| \leq \mu(t) - \underline{\mu}_c < \mu(t). \tag{38}$$

From the analysis on the two cases, we can conclude by combining (29) and (38) that, there exists a finite positive constant $\underline{\mu}_z = \min\{\underline{\mu}_c, \mu_\infty - \beta_\infty\}$ such that

$$|z(t)| \leq \mu(t) - \underline{\mu}_z < \mu(t), \quad \forall t \in [t_0, t_s), \tag{39}$$

which clearly contradicts (26). Therefore, we have

$$|z(t)| < \mu(t), \quad \forall t \in [t_0, t_f). \tag{40}$$

According to (40), it is not hard to show that there exists a finite positive constant $0 < \underline{\mu} < \mu_\infty$ such that

$$|z(t)| \leq \mu(t) - \underline{\mu} < \mu(t), \quad \forall t \in [t_0, t_f), \tag{41}$$

by following the same procedure as in (29)-(39).

Step 2: We show that the prescribed-time tracking is ensured and the rate of the convergence is user-assignable.

Since $|z(t)| < \mu(t)$ holds for all $t \in [t_0, t_f)$, it is easy to see that (28) also holds for all $t \in [t_0, t_f)$, and then

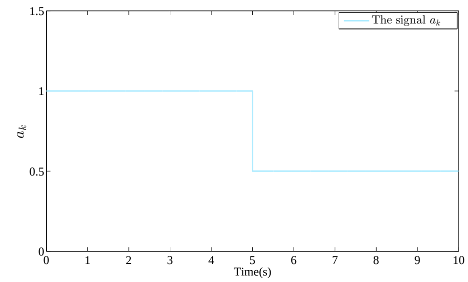
$$\begin{aligned}\|\epsilon(t)\| &= \left\|\eta^\alpha Q(\eta) B^{-1}\bar{w}\right\| \leq \eta^\alpha \bar{q}\left\|B^{-1}\right\|\|\bar{w}\| \leq \eta^\alpha \bar{q}\left\|B^{-1}\right\| \\ &\times \left[c_0 e^{-\lambda_0(t-t_0)}\|r_1(t_0)\| + \frac{c_0}{\lambda_0}\mu_0 + \mu(t)\right],\end{aligned} \tag{42}$$

for all $t \in [t_0, t_f)$, from which we see that $\epsilon(t) \to 0$ as $t \to t_f^-$ with the convergence rate $\eta^\alpha(t)$, and meanwhile, $\epsilon(t)$ is guaranteed to evolve within a performance bound related to $\mu(t)$, which can be well shaped by adjusting $\lambda_1, \ldots, \lambda_{n-1}$ in Λ_{n-1} and μ_0, μ_∞ in $\mu(t)$ properly.

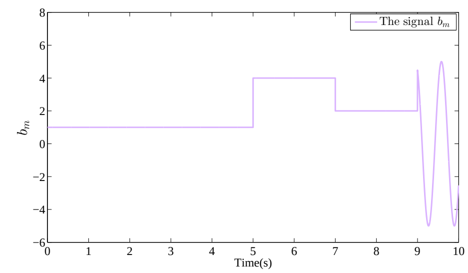
Step 3: We show that all states and control input signals in system (1) remain continuous and bounded on $[t_0, t_f)$.

From (41) and the definition of $H(t)$ in (14), we get that $H(t)$ is continuous and bounded for all $t \in [t_0, t_f)$. Since $|z(t)| < \mu(t)$ for all $t \in [t_0, t_f)$, it is not hard to show that ζ and $N(\zeta)$ are bounded on $[t_0, t_f)$ by following the same procedure as in (29)-(37). By noting that ζ and $N(\zeta)$ are continuous, we then conclude that $u(t)$ is continuous and bounded for all $t \in [t_0, t_f)$ by the continuousness and boundedness of $H(t)$ and $N(\zeta)$. In addition, (40) also ensures that $x \in L_\infty$ for all $t \in [t_0, t_f)$. Thus, all states and control input signals in the closed-loop system (1) remain continuous and bounded, which completes the proof. ■

