Formation-containment control for high-order linear time-invariant multi-agent systems

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Abstract: Formation-containment control problems for high-order linear time-invariant multi-agent systems with directed interaction topologies are dealt with. Firstly, protocols are presented for leaders and followers respectively to drive the states of leaders to realize the predefined time-varying formation and propel the states of followers to converge to the convex hull formed by the states of leaders. Then formation-containment problems of multi-agent systems are transformed into asymptotic stability problems and an explicit expression of the formation reference function is derived. Sufficient conditions for multi-agent systems to achieve formation-containment are proposed. Furthermore, necessary and sufficient conditions for multi-agent systems to achieve containment and time-varying formation are presented respectively as special cases. An approach to determine the gain matrices in the protocols is given. It is shown that containment problems, formation control problems, consensus problems and consensus tracking problems can all be treated as special cases of formation-containment problems. Finally, numerical simulations are provided to demonstrate theoretical results.

Key Words: Formation-containment, High-order, Multi-agent system

1 INTRODUCTION

The research on cooperative control of multi-agent systems has gained considerable attention from scientific communities. This is partly due to broad potential applications of cooperative control of multi-agent systems in various areas, such as flocking [1], rendezvous [2], formation control [3]-[5] and synchronization [6]-[7], etc. Consensus is known as one of the most important issues in the cooperative control of multi-agent systems, which means that all agents reach an agreement on certain variables of interest. In the past decade, consensus problems for linear time-invariant (LTI) multi-agent systems have been studied extensively (e.g., [8]-[16] and references therein).

Although many results have been obtained for consensus problems of multi-agent systems, most of them only treat leaderless cases, where group reference states are not considered. In some practical applications, there exist one or even multiple leaders providing external control instructions or group reference states for multi-agent systems. Consensus with one leader is often named as coordinated tracking or consensus tracking. Hong *et al.* discussed consensus tracking problems for multi-agent systems with second-order dynamics and switching interaction topologies in [17] and [18]. Consensus tracking problems for high-order LTI multi-agent systems with both fixed and switching interaction topologies were dealt with by Ni and Cheng in [19].

If there exist multiple leaders, containment problems which require that the states of followers converge to the convex hull formed by the states of leaders arise. Containment problems for low-order multi-agent systems have been considered in [20]-[26]. Li *et al.* [27] discussed containment problems for high-order LTI multi-agent systems using a dynamic output protocol. Dong *et al.* [28] investigated containment problems for high-order LTI singular multi-agent systems with time delays. By classifying the agents into boundary agents and internal agents, Liu *et al.* [29] proposed a criterion for the states of internal agents to converge to a convex combination of the states of the boundary agents. However, in [20]-[29], there exists no coordination among leaders. In some practical scenarios, it is possible for some leaders to be the neighbors of other leaders and to coordinate with each other to accomplish certain complicated tasks, such as formation control. The problems that leaders not only provider reference signals, but also keep certain formation, and the states of followers are required to converge to the convex hull formed by the states of leaders are regarded as formation-containment problems. Ferrari-Trecate *et al.* [30] studied formation-containment problems for first-order multi-agent systems with undirected interaction topologies. Dimarogonas *et al.* [31] dealt with formation-containment problems for second-order nonholonomic multi-agent systems with undirected interaction topologies, and presented sufficient conditions for multi-agent systems to achieve formation-containment. To the best of our knowledge, formation-containment problems for high-order LTI multi-agent systems with directed interaction topologies are still open. They has generality and it can be shown that containment problems, formation control problems, consensus problems and consensus tracking problems can be treated as special cases of formation-containment problems.

In the current paper, formation-containment control problems for high-order LTI multi-agent systems with directed interaction topologies are investigated. Firstly, protocols are presented for leaders and followers respectively to ensure that the leaders and followers achieve desired time-varying formation and containment respectively. Then formation-containment problems for multi-agent systems are transformed into asymptotic stability problems and an explicit expression of the time-varying formation reference function is derived. Moreover, sufficient conditions for multi-agent systems to achieve formation-containment are presented, and necessary and sufficient conditions for multi-agent systems to achieve containment and time-varying formation are proposed respectively as special cases. Finally, an approach to determine the gain matrices in protocols is given.

This work was supported by the National Natural Science Foundation of China under Grants 61374034 and 61174067.

Compared with the existing results on containment and formation-containment, the main features of the current paper are threefold. Firstly, leaders are allowed to interact with other leaders to achieve the desired time-varying formation. In [20]-[29], leaders cannot achieve time-varying formations. Secondly, both formation-containment analysis and design problems for high-order LTI multi-agent systems with directed interaction topologies are dealt with. In [30] and [31], the dynamics of each agent is restricted to be of first order or second order. Moreover, in [30] and [31], it is assumed that the interaction topologies were undirected and the formations were time-invariant. Thirdly, necessary and sufficient conditions for high-order LTI multi-agent systems to achieve containment and time-varying formation are proposed respectively as special cases. It is also shown that many existing results in cooperative control of multi-agent systems can be regarded as special cases of formation-containment problems, such as containment problems dealt with in [23] and [25], formation control problems discussed in [4] and [5], consensus problems studied in [15] and consensus tracking problems investigated in [19], etc.

The rest of this paper is organized as follows. In Section 2, basic concepts and useful results in graph theory are introduced and the problem description is presented. In Section 3, formation-containment problems of multi-agent systems are transformed into asymptotic stability problems and primary results are derived. In Section 4, sufficient conditions for multi-agent systems to achieve formation-containment are proposed and an approach to determine the gain matrices in the protocols is presented. Numerical simulations are provided in Section 5. Finally, Section 6 concludes the whole work.

Throughout this paper, for simplicity of notation, let 0 be the zero matrices of appropriate size with zero vectors and zero number as special cases. $\mathbf{1}$ is used to represent a column vector of appropriate size with 1 as its elements. I stands for an identity matrix with appropriate dimension. $\otimes$ denotes Kronecker product.

2 Preliminaries and problem description

In this section, basic concepts and results in graph theory are introduced and the problem description is presented.

