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Unified tracking control under full-state constraints imposed irregularly

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Abstract

It is nontrivial to address the control problem of nonlinear systems with irregular constraining conditions (where the constraints, possibly time-varying, asymmetric and alternating positively and negatively, would occur or disappear in the middle of system operation). In this work, we present a control design framework for uncertain pure-feedback systems under the aforementioned constraining conditions. By making use of auxiliary constraining boundaries, we analytically extend the originally imposed constraints over the entire time interval of system operation, with which we introduce a useful state transformation with unique features. It is such transformation that allows for the development of the control scheme capable of uniformly addressing a variety of constraints formed dynamically and imposed irregularly. It is shown that, for the scenarios where the constraints are partially or fully imposed or removed at any time instant during system operation, one only needs to reset the corresponding extended constraining boundaries in the control scheme instead of altering its structure. Both theoretical analysis and numerical simulation authenticate the effectiveness of the proposed method.

KEYWORDS

asymmetric constraints, pure-feedback systems, transformation function, unified tracking control

1 | INTRODUCTION

Background and motivation: Practical systems are always subject to certain operational constraints, due to, for instance, physical limitation, safety consideration and/or specific performance requirement, and so on, making the underlying control problem much more challenging as compared with the case without constraint,[1-4] which has enticed increasing research attention from control community and resulted in a rich collection of control methods (e.g., References 5-9 and the references therein).

It is worth noting that most existing results are valid only if the constraints imposed satisfy two fundamentally regular conditions: 1) the constraining boundaries are constant, or, if time-varying, the upper boundaries must be strictly positive all the time or both upper and lower boundaries must be always away from zero; and 2) the constraints are imposed from the very beginning till the end of system operation (i.e., present all the time during the entire process of system operation).[5,7-9]

In reality, however, the constraints frequently encountered do not actually satisfy such conditions. In fact, the constraints could be imposed only for some period of time (rather than all the time) or removed at some time instants during system operation; the upper and lower boundaries could be sometimes positive and sometimes negative, or even zero at some time instants (this is particularly true when the constraints are formed/imposed dynamically); or during system operation, certain constraints on some states could occur or disappear in the middle of system operation, rendering the system partially constrained for some period of time and fully constrained for some other period of time, and so on. The typical examples include parts auto-assembling, auto self-parking, dynamic obstacle avoidance, and unmanned aerial vehicle passing through a tunnel.[3,10,11] Apparently, such irregular constraints are ubiquitous and are broad enough to include the regular ones considered by most existing works thus far as special cases. Technically, for systems under irregular constraints, the underlying control problem becomes rather challenging, and those methods based upon regular constraints are not directly applicable to any type of such irregular constraints.

Open problem and contribution: Due to the challenge posed by the irregular constraining conditions, it is nontrivial to tackle the control problem for systems with such constraints, and effort thus far has been rarely made in explicitly addressing those constraining scenarios. The purpose of this work is to present a solution to this open problem such that both regular and irregular constraints are coped with uniformly. The main contribution and features of the work can be summarized as follows:

(1) By introducing the concept of auxiliary boundaries and attaching such boundaries to the actual constraining boundaries, we establish a unified control method applicable not only to systems without constraints, but also to systems with constraints imposed or removed anytime during system operation.
(2) With the help of the carefully designed transformation, the constrained system is transformed into one without constraints, allowing the original constrained control problem to be addressed by stabilizing the transformed one. Furthermore, the constraining upper boundaries or lower boundaries could be dynamically formed and do not need to be strictly positive or negative, making the proposed method capable of dealing with a variety of constraints.
(3) By simply resetting the corresponding extended constraining boundaries, the proposed control is able to cope with partial-state constraints and full-state constraints occurring over different time periods without the need for changing its structure during system operation.

2 | PROBLEM FORMULATION AND PRELIMINARIES

In this work, we consider a class of pure-feedback nonlinear systems

$$\begin{cases} \dot{x}_i = f_i(\bar{x}_i, x_{i+1}), \quad i = 1, \ldots, n-1 \\ \dot{x}_n = f_n(\bar{x}_n, u) \\ y = x_1 \end{cases} \tag{1}$$

subject to the following asymmetric and time-varying full-state constraints

$$\Omega_{x_i} = \{x_i \in \mathbf{R} : -F_{i1}(t) < x_i(t) < F_{i2}(t)\}, \tag{2}$$

where $\bar{x}_i = [x_1, \ldots, x_i]^T \in \mathbf{R}^n$ $(i = 1, \ldots, n)$ denotes the state vector, u and y represent system input and output respectively, $f_i(\bar{x}_i, x_{i+1})$ $(i = 1, \ldots, n-1)$ and $f_n(\bar{x}_n, u)$ are unknown nonlinear functions. $-F_{i1}(t)$ and $F_{i2}(t)$ $(i = 1, \ldots, n)$ are time-varying boundaries possibly formed/imposed dynamically on x_i and satisfy $-F_{i1}(t) < F_{i2}(t)$. (For easy description, function/variable argument(s) is (are) sometimes dropped if no confusion is likely to occur).

For system (1) without any constraint, there are numerous control methods reported in literatures.[1,12-15] However, in reality various state/output constraints are always inevitable, making control design and stability analysis quite difficult.[5,7]

Here in this work, we consider the more interesting yet more challenging constraining situation referred to as irregular constraints, in which the corresponding constraints are formed dynamically, such that the resultant lower and upper constraining boundaries ($-F_{i1}$ and F_{i2}) could be not only time-varying and asymmetric, but also alternating positively and negatively or even zero incessantly. Furthermore, the constraints might occur/appear through various scenarios such as

- Scenario 1 (“Entirely Constrained”): Constraints are always present during system operation.
- Scenario 2 (“Constrained-Free”): The constraining boundaries $-F_{i1}(t)$ and $F_{i2}(t)$ are imposed from the beginning of the system operation and then are removed at $t = T_1 > t_0$.
- Scenario 3 (“Free-Constrained”): There is no constraints initially and later the constraining boundaries $-F_{i1}(t)$ and $F_{i2}(t)$ are imposed at $t = T_1$ and thereafter.
- Scenario 4 (“Free-Constrained-Free”): The constraining boundaries $-F_{i1}(t)$ and $F_{i2}(t)$ are not imposed until $t = T_1 > t_0$ and are removed from the moment $t = T_2 > T_1$.
- Scenario 5 (“Unconstrained”): There is no constraint during the system operation, which can be treated as if the constraining boundaries $-F_{i1}(t) = -\infty$ and $F_{i2}(t) = \infty$.
- Scenario 6 (“Partial-State to Full-State Constrained”): The system is partial-state constrained initially and full-state constrained from $t \geq T_1$.

The first four scenarios are conceptually illustrated with curves in solid lines in Figure 1(A–D), respectively.

Remark 1. It is worth noting that Scenario 1 has been the main focus of most existing works, see for instance, References 5-7,9,16, where the constraining boundaries considered are either constants and symmetric or at most time-varying and

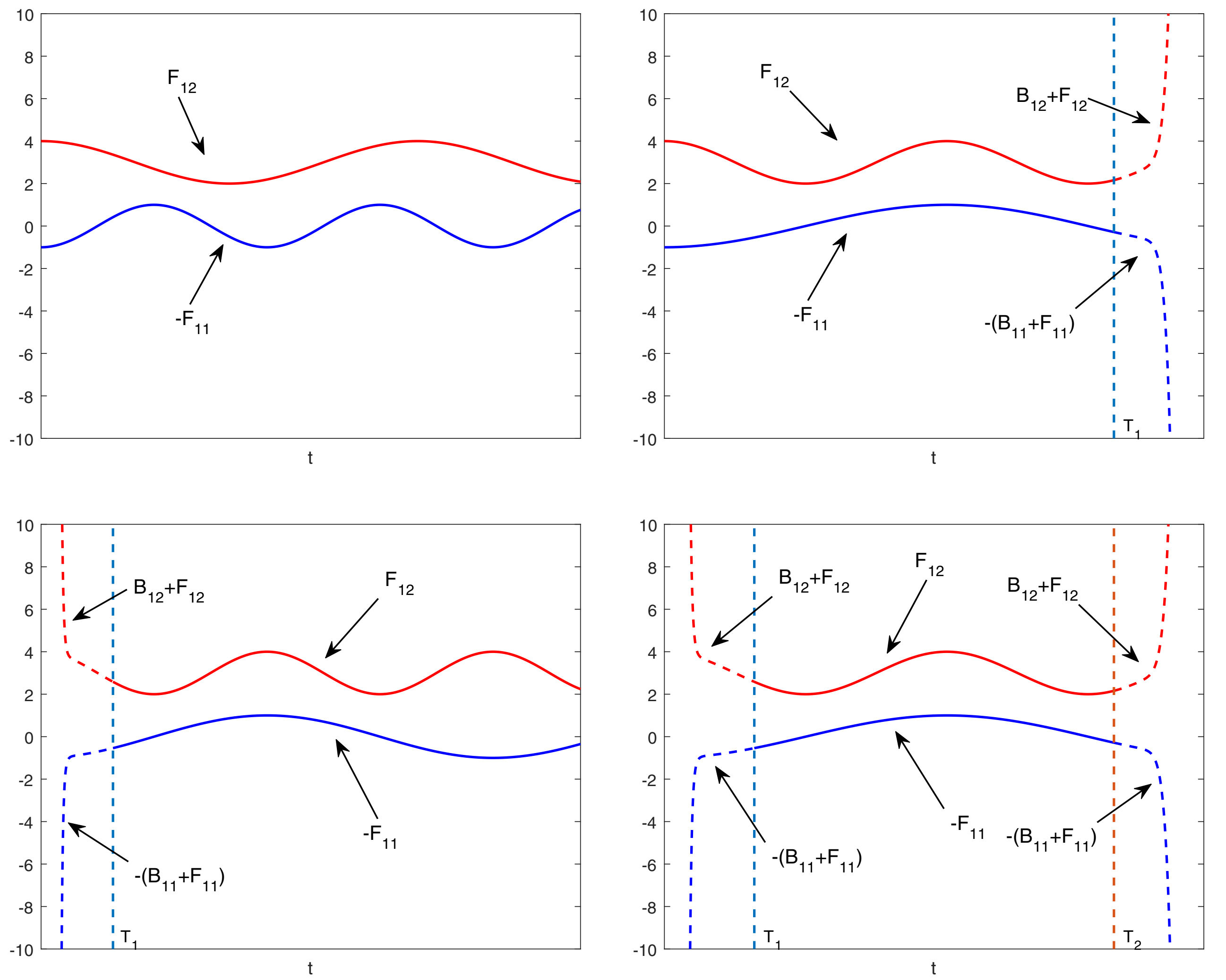

FIGURE 1 Four typical constraining scenarios with different constraining conditions (take one state for example) (Scenario 1: “Entirely Constrained,” Scenario 2: “Constrained-Free,” Scenario 3: “Free-Constrained,” and Scenario 4: “Free-constrained-Free,” as illustrated in solid lines) [Colour figure can be viewed at wileyonlinelibrary.com]