Remark 3: Theoretically, it is guaranteed that z is bounded. But in practical implementation, a numerical problem might occur because $z(t)$ is the product of the multiplication between $\nu(t)$ and $\epsilon(t)$, which eventually becomes the multiplication of infinity and zero for z. For practical implementation of such method, one simple way is employing a deadzone on $\nu(t)$ and another way is to slightly lengthen the control horizon, both of which prevent the gains from going to infinity, while somewhat sacrificing the tracking accuracy.

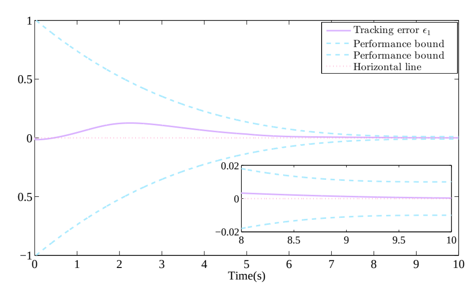


(a) a_k.

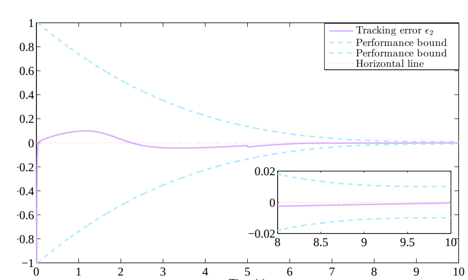


(b) b_m.

Fig. 2. The signals associated with switching.



(a) e_1.



(b) e_2.

Fig. 3. The tracking errors of system (43) with control law (16) under $\lambda = 2$.

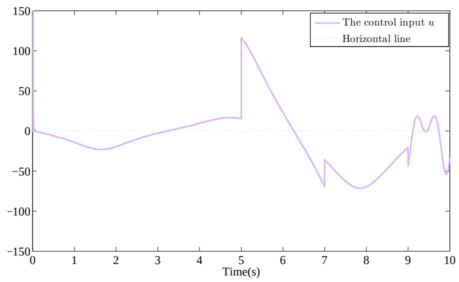


Fig. 4. The control input u of system (43).

IV. Numerical Simulations

For verification, we use the well-known inverted pendulum example modeled by,

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= \sum_{k=1}^{2}\xi_k(t)g_k(x,t)u(x,t) + \sum_{m=1}^{4}\sigma_m(t)f_{dm}(x,t), \end{aligned}\tag{43}$$

in which the switch intervals for $g_k(x,t)$ are $[T_0,T_1) = [0,5)$ and $[T_1,T_2) = [5,10)$; the switch intervals for $f_{dm}(x,t)$ are $[t_0,t_1) = [0,5)$, $[t_1,t_2) = [5,7)$, $[t_2,t_3) = [7,9)$, and $[t_3,t_4) = [9,10)$; the control gain $g_k(x,t) = \frac{\cos(x_1)/(m_c+m_p)}{l[4/3-m_p\cos^2(x_1)/(m_c+m_p)]}a_k$ $(k = 1,2)$ with $a_1 = 1$ and $a_2 = 0.5$; the lumped uncertainties $f_{dm}(x,t) = d_m + \frac{g\sin(x_1)-m_p l x_2^2\cos(x_1)\sin(x_1)/(m_c+m_p)}{l[4/3-m_p\cos^2(x_1)/(m_c+m_p)]}b_m$ $(m = 1,2,3,4)$ with $b_1 = 1$, $b_2 = 4$, $b_3 = 2$, $b_4 = 5\sin(10t)$, $d_1 = 0$, and $d_2 = d_3 = d_4 = g_2(x,t)$, $m_p = 0.1$kg denotes the mass of the pole, $m_c = 1$kg is the mass of the cart, $l = 0.5$m denotes the half length of the pole, $g = 9.8m/s^2$ stands for the gravity coefficient. It should be stated that the patten of $g_k(x,t)$ and $f_{dm}(x,t)$ switch due to, possibly, component malfunction or actuation fault, and the control direction (the signum of $g_k(x,t)$) is unknown. It is seen that the simulation model (43) takes the form of (1) with $n = 2$, $N = 2$, $M = 4$.

