Super-exponential tracking for nonlinear systems with non-vanishing uncertainties

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***Abstract*—In this paper we introduce a general and systematic approach to achieving robust tracking control for nonlinear systems with non-vanishing uncertainties, along with the complete rejection of non-vanishing uncertainties and user-assignable convergence rate. Furthermore, the convergence rate can be pre-specified faster than exponential or nearly as fast as any prescribed finite time if needed. The key design tool is the utilization of a time-varying feedback gain through a time-varying scaling function that satisfies certain conditions. A general way to construct such time-varying rate function is given such that the different yet assignable convergence rates can be achieved.**

***Index Terms*—Nonlinear systems, time-varying feedback, pre-specified convergence rate, super-exponentials.**

I. Introduction

In this paper we present a systematic approach to achieving robust tracking control for nonlinear systems with non-vanishing uncertainties, in which the rate of convergence can be pre-specified by the designer. As is well known, trajectory tracking is the most common goal in control engineering, where the rate of convergence represents the crucial factor to consider in control design for many important applications, such as spacecraft rendezvous and docking, missile interception, machine parts assembling, and vehicle self-parking, etc. [1]–[4]. In fact, for safe and reliable mission accomplishment, it is often required or desired in these applications that the approaching speed to the target be assignable ([5]–[7]).

Earlier efforts that provide inspiration to the approach employed in this work are [8] and [9], in which a speed transformation is employed and then a controller based on the transformed state feedback is designed to stabilize the transformed system, which finally yields the stabilization of the original system with adjustable convergence rate that depends on the selection of the rate function. This idea of using time-varying scaling or speed transformation has been explored by several works, e.g., [10]–[14], but with no clue for achieving arbitrary convergence rate. In this work we address the tracking control problem with assignable convergence rate that can be pre-specified as fast as desired. Further, we do not stop at the special convergence rate, but make the extra step to seek a general and systematic way to construct the time-varying scaling function bearing certain salient features to allow the corresponding tracking control scheme to have user-assignable rate of convergence.

Our method features with complete rejection (rather than partial attenuation) of such uncertainties and regulation of the tracking error to zero at the rate of convergence that can be pre-specified as super exponential or even near-finite time. One of the key designs lies the utilization of time-varying feedback gain, which is based upon the time-varying scaling function satisfying certain conditions. A general and systematic procedure to construct such scaling function is established, which enables the corresponding tracking control scheme with the salient feature of pre-assignable convergence rate.

Notation: Throughout this paper, the initial time t_0 is set as $t_0 = 0$ without loss of generality; $\Re$ and Z denote the set of real numbers and the set of integer numbers, respectively; By C^{∞}, we denote the class of functions that have continuous derivatives of order ∞; By K_{∞}, we denote the class of continuous functions $\alpha : [0, +\infty) \to [0, +\infty)$ that are strictly increasing, with $\alpha(0) = 0$ and $\alpha(r) = \infty$ as $r \to \infty$.

II. Problem Formulation

Our results are developed for the following nonlinear system,

$$\dot{x} = f(x,t) + b(x,t)u, \tag{1}$$

where $x \in \Re$ is the state vector, $u \in \Re$ is the control input, $b(x,t)$ is the (possibly uncertain) control gain, and $f(x,t)$ is the system nonlinearity possibly involving non-parametric or non-vanishing uncertainties.

The following assumptions are in order.

Assumption 1 (global controllability): For system (1), there exists a known $\underline{b} > 0$ such that $\underline{b} \leq |b(x,t)| < \infty$, and $b(x,t)$ is sign-definite (w.l.o.g., sign$(b) = +1$) for all $x \in \Re$ and $t \in \Re_+$.

Assumption 2: (bound on matched but possibly non-vanishing uncertainty) The nonlinearity f in (1) obeys

$$|f(x,t)| \leq d(t)\psi(x) \tag{2}$$

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where $d(t) \geq 0$ is an unknown disturbance bound and $\psi(x) \geq 0$ is a known scalar-valued continuous function.