2.1 Basic concepts and results in graph theory

A weighted directed graph of order N can be denoted by $G = \{V, \varepsilon, W\}$, with the set of nodes $V = \{v_1, v_2, \cdots, v_N\}$, the set of edges $\varepsilon \subseteq \{(v_i, v_j) : v_i, v_j \in V\}$ and a weighted adjacency matrix $W = [w_{ij}] \in \mathbb{R}^{N\times N}$ with nonnegative adjacency matrix elements w_{ij}. An edge of G is denoted by $e_{ij} = (v_i, v_j)$. The adjacency elements associated with the edges of G are positive, i.e., $w_{ji} > 0$ if and only if $e_{ij} \in \varepsilon$. Besides, $w_{ii} = 0$ for all $i \in \{1, 2, \cdots, N\}$. The set of *neighbors* of node v_i is defined as $N_i = \{v_j \in V : (v_j, v_i) \in \varepsilon\}$. The *in-degree* of node v_i is denoted by $\deg_{in}(v_i) = \sum_{j=1}^{N} w_{ij}$. The *degree matrix* of G is represented by $D = \operatorname{diag}\{\deg_{in}(v_i), i = 1, 2, \cdots, N\}$. The *Laplacian matrix* of G is defined as $L = D - W$. A *directed path* from node v_{i_1} to v_{i_l} is a sequence of ordered edges with the form of $(v_{i_k}, v_{i_{k+1}})$, where $v_{i_k} \in V$ $(k = 1, 2, \cdots, l)$. A directed graph is said to have a *spanning tree* if there exists at least one node having a directed path to all the other nodes. More details on graph theory can be found in [32]. The following lemma is useful in analyzing the formation-containment problems of multi-agent systems.

Lemma 1 ([10]) *Let $L \in \mathbb{R}^{N\times N}$ be the Laplacian matrix of a directed graph G, then*

(i) L has at least one zero eigenvalue, and $\mathbf{1}$ is the associated eigenvector; that is, $L\mathbf{1} = 0$;

(ii) If G has a spanning tree, then 0 is a simple eigenvalue of L, and all the other $N - 1$ eigenvalues have positive real parts.

2.2 Problem description

Consider a high-order LTI multi-agent system with N agents described by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \tag{1}$$

where $i = 1, 2, \cdots, N$, $x_i(t) \in \mathbb{R}^n$ is the state, $u_i(t) \in \mathbb{R}^m$ is the control input. The interaction topology of multi-agent system (1) is denoted by G. Each agent is regarded as a node in G. For $i, j \in \{1, 2, \cdots, N\}$, the available interaction from agent i to agent j can be represented by edge e_{ij} and the interaction strength of e_{ij} can be denoted by the nonnegative adjacency element w_{ji}.

Definition 1 A formation is specified by a vector $h(t) = [h_1^T(t), h_2^T(t), \cdots, h_N^T(t)]^T \in \mathbb{R}^{nN}$. Multi-agent system (1) is said to *achieve formation* $h(t)$ if for any given bounded initial states, there exists a vector-valued function $r(t) \in \mathbb{R}^n$ such that

$$\lim_{t\to\infty} (x_i(t) - h_i(t) - r(t)) = 0 \; (i = 1, 2, \cdots, N), \tag{2}$$

where $r(t)$ is called a *formation reference function*.

Definition 2 Agents in the multi-agent system are classified into two categories: leaders and followers. An agent is called a *leader* if its neighbors are only leaders and it coordinates with its neighbors to achieve a formation. An agent is called a *follower* if it has at least one neighbor in the multi-agent system and coordinates with its neighbors to achieve containment.

Assume that in multi-agent system (1) there are M $(M < N)$ followers with states $x_k(t)$ $(k = 1, 2, \cdots, M)$ and $N - M$ leaders with states $x_i(t)$ $(i = M + 1, M + 2, \cdots, N)$. Let $F = \{1, 2, \cdots, M\}$ and $E = \{M + 1, M + 2, \cdots, N\}$ be the follower set and leader set respectively.

Definition 3 Multi-agent system (1) is said to achieve containment if for any given bounded initial states, and any $k \in F$, there exist nonnegative constants α_j $(j \in F)$ satisfying $\sum_{j=M+1}^{N} \alpha_j = 1$ such that

$$\lim_{t\to\infty} \left(x_k(t) - \sum_{j=M+1}^{N} \alpha_j x_j(t) \right) = 0. \tag{3}$$

Definition 4 Multi-agent system (1) is said to achieve formation-containment if for any given bounded initial states, there exist a vector-valued function $r(t) \in$

$\mathbb{R}^n$ and nonnegative constants α_j $(j \in F)$ satisfying $\sum_{j=M+1}^{N} \alpha_j = 1$ such that for any $i \in E$ and $k \in F$, (2) and (3) hold simultaneously.

Definition 5 Multi-agent system (1) is said to achieve consensus if for any given bounded initial states, there exists a vector-valued function $c(t) \in \mathbb{R}^n$ such that

$$\lim_{t \to \infty} (x_i(t) - c(t)) = 0 \; (i = 1, 2, \cdots, N), \tag{4}$$

where $c(t)$ is called a *consensus function*.

Remark 1 From Definitions 1, 3, 4 and 5, one can see that if $M = 0$, formation-containment problems become the formation problems. If $M = N - 1$ or $M = 0$ and $h(t) \equiv 0$ formation-containment problems are just the consensus tracking problems or consensus problems respectively, and the formation reference function is just the consensus function. If $h(t) \equiv 0$ and leaders have no neighbors; that is, for all $i, j \in E$, $w_{ij} = 0$, the formation-containment problem becomes the containment problem. Therefore, formation problems, consensus tracking problems, consensus problems and containment problems all can be regarded as special cases of formation-containment problems.

