Adaptive Nonlinear Regulation: Estimation from the Lyapunov Equation

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Abstract—**This paper presents a stabilizing adaptive controller for a nonlinear system depending affinely on some unknown parameters. We assume only this system is feedback stabilizable. A key feature of our method is the use of the Lyapunov equation to design the adaptive law. We give a result on local stability, two different conditions for global stability, and a local result where the initial conditions of the state of the system only are restricted.**

I. Introduction

We consider a family of nonlinear systems, indexed by a parameter vector p in $\mathbb{R}^l$

$$p = (p_1 \cdots p_l)^{\mathrm{T}}. \tag{1}$$

For each value of p, the corresponding system in this family is called $\mathscr{S}_p$. It is described by

$$\begin{aligned}\mathscr{S}_p\colon \dot{x} &= a^0(x,u) + \sum_{i=1}^{l} p_i a^i(x,u)\\ &= a^0(x,u) + A(x,u)p \end{aligned} \tag{2}$$

where the state x, living in an n-dimensional smooth manifold M, is assumed to be completely measured, the control u is in $\mathbb{R}^m$, and the a^i's are *known* smooth controlled vector fields. The way the system $\mathscr{S}_p$ is written in (2) expresses the assumption that the controlled vector field depends linearly, or, to be more precise, affinely, on the parameter p.

One particular vector p^* in $\mathbb{R}^l$ will be called the "true value of the parameter p." The problem is to stabilize the equilibrium point 0 of the system $\mathscr{S}_{p*}$, p^* being unknown.

Several answers to this stabilization problem have been proposed in the literature. In [16], [7], and [17] the problem is particularized to specific systems: robot arms for [16], [7] and a continuous stirred tank reactor for [17]. More general feedback linearizable systems are considered in [20], [8], [5], and [2]. Finally, Sastry and Isidori [15] study the case of exponentially minimum-phase systems with globally Lipshitz nonlinearities.

Here, we shall address the above stabilization problem without restricting our attention only to linearizable systems. The basic assumption we make instead is that, for every possible p, the system $\mathscr{S}_p$ may be stabilized by means of a feedback law, depending continuously on p. This is made precise in assumption UFS (uniform feedback stabilizability, see Section II). This usually makes sense only if we restrict the possible p's to lie in an open subset Π of $\mathbb{R}^l$. Such an assumption provides for any p in Π a control law $u = u_n(p, x)$ which stabilizes the system $\mathscr{S}_p$. We call these control laws the nominal control laws. Would p^* be known, the controller $u = u_n(p^*, x)$ could be used, and would give local or global stability for the closed-loop system, depending on whether the basic assumption UFS is locally or globally satisfied.

Since p^* is not known, we cannot use $u = u_n(p^*, x)$. Our solution consists of designing from the family of nominal control laws, a dynamic controller of the following form:

$$\left.\begin{aligned} u &= u_n(\hat{p}, x)\\ \dot{\hat{p}} &= \text{function of } (\hat{p}, x, \eta); \qquad \hat{p} \in \Pi\\ \dot{\eta} &= \text{function of } (\hat{p}, x, \eta)\end{aligned}\right\} \tag{3}$$

where η is in $\mathbb{R}^q$, $q \geq 0$. Note that u is defined only if $\hat{p}(t)$ remains in Π. The design of the dynamic controller will consist of the design of these "function of."

Our adaptive controllers will always guarantee a local stabilization. However, in the case where the basic assumption UFS is global, we are interested in designing an adaptive controller which also gives global stability. This will require some additional assumptions: either limiting the growth at infinity of the uncertainties, or not allowing the Lyapunov functions to depend on p. This extends previous results known only for feedback linearizable systems (see the details in Section VII). In particular, we prove the global adaptive stabilization of some systems for which no globally stabilizing control laws existed before. Novelty is even brought to the field of adaptive control of linearizable systems. Indeed, we shall exhibit a globally stabilizing adaptive controller for the feedback linearizable system of Example 2 below, for which no linearizing adaptive controller has yet been proved to be able to globally stabilize. This indicates that it might be very productive not to restrict our attention to feedback linearizing control, even for feedback linearizable systems.

In the case where the Lyapunov function used in the basic assumption does not depend on the parameter p, we are able to give a local result of a new and very interesting kind, in the sense that the initial condition of $\hat{p}$ in (3) is not restricted to be "close enough" to the true value p^*. This is a first attempt to give an adaptive controller yielding nonglobal results with a stability domain being explicitly proved to be larger than the one obtained with nonadaptive control. The required additional assumption makes this new result valid

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only for a rather restrictive class of systems, which, however, includes for example, in the case of feedback linearization, the case considered in [20].

The paper is organized as follows: in Section II, we state precisely the problem, assumptions, and objectives; in Section III, we present a general way of designing some adaptive laws; in Section IV, we explicitly write the adaptive controller proposed in (3) above, with an extra state η of dimension 1. In Section V, we state our main stabilization results; Section VI gives three illustrative examples; in Section VII, we conclude this paper with discussing these results and comparing them to others. The proofs are given in the Appendix.

II. Problem Statement

Recall that the family of systems we are interested in is the one, indexed by $p \in \mathbb{R}^l$, where the system $\mathscr{S}_p$ is described by (2). The system to be controlled is $\mathscr{S}_{p*}$, p^* being unknown.

A. Assumptions

Let Π be an open subset of $\mathbb{R}^l$, containing the "possible" values of the parameter p. It might be all $\mathbb{R}^l$. p^* will be assumed to be in Π.

The following assumption states in details that all the systems $\mathscr{S}_p$, for p in Π, are feedback stabilizable "uniformly" with respect to p. The first assumption, UFS, is global, i.e., states that the systems $\mathscr{S}_p$ are globally feedback stabilizable, and the second one, UFS(Ω), is the local version of UFS, valid only for x lying in a neighborhood Ω of the origin.

Uniform Feedback Stabilizability (UFS) Assumption: There exist *known* functions u_n, later called the "nominal control law," and V such that

$u_n : \Pi \times M \to \mathbb{R}^m$ is of class C^1, $V : \Pi \times M \to \mathbb{R}$ is of class C^2, and the following hold.

1) For all p in Π, $V(p, x)$ is positive for all x in M and zero if and only if x is 0.

2) For any positive real number K and any vector p in Π, the set

$$\{x \mid V(p, x) \le K, \quad x \in M\}$$

is a compact subset of M.

3) For all (p, x) in $\Pi \times M$, we have

$$\dot{V}\big|_{\dot{x}=s(p,x);\, \dot{p}=0} \triangleq \frac{\partial V}{\partial x}(p, x) \cdot s(p, x) \le -cV(p, x) \tag{4}$$

where c is a strictly positive constant and s denotes the "nominal closed loop field":

$$s(p, x) = a^o(x, u_n(p, x)) + A(x, u_n(p, x))p. \tag{5}$$

Local Uniform Feedback Stabilizability [UFS(Ω)] Assumption: With Ω being an open neighborhood of 0 in M, there exist *known* functions u_n and V such that

$u_n : \Pi \times \Omega \to \mathbb{R}^m$ is of class C^1, $V : \Pi \times M \to \mathbb{R}$ is of class C^2, and the following hold.

1) For all p in Π, $V(p, x)$ is positive for all x in Ω and zero if and only if x is 0.

2) For any positive real number K and any vector p in Π, the set

$$\{x \mid V(p, x) \le K, \quad x \in M\}$$

is a compact subset of M.

3) For all (p, x) in $\Pi \times \Omega$, inequality (4) holds.

Note that point 1) in these two assumptions implies that V is minimal at $(p, 0)$ for all p in Π. Hence, (4) means in particular that a nominal control law u_n can be found to satisfy

$$s(p, 0) = a^o(0, u_n(p, 0)) + A(0, u_n(p, 0))p = 0, \qquad \forall p \in \Pi. \tag{6}$$

The following assumption is about the open set Π, containing the values of p, for which assumption UFS holds. It is natural to ask that this set be connected and contain p^*. In addition, we ask that it be convex, in order to be able to design a projection onto Π and take advantage of the *a priori* knowledge: "the parameter vector p belongs to the open subset Π of $\mathbb{R}^l$." To make this projection smooth, we require the following precise property.