asymmetric.[5-7,9] The work by Tee et al.[5] is among the first that explicitly considered asymmetrical and time-varying output constraints, which, however, requires that the upper boundary be positive and that the ideal trajectory be within positive boundaries, which is rather restrictive in practice. And in the recent work by Jin,[7] the upper and lower constraining boundaries are not allowed to be zero at any time. Although the time-varying and asymmetric full-state constraints are studied in Reference 9, the constraining boundaries need to be strictly positive, excluding the situation where the signs of upper and lower bounds alternate. The work by Zhao[16] is by far the first one that allows the lower and upper boundaries to be zero at some time instants, but the proposed control only works for Scenario 1 where such constraints must be always imposed during system operation. To our best knowledge, there is no feasible solution capable of uniformly dealing with aforementioned irregular constraints.

The objective of this work is to develop a unified control design framework for system (1) under the aforementioned constraining conditions such that

1) asymmetric and time-varying constraints imposed in any of the six scenarios are never violated;
2) the system output y tracks the desired trajectory y_d closely;
3) all closed-loop signals are bounded.

In order to achieve the above goals, the following assumptions are necessary.

Assumption 1. The desired trajectory y_d and its derivatives up to n-order are continuous and bounded, that is, there exists unknown positive constant $Y_i > 0\,(i = 0, 1, \dots, n)$ such that $|y_d^{(i)}| \le Y_i < \infty$.

Assumption 2. The constraining boundaries F_{ij} $(i = 1, \dots, n, j = 1, 2)$ and its derivatives up to n-order are continuous and bounded, that is, there exists unknown positive constant $\varpi_{ij,k} > 0$ such that $F_{ij}^{(k)} \le \varpi_{ij,k} < \infty$ for $k = 0, 1, \dots, n$.

Assumption 3 (17). For nonlinear functions $f_i(\overline{x}_i, x_{i+1})$ $(i = 1, \dots, n-1)$ and $f(\overline{x}_n, u)$, there exist some unknown yet bounded functions $g_i(\overline{x}_i, x_{i+1}^*)$ and $g_n(\overline{x}_n, u^*)$, $h_i(\overline{x}_i, x_{i+1}^*)$ and $h_n(\overline{x}_n, u^*)$ such that

$$\begin{cases} \dot{x}_i = f_i(\overline{x}_i, 0) + g_i(\overline{x}_i, x_{i+1}^*)x_{i+1} + h_i(\overline{x}_i, x_{i+1}^*), \\ i = 1, \dots, n-1 \\ \dot{x}_n = f_n(\overline{x}_n, 0) + g_n(\overline{x}_n, u^*)u + h_n(\overline{x}_n, u^*) \\ y = x_1 \end{cases}, \tag{3}$$

where $x_{i+1}^* \in [0, x_{i+1}]$ and $u^* \in [0, u]$.

Assumption 4. $g_i(\overline{x}_i, x_{i+1}^*)$ and $g_n(\overline{x}_n, u^*)$ are time-varying yet sign-definite, without loss of generality, it is assumed that $sgn(g_i) = +1$ and $sgn(g_n) = +1$. And there exist unknown and positive constants $\overline{g}_i$, $\underline{g}_i$, a_i and known scalar function $\phi_i(\overline{x}_i)$ $(i = 1, \dots, n)$ such that

$$\begin{aligned} \underline{g}_i &\le g_i(\overline{x}_i, x_{i+1}^*) \le \overline{g}_i, \quad i = 1, \dots, n-1, \\ \underline{g}_n &\le g_n(\overline{x}_n, u^*) \le \overline{g}_n, \\ |f_i(\overline{x}_i, 0) + h_i(\overline{x}_i, x_{i+1}^*)| &\le a_i\phi_i(\overline{x}_i), \quad i = 1, \dots, n. \end{aligned} \tag{4}$$

Assumption 5. There exists positive constant β_i such that $F_{i2}(t) + F_{i1}(t) \ge \beta_i > 0$ $(i = 1, \dots, n)$.

Remark 2. Assumption 1 is rather standard in the literature.[5,18,19] Assumption 2 is also reasonable because in practice the constraining boundaries and their variations must be bounded. As for Assumption 3, unlike the commonly utilized assumption on the nonaffine functions $f_i(\overline{x}_i, x_{i+1})$ $(i = 1, \dots, n-1)$ and $f(\overline{x}_n, u)$, as imposed in most existing works[14,18] that the partial derivatives $\partial f_i(\overline{x}_i, x_{i+1})/\partial x_{i+1}$ and $\partial f_n(\overline{x}_n, u)/\partial u$ are required to exist and be strictly positive, here in this assumption these restrictive conditions are relaxed. In Assumption 4 it should be stressed that as $\overline{g}_i, \underline{g}_i$ and a_i $(i = 1, \dots, n)$ are unknown, thus cannot be used for controller design, those parameters are used only for stability analysis in this work. The way that the nonlinear uncertain function $f_i(\overline{x}_i, 0) + h_i(\overline{x}_i, x_{i+1}^*)$ is handled is commonly adopted in robust adaptive control,[3,20,21] with $\phi_i(\overline{x}_i)$ $(i = 1, \dots, n)$ being the core information extracted from the nonlinearities of the system. For practical systems with only crude model information, $\phi_i(\overline{x}_i)$ can be readily obtained by performing upper bounding on

$f_i(\bar{x}_i, 0) + h_i(\bar{x}_i, x_{i+1}^*)$.[21] Assumption 5 implies that the constraining upper boundaries and lower boundaries do not intercross at any time instant, which is necessary because otherwise it would leave no room for the constrained states to evolve.

3 | MAIN RESULTS

3.1 | Extended constraining boundaries

As one of the key steps toward the development of the control scheme capable of dealing with the aforementioned constraining scenarios, we first introduce the following extended boundaries $-\eta_{i1}$ and η_{i2}

$$-\eta_{i1} = -(F_{i1} + B_{i1}), \quad \eta_{i2} = F_{i2} + B_{i2}, \quad i = 1, \dots, n, \tag{5}$$

where $-F_{i1}$ and F_{i2} are respectively the original lower and upper boundaries, and $-B_{i1}$ and B_{i2} are the auxiliary lower and upper boundaries. Note that for each of the six scenarios there are many possible ways to construct B_{i1} and B_{i2}, one of which (for each scenario) is as follows,

1) *Scenario 1:*

$$B_{i1} = B_{i2} = 0. \tag{6}$$

2) *Scenario 2:*

$$B_{i1} = B_{i2} = \begin{cases} 0, & t_0 \le t < T_1 \\ (t - T_1)^{2m}, & t \ge T_1 \end{cases}, \tag{7}$$

where m is positive constant satisfying $2m > n$.

3) *Scenario 3:*

$$B_{i1} = B_{i2} = \begin{cases} \left(\frac{1}{t-t_0+\tau} - \frac{1}{T_1-t_0+\tau}\right)^{2m}, & t_0 \le t < T_1 \\ 0, & t \ge T_1 \end{cases}, \tag{8}$$

where m is positive constant satisfying $2m > n$, τ is a user-design constant small enough to make $-\eta_{i1}(t_0) < x_i(t_0) < \eta_{i2}(t_0)$ $(i = 1, \dots, n)$ hold for any bounded $x_i(t_0)$.

1. *Scenario 4:*

$$B_{i1} = B_{i2} = \begin{cases} \left(\frac{1}{t-t_0+\tau} - \frac{1}{T_1-t_0+\tau}\right)^{2m}, & t_0 \le t < T_1 \\ 0, & T_1 \le t < T_2 \\ (t - T_2)^{2m}, & t \ge T_2 \end{cases}, \tag{9}$$

where m is positive constant satisfying $2m > n$, τ is a user-design constant small enough to make $-\eta_{i1}(t_0) < x_i(t_0) < \eta_{i2}(t_0)$ $(i = 1, \dots, n)$ hold for any bounded $x_i(t_0)$.

2. *Scenario 5:*

$$B_{i1} = B_{i2} = \infty. \tag{10}$$

3. *Scenario 6:*

For $t_0 \le t < T_1$,

$$B_{i1} = B_{i2} = \begin{cases} 0, & i = 1, \dots, r \\ \left(\frac{1}{t-t_0+\tau} - \frac{1}{T_1-t_0+\tau}\right)^{2m}, & i = r+1, \dots, n \end{cases}, \tag{11}$$

where $r < n$ is a positive integer representing the number of states being constrained, τ is a user-design constant small enough to make $-\eta_{i1}(t_0) < x_i(t_0) < \eta_{i2}(t_0)$ $(i = r+1, \dots, n)$ hold for any bounded $x_i(t_0)$. For $t \geq T_1$,

$$B_{i1} = B_{i2} = 0, \quad i = 1, \dots, n. \tag{12}$$

The continuity and differentiability of η_{i1} and η_{i2} are ensured as proven in the Appendix.

