Automatica 45 (2009) 2605–2611

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Finite-time formation control for multi-agent systems☆

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ARTICLE INFO

Article history:
Received 18 January 2009
Received in revised form
29 June 2009
Accepted 7 July 2009
Available online 19 August 2009

Keywords:
Multi-agent systems
Decentralized control
Finite-time formation
Finite-time consensus

ABSTRACT

In this paper, we develop a new finite-time formation control framework for multi-agent systems with a large population of members. In this framework, we divide the formation information into two independent parts, namely, the global information and the local information. The global formation information decides the geometric pattern of the desired formation. Furthermore, it is assumed that only a small number of agents, which are responsible for the navigation of the whole team, can obtain the global formation information, and the other agents regulate their positions by the local information in a distributed manner. This approach can greatly reduce the data exchange and can easily realize various kinds of complex formations. As a theoretical preparation, we first propose a class of nonlinear consensus protocols, which ensures that the related states of all agents will reach an agreement in a finite time under suitable conditions. And then we apply these consensus protocols to the formation control, including time-invariant formation, time-varying formation and trajectory tracking, respectively. It is shown that all agents will maintain the expected formation in a finite time. Finally, several simulations are worked out to illustrate the effectiveness of our theoretical results.

1. Introduction

In this paper, we study the formation control problem for a group of agents, which is a popular research topic in decentralized control. Traditional methods in the study of formation control generally fall into three categories, namely, the leader-following approach (Egerstedt & Hu, 2001; Egerstedt, Hu, & Stotsky, 2001; Leonard & Fiorelli, 2001; Shao, Xie, & Wang, 2007), the behavior-based approach (Arkin, 1998; Balch & Arkin, 1998; Lumelsky & Harinarayan, 1997), and the virtual structure approach (Lewis & Tan, 1997). In addition, the techniques from algebraic graph theory are usually used to tackle the modeling problem of information flow or interaction topology (Fax & Murray, 2004).

Up to now, most of the existing work on formation control has been based on a common assumption that each agent knows the formation information, namely, each agent can obtain the information about its position in global coordinates in the expected formation. However, this assumption is impractical or unnecessary in many practical situations. For instance, when the number of agents is very large, it is costly to convey the formation data to each agent. Furthermore, there are many uncertainties and disturbances in the real environment, which may result in information transmission failure. Also, in many cases, it is enough to pay more attention to the positions of some key members in formation maintenance, in other words, some formations do not have strict requirements on all agents' positions. One simple example is the line formation with loose space constraint between agents.

In this paper, we present a new formation control framework, applicable in the above mentioned cases. Our work is somewhat motivated by the swarming behavior in the real world (Couzin, Krause, Franks, & Levin, 2005) and by the work on the formation control (Lafferriere, Williams, Caughman, & Veerman, 2005; Ren, 2006, 2007; Shao et al., 2007). The basic idea is given as follows:

(1) the formation information is divided into two independent parts, namely, the global one and the local one;
(2) the global information decides the geometric pattern of the desired formation, and a small number of agents, which are called leaders and responsible for navigating the whole team, can obtain the global formation information;
(3) the local information decides the relative positions of followers with respect to the frame decided by leaders; and the local information may not be strictly specified in applications;
(4) all members adjust their positions in a distributed manner.

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Andrey V. Savkin under the direction of Editor Ian R. Petersen.

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doi:10.1016/j.automatica.2009.07.012

It is of importance to note that the definition of leaders here is different from that in the leader-following control, which is defined by the topology of information flow, but not by the types of obtained information.

This treatment of information classification has its obvious superiorities and can greatly reduce the information exchange when the number of agents is large and the formation is changing. Moreover, it can easily realize various kinds of complex and even time-varying geometric patterns of formation. Furthermore, this treatment also improves the robustness of proposed protocols against data transmission failures. We will give several examples to demonstrate these advantages.

As an important theoretical preparation for the development of the main results, consensus theory is another interesting topic in decentralized control (Martínez, Cortés, & Bullo, 2007; Olfati-Saber, Fax, & Murray, 2007; Ren, Beard, & Atkins, 2007; Xiao, 2008; Xiao & Wang, 2008a,b) and has been successfully applied in the formation control (Lafferriere et al., 2005; Ren, 2006). One main research issue in consensus problems is how to design effective consensus protocols, which are distributed interaction rules among agents and are aimed at ensuring that the concerned states of agents converge to a common value. This paper discusses the design problem of finite-time consensus protocols. It is well known that finite-time convergence systems possess the advantages of higher control accuracy, better disturbance rejection and robustness against uncertainties (Bhat & Bernstein, 2000; Haimo, 1986). So far several kinds of finite-time consensus protocols have been presented, see (Cortés, 2006; Sundaram & Hadjicostis, 2007; Wang & Hong, 2008; Xiao & Wang, 2007). Compared with the existing ones, our proposed protocols are in different forms and can guarantee the states of agents to reach consensus in finite time under more relaxable conditions.

The contributions of this paper are twofold. First, this paper presents a class of new nonlinear consensus protocols for state consensus of multi-agent systems. These protocols have the form of a continuous state feedback and bridge the gap between asymptotical consensus protocols and discontinuous finite-time consensus protocols (Cortés, 2006). Second, by applying the finite-time convergence protocols, this paper develops a new finite-time formation framework for multi-agent systems with a large population of members and considers time-invariant formation, time-varying formation and trajectory tracking problem respectively.

An outline of this paper is as follows: in Section 2, some preliminary notions in graph theory and a preliminary lemma are assembled; in Section 3, the problem is formulated; in Section 4, the main results are presented; in Section 5, numerical simulations are given to illustrate the effectiveness of the theoretical results; finally, concluding remarks are made in Section 6.