Imbedded Convex Sets (ICS) Assumption: There exists a *known* C^2 function $\mathscr{P}$ from Π to $\mathbb{R}$ such that the following hold.

1) For each real number λ in [0, 1], the set

$$\{p \mid \mathscr{P}(p) \le \lambda\}$$

is convex and contained in Π.

2) The row vector $(\partial \mathscr{P} / \partial p)(p)$ is nonzero for all p such that $\mathscr{P}(p) \in [0, 1]$.

3) The parameter vector p^* of the particular system to be actually controlled satisfies

$$\mathscr{P}(p^*) \le 0. \tag{7}$$

From this assumption, we call Π_c the closed convex subset of Π defined by

$$\Pi_c = \{p \mid \mathscr{P}(p) \le 1\}. \tag{8}$$

Equation (7) in assumption ICS requires that p^* be in the interior of Π_c, which is more restrictive than requiring only that p^* be in Π. As seen in the following example, as soon as Π is open and convex, it is not difficult to construct $\mathscr{P}$, and it can even be constructed such that Π_c is arbitrarily close to Π.

Example 1: Consider the case where the set Π is

$$p = (p_1, \cdots, p_l)^{\mathrm{T}} \in \Pi \Leftrightarrow |p_i - \rho_i| < \sigma_i, \qquad \forall i \in \{1, l\} \tag{9}$$

with ρ_i and σ_i some given real numbers. In such a case, ICS is satisfied if we choose the function $\mathscr{P}$ as, for example,

$$\mathscr{P}(p) = \frac{2}{\epsilon}\left[\sum_{i=1}^{l} \left|\frac{p_i - \rho_i}{\sigma_i}\right|^q - 1 + \epsilon\right] \tag{10}$$

with $0 < \epsilon < 1$ and $q \ge 2$ two real numbers. In this case,

Π_c is

$$\Pi_c = \left\{ p \left| \sum_{i=1}^{l} \left| \frac{p_i - \rho_i}{\sigma_i} \right|^q \le 1 - \frac{\epsilon}{2} \right. \right\} \tag{11}$$

and Π_c approaches Π when ϵ decreases and q increases. ■

Given the function $\mathscr{P}$, we may now define the "smooth projection" Proj, which we shall use in the adaptive controller (3) to maintain the estimate $\hat{p}$ in Π

$$\operatorname{Proj}(p, y) = \begin{cases} y & \text{if } \mathscr{P}(p) \le 0 \\ y & \text{if } \mathscr{P}(p) \ge 0 \text{ and } \dfrac{\partial \mathscr{P}}{\partial p}(p) y \le 0 \\ y - \dfrac{\mathscr{P}(p) \dfrac{\partial \mathscr{P}}{\partial p}(p) y}{\left\| \dfrac{\partial \mathscr{P}}{\partial p}(p) \right\|^2} \dfrac{\partial \mathscr{P}}{\partial p}(p)^T & \text{if not.} \end{cases} \tag{12}$$

Namely, $\operatorname{Proj}(p, \cdot)$ does not alter y if p belongs to the set $\{\mathscr{P}(p) \le 0\}$. In the set $\{0 \le \mathscr{P}(p) \le 1\}$, it subtracts a vector normal to the boundary $\{\mathscr{P}(p) = \lambda\}$ so that we get a smooth transformation from the original vector field y for $\lambda = 0$ to an inward or tangent vector field for $\lambda = 1$. Note that, thanks to the convexity assumption in ICS, we have the following useful properties:

$$\|\operatorname{Proj}(p, y)\| \le \|y\| \tag{13}$$

and, from (7)

$$(p^* - p)^{\mathrm{T}} \operatorname{Proj}(p, y) \ge (p^* - p)^{\mathrm{T}} y. \tag{14}$$

B. Objectives

With assumption UFS, we will derive a dynamic controller of the form (3), where the "function of" will be designed to the following goal: the solutions of the closed-loop system composed of the (adaptive) controller in feedback with $\mathscr{S}_{p^*}$ have to satisfy one of the following Properties P1, P2, P3.

Property P1: If UFS(Ω) holds for some open neighborhood Ω of 0 in M, there exists a neighborhood of $(p^*, 0, 0)$ in $\Pi \times M \times \mathbb{R}^q$ such that any solution $(\hat{p}(t), x(t), \eta(t))$, with $(\hat{p}(0), x(0), \eta(0))$ in this neighborhood, is bounded and $x(t)$ tends to 0.

Property P2: If UFS(Ω) holds for some open neighborhood Ω of 0 in M, there exists a neighborhood of 0 in M such that any solution $(\hat{p}(t), x(t), \eta(t))$, with $x(0)$ in this neighborhood, is bounded and $x(t)$ tends to 0.

Property P3: If UFS holds, all the solutions $(\hat{p}(t), x(t), \eta(t))$, are bounded and $x(t)$ tends to 0.

Property P1 will always be satisfied by our adaptive controller. In this property, we are concerned with two aspects: local boundedness and local convergence. The local boundedness is rather a weak property which is already satisfied by the control $u_n(p, x)$ with p chosen constant and close enough to the true value p^*. This is stated in the following proposition.

Proposition 1 (Robust Control) [4, Section X.5]: Consider the closed-loop system (state: $(\hat{p}, x)$) composed of $\mathscr{S}_{p^*}$ in feedback with the controller.

$$\begin{aligned} u &= u_n(\hat{p}, x) \\ \dot{\hat{p}} &= 0. \end{aligned} \tag{15}$$

There exists a neighborhood of $(p^*, 0)$ such that all the solutions starting in this neighborhood are bounded. Moreover, if there exists a neighborhood of $(p^*, 0)$ and a positive constant K such that, for (p, x) in this neighborhood.

$$\left\| \frac{\partial V}{\partial x}(p, x) A(x, u_n(p, x)) \right\| \le K V(p, x) \tag{16}$$

then, for solutions with initial condition $(\hat{p}(0), x(0))$ in a sufficiently small neighborhood of $(p^*, 0)$, $x(t)$ tends to 0.

According to this proposition, not only the local boundedness but also the local convergence involved in Property P1 are given by the nonadaptive controller [15]. However, we note that, with point 1) in property UFS, (16) implies $A(0, u_n(p, 0)) = 0$ for all p [compare to (6)]. Hence, the only interest of an adaptive controller meeting Property P1 is to guarantee the local convergence property, i.e., to guarantee the convergence of $x(t)$ to 0, although (16) may not be satisfied.

The difference between Property P1 and Property P2 is that, in the latter, there is no restriction on the initial value $\hat{p}(0)$. We shall design an adaptive controller satisfying Property P2 under the assumption that the function V given by assumption UFS(Ω) does not depend on p (Theorem 4). Clearly, Property P2 is not satisfied by the nonadaptive controller (15) if there exists some p such that $u_n(p, \cdot)$ does not stabilize (locally) the equilibrium point 0 of $\mathscr{S}_{p^*}$. Therefore, when Property P2 holds, the adaptive controller yields a larger attraction domain than the nonadaptive controller. This is a step in proving that, in general, adaptive control gives a larger attraction domain than nonadaptive control. See an illustration of this in Example 3.

Property P3 is stronger than the previous Properties P1 and P2 since there is no constraint at all on the initial conditions. However, it asks for the global assumption UFS. This may not be satisfied in practice. In fact, as mentioned above, Property P1 is already satisfied by nonadaptive controllers (see Proposition 1). And Property P2 seems to be very hard to establish. In this context, Property P3 is introduced as an easier criterion to decide on whether an adaptive control law is better than another one. We will obtain Property P3 in two different cases (Theorems 2 and 3) according to the dependence of the systems on the parameter vector p.

To illustrate the topic of this paper, we consider the following example.

Example 2: Let p be in $\mathbb{R}$, x in $\mathbb{R}^3$, and $\mathscr{S}_p$ be given by

$$\begin{aligned} \dot{x}_1 &= x_2 + p x_1^2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u. \end{aligned} \tag{17}$$

Clearly $\mathscr{S}_p$ is linear in p and, for any value of p, it can be globally linearized by feedback and diffeomorphism. This implies that UFS is satisfied. So this family of systems is a good candidate for Property P3. However, except in the very recent paper of Kanellakopoulos *et al.* [6], none of the adaptive controllers proposed in the literature is proved to satisfy this global property.