Let $h_E(t) = [h_{M+1}^T(t), h_{M+2}^T(t), \cdots, h_N^T(t)]^T$. Consider the formation $h_E(t)$ with $h_i(t)$, $i \in E$ piecewise continuously differentiable and consider the following formation-containment protocols

$$u_i(t) = K_1 x_i(t) + K_2 \sum_{j \in N_i} w_{ij} \left(x_i(t) - x_j(t)\right), \; i \in F, \tag{5}$$

$$u_i(t) = K_1 x_i(t) + K_3 \sum_{j \in N_i} w_{ij} \left(\tilde{x}_i(t) - \tilde{x}_j(t)\right), \; i \in E, \tag{6}$$

where $\tilde{x}_i(t) = x_i(t) - h_i(t)$, $i \in E$ and $K_i \in \mathbb{R}^{m \times n}$ $(i = 1, 2, 3)$ are constant gain matrices.

Remark 2 In protocols (5) and (6), the gain matrix K_1 will be used to specify the motion modes of the formation reference. K_2 and K_3 will be used to drive the states of followers to converge to the convex hull formed by those of leaders and propel the states of leaders to achieve the desired formation respectively. The gain matrices K_i $(i = 1, 2, 3)$ can be designed in the following procedure. Firstly, design K_1 to assign the motion modes of the formation reference. Then design K_2 and K_3 to achieve the formation-containment.

In the current paper, the following two problems for multi-agent system (1) with protocols (5) and (6) are mainly investigated: (i) under what conditions formation-containment can be achieved; and (ii) how to determine the gain matrices in protocols (5) and (6) to achieve formation-containment.

3 Problem transformation and primary results

In this section, formation-containment problems for multi-agent system (1) under protocols (5) and (6) are transformed into asymptotic stability problems and an explicit expression of the formation reference function is given.

Denote the interaction topology among leaders as G_E. It is assumed that G_E has a spanning tree. Under Definition 2, the Laplacian matrix corresponding to G has the following form

$$L = \begin{bmatrix} L_1 & L_2 \\ 0 & L_3 \end{bmatrix},$$

where $L_1 \in \mathbb{R}^{M \times M}$, $L_2 \in \mathbb{R}^{M \times (N-M)}$, and $L_3 \in \mathbb{R}^{(N-M) \times (N-M)}$ which is the Laplacian matrix corresponding to G_E.

Assumption 1 For each follower, there exists at least one leader that has a directed path to it.

Under Assumption 1, the following lemma can be obtained.

Lemma 2 ([21]) *If the directed interaction topology G satisfies Assumption 1, then all the eigenvalues of L_1 have positive real parts, each entry of $-L_1^{-1}L_2$ is nonnegative, and each row of $-L_1^{-1}L_2$ has a sum equal to one.*

Let $x_F(t) = [x_1^T(t), x_2^T(t), \cdots, x_M^T(t)]^T$ and $x_E(t) = [x_{M+1}^T(t), x_{M+2}^T(t), \cdots, x_N^T(t)]^T$. Under protocols (5) and (6), multi-agent system (1) can be written in a compact form as follows

$$\begin{aligned}\dot{x}_F(t) = &\left(I_M \otimes (A + BK_1) + L_1 \otimes BK_2\right) x_F(t) \\ &+ (L_2 \otimes BK_2) x_E(t),\end{aligned} \tag{7}$$

$$\begin{aligned}\dot{x}_E(t) = &\left(I_{N-M} \otimes (A + BK_1) + L_3 \otimes BK_3\right) x_E(t) \\ &- (L_3 \otimes BK_3)\, h_E(t).\end{aligned} \tag{8}$$

Let $\tilde{x}_E(t) = [\tilde{x}_{M+1}^T(t), \tilde{x}_{M+2}^T(t), \cdots, \tilde{x}_N^T(t)]^T$. Then system (8) can be rewritten as follows

$$\begin{aligned}\dot{\tilde{x}}_E(t) = &\left(I_{N-M} \otimes (A + BK_1) + L_3 \otimes BK_3\right) \tilde{x}_E(t) \\ &+ (I_{N-M} \otimes (A + BK_1)) h_E(t) - (I_{N-M} \otimes I) \dot{h}_E(t).\end{aligned} \tag{9}$$

The following lemma holds directly.

Lemma 3 *Multi-agent system (8) achieves formation $h_E(t)$ if and only if multi-agent system (9) achieves consensus.*

The eigenvalues of the matrix L_3 are denoted by λ_i $(i \in E)$, where $\lambda_{M+1} = 0$ with the associated eigenvector $\bar{u}_{M+1} = \mathbf{1}$ and $0 < \mathrm{Re}(\lambda_{M+2}) \le \cdots \le \mathrm{Re}(\lambda_N)$. Let $U_E^{-1} L_3 U_E = J_E$, where $U_E = [\bar{u}_{M+1}, \bar{u}_{M+2}, \cdots, \bar{u}_N]$, $U_E^{-1} = [\tilde{u}_{M+1}^H, \tilde{u}_{M+2}^H, \cdots, \tilde{u}_N^H]^H$ and J_E is the Jordan canonical form of L_3.

Let $c_j \in \mathbb{R}^n$ $(j = 1, 2, \cdots, n)$ be linear independent vectors and $q_j = \bar{u}_i \otimes c_p$ $(j = (i-1)n + p; i = M+1, M+2, \cdots, N; p = 1, 2, \cdots, n)$. The subspace $\mathbb{C}(U)$ spanned by $q_{Mn+1}, q_{Mn+2}, \cdots, q_{Mn+n}$ is called a *consensus subspace* (CS) and the subspace $\bar{\mathbb{C}}(U)$ spanned by $q_{Mn+n+1}, q_{Mn+n+2}, \cdots, q_{Nn}$ is called a *complement consensus subspace* (CCS). Since q_j $(j = Mn+1, Mn+2, \cdots, Nn)$ are linear independent, the following lemma can be obtained.

Lemma 4 $\mathbb{C}(U) \oplus \bar{\mathbb{C}}(U) = \mathbb{C}^{(N-M)n}$.

