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Research article

Prescribed-time cluster lag consensus control for second-order non-linear leader-following multiagent systems ☆

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ABSTRACT

Prescribed-time Lag consensus, as a special case of prescribed-time cluster lag consensus, is first investigated. The task is to design a control protocol for each follower so that the multiagent system (MAS) achieves lag consensus in any specified time. To achieve this goal, we propose a new distributed controller, in which the control gains are designed as time-varying functions related to the pre-specified time. In addition, a state transformation is introduced to tackle the technical difficulty caused by time-varying functions of different powers in the theoretical proof process. Then, a solution for the cluster lag consensus problem of the MAS is provided, so that under the proposed control protocol, each subsystem composed of followers from the same group and the leader achieves lag consensus with a different lag time in the specified time. By using a state transformation, Graph theory and generalized Lyapunov stability theory, the validity of the designed schemes is verified theoretically and sufficient conditions for the two conclusions are given respectively. Finally, we give two simulation examples to show performances of the proposed solutions.

1. Introduction

It is noticed that in recent decades, many research results on the subject of distributed consensus control of multiagent systems (MASs) have been obtained. These results provide a solid theoretical foundation for solving various engineering problems, such as cooperative controls of robotic systems, state estimations of wireless networks and optimal controls of smart power grids (see [1–4] and references therein). Early research in this area mainly focused on the asymptotic stability, with the purpose of designing control protocols so that in theory, when the time t approaches infinity, the defined consensus for the multiagent system (MAS) can be achieved (see [5–12]). In recent years, it has been found that compared with asymptotic stability, finite-time stability which has the ability to stabilize a system in limited time, can provide faster convergence speed, stronger robustness to uncertainty and better anti-disturbance ability (see [13]). Therefore, considerable efforts have been made to design effective solutions for finite-time consensus problems of MASs.

Most of the existing studies on finite-time consensus, for instance, literatures [13–17], are based on Theorem 4.2 in [13], and an estimate on the upper bound of the time required for the MAS to achieve consensus (for brevity, referred to as the upper bound of the settling time (UBST)) is explicitly given by $V^{(1-\alpha)}(x_0)/[c(1-\alpha)]$. Obviously the calculation of the UBST depends on the initial value and designed parameters of the MAS. Hence, in the case where the initial value cannot be acquired beforehand, the finite-time control method is no longer applicable. Fixed-time consensus, in which the estimate of the UBST (EUBST) is unrelated to system initial conditions, has been investigated in [18–20]. Specifically, by applying fractional power state feedbacks to the controller, fixed-time consensus can be achieved, and the UBST is usually estimated as: $T_{\max} := 1/[c(1-\alpha)]+1/[d(\gamma-1)]$ (see [20]). It can be seen that although the EUBST has nothing to do with the initial conditions, it is still related to the system parameters and cannot be prescribed arbitrarily.

Recently, prescribed-time consensus has received increasing attention because it can provide a solution that enables the MAS to achieve consensus within any specified time. It should be emphasized that the UBST in prescribed-time consensus can be pre-assigned offline, which obviously meets the requirements of some real-time systems. Two scaling functions, one for constructing an observer and the other for constructing a finite-time

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compensator, were used for the controller design in [21], and the research results showed that leader-following consensus for high-order MASs (HOMASs) was achieved in a pre-specified time under directed topology. The authors in [22] proposed a novel solution to prescribed-time consensus problems of first-order MASs (FOMASs), and the constraint conditions of the designed controller parameters were given separately for the MAS to achieve average consensus, leader-following consensus and containment consensus. The prescribed-time consensus problem for second-order MASs (SOMASs) was investigated in [23], where a motion-planning approach was applied to design the control algorithm. It was demonstrated in [23] that all the system trajectories finally reached a common consensus value within a specified time under periodically time-varying and directed topologies. It is worth pointing out that non-linear terms are not considered in the system model in [23], and the proposed framework is no longer applicable for leader-following consensus problems.

Although some achievements have been made in the research on prescribed-time consensus problems of MASs, in general, the existing research results are relatively few, and most of them focus on relatively simple FOMASs. As we know, it is impossible to directly extend the conclusions of FOMASs to SOMASs, especially when the controller gains also contain time-varying terms (see explanations in Remark 3). On the other hand, many engineering applications, such as the hovercraft system described in [24], can be modelled as non-linear SOMASs with a leader-following configuration. Thus it is meaningful to design control strategies for prescribed-time leader-following consensus problems of non-linear SOMASs.

Lag consensus, which includes consensus as a special case, is widely applied to avoid congestion in practical applications (see [25–28]). In the lag consensus problem, when the states of all followers at time instant t are consistent with those of the leader at time instant $t-\tau$, and this state consistency is maintained after time t, it is said that the lag consensus is achieved and τ is called the lag time. When τ is equal to zero, it becomes a consensus problem. Sometimes, agents may be divided into several groups to carry out a task cooperatively. When they move on a capacity-limited path at the same time, for example, a narrow bridge, confusion often occurs. Then, by developing a cluster lag consensus framework which ensures that followers in different clusters achieve lag consensus at a different lag time, the congestion problem is solved. However, it is found that few studies have focused on cluster lag consensus problem of MASs, except a few literatures such as [29–31]. Furthermore, if lag consensus can be achieved within any time specified by the user, it will be more consistent with the needs of actual application scenarios. Unfortunately, there is currently no research on this subject, which motivates the present research.

Inspired by above-mentioned observations, in this work, we discuss prescribed-time lag consensus problems of SOMASs with a leader-following configuration under directed topology. A novel distributed control law is proposed, where a time-varying function related to the specified UBST is introduced. Although the value of the time-varying function in control gains tends to infinity as time approaches T (the UBST specified by the user), the control input signal of each follower is still bounded over the whole time period. The contributions could be listed as follows.

(1) Different from the existing research on lag consensus problem for MASs such as [25–28], a prescribed-time lag consensus solution is proposed for the first time in this paper so that lag consensus will be achieved in any user-specified time.