Here to meet Property P3, we first observe that UFS can be satisfied by using control laws whose main objective is robust stabilization instead of linearization. Namely, following the Lyapunov design proposed in [14], we obtain the following nominal control law:

$$\begin{aligned} u_n(p,x) = &-(c_1+c_2+2p\xi_1)(\xi_3-c_2\xi_2-\xi_1^{2k-1}) \\ &+\left[c_1^2+4c_1p\xi_1-(2k-1)\xi_1^{2k-2}-2p\xi_2\right] \\ &\cdot(\xi_2-c_1\xi_1)-c_3\xi_3 \\ &-\xi_2\left(\frac{\xi_2^2}{2}+\frac{\xi_1^{2k}}{2k}\right)^{j-1} \end{aligned} \tag{18}$$

where k and j are integers larger than or equal to 1, c_1, c_2, and c_3 are positive real numbers, and $\xi = (\xi_1, \xi_2, \xi_3)$ is given by the following p-dependent diffeomorphism φ:

$$\begin{aligned} \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} &= \varphi(p,x) \\ &= \begin{pmatrix} x_1 \\ x_2+c_1x_1+px_1^2 \\ x_3+c_2(x_2+c_1x_1+px_1^2) \\ \quad +(c_1+2px_1)(x_2+px_1^2)+x_1^{2k-1} \end{pmatrix}. \end{aligned} \tag{19}$$

The associated positive function V is

$$V(p,x) = U(\xi) = \frac{\xi_3^2}{2}+\frac{1}{j}\left(\frac{\xi_2^2}{2}+\frac{\xi_1^{2k}}{2k}\right)^j. \tag{20}$$

The nominal closed-loop field s, defined in (5) is, when written with the coordinates ξ

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 - c_1\xi_1 \\ \dot{\xi}_2 &= \xi_3 - c_2\xi_2 - \xi_1^{2k-1} \\ \dot{\xi}_3 &= -c_3\xi_3 - \xi_2\left(\frac{\xi_2^2}{2}+\frac{\xi_1^{2k}}{2k}\right)^{j-1}. \end{aligned} \tag{21}$$

Notice that if $k = j = 1$, u_n is a linearizing feedback.

Here, since the stabilizing laws exist for any value of p in $\mathbb{R}$, we have $\Pi = \mathbb{R}$. ∎

III. Adaptive Laws

Let us design adaptive laws [the $\dot{\hat{p}}$ and $\dot{\eta}$ equations in (3)] by applying, and extending, techniques from linear estimation theory.

We choose a function h of class C^2 from $\Pi \times M$ to $\mathbb{R}^q$. Different choices for h lead to different adaptive laws. It is only when analyzing the closed-loop system that one choice appears better than another one.

If $\hat{p}(t)$ is some C^1 time function and $u(t)$ some control, with $x(t)$ the corresponding solution of the closed-loop system $u\text{-}S_{p^*}$, we obtain the time function

$$h(t) = h(\hat{p}(t), x(t)) \tag{22}$$

whose derivative with respect to time is

$$\dot{h} = \frac{\partial h}{\partial x}(\hat{p},x)\left[a^o(x,u)+A(x,u)p^*\right]+\frac{\partial h}{\partial p}(\hat{p},x)\dot{\hat{p}}. \tag{23}$$

This equation is linear in p^*, namely it can be rewritten in

$$z(\dot{h}(t),\hat{p}(t),x(t),u(t)) = Z(\hat{p}(t),x(t),u(t))p^* \tag{24}$$

with

$$Z(\hat{p},x,u) = \frac{\partial h}{\partial x}(\hat{p},x)A(x,u) \tag{25}$$

$$z(\dot{h},\hat{p},x,u) = \dot{h}-\frac{\partial h}{\partial x}(\hat{p},x)a^o(x,u)-\frac{\partial h}{\partial p}(\hat{p},x)\dot{\hat{p}}. \tag{26}$$

Consequently, an estimate of p^* can be obtained by using an algorithm from linear estimation theory. For instance, the classical gradient estimator gives

$$\dot{\hat{p}} = -Z^{\mathrm{T}}(Z\hat{p}-z). \tag{27}$$

Unfortunately, since z depends on $\dot{h}$ which is not measured, such an equation cannot be implemented. To round this problem, we use e instead of $Z\hat{p}-z$ in (27)

$$\dot{\hat{p}} = -Z^{\mathrm{T}}e \tag{28}$$

where e is the following filtered version of $Z\hat{p}-z$:

$$\begin{aligned} \dot{e}+r(\hat{p},e,x,u)e &= Z\hat{p}-z \\ &= -\dot{h}+\frac{\partial h}{\partial p}(\hat{p},x)\dot{\hat{p}}+\frac{\partial h}{\partial x}(\hat{p},x) \\ &\quad\cdot\left[a^o(x,u)+A(x,u)\hat{p}\right] \end{aligned} \tag{29}$$

with r a smooth positive function. We check that (28) can actually be implemented by realizing (29) in

$$\begin{aligned} \dot{\eta}+r\eta &= rh(\hat{p},x)+\frac{\partial h}{\partial p}(\hat{p},x)\dot{\hat{p}} \\ &\quad+\frac{\partial h}{\partial x}(\hat{p},x)\left[a^o(x,u)+A(x,u)\hat{p}\right] \\ e &= \eta-h(\hat{p},x) \end{aligned} \tag{30}$$

with the state η in $\mathbb{R}^q$. This method is referred to as error filtering.

It is to be noticed that using a filtered version e of $Z\hat{p}-z$ is not the only way of rounding the problem arising from the fact that $\dot{h}$ is not measured. Another possibility is to filter both Z and z in (24), i.e., define Z_f and z_f by

$$\dot{z}_f+rz_f = z; \quad \dot{Z}_f+rZ_f = Z \tag{31}$$

which allows us to transform (24) into

$$z_f = Z_f p^* + \delta \tag{32}$$

where δ satisfies

$$\dot{\delta} + r\delta = 0, \quad \delta(0) = z_f(0) + Z_f(0)p^* \tag{33}$$

and z_f and Z_f can be computed, without using $\dot{h}$, via the same method used in (30). In this case, the gradient estimator would be

$$\dot{\hat{p}} = -Z_f^{\mathrm{T}}(Z_f\hat{p} - z_f). \tag{34}$$

This method, referred as "regressor filtering," will not be considered here, see point 4) of the discussion in Section VII.

To guarantee that the estimate $\hat{p}$ remains in the set Π we further change (28) into (35), using the smooth projection "Proj," the locally Lipschitz function defined by (12). We finally obtain the following adaptive law:

$$\dot{\hat{p}} = \mathrm{Proj}\,(\hat{p}, -Z^{\mathrm{T}}e) \tag{35}$$

$$e = \eta - h(\hat{p}, x) \tag{36}$$

$$\dot{\eta} + r\eta = rh(\hat{p}, x) + \frac{\partial h}{\partial p}(\hat{p}, x)\mathrm{Proj}\,(\hat{p}, -Z^{\mathrm{T}}e) + \frac{\partial h}{\partial x}(\hat{p}, x)\big[a^o(x, u) + A(x, u)\hat{p}\big]. \tag{37}$$

Let us stress that an actual controller is obtained by choosing a function h and a positive function r. It turns out that an appropriate choice of both r and h is crucial for being able to prove the global result of Property P3. See points 3) and 5) of the discussion in Section VII.

The main properties of the subsystem (35)–(37) are given by the following lemma.

Lemma 1: Assume that assumption ICS is satisfied. Let u be an arbitrary continuous-time function and h be an arbitrary C^2 function. For any solution $(x(t), \hat{p}(t), \eta(t))$, defined on $[0, T)$, of the closed-loop system u-S_{p^*} (35)–(37), with $\hat{p}(0)$ chosen in Π_c, we have, for all t in $[0, T)$

1) $\hat{p}(t) \in \Pi_c$,

2) $$\tfrac{1}{2}\|p^* - \hat{p}(t)\|^2 + \tfrac{1}{2}|e(t)|^2 + \int_0^t r|e|^2 \le \tfrac{1}{2}\|p^* - \hat{p}(0)\|^2 + \tfrac{1}{2}|e(0)|^2 \tag{38}$$

with e defined in (36).