Notation: In this paper, $\mathcal{I}_n$ denotes the index set $\{1, 2, \ldots, n\}$; I_N is the $N \times N$ identity matrix; and $\otimes$ denotes the Kronecker product. If b is a vector, then $\mathrm{diag}(b)$ denotes the diagonal matrix with the ith diagonal entry being the ith element of vector b. Let $\mathrm{sig}(r)^\alpha = \mathrm{sign}(r)|r|^\alpha$, where $\alpha > 0$, $r \in \mathbb{R}$ and $\mathrm{sign}(\cdot)$ is the sign function. If r is a vector, $\mathrm{sig}(r)^\alpha$ denotes the vector with the same dimensions as r, obtained by operating $\mathrm{sig}(\cdot)^\alpha$ on each entry of r.

2. Preliminaries

Directed graph is used to model the interaction topology among agents. A directed graph $\mathcal{G}$ consists of a vertex set $\mathcal{V}(\mathcal{G}) = \{v_i : i \in \mathcal{I}_n\}$ and an edge set $\mathcal{E}(\mathcal{G}) \subset \{(v_i, v_j) : v_i, v_j \in \mathcal{V}(\mathcal{G})\}$. Edges such as (v_i, v_i) are called *self-loops*. If $(v_i, v_j) \in \mathcal{V}(\mathcal{G})$, v_i is called the *parent* vertex of v_j. A *path* in directed graph $\mathcal{G}$ from v_{i_1} to v_{i_k} is a sequence $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ of finite vertices such that $(v_{i_j}, v_{i_{j+1}}) \in \mathcal{E}(\mathcal{G})$ for $j = 1, 2, \ldots, k-1$. If $i_1 = i_k$, then this path is also called a *loop*. Directed graph $\mathcal{G}$ is *strongly connected* if between every pair of distinct vertices v_i, v_j, there exists a path that begins at v_i and ends at v_j. Directed graph $\mathcal{G}$ is said to *have a spanning tree* if there exists a vertex that is called the *root* which can be connected to any others through paths.

A *subgraph* $\mathcal{G}_s$ of directed graph $\mathcal{G}$ is a directed graph such that $\mathcal{V}(\mathcal{G}_s) \subset \mathcal{V}(\mathcal{G})$ and $\mathcal{E}(\mathcal{G}_s) \subset \mathcal{E}(\mathcal{G})$. If for any $v_i, v_j \in \mathcal{V}(\mathcal{G}_s)$, $(v_i, v_j) \in \mathcal{E}(\mathcal{G}_s) \iff (v_i, v_j) \in \mathcal{E}(\mathcal{G})$, then $\mathcal{G}_s$ is called an *induced subgraph*. In this case, $\mathcal{G}_s$ is also said to be induced by $\mathcal{V}(\mathcal{G}_s)$. A *strongly connected component* is an induced subgraph that is maximal, subject to being strongly connected. Suppose that all strongly connected components of $\mathcal{G}$ are $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_p$. We define another directed graph $\mathcal{G}^c$ related to $\mathcal{G}$ with vertex set $\{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_p\}$, satisfying the assumption that no self-loop exists in $\mathcal{G}^c$ and satisfying the property that for any $i \neq j$, $(\mathcal{G}_i, \mathcal{G}_j) \in \mathcal{E}(\mathcal{G}^c)$ if and only if there exist a vertex $v_{i'} \in \mathcal{V}(\mathcal{G}_i)$ and a vertex $v_{j'} \in \mathcal{V}(\mathcal{G}_j)$ such that $(v_{i'}, v_{j'}) \in \mathcal{E}(\mathcal{G})$.

A *weighted directed graph* $\mathcal{G}(A)$ is a directed graph $\mathcal{G}$ with a nonnegative weight matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, such that $(v_i, v_j) \in \mathcal{E}(\mathcal{G}) \Leftrightarrow a_{ji} > 0$. In this paper, it is assumed that the diagonal entries of matrix A are all zeros. Let $L(A) = [l_{ij}] \in \mathbb{R}^{n\times n}$ denote the *graph Laplacian* of $\mathcal{G}(A)$, which is defined by

$$l_{ij} = \begin{cases} \sum_{k=1,k\neq i}^{n} a_{ik}, & i = j \\ -a_{ij}, & i \neq j. \end{cases}$$

We have

Lemma 1 (*Olfati-Saber & Murray, 2004; Ren et al., 2007; Xiao, 2008*). *Graph Laplacian $L(A)$ of $\mathcal{G}(A)$ has the following properties:*

(1) *if $\mathcal{G}(A)$ has a spanning tree, then eigenvalue 0 is algebraically simple and all other eigenvalues are with positive real parts;*
(2) *if $\mathcal{G}(A)$ is strongly connected, then there exists a positive column vector $\omega \in \mathbb{R}^n$ such that $\omega^{\mathrm{T}} L(A) = 0$.*

3. Problem formulation

Suppose that the multi-agent system, studied in this section, consists of n agents, e.g., vehicles, robots, etc., labeled 1–n. All these agents interact with each other via local information transmission and share a common state space $\mathbb{R}^N$, representing the positions of agents, where $N = 1, 2$, or 3. Directed graph $\mathcal{G}$ with vertex set $\mathcal{V}(\mathcal{G}) = \{v_i, i \in \mathcal{I}_n\}$ is used to model the interaction topology. Vertex v_i represents agent i and edge (v_i, v_j) corresponds to an available information channel from agent i to agent j. We further assume that there exist no self-loops in $\mathcal{G}$. The *neighbors* of agent i are those agents whose information can be received by agent i. Denote the associated index set by $\mathcal{N}_i$. Clearly, $\mathcal{N}_i = \{j : (v_j, v_i) \in \mathcal{E}(\mathcal{G})$ and $j \neq i\}$.

