Proceedings of the 35th Chinese Control Conference
July 27-29, 2016, Chengdu, China

Group Consensus of Multi-agent System with Double-integrator Dynamics under Directed Topology

LIN Yuyang, LI Zhongshen*

College of Mechanical Engineering and Automation, Huaqiao University, Xiamen 361021
E-mail: lzscyw@hqu.edu.cn

Abstract: Without the in-degree balanced assumption, the group consensus protocol design problem is discussed for multi-agent systems with double-integrator dynamics. Based on the dynamic model of double-integrator system, a group consensus control protocol containing the couple coefficient γ is designed. The convergence analysis is discussed and the sufficient conditions of group convergence for double-integrator system are obtained. And we know that the sufficient conditions of group convergence are depended on the interactive topology and the couple coefficient γ. Meanwhile, the couple coefficient γ can affect the convergence speed of the system. Finally some numerical simulations are given to show the effectiveness of the theoretical analyze.

Key Words: Double-integrator systems, Directed topology, Group consensus, Couple coefficient

1 Introduction

The consensus problem of multi-agent systems has become an attractive topic in recent years, which have drawn much attention from researchers in many fields such as computer sciences, control engineering, robotics, biology, and social science Consensus of a multi-agent system means that all the agents in the system reach an agreement upon some certain values of interest via local interaction.

Many dramatic achievements about the consensus problem of multi-agent systems were made in the past few years. A simple model for phase transition of a set of self-driven particles was proposed and the complex systematic dynamics of the simple model was presented numerically by Vicsek *et al* [1]. Olfati-Saber and Murray[2] introduced a systematical frame structure for the consensus problem in network of agents with single-integrator dynamics. Since 2007, the consensus problem for agents with double-integrator dynamics has been studied from diverse perspectives[3-5].

In recent years, group consensus which is a more general kind of consensus, has been taking into account and many researchers have been attracted increasingly. In a complex multi-agent system which the agents are divided into multiple subgroups, group consensus of a multi-agent system means that the agents in the same subgroup reach consensus while the consistent values of different subgroups may be different. In reality, the research on group consensus is more valuable for human society and nature, for the agreements in some actual conditions, such as the formation of unmanned aerial vehicle, the cooperative search or the traffic control. Therefore, many significant results of group consensus have been obtained. By using the semi-tensor product of matrices, Wang *et al.*[6] studied the vertex coloring problems of graphs and maximum stable set which were applied to the group consensus problem, and obtained some new results and algorithms. To guarantee the group consensus with fix and switching topology, some sufficient conditions in accordance with linear matrix inequalities were presented by Yu and Wang[7-8]. Furthermore, Qin and Yu[9] discovered that the group consensus can be achieved asymptotically under directed acyclic topology regardless of the magnitudes of the couple coefficients among the agents. Xie *et al.*[10] proposed some necessary and sufficient conditions for group consensus of continuous-time multi-agent systems by using a distributed protocol with Cartesian coordinate coupling matrix. Wang *et al.*[11] investigated the group consensus protocol

*Corresponding author. E-mail address: lzscyw@hqu.edu.cn

This research is supported by the Natural Science Foundation of Fujian Province of China (2011J01350), the Fundamental Research Funds for the Central Universities (JB-ZR1107) and the Subsidized Project for Cultivating Postgraduates' Innovative Ability in Scientific Research of Huaqiao University

problem for multi-agent systems with and without time delay respectively under the connected bipartite graph topology.

Note that most results of group consensus above relied on the in-degree balanced assumption. The in-degree balanced assumption is a rigorous condition which requires that the sum of adjacent weights from every node in one subgroup to all nodes in another subgroup is equal to zero at any time. As we know, the discuss without the in-degree balanced assumption has more practical significance. Inspired mainly by the aforementioned results, this paper concentrates on the group consensus problem without the in-degree balanced assumption for double-integrator dynamics under the directed interaction topology. The rest of the paper is organized as follows. First, some preliminary results, including the definition of group consensus for double-integrator dynamics, are presented. Second, a group consensus protocol for double-integrator dynamics under the directed topology is proposed and some necessary and sufficient conditions for group consensus of the double-integrator multi-agent systems are established Finally, some numerical simulations are provided to validate the theoretical analysis.