The initial conditions are $t_0 = 0$, $x_1(t_0) = x_2(t_0) = 0$ and $\zeta(t_0) = 1$. The reference signal is $x_1^* = \sin(t)$ and $x_2^*(t) = \cos(t)$. In the simulation, the MATLAB is used and controller (16) is applied, with the design parameters: $\lambda = 2$, $\lambda_1 = 0.1$, $\mu(t) = (1.01-0.01)\varsigma^{-1}(t) + 0.01$, $\beta(t) = (1.01-0.01)\varsigma^{-1}(t) + 0.0085$, with $\varsigma^{-1}(t) = (1-(t/10))^3$, and $\nu(t) = (10/(10-t))^3$.

The signals associated with switching a_k $(k = 1,2)$ and b_m $(m = 1,2,3,4)$ are shown in Fig. 2. The tracking results are represented in Fig.3, where Fig.3(a) and Fig.3(b) are the tracking errors for x_1 and x_2, i.e., $\epsilon_1 = x_1 - x^*$ and $\epsilon_2 = x_2 - \dot{x}^*$, respectively. The control input signal is shown in Fig. 4. It is seen from Fig. 3 that the proposed method is able to achieve precise tracking within a prescribed time by employing the rate function that belongs to K^{**}, and the control input is bounded under the proposed control method as can be seen in Fig. 4, both of which agree well with the theoretical prediction.

To show the better performance and efficiency of the proposed method, we compare the proposed control with the control scheme established in [10]. We also take the above inverted pendulum model and the same system parameters. In this simulation, the MATLAB is used and proposed control (16) with the same design parameters as in the above simulation is applied. The design parameters in [10] are taken as: $\lambda_1 = \lambda_2 = 2$, $\rho_{10} = 0.5$, $\rho_{20} = 2$, $k_1(t) = (1.01-0.01)\exp(-0.5t)+0.01$, and $k_2(t) = (4-0.1)\exp(-0.15t)+0.1$. The simulation results are shown in Fig. 5, from which we see that indeed better control performance in terms of tracking accuracy and rate of convergence is obtained with the proposed method.

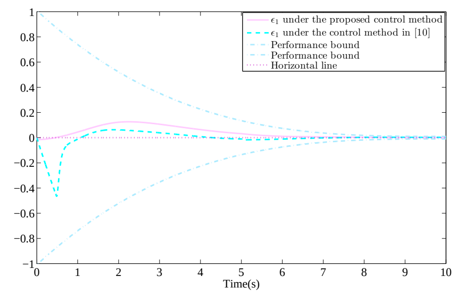


Fig. 5. The tracking error of system (43) under two different control schemes.

V. Conclusions

We introduce a new and systematic approach for zero steady-state error tracking control of a class of self-switching nonlinear systems subject to non-vanishing uncertainties and unknown control directions. Our method features with the regulation of the tracking error to zero with assignable convergence rate and the assurance of the tracking error within the prescribed performance bound. Two interesting topics for future research are: 1) extending the method to more general nonlinear systems with unmatched uncertainties; 2) extending the method to networked multi-agent systems under directed topology.

Appendix

We first prove that $\zeta(t)$ is bounded on $[t_0,t_f)$ by seeking a contradiction. Suppose that $\zeta(t)$ is unbounded, then it must be either case 1) $\zeta(t)$ has no upper bound for $t\in[t_0,t_f)$; or case 2) $\zeta(t)$ has no lower bound for $t\in[t_0,t_f)$. Now we show there is a contradiction for either case, thus $\zeta(t)$ must be bounded.

