Time axis shifting finite-gain based prescribed-time tracking control under non-vanishing disturbances

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***Abstract*— In this paper, the prescribed-time tracking control problem for a class of strict-feedback systems with non-vanishing time-varying disturbances is investigated. Firstly, a group of novel disturbance observers are established to observe the disturbances in prescribed time based on time axis shifting finite time-varying gains, and then the state feedback based control scheme is proposed to realize the prescribed-time tracking control objective upon using the system transformation and adding power integrator technique. The proposed control scheme allows the output of the system to track the reference trajectory in prescribed time and the tracking error maintains zero after that prescribed settling time, making itself different from those most existing prescribed-time control results where the system only exists on the finite time interval based on infinity time-varying gains. The effectiveness of the theoretical results are confirmed by the numerical simulation.**

***Index Terms*— Prescribed-time control, Disturbance observer, Nonlinear system, State feedback**

I. Introduction

During the past few decades, finite-time control of dynamic systems has become an appealing research area [1]-[12] due to practical needs. The definition and sufficient conditions for finite-time stability were first presented in the classical work [4]. The finite-time control result is achieved if the equilibrium state is stable within a finite settling time that depends on the initial condition and a set of design parameters, which also represents the upper bound of the convergence time of system. Since finite-time control allows the closed-loop system to be stable in finite time and with better perturbation rejection, increasing attention has been put on studying the problems of finite-time control. As a breakthrough research of finite-time control, fixed-time control method [6] allows the settling time function to be independent of initial conditions. However, either in finite-time control or fixed-time control method the settling time function is conservative where only the upper bound of the actual convergence time can be given, making it difficult to set the convergence time precisely.

For some application scenarios, such as missile guidance field, it is necessary to give the precise convergence time [13]. In [14], the convergence rate of the states is prescribed in advance by accelerated twisting controller that uses a positive-definite strictly growing convergence-rate function. Motivated by the idea of positive-definite strictly growing function, in [10] a time-varying strictly growing function was introduced into the controller design to allow the system state to converge to zero as time tends to a prescribed finite time T, even if in the presence of non-vanishing uncertainties. The novelty of the control scheme proposed in [10] lies in that the convergence time T can be prescribed by the designer in advance. Upon utilizing the time-varying scaling function based prescribed-time control technique, a prescribed-time consensus tracking control algorithm was presented for a class of multi-agent systems under directed communication topology in [11]. The sufficient conditions for how to choose the time-varying scaling functions were first given in work [12].

Other extensions of prescribed finite time control methods can be found in [15]-[21]. Upon using the time-varying gain technique and backstepping method, the work [15] investigated the prescribed-time tracking control problem for a class of MIMO strick-feedback-like systems subject to non-vanishing uncertainties. The work [16], provides a prescribed-time output feedback control solution for linear systems in controllable canonical form. A novel adaptive prescribed-time stability theorem was first proposed in [17], in which the time-varying function scaling function has not been used for the state transformation but for the controller design directly. By establishing a new non-scaling backstepping design method, the prescribed-time mean-square stabilization and inverse optimality control problems was addressed for stochastic strict-feedback nonlinear systems in [18]. Additionally, the linear time-varying feedback was introduced into the controller design to solve the global prescribed-time stabilization problem for a class of nonlinear systems satisfying the linear growth condition in [19]. A dynamic surface control (DSC) based adaptive prescribed-time control algorithm was provided for strict-feedback systems in [20]. Upon using the L_GV-backstepping method, the work [21] proposed a prescribed-time tracking control scheme for a class of nonlinear systems with guaranteed performance that is valid for the whole time interval but not only the finite time interval.

It is worth noting that in the aforementioned [10], [11], and [15]-[21], the time-varying scaling function that would approach to infinity as time approaches to the prescribed

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time T is used, which would inevitably bring the high-gain challenge for the controller implementation. To avoid the controller implementation problem, the works [22] and [23] proposed a time space deformation based method to achieve the prescribed-time stabilization, however, only the stabilization problem for integrator systems are considered. It is unclear how to apply the time space deformation based method to the tracking control problem for nonlinear systems with non-vanishing uncertainties. Based on the aforementioned discussions, the prescribed-time tracking control strategy for a class of strict-feedback systems subject to time-varying disturbances. The main contributions of this work are summarized as follows.

1) Different from most existing finite-time slide-mode observer [1]-[3], [27]-[29], where the converge time of the observation error can not be given precisely, the upper bound of the convergence time for the observation error can be provided precisely in advance under the prescribed-time disturbance observer proposed in this work.
2) A novel switching control algorithm is employed in this work, which allows the control gains not to escape to infinity and the system to operate on the whole time interval but not only on the finite time interval, distinguishing itself from most existing prescribed-time control works [10], [11], and [15]-[21]. In addition, the tracking error is driven to zero within the prescribed time and maintains zero after that time constant.
3) To facilitate the stability analysis for the proposed prescribed-time tracking control scheme for strict-feedback systems, an important lemma (Lemma 5) is established, which allows the successful integration of the time axis shifting finite-gain based method and adding power integrator technique.

Notation: In this work, R, R^i, and $R^{n\times m}$ stands for the set of real numbers, the set of i-dimensional column vectors, and the set of $n \times m$ real matrices, respectively; Denote by $\lceil y\rfloor^{\alpha} = |y|^{\alpha}\mathrm{sign}(y)$ for any $y, \alpha \in R, y \neq 0$; Let $\lceil y\rfloor^{0} = \mathrm{sign}(y)$; The symbol C^i $(i \geq 0)$ stands for the set of all differentiable functions whose ith order derivatives are continuous. The symbol $\mathcal{L}_\infty$ represents the set of all signals whose infinity-norms are bounded.

II. Problem formulation and preliminaries

We consider a class of high-order nonlinear systems subject to time-varying disturbances as follows,

$$\begin{aligned}&\dot{x}_i(t) = x_{i+1} + f_i(t,\bar{x}_i) + d_i(t), \quad 1 \leq i \leq n-1,\\ &\dot{x}_n(t) = u + f_n(t,\bar{x}_n) + d_n(t),\\ &y(t) = x_1(t),\end{aligned} \tag{1}$$

where $\bar{x}_i = [x_1, x_2, \cdots, x_i]^T \in R^i$ $(i = 1, \cdots, n)$, $x_i \in R$, $u \in R$ and $y \in R$ are system state, control input and control output of the system, respectively; $f(t,\bar{x}_i) \in R$ is smooth function, $d_i \in R$ denotes the non-vanishing time-varying disturbance.

To facilitate the main results of this paper, the following assumptions, definitions and lemmas should be given.

Assumption 1: For $i = 1, 2, \cdots, n$, there exists a positive known function $\gamma_i(\bar{x}_i, \bar{y}_i)$ such that

$$|f_i(t,\bar{x}_i) - f_i(t,\bar{y}_i)| \leq \gamma_i(\bar{x}_i,\bar{y}_i)\sum_{\mathrm{j}=1}^{i}|x_j - y_j|^{\frac{r_i+\tau}{r_j}}, \tag{2}$$

where $r_1 = 1, r_{i+1} = r_i + \tau$ with $\tau \in (-\frac{1}{n}, 0)$ being the homogeneous degree.

Assumption 2: The non-vanishing time-varying disturbance satisfies $d_i(t) \in C^m$ for $m = n+1-i$. Furthermore, $\mathrm{d}_i^{(j)}(t)$ $(0 \leq j \leq m)$ has a Lipschitz constant L_i.

Assumption 3: The time-varying reference signal $r(t)$ and its jth derivative are assumed to be bounded by $\sup_{t\geq 0}|r^{(j)}(t)| \leq L$ for $0 \leq j \leq n$, where L is a known positive constant.

Remark 1: It is worth noting that Assumption 1 satisfies the homogeneous growth condition in [26] as $|f_i(t,\bar{x}_i)| \leq c\sum_{\mathrm{j}=1}^{i}|x_j|^{\frac{r_i+\tau}{r_j}}$, which relaxes the restrictions on nonlinear functions of existing works [8], [9], [19], and [24]. For any continuous function $f_i(\bar{x}_i)$, there exists a constant $-1 < \tau < 0$ and a continuous positive function $\gamma_i(\bar{x}_i, \bar{y}_i)$ such that the inequality (2) in Assumption 1 is satisfied. In addition, Assumption 1 allows the non-vanishing uncertainties to be involved in system model (1). For example, $f_1(t, x_1) = \sin(t)x_1 + x_1^2 + \frac{1}{2}$. In addition, by choosing $\tau = -\frac{1}{2}$, $r_1 = 1$, and $\gamma_1(x_1, y_1) = |x_1 - y_1|^{\frac{1}{2}} + |x_1 + y_1| \cdot |x_1 - y_1|^{\frac{1}{2}}$, the condition (2) in Assumption 1 is satisfied. It should be mentioned that there exists a large class of exogenous signals of disturbances satisfy Assumption 2, such as constant disturbances, ramp disturbances, sinusoidal disturbances. Furthermore, Assumption 2 includes the higher-order sliding mode observers in [1]-[3] as special cases.