3.2 | State transformation

In order to develop a unified control scheme capable of addressing a variety of constraints imposed/removed during system operation, we construct the following transformation by using η_{i1} and η_{i2},

$$\xi_i(x_i, \eta_{i1}, \eta_{i2}) = \kappa_1 \frac{\eta_{i1} x_i + c_i}{\eta_{i1} + x_i} + \kappa_2 \frac{\eta_{i2} x_i + c_i}{\eta_{i2} - x_i}, \tag{13}$$

where $-\eta_{i1}$ and η_{i2} are the extended lower boundary and upper boundary imposed on x_i, respectively, which are related to the actual constraining boundaries ($-F_{i1}$ and F_{i2}) and the auxiliary boundaries ($-B_{i1}$ and B_{i2}) as illustrated conceptually in Figure 1 and specified analytically in (6)–(12), κ_1 and κ_2 are user-chosen positive constants satisfying $\kappa_1 + \kappa_2 = 1$ (the reason for choosing κ_1 and κ_2 this way becomes clear shortly). And c_i is a positive constant satisfying that

$$-\eta_{i2}^2 < c_i < \eta_{i1}^2 \tag{14}$$

Note that the transformation (13), motivated by but slightly different from,[22] is well defined for any $x_i \in (-\eta_{i1}, \eta_{i2})$ $(i = 1, \dots, n)$. Furthermore, (13) exhibits the following interesting properties:

1) $\xi_i \to \infty$ if $x_i \to -\eta_{i1}$ or $x_i \to \eta_{i2}$;
2) $\xi_i = x_i$ if $-\eta_{i1} = -\infty$ and $\eta_{i2} = \infty$.

Remark 3. The important Property 1) means that if ξ_i is bounded, then x_i does not evolve beyond $-\eta_{i1}$ or η_{i2}, in other words, x_i is confined within $(-\eta_{i1}, \eta_{i2})$ if ξ_i is made bounded; Property 2) implies that whenever $-\eta_{i1} = -\infty$ and $\eta_{i2} = \infty$ (i.e., no constraint imposed), it holds that $x_i = \xi_i$ (readily derived from (13) with the help of $\kappa_1 + \kappa_2 = 1$, and one sees the reason for choosing κ_1 and κ_2 this way), which corresponds to the situation that the system state is free from any constraint. The important implication of these properties is that if the control scheme is built upon such transformation, it is naturally capable of dealing with a variety of constraints in a unified manner, as pursued in the sequel.

3.3 | System transformation

To go on, we make use of the state transformation as defined in (13) to get the following ξ_i-dynamics

$$\dot{\xi}_i = \mu_i \dot{x}_i + \omega_i \tag{15}$$

with

$$\begin{cases} \mu_i = \kappa_1 \frac{\eta_{i1}^2 - c_i}{(x_i+\eta_{i1})^2} + \kappa_2 \frac{\eta_{i2}^2 + c_i}{(\eta_{1i}-x_i)^2} \\ \omega_i = \kappa_1 \frac{\dot{\eta}_{i1} x_i^2 - c_i \dot{\eta}_{i1}}{(x_i+\eta_{i1})^2} - \kappa_2 \frac{\dot{\eta}_{i2} x_i^2 - c_i \dot{\eta}_{i2}}{(\eta_{i2}-x_i)^2} \end{cases}. \tag{16}$$

[Correction added on 15 February 2021, after first online publication: the first paragraph under sub-section 3.2, and Equation (13) have been corrected in this version. Also, the '(14)' below Equation (14) were changed to '(13)'.]

Interestingly, it is seen from (15) that for x_i within $(-\eta_{i1}, \eta_{i2})$ $(i = 1, \ldots, n)$, the virtual control gain μ_i as computed by (16) is always positive no matter how the boundaries $-\eta_{i1}(-F_{i1})$ and $\eta_{i2}(F_{i2})$ would evolve, allowing for the development of the corresponding control algorithms capable of gracefully handling those constraining boundaries being sometimes positive or negative, or even zero at some time instants, from which one sees the novelty of the proposed transformation function (13).

With (13), the original system (3) with the constraints as described in the six scenarios is converted into the following transformed and unconstrained system

$$\begin{cases} \dot{\xi}_i = \mu_i(g_i x_{i+1} + f_i + h_i) + \omega_i, \quad i = 1, \ldots, n-1 \\ \dot{\xi}_n = \mu_n(g_n u + f_n + h_n) + \omega_n \end{cases}. \tag{17}$$

Consequently, the original control problem for the constrained system (1) boils down to stabilizing the transformed ξ-system (17).

3.4 | Unified control design and stability analysis

To begin, we introduce the following coordinate transformation

$$z_i = \xi_i - \alpha_{if}, \quad i = 1, \ldots, n, \tag{18}$$

where α_{1f} is generated by

$$\alpha_{1f}(y_d, \eta_{11}, \eta_{12}) = \kappa_1 \frac{\eta_{11} y_d + c_1}{\eta_{11} + y_d} + \kappa_2 \frac{\eta_{12} y_d + c_1}{\eta_{12} - y_d}. \tag{19}$$

And other α_{if} is the output of following first-order filter

$$\varepsilon_i \dot{\alpha}_{if} + \alpha_{if} = \alpha_{i-1}, \quad i = 2, \ldots, n, \tag{20}$$

where $\varepsilon_i > 0$ is the designer-chosen constant. α_{i-1} serves as the input of filter.

Define the filter error as

$$y_i = \alpha_{if} - \alpha_{i-1}, \quad i = 2, \ldots, n. \tag{21}$$

By combining the results shown in (18), (20), and (21), it follows that

$$\xi_i = z_i + y_i + \alpha_{i-1}, \quad i = 2, \ldots, n. \tag{22}$$

And the tracking error is defined as $e = x_1 - y_d$. By means of backstepping design technique (see Reference 23), we have the following n design steps:

Step 1: From the definition of z_1 and (22), taking the time derivative of z_1 yields

$$\begin{aligned} \dot{z}_1 &= \mu_1(g_1 x_2 + f_1 + h_1) + \omega_1 - \dot{\alpha}_{1f} \\ &= \mu_1(g_1(\xi_2 + x_2 - \xi_2) + f_1 + h_1) + \omega_1 - \dot{\alpha}_{1f} \\ &= \mu_1 g_1(z_2 + y_2 + \alpha_1) + \mu_1 g_1 x_2 - \mu_1 g_1 \xi_2 + \mu_1 f_1 \\ &\quad + \mu_1 h_1 + \omega_1 - \dot{\alpha}_{1f}. \end{aligned} \tag{23}$$

Thus, it follows that

$$\begin{aligned} z_1 \dot{z}_1 &= \mu_1 g_1 z_1 \alpha_1 + \mu_1 g_1 z_1 y_2 + \mu_1 g_1 z_1 z_2 + \mu_1 g_1 z_1 x_2 \\ &\quad - \mu_1 g_1 z_1 \xi_2 + \mu_1 z_1 f_1 + \mu_1 z_1 h_1 + z_1 \omega_1 - z_1 \dot{\alpha}_{1f} \\ &= \mu_1 g_1 z_1 \alpha_1 + L_1, \end{aligned} \tag{24}$$

where $L_1 = \mu_1 g_1 z_1 y_2 + \mu_1 g_1 z_1 z_2 + \mu_1 g_1 z_1 x_2 - \mu_1 g_1 z_1 \xi_2 + \mu_1 z_1 f_1 + \mu_1 z_1 h_1 + z_1 \omega_1 - z_1 \dot{\alpha}_{1f}$. Applying Young inequality yields

$$
\begin{aligned}
\mu_1 g_1 z_1 y_2 &\leq \underline{g}_1 \frac{\bar{g}_1^2}{\underline{g}_1} \mu_1^2 z_1^2 + \frac{1}{4} y_2^2, \\
\mu_1 g_1 z_1 z_2 &\leq \underline{g}_2 \mu_1^2 z_1^2 z_2^2 + \frac{\bar{g}_1^2}{4\underline{g}_2}, \\
\mu_1 g_1 z_1 x_2 &\leq \underline{g}_1 \mu_1^2 z_1^2 x_2^2 + \frac{\bar{g}_1^2}{4\underline{g}_1}, \\
-\mu_1 g_1 z_1 \xi_2 &\leq \underline{g}_1 \mu_1^2 z_1^2 \xi_2^2 + \frac{\bar{g}_1^2}{4\underline{g}_1}, \\
\mu_1 z_1 (f_1 + h_1) \leq \mu_1 z_1 a_1 \phi_1 &\leq \underline{g}_1 \mu_1^2 a_1^2 \phi_1^2 z_1^2 + \frac{1}{4\underline{g}_1} \\
z_1 \omega_1 &\leq \underline{g}_1 z_1^2 \omega_1^2 + \frac{1}{4\underline{g}_1}, \\
-z_1 \dot{\alpha}_{1f} &\leq \underline{g}_1 \dot{\alpha}_{1f}^2 z_1^2 + \frac{1}{4\underline{g}_1}.
\end{aligned} \tag{25}
$$

Integrating above results, it follows that

$$
\begin{aligned}
L_1 &\leq \underline{g}_1 \theta_1 (\mu_1^2 + \mu_1^2 x_2^2 + \mu_1^2 \xi_2^2 + \mu_1^2 \phi_1^2 + 1 + \omega_1^2 + \dot{\alpha}_{1f}^2) z_1^2 + \underline{g}_2 \mu_1^2 z_1^2 z_2^2 + \frac{1}{4} y_2^2 + \frac{\bar{g}_1^2}{4\underline{g}_2} + \frac{\bar{g}_1^2}{2\underline{g}_1} + \frac{1}{\underline{g}_1} \\
&\leq \underline{g}_1 \theta_1 \Phi_1 z_1^2 + \underline{g}_2 \mu_1^2 z_1^2 z_2^2 + \frac{1}{4} y_2^2 + \Delta_1
\end{aligned} \tag{26}
$$

with $\theta_1 = \max\{1, \frac{\bar{g}_1^2}{\underline{g}_1}, a_1^2\}$, $\Phi_1 = \mu_1^2 + \mu_1^2 x_2^2 + \mu_1^2 \xi_2^2 + \mu_1^2 \phi_1^2 + 1 + \omega_1^2 + \dot{\alpha}_{1f}^2$, and $\Delta_1 = \frac{\bar{g}_1^2}{4\underline{g}_2} + \frac{\bar{g}_1^2}{2\underline{g}_1} + \frac{1}{\underline{g}_1}$.