Case 1: If $\zeta(t)$ had no upper bound on $[t_0,t_f)$. In this case, there must exist a monotone increasing variable $\theta_i = \theta(t_i) = \zeta(t_i)$ $(i = 1,2,\cdots)$ with $\theta_0 = |\zeta(t_0)| > 0$, $\lim_{i\to+\infty} t_i = t_f$, and $\lim_{i\to+\infty}\theta_i = +\infty$. Define

$$\begin{aligned} N_g(\theta_i,\theta_j) = \int_{\theta_i}^{\theta_j} & [g(x(\tau))N(\theta(\tau)) + 1] \\ & \times e^{-\int_{\tau}^{t_j}\nu(s)ds}\nu(\tau)d\theta(\tau) \end{aligned}\tag{44}$$

where $\theta_i = \theta(t_i) \le \theta(t_j) = \theta_j$ and $\tau\in[t_i,t_j]$. Let $g_{\max} = \max\{|\underline{l}|,|\bar{l}|\}$. Notice that $|g(x(t))| \le g_{\max}$ and

$0 \le e^{-\int_{\tau}^{t_j}\nu(s)ds} \le 1$ for $\tau \in [t_i, t_j]$, we then have

$$\begin{aligned}
&|N_g(\theta_i,\theta_j)| \\
&\le \int_{\theta_i}^{\theta_j} \left[|g(x(\tau))|\ |N(\theta(\tau))| + 1\right] e^{-\int_{\tau}^{t_j}\nu(s)ds}\nu(\tau)d\theta(\tau) \\
&\le \left[g_{\max}\sup_{\theta\in[\theta_i,\theta_j]}|N(\theta(\tau))| + 1\right]\nu(t_j)\int_{\theta_i}^{\theta_j} 1\ d\theta(\tau) \\
&= \left[g_{\max}\sup_{\theta\in[\theta_i,\theta_j]}|N(\theta(\tau))| + 1\right]\nu(t_j)(\theta_j - \theta_i). \qquad (45)
\end{aligned}$$

Here we choose the Nussbaum function $N(\theta) = e^{\theta^2}\cos\left(\frac{\pi}{2}\theta\right)$ (it is positive for $\theta \in (4m-1, 4m+1)$ and negative for $\theta \in (4m+1, 4m+3)$ with $m \in Z$), upon using which, we get from (45) that

$$\begin{aligned}
|N_g(\theta_i,\theta_j)| &\le \left(g_{\max}e^{\theta_j^2} + 1\right)\nu(t_j)(\theta_j - \theta_i) \\
&< (g_{\max}+1)\, e^{\theta_j^2}\nu(t_j)(\theta_j - \theta_i). \qquad (46)
\end{aligned}$$

We first consider the case $g(x) > 0$. It is notable that there exists some constant m such that $\theta_0 < 4m-1$. Let $\theta_{m_1} = 4m-1$. We now consider the interval $[\theta_0, \theta_{m_1}]$. It then follows from (46) that

$$|N_g(\theta_0,\theta_{m_1})| \le (g_{\max}+1)\, e^{(4m-1)^2}\nu(t_{m_1})(4m-1-\theta_0). \qquad (47)$$