Let $x_i \in \mathbb{R}^N$ denote the state of agent i and suppose that the dynamics of agent i is described by the following continuous-time equation

$$\dot{x}_i = u_i, \tag{1}$$

where u_i is a state feedback, called *protocol*, to be designed based on the local information obtained by agent i from its neighbors. For simplicity, let $x = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \ldots, x_n^{\mathrm{T}}]^{\mathrm{T}}$.

Definition 2 (*Finite-Time Formation*). A formation (h, F) of n agents consists of a time-dependent column vector $h = [h_1^{\mathrm{T}}, \ldots, h_m^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nm}$, $m \leq n$, representing the global formation information, and a time-dependent nonnegative matrix $F = [F_{m+1}^{\mathrm{T}}, \ldots, F_n^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{(n-m)\times n}$, representing the local formation information, with the property that the entry sum of matrix F_i is 1 and the ith entry of F_i is zero, $i = m+1, \ldots, n$, where $h_1, \ldots, h_m \in \mathbb{R}^N$ and $F_{m+1}, \ldots, F_n \in \mathbb{R}^{1\times n}$:

(1) the n agents are in formation (h, F) at time t if there exists a vector $h^c \in \mathbb{R}^N$ such that $x_i(t) = h_i(t) + h^c$, $i \in \mathcal{I}_m$, and $x_i(t) = (F_i(t) \otimes I_N)x(t)$, $i = m+1, \ldots, n$;
(2) the n agents converge asymptotically to formation (h, F) if there exists a $\mathbb{R}^N$-valued function $h^c(t)$ such that $x_i(t) \to h_i(t) + h^c(t)$, $i \in \mathcal{I}_m$, and $x_i(t) \to (F_i(t) \otimes I_N)x(t)$, $i = m+1, \ldots, n$, as $t \to \infty$;
(3) the n agents will be in formation (h, F) in finite time if there exist time $t^* > 0$ and a $\mathbb{R}^N$-valued function $h^c(t)$ such that $x_i(t) = h_i(t) + h^c(t)$, $i \in \mathcal{I}_m$, and $x_i(t) = (F_i(t) \otimes I_N)x(t)$, $i = m+1, \ldots, n$, for all $t \geq t^*$.

In the above three cases, the first m agents are called *leaders* and the rest are called *followers*.

In formation (h, F), vector h defines the basic frame of the expected formation formed by leaders and nonnegative matrix F specifies the local position-restrictions of followers related to their neighbors, in other words, matrix F determines the space distribution of followers. Since each row entry sum of F equals 1, followers in the formation should lie in the convex region covered by leaders. Column vector $h^c(t)$ decides the position of the formation and may be dependent on initial states or may be an external input, used to guide the group of agents to track an anticipated trajectory.

4. Main results

4.1. Finite-time consensus

Before presenting the control strategies for finite-time formation, we study the simplest case of the formation (h, F), namely, agreement over the location of agents or rendezvous problem. In this case, $h_1(t) = h_2(t) = \cdots = h_m(t)$ and $m = n$.

Given protocol u_i, $i \in \mathcal{I}_n$, u_i or this multi-agent system *solves a consensus problem asymptotically* if for any initial states, there exists an asymptotically stable equilibrium $x^* \in \mathbb{R}^N$ such that $x_i(t) \to x^*$ as $t \to \infty$ for any $i \in \mathcal{I}_n$ (Olfati-Saber & Murray, 2004); u_i or this multi-agent system *solves a finite-time consensus problem* if for any initial states, there exist a finite time t^* and an asymptotically stable equilibrium $x^* \in \mathbb{R}^N$ such that $x_i(t) = x^*$ for all $t \geq t^*$ and all $i \in \mathcal{I}_n$.

Next, we provide a class of new nonlinear finite-time consensus protocols:

$$u_i = \beta \operatorname{sig}\left(\sum_{j\in\mathcal{N}_i} W_{ij}(x_j - x_i)\right)^{\alpha} + \gamma \sum_{j\in\mathcal{N}_i} W_{ij}(x_j - x_i), \tag{2}$$

where $0 < \alpha < 1$, $\beta > 0$, $\gamma \geq 0$, and $W_{ij} > 0$ are called *weighting factors* (Ren & Beard, 2005).

Protocol (2) represents a class of protocols with free parameters α, β and γ. If $\beta = 0$, $\gamma = 1$, then the above protocol becomes the linear protocol, which was studied by Olfati-Saber and Murray (2004) and was proven to solve a consensus problem asymptotically under appropriate conditions. If $\alpha = 0$, $\beta \neq 0$, the above protocol becomes discontinuous. This case is beyond the scope of our research. However, it is worth mentioning that discontinuous protocols were studied by Cortés. Interested readers can be referred to Cortés (2006).

Remark. Note that under the above protocol, the dynamics of each dimension of position vectors is independent of others. We assume that $N = 1$ without loss of generality. In this case, each agent becomes a single integrator. Networks of multiple integrators have been studied by many researchers and there still exist a large number of interesting but unsolved problems. Some researchers have also investigated more realistic and complex agent dynamics in different frameworks (Johansson, Speranzon, Johansson, & Johansson, 2008; Keviczky & Johansson, 2008), which is one of our future research topics.

Since function $\operatorname{sig}(\cdot)$ is continuous, by Peano's Existence Theorem and Extension Theorem (Hartman, 1982), it can be obtained that for any initial state, there exists at least one solution of differential equations (1) under protocol (2) on time interval $[0, \infty)$.

The following theorem characterizes the finite-time convergence property of protocol (2):

Theorem 3. *If the interaction topology $\mathcal{G}$ has a spanning tree, then protocol (2) solves a finite-time consensus problem.*

Proof. We first define a new matrix $A = [a_{ij}]$ by

$$a_{ij} = \begin{cases} W_{ij}, & \text{if } j \in \mathcal{N}_i \\ 0, & \text{otherwise.} \end{cases}$$

This theorem is proven through the following three steps.