From Lemma 1 and the structure of U_E, one can set $J_E = \mathrm{diag}\left\{0, \bar{J}_E\right\}$ where $\bar{J}_E$ consists of Jordan blocks corresponding to λ_i $(i = M+2, M+3, \cdots, N)$. Let $\tilde{U}_E = [\tilde{u}_{M+2}^H, \tilde{u}_{M+3}^H, \cdots, \tilde{u}_N^H]^H$, $\theta(t) = (\tilde{u}_{M+1} \otimes I)\tilde{x}_E(t)$ and $\varsigma(t) = (\tilde{U}_E \otimes I)\tilde{x}_E(t)$, then system (9) can be transformed into

$$\begin{aligned}\dot{\theta}(t) = &(A + BK_1)\, \theta(t) + (\tilde{u}_{M+1} \otimes (A + BK_1))\, h_E(t) \\ &- (\tilde{u}_{M+1} \otimes I) \dot{h}_E(t),\end{aligned} \tag{10}$$

$$\begin{aligned}\dot{\varsigma}(t)=&\left(I_{N-M-1}\otimes(A+BK_1)+\bar{J}_E\otimes BK_3\right)\varsigma(t)\\&+(\tilde{U}_E\otimes(A+BK_1))h_E(t)-(\tilde{U}_E\otimes I)\dot{h}_E(t).\end{aligned}\tag{11}$$

From Lemma 4, the states of system (9) can be uniquely projected onto $\mathbb{C}(U)$ and $\bar{\mathbb{C}}(U)$; that is, there exist $\alpha_j(t)$ $(j = Mn+1, Mn+2, \cdots, Nn)$ such that $\tilde{x}_E(t) = \tilde{x}_{EC}(t) + \tilde{x}_{E\bar{C}}(t)$ with $\tilde{x}_{EC}(t) = \sum_{j=Mn+1}^{Mn+n}\alpha_j(t)q_j$ and $\tilde{x}_{E\bar{C}}(t) = \sum_{j=Mn+n+1}^{Nn}\alpha_j(t)q_j$. Considering that $\varsigma(t) = \left[0, I_{(N-M-1)n}\right]\left(U_E^{-1}\otimes I\right)\tilde{x}_E(t)$, $\theta(t) = [I_n, 0]\left(U_E^{-1}\otimes I\right)\tilde{x}_E(t)$ and $\bar{u}_i\otimes c_j = (U_E\otimes I_n)(e_i\otimes c_j)$ where $i\in E$, $j\in\{1,2,\cdots,n\}$ and $e_i\in\mathbb{R}^{N-M}$ with 1 as its $i-M$th component and 0 elsewhere, one can obtain that

$$\tilde{x}_{E\bar{C}}(t) = (U_E\otimes I_n)\left[0, \varsigma^H(t)\right]^H, \tag{12}$$

$$\tilde{x}_{EC}(t) = (U_E\otimes I_n)\left[\theta^H(t), 0\right]^H = \bar{u}_{M+1}\otimes\theta(t). \tag{13}$$

Based on the structures of q_j $(j = Mn+1, Mn+2, \cdots, Nn)$, the subsystems with states $\tilde{x}_{EC}(t)$ and $\tilde{x}_{E\bar{C}}(t)$ describe the consensus dynamics and disagreement dynamics of system (9) respectively. From Lemma 3 and (10)-(13), the following lemma can be obtained.

Lemma 5 *Multi-agent system (8) achieves formation $h_E(t)$ if and only if for any given bounded initial states,*

$$\lim_{t\to\infty}\varsigma(t) = 0.$$

Let

$$\xi_i(t) = \sum_{j\in N_i} w_{ij}\left(x_i(t)-x_j(t)\right)\ (i\in F),$$

and $\xi(t) = [\xi_1^T(t), \xi_2^T(t), \cdots, \xi_M^T(t)]^T$, then one has

$$\xi(t) = (L_2\otimes I)\,x_E(t) + (L_1\otimes I)\,x_F(t). \tag{14}$$

If $\lim_{t\to\infty}\xi(t) = 0$, one can obtain

$$\lim_{t\to\infty}\left(x_F(t) - \left(-L_1^{-1}L_2\otimes I\right)x_E(t)\right) = 0. \tag{15}$$

From (15) and Lemmas 2 and 5, the following lemma holds.

Lemma 6 *Multi-agent systems (1) under protocols (5) and (6) achieves formation-containment if for any given bounded initial states,*

$$\begin{cases}\lim\limits_{t\to\infty}\varsigma(t) = 0,\\ \lim\limits_{t\to\infty}\xi(t) = 0.\end{cases}$$

Based on the above analysis, an explicit expression of the formation reference function can be presented.

Lemma 7 *If multi-agent systems (1) under protocols (5) and (6) achieves formation-containment, then the formation reference function $r(t)$ satisfies*

$$\lim_{t\to\infty}\left(r(t) - r_0(t) - r_h(t)\right) = 0,$$

with $r_0(t) = e^{(A+BK_1)t}\left(\tilde{u}_{M+1}\otimes I\right)x_E(0)$ and $r_h(t) = -(\tilde{u}_{M+1}\otimes I)h_E(t)$.

Proof: If multi-agent systems (1) under protocols (5) and (6) achieves formation-containment, one knows that multi-agent system (8) achieves formation $h_E(t)$. Then subsystem (10) determines the formation reference function. From Lemma 4, one has

$$\tilde{x}_{EC}(0) = P_{\mathbb{C}(U),\bar{\mathbb{C}}(U)}\tilde{x}_E(0), \tag{16}$$

where

$$P_{\mathbb{C}(U),\bar{\mathbb{C}}(U)} = [q_{Mn+1}, q_{Mn+2}, \cdots, q_{Mn+n}, 0, \cdots, 0]P^{-1}$$

is an oblique projector onto $\mathbb{C}(U)$ along $\bar{\mathbb{C}}(U)$ and $P = [q_{Mn+1}, q_{Mn+2}, \cdots, q_{Nn}]$. Let $Q = [c_1, c_2, \cdots, c_n]$, then it can be obtained that $P = U_E\otimes Q$. Because $[q_{Mn+1}, q_{Mn+2}, \cdots, q_{Mn+n}, 0, \cdots, 0] = [\mathbf{1}, 0]\otimes Q$, one has

$$P_{\mathbb{C}(U),\bar{\mathbb{C}}(U)} = \left([\mathbf{1}, 0]\,U_E^{-1}\right)\otimes I. \tag{17}$$