Assumption 3: For all $t \in [0, \infty)$, the desired trajectory $x^*(t)$ is known and bounded, and $\dot{x}^*$ can be unknown but bounded.

The projective of this work is to present a general and systematic approach to achieving robust tracking control for nonlinear system (1) with non-vanishing uncertainties, in which the unknown non-vanishing uncertainties can be completely rejected rather than partially attenuated. Further, the tracking error to zero at the rate of convergence can be pre-specified as fast as desired (e.g. super exponential or even closer to a pre-specified finite time).

III. General way to construct rate functions

To make the tracking rate as 'fast' as desired, the key is to find suitable rate functions to incorporate into the control scheme. In this section, we introduce one class of functions: K^*, which is associated with the controller design for system (1). In particular, we will introduce a class of super-exponential functions, which belongs to K^* and can be made even closer to any pre-specified finite time.

A. K^* functions

We first give the definition for K^*.

Definition 1: We denote by K^+ the class of non-decreasing C^∞ functions $\nu : \Re_+ \to \Re$ with $\nu(0) = 1$ and $\nu(+\infty) = +\infty$, and we denote by $K^* \subset K^+$ the class of C^∞ functions that belong to K^+ and satisfy $\dot{\nu}/\nu^2 \in L_\infty$.

It is worth noting that, although Definition 1 has explicitly specified the features for K^*, it still unclear what kind of functions belong to K^*, The next lemma establishes a general and systematic procedure for constructing various rate functions that belong to K^* by starting from any K^+ function.

Lemma 1: For every $\phi_1, \phi_2 \in K^*$, and for all constants $h \geq 1$, it holds that the functions $\phi_1 + \phi_2$, $\phi_1\phi_2$ and ϕ_1^h are of class K^* as well. Furthermore, for every function $\phi \in K^+$, there exists a constant $\beta > 0$ and a function $\tilde{\phi} \in K^*$, such that $\phi(t) \leq \beta\tilde{\phi}(t)$ for all $t \geq 0$.

Proof: The implications that $\phi_1 + \phi_2$, $\phi_1\phi_2$ and ϕ_1^h are of class K^* are obvious. It only needs to prove the last statement of this lemma. Define the function

$$\mu(s) := \begin{cases} \phi(\log s), & \text{if } \ s \geq 1 \\ \phi(0)s, & \text{if } \ 0 \leq s < 1. \end{cases} \tag{3}$$

It is clear that $\mu(s)$ belongs to K_∞ and satisfies

$$\phi(t) \leq \mu(e^t), \quad \forall t \geq 0. \tag{4}$$

Define the function

$$\rho(s) := \begin{cases} 0, & \text{if } \ s = 0 \\ \frac{1}{\mu^{-1}(1/s)}, & \text{if } \ s > 0. \end{cases} \tag{5}$$

We have that $\rho(s) \in K_\infty$. Let $\tilde{\rho}(s) \in K_\infty \bigcap C^\infty\big((0, \infty)\big)$ be a function with $\frac{d\tilde{\rho}}{ds}(s) \geq 1$ for all $s > 0$ and $\lim_{s \to 0^+} \frac{\tilde{\rho}(s)}{s} \geq 1$, that satisfies $\tilde{\rho}(s) \geq \rho(s)$ for all $s \geq 0$. Thus by (5) we have

$$\frac{1}{\mu^{-1}(1/s)} \leq \tilde{\rho}(s), \ \forall s > 0 \Rightarrow \mu(e^t) \leq \frac{1}{\tilde{\rho}^{-1}(e^{-t})}, \ \forall t \geq 0. \tag{6}$$

Define

$$\tilde{\phi}(t) := \frac{\tilde{\rho}^{-1}(1)}{\tilde{\rho}^{-1}(e^{-t})}. \tag{7}$$