Using (25), we can further express (24) as

$$
z_1 \dot{z}_1 \leq \mu_1 g_1 z_1 \alpha_1 + \underline{g}_1 \theta_1 \Phi_1 z_1^2 + \underline{g}_2 \mu_1^2 z_1^2 z_2^2 + \frac{1}{4} y_2^2 + \Delta_1. \tag{27}
$$

Since $\mu_1 > 0$ as shown in (16) and $\underline{g}_1 > 0$, the virtual control law α_1 can be designed as

$$
\alpha_1 = -\frac{1}{\mu_1}(k_1 z_1 + \hat{\theta}_1 \Phi_1 z_1) \tag{28}
$$

with adaptive law

$$
\dot{\hat{\theta}}_1 = -\sigma_1 \hat{\theta}_1 + \lambda_1 \Phi_1 z_1^2, \tag{29}
$$

where $k_1 > 0$, $\sigma_1 > 0$ and $\lambda_1 > 0$ are determined by the designer.

For stability analysis, we choose Lyapunov function candidate $V_1 = \frac{1}{2} z_1^2 + \frac{1}{2} y_2^2 + \frac{\underline{g}_1}{2\lambda_1} \tilde{\theta}_1^2$ with $\tilde{\theta}_1$ being the estimation of θ_1. And $\tilde{\theta}_1 = \theta_1 - \hat{\theta}_1$ is satisfied. Substituting (28) and (29) into (27) yields

$$
\begin{aligned}
\dot{V}_1 &= z_1 \dot{z}_1 + y_2 \dot{y}_2 + \frac{\underline{g}_1}{\lambda_1} \tilde{\theta}_1 \dot{\tilde{\theta}}_1 \\
&\leq -k_1 \underline{g}_1 z_1^2 + \frac{1}{4} y_2^2 + \underline{g}_2 \mu_1^2 z_1^2 z_2^2 + y_2 \dot{y}_2 + \frac{\underline{g}_1}{\lambda_1} \sigma_1 \hat{\theta}_1 \tilde{\theta}_1 + \Delta_1 \\
&\leq -k_1 \underline{g}_1 z_1^2 - \frac{\underline{g}_1}{2\lambda_1} \sigma_1 \tilde{\theta}_1^2 + \underline{g}_2 \mu_1^2 z_1^2 z_2^2 + \frac{1}{4} y_2^2 + y_2 \dot{y}_2 + \frac{\underline{g}_1}{2\lambda_1} \sigma_1 \theta_1^2 + \Delta_1.
\end{aligned} \tag{30}
$$

Combining the definition of y_2 (21) and the first-order filter (20), it follows that

$$y_2\dot{y}_2 = y_2\left(-\frac{y_2}{\varepsilon_2} - \dot{\alpha}_1\right) = -\frac{y_2^2}{\varepsilon_2} + y_2\ell_2 \le -\frac{y_2^2}{\varepsilon_2} + \frac{y_2^2}{4} + \ell_2^2, \tag{31}$$

where $\ell_2(z_1, z_2, z_3, \overline{\eta}_{11}^{(2)}, \overline{\eta}_{12}^{(2)}, \overline{y}_d^{(2)}) = -\dot{\alpha}_1 = -\frac{\partial\alpha_1}{\partial\mu_1}\dot{\mu}_1 - \frac{\partial\alpha_1}{\partial z_1}\dot{z}_1 - \frac{\partial\alpha_1}{\partial\hat{\theta}_1}\dot{\hat{\theta}}_1 - \frac{\partial\alpha_1}{\partial\Phi_1}\dot{\Phi}_1$.

From (31), $\dot{V}_1$ is rewritten as

$$\begin{aligned}\dot{V}_1 &\le -k_1\underline{g}_1 z_1^2 - \frac{\underline{g}_1}{2\lambda_1}\sigma_1\tilde{\theta}_1^2 + \underline{g}_2\mu_1^2 z_1^2 z_2^2 + \frac{1}{4}y_2^2 + \frac{\underline{g}_1}{2\lambda_1}\sigma_1\theta_1^2 - \frac{y_2^2}{\varepsilon_2} + \frac{y_2^2}{4} + \ell_2^2 + \Delta_1 \\ &\le -k_1\underline{g}_1 z_1^2 - \frac{\underline{g}_1}{2\lambda_1}\sigma_1\tilde{\theta}_1^2 - \varepsilon_2^* y_2^2 + \underline{g}_2\mu_1^2 z_1^2 z_2^2 + \frac{\underline{g}_1}{2\lambda_1}\sigma_1\theta_1^2 + \ell_2^2 + \Delta_1 \\ &\le -k_1\underline{g}_1 z_1^2 - \frac{\underline{g}_1}{2\lambda_1}\sigma_1\tilde{\theta}_1^2 - \varepsilon_2^* y_2^2 + \underline{g}_2\mu_1^2 z_1^2 z_2^2 + \ell_2^2 + \overline{\Delta}_1,\end{aligned} \tag{32}$$

where $-\frac{1}{\varepsilon_2} + \frac{1}{2} \le -\varepsilon_2^*$ and $\overline{\Delta}_1 = \frac{\underline{g}_1}{2\lambda_1}\sigma_1\theta_1^2 + \Delta_1$.

Step 2: Following the line similar to (23)–(27), it is obtained that

$$z_2\dot{z}_2 \le \mu_2 g_2 z_2\alpha_2 + \underline{g}_2\theta_2\Phi_2 z_2^2 + \underline{g}_3\mu_2^2 z_2^2 z_3^2 + \frac{1}{4}y_3^2 + \Delta_2 \tag{33}$$

with $\theta_2 = \max\{1, \frac{\overline{g}_2^2}{\underline{g}_2}, a_2^2\}$, $\Phi_2 = \mu_2^2 + \mu_2^2 x_3^2 + \mu_2^2\xi_3^2 + \mu_2^2\phi_2^2 + 1 + \omega_2^2 + \dot{\alpha}_{2f}^2$ and $\Delta_2 = \frac{\overline{g}_2^2}{4\underline{g}_3} + \frac{\overline{g}_2^2}{2\underline{g}_2} + \frac{1}{\underline{g}_2}$.

From (33), we design the virtual control α_2 as

$$\alpha_2 = -\frac{1}{\mu_2}(k_2 z_2 + \hat{\theta}_2\Phi_2 z_2 + \mu_1^2 z_1^2 z_2) \tag{34}$$

with adaptive law

$$\dot{\hat{\theta}}_2 = -\sigma_2\hat{\theta}_2 + \lambda_2\Phi_2 z_2^2, \tag{35}$$

where k_2, σ_2, and λ_2 are any positive constants to be chosen by the designer.

Then choose the Lyapunov function candidate as $V_2 = \frac{1}{2}z_2^2 + \frac{1}{2}y_3^2 + \frac{\underline{g}_2}{2\lambda_2}\tilde{\theta}_2^2$, where $\tilde{\theta}_2$ being the estimation of θ_2, satisfying $\tilde{\theta}_2 = \theta_2 - \hat{\theta}_2$. Upon taking time derivative of V_2 and using (34) and (35), it yields that

$$\dot{V}_2 \le -k_2\underline{g}_2 z_2^2 - \frac{\underline{g}_2}{2\lambda_2}\sigma_2\tilde{\theta}_2^2 + \underline{g}_3\mu_2^2 z_2^2 z_3^2 + \frac{1}{4}y_3^2 + y_3\dot{y}_3 + \frac{\underline{g}_2}{2\lambda_2}\sigma_2\theta_2^2 + \Delta_2. \tag{36}$$

According to (31), it follows that

$$y_3\dot{y}_3 \le -\frac{y_3^2}{\varepsilon_2} + \frac{y_3^2}{4} + \ell_3^2, \tag{37}$$

where $\ell_3(z_2, z_3, z_4, \overline{F}_{21}^{(2)}, \overline{F}_{22}^{(2)}, \overline{y}_d^{(2)}) = -\dot{\alpha}_2 = -\frac{\partial\alpha_2}{\partial\mu_2}\dot{\mu}_2 - \frac{\partial\alpha_2}{\partial z_2}\dot{z}_2 - \frac{\partial\alpha_2}{\partial\hat{\theta}_2}\dot{\hat{\theta}}_2 - \frac{\partial\alpha_2}{\partial\Phi_2}\dot{\Phi}_2$. Thus, $\dot{V}_2$ is reexpressed as

$$\dot{V}_2 \le -k_2\underline{g}_2 z_2^2 - \frac{\underline{g}_2}{2\lambda_2}\sigma_2\tilde{\theta}_2^2 - \varepsilon_3^* y_3^2 + \underline{g}_3\mu_2^2 z_2^2 z_3^2 + \ell_3^2 + \overline{\Delta}_2 \tag{38}$$

with $-\frac{1}{\varepsilon_3} + \frac{1}{2} \le -\varepsilon_3^*$ and $\overline{\Delta}_2 = \frac{\underline{g}_2}{2\lambda_2}\sigma_2\theta_2^2 + \Delta_2$.