Proof: $\hat{p}(t)$ cannot leave Π_c because, thanks to the projection [see (12)], $\dot{\hat{p}}$ is pointing toward the inside of the set Π_c when evaluated at the boundary of this set. To prove (38), we consider the following comparison function:

$$W(e, \hat{p}) = \tfrac{1}{2}(e^2 + \|p^* - \hat{p}\|^2). \tag{39}$$

Along the solutions of u-S_{p^*} (35)–(37), for any t in $[0, T)$, we have, from (29), (44), and (14)

$$\dot{W} \le -r|e|^2. \tag{40}$$

□

Notice that $\hat{p}$ is not proved to converge to p^*.

IV. An Adaptive Controller

In the adaptive law (35)–(37) described in the previous section, we choose

$$h(p, x) = V(p, x) \tag{41}$$

and

$$r = 1 + \|Z\|\left\|\frac{\partial V}{\partial p}\right\|. \tag{42}$$

Taking the control itself as $u = u_n(\hat{p}, x)$ we obtain the following dynamic controller:

Adaptive Controller $\mathscr{AC}(V)$:

$$u = u_n(\hat{p}, x) \tag{43}$$

$$\dot{\hat{p}} = \mathrm{Proj}\,(\hat{p}, -Z^{\mathrm{T}}e), \qquad \hat{p}(0) \in \Pi_c \tag{44}$$

$$\dot{\eta} = -re + \frac{\partial V}{\partial x}(\hat{p}, x)\cdot s(\hat{p}, x) + \frac{\partial V}{\partial p}\dot{\hat{p}} \tag{45}$$

where

$$e = \eta - V(\hat{p}, x) \tag{46}$$

$$Z = \frac{\partial V}{\partial x}(\hat{p}, x)A(x, u_n(\hat{p}, x)) \tag{47}$$

$$r = 1 + \|Z\|\left\|\frac{\partial V}{\partial p}\right\|. \tag{48}$$

This controller explicitly incorporates the function V. A different choice of V yields a different controller. This explains the notation "V" in $\mathscr{AC}(V)$ and will be exploited in Theorem 4.

Example 2 (Continued): For our example (17), the adaptive controller is

$$\dot{\hat{p}} = -Z^{\mathrm{T}}(V(\hat{p}, x) - \eta) \tag{49}$$

$$\begin{aligned}\dot{\eta} = (V(\hat{p}, x) - \eta)&\left[1 + |Z|\left\|\frac{\partial U}{\partial \xi}(\hat{\xi})\frac{\partial \varphi}{\partial p}(\hat{p}, x)\right\|\right] \\ &- c_3\hat{\xi}_3^2 - \left(\frac{\hat{\xi}_2^2}{2} + \frac{\hat{\xi}_1^{2k}}{2k}\right)^{j-1}(c_2\hat{\xi}_2^2 + c_1\hat{\xi}_1^{2k}) \\ &+ \Bigg[\left(\frac{\hat{\xi}_2^2}{2} + \frac{\hat{\xi}_1^{2k}}{2k}\right)^{j-1}\hat{\xi}_2 x_1^2 \\ &+ \hat{\xi}_3((c_1 + c_2 + 4\hat{p}x_1)x_1^2 + 2x_1x_2)\Bigg] \\ &\cdot Z^{\mathrm{T}}(V(\hat{p}, x) - \eta)\end{aligned} \tag{50}$$

$$Z = \frac{\partial U}{\partial \xi}(\hat{\xi})\frac{\partial \varphi}{\partial x_1}(\hat{p}, x)x_1^2 \tag{51}$$

$$u = u_n(\hat{p}, x) \tag{52}$$

with U given in (20), u_n given in (18), and, using φ in (19)

$$\hat{\xi} = (\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3)^{\mathrm{T}} = \varphi(\hat{p}, x). \tag{53}$$

Note that, since u_n is defined for all p in $\mathbb{R}$, no "Proj" is needed here. ■

V. The Stabilization Properties

Applying the adaptive controller $\mathscr{A}\mathscr{C}(V)$ to the system S_{p*} leads to an autonomous nonlinear locally Lipschitz continuous system living in $\Pi \times M \times \mathbb{R}$. Its solutions $(\hat{p}(t), x(t), \eta(t))$ are locally well defined, continuously differentiable, and unique. Assuming that assumptions ICS and UFS or UFS(Ω) are met, we have the following results on the behavior of these solutions. All the proofs are given in the Appendix.

Theorem 1 (Property P1): If ICS and UFS(Ω) hold for some open neighborhood Ω of 0 in M, there exists an open neighborhood of $(p^*, 0, 0)$ in $\Pi_c \times M \times \mathbb{R}^q$ such that any solution $(\hat{p}, x, \eta)$ with initial condition $(\hat{p}(0), x(0), \eta(0))$ in this neighborhood, exists on $[0, \infty)$, remains in a compact set, and its (x, η)-component tends to $(0, 0)$. In addition, the point $(p^*, 0, 0)$ is a Lyapunov stable equilibrium point.

This local result requires no additional assumption to UFS(Ω) and ICS. The following Theorems 2 and 3 give global adaptive stability in the case where the global assumption UFS holds. They both require additional assumptions: (54) limits the growth at infinity of the dependence on the parameters, whereas Theorem 3 requires that the Lyapunov function V do not depend on the parameters.

Theorem 2 (Property P3): If ICS and UFS hold and if there exists a C^0 function d on Π such that, for all (p, x) in $\Pi \times M$

$$\left\| \frac{\partial V}{\partial x}(p, x) A(x, u_n(p, x)) \right\| \left\| \frac{\partial V}{\partial p}(p, x) \right\| \leq d(p)\left(1 + V(p, x)^2\right) \quad (54)$$

then all the solutions are defined on $[0, \infty)$, remain in a compact set, and their (x, η)-component tends to $(0, 0)$ as t tends to infinity.

Theorem 3 (Property P3): If ICS and UFS hold and V does not depend on p, then all the solutions are defined on $[0, \infty)$, remain in a compact set, and their (x, η)-component tends to $(0, 0)$ as t tends to infinity.

As stressed in the Introduction, Theorem 1 states a rather weak property which is already almost satisfied by nonadaptive controllers: see Proposition 1. On the other hand, we establish in Theorems 2 and 3 some global stabilization results which in general are not given by nonadaptive controllers, but this requires the strong global assumption UFS. In the case when only the local assumption UFS(Ω) is satisfied, one would like a stronger local result than Theorem 1. Indeed, we would like to have an estimate of the attraction domain good enough to prove that this attraction domain is larger than the one obtained by using a nonadaptive controller. Unfortunately, as far as we know, such a nice property has never been established. This shows the interest of the following theorem where this property is proved in the case when the Lyapunov function does not depend on p. This result follows from replacing V by an appropriate other function in the adaptive controller $\mathscr{A}\mathscr{C}(V)$. In this theorem, V_o being a strictly positive real number, $\mathscr{A}\mathscr{C}((V/V_o - V))$ is the controller defined by $u = u_n(\hat{p}, x)$ and the estimator (35)–(37) with

$$h(p, x) = \frac{V(p, x)}{V_o - V(p, x)}. \quad (55)$$

Theorem 4 (Property P2): Assume V does not depend on p and ICS and UFS(Ω) hold for some open neighborhood Ω of 0 in M and let $V_o > 0$ be such that

$$V(x) < V_o \Rightarrow x \in \Omega. \quad (56)$$

Under these conditions, by using the adaptive controller $\mathscr{A}\mathscr{C}((V/V_o - V))$, the closed-loop system is such that all the solutions with initial condition satisfying

$$V(x(0)) < V_o, \qquad \eta(0) \text{ arbitrary and } \hat{p}(0) \in \Pi_c \quad (57)$$

are defined on $[0, \infty)$, remain in a compact set, and their (x, η)-component tends to $(0, 0)$ as t tends to infinity.