2 Preliminaries and Problem Statement

Algebraic graph theory is used to analyze consensus problems. The interaction topology of agents is denoted by a graph if each agent in the system is regarded as a node of the graph. Consider a graph $G=\{V,E,A\}$ as a weighted directed graph, indexed by agents $0,1,...,n$, where $V=\{v_1,v_2,...,v_n\}$ denotes the set of nodes and $E\subseteq V\times V$ denotes the set of edges. An edge of G is described by (i,j) which means that agent j can obtain information from agent i, but not necessarily vice versa. Consider a set $N_i=\{v_j\in V:(v_i,v_j)\in E\}$ as the set of neighbors of node i. The adjacency matrix of the directed graph is denoted by $A=\{a_{ij}\}\in R^{n\times n}$ where $a_{ij}=1$ if $(i,j)\in E$, and $a_{ij}=0$ if $(i,j)\notin E$. Self-edges are not considered in the paper, that is, for $\forall i\in N, a_{ii}=0$. Define the Laplacian matrix $L=\{l_{ij}\}\in R^{n\times n}$ as: $\begin{cases} l_{ij}=\sum\limits_{j=1,j\neq i}^{n} a_{ij} & i=j \\ l_{ij}=-a_{ij} & i\neq j \end{cases}$. Note that if $(i,j)\notin E$ then $l_{ij}=-a_{ij}=0$. Laplacian Matrix L satisfies $l_{ij}\le 0, i\neq j$ and $\sum_{j=1}^{n} l_{ij}=0$, $i=1,2,...n$. L has a zero eigenvalue corresponding to the right eigenvector $\mathbf{1}_n=[1,1,...,1]^{\mathrm{T}}\in R^n$ for All row-sums of L are 0.

Suppose that every agent has double-integrator dynamics given by

$$\begin{cases} \dot{x}_i=v_i \\ \dot{v}_i=u_i \end{cases} \quad i=1,2,...,n \tag{1}$$

Where $x_i\in R^n$, $v_i\in R^n$ and $u_i\in R^n$ denote the position, velocity and control input, respectively, of agent i.

Note that $\{X_1,X_2,...,X_k\}(k>1)$ is a partition of the set $X=\{x_1,x_2,...,x_n\}$ if $X_i=\varnothing$, $X_i\bigcap X_j\neq\varnothing$ and $\bigcup_{i=1}^{k}X_i=X$. Index $\tilde{i}$ is used to denote the subset of the partition in which the number i lies, *i.e.*, $x_i\in X_{\tilde{i}}$. Evidently, $1\le\tilde{i}\le k$. Note that agent i and j are in the same subgroup if $\tilde{i}=\tilde{j}$.

Consider the double-integrator system (1), suppose that $\{X_1,X_2,...,X_k\}$ is one of the partitions of all agents, the definition of group consensus for double-integrator dynamics is presented as follow.

Definition 1. For given initial conditions $x(0)=[x_1(0),x_2(0),...,x_n(0)]^{\mathrm{T}}$, $v(0)=[v_1(0),v_2(0),...,v_n(0)]^{\mathrm{T}}$, where $x_i(0)\in R$, $v_i(0)\in R$, system (1) is said to realize group consensus with the partition $\{X_1,X_2,...,X_k\}$ if $\lim_{t\to\infty}\left|x_i(t)-x_j(t)\right|=0$, $\lim_{t\to\infty}\left|v_i(t)-v_j(t)\right|=0$ for $\tilde{i}=\tilde{j}$ and $\lim_{t\to\infty}\left|x_i(t)-x_j(t)\right|\neq 0$, $\lim_{t\to\infty}\left|v_i(t)-v_j(t)\right|\neq 0$ for $\tilde{i}\neq\tilde{j}$.

Consider the double-integrator system (1), a group consensus protocol is proposed as

$$\begin{cases} \dot{x}_i(t) = v_i(t) - \bar{v}_i \\ \dot{v}_i(t) = \sum_{j=1}^{n} a_{ij}[(x_j(t) - x_i(t)) + \gamma(v_j(t) - v_i(t))] + \sum_{j=1}^{n} l_{ij}[\bar{x}_j + \bar{v}_j] \end{cases} \quad i = 1, 2, ..., n \tag{2}$$

where $\bar{x}_j$, $\bar{v}_j$ denote the convergent values of the position and velocity respectively for the partition where the agent j lies, and $\gamma > 0$ denotes the couple coefficient.