Step i ($i=3, \dots, n-1$): Following the design procedure similar to that in (23)–(27), it is derived that

$$z_i \dot{z}_i \leq \mu_i g_i z_i \alpha_i + \underline{g}_i \theta_i \Phi_i z_i^2 + \underline{g}_{i+1} \mu_i^2 z_i^2 z_{i+1}^2 + \frac{1}{4} y_{i+1}^2 + \Delta_i \tag{39}$$

with $\theta_i = \max\left\{1, \frac{\overline{g}_i^2}{\underline{g}_i}, a_i^2\right\}$, $\Phi_i = \mu_i^2 + \mu_i^2 x_{i+1}^2 + \mu_i^2 \xi_{i+1}^2 + \mu_i^2 \phi_i^2 + 1 + \omega_i^2 + \dot{\alpha}_{if}^2$, and $\Delta_i = \frac{\overline{g}_i^2}{4\underline{g}_{i+1}} + \frac{\overline{g}_i^2}{2\underline{g}_i} + \frac{1}{\underline{g}_i}$.

From (39), we design the virtual control α_i as

$$\alpha_i = -\frac{1}{\mu_i}(k_i z_i + \hat{\theta}_i \Phi_i z_i + \mu_{i-1}^2 z_{i-1}^2 z_i) \tag{40}$$

with adaptive law

$$\dot{\hat{\theta}}_i = -\sigma_i \hat{\theta}_i + \lambda_i \Phi_i z_i^2, \tag{41}$$

where $k_i > 0, \sigma_i > 0$ and $\lambda_i > 0$ are any user-determined constants.

Choose the Lyapunov function candidate $V_i = \frac{1}{2} z_i^2 + \frac{1}{2} y_{i+1}^2 + \frac{\underline{g}_i}{2\lambda_i} \tilde{\theta}_i^2$. And $\tilde{\theta}_i = \theta_i - \hat{\theta}_i$ is established. Based on the process similar to (30) and (31), it follows that

$$\dot{V}_i \leq -k_i \underline{g}_i z_i^2 - \frac{\underline{g}_i}{2\lambda_i} \sigma_i \tilde{\theta}_i^2 - \varepsilon_{i+1}^* y_{i+1}^2 + \underline{g}_{i+1} \mu_i^2 z_i^2 z_{i+1}^2 + \ell_{i+1}^2 + \overline{\Delta}_i. \tag{42}$$

Step n: From (17) and (18) for $i=n$, it follows that

$$\begin{aligned} z_n \dot{z}_n &\leq \mu_n g_n z_n u + \mu_n z_n f_n + \mu_n z_n h_n + z_n \omega_n - z_n \dot{\alpha}_{nf} \\ &\leq \mu_n g_n z_n u + L_n, \end{aligned} \tag{43}$$

where $L_n = \mu_n z_n f_n + \mu_n z_n h_n + z_n \omega_n - z_n \dot{\alpha}_{nf}$.

By means of Young inequality, it holds that

$$\begin{aligned} \mu_n z_n (f_n + h_n) &\leq \underline{g}_n \mu_n^2 a_n^2 \phi_n^2 z_n^2 + \frac{1}{4\underline{g}_n} \\ z_n \omega_n &\leq \underline{g}_n \omega_n^2 z_n^2 + \frac{1}{4\underline{g}_n} \\ -z_n \dot{\alpha}_{nf} &\leq \underline{g}_n \dot{\alpha}_{nf}^2 z_n^2 + \frac{1}{4\underline{g}_n}. \end{aligned} \tag{44}$$

Thus L_n is upper bounded by

$$\begin{aligned} L_n &\leq \underline{g}_n \theta_n (\mu_n^2 \phi_n^2 + \mu_n^2 + \omega_n^2 + \dot{\alpha}_{nf}^2) z_n^2 + \frac{3}{4\underline{g}_n} \\ &\leq \underline{g}_n \theta_n \Phi_n z_n^2 + \Delta_n \end{aligned} \tag{45}$$

Then (43) can be reexpressed

$$z_n \dot{z}_n \leq \mu_n g_n z_n u + \underline{g}_n \theta_n \Phi_n z_n^2 + \Delta_n \tag{46}$$

with $\theta_n = \max\{1, a_n^2\}$, $\Phi_n = \mu_n^2 \phi_n^2 + \omega_n^2 + \dot{\alpha}_{nf}^2$ and $\Delta_n = \frac{3}{4\underline{g}_n}$.

The actual control law is designed as

$$u = -\frac{1}{\mu_n}(k_n z_n + \hat{\theta}_n \Phi_n z_n + \mu_{n-1}^2 z_{n-1}^2 z_n) \tag{47}$$

with adaptive law

$$\dot{\hat{\theta}}_n = -\sigma_n \hat{\theta}_n + \lambda_n \Phi_n z_n^2, \tag{48}$$

where $k_n > 0$, $\sigma_n > 0$ and $\lambda_n > 0$ are any constants chosen by the designer.

The Lyapunov function candidate is selected as $V_n = \frac{1}{2} z_n^2 + \frac{g_n}{2\lambda_n}\tilde{\theta}_n^2$. Similarly, $\tilde{\theta}_n = \theta_n - \hat{\theta}_n$.

According to (46) and the definition of V_n, we have

$$\dot{V}_n \leq -k_n \underline{g}_n z_n^2 - \frac{\underline{g}_n}{2\lambda_n}\sigma_n \tilde{\theta}_n^2 - \underline{g}_n z_{n-1}^2 z_n^2 + \overline{\Delta}_n, \tag{49}$$

where $\overline{\Delta}_n = \Delta_n + \frac{g_n}{2\lambda_n}\sigma_n \theta_n^2$.

Define a Lyapunov function as $V = \sum_{i=1}^{i=n} V_i$. Based on (30), (36), (42), and (49), the time derivative of V is

$$\dot{V} \leq -\sum_{i=1}^{n-1} k_i \underline{g}_i z_i^2 - \sum_{i=1}^{n} \frac{\underline{g}_i}{2\lambda_i}\sigma_i \tilde{\theta}_i^2 - \sum_{j=2}^{n} \varepsilon_j^* y_i^2 + \sum_{i=2}^{n} \ell_i^2 + \sum_{i=1}^{n} \overline{\Delta}_i. \tag{50}$$

Define the compact sets $\Omega_d := \{[y_d, \dot{y}_d, \ddot{y}_d]^T : y_d^2 + \dot{y}_d^2 + \ddot{y}_d^2 \leq B_d \subset \mathbf{R}^3\}$ and $\Omega_\eta := \{[\eta_{ij}, \dot{\eta}_{ij}, \ddot{\eta}_{ij}]^T : \eta_{ij}^2 + \dot{\eta}_{ij}^2 + \ddot{\eta}_{ij}^2 \leq B_\eta \subset \mathbf{R}^{6n}\}$ as well as $\Omega_V := \{\sum_{i=1}^{n} z_i^2 + \sum_{i=1}^{n} \tilde{\theta}_i^2 + \sum_{i=2}^{n} y_i^2 \leq 2B_V \subset \mathbf{R}^{3n-1}\}$, where B_d, B_η, and B_V are positive constants, then there exists positive constant l_i $(i = 2, \ldots, n)$ such that $|\ell_i| \leq l_i$ on $\Omega_d \times \Omega_\eta \times \Omega_V$. Thus, (50) is rewritten as

$$\dot{V} \leq -\lambda V + \Lambda, \tag{51}$$

where $\lambda = \min_{i=1,\ldots,n}\{2k_i \underline{g}_i, \sigma_i, 2\varepsilon_j^*\}$ and $\Lambda = \sum_{i=1}^{n} \overline{\Delta}_i + \sum_{i=2}^{n} l_i^2$.

Now we are at the position to state the following theorem.

Theorem 1. *Consider the uncertain nonlinear system (1), under Assumption 1–Assumption 5 and (14), then for any* $-\eta_{i1}(t_0) < x_i(t_0) < \eta_{i2}(t_0)$ $(i = 1, \ldots, n)$, *if control algorithms (47)–(48) are applied to each of the six scenarios, then the following results are obtained:*

1) *the imposed constraints are never violated;*
2) *the system output y follows the desired trajectory y_d closely in that the output tracking error is ensured to uniformly ultimately bounded and can be adjusted as needed by properly choosing some design parameters;*
3) *all closed-loop signals are guaranteed bounded.*

Proof.

i) First we prove that the constraints imposed are maintained. Clearly from (51) it is true that $V(t) \in L_\infty$ for any $t \geq t_0$. Therefore, from the definition of V, it can be inferred that $z_i \in L_\infty$, $\tilde{\theta}_i \in L_\infty$ $(i = 1, \ldots, n)$ and $y_i \in L_\infty$ $(i = 2, \ldots, n)$ for any $t \geq t_0$. Since $z_1 \in L_\infty$, $\alpha_{1f} \in L_\infty$ and $z_1 = \xi_1 - \alpha_{1f}$, it certainly means that $\xi_1 \in L_\infty$. In view of the relation between ξ_1 and x_1, it is readily established that x_1 is confined within the boundaries $(-F_{11}, F_{12})$. This can easily be proven via contradiction. Assume that there exists one such time instant $t_1 > t_0$ at which $x_1(t_1) \notin (-F_{11}, F_{12})$ holds. Since $-F_{11}(t_0) < x_1(t_0) < F_{12}(t_0)$, $x_1(t_0)$ is initially within $(-F_{11}, F_{12})$, therefore, for $x_1(t)$ $(t_0 < t < t_1)$ to move from $x_1(t_0)$ to $x_1(t_1)$, $x_1(t_1) \notin (-F_{11}, F_{12})$, it must first pass through the point $-F_{11}$ or F_{12}, which then means that $x_1 \to -F_{11}$ or $x_1 \to F_{12}$, implying that $\xi_1 \to \infty$ by (13), which contradicts with the fact that ξ_1 is bounded for any $t \geq t_0$. Therefore, it is indeed true that $-F_{11} < x_1 < F_{12}$ for any $t \geq t_0$. From the expression of $\dot{\hat{\theta}}_1$ and $\hat{\theta}_1 \in L_\infty$, it follows that $\Phi_1 \in L_\infty$. Then based on the boundedness of μ_1, z_1, $\hat{\theta}_1$ and Φ_1, it means that $\alpha_1 \in L_\infty$. Then from $y_2 = \alpha_{2f} - \alpha_1$ and $\alpha_1 \in L_\infty$, it yields that $\alpha_{2f} \in L_\infty$. According to $z_2 = \xi_2 - \alpha_{2f}$ and $z_2 \in L_\infty$, it follows that $\xi_2 \in L_\infty$. From *Remark* 3, it gets that x_2 is bounded in $(-F_{21}, F_{22})$. Similarly, x_i $(i = 3, \ldots, n)$ is confined in specific constraints $(-F_{i1}, F_{i2})$.