(2) The prescribed-time leader-following consensus control for SOMASs is a special case of our research, so it can also be solved by the proposed framework in this work. Moreover, compared with [32] that studies the predefined-time stability of a single second-order system and the literature [33] which investigates the prescribed-time leader-following consensus of SOMASs, the controller designed in this work has a simpler form (rather than a fractional-order state feedback form).

(3) The research results are extended to the cluster lag consensus problem by designing another proper controller for each follower, and some sufficient conditions for the MAS to achieve cluster lag consensus are also derived.

In the following, Section 2 shows some preliminaries, including descriptions of the symbols, the communication topology and the model of the MAS, and the introduction of some assumptions and lemmas. Section 3 gives the main results, that is, two control protocols that drive the MAS to achieve prescribed-time lag consensus and prescribed-time cluster lag consensus respectively. Section 4 presents the results and some analyses of the simulation examples. The conclusions and the plan for our next work are presented in Section 5.

2. Problem formulation and preliminaries

2.1. Notations

$\mathbf{1}_m$ refers to the m-element column vector with each element taking the value 1. $\mathbf{0}$ denotes the vectors or matrices of proper dimensions with all elements being 0. R denotes the real number space. $I_m \in R^{m\times m}$ represents the identity matrix. The binary operator $\otimes$ represents the Cartesian product of two matrices. The symbol $\|Q\|$ denotes the Euclidean norm of Q (when Q is a vector) or the induced 2-norm of Q (when Q is a matrix). We use $\lambda_{\max}(\mathcal{H})$ and $\lambda_{\min}(\mathcal{H})$ to represent the largest value and the smallest value among all the eigenvalues of a symmetric matrix $\mathcal{H}$, respectively.

2.2. Communication topology

Consider a MAS consisting of $n+1$ agents with a leader–follower configuration, in which n followers are indexed as $1,\ldots,n$ and the remaining one whose index is $n+1$ plays the role of leader. Let $\mathcal{V}=\{1,\ldots,n+1\}$ and $\mathcal{F}=\{k \mid k\in\mathcal{V}\setminus\{n+1\}\}$ be the agent set and the set of all followers, respectively. We use a digraph (directed graph) $G=\{\mathcal{V},\varepsilon\}$ to describe the communication topology among these $n+1$ agents with $\varepsilon=\{(k,s)\mid k\in\mathcal{V}, s\in\mathcal{F}\}$ being the set of edges. Here (k,s) is an edge of G and indicates that node s can obtain information from node k. For convenience, let $\mathcal{N}_k=\{j\in\mathcal{F}\mid (j,k)\in\varepsilon\}$ be the neighbour set of agent k.

The adjacency matrix of G, which is a $(n+1)\times(n+1)$ matrix, is used to store the connectivity information between adjacent nodes, and it can be described as $\mathcal{A}=[a_{ks}]$. $a_{ks}=0$ means that edge (s,k) does not exist in G, otherwise it implies that there exists an edge (s,k), and the value of a_{ks} represents the connection weight value of edge (s,k). $L=[l_{ks}]\in R^{(n+1)\times(n+1)}$ represents the Laplacian matrix with $l_{kk}=\sum_{s=1}^{n+1}a_{ks}$ and $l_{ks}=-a_{ks}$. The block matrix form of L is given as follow:

$$L=\begin{bmatrix} L^f & L^l \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix}, \tag{1}$$

where $L^f\in R^{n\times n}$ and $L^l\in R^{n\times 1}$.

2.3. Model description

Before giving the mathematical model of agent k, we first introduce the following symbols: $x_k(t)$, $v_k(t)$, and $u_k(t)$, which are all m-dimensional vectors, and denote the position, velocity and control input of agent k respectively. In the following, the term (t) in a symbol is sometimes omitted (for example, $x_k(t)$ is sometimes abbreviated as x_k) if there is no ambiguity. Now we consider the non-linear model of the follower k ($k \in \mathcal{F}$) being expressed as

$$\begin{cases}\dot{x}_k = v_k \\ \dot{v}_k = g(t,\ x_k,\ v_k) + u_k,\end{cases} \tag{2}$$

where $g(\cdot,\cdot,\cdot): R \times R^m \times R^m \to R^m$ describes the non-linear characteristics of the system and satisfies Assumption 1 to be given later. The leader's model has the form

$$\begin{cases}\dot{x}_{n+1} = v_{n+1} \\ \dot{v}_{n+1} = g(t,\ x_{n+1},\ v_{n+1}).\end{cases} \tag{3}$$

Remark 1. Many practical systems, for example, the forced pendulums and Chua's oscillator described in [6], present inherently non-linear dynamics, and their system dynamics can be completely determined by the non-linear characteristics, so the external input is not considered in the leader's model in this paper. For the case where there is a bounded input and disturbances in the model of the leader, an effective solution is to design controllers based upon sliding mode control schemes, as shown in [34].

2.4. Preliminaries

Some common assumptions and lemmas required for obtaining conclusions of both the lag consensus problem and the cluster lag consensus problem are listed here.

Assumption 1. For any vectors $\mathbb{y}, \mathbb{z}, \hat{\mathbb{y}}, \hat{\mathbb{z}} \in R^m$, two positive real numbers c_x, c_v can always be found to satisfy

$$\begin{aligned}&\| g(t,\ \mathbb{y},\ \mathbb{z}) - g(t,\ \hat{\mathbb{y}},\ \hat{\mathbb{z}}) \| \\ &\qquad \le c_x \|\mathbb{y} - \hat{\mathbb{y}}\| + c_v \|\mathbb{z} - \hat{\mathbb{z}}\|.\end{aligned} \tag{4}$$

Lemma 1 (*[28]*). *The following inequality holds*

$$\|\mathbb{x}\|\,\|\mathbb{v}\| \le \ell \|\mathbb{x}\|^2 + \frac{1}{4\ell}\|\mathbb{v}\|^2, \tag{5}$$

in which $\mathbb{x}$, $\mathbb{v}$ are any given vectors with proper dimensions and ℓ is a positive constant.