To appreciate the interest of Theorem 4, we note that, if in Π_c, there exists a value p_1 of the parameter vector such that the control $u = u_n(p_1, x)$ does not guarantee (local) stabilization for the system $\mathscr{S}_{p*}$, then the adaptive controller $\mathscr{A}\mathscr{C}((V/V_o - V))$ is better than the nonadaptive controller $u = u_n(p, x)$ (p constant). Indeed, its domain of attraction is larger since it is $\{x \mid V(x) < V_0\} \times \Pi_c$.

Let us comment on the additional assumptions invoked in Theorems 2, 3, and 4.

- Assumption (54) limits the growth at infinity of functions measuring the dependence on p. This assumption is always satisfied for linear systems since, in this case, V is a quadratic definite positive function of x. $(\partial V/\partial p)$ is a quadratic function of x, and $(\partial V/\partial x)$ and A are linear functions of x. In the general case, it is difficult to understand the meaning of this assumption although it is easy to check when given the system and V. Example 2 below shows that it can be satisfied, by using appropriately chosen nominal control laws, for a system which had never been proved to be globally adaptively stabilizable except in the very recent report of Kanellakopoulos, Kokotovic, and Morse [6].
- The assumption that V does not depend on p is also difficult to understand. It is not satisfied in general by linear systems when linear control is used. However, it holds if the systems of the family $\{\mathscr{S}_p\}_{p \in \Pi_c}$ are all equivalent via regular state feedback, i.e., for any pair (p_1, p_2) in $\Pi_c \times \Pi_c$, there exists a control $u(p_1, p_2, x, v)$ such that, for all (x, v),

$$a^0(x, v) + A(x, v)p_1 = a^0(x, u(p_1, p_2, x, v)) + A(x, u(p_1, p_2, x, v))p_2 \quad (58)$$

with $u(p_1, p_2, x, v)$ smooth. This equivalence is characterized by Pomet and Kupka in [13]. In particular, it is implied by the "strict matching assumption" introduced in [20] in the case of linearizable systems. Nevertheless, in Example 4 below, we show that this equivalence, although sufficient, is not necessary for the existence of a function V not depending on p.

In view of these comments, one way to solve the adaptive stabilization problem is to use these additional assumptions—growth condition (54) or V independent of p —as guidelines in the design of the functions u_n and V we need to satisfy assumption UFS. This leads to the following two-stage procedure.

1) Find a class of stabilizing control laws with sufficient degrees of freedom to significantly change the behavior at infinity or the dependence on p of the Lyapunov function. Unfortunately, *a priori* given stabilizing control laws, like linearization by feedback and diffeomorphism, are usually inappropriate (see Example 2 continued below). Also, we do not know any systematic way to achieve stabilization for general systems. Nevertheless, if the systems S_p are in the following restricted pure-feedback form (see [19]), namely:

$$\begin{aligned}\dot{x}_1 &= x_1 + f_1(p, x_1)\\ \dot{x}_i &= x_{i+1} + f_i(p, x_1, \cdots, x_{i-1}), \qquad 2 \le i \le n-1\\ \dot{x}_n &= u + f_n(p, x_1, \cdots, x_n)\end{aligned} \tag{59}$$

where the f_i's and their derivatives have polynomial growth, then the Lyapunov design of [14] is an efficient tool. For instance, this is applied in Example 2 with the introduction of the integer numbers k and j in (18) (see below).

2) Specify these degrees of freedom to meet (54) or to make V independent of p.

This design procedure will be illustrated in the next section by continuing Example 2 for which we have already done the first stage with (18) and (20).

VI. Some Examples

Example 2 (Continued): For the system (17), Theorem 1 gives the local convergence and boundedness properties of Property P1. But, since every system $\mathscr{S}_p$ is globally feedback linearizable, one might expect a global result. However, no adaptive controller, given in the literature and based on feedback linearization, is proved to give global stabilization. To apply Theorem 2 here, we have to check whether (54) is satisfied or not. Let us prove that if we use linearizing control laws u_n, i.e., if we choose $k = j = 1$ in (18), (54) cannot be satisfied, although it is satisfied if $j \ge 2$ and $k \ge 3$.

We have to compare the product of the norms of

$$Z = \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial x_1}x_1^2 \quad \text{and} \quad \frac{\partial V}{\partial p} = \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial p}$$

to a power of $V = U$. For this comparison, we use the ξ coordinates to simplify the expression of V. We have

$$\frac{\partial U}{\partial \xi} = \left(\left(\frac{\xi_2^2}{2} + \frac{\xi_1^{2k}}{2k}\right)^{j-1}\xi_1^{2k-1},\ \left(\frac{\xi_2^2}{2} + \frac{\xi_1^{2k}}{2k}\right)^{j-1}\xi_2,\ \xi_3\right), \tag{60}$$

$$\frac{\partial \xi}{\partial x_1} = \begin{pmatrix} 1\\ c_1 + 2p\xi_1\\ (c_1 + 2p\xi_1)(c_2 + 2p\xi_1) + 2p(\xi_2 - c_1\xi_1) + (2k-1)\xi_1^{2k-2}\end{pmatrix}, \tag{61}$$

$$\frac{\partial \xi}{\partial p} = \begin{pmatrix} 0\\ \xi_1^2\\ c_2\xi_1^2 + 2\xi_1(\xi_2 - c_1\xi_1) + (c_1 + 2p\xi_1)\xi_1^2\end{pmatrix}. \tag{62}$$

To obtain our inequalities, we remark

$$|\xi_1| \le (\gamma U)^{\frac{1}{2kj}}, \quad |\xi_2| \le (\gamma U)^{\frac{1}{2j}}, \quad |\xi_3| \le (\gamma U)^{\frac{1}{2}} \tag{63}$$

with

$$\gamma = \sup\{j(2k)^j, j2^j, 2\}. \tag{64}$$

Also, for any positive real number α, we have

$$|a + bU^\alpha| \le (|a| + |b|)\sup\{1, U^\alpha\}. \tag{65}$$

We get

$$\|Z\| \le d_1(p)\sup\{1, U^{\alpha_1}\}, \qquad \left\|\frac{\partial V}{\partial p}\right\| \le d_2(p)\sup\{1, U^{\alpha_2}\} \tag{66}$$

with

$$d_1(p) = \left[(c_1 + 2|p|) + (c_1c_2 + 2c_2|p| + 4p^2) + 2|p| + (2k-1)\right]\gamma^{\alpha_1}, \tag{67}$$

$$\alpha_1 = \sup\left\{1 + \frac{1}{2kj}, 1 - \frac{1}{2j} + \frac{3}{2kj}, \frac{1}{2} + \frac{2}{kj}, \frac{1}{2} + \frac{1}{2j} + \frac{1}{kj}, \frac{1}{2} + \frac{1}{j}\right\}, \tag{68}$$

$$d_2(p) = \left[2 + |c_1 - c_2| + 2|p|\right]\gamma^{\alpha_2}, \tag{69}$$

$$\alpha_2 = \sup\left\{1 - \frac{1}{2j} + \frac{1}{kj}, \frac{1}{2} + \frac{3}{2kj}, \frac{1}{2} + \frac{k+1}{2kj}\right\}. \tag{70}$$

It follows that:

$$\left\|\frac{\partial V}{\partial x}(p, x)A(x, u_n(p, x))\right\|\left\|\frac{\partial V}{\partial p}(p, x)\right\| \le d_1(p)d_2(p)(1 + V(p, x)^\alpha) \tag{71}$$

where α depending on j and k is given in Table I.

Feedback linearization is obtained with $k = j = 1$. In this case, as stated in Table I, we cannot find any α smaller than $9/2$ such that

$$\left\|\frac{\partial V}{\partial x}(p, x)A(x, u_n(p, x))\right\|\left\|\frac{\partial V}{\partial p}(p, x)\right\| \le d(p)(1 + V(p, x)^\alpha). \tag{72}$$

This prevents us from applying Theorem 2 and therefore from getting a global result from feedback linearizing nominal control laws.