3 Main Results

To investigate the conditions for group consensus for double-integrator dynamics we have the following theorem for the group consensus protocol (2):

Theorem 1: Protocol (2) achieves group consensus asymptotically if the directed topology is connected graph and the nonzero eigenvalues of Θ have negative real parts.

where $\Phi = \begin{bmatrix} 0_n & I_n \\ -L & -\gamma L \end{bmatrix}, \gamma > 0$.

Proof: Noting that

$$\dot{v}_i(t) = \sum_{j=1}^{n} a_{ij}[(x_j(t) - x_i(t)) + \gamma(v_j(t) - v_i(t))] + \sum_{j=1}^{n} l_{ij}[\bar{x}_j + \bar{v}_j]$$
$$= -\sum_{j=1}^{n} l_{ij}(x_j(t) - \bar{x}_j) - \gamma\sum_{j=1}^{n} l_{ij}(v_j(t) - \bar{v}_j)$$

Define the errors: $e_{x_i(t)} = x_i(t) - \bar{x}_i$, $e_{v_i(t)} = v_i(t) - \bar{v}_i$, let

$e_x = (e_{x_1}^{\mathrm{T}}, e_{x_2}^{\mathrm{T}}, ..., e_{x_n}^{\mathrm{T}})$, $e_v = (e_{v_1}^{\mathrm{T}}, e_{v_2}^{\mathrm{T}}, ..., e_{v_n}^{\mathrm{T}})$, $e = (e_x^{\mathrm{T}} \quad e_v^{\mathrm{T}})^{\mathrm{T}}$.

After some manipulation, protocol (2) can be written as:

$$\dot{e} = \Phi e \tag{3}$$

where $\Phi = \begin{bmatrix} 0_n & I_n \\ -L & -\gamma L \end{bmatrix}, \gamma > 0$.

According to definition 1, the fact that protocol (2) achieves group consensus asymptotically and the equation (4) are equivalent.

$$\lim_{t\to\infty} \left|e_{x_i(t)}\right| = 0, \quad \lim_{t\to\infty} \left|e_{v_i(t)}\right| = 0 \tag{4}$$

Consider a Lyapunov function candidate:

$$V(t) = \frac{1}{2} e^{\mathrm{T}}(t) e(t) \tag{5}$$

Taking the derivative of V gives

$$\dot{V}(t) = e^{\mathrm{T}} \dot{e}(t) = e^{\mathrm{T}}(t) \Phi e(t) \tag{6}$$

Let $y(t) = e^{\mathrm{T}}(t)$. If the nonzero eigenvalues of Φ have negative real parts, $\dot{V}(t) \le 0$. Moreover $\dot{V}(t) = 0$ if and only if $y(t) = \mathbf{0}_n$. Therefore, $M = \{y(t) \mid y(t) = 0_n\}$ is the maximum invariant set of $D = \{y(t) \mid \dot{V}(t) = 0_n\}$. According to the LaSalle's invariance principle, when $t \to 0$, each solution of system (5) is attracted to M, which means $\|e_i(t)\| \to 0$ for any given initial conditions. So condition (4) holds, and the double- integrator system can achieve group consensus ■

Note that theorem 1 gives the sufficient conditions for the group consensus. However, the nonzero eigenvalues of Φ are difficult to obtain. We propose the following theorem to build the connection between matrix Φ and Laplacian matrix L corresponding to the directed topology.

Theorem 2: Given the group consensus protocol (2), the directed topology has a directed spanning tree. Let μ_i be the *ith* eigenvalue of Laplacian matrix L. Also let $\Phi = \begin{bmatrix} 0_n & I_n \\ -L & -\gamma L \end{bmatrix}$, where γ is a positive scalar. Then the eigenvalues of Φ are given by $\lambda_{i\pm} = \dfrac{-\gamma\mu_i \pm \sqrt{\gamma^2 \mu_i^2 - 4\mu_i}}{2}$, $i = 1, 2, ..., n$. When $\mathrm{Re}(\mu_i) > 0$, $\mathrm{Re}(\lambda_{i\pm}) < 0$ if and only if $\lambda > \max_{\forall \mu_i \ne 0} \dfrac{\left|\mathrm{Im}^2(\mu_i)\right|}{\sqrt{\mathrm{Re}(\mu_i)}\left|\mu_i\right|}$.