Since $\tilde{\rho}(s) \in K_\infty \bigcap C^\infty\big((0, \infty)\big)$ with $\frac{d\tilde{\rho}}{ds}(s) \geq 1$ for all $s > 0$, we easily establish that $\tilde{\phi}(t) \in C^\infty(\Re R_+)$ and that $\tilde{\phi}$ is non-decreasing with $\tilde{\phi}(0) = 1$. Furthermore, since $\frac{d\tilde{\rho}}{ds}(s) \geq 1$ for all $s > 0$ and $\lim_{s \to 0^+} \frac{\tilde{\rho}(s)}{s} \geq 1$, it follows that $0 \leq \frac{d\tilde{\rho}^{-1}}{ds}(s) \leq 1$ for all $s \geq 0$. This fact and the definition of $\tilde{\phi}(t)$ in (7) imply

$$\frac{d\tilde{\phi}}{dt}(t) = \frac{\tilde{\phi}^2(t)}{\tilde{\rho}^{-1}(1)} \frac{d\tilde{\rho}^{-1}}{ds}(e^{-t})e^{-t} \leq \frac{\tilde{\phi}^2(t)}{\tilde{\rho}^{-1}(1)} e^{-t}. \tag{8}$$

The latter inequality gives

$$\frac{1}{\tilde{\phi}^2(t)} \frac{d\tilde{\phi}}{dt}(t) \leq \frac{e^{-t}}{\tilde{\rho}^{-1}(1)} \in L_\infty. \tag{9}$$

Consequently, we have $\tilde{\phi}(t) \in K^*$. By combining (4), (6) and (7), we get $\phi(t) \leq \frac{1}{\tilde{\rho}^{-1}(1)}\tilde{\phi}(t) = \beta\tilde{\phi}(t)$ for all $t \geq 0$ with $\beta = \frac{1}{\tilde{\rho}^{-1}(1)}$. ■

Remark 1: The result of Lemma 1, provides a useful procedure to construct a new scaling function $\nu(t) \in K^* \subset K^+$ (with faster rate of convergence) by starting with any $\nu(t) \in K^+$, which can be made as a rate function for the nonlinear system (1) with non-vanishing uncertainties. This result makes it possible to develop the general approach for robust tracking with user-assignable rate of convergence.

The next subsection represents a class of super-exponential functions, *nested exponentials*, with which we can derive the super-exponential tracking control scheme for the system.

B. Super-exponential functions

The sequence of *nested exponential functions* of base a ($a > 1$), which possess the property of having the value of one at the origin, is defined as

$$\nu_{m+1}(t) = \exp_a(\nu_m(t) - 1), \tag{10}$$

with $\nu_1(t) = \exp_a(t) = a^t$ for all $t \in [0, +\infty)$, where $m \in Z_+$. When $a = e$, (10) defines the *natural nested exponential function*, and in such case

$$\nu_{m+1}(t) = \exp(\nu_m(t) - 1) \tag{11}$$

with $\nu_1(t) = \exp(t) = e^t$.

In addition, we introduce the *generalized nested exponential function* of base a ($a > 1$) as follows,

$$\nu_{dm}(t) = \exp_a(d_m(\ldots \exp_a(d_2(\exp_a(d_1 t) - 1)) \ldots - 1)) \tag{12}$$

where $d_i > 0$ ($i = 1, \ldots, m$) denotes the parameter of the mth *nested exponential function*. When $d_i = 1$ ($i = 1, \ldots, m$), the

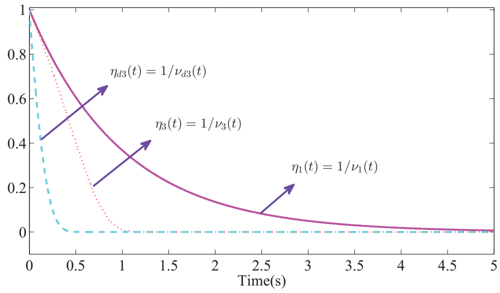


Fig. 1. $\eta_1(t) = 1/\nu_1(t)$, $\eta_3(t) = 1/\nu_3(t)$ and $\eta_{d3}(t) = 1/\nu_{d3}(t)$ with $\nu_1(t) = \exp(t)$, $\nu_3(t) = \exp(\exp(\exp(t) - 1) - 1)$ and $\nu_{d3}(t) = \exp(d_3(\exp(d_2(\exp(d_1 t) - 1)) - 1))$ (here $d_1 = 1$, $d_2 = 2$, $d_3 = 3$).