From (13), one can obtain that

$$\theta(0) = [I, 0, \cdots, 0]\,\tilde{x}_{EC}(0). \tag{18}$$

Since $[\mathbf{1}, 0]\,U_E^{-1} = (\mathbf{1}\otimes\tilde{u}_{M+1})$, from (16) to (18), one has

$$\theta_{M+1}(0) = (\tilde{u}_{M+1}\otimes I)\,\tilde{x}_E(0). \tag{19}$$

From (10), (19) and the fact that

$$\begin{aligned}&\textstyle\int_0^t e^{(A+BK_1)(t-\tau)}(\tilde{u}_{M+1}\otimes I)\dot{h}_E(\tau)d\tau\\&= (\tilde{u}_{M+1}\otimes I)h_E(t) - e^{(A+BK_1)t}(\tilde{u}_{M+1}\otimes I)h_E(0)\\&+ \textstyle\int_0^t e^{(A+BK_1)(t-\tau)}\left(\tilde{u}_{M+1}\otimes(A+BK_1)\right)h_E(\tau)d\tau,\end{aligned}$$

the conclusion of Lemma 7 can be obtained.□

Remark 3 In Lemma 7, $r_0(t)$ is said to be the consensus function which describes the formation reference function of the multi-agent system with formation $h_E(t)\equiv 0$. $r_h(t)$ describes the impact of nonzero formation $h_E(t)$. From Lemma 7, one can see that the gain matrix K_1 can be used to assign the motion modes of the formation reference by specifying the eigenvalues of $A+BK_1$.

4 Formation-containment analysis and protocols design

In this section, for multi-agent system (1) under protocols (5) and (6), sufficient conditions to achieve formation-containment are first proposed, and then necessary and sufficient conditions to achieve containment and formation are presented respectively as special cases of formation-containment. Finally, an approach to determine the gain matrices in the protocols is given.

Let $U_F\in\mathbb{C}^{M\times M}$ be a nonsingular matrix such that $U_F^{-1}L_1U_F = \Lambda_F$, where Λ_F is an upper-triangular matrix with λ_i $(i\in F)$ as its diagonal entries and $\mathrm{Re}(\lambda_1)\leqslant\mathrm{Re}(\lambda_2)\leqslant\cdots\leqslant\mathrm{Re}(\lambda_M)$.

Theorem 1 *Multi-agent system (1) under protocols (5) and (6) achieves formation-containment if the following conditions hold simultaneously*

(i) For $\forall i\in E$,

$$\lim_{t\to\infty}\left((A+BK_1)h_{ij}(t) - \dot{h}_{ij}(t)\right) = 0, j\in N_i, \tag{20}$$

where $h_{ij}(t) = h_i(t) - h_j(t)$;

(ii) For all $i\in F$, $A+BK_1+\lambda_iBK_2$ are Hurwitz;

(iii) For all $i\in\{M+2, M+3, \cdots, N\}$, $A+BK_1+\lambda_iBK_3$ are Hurwitz.

Proof: If condition (i) holds, one can obtain that

$$\lim_{t\to\infty}\Big(\left(L_3\otimes(A+BK_1)\right)h_E(t)-\left(L_3\otimes I\right)\dot{h}_E(t)\Big)=0. \quad (21)$$

Substitute $L_3=U_EJ_EU_E^{-1}$ into (21) and then pre-multiply the left and right sides of (21) by $U_E^{-1}\otimes I$. It follows that

$$\lim_{t\to\infty}\Big(\Big(\bar{J}_E\tilde{U}_E\otimes(A+BK_1)\Big)h_E(t)-\Big(\bar{J}_E\tilde{U}_E\otimes I\Big)\dot{h}_E(t)\Big)=0. \quad (22)$$

Since G_E has a spanning tree, by Lemma 1 and the structure of J_E, one knows that $\bar{J}_E$ is nonsingular. Pre-multiplying the left and right sides of (22) by $\bar{J}_E^{-1}\otimes I$, one has

$$\lim_{t\to\infty}\Big(\Big(\tilde{U}_E\otimes(A+BK_1)\Big)h_E(t)-\Big(\tilde{U}_E\otimes I\Big)\dot{h}_E(t)\Big)=0. \quad (23)$$

Consider the following $N-M-1$ subsystems

$$\dot{\tilde{\theta}}_i(t)=(A+BK_1+\lambda_iBK_3)\,\tilde{\theta}_i(t), \quad (24)$$

where $i=M+2,M+3,\cdots,N$. If condition (iii) holds, one can obtain that the $N-M-1$ subsystems described by (24) are asymptotically stable. By (24) and the structure of $\bar{J}_E$, it holds that the system described by

$$\dot{\tilde{\varsigma}}(t)=\left(I_{N-M-1}\otimes(A+BK_1)+\bar{J}_E\otimes BK_3\right)\tilde{\varsigma}(t), \quad (25)$$

is asymptotically stable. From (23) and (25), one knows that

$$\lim_{t\to\infty}\varsigma(t)=0. \quad (26)$$

By (26) and Lemma 5, multi-agent system (8) achieves formation $h_E(t)$.