A time-varying function which was first introduced in [22] will be applied to our control laws, and it has the form

$$\eta(t) = \left(\frac{T}{t_0 + T - t}\right)^p, \quad t \in [t_0,\ t_0 + T), \tag{6}$$

in which $p > 1$, and $T > 0$ is a user-specified constant.

Lemma 2. *Consider a system described by*

$$\dot{q}(t) = f(t,\ q(t)), \tag{7}$$

where $q(t) \in R^m$ is the state and $f(\cdot,\cdot)$ is a vector field bounded in time. Let $V(t, q(t))$ with $V(t, 0) = 0$ be a valid Lyapunov function of (7). For simplicity, we use V to denote $V(t, q(t))$. If it holds that

$$\dot{V} \le -bV - \Bbbk\,\psi(t)V \tag{8}$$

with $b \ge 0$, $\Bbbk > 0$ being two constants and $\psi(t)$ being defined in (16), then for $t \in [t_0, t_0 + T)$, it yields

$$V \le \eta^{-\Bbbk}(t)\exp^{-b(t-t_0)} V(t_0). \tag{9}$$

Proof. Multiplying $\eta^{\Bbbk}(t)$ on both sides of (8), and noting $\eta^{\Bbbk}(t) > 0$, $\forall t \in [t_0, t_0 + T)$, one concludes that

$$\eta^{\Bbbk}(t)\dot{V} \le -\eta^{\Bbbk-1}(t)\,[b\,\eta(t) + \Bbbk\,\dot{\eta}(t)]\,V. \tag{10}$$

By deriving $\eta^{\Bbbk}(t)V$ and according to (10), it yields

$$\frac{d(\eta^{\Bbbk}(t)V)}{dt} = \eta^{\Bbbk-1}(t)\left[\eta(t)\dot{V} + \Bbbk\,\dot{\eta}(t)V\right] \le -b\,\eta^{\Bbbk}(t)V. \tag{11}$$

Then, by solving the above differential equation, we can easily get

$$\eta^{\Bbbk}(t)V \le \exp^{-b(t-t_0)} V(t_0)\,\eta^{\Bbbk}(t_0) = \exp^{-b(t-t_0)} V(t_0), \tag{12}$$

which yields (9) immediately.

Remark 2. Lemma 2 is very useful for subsequent theorem proofs. Compared with Lemma 1 in [22], we get a more general conclusion by introducing a parameter $\Bbbk$. Lemma 1 in [22] is a special case of our Lemma 2 with $\Bbbk = 2$. If we take the value of $\Bbbk$ less than 2, just as we will do in our subsequent simulations, the conservativeness of the system will be further reduced.

3. Main results

We aim to give two control laws to drive the MAS (2)–(3) to achieve lag consensus (when all followers form one group) and cluster lag consensus (when all followers are divided into at least two groups) in the prescribed time T, respectively.

3.1. Controller design for prescribed-time lag consensus

To obtain the main results for lag consensus, these additional assumptions, definitions and lemmas are needed.

Assumption 2. G has a spanning tree, and the leader node $n+1$ is the root node of this tree, that is, for each follower $k \in \mathcal{F}$, there exists an edge $(n+1, k)$ or a path formed of a series of edges $(n+1, j_1), (j_1, j_2), \ldots, (j_\kappa, j_{\kappa+1}), (j_{\kappa+1}, k)$.

Definition 1. For a user-assignable finite time T, a positive constant τ and all $k \in \mathcal{F}$, if the solution of (2)–(3) with any initial states satisfies

$$\begin{aligned}&\lim_{t\to t_0+T} \|x_k(t) - x_{n+1}(t-\tau)\| = 0, \\ &\lim_{t\to t_0+T} \|v_k(t) - v_{n+1}(t-\tau)\| = 0,\end{aligned} \tag{13}$$

and

$$\begin{aligned}&\|x_k(t) - x_{n+1}(t-\tau)\| = 0, \\ &\|v_k(t) - v_{n+1}(t-\tau)\| = 0, \quad \forall\, t \ge t_0 + T,\end{aligned} \tag{14}$$

then the prescribed-time lag consensus with the lag time τ is said to be achieved for the leader-following MAS (2)–(3).

Lemma 3 (*[35]*). *Under Assumption 2, L^f defined in (1) is a non-singular M-matrix. Further, a vector $w = (w_1, \ldots, w_n)^T$ ($w_i \in R$, $i = 1, \ldots, n$) can be found such that $L^f w = \mathbf{1}_n$. Denote $W = \mathrm{diag}\{1/w_1, \ldots, 1/w_n\}$ and $\Gamma = WL^f + \left(L^f\right)^T W$, then W and Γ are positive definite.*

To choose a controller for each follower k so that (13) and (14) hold for MAS (2)–(3) is the task of this subsection. Let $x^\tau_{n+1} \triangleq x_{n+1}(t-\tau)$ and $v^\tau_{n+1} \triangleq v_{n+1}(t-\tau)$, then the distributed control law for follower k is proposed as

$$\begin{aligned}u_k = -\psi(t)\Big[&k_1\psi(t)\sum_{s\in\mathcal{N}_k} a_{ks}\,(x_k - x_s) \\ &+ k_1\psi(t)\,a_{k(n+1)}\left(x_k - x^\tau_{n+1}\right) \\ &+ k_2\sum_{s\in\mathcal{N}_k} a_{ks}\,(v_k - v_s) \\ &+ k_2\,a_{k(n+1)}\left(v_k - v^\tau_{n+1}\right)\Big],\end{aligned} \tag{15}$$

where

$$\psi(t)=\begin{cases}\frac{\dot{\eta}(t)}{\eta(t)}, & t_0\le t<t_0+T,\\ \frac{p}{T}, & t\ge t_0+T\end{cases} \tag{16}$$

with $\eta(t)$ being defined in (6), $a_{ks}\ge 0$ is the element in weighted matrix $\mathcal{A}$, and k_1, k_2 are positive constant control gains.