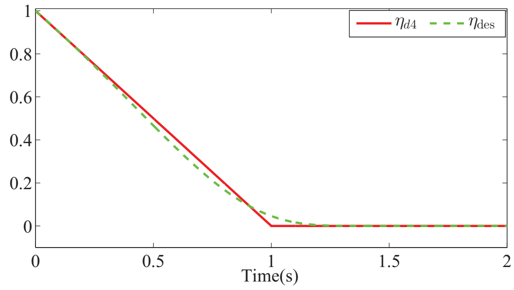


Fig. 2. Approximation of $\eta_{\rm des}(t)$ with $\eta_{d4}(t)$.

generalized nested exponentials reduce to the *nested exponentials*. In addition, when $a = e$, it becomes the *generalized natural nested exponential function*, and in such case

$$\nu_{dm}(t) = \exp(d_m(\dots\exp(d_2(\exp(d_1 t) - 1))\dots - 1)). \tag{13}$$

To gain insight into the distinct growth profiles among *exponential*, *nested-exponential*, and *generalized nested-exponential* functions, we plot in Fig. 1 the following three functions: $\eta_1(t) = 1/\nu_1(t)$; $\eta_3(t) = 1/\nu_3(t)$; and $\eta_{d3}(t) = 1/\nu_{d3}(t)$, where $\nu_1(t) = \exp(t)$ is the *exponential*, $\nu_3(t) = \exp(\exp(\exp(t) - 1) - 1)$ is the *triply-natural nested exponential*, and $\nu_{d3}(t) = \exp(d_3(\exp(d_2(\exp(d_1 t) - 1)) - 1))$ is the *triply-generalized natural nested exponential* with $d_1 = 1$, $d_2 = 2$, and $d_3 = 3$, from which it is interesting to see that super fast convergence can be obtained by using different time-varying scaling function $\nu(t)$.

If the user of the control system has finite-time convergence as the objective, super-exponential functions can be used to approximate functions that converge to zero in finite time. In the following we illustrate with the example of a finite-time converging function how such a function can be approximated, to an acceptable accuracy, with a nested exponential function. Consider the finite-time convergent function

$$\eta_{\rm des} = \begin{cases} \left(1 - \frac{t}{T}\right)^h, & t \in [0, T), \\ 0, & t \in [T, \infty), \end{cases} \tag{14}$$

with T being a “user-specified” interval, and $h \geq 1$. It is interesting to note that such a function can be approximated closely by a generalized nested exponential function with a proper order. For example, for $T = 1s$ and $h = 1$, this function can be approximated by the following *generalized natural nested exponential function* of 4-th order:

$$\begin{aligned} &\eta_{d4}(t) \\ &= \exp(-d_4(\exp(d_3(\exp(d_2(\exp(d_1 t) - 1)) - 1)) - 1)) \end{aligned} \tag{15}$$

for $t \in [0, \infty)$, where $d_1 = 0.6$, $d_2 = 0.8$, $d_3 = 1$, and $d_4 = 2$, as can be seen in Fig. 2. The point of this example is that super-exponential rates of convergence, in addition to being interesting in their own right, can serve as acceptable approximations to commonly-utilized forms of finite-time convergence (such as linear decays to zero under sliding mode control).

Several important properties of the *generalized nested exponential functions* are summarized in the following lemma, which is crucial to our later control design and stability analysis.

Lemma 2: The *generalized nested exponentials*, ν_{dm}, defined in (12), possess the following properties:

i) $\nu_{dm}(t) : \Re_+ \to \Re$ is non-decreasing and C^∞;
ii) $\nu_{dm}(0) = 1$ and $\nu_{dm}(+\infty) = +\infty$;
iii) $\nu_{dm}(t)^h$ is non-decreasing and $\nu_{dm}(t)^{-h}$ is non-increasing for any $h > 0$;
iv) For all $t \geq 0$, there exist a finite positive constant γ_{dm}, such that

$$\dot{\nu}_{dm}/\nu_{dm}^2 \leq \gamma_{dm}; \tag{16}$$

v) ν_{dm} belongs to class K^*.