Next, let $\theta_{m_2} = 4m+1$ and we observe variation in the interval $[\theta_{m_1}, \theta_{m_2}]$. Note that $N(\theta) \ge 0$, $\forall\theta \in [\theta_{m_1}, \theta_{m_2}]$, we have the following inequality

$$\begin{aligned}
&N_g(\theta_{m_1},\theta_{m_2}) = N_g(4m-1, 4m+1) \\
&\ge \int_{4m-\iota_1}^{4m+\iota_1} \left[g(x(\tau))N(\theta(\tau)) + 1\right] e^{-\int_{\tau}^{t_{m_2}}\nu(s)ds}\nu(\tau)d\theta(\tau) \\
&\ge \int_{4m-\iota_1}^{4m+\iota_1} g(x(\tau))N(\theta(\tau))e^{-\int_{\tau}^{t_{m_2}}\nu(s)ds}\nu(\tau)d\theta(\tau) \\
&\ge g_{\min}\inf_{\theta\in[4m-\iota_1,4m+\iota_1]} N(\theta(\tau))e^{-\int_{t_{m_1}}^{t_{m_2}}\nu(s)ds}\nu(t_{m_1})2\iota_1 \\
&= c_{b1}e^{(4m-\iota_1)^2}\nu(t_{m_1}) \qquad (48)
\end{aligned}$$

with $\iota_1 \in (0,1)$, $g_{\min} = \min\{|\underline{l}|, |\bar{l}|\}$, and $c_{b1} = 2\iota_1 g_{\min}\cos(\frac{\pi}{2}\iota_1)e^{-\int_{t_{m_1}}^{t_{m_2}}\nu(s)ds}$, in which we have used the fact that $g(x(\tau)) \ge g_{\min}$, $e^{-\int_{\tau}^{t_{m_2}}\nu(s)ds} \ge e^{-\int_{t_{m_1}}^{t_{m_2}}\nu(s)ds}$, and $\inf_{\theta\in[4m-\iota_1,4m+\iota_1]} N(\theta(\tau)) = e^{(4m-\iota_1)^2}\cos(\frac{\pi}{2}\iota_1)$. By combining (47) and (48), we arrive at

$$\begin{aligned}
N_g(\theta_0,\theta_{m_2}) =& N_g(\theta_0,\theta_{m_1}) + N_g(\theta_{m_1},\theta_{m_2}) \\
\ge& c_{b1}e^{(4m-\iota_1)^2}\nu(t_{m_1}) - (g_{\max}+1) \\
&\times(4m-1-\theta_0)\nu(t_{m_1})e^{(4m-1)^2} \\
=& e^{(4m-1)^2}\nu(t_{m_1})\Big[c_{b1}e^{(1-\iota_1)(8m-\iota_1-1)} \\
&- (g_{\max}+1)(4m-1-\theta_0)\Big]. \qquad (49)
\end{aligned}$$

Diving both sides of (49) by θ_{m_2} yields

$$\begin{aligned}
\frac{N_g(\theta_0,\theta_{m_2})}{\theta_{m_2}} \ge& e^{(4m-1)^2}\nu(t_{m_1})\Bigg[\frac{c_{b1}e^{(1-\iota_1)(8m-\iota_1-1)}}{4m+1} \\
&- \frac{(g_{\max}+1)(4m-1-\theta_0)}{4m+1}\Bigg]. \qquad (50)
\end{aligned}$$

Upon using the L'Hospital's rule we see that

$$\begin{aligned}
&\lim_{m\to+\infty}\left[\frac{c_{b1}e^{(1-\iota_1)(8m-\iota_1-1)}}{4m+1} - \frac{(g_{\max}+1)(4m-1-\theta_0)}{4m+1}\right] \\
&= \lim_{m\to+\infty} 2c_{b1}(1-\iota_1)e^{(1-\iota_1)(8m-\iota_1-1)} - \lim_{m\to+\infty}(g_{\max}+1) \\
&= +\infty. \qquad (51)
\end{aligned}$$

Notice that $\lim_{m\to+\infty} e^{(4m-1)^2}\nu(t_{m_1}) = +\infty$, it then follows from (50)-(51) that

$$\lim_{m\to+\infty}\frac{N_g(\theta_0,\theta_{m_2})}{\theta_{m_2}} = +\infty. \qquad (52)$$