However, choosing the control law corresponding to $k = 3$ and $j = 2$ in (18), it is clear from this table that (54) is satisfied. Then Theorem 2 applies and gives the global con-

TABLE I
$\alpha(k, j)$

k \ j	1	2	3	4	5
1	9/2	11/4	5/2	19/8	23/10
2	11/4	17/8	25/12	33/16	41/20
3	8/3	2	2	2	2
4	21/8	31/16	47/24	63/32	79/40
5	13/5	19/10	29/15	39/20	49/25

vergence and boundedness properties of Property P3 for the corresponding adaptive controller. ■

Example 3: Let us consider the family of one-dimensional systems in which $\mathscr{S}_p$ is described by

$$\mathscr{S}_p : \dot{x} = px + (1 - x^2)u. \tag{73}$$

For any $p > 0$, the origin is only *locally* asymptotically stabilizable with a domain of attraction

$$\Omega = (-1, 1). \tag{74}$$

A stabilizing (locally linearizing) feedback law is

$$u_n(p, x) = -(p+1)\frac{x}{1 - x^2}. \tag{75}$$

A corresponding positive function V may be chosen independent of p

$$V(x) = x^2. \tag{76}$$

Following the idea of Theorem 4, we propose the controller $\mathscr{A}\mathscr{C}((V/1 - V))$

$$u = -(\hat{p} + 1)\frac{x}{1 - x^2} \tag{77}$$

$$\dot{\hat{p}} = -\frac{2x^2}{1 - x^2}\left(\eta - \frac{x^2}{1 - x^2}\right) \tag{78}$$

$$\dot{\eta} = -\eta + \frac{x^2}{1 - x^2} - \frac{2x^2}{(1 - x^2)^2}. \tag{79}$$

Notice that u and $\dot{\hat{p}}$ tend to infinity as x tends to 1 or -1.

For any value of p^* in $\mathbb{R}$, considering the dynamical system in $\mathbb{R}^3$ composed of $\mathscr{S}_{p^*}$ in feedback with the controller (77)–(79), Theorem 4 proves that each solution $(x(t), \hat{p}(t), \eta(t))$ with $x(0)^2 < 1$ is defined for all positive time, bounded, and such that $(x(t), \eta(t))$ goes to $(0, 0)$ as t tends to infinity.

Since each system $\mathscr{S}_p$ given by (73) is feedback linearizable and satisfies the "matching assumption" given in [20], we might apply the design proposed in [20]; the adaptive controller would be

$$u = -(\hat{p} + 1)\frac{x}{1 - x^2} \tag{80}$$

$$\dot{\hat{p}} = x^2 \tag{81}$$

but the stability domain obtained in [20] is only

$$x(0)^2 + (p(0) - p^*)^2 < 1 \tag{82}$$

which does not allow $p(0)$ to be far away from p^*.

In fact, using the same trick of changing the Lyapunov function V into $(V/V_o - V)$, or here x^2 into $x^2/1 - x^2)$, the Lyapunov design used in [20] would lead to exactly the same result as Theorem 4. ■

Example 4: This is an example of a situation where it is possible to find a family of control laws yielding a Lyapunov function independent of p, whereas the systems of the family are not equivalent to one another via pure feedback transformations.

Consider the family of systems described by

$$\begin{aligned} \dot{x}_1 &= x_2 + px_1x_3 \\ \dot{x}_2 &= x_3 + 2px_1(2x_1 + x_2) \\ \dot{x}_3 &= u. \end{aligned} \tag{83}$$

It is clear that the systems in this family indexed by p are not feedback equivalent [see (58)], even locally—a feedback transformation will only alter $\dot{x}_3$.

However, we shall find functions u_n and V satisfying the assumptions of Theorem 3 in the following way. We first define ξ_1, ξ_2, ξ_3 by the following linear change of coordinates, not depending on p:

$$\begin{aligned} \xi_1 &= x_1 & x_1 &= \xi_1 \\ \xi_2 &= x_2 + x_1 \qquad , & x_2 &= \xi_2 - \xi_1 \quad . \\ \xi_3 &= x_3 + 2x_1 + 2x_2 & x_3 &= \xi_3 - 2\xi_2 \end{aligned} \tag{84}$$

The equations (83) read

$$\begin{aligned} \dot{\xi}_1 &= -\xi_1 + \xi_2 + p\xi_1(-2\xi_2 + \xi_3) \\ \dot{\xi}_2 &= -\xi_2 - \xi_1 + \xi_3 + p\xi_1(2\xi_1 + \xi_3) \\ \dot{\xi}_3 &= u - 2\xi_1 - 2\xi_2 + 2\xi_3 + p\xi_1(4\xi_1 + 2\xi_3). \end{aligned} \tag{85}$$

Then, by choosing V as

$$V = \tfrac{1}{2}\xi_1^2 + \tfrac{1}{2}\xi_2^2 + \tfrac{1}{2}\xi_3^2 \tag{86}$$

we have

$$\begin{aligned} \dot{V}|_{\dot{x}=s(p,x);\,\dot{p}=0} &= -\xi_1^2 - \xi_2^2 \\ &\quad + \xi_3(u - 2\xi_1 + 5p\xi_1^2 - \xi_2 + p\xi_1\xi_2 + 2\xi_3 + 2p\xi_1\xi_3). \end{aligned} \tag{87}$$

Therefore, defining the nominal control law by

$$u_n(p, x) = 2\xi_1 + \xi_2 - 3\xi_3 - p\xi_1(5\xi_1 + \xi_2 + 2\xi_3) \tag{88}$$

yields (4) with $c = 2$. V obviously does not depend on p. Theorem 3 applies and our adaptive controller $\mathscr{A}\mathscr{C}(V)$ yields global regulation in this case. ■

VII. Discussion and Conclusion

A. What Makes Our Controller Differ from Others?

1) We do not specify the nominal control laws u_n. Our contribution may be viewed as a general method to make a stabilizing control law adaptive, whereas most proposed designs deal with "adaptive linearization" in which the nominal control laws are supposed to be feedback linearization.

2) As in [2], [7], [8], [15], and [11], our controller is

based on estimation, as opposed to the Lyapunov design introduced by Parks in [9] and used, in the nonlinear case, in [5], [16], [17], and [20].

3) Our adaptive controller is obtained from the general adaptive law of Section III by choosing $h = V(\hat{p}(t), x(t))$. In fact, one may choose instead any vector or scalar function of x and $\hat{p}$. In [2] and [8], where the system is supposed to be feedback linearizable, this function is chosen as $\varphi(\hat{p}, x)$, where $\varphi(p, \cdot)$ is, for any p, the change of coordinates which allows the system $\mathscr{S}_p$ to be written in linear form. In [15], where output linearization is considered, this function is a linear combination of the output and its derivatives. Here, choosing $V(\hat{p}, x)$ seems more natural to deal with the stabilization problem. It turns out to be a better choice when the nominal control laws are not feedback linearization (see [10] for a discussion of the possible choices).

4) We benefit by the fact that Z in (47), or (25), is measured and we just "filter" $Z\hat{p} - z$ instead of filtering both z and Z as in [8], [15], or [11].

5) Pursuing the idea of [11], we introduce into the estimation some information on the stabilization; this is done both through choosing the Lyapunov equation to base adaptation (see point 3) above), and through the term r in (48) which acts more or less as a "normalization" introduced in the above-mentioned filtering. This allows us to enforce more robust properties to the filtered equation error e in (29).

B. Comparison to Other Results

As mentioned above, most of the controllers in the literature are proved to satisfy the local convergence and boundedness properties of Property P1 just like our Theorem 1. We discuss here the possibility to get the global result of Property P3.

The first kind of assumptions used, in the literature, to prove global results is often called *matching assumptions*. It allows the authors to state the global Property P3 in [20], [5], and [2]. Note that the results in [20] and [5] are based on a Lyapunov design and can be easily extended from the case of feedback linearization, considered in these papers, to the case of assumption UFS. The assumption in [20] implies that we can choose V not depending on p, and the assumptions in [5], [2], or [12] more or less imply that one can annihilate the p-dependence of V through the control. This is further discussed in [10] and in [13]. On the other hand, the controller proposed in [2], which is based on estimation, is proved to always give the local Property P1. However, its generalization to the case when the control law u_n is not feedback linearization would not give the global Property P3, even if V does not depend on p, without adding an assumption on the nominal closed-loop field s in (5) (see [10]). Using the Lyapunov equation in the estimation (point 3) above) allows us to avoid this problem. Our controller gives the local Property P1 always, and the global Property P3 when V does not depend on p, no matter what type of stabilizing feedback the u_n is.