The tracking errors of follower k is given by

$$\tilde{x}_k=x_k-x_{n+1}^{\tau},\ \ \tilde{v}_k=v_k-v_{n+1}^{\tau}. \tag{17}$$

In order to eliminate the cross terms in the derivative of subsequent Lyapunov function, an important state transformation is introduced as

$$\breve{x}_k=\psi(t)\tilde{x}_k,\ \ \ \breve{v}_k=\tilde{v}_k. \tag{18}$$

Taking the derivatives of $\breve{x}_k$ and $\breve{v}_k$ along (2)–(3), we get

$$\begin{aligned}\dot{\breve{x}}_k&=\psi(t)\breve{v}_k+\breve{\psi}(t)\breve{x}_k,\\ \dot{\breve{v}}_k&=g(t,\ x_k,\ v_k)-g(t,\ x_{n+1}^{\tau},\ v_{n+1}^{\tau})\\ &\quad-\ k_1\psi(t)\sum_{s=1}^{n}l_{ks}\breve{x}_s-k_2\,\psi(t)\sum_{s=1}^{n}l_{ks}\,\breve{v}_s,\end{aligned} \tag{19}$$

where

$$\breve{\psi}(t)=\begin{cases}\frac{\psi(t)}{p}, & t_0\le t<t_0+T,\\ 0, & t\ge t_0+T.\end{cases} \tag{20}$$

Let $\breve{x}$ and $\breve{v}$ be the stack vector of $\breve{x}_1,\dots,\breve{x}_n$ and $\breve{v}_1,\dots,\breve{v}_n$ respectively. In this paper, for a set of vectors $z_1,\dots,z_n$, the stack vector composed of them is defined as $[z_1^T,\dots,z_n^T]^T$. Then the compact form of (19) can be rewritten as follows:

$$\begin{aligned}\dot{\breve{x}}&=\psi(t)\,\breve{v}+\breve{\psi}(t)\breve{x},\\ \dot{\breve{v}}&=\mathscr{F}(t)-\psi(t)\left(L^f\otimes I_m\right)\left(k_1\breve{x}+k_2\breve{v}\right),\end{aligned} \tag{21}$$

where $\mathscr{F}(t)=\mathcal{F}(t)-\mathbf{1}_n\otimes g(t,\ x_{n+1}^{\tau},\ v_{n+1}^{\tau})$, $\mathcal{F}(t)$ is the stack vector of $g(t,\ x_1,\ v_1),\ \dots,\ g(t,\ x_n,\ v_n)$, and x, v are the stack vector of $x_1,\dots,x_n$ and $v_1,\dots,v_n$ respectively.

Now, the sufficient condition to ensure prescribed-time lag consensus for MAS (2)–(3) is given in the following theorem.

Theorem 1. *Assume Assumptions 1–2 hold. For a given finite time T, under the controller (15), the prescribed-time lag consensus for the leader-following MAS (2)–(3) can be achieved within time T, if there exist parameters $\Bbbk\ge\frac{2}{p}$, $b\ge 0$, $\beta>0$, $\gamma>0$, and $\ell>0$ which satisfy the following conditions:*

$$2\beta^2-\gamma\,(k_1\gamma+k_2\beta)\,\lambda_{\min}\,(\,\Gamma\,)\,w_{\min}<0, \tag{22}$$

$$\left(-\frac{k_1\beta}{2}+\Delta_1\right)w_{\min}\,\lambda_{\min}(\,\Gamma\,)+\frac{\beta\ell}{p}+\Bbbk\beta\ell+\frac{T}{p}\,\Delta_2<0, \tag{23}$$

$$-\frac{k_2\gamma}{2}w_{\min}\lambda_{\min}(\,\Gamma\,)+\beta+\frac{\Bbbk\gamma}{2}+\frac{(\beta+\Bbbk\beta p)}{4\ell p}+\frac{T}{p}\,\Delta_4<0, \tag{24}$$

where

$$\begin{aligned}\Delta_1&=\frac{(k_1\gamma+k_2\beta)(\Bbbk p+bT+2)}{4p},\\ \Delta_2&=\frac{T}{p}\,\beta c_x+\Delta_3+\ell\beta b,\\ \Delta_3&=\frac{T\gamma\ell c_x+p\beta\ell c_v}{p},\\ \Delta_4&=\gamma C_v+\frac{\Delta_3+\beta b\ell}{4\ell^2}+\frac{b\gamma}{2},\end{aligned} \tag{25}$$

Γ is defined in Lemma 3, and $w_{\min}=\min\{w_i|i=1,\ \dots,n\}$ with w_i being defined in Lemma 3. Furthermore, for each follower k, the input u_k is bounded.

Proof. Let $\delta=[\breve{x}^T,\ \breve{v}^T]^T$. The Lyapunov function is taken as

$$V=\frac{1}{2}\,\delta^T\,(\Omega\otimes I_m)\,\delta, \tag{26}$$

where $\Omega=\begin{bmatrix}\frac{k_1\gamma+k_2\beta}{2}\,\Gamma & \beta W\\ \beta W & \gamma W\end{bmatrix}$, and W is a positive definite matrix defined in Lemma 3.

First of all, we need to ensure the validity of $V(t)$. According to Schur's Complement Lemma, $\Omega>0$ iff the followings hold: (i) $\gamma W>0$, (ii) $\frac{k_1\gamma+k_2\beta}{2}\Gamma-\beta W(\gamma W)^{-1}\beta W>0$. (i) is obviously true. Moreover, if condition (22) is satisfied, then (ii) holds. Thus V is valid under the constraint (22).

Next, we prove that the lag consensus for MAS (2)–(3) can be achieved with the lag time τ in the prescribed time T.