Proof: For the first part, assume that λ is an eigenvalue of Φ with an associated right eigenvector $[f, g]^{\mathrm{T}}$, where $f, g \in C^n$. It follows that $\begin{bmatrix} 0 & I \\ -L & -\gamma L \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \lambda \begin{bmatrix} f \\ g \end{bmatrix}$, which implies $g = \lambda f$ and $-Lf - \gamma L = \lambda g$. It thus follows that $\lambda^2 f + \gamma\lambda Lf + Lf = 0$. Noting that $Lq_i = \mu_i q_i$, let $f = q_i$ and $\lambda^2 + \gamma\mu_i\lambda + \mu_i = 0$. That is, each eigenvalue of L,

λ_i, corresponds to two eigenvalues of Φ, denoted by $\lambda_{i\pm} = \frac{-\gamma\mu_i \pm \sqrt{\gamma^2\mu_i^2 - 4\mu_i}}{2}$.

For the second part, note that $\sqrt{\gamma^2\mu_i^2 - 4\mu_i}$ has a nonnegative real part. For $\lambda_{i+} = \frac{-\gamma\mu_i + \sqrt{\gamma^2\mu_i^2 - 4\mu_i}}{2}$ and $\lambda_{i-} = \frac{-\gamma\mu_i - \sqrt{\gamma^2\mu_i^2 - 4\mu_i}}{2}$, it follows that $\mathrm{Re}(\lambda_{i-}) < 0$ if $\gamma > 0$. To investigate the conditions for which $\mathrm{Re}(\lambda_{i+}) < 0$. When λ_{i+} is on the imaginary axis, γ^* is used to denote the critical value for γ. Let $\lambda_{i+} = \eta_i \boldsymbol{\imath}$, where $\eta_i \in R$ and $\boldsymbol{\imath}$ denotes the imaginary unit. After some manipulation, it follows that $\gamma^* = \frac{|\mathrm{Im}^2(\mu_i)|}{\sqrt{\mathrm{Re}(\mu_i)}|\mu_i|}$. Note that $\mathrm{Re}(\mu_i) > 0$. It is obvious that if $\gamma > \gamma^*$ (respectively, $\gamma < \gamma^*$), $\mathrm{Re}(\lambda_{i+}) < 0$ (respectively, $\mathrm{Re}(\lambda_{i+}) > 0$), Therefore, when $\mathrm{Re}(\mu_i) > 0$, $\mathrm{Re}(\lambda_{i\pm}) < 0$ if and only if $\lambda > \max_{\forall\mu_i \neq 0} \frac{|\mathrm{Im}^2(\mu_i)|}{\sqrt{\mathrm{Re}(\mu_i)}|\mu_i|}$. ■

To illustrate the connection between the directed topology of agents and the group consensus, we propose a corollary based on the theorems above. Before that, we present a lemma as follow.

Lemma 1[12]: Suppose that $x = [x_1, x_2, ..., x_n]^{\mathrm{T}}$ with $x_i \in R^n$ and Laplacian matrix $L \in R^{n\times n}$ associated with directed graph *G*. There are five equivalent conditions.

1) The rank of *L* is *n-1*,
2) There is a directed spanning tree associated with *L* in the directed graph,
3) for the system $\dot{x} = -Lx$, consensus can be achieved asymptotically,
4) $Lx = 0$ means that $x_1 = x_2 = \cdots = x_n$,
5) *L* has a simple zero eigenvalue corresponding to a right eigenvector $\mathbf{1}_n = [1,1,...,1]^{\mathrm{T}}$ and all other eigenvalues have positive real parts.

Corollary 1: The multi-agent system can achieve group consensus asymptotically with the group consensus protocol (2) if and only if directed graph G_n has a directed spanning tree and

$$\gamma > \gamma^* = \max_{\forall\mu_i \neq 0} \frac{|\mathrm{Im}^2(\mu_i)|}{\sqrt{\mathrm{Re}(\mu_i)}|\mu_i|} \quad i = 1,2,...,n \qquad (7)$$

where μ_i denotes the *ith* eigenvalue of Laplacian matrix *L* and γ^* denotes the critical value of the couple coefficient γ.