Step 1: Suppose that $\mathcal{G}(A)$ is strongly connected.

By Lemma 1, there exists a positive vector $\omega = [\omega_1, \omega_2, \ldots, \omega_n]^{\mathrm{T}} \in \mathbb{R}^n$ such that $\omega^{\mathrm{T}} L(A) = 0$, where $L(A) = [l_{ij}]$ denotes the graph Laplacian of $\mathcal{G}(A)$, defined in Section 2. Let $y_i = \sum_{j=1}^n a_{ij}(x_j - x_i)$ and $y = [y_1, y_2, \ldots, y_n]^{\mathrm{T}}$. Then $\omega \perp y$ and

$$\dot{y} = -\beta L(A)\operatorname{sig}(y)^{\alpha} - \gamma L(A) y.$$

Choose Lyapunov candidate

$$V_1(t) = \sum_{i=1}^n \omega_i \left(\frac{\beta}{1+\alpha}|y_i|^{1+\alpha} + \frac{\gamma}{2} y_i^2\right).$$

Obviously $V_1(t) \geq 0$ and

$$\frac{\mathrm{d}V_1(t)}{\mathrm{d}t} = -\left(\beta\operatorname{sig}(y)^{\alpha} + \gamma y\right)^{\mathrm{T}} \operatorname{diag}(\omega) L(A) \left(\beta\operatorname{sig}(y)^{\alpha} + \gamma y\right).$$

Claim: Given initial state $x(0)$, there exists $K_1 > 0$ such that

$$\frac{\mathrm{d}V_1(t)}{\mathrm{d}t} \leq -K_1 V_1(t)^{\frac{2\alpha}{1+\alpha}}.$$

By the above claim, $V_1(t)$ will reach zero in finite time $t^* = \frac{(1+\alpha)V_1(0)^{\frac{1-\alpha}{1+\alpha}}}{K_1(1-\alpha)}$, which implies that y will be zero. Since $y = -L(A)x$ and $\operatorname{rank}(L(A)) = n - 1$ (by Lemma 1), $y = 0$ implies that $x_1 = x_2 = \cdots = x_n$ and $\dot{x}(t) = 0$. Therefore this system solves a finite-time consensus problem.

It is left to us to prove that the above claim holds.

Suppose that $V_1(t) \neq 0$. Let $\Delta_1(y) = -\frac{\frac{\mathrm{d}V_1(t)}{\mathrm{d}t}}{V_1(t)^{\frac{2\alpha}{1+\alpha}}}$, let $M = \frac{1}{2}\left(\operatorname{diag}(\omega)L(A) + L(A)^{\mathrm{T}}\operatorname{diag}(\omega)\right)$ and let $\mathcal{U} = \{\xi \in \mathbb{R}^n : \xi^{\mathrm{T}}\xi = 1$ and $\xi = \beta\operatorname{sig}(\zeta)^{\alpha} + \gamma\zeta$ for some $\zeta \perp \omega\}$. Then $\mathcal{U}$ is a bounded closed set. Since function $\xi^{\mathrm{T}} M \xi$ is continuous with respect to ξ and for any $\xi \in \mathcal{U}$, $\xi^{\mathrm{T}} M \xi \neq 0$ (by Lemma 1), we have that $\min_{\xi\in\mathcal{U}} \xi^{\mathrm{T}} M \xi$, denoted by k_1, exists and is larger that zero. It follows from $\omega \perp y$ that

$$\frac{\left(\beta\operatorname{sig}(y)^{\alpha} + \gamma y\right)^{\mathrm{T}} M \left(\beta\operatorname{sig}(y)^{\alpha} + \gamma y\right)}{\left(\beta\operatorname{sig}(y)^{\alpha} + \gamma y\right)^{\mathrm{T}} \left(\beta\operatorname{sig}(y)^{\alpha} + \gamma y\right)} \geq k_1.$$

Therefore,

$$\begin{aligned} \Delta_1(y) &\geq \frac{k_1\left(\beta\operatorname{sig}(y)^{\alpha} + \gamma y\right)^{\mathrm{T}}\left(\beta\operatorname{sig}(y)^{\alpha} + \gamma y\right)}{V_1(t)^{\frac{2\alpha}{1+\alpha}}} \\ &\geq \frac{k_1 \sum_{i=1}^n \left(\beta\operatorname{sig}(y_i)^{\alpha} + \gamma y_i\right)^2}{\sum_{i=1}^n \left(\frac{\omega_i\beta}{1+\alpha}\right)^{\frac{2\alpha}{1+\alpha}} |y_i|^{2\alpha} + \sum_{i=1}^n \left(\frac{\omega_i\gamma}{2}\right)^{\frac{2\alpha}{1+\alpha}} |y_i|^{\frac{4\alpha}{1+\alpha}}} \end{aligned}$$

$$\geq \frac{k_1\beta^2\sum_{i=1}^{n}|y_i|^{2\alpha}}{\sum_{i=1}^{n}\left(\frac{\omega_i\beta}{1+\alpha}\right)^{\frac{2\alpha}{1+\alpha}}|y_i|^{2\alpha}+\sum_{i=1}^{n}\left(\frac{\omega_i\gamma}{2}\right)^{\frac{2\alpha}{1+\alpha}}|y_i|^{\frac{4\alpha}{1+\alpha}}}$$
$$\triangleq \Delta_2(y).$$

The third inequality follows from the fact that y_i and $\mathrm{sig}(y_i)^\alpha$ are with the same sign.