Proof: Claims i)-iii) are immediate from the function's definition.

iv) To help understand the fundamental idea and technical development of the claimed property, we start with the analysis for *nested exponential functions*, and the results for the *generalized nested exponentials* can be derived by following the same line.

We first prove by induction that, for the nested exponential function $\nu_m(t)$ ($m \in Z_+$) defined as in (12), there exist finite positive constant γ_m such that

$$\dot{\nu}_m \leq \gamma_m \nu_m^2. \tag{17}$$

Note that

$$\dot{\nu}_1 = \ln a \nu_1 \leq \gamma_1 \nu_1^2 \tag{18}$$

with $\gamma_1 = \ln a$. Suppose there exist some finite constant γ_{m-1} ($m \in Z_+/\{1\}$) such that

$$\dot{\nu}_{m-1} \leq \gamma_{m-1}\nu_{m-1}^2. \tag{19}$$

Note that

$$\dot{\nu}_m = (\ln a)\nu_m \dot{\nu}_{m-1}. \tag{20}$$

By inserting (20) into (19), we get

$$\dot{\nu}_m \leq (\ln a)\gamma_{m-1}\nu_m \nu_{m-1}^2. \tag{21}$$

Upon using the fact that

$$\begin{aligned}\nu_m =& a^{\nu_{m-1}-1} = a^{-1}a^{\nu_{m-1}} \\ =& a^{-1}\left[1+\frac{(\ln a)\nu_{m-1}}{1!}+\frac{(\ln a)^2\nu_{m-1}^2}{2!}+\cdots\right], \quad (22)\end{aligned}$$

we have $\nu_{m-1}^2 \leq \frac{2a}{(\ln a)^2}\nu_m$, by inserting which into (21), we then arrive at

$$\dot{\nu}_m \leq \frac{2a}{\ln a}\gamma_{m-1}\nu_m^2. \quad (23)$$

Let $\gamma_m = \frac{2a}{\ln a}\gamma_{m-1}$, we then arrive at (17) by induction.

The results for the *generalized nested exponentials* can be derived by following the same line as in the above analysis.

v) From i)-iii) of this lemma, it is clear that $\nu_{dm} \in K^+$ according to the definition of K^+ given in Definition 1. From the result derived in iv) of this lemma, it is straightforward that $\nu_{dm} \in K^*$ according to the definition of K^*. ■

The next lemma presents another fundamental property related to the generalized nested exponential functions. To proceed, three sets of functions: Σ, $\bar{\Sigma}$ and $\tilde{\Sigma}$ are in need, where Σ contains all the basic elementary functions and hyperbolic functions, and

$$\bar{\Sigma} = \left\{\phi_1+\phi_2, \phi_1\phi_2, M\phi_1^h \middle| \phi_1,\phi_2 \in \Sigma \text{ and } M,h>0\right\}, \quad (24)$$

$$\tilde{\Sigma} = \left\{\phi_1+\phi_2, \phi_1\phi_2, M\phi_1^h \middle| \phi_1,\phi_2 \in \bar{\Sigma} \text{ and } M,h>0\right\}. \quad (25)$$

Lemma 3: For $\forall\phi \in \tilde{\Sigma}\cap K^+$, there exists a generalized nested exponential function $\bar{\phi}$ such that $\phi \leq \bar{\phi}$.

Proof: We first prove that for any basic power function that also belongs to K^+: t^a+1 ($a \geq 1$), there exists a generalized exponential function that is not less than it. Indeed, by noting that

$$\exp(t) = 1+t+\frac{t^2}{2!}+\cdots+\frac{t^{[a]+1}}{([a]+1)!}+\cdots \quad (26)$$

we have

$$1+t^a \leq 1+t^{[a]+1} \leq ([a]+1)!\exp(t) \quad (27)$$

which means that there exists the nested exponential function $([a]+1)!\exp(t)$ that is not less than the basic power function $1+t^a$. Next, we consider the basic logarithmic function $\log_a(t+a)$ that also belongs to K^+. Note that