[Correction added on 15 February 2021, after first online publication: in the fourth line of *Proof* section, '$-F_{11}, F_{12}$' was changed to '$(-F_{11}, F_{12})$' in this version.]

ii) Second, we show that UUB output tracking is ensured. From the definition of z_1, it is not difficult to derive from (18) and (19) that $e = \epsilon z_1$ with $\epsilon = \kappa_1 \frac{\eta_{11}^2 - c_1}{(\eta_{11}+x_1)(\eta_{11}+y_d)} + \kappa_2 \frac{\eta_{12}^2 + c_1}{(\eta_{12}-x_1)(\eta_{12}-y_d)}$. Note that x_1 and y_d are bounded and $x_1 \in (-\eta_{11}, \eta_{12})$, it is readily established that there exist positive constants $\underline{\sigma}$ and $\overline{\sigma}$ such that $0 < \underline{\sigma} \le \epsilon \le \overline{\sigma}$. In addition, from (51), it follows that $\dot{V} \le -\underline{g}_1 k_1 z_1^2 + \Lambda$, implying that $\dot{V}$ will be negative if $|z_1| > \sqrt{\frac{\Lambda + \varrho_1}{\underline{g}_1 k_1}}$ with ϱ_1 being a small constant, then z_1 will enter into and remain within the compact set $\Omega_{z_1} = \left\{ z_1 \in \mathbf{R} \middle| |z_1| \le \sqrt{\frac{\Lambda + \varrho_1}{\underline{g}_1 k_1}} \right\}$, thus it means that e will enter into and remain within the compact set $\Omega_e = \left\{ e \in \mathbf{R} \middle| |e| \le \frac{1}{\underline{\sigma}} \sqrt{\frac{\Lambda + \varrho_1}{\underline{g}_1 k_1}} \right\}$. Thus the tracking error can be made arbitrarily small by choosing the large design parameter k_1.

iii) Finally, we prove that all the internal signals are bounded. From the expression in (28) and the boundedness of z_1, μ_1 and Φ_1, it can be inferred that α_1 is bounded. Then according to the second equation in (18), $x_2 \in L_\infty$ follows. By applying the analysis similar to the above recursively, the boundedness of α_i, x_i $(i = 2, \ldots, n-1)$, u and other internal signals can be established. Therefore, it is proven that all the closed-loop signals are bounded.

■

It is interesting to note that because of the novel transformation (13), the control algorithms (47)–(48) are applicable to the six scenarios without the need for changing its structure.

3.5 | Merits and discussion

To close this section, we briefly comment on the merits of the proposed method with comparison to the related works.

Firstly, we recap the design procedure of the proposed unified control scheme as in Figure 2. It is interesting to note that without the need for changing control structure, but by simply resetting the extended boundaries, the control objectives for each scenario are achieved. In fact, in order to apply the control scheme to each case, one only needs to do each of the following:

1) For Scenario 1: The extended boundaries η_{i1} and η_{i2} are set as (5) with (6);
2) For Scenario 2: The extended boundaries η_{i1} and η_{i2} are set as (5) with (7);
3) For Scenario 3: The extended boundaries η_{i1} and η_{i2} are set as (5) with (8);
4) For Scenario 4: The extended boundaries η_{i1} and η_{i2} are set as (5) with (9);
5) For Scenario 5: The extended boundaries η_{i1} and η_{i2} are set as (5) with (10). Correspondingly, it follows that $\mu_i = 1$ and $\omega_i = 0$;
6) For Scenario 6: The extended boundaries η_{i1} and η_{i2} are set as (5) with (11) and (12).

Secondly, it is worth mentioning that asymmetric and time-varying constraints are studied in literatures (see, for instance, References 5,7,8,24, etc.), however, due to the use of the piecewise log-type barrier Lyapunov function (BLF), extra effort is required to ensure the continuity and differentiability of the virtual controllers, which complicates the control structure and stability analysis. In the recent work by Jin,[7] a new BLF is utilized to circumvent such issue, but it works only for certain type of constraints that do not alternate positively and negatively. Furthermore, they all are only applicable to Scenario 1.

Finally, although the method by Zhao et al.[16] considered the constraints that could alternate positively and negatively, it works only for Scenario 1, whereas the proposed method works uniformly for all those cases.

4 | SIMULATION

To validate the effectiveness of the proposed control method, we conduct two simulations on the following single link robotic manipulator system and nonaffine system respectively.

4.1 | Simulation on single link robotic manipulator system

The single link robotic manipulator system is descried as follows

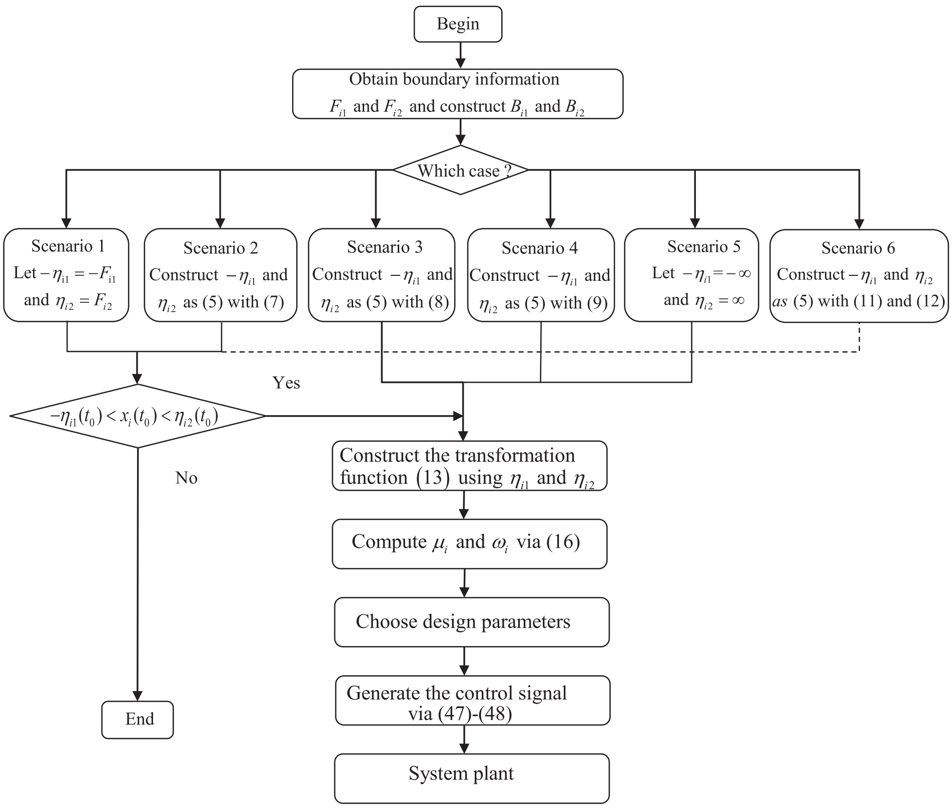

FIGURE 2 Unified control design framework for six scenarios (for Scenarios 1–5, $i=1, \dots, n$, and Scenario 6, $i=1, \dots, r$)

$$J\ddot{q}(t) + D\dot{q}(t) + MgL\sin(q(t)) = u, \tag{52}$$

where J is the rotation inertia of the servo motor, D is the damping coefficient, L represents the length from the axis of joint to the mass center, M denotes the mass of the link and g represents the gravitational acceleration. Here q and $\dot{q}$ denote the angle and angular velocity of the joint, respectively. For all the six cases, the design parameters are set as $k_1=10, k_2=100$ and the initial time is $t_0=0$. For simulation purpose, we test the state constraining boundaries generated dynamically by

$$\begin{aligned}
\ddot{F}_{11} &= -0.25F_{11} - 0.025,\\
\ddot{F}_{12} &= -F_{12} - 0.1,\\
\ddot{F}_{21} &= -F_{21} + 0.1,\\
\ddot{F}_{22} &= -F_{22} + 0.1
\end{aligned} \tag{53}$$

with $F_{11}(0)=0.1$ rad, $\dot{F}_{11}(0)=0$ rad/s, $F_{12}(0)=1.1$ rad, $\dot{F}_{12}(0)=0$ rad/s, $F_{21}(0)=0.2$ rad, $\dot{F}_{21}(0)=0$ rad/s, $F_{22}(0)=0.8$ rad, $\dot{F}_{22}(0)=0.1$ rad/s, resulting in $F_{11}=0.1\cos(0.5t)$ rad, $F_{12}=1+0.1\cos(t)$ rad, $F_{21}=0.2+0.1\sin(t)$ rad/s, and $F_{22}=0.8+0.1\sin(t)$ rad/s. And the initial states are $q(0)=0.15$ rad and $\dot{q}(0)=0.1$ rad/s, which is within the lower and upper boundaries. In simulation, we choose $\kappa_1=0.5$, $\kappa_2=0.5$, $\tau=0.001$ quite arbitrarily. The constants c_1 and c_2 contained in (13) are chosen as $c_1=-0.001$ and $c_2=0$. The desired trajectory $y_d=0.1\sin(t)+0.2$ rad.