Let $\mathcal{W} = \{\xi \in \mathbb{R}^n : 1 \leq \|\xi\|_\infty \leq \|L(A)\|_{i\infty}\|x(0)\|_\infty\}$, where $\|\cdot\|_{i\infty}$ is the induced matrix norm of maximum norm $\|\cdot\|_\infty$. We first suppose $\mathcal{W} \neq \emptyset$. Then it is compact, and for any $\xi \in \mathcal{W}$, $\Delta_2(\xi) \neq 0$. Thus $\min_{\xi\in\mathcal{W}}\Delta_2(\xi)$, denoted by k_2, exists, and is larger than zero. Therefore, if $y \in \mathcal{W}$, then $\Delta_2(y) \geq k_2$. On the other hand, if $y \notin \mathcal{W}$ or if $\mathcal{W} = \emptyset$, it follows from $\|y(t)\|_\infty = \|-L(A)x(t)\|_\infty \leq \|L(A)\|_{i\infty}\|x(t)\|_\infty \leq \|L(A)\|_{i\infty}\|x(0)\|_\infty$ that $0 < \|y\|_\infty < 1$. In this case,

$$\Delta_2(y) > \frac{k_1\beta^2\sum_{i=1}^{n}|y_i|^{2\alpha}}{2\omega_0\sum_{i=1}^{n}|y_i|^{2\alpha}} = \frac{k_1\beta^2}{2\omega_0},$$

where $\omega_0 = \max\left\{\left(\frac{\omega_i\beta}{1+\alpha}\right)^{\frac{2\alpha}{1+\alpha}}, \left(\frac{\omega_i\gamma}{2}\right)^{\frac{2\alpha}{1+\alpha}}, i \in \mathcal{I}_n\right\}$.

Let $K_1 = \min\left\{k_2, \frac{k_1\beta^2}{2\omega_0}\right\}$. Then the claim holds.

Step 2: Next, we suppose that $\mathcal{G}(A)$ has a spanning tree, the associated root vertex has no parent vertex, and the subgraph induced by the other vertices is strongly connected.

Without loss of generality, assume that the root vertex is v_n. Then we get that $a_{n1} = a_{n2} = \cdots = a_{nn} = 0$ and $a_{1n}, a_{2n}, \ldots, a_{n-1,n}$ are not all zeros. For simplicity, let $b = [a_{1n}, a_{2n}, \ldots, a_{n-1,n}]^{\mathrm{T}}$, let $A^0 = [a_{ij}]_{1\leq i,j\leq n-1}$, let $y_i = \sum_{j=1}^{n} a_{ij}(z_j - z_i)$, $i = 1, 2, \ldots, n-1$, and let $y^0 = [y_1, y_2, \ldots, y_{n-1}]^{\mathrm{T}}$.

Because the subgraph induced by $v_1, v_2, \ldots, v_m$ is strongly connected, i.e., $\mathcal{G}(A^0)$ is strongly connected, by Lemma 1, there exists a positive column vector $\omega^0 = [w_1, w_2, \ldots, w_m]^{\mathrm{T}} \in \mathbb{R}^m$ such that $\omega^0 L(A^0) = 0$. Differentiate Lyapunov candidate

$$V_2(t) = \sum_{i=1}^{m}\omega_i\left(\frac{\beta}{1+\alpha}|y_i|^{1+\alpha} + \frac{\gamma}{2}y_i^2\right),$$

with respect to time and we have

$$\begin{aligned}\frac{\mathrm{d}V_2(t)}{\mathrm{d}t} &= \left(\beta\mathrm{sig}(y^0)^\alpha + \gamma y^0\right)^{\mathrm{T}} \\ &\quad\times\left(-\mathrm{diag}(w^0)L(A^0) - \mathrm{diag}(\omega^0)\mathrm{diag}(b)\right) \\ &\quad\times\left(\beta\mathrm{sig}(y^0)^\alpha + \gamma y^0\right).\end{aligned}$$

Given initial state $x(0)$, with the same arguments as in the first step, we can prove that there exists $K_2 > 0$ such

$$\frac{\mathrm{d}V_2(t)}{\mathrm{d}t} \leq -K_2 V_2(t)^{\frac{2\alpha}{1+\alpha}},$$

and thus $V_2(t)$ will reach zero in finite time. Therefore, in this case, the system solves a finite-time consensus problem and the final state is x_n.

Step 3: Finally, by induction we prove the correctness of the general case, namely, the case when $\mathcal{G}(A)$ has a spanning tree.

Consider the directed graph $\mathcal{G}^c(A)$ with the strongly connected components of $\mathcal{G}(A)$ as its vertices. $\mathcal{G}^c(A)$ is defined in Section 2 and by its definition, $\mathcal{G}^c(A)$ contains only one root vertex.

(1) The dynamics of agents corresponding to the vertex set of the root of $\mathcal{G}^c(A)$ is not affected by others and by the conclusion of the first step, the states of them will reach consensus in finite time. Denote their final state by x_0.

(2) Consider the agents that constitute some strongly connected component of the interaction topology $\mathcal{G}$, but not the root vertex of $\mathcal{G}^c(A)$. Obviously, their dynamics is only affected by the other agents that can be connected to them by paths. Suppose that the states of the "affecting" agents have already reached an agreement and maintain their current values as time evolves. Suppose that the common state is also x_0 and denote the index set of the "affecting" agents by $\mathcal{L}$. Consider the dynamics of "affected" agent i. We have

$$\begin{aligned}u_i &= \beta\mathrm{sig}\left(\sum_{j\in\mathcal{N}_i\setminus\mathcal{L}} W_{ij}(x_j - x_i) + \left(\sum_{j\in\mathcal{N}_i\cap\mathcal{L}} W_{ij}\right)(x_0 - x_i)\right)^\alpha \\ &\quad+ \gamma\left(\sum_{j\in\mathcal{N}_i\setminus\mathcal{L}} W_{ij}(x_j - x_i) + \left(\sum_{j\in\mathcal{N}_i\cap\mathcal{L}} W_{ij}\right)(x_0 - x_i)\right).\end{aligned}$$

By the above expression, for the "affected" agents, the "affecting" agents, as a whole, can be seen as one virtual agent. By the conclusion obtained in the second step, the states of those "affected" agents will reach consensus in finite time and the final state of them is x_0.