We now prove that $\lim_{m\to+\infty}\frac{N_g(\theta_0,4m+3)}{4m+3} = -\infty$. To this end, we first observe the interval $[\theta_0, \theta_{m_2}] = [\theta_0, 4m+1]$. By letting $\theta_i = \theta_0$ and $\theta_j = 4m+1$, we then get from (46) that

$$|N_g(\theta_0,\theta_{m_2})| \le (g_{\max}+1)(4m+1-\theta_0)\nu(t_{m_2})e^{(4m+1)^2}. \qquad (53)$$

Let us consider the next immediate interval $[\theta_{m_2}, \theta_{m_3}] = [4m+1, 4m+3]$. Noting that $N(\theta) \le 0$, $\forall\theta \in [\theta_{m_2}, \theta_{m_3}]$. As for $\theta \in [\theta_{m_2}, \theta_{m_3}]$, we have

$$\begin{aligned}
&N_g(\theta_{m_2},\theta_{m_3}) \\
&= \int_{\theta_{m_2}}^{\theta_{m_3}} \left[g(x(\tau))N(\theta(\tau)) + 1\right] e^{-\int_{\tau}^{t_{m_3}}\nu(s)ds}\nu(\tau)d\theta(\tau) \\
&\le \int_{4m+2-\iota_2}^{4m+2+\iota_2} \underbrace{g(x(\tau))N(\theta(\tau))e^{-\int_{\tau}^{t_{m_3}}\nu(s)ds}\nu(\tau)}_{\Delta_1(\tau)}d\theta(\tau) \\
&\quad + \int_{\theta_{m_2}}^{\theta_{m_3}} \underbrace{e^{-\int_{\tau}^{t_{m_3}}\nu(s)ds}\nu(\tau)}_{\Delta_2(\tau)}d\theta(\tau). \qquad (54)
\end{aligned}$$

For $\theta \in [4m+2-\iota_2, 4m+2+\iota_2]$, we see that $g(x(\tau)) \ge g_{\min}$, $\nu(\tau) \ge \nu(t_{m_2})$, $e^{-\int_{\tau}^{t_{m_3}}\nu(s)ds} \ge e^{-\int_{t_{m_2}}^{t_{m_3}}\nu(s)ds}$, and $N(\theta(\tau)) \le -e^{(4m+2-\iota_2)^2}\cos\left(\frac{\pi}{2}\iota_2\right) < 0$, and then we have

$$\begin{aligned}
\Delta_1(\tau) =& g(x(\tau))N(\theta(\tau))e^{-\int_{\tau}^{t_{m_3}}\nu(s)ds}\nu(\tau) \\
\le& -c_{b2}e^{(4m+2-\iota_2)^2}\nu(t_{m_2}) \qquad (55)
\end{aligned}$$

with $c_{b2} = g_{\min}\cos\left(\frac{\pi}{2}\iota_2\right)e^{-\int_{t_{m_2}}^{t_{m_3}}\nu(s)ds}$. On the other hand, we see that $0 < e^{-\int_{\tau}^{t_{m_3}}\nu(s)ds} \le 1$ for $\theta \in [\theta_{m_2}, \theta_{m_3}]$, we then have $\Delta_2(\tau) \le \nu(t_{m_3})$. It then follows,

$$N_g(\theta_{m_2},\theta_{m_3}) \le -2\iota_2 c_{b2}e^{(4m+2-\iota_2)^2}\nu(t_{m_2}) + 2\nu(t_{m_3}). \qquad (56)$$