The simulation results on the time evolution history of q are presented in Figures 3 and 4 for each scenario, and $T_1=2$ s, $T_2=10$ s, confirming that good tracking precision is achieved, whereas the imposed output constraints are never violated. Furthermore, when q is free from any constraint, the controller designed under constraints still works well, making q closely track the desired signal y_d without the need for changing its structure. Note that in Figure 4, there are two figures in Scenario 6, showing that partial-state constraints and full-state constraints are uniformly addressed by the same control. Due to page limit, time histories of $\dot{q}$ in other scenarios are omitted.

It is worth noting that the constraining lower boundary F_{11} is sometimes positive and sometimes negative, or even zero periodically, rendering existing works inapplicable,[5,7-9] and since the constraints are not imposed all the time during the system operation, the method by Zhao et al.[16] is not applicable either.

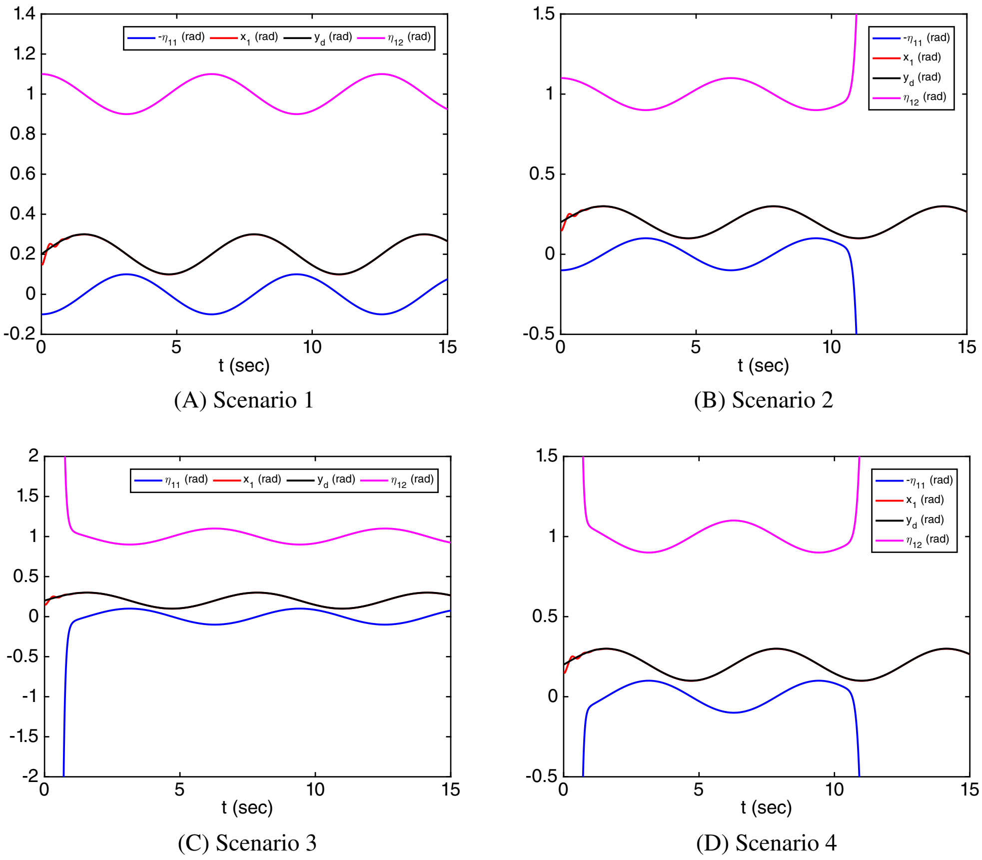

FIGURE 3 The evolution of q tracking q_d for single link robotic manipulator system under Scenarios 1–4 [Colour figure can be viewed at wileyonlinelibrary.com]

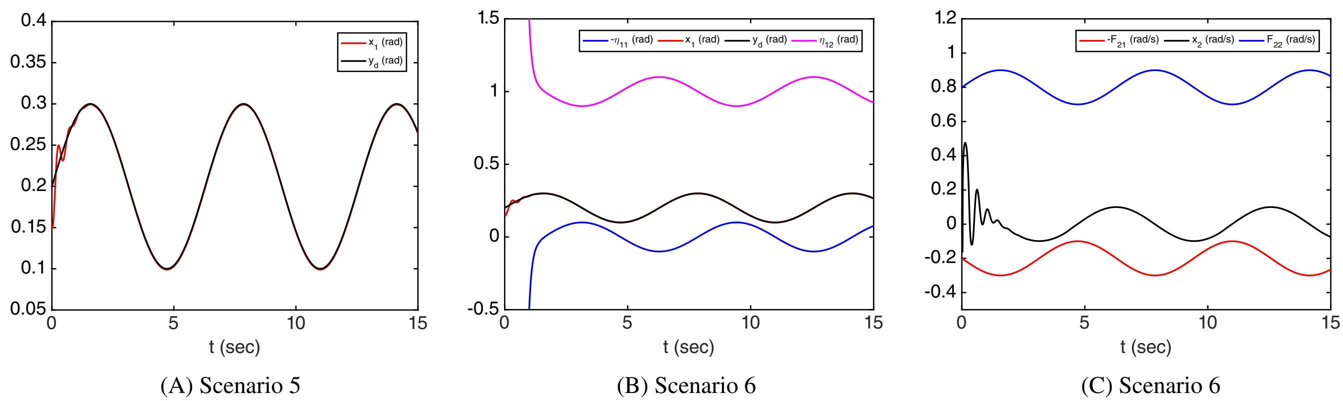

FIGURE 4 The evolution of q tracking q_d under Scenarios 5–6 and the evolution of $\dot{q}$ for single link robotic manipulator system under Scenario 6 [Colour figure can be viewed at wileyonlinelibrary.com]

4.2 | Simulation on nonaffine system

We consider the following nonaffine system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u + 0.1\sin(u)/(1+(x_1+x_2)^2) + f_d \end{cases}, \tag{54}$$

where $f_d = a_0 + a_1 x_1^2 + a_2 x_2^2$ with $a_0 = 0.1$ and $a_1 = a_2 = \sin(t/2)$. The initial values are set as $x_1(0) = 0.1$ and $x_2(0) = 0.3$. The constraining boundaries are $F_{11} = 0.1\cos(t)$, $F_{12} = 1 + 0.1\cos(0.1t)$, $F_{21} = 0.2 + 0.1\sin(t)$, and

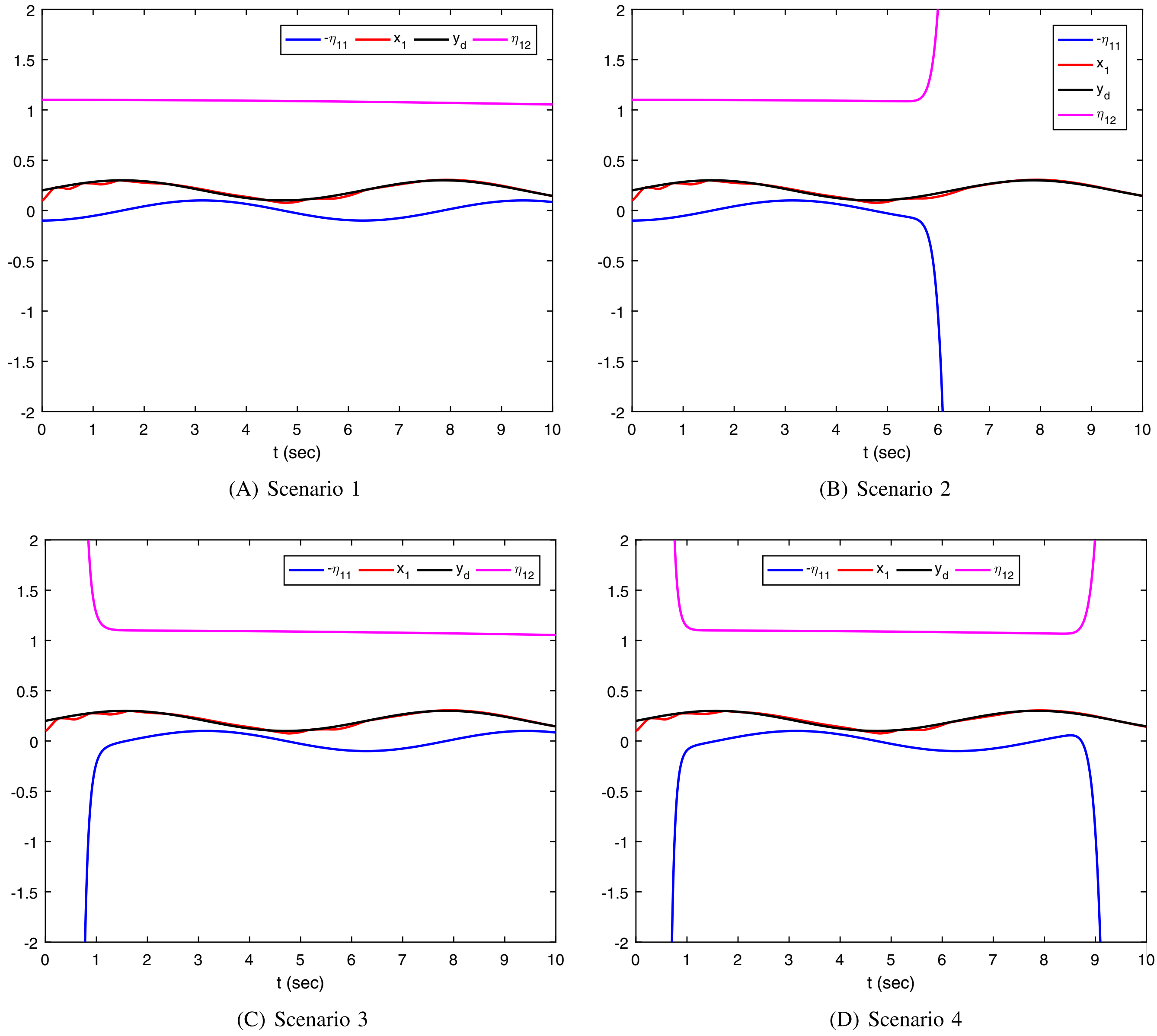

FIGURE 5 The evolution of q tracking q_d for nonaffine system under Scenarios 1–4 [Colour figure can be viewed at wileyonlinelibrary.com]

$F_{22} = 0.8 + 0.1\sin(0.5t)$. The constants c_1 and c_2 contained in (13) are chosen as $c_1 = -0.001$ and $c_2 = 0$. The simulation results are presented in Figures 5 and 6, from which it is observed that although there exist six types of irregular constraints, none is violated with the proposed control method during system operation.