Proof: If there is a directed spanning tree in the directed graph G_n, $\mathrm{Re}(\mu_i) > 0$, $i = 1,2,...,n$ for $\forall\mu_i \neq 0$ by lemma 1. Note from theorem 2 that the nonzero eigenvalues of matrix Φ have negative real part if and only if (7) holds. As a result, the group consensus protocol (2) can achieve group consensus by the theorem 1. ■

Note 1: The couple coefficient γ describes the influence of the velocity. The system can achieve group consensus with protocol (2) when $\gamma > \gamma^*$, while the convergent speed and γ are positively correlated.

4 Numerical Simulations

The numerical simulations following are given to prove the validity of the proposed theoretical analyze. Without loss of generality, we take a multi-agent system consisting of five agents into consideration. The directed topology associated with the five agents is shown in Fig 1.

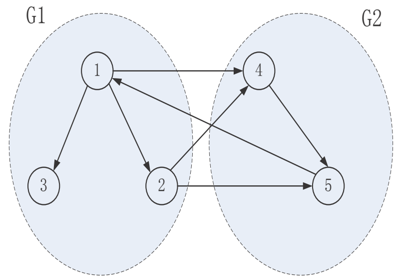

Fig 1. The directed topology associated with five agents

Note from the Fig 1 that the system is divided into two groups. The agent *1,2,3* belong to group *G1* while the agent *4,5* belong to group *G2*. The communication weights of the agents are 1. It is straightforward to know that the directed topology contains a directed spanning tree. The adjacency matrix and the Laplacian matrix are shown as follows respectively.

$$A=\begin{bmatrix}0&0&0&0&1\\1&0&0&0&0\\1&0&0&0&0\\1&1&0&0&0\\0&1&0&1&0\end{bmatrix}\quad L=\begin{bmatrix}1&0&0&0&-1\\-1&1&0&0&0\\-1&0&1&0&0\\-1&-1&0&2&0\\0&-1&0&-1&2\end{bmatrix}$$

After some calculation, the eigenvalues of the Laplacian matrix L corresponding to the directed topology are $\mu=[1\quad 0\quad 2+i\quad 2-i\quad 2]$, and the critical couple coefficient $\gamma^*=0.3162$. The initial states are chosen randomly as $x^{\mathrm{T}}(0)=[3\quad 4\quad 7\quad 1\quad 5]$, $v^{\mathrm{T}}(0)=[0.5\quad 1\quad 1.5\quad 2\quad 4]$. Letting $\gamma=1$, $\gamma=0.5$, $\gamma=0.2$ respectively, the simulation results are shown in Fig.2-4, respectively.

According to Fig. 2 and Fig.3, the system can achieve group consensus by corollary 1 when $\gamma=1$ and $\gamma=0.5$. So corollary 1 is validated. By comparing Fig. 2 and Fig.3, the convergent efficiency is improved with the increase of the couple coefficient γ, which is in line with Note 1. In contrast, the system can not achieve group consensus with the condition $\gamma\le\gamma^*$ when $\gamma=0.2$, as shown in Fig. 4. Corollary 1 is validated as well.