By combining (53) and (56), we arrive at

$$\begin{aligned}N_g(\theta_0,\theta_{m_3}) &= N_g(\theta_0,\theta_{m_2})+N_g(\theta_{m_2},\theta_{m_3})\\ \le&\nu(t_{m_3})e^{(4m+1)^2}\Bigg[(g_{\max}+1)(4m+1-\theta_0)\frac{\nu(t_{m_2})}{\nu(t_{m_3})}\\ &-2\iota_2 c_{b2}e^{(1-\iota_2)(8m+3-\iota_2)}\frac{\nu(t_{m_2})}{\nu(t_{m_3})}+2e^{-(4m+1)^2}\Bigg]\\ \le&\nu(t_{m_3})e^{(4m+1)^2}\Bigg[(g_{\max}+1)(4m+1-\theta_0)\\ &-2\iota_2 c_{b2}e^{(1-\iota_2)(8m+3-\iota_2)}\frac{\nu(t_{m_2})}{\nu(t_{m_3})}+2\Bigg] \end{aligned}\tag{57}$$

in which we have used the fact that $0<\frac{\nu(t_{m_2})}{\nu(t_{m_3})}<1$ and $0<e^{-(4m+1)^2}<1$. Dividing both sides of (57) by θ_{m_3},

$$\frac{N_g(\theta_0,\theta_{m_3})}{\theta_{m_3}}\le\nu(t_{m_3})e^{(4m+1)^2}\Bigg[\frac{(g_{\max}+1)(4m+1-\theta_0)}{4m+3}-\frac{2\iota_2 c_{b2}e^{(1-\iota_2)(8m+3-\iota_2)}}{4m+3}\frac{\nu(t_{m_2})}{\nu(t_{m_3})}+\frac{2}{4m+3}\Bigg].\tag{58}$$

Upon using the L'Hopital's Rule, we can obtain that

$$\lim_{m\to+\infty}\left[\frac{(g_{\max}+1)(4m+1-\theta_0)}{4m+3}+\frac{2}{4m+3}\right]=g_{\max}+1,$$

$$\begin{aligned}&\lim_{m\to+\infty}-\frac{2\iota_2 c_{b2}e^{(1-\iota_2)(8m+3-\iota_2)}}{4m+3}\\ &=-\lim_{m\to+\infty}4\iota_2 c_{b2}e^{(1-\iota_2)(8m+3-\iota_2)}(1-\iota_2)=-\infty.\end{aligned}\tag{59}$$

By noting that $\lim_{m\to+\infty}\nu(t_{m_3})e^{(4m+1)^2}=+\infty$ and $0<\frac{\nu(t_{m_2})}{\nu(t_{m_3})}<1$, it then follows from (58) and (59) that

$$\lim_{m\to+\infty}\frac{N_g(\theta_0,\theta_{m_3})}{\theta_{m_3}}=-\infty.\tag{60}$$

By recalling (52) and (60), we then conclude that

$$\lim_{\theta_j\to+\infty}\sup\frac{N_g(\theta_0,\theta_j)}{\theta_j}=+\infty,\quad \lim_{\theta_j\to+\infty}\inf\frac{N_g(\theta_0,\theta_j)}{\theta_j}=-\infty.\tag{61}$$

By following the similar procedure to the proof from (47) to (60), we can derive that (61) still holds for $g(x)<0$. Dividing both sides of (19) in Lemma 2 by $\theta_i=\zeta(t_i)>0$ and taking the limitation, yields

$$\begin{aligned}0\le&\lim_{i\to+\infty}\frac{V(t_i)}{\theta(t_i)}\le\lim_{i\to+\infty}\left[\frac{C_0}{\theta(t_i)}+\frac{N_g(\theta(t_0),\theta(t_i))}{\theta(t_i)}\right]\\ =&\lim_{i\to+\infty}\frac{N_g(\theta(t_0),\theta(t_i))}{\theta(t_i)}\end{aligned}\tag{62}$$

which contradicts (61). Therefore, $\zeta(t)$ is upper bounded on $[t_0,t_f)$.