When the leaders achieve formation $h_E(t)$, then

$$\begin{aligned}&\lim_{t\to\infty}\left((L_2L_3)\otimes(BK_3)\right)(x_E(t)-h_E(t))\\&=\lim_{t\to\infty}\left((L_2L_3)\otimes(BK_3)\right)(\mathbf{1}\otimes c(t)).\end{aligned} \quad (27)$$

Since $L_3\mathbf{1}=0$, one has

$$\lim_{t\to\infty}\left((L_2L_3)\otimes(BK_3)\right)(x_E(t)-h_E(t))=0. \quad (28)$$

If condition (ii) holds, then it can be shown that the following M subsystems

$$\dot{\tilde{\xi}}_i(t)=(A+BK_1+\lambda_iBK_2)\,\tilde{\xi}_i(t)\ (i=1,2,\cdots,M), \quad (29)$$

are asymptotically stable; that is, the following system

$$\dot{\bar{\xi}}(t)=\left(I_M\otimes(A+BK_1)+(L_1\otimes BK_2)\right)\bar{\xi}(t), \quad (30)$$

is asymptotically stable. From (7), (8) and (14), it can be derived that

$$\begin{aligned}\dot{\xi}(t)=&\left(I_M\otimes(A+BK_1)+(L_1\otimes BK_2)\right)\xi(t)\\&+\left((L_2L_3)\otimes(BK_3)\right)(x_E(t)-h_E(t)).\end{aligned} \quad (31)$$

From (28) and (30), it holds that

$$\lim_{t\to\infty}\xi(t)=0. \quad (32)$$

From (26), (32) and Lemma 6, one sees that multi-agent system (1) under protocols (5) and (6) achieves formation-containment. The proof of Theorem 1 is completed. □

Theorem 1 presents sufficient conditions to achieve formation-containment. In the case that leaders have no neighbors; that is, for all $i,j\in E$, $w_{ij}=0$, formation-containment problems become containment problems, and the following Corollary can be obtained.

Corollary 1 *In the case that leaders have no neighbors, multi-agent system (1) under protocols (5) and (6) achieves containment if and only if condition (ii) in Theorem 1 holds.*

Proof: Sufficiency can be obtained directly from the proof of Theorem 1.

Necessity: The necessary is proven by contradiction. Suppose that multi-agent system (1) under protocols (5) and (6) achieves containment for any given bounded initial states and condition (ii) in Theorem 1 does not hold. Since leaders have no neighbors, then $L_3=0$. Let $x(t)=[x_F^T(t),x_E^T(t)]^T$. Systems (7) and (8) can be rewritten in a compact form as follows

$$\dot{x}(t)=\Phi x(t), \quad (33)$$

where

$$\Phi=\begin{bmatrix}\Xi_1 & \Xi_2\\ 0 & \Xi_3\end{bmatrix},$$

with

$$\Xi_1=I_M\otimes(A+BK_1)+L_1\otimes BK_2,$$
$$\Xi_2=L_2\otimes BK_2,\ \Xi_3=I_{N-M}\otimes(A+BK_1).$$

The solution to equation (33) can be written as

$$x(t)=\begin{bmatrix}e^{\Xi_1t} & \Psi_1\\ 0 & e^{\Xi_3t}\end{bmatrix}\begin{bmatrix}x_F(0)\\ x_E(0)\end{bmatrix}, \quad (34)$$

where

$$\Psi_1=\mathcal{L}^{-1}\left((sI-\Xi_1)^{-1}\Xi_2(sI-\Xi_3)^{-1}\right),$$

with $\mathcal{L}^{-1}$ being the inverse Laplace transform and s being a complex parameter. It can be shown that

$$\begin{aligned}&(sI-\Xi_1)^{-1}\Xi_2(sI-\Xi_3)^{-1}\\&=(sI-\Xi_1)^{-1}\left(L_1^{-1}L_2\otimes I\right)+\left(-L_1^{-1}L_2\otimes I\right)(sI-\Xi_3)^{-1}.\end{aligned}$$

So one has

$$x(t)=\begin{bmatrix}e^{\Xi_1t} & e^{\Xi_1t}\Omega-\Omega e^{\Xi_3t}\\ 0 & e^{\Xi_3t}\end{bmatrix}\begin{bmatrix}x_F(0)\\ x_E(0)\end{bmatrix}, \quad (35)$$

where $\Omega=L_1^{-1}L_2\otimes I$. From (35), one has

$$x_F(t)=e^{\Xi_1t}x_F(0)+e^{\Xi_1t}\Omega x_E(0)-\Omega e^{\Xi_3t}x_E(0). \quad (36)$$

By the structure of Λ_F and U_F, condition (ii) in Theorem 1 does not hold implies that Ξ_1 is not Hurwitz. Choose $x_E(0)=0$ and $x_F(0)\neq0$. Then $x_E(t)=0$ but $\lim_{t\to\infty}x_F(t)\neq0$, which means that multi-agent system (1) under protocols (5) and (6) does not achieve containment. This results a contradiction. The proof of Corollary 1 is completed.□

Remark 4 Corollary 1 presents necessary and sufficient conditions for high-order LTI multi-agent systems with multiple leaders to achieve containment. To the best of our knowledge, for high-order multi-agent systems with multiple leaders only sufficient conditions have been obtained before, such as the results obtained in [27]-[29]. Necessary and sufficient conditions for first-order and second-order multi-agent systems to achieve containment which include Theorem 3.1 in [23] and Theorems 1 and 2 in [25] can be regard as special cases of Corollary 1.

If $M = 0$, formation-containment problems become formation problems, and the following corollary can be obtained.

Corollary 2 *In the case that $M = 0$, multi-agent system (1) under the protocol (6) achieves formation $h_E(t)$ if and only if conditions (i) and (iii) in Theorem 1 hold.*

Proof: Sufficiency can be obtained directly from the proof of Theorem 1.

Necessity: If multi-agent system (1) under protocol (6) achieves formation $h_E(t)$, from Lemma 5, one has that system (25) is asymptotically stable and equation (23) holds. Because the stability of system (25) is equivalent to that of subsystems (24), the condition (iii) in Theorem 1 is required.