$$\frac{d\log_a(t+a)}{dt} = \frac{1}{t+a}\ln a \leq a^t \ln a = \frac{da^t}{dt} \quad (28)$$

from which we get

$$\log_a(t+a) \leq a^t \quad (29)$$

meaning that there exists a nested exponential function a^t that is not less than the basic logarithmic function $\log_a(t+a)$. Note that the trigonometric functions are not belong to K^+. We now consider the hyperbolic functions, in which only $\sinh(t)+1$ and $\cosh(t)$ are belong to K^+. Apparently,

$$\begin{aligned}\sinh(t)+1 =& \frac{\exp(t)-\exp(-t)}{2}+1 \\ \leq& 1+\exp(t) \leq \exp(\exp(t)), \quad (30)\end{aligned}$$

$$\cosh(t) \leq \frac{\exp(t)+\exp(-t)}{2} \leq \exp(t), \quad (31)$$

both of which imply that there exist nested exponential functions that are not less than the hyperbolic functions that are also belong to K^+.

By now, for any function ϕ that belongs to $\Sigma \cap K^+$, we can conclude that there exists a generalized nested exponential function $\bar{\phi}$ such that $\phi \leq \bar{\phi}$. For functions that belong to $\tilde{\Sigma}\cap K^+$, the claims are immediate from the definition of the function. ■

Remark 2: On the basis of Lemma 2 and Lemma 3, one can start with any $\nu \in \tilde{\Sigma}\cap K^+$ to choose a generalized nested exponential function that converges at the rate faster than that of the original $\nu(t)$, making the selection for rate functions be a simple task and the results more generalized. With the help of Lemmas 1-3, a general approach for tracking control with assignable convergence rate (including super-exponential or near-finite time as special cases) can be developed in what follows.

IV. Tracking with pre-assignable convergence for first-order nonlinear systems

In this section, we develop a tracking control law based on the time-varying scaling function that belongs to K^* for system (1). To derive the result, the following lemma is needed.

Lemma 4: ([15]) For the constant $l>0$ and time-varying function $\nu(t) \geq 1$, it holds that

$$\int_{t_0}^{t} \exp^{-l\int_\tau^t \nu(s)ds}\nu(\tau)d\tau \leq 1/l. \quad (32)$$

Define the tracking error as $\epsilon = x - x^*$. We make use of the time-varying scaling function $\nu \in K^*$ to perform the transformation

$$\xi = \nu\epsilon. \quad (33)$$

With ξ the original model (1) is converted into

$$\dot{\xi} = \nu\dot{\epsilon}+\dot{\nu}\epsilon = \nu\big[bu+f-\dot{x}^*+(\dot{\nu}/\nu)\epsilon\big]. \quad (34)$$

The controller is designed as

$$u = -\frac{1}{b}(k+\theta+k_1\psi^2)\xi, \quad k>k_1>0 \quad (35)$$

where $\theta>0$ is a finite constant such that $|\dot{\nu}/\nu^2| \leq \theta$ according to Definition 1.

Theorem 1: For system (1) with control (35),

$$|\epsilon(t)| \leq \eta(t)\left(\exp^{-(k-k_1)\int_{t_0}^t \nu(\tau)d\tau}|\epsilon(t_0)| + \frac{\sqrt{\|\bar{d}\|_{[t_0,t]}}}{2\sqrt{k_1(k-k_1)}}\right) \quad (36)$$

for all $t \in [0,\infty)$, where $\eta = 1/\nu$ and

$$\|\bar{d}\|_{[t_0,t]} := \sup_{\tau\in[t_0,t]} \left(d(\tau)^2 + \dot{x}^*(\tau)^2\right). \tag{37}$$

Furthermore, the control input u remains uniformly bounded.