Case 2: If $\zeta(t)$ **had no lower bound on** $[t_0,t_f)$**.** For such case, there must exist a monotone decreasing variable $\{\theta_i=\theta(\tilde t_i)=\zeta(\tilde t_i)\}$ $(i=1,2,\cdots)$ with $\theta_0=-|\zeta(t_0)|\le 0$, $\lim_{i\to+\infty}\tilde t_i=t_f$, and $\lim_{i\to+\infty}\theta_i=-\infty$. For $\tilde t_i<\tilde t_j$, $\theta_i=\theta(\tilde t_i)\ge\theta(\tilde t_j)=\theta_j$. We can prove that $\zeta(t)$ is lower bounded on $[t_0,t_f)$ using contradiction by following the procedure similar to that in case 1) upon defining $N_g(\theta_i,\theta_j)=-\int_{\theta_j}^{\theta_i}[g(x(\tau))N(\theta(\tau))+1]\times e^{-\int_\tau^{\tilde t_j}\nu(s)ds}\nu(\tau)d\theta(\tau)$ (the detailed process of proof is omitted due to page limit).

Consequently, $\zeta(t)$ must be bounded on $[t_0,t_f)$, which further implies that $N(\zeta)$ is bounded on $[t_0,t_f)$. Now the rest of the proof is to show that $V(t)$ is bounded on $[t_0,t_f)$. By noting that $g(x(\tau))\in L_\infty$, $\dot\zeta\in L_\infty$ and $N(\zeta)\in L_\infty$, we then get from (19) that

$$\begin{aligned}V(t)\le&C_0+\int_{t_0}^{t}(g(x(\tau))N(\zeta)+1)e^{-\int_\tau^t\nu(s)ds}\dot\zeta\nu(\tau)d\tau\\ \le&C_0+\bar N e^{-\int_{t_0}^t\nu(s)ds}\int_{t_0}^{t}e^{\int_{t_0}^{\tau}\nu(s)ds}d\left(\int_{t_0}^{\tau}\nu(s)ds\right)\\ =&C_0+\bar N\left(1-e^{-\int_{t_0}^t\nu(s)ds}\right)\le C_0+\bar N\in L_\infty\end{aligned}\tag{63}$$

with $\bar N=\sup_{\tau\in\{t_0,t\}}\left\{\left|(g(x(\tau))N(\zeta)+1)\dot\zeta\right|\right\}\in L_\infty$, which then implies that $V(t)\in L_\infty$.

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Yujuan Wang received her Ph.D. degree in control theory and control engineering from Chongqing University, China, in 2016. She is currently a Professor with the School of Automation in Chongqing University. She was a Research Associate with the Department of Electrical and Electronic Engineering in the University of Hong Kong during 2017-2018. Her research interests include cooperative control, adaptive control, finite-time control, fault-tolerant control.

Yongduan Song (Fellow, IEEE) received the Ph.D. degree in electrical and computer engineering from Tennessee Technological University, Cookeville, TN, USA, in 1992.

He was granted a Tenured Full Professor position with North Carolina Agricultural and Technical State University, Greensboro, NC, USA, in 2004, and he was one of the six Langley Distinguished Professors with the National Institute of Aerospace (NIA), Hampton, VA, USA, and the Founding Director of the Center for Cooperative Systems, NIA from 2005 to 2008. He is currently the Dean of the School of Automation, Chongqing University, Chongqing, China, and the Founding Director of the Institute of Smart Engineering, Chongqing University.

Dr. Song is the Editor-in-Chief of IEEE TRANSACTIONS ON NEURAL NETWORKS AND LEARNING SYSTEMS. He has served/been serving as an Associate Editor for several top IEEE journals, including IEEE TRANSACTIONS ON AUTOMATIC CONTROL, IEEE TRANSACTIONS ON SYSTEMS, MAN, AND CYBERNETICS, IEEE TRANSACTIONS ON INTELLIGENT TRANSPORTATION SYSTEMS, and IEEE TRANSACTIONS ON COGNITIVE AND DEVELOPMENTAL SYSTEMS.

Xiang Chen is currently a Ph. D. student with the School of Automation in Chongqing University. His current research interests include network systems, finite-time control, adaptive control, and game theory.