(3) With the above conclusions and by the fact that $\mathcal{G}^c(A)$ has no loop, we can employ the mathematical induction to derive the conclusion that the system solves a finite-time consensus problem. ■

Remark. The convergence rate of the proposed algorithm is closely related to the interaction topology and protocol parameter α. Especially, in the bidirectional interaction case, i.e., when $A^{\mathrm{T}} = A$, we can show that larger algebraic connectivity of $\mathcal{G}(A)$, namely, the second smallest eigenvalue of $L(A)$, will lead to shorter convergence time and we also can shown that smaller α can lead to a higher convergence rate when agents' states differ a little from each other, and larger α can lead to a higher convergence rate when agents states differ a lot from each other. Moreover, by some straightforward arguments, it can be shown that the system under protocol (2) converges faster to some extent than that under the linear counterpart of protocol (2).

We present the following corollary and omit the proof details due to the limit of paper length.

Corollary 4. *Consider system (1) under protocol (2). Given any initial states, there exists a finite time, after which the states of all agents will remain constant, and the state of each agent will reach a convex combination of its neighbors' states. More explicitly,*

$$x_i = \frac{\sum_{j\in\mathcal{N}_i} W_{ij}x_j}{\sum_{j\in\mathcal{N}_i} W_{ij}}, \quad i \in \mathcal{I}_n.$$

4.2. Time-invariant formation

In this subsection, we first assume that the expected formation (h, F) is time-invariant and make the following assumptions about the multi-agent system:

(A1) the local interaction topology among the leaders has a spanning tree and the leaders' dynamics is not affected by the followers;

(A2) the neighbors of follower i, $i \in \{m+1, m+2, \ldots, n\}$, should include all such agent j that the jth entry of vector F_i is nonzero; the followers adjust their positions based on the information received from their neighbors;

(A3) leader i, $i \in \mathcal{I}_m$, can get the global formation information h_i and the local information $x_j - h_j$, $j \in \mathcal{N}_i$, and send the information $x_i - h_i$ to other agents;

(A4) follower i, $i \in \{m+1, m+2, \ldots, n\}$, can get the local formation information F_i and can obtain the relative positions related to its neighbors.

Denote $F_i = [f_1^i, f_2^i, \ldots, f_n^i]$. Based on the above assumption, the presented control law is

$$u_i = \begin{cases} \beta \operatorname{sig}\Big(\sum_{j\in\mathcal{N}_i} W_{ij}\big((x_j-h_j)-(x_i-h_i)\big)\Big)^{\alpha} \\ \quad +\gamma \sum_{j\in\mathcal{N}_i} W_{ij}\big((x_j-h_j)-(x_i-h_i)\big), \\ \qquad\qquad i = 1,\ldots,m, \\ \beta \operatorname{sig}\Big(\sum_{j=1}^{n} f_j^i (x_j - x_i)\Big)^{\alpha} + \gamma \sum_{j=1}^{n} f_j^i (x_j - x_i), \\ \qquad\qquad i = m+1,\ldots,n, \end{cases} \tag{3}$$

where weighting factors $W_{ij} > 0$ and parameters $0 < \alpha < 1$, $\beta > 0, \gamma \geq 0$.

Corollary 5. *Under protocol (3) and Assumption (A1)–(A4), the n agents will be in time-invariant formation (h, F) in finite time, where the first m agents are leaders and the rest are followers.*

Proof. By observing Eq. (3) and by Theorem 3, $x_i - h_i$, $i \in \mathcal{I}_m$, will reach consensus in finite time t_1^*. Let the consensus state be h^c. Then after time $t_1^*, x_i(t) = h_i + h^c, i \in \mathcal{I}_m$.

Consider the dynamics of followers after time t_1^*. Since the positions of leaders are constant, by Corollary 4, there exists a finite time t_2^*, after which the position of follower i is $F_i x$.

To conclude, the n agents will be in formation (h, F) in finite time. ■

4.3. Time-varying formation

In many practical situations, such as to avoid obstacles, the formation is required to be changing with time. In this case, we assume that the studied multi-agent system satisfies the following assumptions together with Assumptions (A1, A2):

(A3') leader i, $i \in \mathcal{I}_m$, can get the global formation information $h_i, \dot{h}_i$ and the local information $x_j - h_j, j \in \mathcal{N}_i$, and send the information $x_i - h_i$ and u_i to other agents;

(A4') follower i, $i \in \{m+1, m+2, \ldots, n\}$, can get the local formation information F_i, $\dot{F}_i$, and can obtain the relative positions and control inputs of its neighbors.

The allowable time-varying formation control protocol is

$$u_i = \begin{cases} \dot{h}_i + \beta \operatorname{sig}\Big(\sum_{j\in\mathcal{N}_i} W_{ij}\big((x_j-h_j)-(x_i-h_i)\big)\Big)^{\alpha} \\ \quad +\gamma \sum_{j\in\mathcal{N}_i} W_{ij}\big((x_j-h_j)-(x_i-h_i)\big), \\ \qquad\qquad i = 1,2,\ldots,m, \\ \sum_{j=1}^{n} (f_j^i u_j + \dot{f}_j^i x_j) + \beta \operatorname{sig}\Big(\sum_{j=1}^{n} f_j^i (x_j - x_i)\Big)^{\alpha} \\ \quad + \gamma \sum_{j=1}^{n} f_j^i (x_j - x_i), \quad i = m+1, m+2,\ldots,n, \end{cases} \tag{4}$$

where weighting factors $W_{ij} > 0$, parameters $0 < \alpha < 1, \beta > 0$, $\gamma \geq 0$.