5 | CONCLUSIONS

This article studied the tracking control problem for uncertain pure-feedback systems under irregular constraints. A unified control design framework capable of dealing with a variety of constraints without the need for altering the control structure is developed, which, compared with most existing methods, is able to handle a larger type of constraints formed dynamically and imposed irregularly. Indeed, to ensure the existence of the positive constants ℓ_i, the parameters in the DSC filters should be properly selected. Currently, certain trial and error process has to be used in choosing these parameters. How to systematically determine these parameters represents an interesting topic for future study.

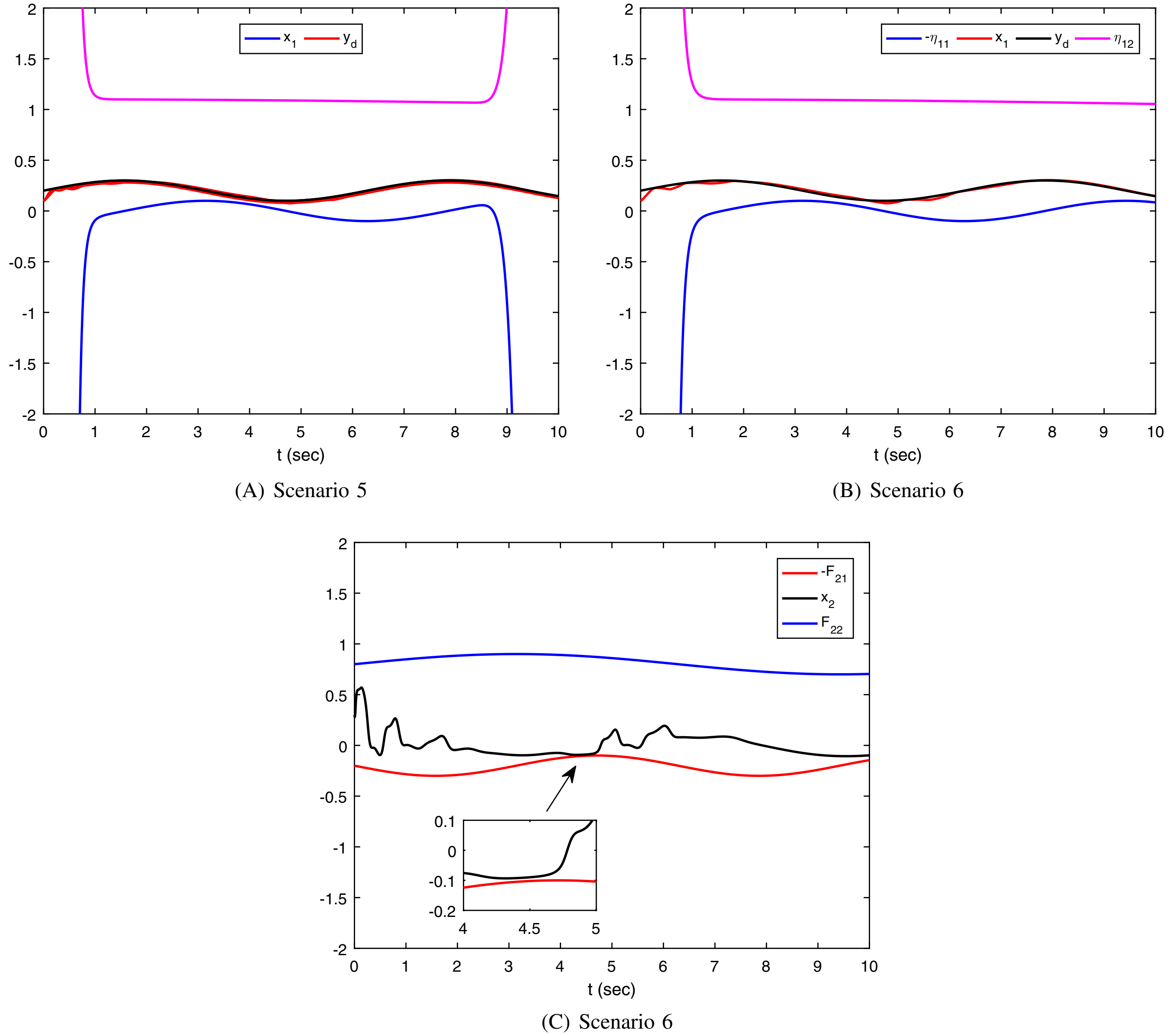

FIGURE 6 The evolution of q tracking q_d for nonaffine system under Scenarios 5–6 and the evolution of $\dot{q}$ under Scenario 6 [Colour figure can be viewed at wileyonlinelibrary.com]

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CONFLICT OF INTEREST

The authors confirm that there is no conflict of interest involved here

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APPENDIX A. THE CONTINUITY AND DIFFERENTIABILITY OF η_{i1} AND η_{i2}

Upon examining (9), it becomes obvious that η_{ij} $(i = 1, \dots, n, j = 1, 2)$ is continuous and differentiable for $t \in [t_0, T_1), (T_1, T_2), (T_2, \infty)$. We show that this is also true for $t = T_1$ and $t = T_2$. The continuity of η_{i1} and η_{i2} can be easily verified.

Next we prove that $\dot{\eta}_{i1}$ exists at $t = T_1$ and $t = T_2$. From (9), using the definition on the time derivative of η_{i1} over the time interval $[T_1, t]$ and $t \in [T_1, \infty)$, yields

$$
\begin{aligned}
\dot{\eta}_{i1}(T_1^-) &= \lim_{t\to T_1^-} \frac{\left(\frac{1}{t-t_0+\tau} - \frac{1}{T_1-t_0+\tau}\right)^{2m} + F_{i1}(t) - F_{i1}(T_1)}{t-T_1} \\
&= \lim_{t\to T_1^-} - \frac{(T_1-t)^{2m-1}}{((t-t_0+\tau)(T_1-t_0+\tau))^{2m}} + \dot{F}_{i1}(T_1) \\
&= \dot{F}_{i1}(T_1) = \dot{\eta}_{i1}(T_1), \\
\dot{\eta}_{i1}(T_1^+) &= \lim_{t\to T_1^+} \frac{F_{i1}(t) - F_{i1}(T_1)}{t-T_1} = \dot{F}_{i1}(T_1) = \dot{\eta}_{i1}(T_1), \\
\dot{\eta}_{i1}(T_2^-) &= \lim_{t\to T_2^-} \frac{F_{i1}(t) - F_{i1}(T_2)}{t-T_2} = \dot{F}_{i1}(T_2) = \dot{\eta}_{i1}(T_2), \\
\dot{\eta}_{i1}(T_2^+) &= \lim_{t\to T_2^+} \frac{(t-T_2)^{2m} + F_{i1}(t) - F_{i1}(T_2)}{t-T_2} \\
&= \lim_{t\to T_2^+} (t-T_2)^{2m-1} + \dot{F}_{i1}(T_2) \\
&= \dot{F}_{i1}(T_2) = \dot{\eta}_{i1}(T_2).
\end{aligned} \tag{A1}
$$

Thus from *Assumption* 2, we have $\lim_{t\to T_1^-}\dot{\eta}_{i1}(t) = \lim_{t\to T_1^+}\dot{\eta}_{i1}(t) = \dot{\eta}_{i1}(T_1)$ and $\lim_{t\to T_2^-}\dot{\eta}_{i1}(t) = \lim_{t\to T_2^+}\dot{\eta}_{i1}(t) = \dot{\eta}_{i1}(T_2)$, confirming that the derivatives of η_{i1} at the time instants T_1 and T_2 are continuous.

Similarly, we can show that η_{i2} and $\dot{\eta}_{i2}$ $(i = 1, \dots, n)$ are continuous at time instants T_1 and T_2.

Also, it can be derived that

$$
\eta_{i1}^{(k)}(t) = \begin{cases} \frac{(-1)^k (T_1-t)^{2m-k}}{((t-t_0+\tau)(T_1-t_0+\tau))^{2m}} + F_{i1}^{(k)}, & t_0 \le t < T_1 \\ F_{i1}^{(k)}, & T_1 \le t < T_2 \\ (t-T_2)^{2m-k} + F_{i1}^{(k)}, & t \ge T_2 \end{cases} \tag{A2}
$$

with $k = 0, 1, 2, \dots, n$.

Therefore, from *Assumption* 2, it can be shown that $\eta_{i1}^{(k)}$ and $\eta_{i2}^{(k)}$ $(k = 0, 1, 2, \dots, n)$ is continuous as long as $2m > n$ is satisfied.