Consider the time interval $[t_0,\ t_0+T)$. In this case, $\breve{\psi}(t)=\frac{1}{p}\psi(t)$. Calculating the derivative of V given in (26) along (21), one has

$$\begin{aligned}\dot{V}=&\frac{k_1\gamma+k_2\beta}{2}\,\psi(t)\breve{x}^T\,(\,\Gamma\otimes I_m)\,\breve{v}\\ &+\beta\,\psi(t)\,\breve{v}^T\,(W\otimes I_m)\,\breve{v}\\ &+\frac{\beta}{p}\,\psi(t)\,\breve{v}^T\,(W\otimes I_m)\,\breve{x}\\ &+\frac{k_1\gamma+k_2\beta}{2p}\,\psi(t)\breve{x}^T\,(\,\Gamma\otimes I_m)\,\breve{x}\\ &-\psi(t)\left(\beta\breve{x}^T+\gamma\breve{v}^T\right)(W\otimes I_m)\,\big(L^f\\ &\otimes I_m\,\big)\left(k_1\breve{x}+k_2\breve{v}\right)\\ &+\left(\beta\breve{x}^T+\gamma\breve{v}^T\right)(W\otimes I_m)\,\mathscr{F}(t).\end{aligned} \tag{27}$$

Based on Assumption 1, Lemma 1 and the fact $\frac{1}{\psi(t)}\le\frac{T}{p}$, one has

$$\begin{aligned}&\left(\,\beta\breve{x}^T+\gamma\breve{v}^T\,\right)\left(W\otimes I_m\right)\mathscr{F}(t)\\ &=\sum_{k=1}^{n}\frac{1}{w_k}\left(\beta\breve{x}_k^T+\gamma\breve{v}_k^T\right)\left[\ g(t,\ x_k,\ v_k)\right.\\ &\quad\left.-\ g(t,\ x_{n+1}^{\tau},\ v_{n+1}^{\tau})\ \right]\\ &\le\sum_{k=1}^{n}\frac{1}{w_k}\left[\ \frac{\beta c_x}{\psi(t)}\left\|\breve{x}_k\right\|^2+\gamma c_v\left\|\breve{v}_k\right\|^2\right.\\ &\quad\left.+\left(\frac{\gamma c_x}{\psi(t)}+\beta c_v\right)\left\|\breve{x}_k\right\|\left\|\breve{v}_k\right\|\ \right]\\ &\le\left(\frac{T}{p}\,\beta c_x+\Delta_3\right)\breve{x}^T\,(W\otimes I_m)\,\breve{x}\\ &\quad+\left(\gamma c_v+\frac{\Delta_3}{4\ell^2}\right)\breve{v}^T\,(W\otimes I_m)\,\breve{v}.\end{aligned} \tag{28}$$

By simple calculations, we obtain

$$\begin{aligned}&\left(\beta\breve{x}^T+\gamma\breve{v}^T\right)(W\otimes I_m)\left(L^f\otimes I_m\right)\left(k_1\breve{x}+k_2\breve{v}\right)\\ &=k_1\beta\,\breve{x}^T\left[\frac{1}{2}\left(WL^f+(L^f)^T\,W\right)\otimes I_m\right]\breve{x}\\ &\quad+k_2\gamma\,\breve{v}^T\left[\frac{1}{2}\left(WL^f+(L^f)^T\,W\right)\otimes I_m\right]\breve{v}\\ &\quad+\frac{k_1\gamma+k_2\beta}{2}\,\breve{x}^T\left[\left(WL^f+(L^f)^T\,W\right)\otimes I_m\right]\breve{v}\\ &=\frac{k_1\beta}{2}\,\breve{x}^T\,(\,\Gamma\otimes I_m)\,\breve{x}+\frac{k_2\gamma}{2}\,\breve{v}^T\,(\,\Gamma\otimes I_m)\,\breve{v}\\ &\quad+\frac{k_1\gamma+k_2\beta}{2}\,\breve{x}^T\,(\,\Gamma\otimes I_m)\,\breve{v}\end{aligned} \tag{29}$$

and

$$\begin{aligned}\breve{x}^T\,(\varGamma\otimes I_m)\,\breve{x} &\geq \lambda_{\min}(\varGamma)\sum_{k=1}^{n}\|\breve{x}_k\|^2\\ &\geq \lambda_{\min}(\varGamma)\,w_{\min}\sum_{k=1}^{n}\frac{1}{w_k}\|\breve{x}_k\|^2\\ &= \lambda_{\min}(\varGamma)\,w_{\min}\,\breve{x}^T\,(W\otimes I_m)\,\breve{x}.\end{aligned}\tag{30}$$

Similarly, one gets

$$\begin{aligned}&\breve{v}^T\,(\varGamma\otimes I_m)\,\breve{v}\\ &\quad\geq \lambda_{\min}(\varGamma)\,w_{\min}\,\breve{v}^T\,(W\otimes I_m)\,\breve{v}.\end{aligned}\tag{31}$$

With the help of Lemma 1, we get

$$\begin{aligned}&\breve{v}^T\,(W\otimes I_m)\,\breve{x}\\ &=\sum_{k=1}^{n}\frac{1}{w_k}\breve{v}_k^T\,\breve{x}_k\\ &\leq\sum_{k=1}^{n}\frac{1}{w_k}\left(\ell\|\breve{x}_k\|^2+\frac{1}{4\ell}\|\breve{v}_k\|^2\right)\\ &=\ell\breve{x}^T\,(W\otimes I_m)\,\breve{x}+\frac{1}{4\ell}\breve{v}^T\,(W\otimes I_m)\,\breve{v}.\end{aligned}\tag{32}$$