Corollary 6. *Suppose that h and F are continuous and piece-wise differentiable. Under protocol (4) and Assumption (A1, A2, A3', A4'), the n agents will be in formation (h, F) in finite time.*

Proof. Also by Theorem 3, $x_i - h_i, i \in \mathcal{I}_m$, will reach consensus in finite time t_1^*. Denote the consensus state by h^c and then after time $t_1^*, x_i(t) = h_i(t) + h^c$.

Rewrite the second equation of protocol (4) as

$$\frac{\mathrm{d}(x_i - \sum_{j=1}^{n} f_j^i x_j)}{\mathrm{d}t} = \beta \operatorname{sig}\Big(\sum_{j=1}^{n} f_j^i (x_j - x_i)\Big)^{\alpha} + \gamma \sum_{j=1}^{n} f_j^i (x_j - x_i),$$
$$i = m+1, m+2, \ldots, n.$$

By Theorem 3, $x_i - \sum_{j=1}^{n} f_j^i x_j$, $i = m+1, m+2, \ldots, n$, will reach zero in finite time t_2^*. Denote $t^* = \max\{t_1^*, t_2^*\}$, after which the n agents are in formation (h, F). ■

4.4. Trajectory tracking

We assume that h^c, representing the expected trajectory of the formation, satisfies the following differential equation

$$\dot{h}^c = f(t, h^c), \tag{5}$$

and the leaders of the studied system satisfy the following assumption

(A5) some root agents in the local interaction topology among the leaders can access the reference trajectory.

The same problem was studied in Ren (2007) in the development of consensus theory. The available protocol in our framework is

$$u_i = \begin{cases} \dot{h}_i + f(t, h^c) \\ \quad + \beta \operatorname{sig}\Big(\sum_{j\in\mathcal{N}_i} W_{ij}\big((x_j-h_j)-(x_i-h_i)\big) \\ \quad + W_i^c\big(h^c - (x_i - h_i)\big)\Big)^{\alpha} \\ \quad + \gamma \Big(\sum_{j\in\mathcal{N}_i} W_{ij}\big((x_j-h_j)-(x_i-h_i)\big) \\ \quad + W_i^c\big(h^c - (x_i - h_i)\big)\Big), \quad i = 1,2,\ldots,m, \\ \sum_{j=1}^{n} (f_j^i u_j + \dot{f}_j^i x_j) + \beta \operatorname{sig}\Big(\sum_{j=1}^{n} f_j^i (x_j - x_i)\Big)^{\alpha} \\ \quad + \gamma \sum_{j=1}^{n} f_j^i (x_j - x_i), \quad i = m+1, m+2,\ldots,n, \end{cases} \tag{6}$$

where parameters $0 < \alpha < 1, \beta > 0, \gamma \geq 0$, weighting factors $W_{ij} > 0$, $W_i^c \geq 0$, and $W_i^c > 0$ if and only if leader i can access the reference trajectory.

Corollary 7. *Suppose that h, F and h^c are continuous and piece-wise differentiable. Under protocol (6) and Assumption (A1, A2, A3', A4', A5), the n agents will be in formation (h, F) in finite time, tracking the trajectory described by Eq. (5).*

Proof. We only consider the dynamics of the leaders. By Theorem 3, there exists a finite time, after which $x_i - h_i - h^c$ will be equal to zero, namely, the leaders will form the formation frame h, tracking the trajectory h^c. ■

5. Simulations

In the following examples, protocol parameters $\alpha = \frac{1}{3}$, $\beta = \gamma = 1$, weighting factors are all equal to 2, and the total number of members is 30. Assume that the first 5 agents are leaders and all the agents move on a plane, namely, $N = 2$. The anticipated formation is a pentagon, specifically,

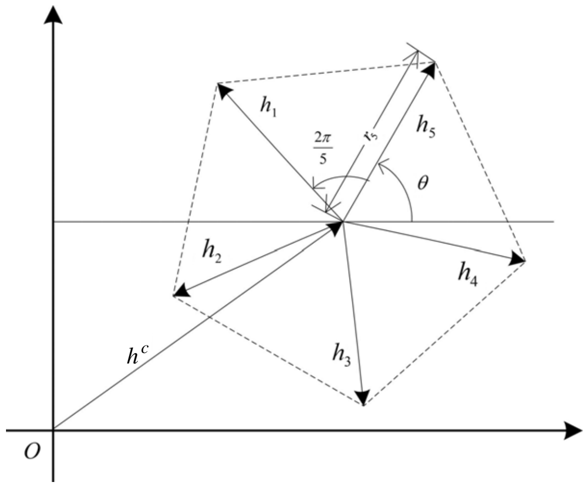

Fig. 1. The expected formation frame h.

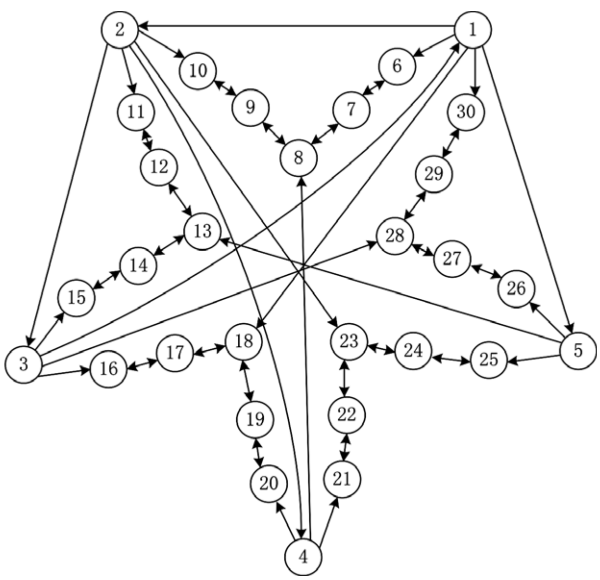

Fig. 2. The interaction topology.