Let $\tilde{U}_E = [\hat{U}_E, \hat{u}_E]$ with $\hat{U}_E \in \mathbb{R}^{(N-1)\times(N-1)}$ and $\hat{u}_E \in \mathbb{R}^{(N-1)\times 1}$. Since $\operatorname{rank}(\tilde{U}_E) = N-1$, without loss of generality, it is assumed that $\operatorname{rank}(\hat{U}_E) = N-1$. From (23), one has

$$\lim_{t\to\infty} \mathrm{T}(t) = 0, \tag{37}$$

where

$$\mathrm{T}(t)=\Big([\hat{U}_E, \hat{u}_E]\otimes(A+BK_1)\Big)h_E(t)-\Big([\hat{U}_E, \hat{u}_E]\otimes I\Big)\dot{h}_E(t).$$

Due to $\tilde{U}_E\mathbf{1}_N = 0$, it follows that

$$\hat{u}_E = -\hat{U}_E\mathbf{1}_{N-1}. \tag{38}$$

Let $\bar{h}_E^T(t) = [h_1^T(t), h_2^T(t), \cdots, h_{N-1}^T(t)]^T$. From (37) and (38), one can obtain

$$\lim_{t\to\infty}\left(\hat{U}_E\otimes I\right)\left(\Gamma_{\bar{h}_E} - \Gamma_{h_N}\right) = 0, \tag{39}$$

where

$$\Gamma_{\bar{h}_E} = \left(I_{N-1}\otimes(A+BK_1)\right)\bar{h}_E(t) - \left(I_{N-1}\otimes I\right)\dot{\bar{h}}_E(t),$$

$$\Gamma_{h_N} = \left(\mathbf{1}_{N-1}\otimes(A+BK_1)\right)h_N(t) - \left(\mathbf{1}_{N-1}\otimes I\right)\dot{h}_N(t).$$

Pre-multiplying the left sides and right sides of (39) by $\hat{U}_E^{-1}\otimes I$, one has

$$\lim_{t\to\infty}\Big((A+BK_1)\,h_{iN}(t) - \dot{h}_{iN}(t)\Big) = 0. \tag{40}$$

where $i = 1, 2, \cdots, N-1$. From (40), one knows that condition (i) in Theorem 1 is required. This completes the proof.□

Remark 5 It should be pointed out that necessary and sufficient conditions for high-order LTI multi-agent systems with directed interaction topologies to achieve time-varying formation have not be obtained in the previous literatures on formation control. Moreover, let $\kappa_i \in \mathbb{R}^q$ and $h_i = \kappa_i\otimes[1,0]^T$ $(i = 1, 2, \cdots, N)$. If $h = [h_1^T, h_2^T, \cdots, h_N^T]^T$, $K_1 = 0$, $B = I_q\otimes[0,1]^T$ and

$$A = \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ a_{21}^1 & a_{22}^1 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 1 \\ a_{21}^q & a_{22}^q \end{bmatrix}\right),$$

Corollary 2 gives the conclusions in Lafferriere *et al*. [4]. If $K_1 = 0$ and $\dot{h}(t) \equiv 0$, then Corollary 2 is equivalent to Theorem 1 in Ma and Zhang [5].

Remark 6 If $M = 0$ and $h(t) \equiv 0$, formation-containment problems in the current paper become the consensus problems discussed in [15] and Theorem 1 in [15] is just a special case of Corollary 2.

Theorem 2 *If condition (i) in Theorem 1 holds and (A, B) is stabilizable, then multi-agent system (1) achieves formation-containment by protocols (5) and (6) with $K_2 = -[\operatorname{Re}(\lambda_1)]^{-1}R_o^{-1}B^TP_o$ and $K_3 = -[\operatorname{Re}(\lambda_{M+2})]^{-1}R_o^{-1}B^TP_o$ where $P_o^T = P_o > 0$ is the solution to the algebraic Riccati equation*

$$P_o(A+BK_1)+(A+BK_1)^TP_o-P_oBR_o^{-1}B^TP_o+Q_o=0, \tag{41}$$

for $R_o^T = R_o > 0$ and $Q_o = D_o^TD_o \geq 0$ with $(A + BK_1, D_o)$ detectable.

Proof: If (A, B) is stabilizable, then $(A + BK_1, B)$ is stabilizable. Thus, for any given $R_o^T = R_o > 0$ and $Q_o = D_o^TD_o \geq 0$ with $(A+BK_1, D_o)$ detectable, algebraic Riccati equation (41) has a unique solution $P_o^T = P_o > 0$. Consider the $N-M-1$ subsystems described by (24). Construct the following Lyapunov function candidates

$$\tilde{V}_i(t) = \tilde{\theta}_i^H(t)P_o\tilde{\theta}_i(t)\ (i = M+2, M+3, \cdots, N). \tag{42}$$

Let

$$K_3 = -[\operatorname{Re}(\lambda_{M+2})]^{-1}R_o^{-1}B^TP_o.$$

Taking the derivative of $\tilde{V}_i(t)$ with respect to t along the solution to subsystems (24), one has

$$\begin{aligned}\dot{\tilde{V}}_i(t)=&-\tilde{\theta}_i^H(t)Q_o\tilde{\theta}_i(t)+(1-2\operatorname{Re}(\lambda_i)[\operatorname{Re}(\lambda_{M+2})]^{-1})\\&\times\tilde{\theta}_i^H(t)P_oBR_o^{-1}B^TP_o\tilde{\theta}_i(t) \leq 0.\end{aligned} \tag{43}$$

Since $(A + BK_1, D_o)$ is detectable and $R_o^T = R_o > 0$, one can obtain that the $N-M-1$ subsystems described by (24) are asymptotically stable.

Consider the M subsystems described by (29). By a similar analysis as for subsystems described by (24), it can be shown that K_2 can make the M subsystems described by (29) asymptotically stable. From the proof of Theorem 1, one can see that multi-agent system (1) under protocols (5) and (6) achieves formation-containment. This completes the proof.□

Remark 7 If $M = N-1$, Corollary 1 presents necessary and sufficient conditions for multi-agent system (1) under protocols (5) and (6) to achieve consensus tracking. More specifically, if $M = N-1$, $K_1 = 0$, $u_N(t) \equiv 0$ and the interaction topologies are undirected, then the formation-containment problems in the current paper become the consensus tracking problems investigated in Ni and Cheng [19]. Moreover, if $R_o^{-1} = 2\lambda_1$ and $D_o = \sqrt{\lambda_1}I$, Theorem 1 in [19] becomes a special case of Theorem 2 in this paper.