Lemma 1: ([26]) If $\alpha \geq 1$ is an integer, then for any $x \in R$, $y \in R$, it holds that $|x+y|^{\alpha} \leq 2^{\alpha-1}|x^{\alpha}+y^{\alpha}|$, $(|x|+|y|)^{1/\alpha} \leq |x|^{1/\alpha}+|y|^{1/\alpha} \leq 2^{(\alpha-1)/\alpha}(|x|+|y|)^{1/\alpha}$. If $\alpha \geq 1$ is an odd integer, then $|x-y|^{\alpha} \leq 2^{\alpha-1}|x^{\alpha}-y^{\alpha}|$, $|x^{1/\alpha}-y^{1/\alpha}| \leq 2^{(\alpha-1)/\alpha}|x-y|^{1/\alpha}$ and $|x-y|^{\alpha} \leq \lambda|x-y|((x-y)^{\alpha-1}+y^{\alpha-1})$ for some constant $\lambda > 0$.

Lemma 2: ([30]) For any $x \in R$, $y \subseteq R$, and real-valued function $\gamma(x,y) > 0$, it holds $|x|^m|y|^n \leq \frac{m}{m+n}\gamma(x,y)|x|^{m+n} + \frac{n}{m+n}\gamma^{-m/n}(x,y)|y|^{m+n}$.

Lemma 3: ([5]) If $\alpha \geq 1$, for any $x_i \in R$, it holds that $N^{1-\alpha}(\sum_{i=1}^{N}|x_i|)^{\alpha} \leq \sum_{i=1}^{N}|x_i|^{\alpha} \leq (\sum_{i=1}^{N}|x_i|)^{\alpha}$

Definition 1: ([12]) The function $\mu(t) : [0, t_f) \to R$ belongs to class K^+ if $\mu(t)$ is non-decreasing C^∞ function with $\mu(0) = 1$ and $\mu(t_f) = \infty$.

Definition 2: ([12]) The function $\mu(t) : [0, t_f) \to R$ belongs to class K^* if $\mu(t) \in K^+$ and satisfies $\dot{\mu}/\mu^2 \in \mathcal{L}_\infty$.

In this work, we introduce the time-varying function $\mu(t) = \frac{T}{T-t}$, where $T = t_f$ represents the prescribed convergence time given by the user. It is readily verified that the function $\mu(t)$ in K^*.

In addition, by integrating the time-varying function $\mu(t) = \frac{T}{T-t} : [0, T) \to [1, \infty)$, we obtain a scaling time transformation

$$s(t) = -T\ln\frac{T-t}{T} : [0, T) \to [0, \infty), \tag{3}$$

which has a continuous inverse

$$t(s) = T(1 - e^{-s/T}) : [0, \infty) \to [0, T). \tag{4}$$

Consider a nonlinear system

$$\dot{x} = f(t, x), \quad x(0) = x_0, \tag{5}$$

where $x \in R^n$ and $f : R_+ \times R^n \to R^n$ is a nonlinear function. For system (5) there exists a Filippov solution [34] of this system if $f(t, x)$ is discontinuous. Suppose that the origin is an equilibrium point of this system.

Lemma 4: ([4]) If there exists a C^1 function $V(x) : R^n \to R_+ \cup \{0\}$ such that $\dot{V}(x) \leq -\lambda V(x)^\alpha$, where $\lambda > 0$ and $\alpha \in (0, 1)$. Then the origin is a globally finite-time stable equilibrium of system (5) and the settling time satisfies $T(x_0) \leq \frac{1}{\lambda(1-\alpha)} V(x_0)^{(1-\alpha)}$, which depends on the initial state $x(0) = x_0$.

Lemma 5: Consider the function $\mu(t) = \frac{T}{T-t}$ on $[0, T)$. If there exists a continuous radially unbounded $V(x) : R^n \to R_+ \cup \{0\}$ such that

1) $V(x) = 0 \Leftrightarrow x = 0$;

2) any solution $x(t)$ of system (5) satisfies $\dot{V}(x) \leq -\mu(t)V(x)^\alpha$, where $0 < \alpha < 1$.

Then the origin of system (5) is globally prescribed-time stable and the settling time function T^* can be estimated as

$$T^* \leq T\left(1 - e^{-\frac{1}{kT(1-\alpha)} V(x_0)^{(1-\alpha)}}\right). \tag{6}$$

Moreover, it is clear that the convergence time T^* meets a prescribed upper bound T.

Proof: We consider following auxiliary differential equation,

$$\dot{y}(t) = -k\mu(t)y^\alpha, \tag{7}$$

where $0 < \alpha < 1$, k is a positive constant, $y(t) \geq 0$.

Upon using the separation of variables, the equation (7) can be rewritten as

$$\int \frac{T}{T-t} dt = -\int \frac{1}{ky^\alpha} dy. \tag{8}$$

Solving equation (8) yields

$$T \ln(\frac{T-t}{T}) + c_0 = \frac{1}{k(1-\alpha)} y^{1-\alpha}, \tag{9}$$

where

$$c_0 = \frac{1}{k(1-\alpha)} y(0)^{(1-\alpha)}. \tag{10}$$

For $y(t) = 0$, we have

$$t = T\left(1 - e^{-\frac{1}{kT(1-\alpha)} y(0)^{(1-\alpha)}}\right) < T, \tag{11}$$

hence the solution of (7) reaches the equilibrium in prescribed-time. This implies that (6) holds. ■

Corollary 1: Consider the function $\mu(t) = \frac{T}{T-t}$ for $t \in [0, T)$. If there exists a continuous radially unbounded $V(x) : R^n \to R_+ \cup \{0\}$ such that

$$\dot{V}(x) \leq -k\mu(t)V(x)^\alpha + \mu(t)\zeta \tag{12}$$

with $0 < \alpha < 1$, $0 < \theta < 1$, and ζ being a constant. Then there exists a finite time T^*

$$T^* \leq T\left(1 - e^{-\frac{1}{kT(1-\theta)(1-\alpha)} V(x_0)^{(1-\alpha)}}\right), \tag{13}$$

and the residual set of the solution of system (5) is given by

$$x \in \{V(x) \leq (\zeta/\theta)^{\frac{1}{\alpha}}\}. \tag{14}$$

Proof: Note that $0 < \theta < 1$, the equation (12) can be rewritten as

$$\dot{V}(x) \leq -k\mu(t)(1-\theta)V(x)^\alpha - k\mu(t)\theta V(x)^\alpha + \mu(t)\zeta. \tag{15}$$

If $V(x) > (\zeta/k\theta)^{\frac{1}{\alpha}}$, we have

$$\dot{V}(x) \leq -k\mu(t)(1-\theta)V(x)^\alpha. \tag{16}$$

Then, by Lemma 5, the setting time is bounded by

$$T^* \leq T\left(1 - e^{-\frac{1}{kT(1-\theta)(1-\alpha)} V(x_0)^{(1-\alpha)}}\right). \tag{17}$$