$$
\begin{cases}
h_i = r_i(t)\begin{pmatrix}\cos\left(\theta + \dfrac{2i\pi}{5}\right)\\ \sin\left(\theta + \dfrac{2i\pi}{5}\right)\end{pmatrix}, \quad i = 1, 2, \ldots, 5;\\
f_i^{5i+1} = f_{5i+2}^{5i+1} = f_{5i+1}^{5i+2} = f_{5i+3}^{5i+2} = f_{5i+3}^{5i+4} = f_{5(i+1)}^{5i+4}\\
\quad = f_{5i+4}^{5(i+1)} = f_{i\bmod 5+1}^{5(i+1)} = 0.5,\\
f_{5i+2}^{5i+3} = f_{5i+4}^{5i+3} = 0.45, f_{(i+2)\bmod 5+1}^{5i+3} = 0.1,\\
\qquad\qquad i = 1, 2, \ldots, 5;\\
f_j^i = 0 \text{ in other cases.}
\end{cases}
$$

The expected formation frame h is depicted in Fig. 1 and the interaction topology is depicted in Fig. 2.

Fig. 3 shows the simulation for time-invariant formation, where the initial positions of agents are randomly generated in a given bounded region, $r_i(t) \equiv 10$ and $\theta = 0$.

In the trajectory tracking formation, we assume

$$h^c(t) = \begin{bmatrix} t \\ 20 \end{bmatrix}, \quad t \geq 0,$$

which guides the whole group to move along a straight line. We further assume that the formation frame h is also time-varying, explicitly, $\theta = \frac{\pi t}{30}$. Fig. 4 shows the trajectories of agents in this case.

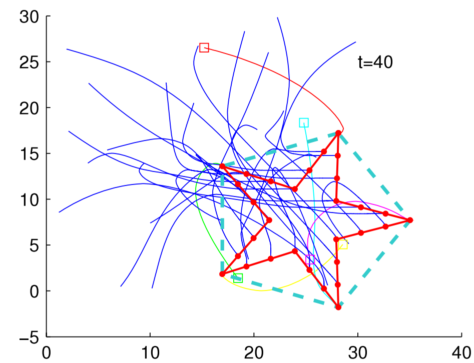

Fig. 3. The trajectories of agents in the time-invariant formation.

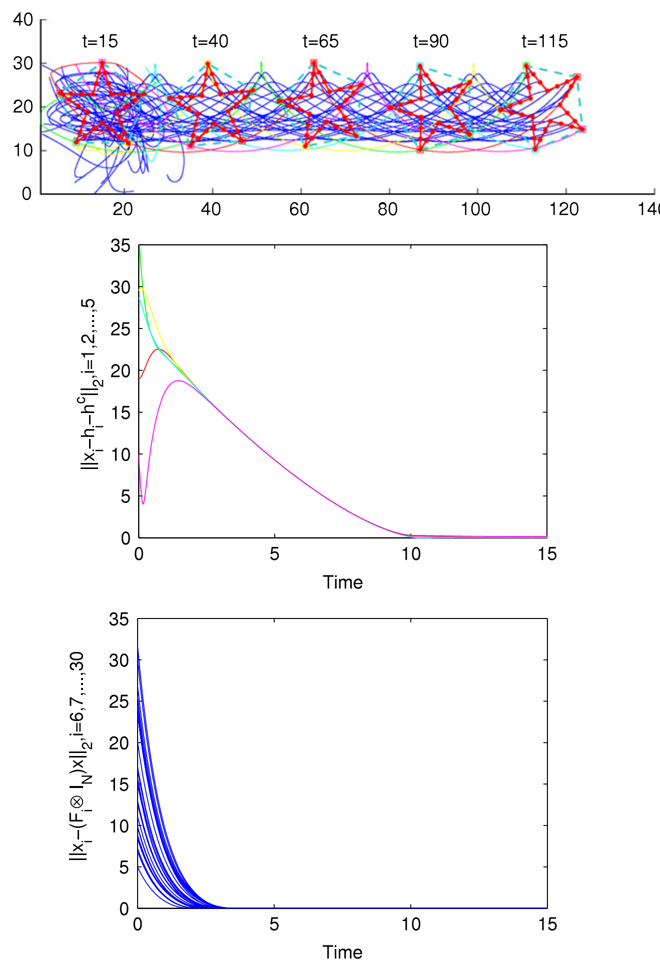

Fig. 4. The trajectories of agents and formation errors $\|x_i - h_i - h^c\|_2$, $i = 1, 2, \ldots, 5$, and $\|x_i - (F_i \otimes I_N)x\|_2$, $i = 6, 7, \ldots, 30$, in the time-varying formation.

6. Conclusion

This paper presented a class of new finite-time consensus protocols and developed a new framework for formation control of multi-agent systems with a large number of agents. In the framework, the problems of time-invariant formation, time-varying formation and trajectory tracking were discussed respectively. Finally, the effectiveness of the proposed control strategies was illustrated by simulations. Nevertheless, in our framework, there are still some other problems that need to be addressed, such as the design of distributed protocols for leaders, which can lead the whole group of agents to get across obstacles to their destination without a reference trajectory, the design of obstacle avoidance strategies with bounded repulsive force, ensuring inter-agent collision avoidance, and the study on more realistic and complex agent dynamics.

Acknowledgements

This work was supported by the Basic Research Foundation of BIT(20080642002), the Beijing Education Committee Cooperation Building Foundation Project, and NSFC (60674050, 60736022 and 60528007).

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