In order to get the inequality (8) in Lemma 2, we consider the item $H(t)=\dot{V}+(\Bbbk\psi(t)+b)V$. Based on (27)–(32), we get

$$\begin{aligned}H(t)\leq&\left[\frac{-2k_1\beta p+(k_1\gamma+k_2\beta)(\Bbbk p+bT+2)}{4p}w_{\min}\lambda_{\min}(\varGamma)\right.\\ &+\frac{\beta\ell}{p}+\Bbbk\beta\ell+\frac{T}{p}\left(\frac{T}{p}\beta c_x+\varDelta_3\right.\\ &\left.\left.+\ell\beta b\right)\right]\psi(t)\breve{x}^T\,(W\otimes I_m)\,\breve{x}\\ &+\left[-\frac{k_2\gamma}{2}w_{\min}\lambda_{\min}(\varGamma)+\beta+\frac{\Bbbk\gamma}{2}\right.\\ &+\frac{\beta+\Bbbk\beta p}{4\ell p}+\frac{T}{p}\left(\gamma C_v+\frac{\varDelta_3+\beta b\ell}{4\ell^2}\right.\\ &\left.\left.+\frac{b\gamma}{2}\right)\right]\psi(t)\,\breve{v}^T\,(W\otimes I_m)\,\breve{v}.\end{aligned}\tag{33}$$

If conditions (23) and (24) are satisfied, then we can get $H(t)\leq 0$, that is

$$\dot{V}\leq-(\Bbbk\,\psi(t)+b)\,V.\tag{34}$$

Thus, based on Lemma 2, one gets

$$V\leq\eta^{-\Bbbk}(t)\exp^{-b(t-t_0)}V(t_0).\tag{35}$$

Then the following inequality holds

$$\begin{aligned}\|\breve{x}\|^2+\|\breve{v}\|^2&=\psi^2(t)\left\|\tilde{x}\right\|^2+\|\tilde{v}\|^2\\ &\geq h\left(\|\tilde{x}\|^2+\|\tilde{v}\|^2\right),\end{aligned}\tag{36}$$

where $h=\min\{p^2/T^2,1\}$. Note that $\psi(t)\geq\frac{p}{T}$ is used to deal with the above inequality. Moreover, based on (35), one has

$$\begin{aligned}&\|\breve{x}\|^2+\|\breve{v}\|^2\\ &\quad\leq\frac{1}{\lambda_{\min}(\varOmega)}\eta^{-\Bbbk}(t)\exp^{-b(t-t_0)}V(t_0).\end{aligned}\tag{37}$$

By considering inequalities (36) and (37), we get

$$\left(\|\tilde{x}(t)\|^2+\|\tilde{v}(t)\|^2\right)\leq\frac{1}{h\lambda_{\min}(\varOmega)}\eta^{-\Bbbk}(t)\exp^{-b(t-t_0)}V(t_0).\tag{38}$$

Thus, when $t\to t_0+T$, it yields $\|\tilde{x}(t)\|\to 0$ and $\|\tilde{v}(t)\|\to 0$ by noting $\lim_{t\to t_0+T}\eta^{-\Bbbk}(t)=0$, that is, (13) in Definition 1 holds.

For $t\geq t_0+T$, by following the steps in (27)–(33), it yields $\dot{V}\leq-\Bbbk\psi(t)V-bV=-(kp/T+b)V\leq 0$ if conditions (23)–(24) are all satisfied. By noting that V is continuous, we have

$$0\leq V(t)\leq V(t_0+T)=0,\tag{39}$$

that is, $V(t)\equiv 0$ for all $t\geq t_0+T$. Hence, (14) in Definition 1 holds.

We have now proved that both (13) and (14) hold, thus the prescribed-time lag consensus for the leader-following MAS (2)–(3) is achieved according to Definition 1.

Next, we need to show that the input is bounded. From (15), (17) and (18), the compact form of u_k is given by

$$U=-\psi(t)\left(L^f\otimes I_m\right)\left(k_1\breve{x}+k_2\breve{v}\right).\tag{40}$$

For $t\in[t_0,t_0+T)$, based on Lemma 1 and the inequality (37), we get

$$\begin{aligned}&\left\|k_1\breve{x}+k_2\breve{v}\right\|\\ &\leq\left[\left(k_1^2+k_1k_2\right)\|\breve{x}\|^2+\left(k_2^2+k_1k_2\right)\|\breve{v}\|^2\right]^{\frac{1}{2}}\\ &\leq\max\left\{\sqrt{k_1^2+k_1k_2},\sqrt{k_2^2+k_1k_2}\right\}\left(\|\breve{x}\|^2+\|\breve{v}\|^2\right)^{\frac{1}{2}}\\ &\leq m\,\eta^{-\frac{1}{2}\Bbbk}\exp^{-\frac{b}{2}(t-t_0)}V^{\frac{1}{2}}(t_0),\end{aligned}\tag{41}$$

where $m=\max\left\{\sqrt{k_1^2+k_1k_2},\sqrt{k_2^2+k_1k_2}\right\}\lambda_{\min}(\varOmega)^{-\frac{1}{2}}$. It is known that $0<\eta^{-\frac{1}{2}\Bbbk},\exp^{-\frac{b}{2}(t-t_0)}\leq 1$ and

$$\begin{aligned}\psi(t)\,\eta^{-\frac{1}{2}\Bbbk}&=\frac{p}{t_0+T-t}\frac{(t_0+T-t)^{\frac{1}{2}p\Bbbk}}{T^{\frac{1}{2}p\Bbbk}}\\ &=\frac{p}{T}\left(\frac{T+t_0-t}{T}\right)^{\frac{p\Bbbk-2}{2}}\leq\frac{p}{T},\end{aligned}\tag{42}$$

where the last inequality holds when $p\Bbbk\geq 2$. Then, combining (40), (41) and (42), it yields

$$\begin{aligned}\|U\|&\leq\psi(t)\left\|L^f\otimes I_m\right\|\left\|k_1\breve{x}+k_2\breve{v}\right\|\\ &\leq m\left\|L^f\otimes I_m\right\|\psi(t)\,\eta^{-\frac{1}{2}\Bbbk}\exp^{-\frac{b}{2}(t-t_0)}V^{\frac{1}{2}}(t_0)\\ &\leq\frac{mp}{T}\left\|L^f\otimes I_m\right\|V^{\frac{1}{2}}(t_0).\end{aligned}\tag{43}$$

For $t\geq t_0+T$, we have proved that $\breve{x}(t)=0$, $\breve{v}(t)=0$ hold, and then $\mathrm{U}\equiv 0$ holds by observing from (40), which together with (43) demonstrates that the input for each follower is bounded.