Meanwhile, the state x will be driven into the region

$$x \in \{V(x) \leq (\zeta/k\theta)^{\frac{1}{\alpha}}\}. \tag{18}$$

■

III. MAIN RESULTS

A. Prescribed-time disturbance observer

In this paper, suppose that the non-vanishing disturbances $d_i(t)$ $(i = 1, \ldots, n)$ satisfy Assumption 2. Inspired by [2] and [25], a set of prescribed-time observers are designed to estimate the disturbances and their derivatives in systems (1), which are designed as

$$\begin{cases} \dot{\sigma}_0^i = v_0^i + x_{i+1} + f_i(t, \bar{x}_i), \\ v_0^i = \sigma_1^i + q_0^i(\tilde{\sigma}_0^i, t), \\ \dot{\sigma}_1^i = v_1^i, \\ \dot{v}_1^i = \sigma_2^i + q_1^i(\tilde{\sigma}_0^i, t), \\ \vdots \\ \dot{\sigma}_{m-1}^i = v_{m-1}^i, \\ v_{m-1}^i = \sigma_m^i + q_{m-1}^i(\tilde{\sigma}_0^i, t), \\ \dot{\sigma}_m^i = q_m^i(\tilde{\sigma}_0^i, t) \end{cases} \tag{19}$$

for $i = 1, 2, \ldots, n$, $m = n + 1 - i$, where $\tilde{\sigma}_0^i = x_i - \sigma_0^i$, $x_{n+1} = u$, $\sigma_0^i = \hat{x}_i$, $\sigma_k^i = \widehat{d_i^{(k-1)}}$, $k = 1, 2, \ldots, m$. Let $Q^i(\tilde{\sigma}_0^i, t) = [q_0^i(\tilde{\sigma}_0^i, t), q_1^i(\tilde{\sigma}_0^i, t), \ldots, q_m^i(\tilde{\sigma}_0^i, t)]^T$, given by

$$Q^i(\tilde{\sigma}_0^i, t) = \begin{cases} \mathcal{M}^i(t)P^iF^i(\mu_1^m(t)\tilde{\sigma}_0^i), & \text{if } |\text{sign}(\tilde{\sigma}_0^i)| = 1, \\ H^i(\tilde{\sigma}_0^i), & \text{otherwise}, \end{cases} \tag{20}$$

where $H^i(\tilde{\sigma}_0^i) = [h_0^i(\tilde{\sigma}_0^i), h_1^i(\tilde{\sigma}_0^i), \ldots, h_m^i(\tilde{\sigma}_0^i, t)]^T$, and

$$h_j^i(\tilde{\sigma}_0^i) = \lambda_j^i L_i^{\frac{j+1}{m+1}} \lceil \tilde{\sigma}_0^i \rfloor^{\frac{m-j}{m+1}} \tag{21}$$

for $j = 0, 1, \ldots, m$, with L_i being the upper bound of the $(n+1-i)$th order derivative disturbance magnitude, and λ_j^i $(i = 1, \ldots, n)$ are design parameters. The detailed selection of observer parameters λ_j^i can be seen as in [1].

The diagonal matrix $\mathcal{M}^i$ is designed as

$$\mathcal{M}^i = \text{diag}(\mu_1^{1-m}(t), \mu_1^{2-m}(t), \ldots, \mu_1^{m-m}(t), \mu_1(t)), \tag{22}$$

where $\mu_1(t) = \frac{T_{ob}}{T_{ob}-t}$, T_{ob} denotes the observing time that is designed as user's need, $0 \leq t < T_{ob}$.

The design for P^i and F^i is given as follows.

Define

$$A^i = \mathrm{diag}(\delta_j) + \Lambda^i, \quad i = 1, 2 \ldots, n, \tag{23}$$

where $\delta_j = \frac{m-j}{T_{ob}}$, $\Lambda^i = \begin{bmatrix} 0 & I_m \\ \vdots & \\ 0 & \ldots 0 \end{bmatrix}$, for $m = n+1-i$, $j = 0, 1, \ldots, m$.

Upon using A^i and $C^i = [1, 0, \ldots, 0] \in R^{1\times(m+1)}$, we define $\mathcal{O}_1^i = [C^i, C^i A^i, \ldots, C^i (A^i)^m]^T \in R^{(m+1)\times(m+1)}$.

Define

$$\mathcal{A}^i = \begin{bmatrix} -a_1 & 0 & \ldots & 0 \\ -a_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{m+1} & 0 & \ldots & 0 \end{bmatrix} + \Lambda^i, \tag{24}$$

where $a_1, a_2, \ldots, a_{m+1}$ are the coefficients of the following characteristic equation

$$det(\lambda I_{m+1} - A^i) = \lambda^{m+1} + a_1 \lambda^m + \ldots + a_m \lambda + a_{m+1} = 0. \tag{25}$$

We then define $F^i(\mu_1^m(t)\tilde{\sigma}_0^i)$ as follows,

$$F^i = (-a_1, -a_2, \ldots, -a_{m+1})^T \mu_1^m(t)\tilde{\sigma}_0^i + H^i(\mu_1^m(t)\tilde{\sigma}_0^i), \tag{26}$$

where the vector function $H^i(\tilde{\sigma}_0^i)$ is given as in (21).

By matrices C^i and $\mathcal{A}^i$, we introduce another new matrix $\mathcal{O}_2^i = [C^i, C^i \mathcal{A}^i, \ldots, C^i (\mathcal{A}^i)^m]^T \in R^{(m+1)\times(m+1)}$.

Using $\mathcal{O}_1^i$ and $\mathcal{O}_2^i$, we define

$$P^i = (\mathcal{O}_1^i)^{-1}\mathcal{O}_2^i. \tag{27}$$

Given the estimation error variables as,

$$\tilde{\sigma}_0^i = x_i - \sigma_0^i, \tilde{\sigma}_1^i = d_i(t) - \sigma_1^i, \ldots, \tilde{\sigma}_m^i = d_i^{(m-1)}(t) - \sigma_m^i. \tag{28}$$

The prescribe-time estimation of the disturbances and their derivatives is presented by the following lemma.

Lemma 6: Under Assumption 1, consider system (1) and its the disturbance observers (19)-(20) that specified with (21), (22), (26), and (27), where prescribed-time T_{ob} independent of initial conditions. By choosing appropriate λ_j^i, it holds that

1) all the error signals $\tilde{\sigma}_k^i$ and σ_k^i , $i = 1, 2, \ldots, n$, $k = 0, 1, \ldots, n+1-i$, are ensured to be globally uniformly bounded.

2) The error of observation (28) converges to zero in a prescribed-time T_{ob}^*, which satisfies $T_{ob}^* < T_{ob}$, with $T_{ob} > 0$ being independent of initial conditions and other design parameters that can be pre-specified by the designer.

Proof: It is noted that the disturbance observer (19)-(20) is composed of time-varying counterpart $\mathcal{M}^i(t)P^iF^i(\mu_1^m(t)\tilde{\sigma}_0^i)$ and the finite-time stabilizing components (21) with the injection term $H^i(\tilde{\sigma}_0^i)$. When $|\mathrm{sign}(\tilde{\sigma}_0^i)| = 1$, the time-varying observer component $F^i(\mu_1^m(t)\tilde{\sigma}_0^i)$ can enforce the estimation error $\tilde{\sigma}_j^i$ $(j = 0, 1, \ldots, m)$ to reset to zero in a finite time faster than the prescribed time T_{ob}. When $|\mathrm{sign}(\tilde{\sigma}_0^i)| \neq 1$, the observer then switches to the autonomous $H^i(\tilde{\sigma}_0^i)$ that can keep the estimation error $\tilde{\sigma}_j^i$ in the sliding mode that occurring in the origin $\tilde{\sigma}_j^i = 0$.

By combining (19) and (20), we arrive at

$$\dot{\tilde{\sigma}}^i = \Lambda\tilde{\sigma}^i + B^i d_i^{(m)} - Q^i(\tilde{\sigma}_0^i, t) \tag{29}$$

for $i = 1, 2, \ldots, n$, where $\dot{\tilde{\sigma}}^i = [\tilde{\sigma}_0, \tilde{\sigma}_1, \ldots, \tilde{\sigma}_m]^T \in R^{m+1}$, $B^i = [0, 0, \ldots, 1]^T \in R^{m+1}$.

When $|sign(\tilde{\sigma}_0^i)| = 1$,

$$\dot{\tilde{\sigma}}^i = \Lambda\tilde{\sigma}^i + B^i d_i^{(m)} - \mathcal{M}^i(t)P^iF^i(\mu_1^m(t)\tilde{\sigma}_0^i). \tag{30}$$

Let $\tilde{\zeta}_j^i = \mu_1^{m-j}\tilde{\sigma}_j^i$ with $m = n+1-i$, $j = 0, 1, \ldots, m$. The equation (30) can be rewritten as

$$\dot{\tilde{\zeta}}^i = \mu_1 \left(A^i\tilde{\zeta}^i + \mu_1^{-1}B^i d_i^{(m)} - P^iF^i(\tilde{\zeta}_0^i) \right), \tag{31}$$

where A^i is given as in (23). Upon using the temporal scale transformation (3)-(4), we arrive at

$$\frac{d\tilde{\zeta}^i}{ds} = A^i\tilde{\zeta}^i + B^i d_i^{(m)} e^{-s/T_{ob}} - P^iF^i(\tilde{\zeta}_0^i). \tag{32}$$

Let $z(s)^i = (P^i)^{-1}\tilde{\zeta}(s)^i$. By combining $(P^i)^{-1}B^i = B^i$, $C^iP^i = C$, $(P^i)^{-1}A^iP^i = \mathcal{A}^i$ that can also be seen in it is shown [25, Section 12.7], we get

$$\frac{d\tilde{z}^i}{ds} = \Lambda^i\tilde{z}^i + B^i d_i^{(m)} e^{-s/T_{ob}} - H^i(\tilde{z}_0^i). \tag{33}$$

From Theorem 2 of [3], the error signals $z_j^i(s)$ $(i = 1, 2, \ldots, n)$ are uniformly globally bounded and are convergent to zero within finite time. By $t(s) = T_{ob}(1 - e^{-s/T_{ob}})$, observing time T_{ob}^* satisfies that $T_{ob}^* < T_{ob}$. When the time instant $t \geq T_{ob}$, the corrective term $Q^i(\tilde{\sigma}_0^i, t)$ of observer switches to $H^i(\tilde{\sigma}_0^i)$, which keeps the error σ_j^i in the origin. This completes the proof. ■

Remark 2: In contrast to the finite-time disturbance observers used in [2], [27]-[29], where the convergence time is dependent on a set of design parameters and initial conditions. and thus can not be explicitly given, here we propose a prescribed-time disturbance observer where the upper bound of the convergence time can be pre-specified by the designer and is independent of any initial conditions and design parameters.