Another kind of assumption is called the *growth conditions*. They limit the growth at infinity of functions measuring the dependence on the unknown parameters. For the case when the u_n is a feedback linearizing law, this growth at infinity is given by the global Lipschitz assumptions in [15] and [8]. Here, thanks to basing the adaptation on the Lyapunov equation and to using a normalization, the less conservative assumption (54) is sufficient. As an aside result, we have shown with Example 2 that, even dealing with feedback linearizable systems, it has an interest to consider other possible stabilizing control laws than feedback linearization: for adaptive stabilization, adaptive linearization may not be appropriate due to its lack of robustness far from the equilibrium.

C. Extensions

Nonsmooth Stabilizing Feedback Laws: Stabilizing control laws may often be guaranteed to be smooth everywhere but at the origin (see [18]). In such a case, replacing V in $\mathscr{A}\mathscr{C}(V)$ by, say, $(\sup\{V - \epsilon, 0\})^2$ with ϵ some strictly positive constant, would give similar results with the equilibrium point 0 replaced by the set $\{V \leq \epsilon\}$ (see [1]).

Stabilizing Dynamical Feedback Laws: It may also occur that the systems $\mathscr{S}_p$ are not stabilizable via static-state feedback $u = u_n(p, x)$ but via dynamic state feedback

$$\begin{cases} \dot{\chi} = \psi(p, x, \chi) \\ u = u_n(p, x, \chi) \end{cases} \tag{89}$$

where χ belongs to some manifold, independent of p. Then, defining X by

$$X = \begin{pmatrix} x \\ \chi \end{pmatrix} \tag{90}$$

and replacing x by X in this paper, the extension of the presented method and results is straightforward.

Nonlinear Parametrization: The fact that the systems depend linearly in the parameter p [in (2)] is very important here. A possible extension is to consider "implicitly linearly parametrized" systems, i.e.,

$$\mathscr{S}_p: \left[J^o(x) + \sum_{i=1}^{l} p_i J^i(x)\right]\dot{x} = a^0(x, u) + \sum_{i=1}^{l} p_i a^i(x, u) \tag{91}$$

where $J^o(x) + \sum_{i=1}^{l} p_i J^i(x)$ is an invertible matrix for any p and x (the systems considered in this paper are a particular case of (91) with $J^o \equiv I$ and $J^i \equiv 0$). This has been extensively studied in the case of robot arms (see [16] and [7], for example), and in a more general case, in [11]. The methods presented here do not directly apply.

VIII. Conclusion

We believe that the kind of adaptive control we propose fits well to the adaptive stabilization problem as we state it here. It is the only method we know that gives as good results as the direct Lyapunov method used in [20] or [5] when some "matching assumptions" are satisfied. It works also when these assumptions fail, without restricting to linearizing feedback controls.

Appendix

Proof of the Theorems

With the help of (4) in assumption UFS, the main idea of these proof is to observe that the evolution of $V(p(t), x(t))$ can be written [with $h = V$ in (23)]

$$\dot{V} = \frac{\partial V}{\partial x}(\hat{p}, x) \cdot s(\hat{p}, x) + Z(\hat{p}, x)(p^* - \hat{p}) + \frac{\partial V}{\partial p}(\hat{p}, x)\dot{\hat{p}} \tag{92}$$

$$\leq -cV + \dot{e} + re + \left\| \frac{\partial V}{\partial p} \right\| \|Z\| \, |e|. \tag{93}$$

We view (93) as a "nominal" evolution $\dot{V} \leq -cV$ perturbed by the other terms, Lemma 1 tells us that e, $\sqrt{r}\,e$, and $\sqrt{\|(\partial V/\partial p)\| \, \|Z\| \, e}$ are small—in the L^2-sense. However, since we have no bound on $\dot{e}$, which comes into (93) as a perturbation, we shall study, rather than (93), the evolution of η defined by $V = \eta + e$ [see (46)]. We get

$$\dot{\eta} \leq -c\eta + (c + r)e + \left\| \frac{\partial V}{\partial p} \right\| \|Z\| \, |e| \tag{94}$$

or, from (48),

$$\dot{\eta} \leq -c\eta + \left(1 + c + 2\|Z\| \left\| \frac{\partial V}{\partial p} \right\|\right) |e|. \tag{95}$$

We shall use the following well-known lemma (see [3]).

Lemma 2: Let U be a C^1 time function defined on $[0, T)$ $(0 < T \leq +\infty)$, satisfying

$$\dot{U} \leq -cU + \alpha(t)U(t) + \beta(t) \tag{96}$$

where c is a strictly positive constant and α and β are two positive time functions belonging to $L^2(0, T)$

$$\int_0^T \alpha^2 \leq S_1 < +\infty, \quad \int_0^T \beta^2 \leq S_2 < +\infty. \tag{97}$$

Under this assumption, $U(t)$ is upperbounded on $[0, T)$ and, precisely,

$$U(t) \leq e^{\frac{S_1}{c}} \left[U(0) + \sqrt{\frac{2}{c}} \sqrt{S_2} \right] \qquad \forall t \in [0, T). \tag{98}$$

Moreover, if T is infinite then

$$\limsup_{t \to \infty} U(t) \leq 0. \tag{99}$$

Proof: This is a straightforward consequence of a known result on differential inequalities: from (96), one derives (see [4, Theorem I.6.1]):

$$U(t) \leq U(0)e^{-ct + \int_0^t \alpha} + \int_0^t e^{[-c(t-\tau) + \int_\tau^t \alpha]}\beta(\tau)\, d\tau. \tag{100}$$

But (97) and Cauchy–Schwartz inequality yield, for any positive t and τ

$$-c(t - \tau) + \int_\tau^t \alpha \leq -c(t - \tau) + \sqrt{S_1}\sqrt{t - \tau} \leq -\frac{c}{2}(t - \tau) + \frac{S_1}{c}. \tag{101}$$

Then substituting (101) in (100), and applying Cauchy–Schwartz inequality again, one gets

$$U(t) \leq U(0)e^{-\frac{c}{2}t + \frac{S_1}{c}} + \left(\frac{2}{c} e^{\frac{S_1}{c}}\right)^{\frac{1}{2}} \cdot \left(e^{\frac{S_1}{c}} \int_0^t e^{-\frac{c}{2}(t-\tau)}\beta(\tau)^2\, d\tau\right)^{\frac{1}{2}}. \tag{102}$$

This gives (98) from (97). Now, since

$$\int_0^t e^{-\frac{c}{2}(t-\tau)}\beta(\tau)^2\, d\tau \leq e^{-\frac{c}{2}\frac{t}{2}} \int_0^{\frac{t}{2}} \beta(\tau)^2\, d\tau + \int_{\frac{t}{2}}^t \beta(\tau)^2\, d\tau \tag{103}$$

we have

$$U(t) \leq e^{\frac{S_1}{c}} \left[U(0)e^{-\frac{c}{2}t} + \sqrt{\frac{2}{c}} \cdot \sqrt{S_2 e^{-\frac{c}{4}t} + \int_{\frac{t}{2}}^T \beta(\tau)^2\, d\tau} \right]. \tag{104}$$

From (97), this gives (99), and the proof is completed. □

Now, let us introduce the following function K and compact subset $\mathcal{O}$ of $\Pi \times M$:

$$\mathcal{O}(v, \pi) = \{(p, x) \mid V(p, x) \leq v, \ \|p - p^*\| \leq \pi \text{ and } p \in \Pi_c\}, \tag{105}$$

$$K(v, \pi) = \sup_{(p,x) \in \mathcal{O}(v,\pi)} \|Z(p, x)\| \left\| \frac{\partial V}{\partial p}(p, x) \right\|. \tag{106}$$

Note that since Z and $(\partial V/\partial p)$ are continuous and $\mathcal{O}(v, \pi)$ is bounded. $K(v, \pi)$ is well defined.

Proof of Theorem 1: We suppose that UFS(Ω) holds and we must find initial conditions such that x remains inside the set Ω.