Proof: We choose the Lyapunov function candidate as $V = \xi^2/2$, whose derivative along (34) is

$$\dot{V} = \xi\nu\left[bu + f - \dot{x}^* + (\dot{\nu}/\nu^2)\xi\right]. \tag{38}$$

Applying Young's inequality yields

$$\xi\nu f \le \nu k_1\xi^2\psi^2 + \nu\frac{d(t)^2}{4k_1} \le \xi\nu b\frac{1}{\underline{b}}k_1\psi^2\xi + \nu\frac{d(t)^2}{4k_1}, \tag{39}$$

$$\xi\nu(-\dot{x}^*) \le \xi\nu b\frac{1}{\underline{b}}k_1\xi + \nu\frac{\dot{x}^*(t)^2}{4k_1}, \tag{40}$$

$$\xi\nu(\dot{\nu}/\nu^2)\xi \le \xi\nu b\frac{1}{\underline{b}}\theta\xi. \tag{41}$$

We insert (39)-(41) into (38) and get

$$\begin{aligned}\dot{V} \le& \xi\nu b\left(u + \frac{1}{\underline{b}}k_1\psi^2\xi + \frac{1}{\underline{b}}k_1\xi + \frac{1}{\underline{b}}\theta\xi\right)\\ &+ \nu\left(\frac{d(t)^2}{4k_1} + \frac{\dot{x}^*(t)^2}{4k_1}\right).\end{aligned} \tag{42}$$

Upon using (35) and (37), we get from (42) that

$$\begin{aligned}\dot{V} \le& -\frac{b}{\underline{b}}(k-k_1)\nu\xi^2 + \nu\frac{\|\bar{d}\|_{[t_0,t]}}{4k_1}\\ \le& -2(k-k_1)\nu V + \nu\frac{\|\bar{d}\|_{[t_0,t]}}{4k_1}.\end{aligned} \tag{43}$$

Solving the differential inequality (43) gives

$$\begin{aligned}V(t) \le& \exp^{-2(k-k_1)\int_{t_0}^{t}\nu(\tau)d\tau} V(t_0) + \frac{\|\bar{d}\|_{[t_0,t]}}{4k_1}\\ &\times\int_{t_0}^{t}\exp^{-2(k-k_1)\int_{\tau}^{t}\nu(s)ds}\nu(\tau)d\tau.\end{aligned} \tag{44}$$

Upon using Lemma 4, we obtain

$$V(t) \le \exp^{-2(k-k_1)\int_{t_0}^{t}\nu(\tau)d\tau} V(t_0) + \frac{\|\bar{d}\|_{[t_0,t]}}{8k_1(k-k_1)}, \tag{45}$$

which implies

$$|\xi(t)| \le \exp^{-(k-k_1)\int_{t_0}^{t}\nu(\tau)d\tau}|\xi(t_0)| + \frac{\sqrt{\|\bar{d}\|_{[t_0,t]}}}{2\sqrt{k_1(k-k_1)}}, \tag{46}$$

and further implies (36) from (33). The claimed property of control input u in (35) follows from the boundedness of ξ and ψ, where the boundedness of ξ is established by (46) and the boundedness of ψ is derived from (36) and Assumption 3. ■

Remark 3: As $\nu(t)$ can be chosen by the designer, which has a variety of forms, the resultant control method is able to achieve exponential tracking, super exponential tracking or even near-finite time tracking by simply specifying ν properly.

V. Numerical Simulations

To further verify the effectiveness of the proposed control method, here we compare the proposed control with the Nussbaum gain based control established in [16], which enables the tracking error to converge to zero for system with bounded disturbances. Here we consider the same simulation model with actuator nonlinearity as in [16],

$$\dot{x} = \mu\tanh(x/2) + \mu_0 u(t) + d(t) \tag{47}$$

where the the actuator nonlinearity is modeled by the following Bouc-Wen hysteresis,

$$u = \mu_1\upsilon + \mu_2\zeta, \tag{48}$$

$$\dot{\zeta} = \dot{\upsilon} - \beta_0|\dot{\upsilon}||\zeta|^{m_0-1}\zeta - \chi\dot{\upsilon}|\zeta|^{m_0}, \tag{49}$$

in which μ_1 and μ_2 are constants with the same sign, β_0 and χ describe the shape and amplitude of the hysteresis, respectively, and $\beta_0 > |\chi|$, $m_0 \ge 1$. Then the system dynamic (47) with Bouc-Wen hysteresis becomes,