B. The estimations of the steady states

According to the nonlinear output regulation theory [31]-[32], the steady states of system (1) can be represented by

$$\begin{aligned} \varpi_1(t) &= r(t), \\ \varpi_i(t) &= \frac{d\varpi_{i-1}}{dt}(t) - f_{i-1}(t, \bar{\varpi}_{i-1}) - d_{i-1}(t) \end{aligned} \tag{34}$$

for $i = 2, 3, \ldots, n+1$. Let $\varpi_1(t) = \phi_1(r_1(t))$, $\varpi_i(t) = \phi_i(\bar{d}_i(t), \bar{r}_i(t), t)$, where $r_1(t) = r(t)$, $\bar{r}_i(t) = \{r, \dot{r}, \ldots, r^{(i-1)}\}$, and $\bar{d}_i(t) = \{\bar{d}_{i-1,0}, \bar{d}_{i-2,1}, \ldots, \bar{d}_{1,i-2}\}$ with $\bar{d}_{i,j} = \{d_1^{(j)}, d_2^{(j)}, \ldots, d_i^{(j)}\}$, $i = 2, 3, \ldots, n+1$.

It is obvious that $\phi_1(r_1(t))$ and $\phi_i(\bar{d}_i(t), \bar{r}_i(t), t)$ are constructed by tracking signals, disturbances and their up to nth derivatives. Based on prescribed-time disturbance observers as given in subsection A, we obtain the steady state estimation of system (1) as follows,

$$\begin{aligned} \hat{\varpi}_1(t) &= \phi_1(r_1(t)), \\ \hat{\varpi}_i(t) &= \phi_i(\bar{\sigma}_i(t), \bar{r}_i(t), t), \quad 2 \leq i \leq n+1, \end{aligned} \tag{35}$$

where $\bar{\sigma}_i(t) = \{\bar{\sigma}_{i-1,0}, \bar{\sigma}_{i-2,1}, \ldots, \bar{\sigma}_{1,i-2}\}$ with $\bar{\sigma}_{i,j} = \{\sigma_1^{(j)}, \sigma_2^{(j)}, \ldots, \sigma_i^{(j)}\}$.

Then, the property of the prescribed-time estimation of the steady states is presented by the following lemma.

Lemma 7: Under Assumption 2-3, and by the disturbance observers (19)-(21), then $\hat{\varpi}_j(t) = \varpi_j(t)$ for $t \geq T_{ob}$, $i = 1, 2, \ldots, n$, $k = 0, 1, \ldots, n+1-i$, $j = 1, 2, \ldots, n+1$.

Proof: By Lemma 6, we can obtain that $\tilde{\sigma}_k^i \in \mathcal{L}_\infty$ and $\sigma_k^i \in \mathcal{L}_\infty$. By $\sigma_k^i \in \mathcal{L}_\infty$ and $\bar{r}_i(t) \in \mathcal{L}_\infty$, it can be concluded that $\hat{\varpi}_j(t) \in \mathcal{L}_\infty$. Similarly, from Lemma 6, the convergence time satisfies that $T_{ob}^* < T_{ob}$ and $\tilde{\sigma}_k^i = 0$ for $t \geq T_{ob}$. Therefore, it is to directly obtain $\hat{\varpi}_j(t) = \varpi_j(t)$ for $t \geq T_{ob}$. ∎

C. Controller design

Firstly, we establish the η-coordinate transformation as

$$\begin{aligned} \eta_i(t) &= x_i(t) - \phi_i(\bar{\sigma}_i(t), \bar{r}_i(t), t), \quad 1 \leq i \leq n-1, \\ \eta_n(t) &= x_n(t) - \phi_n(\bar{\sigma}_n(t), \bar{r}_n(t), t), \end{aligned} \tag{36}$$

where $\phi_i(\bar{\sigma}_i(t), \bar{r}_i(t), t)$ are defined as in (35). Then, by combining (1), (19), (28), (35), and (36), we arrive at

$$\begin{aligned} \dot{\eta}_i &= \eta_{i+1} + f_i(t, \bar{x}_i) - f_i(t, \bar{\hat{\varpi}}_i) + \tilde{\sigma}_1^i, \quad 1 \leq i \leq n-1, \\ \dot{\eta}_n &= u - \hat{\varpi}_{n+1} + f_n(t, \bar{x}_n) - f_n(t, \bar{\hat{\varpi}}_n) + \tilde{\sigma}_1^n. \end{aligned} \tag{37}$$

As another key step, we introduce the following scaling transformation

$$z_i(t) = \mu^{n+1-i}(t)\eta_i(t), \quad 1 \leq i \leq n, \tag{38}$$

where

$$\mu(t) = \begin{cases} \dfrac{T}{T-t}, & t \leq T^*, \\ \dfrac{T}{T-T^*}, & t > T^* \end{cases} \tag{39}$$

with $0 < T^* < T$ (T^* will be given later), and T denotes the prescribed convergence time. By differentiating z_i as given in (38), we have

$$\begin{aligned} \dot{z}_i &= \mu(z_{i+1} + \tilde{f}_i(t, \bar{z}_i) + \mu^{n-i}\tilde{\sigma}_1^i), \quad 1 \leq i \leq n-1, \\ \dot{z}_n &= \mu(u - \hat{\varpi}_{n+1} + \tilde{f}_n(t, \bar{z}_n) + \tilde{\sigma}_1^n), \end{aligned} \tag{40}$$

where

$$\tilde{f}_i(t, \bar{z}_i) = \mu^{n-i}(f_i(t, \bar{x}_i) - f_i(t, \bar{\hat{\varpi}}_i)) + \frac{n+1-i}{T} z_i. \tag{41}$$

Moreover, If Assumption 1 is satisfied, we can get

$$\begin{aligned} |\tilde{f}_i(t, \bar{z}_i)| &\leq \mu^{n-i}\gamma_i(\bar{x}_i, \bar{\varpi}_i) \sum_{j=1}^{i} |x_j - \hat{\varpi}_i|^{\frac{r_i+\tau}{r_j}} \\ &\quad + \frac{n+1-i}{T}|z_i| \\ &\leq (\mu^{n-i}\gamma_i(\bar{x}_i, \bar{\hat{\varpi}}_i) + \frac{n+1-i}{T}|z_i|^{\frac{-\tau}{r_i}}) \\ &\quad \times \sum_{j=1}^{i} |x_j - \hat{\varpi}_j|^{\frac{r_i+\tau}{r_j}}, \end{aligned} \tag{42}$$

where $\tau \in (-\frac{1}{n}, 0)$. According to (38), $0 < \mu^{-1}(t) < 1$ for $t \in [0, T)$, and $0 < r_i + \tau < 1$, we have

$$|x_j - \hat{\varpi}_j|^{\frac{r_i+\tau}{r_j}} \leq |\mu|^{\frac{-(n+1-j)(r_i+\tau)}{r_j}} |z_j|^{\frac{r_i+\tau}{r_j}} \leq |z_j|^{\frac{r_i+\tau}{r_j}}. \tag{43}$$

Hence, (42) can be rewritten as

$$\begin{aligned} |\tilde{f}_i(t, \bar{z}_i)| &\leq (\mu^{n-i}\gamma_i(\bar{x}_i, \bar{\hat{\varpi}}_i) + \frac{n+1-i}{T}|z_i|^{\frac{-\tau}{r_i}}) \\ &\quad \times \sum_{j=1}^{i} |z_j|^{\frac{r_i+\tau}{r_j}} \\ &\leq \bar{\gamma}_i(t, \bar{z}_i) \times \sum_{j=1}^{i} |z_j|^{\frac{r_i+\tau}{r_j}}, \end{aligned} \tag{44}$$

where $\bar{\gamma}_i(t, \bar{z}_i) \geq \mu^{n-i}\gamma_i(\bar{x}_i, \bar{\hat{\varpi}}_i) + \frac{n+1-i}{T}|z_i|^{-\frac{\tau}{r_i}}$ is a non-negative continuous function.