5 Numerical simulations

In this section, a numerical example is given to illustrate the effectiveness of theoretical results obtained in the previous sections.

Consider a multi-agent system with fourteen agents. The dynamics of each agent is described by (1) with

$$A = \begin{bmatrix} -2.1 & 1 & -2.1 \\ 0 & 0 & 1 \\ -2 & 4 & 5 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Assume that there are six leaders and eight followers in the multi-agent system. The interaction topology of the multi-agent system is shown in Fig. 1. For simplicity, it is assumed that the interaction topology has $0-1$ weights.

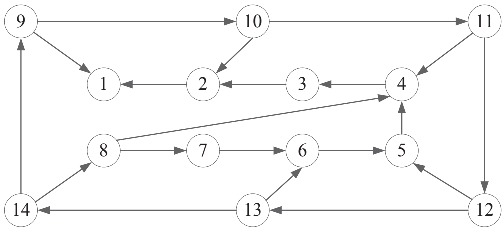


Fig. 1: Directed interaction topology G.

The formation-containment problem for the multi-agent system is described as follows. Firstly, the six leaders are required to preserve a periodic time-varying parallel hexagon formation and keep rotation around the predefined time-varying formation reference. The formation is defined as follows

$$h_i(t) = \begin{bmatrix} 3\sin\left(t + \frac{(i-9)\pi}{3}\right) \\ 3\cos\left(t + \frac{(i-9)\pi}{3}\right) \\ -3\sin\left(t + \frac{(i-9)\pi}{3}\right) \end{bmatrix}, \ (i = 9, 10, \cdots, 14).$$

If the formation specified by above $h_i(t)$ $(i = 9, 10, \cdots, 14)$ is achieved, the six leaders will locate at the six diagonals of a regular hexagon respectively and keep rotation with an angular velocity of $1\mathrm{rad}/s$. Moreover, the edge length of the desired parallel hexagon is periodic time-varying. Secondly, the states of the eight followers are required to converge to the convex hull formed by the states of the six leaders.

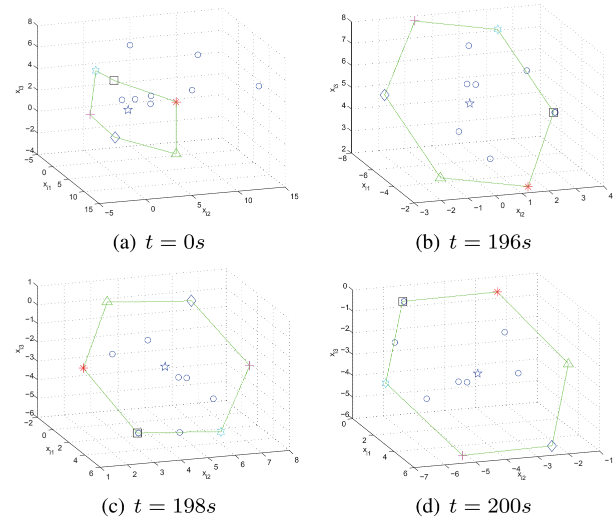
Choose $K_1 = [4, -5, -3]$ to assign the eigenvalues of $(A + BK_1)$ at $1j$, $-1j$ and -0.1 with $j^2 = -1$. In this case, the formation reference will move periodically. It can be verified that condition (i) in Theorem 1 holds and (A, B) is stabilizable. Therefore, from Theorem 2, one can obtain a K_2 and K_3 to make multi-agent system (1) achieves formation-containment by protocols (5) and (6) as follows

$$K_2 = [-0.9663, 0.4601, -1.8541],$$

$$K_3 = [-1.9326, 0.9203, -3.7083].$$

For simplicity of description, let the initial states of each agent be $x_{i1}(0) = \Theta(15 - i)$, $x_{i2}(0) = \Theta(15 - i)$ and $x_{i3}(0) = \Theta(15 - i)$ where $i = 1, 2, \cdots, 14$, and Θ is a pseudorandom value with a uniform distribution on the interval $(0, 1)$.

Fig. 2 shows the trajectories snapshots of the fourteen agents and the predefined formation reference function at different time, where the state trajectories of leaders are denoted by the asterisk, triangle, diamond, hexagram, plus and square, and those of the followers and predefined formation reference function are represented by the circle and pentagram respectively. Moreover, the convex hull formed by the states of leaders is marked by solid lines. Figs. 2(a) and 2(b) indicate that the leaders achieve the desired parallel hexagon formation and followers converge to the convex hull formed by the leaders. Figs. 2(b), 2(c) and 2(d) show



(a) $t = 0s$ (b) $t = 196s$ (c) $t = 198s$ (d) $t = 200s$

Fig. 2: Trajectory snapshots of fourteen agents and $r(t)$ at different time.

that the achieved formation keeps rotation around the predefined formation reference, and both the edge length of formation and the formation reference are time-varying. Therefore, multi-agent system (1) achieves the desired formation-containment. It should be mentioned that in Figs. 2(b), 2(c) and 2(d), the states of some agents are overlapped by the states of others. This is because some followers have only one neighbor, which means that when the multi-agent system achieves formation-containment the states of these followers will reach an agreement with those of the neighbors.

6 Conclusions

Formation-containment analysis and design problems for high-order LTI multi-agent systems were investigated. Protocols were proposed for leaders and followers respectively to propel the states of leaders to achieve predefined time-varying formation and drive the states of followers to converge to the convex hull formed by the states of leaders. Formation-containment problems were transformed into asymptotic stability problems and an explicit expression of the formation reference function was presented. Sufficient conditions for multi-agent systems to achieve formation-containment were proposed. Necessary and sufficient conditions for high-order LTI multi-agent systems to achieve containment and time-varying formation were shown respectively as special cases. An approach to design the gain matrices in the protocols was given. It was shown that containment problems, formation control problems, consensus problems and consensus tracking problems can all be treated as special cases of formation-containment problems.

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