$$\dot{x} = \mu\tanh(x/2) + \rho\upsilon(t) + D_0(t) \tag{50}$$

with $\rho(t) = \mu_0\mu_1$ and $D_0(t) = \mu_0\mu_2\zeta + d(t)$. The desired trajectory is $x^* = \sin(t)$. The control scheme established in [16] based on backstepping technique and Nussbaum gain approach is of the following form,

$$z_1 = x_1 - x^*, \tag{51}$$

$$g(\cdot) = \sqrt{\tanh(x/2) + l_1} + 1, \tag{52}$$

$$\upsilon(t) = N(\lambda)\bar{\upsilon}(t), \tag{53}$$

$$\bar{\upsilon}(t) = (c_1 + k_0)z_1 + \frac{z_1\hat{\varrho}(t)^2 g(\cdot)^2}{z_1\tanh(z_1\varepsilon^{-1})\hat{\varrho}(t)g(\cdot) + \varepsilon}, \tag{54}$$

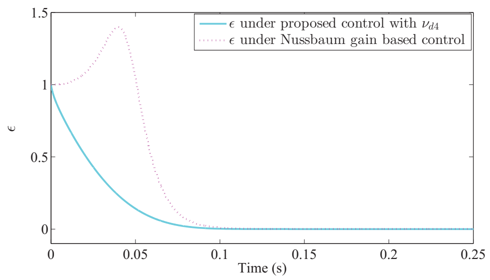
$$\dot{\hat{\varrho}} = \Gamma[|z_1|g(\cdot) - \sigma_0(\hat{\varrho}(t) - \varrho_0)], \tag{55}$$

$$\dot{\lambda} = z_1\bar{\upsilon}(t), \tag{56}$$

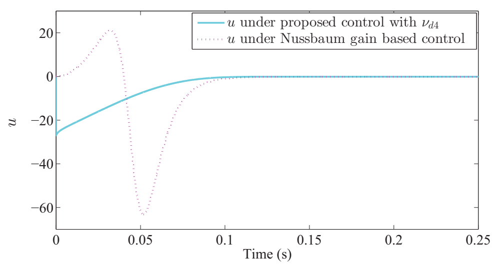
$$N(\lambda) = \lambda^2\cos(\lambda). \tag{57}$$

In the simulation, the actual system parameters are $\mu = 1$ and $\mu_0 = 1$, the disturbance is $d(t) = 0.1\sin(t)$, the initial state is $x(0) = 1$, $\mu_1 = \mu_2 = 1$, $\beta_0 = 1$, $m_0 = 1$, and $\chi = 0$. The design parameters in the control scheme (51)–(57) are taken the same as in [16]: $l_1 = 10$, $c_1 = 15$, $k_0 = 15$, $\varepsilon = 10^4$, $\Gamma = 10$, $\sigma_0 = 10$, $\varrho_0 = 2$, with the initial conditions: $\hat{\varrho}(0) = 0$, $\lambda(0) = 0$, $\zeta(0) = 1$. On the other hand, the design parameters in the proposed control (35) are taken as: $k = 12$, $\theta_0 = 12$, $k_1 = 1$, and $\psi(x) = |\tanh(x/2)|$. For the proposed method, we detect the scaling function ν_{d4} defined the same as in (12) with $m = 4$, $d_1 = d_2 = 0.89$, $d_3 = 2.167$, and $d_4 = 7.77$.

The simulation results are shown in Fig.3, from which we see that the superiority of our method is obvious in terms of transient behavior and steady-state precision as well as the control effort.



(a) e.



(b) u.

Fig. 3. System response under the two control schemes: the proposed control (35) and the Nussbuam gain based control (51)–(57) in [16].

VI. Conclusions

We introduce a general and systematic approach for tracking control with complete rejection of non-vanishing uncertainties and assignable convergence rate. Furthermore, the convergence rate can be pre-specified faster than exponential or nearly as fast as any prescribed finite time if needed. A general way to construct the time-varying rate function is given.

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