To begin with, the following transformation is introduced,

$$\xi_i = \lceil z_i \rceil^{\frac{1}{r_i}} - \lceil z_i^* \rceil^{\frac{1}{r_i}}, \quad z_i^* = -K_{i-1}(t, \bar{z}_{i-1}) \lceil \xi_{i-1} \rceil^{r_i}, \tag{45}$$

where z_i^* is the virtual controller and $z_1^* = 0$, $i = 1, 2, \ldots, n$.

In this work, define the Lyapunov function candidate $V_i(\bar{z}_i) = V_{i-1}(\bar{z}_{i-1}) + W_i(\bar{z}_i)$, where $V_1(z_1) = W_1(z_1)$, and $W_i(\bar{z}_i)$ is given as

$$W_i(\bar{z}_i) = \int_{z_i^*}^{z_i} \left\lceil \lceil \varphi \rceil^{\frac{1}{r_i}} - \lceil z_i^* \rceil^{\frac{1}{r_i}} \right\rceil^{2-r_{i+1}} d\varphi. \tag{46}$$

By following the procedure to that in [8] and [33], we get the following lemma.

Lemma 8: For $i = 1, 2, \ldots, n$, $W_i(\cdot)$ is differentiable and satisfies

$$\begin{cases} \dfrac{\partial W_i}{\partial z_i} = \lceil \xi_i \rceil^{2-r_{i+1}}, \\ \dfrac{\partial W_i}{\partial \chi_j} = -(2 - r_{i+1}) \dfrac{\partial}{\partial \chi_k} \left(\lceil z_i^* \rceil^{\frac{1}{r_i}} \right) \\ \qquad \times \displaystyle\int_{z_i^*}^{z_i} \left\lceil \lceil \varphi \rceil^{\frac{1}{r_i}} - \lceil z_i^* \rceil^{\frac{1}{r_i}} \right\rceil^{1-r_{i+1}} d\varphi, \end{cases} \tag{47}$$

where $\chi_j = z_j$ for $j = 1, 2, \ldots, i-1$ and $\chi_i = t$.

Then, we construct the controller by the following steps.

Step 1: For $V_1 = W_1$ with virtual control $z_1^* = 0$, take derivative of V_1 long (40) yields

$$\dot{V}_1(\bar{z}_1) = \mu \cdot \lceil z_1 \rceil^{2-\tau-r_1}(z_2 + f_1(t, \bar{z}_1) + \mu^{n-1}\tilde{\sigma}_1^1). \tag{48}$$

According to (44) and Lemma 2, the equation (48) is rewritten as

$$\begin{aligned} \dot{V}_1(\bar{z}_1) &\leq \mu \cdot \lceil z_1 \rceil^{2-\tau-r_1}(z_2 - z_2^* + z_2^*) \\ &\quad + \mu \cdot |z_1|^{\frac{2-\tau-r_1}{r_1}}(\bar{\gamma}_1(t, \bar{\eta}_1)|\eta_1|^{\frac{r_1+\tau}{r_1}} + \mu^{n-1}\tilde{\sigma}_1^1) \\ &\leq \mu \lceil z_1 \rceil^{2-\tau-r_1}(z_2 - z_2^* + z_2^*) \\ &\quad + \mu(\bar{\gamma}_1(t, \bar{\eta}_1) + h_1(t)) z_1^{\frac{2}{r_1}} + \mu\frac{1}{n}(\tilde{\sigma}_1^1)^{\frac{2}{r_1+\tau}}, \end{aligned} \tag{49}$$

where $h_1(t)$ is a positive smooth function. We select the control gain $K_1 \geq n + \bar{\gamma}_1(t, \bar{\eta}_1) + h_1(t)$. By inserting the virtual control law z_2^* into (49), we arrive at

$$\dot{V}_1(\bar{z}_1) \leq \mu \left(\lceil \xi_1 \rceil^{2-\tau-r_1}(z_2 - z_2^*) - n\xi_1^2 + \frac{1}{n}(\tilde{\sigma}_1^1)^{\frac{2}{r_1+\tau}} \right). \tag{50}$$

Step i ($i = 2, 3, \ldots, n-1$): Suppose that at step $i-1$, Assumptions 2-3 are satisfied. By employing the control law z_i^*, the derivative of $V_i(\bar{z}_i)$ along (40) is

$$\dot{V}_{i-1}(\bar{z}_{i-1}) \leq \mu\lceil\xi_{i-1}\rceil^{2-r_i}(z_i - z_i^*) - \mu(n+2-i)\sum_{\mathrm{j}=1}^{i-1}\xi_j^2 + \mu\frac{1}{n+2-i}\sum_{\mathrm{j}=1}^{i-1}(\tilde{\sigma}_1^j)^{\frac{2}{r_j+\tau}}. \quad (51)$$

At step i, define a Lyapunov function $V_i(\bar{z}_i) = V_{i-1}(\bar{z}_{i-1}) + W_i(\bar{z}_i)$. By differentiating $V_i(\bar{z}_i)$ and by combining (40), (46), and (47), we have

$$\begin{aligned}\dot{V}_i(\bar{z}_i) \leq{}& \dot{V}_{i-1}(\bar{z}_{i-1}) + \mu\lceil\xi_i\rceil^{2-\tau-r_i}(z_{i+1} - z_{i+1}^*) \\ &+ \mu\lceil\xi_i\rceil^{2-\tau-r_i}(z_{i+1}^* + \tilde{f}_i(t,\bar{z}_i) + \mu^{n-i}\tilde{\sigma}_1^i) \\ &+ \sum_{\mathrm{j}=1}^{i-1}\frac{\partial W_i}{\partial z_j}\dot{z}_j + \frac{\partial W_i}{\partial t}. \end{aligned} \quad (52)$$

By inserting (51) into (52), we get

$$\begin{aligned}\dot{V}_i(\bar{z}_i) \leq{}& \mu\lceil\xi_i\rceil^{2-r_{i+1}}(z_{i+1} - z_{i+1}^*) + \mu\left|\xi_{i-1}^{2-r_i}(z_i - z_i^*)\right| \\ &+ \mu\left|\xi_i^{2-r_{i+1}}(\tilde{f}_i(t,\bar{z}_i) + \mu^{n-i}\tilde{\sigma}_1^i)\right| \\ &- \mu(n+2-i)\sum_{\mathrm{j}=1}^{i-1}\xi_j^2 + \mu\frac{1}{n+2-i}\sum_{\mathrm{j}=1}^{i-1}(\tilde{\sigma}_1^j)^{\frac{2}{r_j+\tau}} \\ &+ \mu\lceil\xi_i\rceil^{2-r_{i+1}}\cdot z_{i+1}^* + \left|\sum_{\mathrm{j}=1}^{i-1}\frac{\partial W_i}{\partial z_j}\dot{z}_j\right| + \left|\frac{\partial W_i}{\partial t}\right|. \end{aligned} \quad (53)$$

By Lemma 1 and 2, the second term on the right hand side of (53) becomes

$$\mu\left|\xi_{i-1}^{2-r_i}(z_i - z_i^*)\right| \leq \mu\left(\frac{1}{4}\xi_{i-1}^2 + b_i\xi_i^2\right), \quad (54)$$

where b_i is a positive constant.