From points 1) and 2) of UFS(Ω), one can find strictly positive $v_{\max}$ and $\pi_{\max}$ such that

$$(p, x) \in \mathcal{O}(v_{\max}, \pi_{\max}) \Rightarrow x \in \Omega. \tag{107}$$

We now consider a solution $(\hat{p}(t), \eta(t), x(t))$ whose initial condition satisfies

$$V(\hat{p}(0), x(0)) \leq \frac{1}{6} \min\left\{ \frac{v_{\max}}{1 + L(v_{\max}, \pi_{\max})}, \pi_{\max} \right\}$$

$$\eta(0) \leq \frac{1}{6} \min\left\{ \frac{v_{\max}}{2 + L(v_{\max}, \pi_{\max})}, \pi_{\max} \right\}$$

$$\| \hat{p}(0) - p^* \| \leq \frac{1}{6} \min \left\{ \frac{v_{\max}}{1 + L(v_{\max}, \pi_{\max})}, \pi_{\max} \right\} \tag{108}$$

where

$$L(v_{\max}, \pi_{\max}) = \sqrt{\frac{1}{c}} \left(1 + c + 2K(v_{\max}, \pi_{\max})\right). \tag{109}$$

To prove that its component $x(t)$ remains in Ω, let us suppose that this is false. Then let t_1 be the infimum of the times t such that

$$\left(\hat{p}(t), x(t)\right) \notin \mathcal{O}(v_{\max}, \pi_{\max}). \tag{110}$$

The constraint (108) on the initial condition implies $t_1 > 0$ and, for all t in $[0, t_1)$, we have

$$\dot{\eta}(t) \leq -c\eta + \left(1 + c + 2K(v_{\max}, \pi_{\max})\right) |e(t)|. \tag{111}$$

Hence, for all t in $[0, t_1)$, Lemma 2 gives (here $S_1 = 0$)

$$\eta(t) \leq \eta(0) + \sqrt{2}\, L(v_{\max}, \pi_{\max}) \sqrt{\int_0^{t_1} |e|^2} \tag{112}$$

with $L(v_{\max}, \pi_{\max})$ defined in (109). On the other hand, V being positive, we have (with $\eta = V + e$)

$$-|e(t)| \leq \eta(t). \tag{113}$$

Finally, note that, since $r \geq 1$ and

$$|e(0)| \leq |\eta(0)| + |V(\hat{p}(0), x(0))| \tag{114}$$

Lemma 1 implies

$$|e(t)| \leq |V(\hat{p}(0), x(0))| + |\eta(0)| + \| p^* - \hat{p}(0) \|$$

$$\sqrt{\int_0^{t_1} |e|^2} \leq \frac{1}{\sqrt{2}} \left(|V(\hat{p}(0), x(0))| + |\eta(0)| + \| p^* - \hat{p}(0) \| \right). \tag{115}$$

It follows

$$\eta(t) \leq \tfrac{1}{2} v_{\max}. \tag{116}$$

We conclude that $V = \eta + e$, (108), (112), and (115) yield

$$V(\hat{p}(t), x(t)) \leq \tfrac{1}{2} v_{\max}. \tag{117}$$

And Lemma 1 together with (108) imply

$$\| p^* - \hat{p}(t) \| \leq \tfrac{1}{2} \pi_{\max}. \tag{118}$$

Continuity of the solutions and the fact that (117) and (118) are true for any t in $[0, t_1)$ contradicts the definition of t_1 [see (110)].

We have proved that any solution with initial condition satisfying (108) is such that $(\hat{p}(t), x(t))$ remains in the compact set $\mathcal{O}(v_{\max}, \pi_{\max})$ and $\eta(t)$ satisfies (116). Therefore, it is defined on $[0, +\infty)$. Lemma 2 and (111) then imply

$$\limsup_{t \to +\infty} \eta(t) \leq 0. \tag{119}$$

On the other hand, from Lemma 1, e is bounded and belongs to $L^2(0, +\infty)$. Also $\dot{e}$ is bounded since [see (24) and (29)]

$$\dot{e} = -re + Z(\hat{p} - p^*) \tag{120}$$

where r and Z are continuous functions with bounded arguments. This altogether implies

$$\lim_{t \to \infty} e(t) = 0. \tag{121}$$

Now, (119) together with (113) and (121) prove

$$\lim_{t \to \infty} \eta(t) = 0. \tag{122}$$

Since $V = \eta - e$, (121) and (122) yield

$$\lim_{t \to \infty} V(\hat{p}(t), x(t)) = 0. \tag{123}$$

The properties of V imply finally that x goes to 0.

Stability of the equilibrium point $(p^*, 0, 0)$ follows from the fact that, for any positive $\bar{v}$, $\bar{\pi}$, and $\bar{\eta}$, with $\bar{v} < v_{\max}$ and $\bar{\pi} < \pi_{\max}$, a solution with initial conditions meeting:

$$V(\hat{p}(0), x(0)) \leq \frac{1}{3} \min \left\{ \frac{\bar{v}}{1 + L(v_{\max}, \pi_{\max})}, \frac{\bar{\eta}}{L(v_{\max}, \pi_{\max})}, \bar{\pi} \right\}$$

$$\eta(0) \leq \frac{1}{3} \min \left\{ \frac{\bar{v}}{2 + L(v_{\max}, \pi_{\max})}, \frac{\bar{\eta}}{1 + L(v_{\max}, \pi_{\max})}, \bar{\pi} \right\}$$

$$\| \hat{p}(0) - p^* \| \leq \frac{1}{3} \min \left\{ \frac{\bar{v}}{1 + L(v_{\max}, \pi_{\max})}, \frac{\bar{\eta}}{L(v_{\max}, \pi_{\max})}, \bar{\pi} \right\} \tag{124}$$

satisfies, for any t,

$$V(\hat{p}(t), x(t)) \leq \bar{v};\ \| p^* - \hat{p}(t) \| \leq \bar{\pi};\ |\eta(t)| \leq \bar{\eta}. \tag{125}$$

□

Proof of Theorem 2: Let $(\hat{p}(t), x(t), \eta(t))$ be a solution of the closed-loop system and $[0, T)$ be its right maximal interval of definition. From Lemma 1, we know that e and

$$\epsilon \triangleq |e| \sqrt{1 + \|Z\| \left\| \frac{\partial V}{\partial p} \right\|} \tag{126}$$

are in $L^2(0, T)$. We know also that $\hat{p}$ is bounded. This implies that the function $d(\hat{p}(t))$ in (54) is bounded, say by $D > 0$. Then (95), (54), and (126) give

$$\dot{\eta} \leq -c\eta + (c - 1)|e| + 2\sqrt{1 + D(1 + V^2)}\, |\epsilon| \tag{127}$$

and, since $V = \eta + e$,

$$\dot{\eta} \leq -c\eta + 2\sqrt{D}\, |\epsilon| \eta + 2\left(\sqrt{D}\, |e| + \sqrt{1 + D}\right) |\epsilon| + (c - 1)|e|. \tag{128}$$

Now Lemma 1 states

$$2\sqrt{D}\,|\epsilon| \in L^2(0, T) \tag{129}$$

and (with $|e|$ bounded)

$$2\left(\sqrt{D}\,|e| + \sqrt{1+D}\right)|\epsilon| + (c-1)|e| \in L^2(0, T). \tag{130}$$

Since Lemma 1 says that $\hat{p}$ and $e = V - \eta$ are bounded, by using Lemma 2, (113), (128), (129), and (130), we conclude that η and, therefore, $V = e + \eta$ are bounded. With point 2) of assumption UFS, we have proved that the solution itself is bounded. This implies that T is infinite, and Lemma 2 finally implies

$$\limsup_{t \to +\infty} \eta(t) \leq 0. \tag{131}$$

We conclude that (x, η) goes to $(0, 0)$ as in the proof of Theorem 1. □

Proof of Theorem 3: Since $(\partial V / \partial p)$ is zero, (95) becomes

$$\dot{\eta} \leq -c\eta + (1 + c)|e|. \tag{132}$$

The proof follows from (132) as the proof of Theorem 2. □

Proof of Theorem 4: Substitute W defined by

$$W(x) = \frac{V(x)}{V_o - V(x)} \tag{133}$$

to V in the proof of Theorem 3. Notice that UFS is also satisfied with W instead of V. □

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