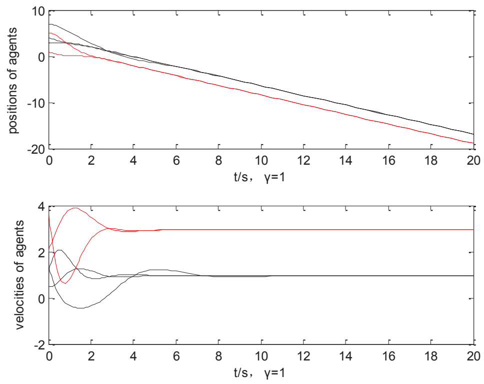

Fig. 2 Simulation results with $\gamma=1$

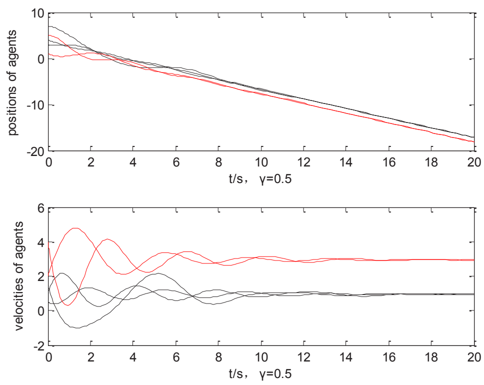

Fig. 3 Simulation results with $\gamma=0.5$

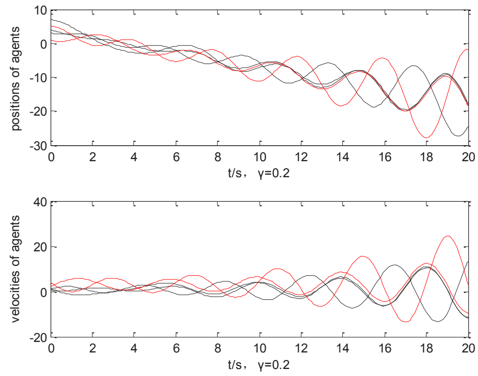

Fig. 4 Simulation results with $\gamma=0.2$

5 Conclusion

In this paper, the group consensus was discussed without the in-degree balanced assumption for multi-agent systems with double-integrator dynamics. The group consensus protocol was implemented based on the position and velocity of each agent's neighbors. We divided the agents into some partitions. By employing matrix theory and graph theory, we derived some results for group consensus. It is shown that the system finally achieves group consensus by the criteria mentioned above. In the future, we will take the undirected interaction topology into consideration and try to extend the work to more general scenarios.

References

[1] T. Vicsek, A. Czirook, E. Ben-Jacob, *et al*. Novel Type of Phase Transition in a System of Self-Driven Paticles[J]. *Physical Review Letters,* 1995, 75(6)：1226-1229.

[2] R. Olfati-Saber, R. M. Murray. Consensus Problems in Networks of Agents with Switching Topology and Time-delays[J]. *IEEE Transactions on Automatic Control,* 2004, 49(9)：1520-1533.

[3] G. Xie, L. Wang. Consensus Control for a Class of Networks of Dynamic Agents[J]. *International Journal of Robust and Nonlinear Control,* 2007, 17(10-11)：941-959.

[4] W. Ren, E. M. Atkins. Distributed Multi-vehicle Coordinated Control via Local Information Exchange[J]. *International Journal of Robust and Nonlinear Control,* 2007, 17(10-11)：1002-1033.

[5] W. Yu, G. Chen, M. Cao, *et al*. Second-Order Consensus for Multi-agent Systems with Directed Topologies and Nonlinear Dynamics[J]. *IEEE Transactions on Systems, Man, and Cybernetics-Part B：Cybernetics,* 2010, 40(3)：

881–891.

[6] Y. Z. Wang, C. H. Zhang, Z. B. Liu. A matrix approach to graph maximum stable set and coloring problems with application to multi-agent system[J]. *Automatica,* 2012, 48(7)：1227-1236.

[7] J. Y. Yu, L. Wang. Group consensus in multi-agent systems with switching topologies[C]. *Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference,* Hong Kong, 2009：2652-2657.

[8] J. Y. Yu, L. Wang. Group consensus in multi-agent systems with switching topologies and communication delays[J]. *System & Control Letters,* 2010, 59(6)：340-348.

[9] J. Qin, C. Yu. Cluster consensus control of generic linear multi-agent systems under directed topology with acyclic partition [J]. *Automatica,* 2013, 49(2)：2898-2905.

[10] D. Xie, Q. Liu. Necessary and sufficient condition for the group consensus of multi-agent systems[J]. *International Journal of Applied Mathematics and Computer Science,* 2014, 243(15)：870-878.

[11] Q. Wang, Y. Z. Wang, R. M. Yang. Design and analysis of group-consensus protocol for a class of multi-agent systems[J]. *Control and Decision,* 2013, 28(3): 369-373, 378.

[12] W. Ren, R. W. Beard. *Distributed Consensus in Multi-vehicle Cooperative Control：Communications and Control Engineering*[M]. London：Springer, 2008.