From (44), the third term on the right hand side of (53) can be rewritten as

$$\begin{aligned}&\mu\left|\xi_i^{2-\tau-r_i}(\tilde{f}_i(t,\bar{z}_i) + \mu^{n-i}\tilde{\sigma}_1^i)\right| \\ &\leq \mu\left|\xi_i^{2-\tau-r_i}\right|\left(\bar{\gamma}_i(t,\bar{z}_i)\sum_{\mathrm{j}=1}^{i}|z_j|^{\frac{r_i+\tau}{r_j}} + \mu^{n-i}|\tilde{\sigma}_1^i|\right). \end{aligned} \quad (55)$$

From (45), we see that

$$|z_i| = |\xi_i - K_{i-1}^{\frac{1}{r_i}}\xi_{i-1}|^{r_i} \leq c_i(t,\bar{z}_{i-1})(|\xi_i|^{r_i} + |\xi_{i-1}|^{r_i}), \quad (56)$$

where $c_i(t,\bar{z}_{i-1})$ is a positive smooth function. Substituting (56) into (55) yields

$$\begin{aligned}&\mu\left|\xi_i^{2-r_{i+1}}(\tilde{f}_i(t,\bar{z}_i) + \mu^{n-i}\tilde{\sigma}_1^i)\right| \\ &\leq \mu\left|\xi_i^{2-r_{i+1}}\right|\left(\bar{\gamma}_i\sum_{\mathrm{j}=1}^{i}(c_j(|\xi_j|^{r_j} + |\xi_{j-1}|^{r_j}))^{\frac{r_i+\tau}{r_j}}\right) \\ &\quad + \mu\left|\xi_i^{2-r_{i+1}}\right|(\mu^{n-i}|\tilde{\sigma}_1^i|) \\ &\leq \frac{1}{3}\mu\sum_{\mathrm{j}=1}^{i-2}\xi_j^2 + \frac{1}{4}\mu\xi_{i-1}^2 + \check{c}_i\mu\xi_i^2 + \mu\frac{1}{n+1-i}(\tilde{\sigma}_1^i)^{\frac{2}{r_{i+1}}}, \end{aligned} \quad (57)$$

where $\check{c}_i(t,\bar{z}_i)$ is a positive smooth function.

The calculation of the last two terms on the right hand side of (53) is provided in the Appendix.

Substituting (54), (57), (75) and (77) into (53) yields

$$\begin{aligned}\dot{V}_i(\bar{z}_i) \leq{}& \mu\lceil\xi_i\rceil^{2-r_{i+1}}(z_{i+1} - z_{i+1}^*) + \mu\lceil\xi_i\rceil^{2-r_{i+1}}z_{i+1}^* \\ &- \mu(n+1-i)\sum_{\mathrm{j}=1}^{i-1}\xi_j^2 + \mu\bar{c}_i(t,\bar{z}_i)\xi_i^2 \\ &+ \mu\frac{1}{n+1-i}\sum_{\mathrm{j}=1}^{i}(\tilde{\sigma}_1^j)^{\frac{2}{r_j+\tau}}, \end{aligned} \quad (58)$$

where $\bar{c}_i(t,\bar{z}_i) = \check{c}_i(t,\bar{z}_i) + \hat{c}_i(t,\bar{z}_{i-1}) + \tilde{c}_i(t,\bar{z}_i)$. By inserting a virtual controller z_{i+1}^* as given in (41), where the control gain $K_i(t,\bar{z}_i) \geq (n+1-i) + \bar{c}_i(t,\bar{z}_i)$, into (57), we arrive at

$$\dot{V}_i(\bar{z}_i) \leq \mu\lceil\xi_i\rceil^{2-r_{i+1}}(z_{i+1} - z_{i+1}^*) - \mu(n+1-i)\sum_{\mathrm{j}=1}^{i}\xi_j^2 + \mu\frac{1}{n+1-i}\sum_{\mathrm{j}=1}^{i}(\tilde{\sigma}_1^j)^{\frac{2}{r_j+\tau}}. \quad (59)$$

For a Lyapunov function $V_n(\bar{z}_n) = V_{n-1}(\bar{z}_{n-1}) + W_n(\bar{z}_i)$, by following the procedure similar to (54), (57), (75), and (77), we get

$$\dot{V}_n(\bar{z}_n) \leq \mu\lceil\xi_n\rceil^{2-\tau-r_n}(u - \hat{\varpi}_{n+1} - z_{n+1}^*) - \mu\sum_{\mathrm{i}=1}^{n}\xi_i^2 + \mu\sum_{\mathrm{i}=1}^{n}(\tilde{\sigma}_1^i)^{\frac{2}{r_i+\tau}} \quad (60)$$

Based on the above analysis, our main result in this paper is presented in the following Theorem.

Theorem 1: Consider system (1) under Assumptions 1-3. If the control input is designed as

$$u = \hat{\varpi}_{n+1} + z_{n+1}^*, \quad (61)$$

where $T^* = T\left(1 - e^{-\frac{2-\tau}{kT(-\tau)}V(x_0)^{\frac{-\tau}{2-\tau}}}\right)$ with $k = \lambda^{\frac{2-\tau}{2}}$, $\lambda > 0$ is a constant, $\tau \in (-\frac{1}{n}, 0)$ is defined as in Assumption 1, $\hat{\varpi}_{n+1}$ is defined as in (35), and z_{n+1}^* is given as

$$z_{n+1}^* = -\left(\sum_{\mathrm{i}=1}^{n}\kappa_i(t,\bar{z}_n)z_i^{\frac{1}{r_i}}\right)^{r_{n+1}} \quad (62)$$

with $\kappa_i(t,\bar{z}_n) = \prod_{l=i}^{n} K_l^{\frac{1}{r_{l+1}}}(t,\bar{z}_l)$ being a positive smooth function. Then,

1) all signals in the closed-loop system are bounded for $t \in [0,\infty)$.
2) The exact output tracking for the time-varying reference signal $r(t)$ within T.
3) The output tracking error η_1 remains in the origin for $t \in [T,\infty)$.

Proof: By inserting the control input u as given in (61) into (60), we have

$$\dot{V}_n(\bar{z}_n) \le -\mu \sum_{i=1}^{n} \xi_i^2 + \mu \sum_{i=1}^{n} (\tilde{\sigma}_1^i)^{\frac{2}{r_i+\tau}}, \tag{63}$$

where $\tilde{\sigma}_1^i$ $(i = 1,2,\ldots,n)$ denotes the disturbance estimation error, which is bounded that can be seen in Lemma 6. By utilizing the definition of V_n, Lemma 1, and Lemma 3, we have

$$V_n(\bar{z}_n) \le \lambda \sum_{i=1}^{n} |\xi_i|^{2-\tau} \le \lambda \left(\sum_{i=1}^{n} \xi_i^2\right)^{\frac{2-\tau}{2}} \tag{64}$$

where $\lambda > 0$ is a constant. Substituting (64) into (63) yields

$$\begin{aligned}\dot{V}_n(\bar{z}_n) &\le -\mu \lambda^{\frac{2-\tau}{2}} V_n^{\frac{2}{2-\tau}} + \mu \sum_{i=1}^{n} (\tilde{\sigma}_1^i)^{\frac{2}{r_i+\tau}} \\ &= -k\mu V_n^{\frac{2}{2-\tau}} + \mu \sum_{i=1}^{n} (\tilde{\sigma}_1^i)^{\frac{2}{r_i+\tau}},\end{aligned} \tag{65}$$

where $k = \lambda^{\frac{2-\tau}{2}}$, $0 < \frac{2}{2-\tau} < 1$.

For $0 \le t < T_{ob}$, by using Corollary 1, we get $\dot{V}_n(\bar{z}_n) \in \mathcal{L}_\infty$. It means that $z_i \in \mathcal{L}_\infty$, $i = 1,2,\ldots,n$. We choose T such that $T > T_{ob}$, where T denotes the prescribed-time and T_{ob} is the upper bound of convergence time of the disturbance observer.

For $T_{ob} \le t \le T^*$, the disturbance estimation error $\tilde{\sigma}_1^i = 0$, then (65) is rewritten as

$$\dot{V}_n(\bar{z}_n) \le -k\mu(t) V_n^{\frac{2}{2-\tau}}. \tag{66}$$

Upon using Lemma 5, it can be concluded that $z_i = 0$ for $t \ge T^* = T\left(1 - e^{-\frac{2-\tau}{kT(-\tau)} V(x_0)^{\frac{-\tau}{2-\tau}}}\right)$. It is clear that $T^* < T$. By recalling that $\mu(t) = \frac{T}{T-t}$ and z_i is bounded for $t \in [0,T^*]$, it can be derived that the control input $u = \hat{\varpi}_{n+1} + z_{n+1}^*$ is bounded. Furthermore, by noting that $z_i(t) = \mu^{n+1-i}(t)\eta_i(t)$ and $\mu(t)$ is bounded for $t \in [0,T^*]$, it is not hard to get that η_i is bounded for $t \in [0,T^*]$. In addition, $z_i(t) = 0$ implies that $\eta_i = 0$ for $t = T^*$.