Remark 3. We borrow the idea from literature [22] to obtain our conclusions. Nevertheless, it is not easy to extend the results for FOMASs to SOMASs. First, how to design time-varying gains for the feedback control items of velocity states is a challenging issue. Second, due to the existence of the time-varying term in the controller, the cross term (the product of position state and velocity state) in the derivative of Lyapunov function cannot be directly eliminated. In this work, by considering the characteristics of function $\eta(t)$, a suitable controller is designed for the second-order MAS. It is noticed that (15) contains both the time-varying term $\psi(t)$ and $\psi^2(t)$, which results in technical difficulties when we deal with the cross term in the derivative of Lyapunov function. However, with the help of the state transformation (18), we get (19) which contains only the term $\psi(t)$, so that it can be easily processed.

Remark 4. It is noticed that $\psi(t)$ in the controller (15) is time-varying and $\psi(t)\to\infty$ when t approaches t_0+T. Then, will the value of (15) at time t_0+T be bounded? In the above theorem, we have proved that in the whole time interval (including the time instant t_0+T), the input control signal of each follower is bounded, so the controller we designed is effective.

3.2. Controller design for prescribed-time cluster lag consensus

Assume that all follower nodes in G are divided into r $(r>1)$ groups, and let $\{q_1, q_2, \ldots, q_r\}$ be a partition of the follower set $\mathcal{F}$ with $q_1=\{1,2,\ldots,s_1\}$, $q_2=\{s_1+1, s_1+2, \ldots, s_2\}$, …, $q_r=\{s_{r-1}+1, s_{r-1}+2, \ldots, s_r\}$ and $s_r=n$. We denote $\mathcal{C}\triangleq\{1,2,\ldots,r\}$, then for $\hat{i}\in\mathcal{C}$, the number of agents in cluster $q_{\hat{i}}$ is given by $n_{\hat{i}}=s_{\hat{i}}-s_{\hat{i}-1}$ with $s_0=0$. Then L^f in (1) can be expressed as

$$L^f=\begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1r}\\ L_{21} & L_{22} & \cdots & L_{2r}\\ \vdots & \vdots & \ddots & \vdots\\ L_{r1} & L_{r2} & \cdots & L_{rr}\end{bmatrix}, \tag{44}$$

where $L_{\hat{i}\hat{i}}\in R^{n_{\hat{i}}\times n_{\hat{i}}}$ and $L_{\hat{i}\hat{j}}\in R^{n_{\hat{i}}\times n_{\hat{j}}}$ with $\hat{i},\hat{j}=1,2,\ldots,r$.

Define the r subgraphs $G_{\hat{i}}$ $(\hat{i}=1,\ldots,r)$ of graph G as $G_{\hat{i}}=\{\mathcal{V}_{\hat{i}}, \varepsilon_{\hat{i}}\}$, where $\mathcal{V}_{\hat{i}}=q_{\hat{i}}\cup\{n+1\}$ and $\varepsilon_{\hat{i}}$ is the corresponding edge set.

Assumption 3. For each subgraph $G_{\hat{i}}$, $\hat{i}=1,\ldots,r$, there is a spanning tree rooted at $n+1$ (the leader node).

Assumption 4. $\sum_{s\in q_{\hat{j}}} a_{ks}=0$, for all $k\in\mathcal{F}\setminus q_{\hat{j}}$; $\hat{j}\in\mathcal{C}$.

Remark 5. In the above discussion on lag consensus in Section 3.1, a_{ks} always takes a non-negative value. Under Assumption 4, a_{ks} could be positive or negative. a_{ks} with a positive value which is called a cooperative coupling can increase consensus ability, whereas a_{ks} with a negative value which is called a competitive coupling will decrease consensus ability (see [29,36,37]).

The goal is to design suitable control protocols to ensure that the agents in each group $q_{\hat{i}}$ lag behind the leader at time $\tau_{\hat{i}}$ within the specified time T. Here, the lag time of different groups relative to the leader can be different. Let $x_{n+1}^{\tau_{\hat{i}}}(t)\triangleq x_{n+1}(t-\tau_{\hat{i}})$ and $v_{n+1}^{\tau_{\hat{i}}}(t)\triangleq v_{n+1}(t-\tau_{\hat{i}})$. The definition of the prescribed-time cluster lag consensus of MAS (2)–(3) is given below.

Definition 2. Let $\tau_{\hat{i}}$ $(\hat{i}=1,\ldots,r)$ with $0\le\tau_1<\ldots<\tau_r$ be the corresponding lag time for cluster $q_{\hat{i}}$, then for follower $k\in q_{\hat{i}}$ and a user-prescribed time T, if the solutions of (2) and (3) with any initial states satisfy:

$$\begin{aligned}&\lim_{t\to t_0+T}\|x_k(t)-x_{n+1}^{\tau_{\hat{i}}}(t)\|=0,\\ &\lim_{t\to t_0+T}\|v_k(t)-v_{n+1}^{\tau_{\hat{i}}}(t)\|=0,\end{aligned} \tag{45}$$

and

$$\begin{aligned}&\|x_k(t)-x_{n+1}^{\tau_{\hat{i}}}(t)\|=0,\\ &\|v_k(t)-v_{n+1}^{\tau_{\hat{i}}}(t)\|=0,\ \forall t\ge t_0+T,\end{aligned} \tag{46}$$

then the prescribed-time cluster lag consensus with the UBST T is said to be achieved for the MAS (2)–(3).