For $t \in (T^*,\infty)$ it is noted that $\mu(t) = \mu(T^*) = \frac{T}{T-T^*}$, then we have

$$\dot{V}_n(\bar{z}_n) \le -k\mu(T^*) V_n^{\frac{2}{2-\tau}}. \tag{67}$$

By the above analysis, $z_i = 0$ for $t = T^*$, hence $V_n(\bar{z}_n)(T^*) = 0$. According to Lemma 4, it can be concluded that the signals z_i $(i = 1,2,\ldots,n)$ can be maintained in the origin for $t \in (T^*,\infty)$. This proof is completed. ∎

IV. Numerical Simulations

To confirm the effectiveness of the proposed prescribed-time tracking control algorithm, we conduct the simulation on the following system

$$\begin{aligned}\dot{x}_1 &= x_2 + f_1(t,x_1) + d_1(t), \\ \dot{x}_2 &= u + f_2(t,\bar{x}_2) + d_2(t),\end{aligned} \tag{68}$$

in which $f_1(t,x_1) = 0.3x_1^{3/5} + 1$, $f_2(t,\bar{x}_2) = x_1^2 + \sin(t)x_2^2$, $d_1 = 1+\sin(\frac{\pi t}{2})$, and $d_2(t) = \cos(\pi t)$. Consider the reference signal $r(t) = 3 + \sin(2t) + \cos(t)$. Choose $\tau = -2/5$, $r_1 = 1$, $r_2 = 3/5$, $r_3 = 1/5$. By brief calculation, we get $\gamma_1(t,z_1) = 0.8\mu^{-\frac{1}{5}} + \frac{2}{T}|z_1|^{\frac{2}{5}}$ and $\gamma_2(t,\bar{z}_2) = |x_1 + \phi_1||\eta_1|^{\frac{4}{5}}\mu^{-\frac{2}{5}} + |x_2 + \phi_2||x_2 - \phi_2|^{\frac{2}{3}}\mu^{-\frac{1}{3}} + \frac{1}{T}|z_2|^{\frac{2}{3}}$. Hence, Assumption 1 is satisfied. The prescribed-time controller is given by $u = -(k_2 + k_{2c}\gamma_2 + k_{2h}\gamma_1^{\frac{10}{9}})\lceil \xi_2 \rfloor^{r_3}$, where $\xi_2 = \lceil z_2 \rfloor^{\frac{1}{r_2}} - \lceil z_2^* \rfloor^{\frac{1}{r_2}}$ with $z_2^* = -(k_1 + k_{1c}\gamma_1(t,z_1))\lceil \xi_1 \rfloor^{r_2}$, $k_1 = 1$, $k_{1c} = 5$, $k_2 = 5.5$, $k_{2c} = 2$, $k_{2h} = 4$.

In simulation, we choose $\mu(t)$ is defined as in (39) with $T = 3s$ (it is the prescribed convergence time), and choose $\mu_1(t) = \frac{T_{ob}}{T_{ob}-t}$ with $T_{ob} = 1.5s$ (it is the prescribed observation time). According to the design procedure proposed in this paper, we get $T^* = 2.5s$. The initial conditions are taken as $x_1(0) = 1$, $x_2(0) = -1$, $\sigma_0^1(0) = \upsilon_0^1(0) = 0$, $\sigma_1^1(0) = 0$, $\sigma_0^2(0) = 0$, and $\upsilon_0^2(0) = 2$. The parameters of observers are given by $L_1 = 3$, $L_2 = 4$, $\lambda_0^1 = 2$, $\lambda_1^1 = 1.5$, $\lambda_2^1 = 1.1$, $\lambda_0^2 = 3$, and $\lambda_1^2 = 1.1$.

The simulation results of tracking error are shown in Fig.1, from which we see that the tracking error converges to zero within the prescribed-time. The estimates of $d_1(t)$, $\dot{d}_1(t)$ and $d_2(t)$ are depicted in Figs 2-4, respectively. The input of the system is depicted in Fig.5. We see from Fig. 1-5 that the tracking is achieved within the prescribed-time and all the internal signals of the closed-loop system are remained bounded. Fig.6 represents the system response under the proposed scheme given in Theorem 1 under different initial conditions with $T = 3s$. For initial conditions $(x_1(0), x_2(0)) = (1,-1)$, the system responses with different prescribed-time $T = 1.8s$, $T = 3s$ and $T = 4s$ are presented in Fig.7. To confirm the benefits and efficiency of the prescribed-time disturbance observer, the estimates with different observation time $T_{ob} = 1.2s$, $T_{ob} = 1.5s$ and $T_{ob} = 2.5s$ are depicted in Fig. 8. From Fig. 6-8 it is seen that both the convergence time and observation time can be pre-assigned by the designer under the proposed method.

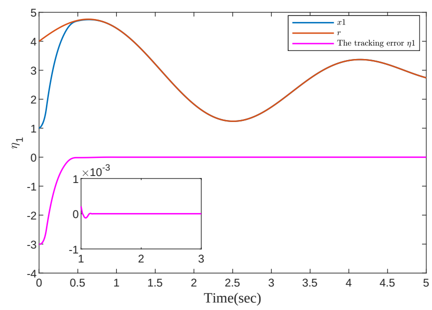


Fig. 1: The tracking error η_1 under the proposed control scheme as given in Theorem 1 with $T = 3s$.

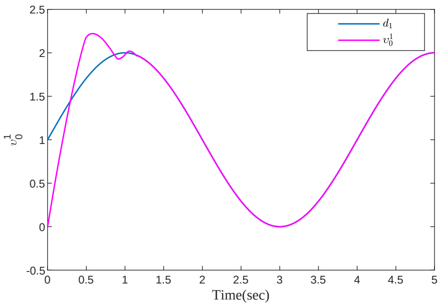


Fig. 2: The estimate of disturbance $d_1(t)$ under the proposed disturbance observer as given in Eqs. (19)-(21) with $T_{ob} = 1.5s$.

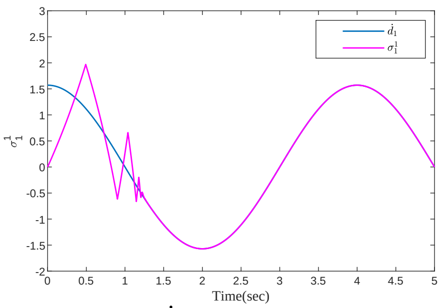


Fig. 3: The estimate of $\dot{d}_1(t)$ under the proposed disturbance observer as given in Eqs. (19)-(21) with $T_{ob} = 1.5s$.

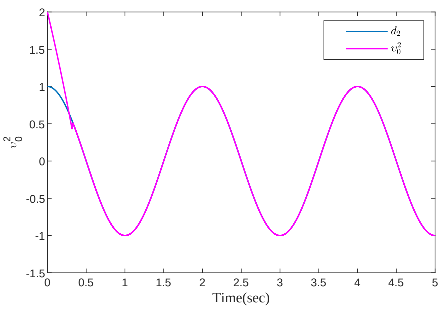


Fig. 4: The estimate of disturbance $d_2(t)$ under the proposed disturbance observer as given in Eqs. (19)-(21) with $T_{ob} = 1.5s$.

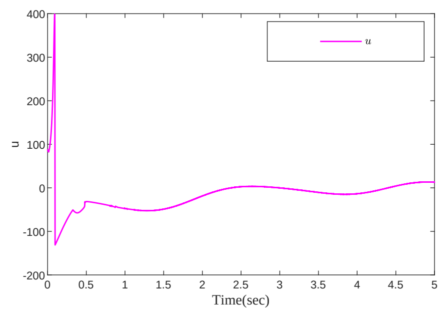
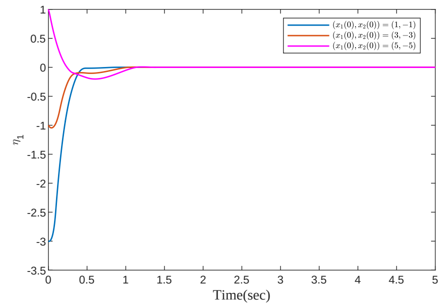


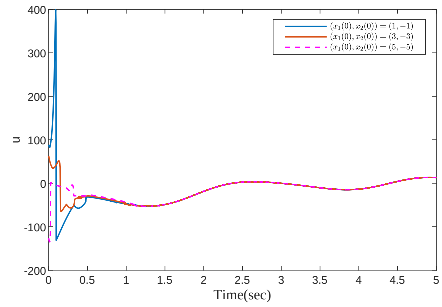
Fig. 5: The response of the control input u under the scheme (60) with $T = 3s$.

V. Conclusions

In this work, the prescribed-time tracking control problem is investigated for a class of strict-feedback systems in presence of time-varying disturbances. By introducing

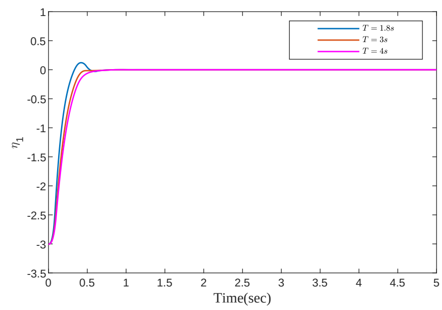


(a) The tracking error η_1

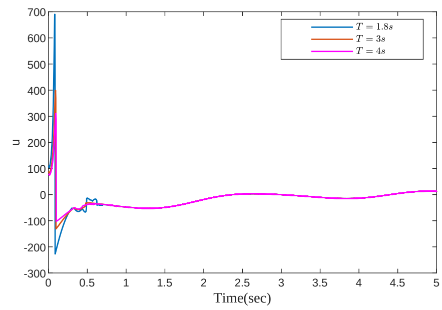


(b) The responses of controller u

Fig. 6: System response under the proposed control scheme in Theorem 1 under different initial conditions with $T = 3s$.



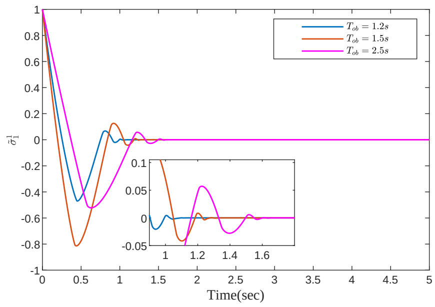
(a) The tracking error η_1



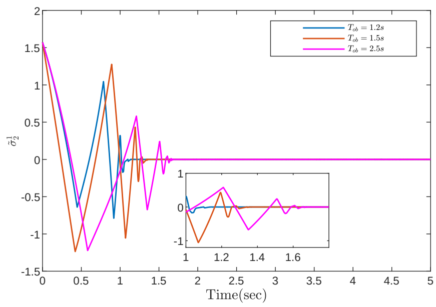
(b) The responses of controller u

Fig. 7: The system response under the proposed control scheme in Theorem 1 under different prescribed time T.

a prescribed-time perturbation observer into the controller design and establishing a new prescribed-time Lyapunov theorem, the precise tracking is achieved within prescribed time, in spite of the uncertainties and disturbances. The proposed control method also prevents the restrictive requirement that the system can only operate on the finite time interval as



(a) $\tilde{\sigma}_1^1 = d_1 - \hat{d}_1$



(b) $\tilde{\sigma}_2^1 = \dot{d}_1 - \hat{\dot{d}}_1$

Fig. 8: The estimate error of disturbances for different prescribed observation time T_{ob}.

imposed in most existing prescribed-time control results. Here the proposed method allows the system to operate on the whole time interval and the control gain does not escape to infinity as time goes to the prescribed time instant. The prescribed-time output tracking control for nonlinear systems with non-vanishing uncertainties represents an interesting topic for the future work.

APPENDIX

From (45), we can get

$$\lceil z_i^* \rfloor^{\frac{1}{r_i}} = -\sum_{\mathrm{j}=1}^{i-1} \kappa_j(t, \bar{z}_{i-1}) z_j^{\frac{1}{r_j}}, \tag{69}$$

where $\kappa_j(t, \bar{z}_{i-1}) = \prod_{\mathrm{l}=\mathrm{j}}^{i-1} K_l^{\frac{1}{r_{l+1}}}(t, \bar{z}_l)$ is a positive smooth function. Then, by taking the derivative of both sides of (69) with respect to z_j, and by combining (56), we arrive at

$$\left|\frac{\partial \lceil z_i^* \rfloor^{\frac{1}{r_i}}}{\partial z_l}\right| = \bar{\kappa}_j(t, \bar{z}_{i-1}) \sum_{\mathrm{j}=1}^{i-1} |\xi_j|^{1-r_l}, \quad 1 \le l \le i-1, \tag{70}$$

where $\bar{\kappa}_j(t, \bar{z}_{i-1})$ is a positive smooth function.

From (38) and (56), it follows that

$$\begin{aligned} |\dot{z}_l| \le \mu & \left(c_{l+1}(t, \bar{z}_l)(|\xi_{l+1}|^{r_{l+1}} + |\xi_l|^{r_{l+1}}) + \mu^{n-l}|\tilde{\sigma}_1^l|\right) \\ & + \mu \tilde{\gamma}_l \sum_{\mathrm{j}=1}^{l} (|\xi_j|^{r_l+\tau} + |\xi_{j-1}|^{r_l+\tau}) \\ \le \mu & \left(\psi_l(t, \bar{z}_l) \sum_{\mathrm{j}=1}^{l+1} |\xi_j|^{r_l+\tau} + \mu^{n-l}|\tilde{\sigma}_1^l|\right) \end{aligned} \tag{71}$$

where $\psi_l(t, \bar{z}_l) > 0$ is a smooth function, and $c_{l+1}(t, \bar{z}_l)$ is defined as in (56). Besides, it can be easily observed from Lemma 2 and 3 that

$$\int_{z_i^*}^{z_i} \left| \lceil \varphi \rfloor^{\frac{1}{r_i}} - \lceil z_i^* \rfloor^{\frac{1}{r_i}} \right|^{1-r_{i+1}} d\varphi \le 2^{1-r_i} |\xi_i|^{1-\tau}. \tag{72}$$

By (47) and Lemma 2, we have

$$\left|\frac{\partial W_i}{\partial z_l} \dot{z}_l\right| \le 2^{1-r_i} \cdot (2 - r_{i+1}) \cdot \mu |\xi_i|^{1-\tau} \left|\frac{\partial \lceil z_i^* \rfloor^{\frac{1}{r_i}}}{\partial z_l}\right| |\dot{z}_l|. \tag{73}$$

By inserting (70) and (71) into (73) we arrive at

$$\begin{aligned} \left|\frac{\partial W_i}{\partial z_l} \dot{z}_l\right| & \le \mu \beta_l(t, \bar{z}_{i-1}) |\xi_i|^{1-\tau} \left(\sum_{\mathrm{j}=1}^{i} |\xi_j|^{1+\tau} + |\tilde{\sigma}_1^l|^{\frac{1+\tau}{r_{l+1}}}\right) \\ & \le \mu \left(\tilde{\beta}_l(t, \bar{z}_{i-1}) \sum_{\mathrm{j}=1}^{i} |\xi_j|^2 + \nu_i (\tilde{\sigma}_1^l)^{\frac{2}{r_{l+1}}}\right), \end{aligned} \tag{74}$$

where $\tilde{\beta}_l(t, \bar{z}_{i-1}) > 0$ is a smooth function, and $\nu_i = \frac{1}{(n+2-i)(n+1-i)}$ is a constant. Hence, the eighth term on the right hand side of (53) becomes

$$\begin{aligned} \left|\sum_{\mathrm{j}=1}^{i-1} \frac{\partial W_i}{\partial z_j} \dot{z}_j\right| \le & \frac{1}{3}\mu \sum_{\mathrm{j}=1}^{i-2} \xi_j^2 + \frac{1}{4}\mu \xi_{i-1}^2 + \hat{c}_i \mu \xi_i^2 \\ & + \nu_i \mu \sum_{\mathrm{j}=1}^{i-1} (\tilde{\sigma}_1^j)^{\frac{2}{r_{j+1}}}, \end{aligned} \tag{75}$$

where $\hat{c}_i(t, \bar{z}_{i-1}) > 0$ is a smooth function.

By taking the derivative of both sides of (69) with respect to t, and by combining (56), we get

$$\left|\frac{\partial \lceil z_i^* \rfloor^{\frac{1}{r_i}}}{\partial t}\right| \le \mu \upsilon_i(t, \bar{z}_{i-1}) \sum_{\mathrm{j}=1}^{i-1} |\xi_j|^{1+\tau}. \tag{76}$$

By combining (72) and (76), we have

$$\left|\frac{\partial W_i}{\partial t}\right| \le \frac{1}{3}\mu \sum_{\mathrm{j}=1}^{i-2} \xi_j^2 + \frac{1}{4}\mu \xi_{i-1}^2 + \tilde{c}_i \mu \xi_i^2, \tag{77}$$

where $\tilde{c}_i(t, \bar{z}_i) > 0$ is a smooth function.

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