Lemma 4 ([29]). *If Assumption 3 holds, then for $\hat{i}=1,\ldots,r$, the matrix $L_{\hat{i}\hat{i}}$ in (44) is nonsingular, and there exists a vector $\hat{w}_{\hat{i}}=[w_{\hat{i}1},\ldots,w_{\hat{i}n_{\hat{i}}}]^T=L_{\hat{i}\hat{i}}^{-1}\mathbf{1}_{n_{\hat{i}}}$, such that $\widehat{W}_{\hat{i}}=\mathrm{diag}\{1/w_{\hat{i}1},\ldots,1/w_{\hat{i}n_{\hat{i}}}\}$ and $\widehat{A}_{\hat{i}}=\widehat{W}_{\hat{i}}L_{\hat{i}\hat{i}}+L_{\hat{i}\hat{i}}^T\widehat{W}_{\hat{i}}$ are positive definite.*

Lemma 5 ([37]). *Let $\lambda_1(\mathscr{P}),\ldots,\lambda_M(\mathscr{P})$ and $\lambda_1(\mathscr{Q}),\ldots,\lambda_M(\mathscr{Q})$ be the eigenvalues of matrices $\mathscr{P}$ and $\mathscr{Q}$ in ascending order, respectively. Let $\lambda_1(\mathscr{C})\le\ldots\le\lambda_M(\mathscr{C})$ be the ordered eigenvalues of $\mathscr{C}=\mathscr{P}+\mathscr{Q}$. Then the following inequalities hold:*

$$\lambda_1(\mathscr{C})\ge\lambda_1(\mathscr{P})+\lambda_1(\mathscr{Q}),\ \ \lambda_M(\mathscr{C})\le\lambda_M(\mathscr{P})+\lambda_M(\mathscr{Q}). \tag{47}$$

Now, we propose the following control law for the follower $k\in q_{\hat{i}}$ $(k\in\mathcal{F},\hat{i}\in\mathcal{C})$:

$$\begin{aligned}u_k=&-\psi^2(t)\Big[\bar{k}_{\hat{i}}\sum_{j\in q_{\hat{i}}}a_{ks}(x_k-x_s)\\ &\quad+\bar{k}_{\hat{i}}a_{k(n+1)}\left(x_k-x_{n+1}^{\tau_{\hat{i}}}\right)\\ &\quad+\sum_{s\in\mathcal{N}_k\setminus q_{\hat{i}}}a_{ks}(x_k-x_s)\Big]\\ &-\psi(t)\Big[\bar{k}_{\hat{i}}\sum_{s\in q_{\hat{i}}}a_{ks}(v_k-v_s)\\ &\quad+\bar{k}_{\hat{i}}a_{k(n+1)}\left(v_k-v_{n+1}^{\tau_{\hat{i}}}\right)\\ &\quad+\sum_{s\in\mathcal{N}_k\setminus q_{\hat{i}}}a_{ks}(v_k-v_s)\Big],\end{aligned} \tag{48}$$

where $\bar{k}_{\hat{i}}$ is a parameter to be designed later, $\tau_{\hat{i}}$ is the lag time of cluster $q_{\hat{i}}$, and $\psi(t)$ is defined in (16).

By Assumption 4, it follows that

$$\begin{aligned}&\bar{k}_{\hat{i}}\sum_{s\in q_{\hat{i}}}a_{ks}(x_k-x_s)+\bar{k}_{\hat{i}}a_{k(n+1)}\left(x_k-x_{n+1}^{\tau_{\hat{i}}}\right)\\ &\quad+\sum_{s\in\mathcal{N}_k\setminus q_{\hat{i}}}a_{ks}(x_k-x_s)\\ &=\bar{k}_{\hat{i}}\sum_{s\in q_{\hat{i}}}l_{ks}x_s+\bar{k}_{\hat{i}}l_{k(n+1)}x_{n+1}^{\tau_{\hat{i}}}+\sum_{s\in\mathcal{N}_k\setminus q_{\hat{i}}}l_{ks}x_s\\ &=\bar{k}_{\hat{i}}\sum_{s\in q_{\hat{i}}}l_{ks}\left(e_{sx}+x_{n+1}^{\tau_{\hat{i}}}\right)+\bar{k}_{\hat{i}}l_{k(n+1)}x_{n+1}^{\tau_{\hat{i}}}\\ &\quad+\sum_{i=1,i\neq\hat{i}}^{r}\sum_{s\in q_i}l_{ks}\left(e_{sx}+x_{n+1}^{\tau_i}\right)\\ &=\bar{k}_{\hat{i}}\sum_{s\in q_{\hat{i}}}l_{ks}e_{sx}+\sum_{s\in\mathcal{N}_k\setminus q_{\hat{i}}}l_{ks}e_{sx}\end{aligned} \tag{49}$$

and

$$\begin{aligned}&\bar{k}_{\hat{i}}\sum_{s\in q_{\hat{i}}}a_{ks}(v_k-v_s)+\bar{k}_{\hat{i}}a_{k(n+1)}\left(v_k-v_{n+1}^{\tau_{\hat{i}}}\right)\\ &\quad+\sum_{s\in\mathcal{N}_k\setminus q_{\hat{i}}}a_{ks}(v_k-v_s)\\ &=\bar{k}_{\hat{i}}\sum_{s\in q_{\hat{i}}}l_{ks}e_{sv}+\sum_{s\in\mathcal{N}_k\setminus q_{\hat{i}}}l_{ks}e_{sv},\end{aligned} \tag{50}$$

where $e_{kx}=x_k-x_{n+1}^{\tau_{\hat{i}}}$ and $e_{kv}=v_k-v_{n+1}^{\tau_{\hat{i}}}$ for $k\in q_{\hat{i}}$.

Next, we make the following transformation:

$$\bar{e}_{kx}=\psi(t)e_{kx},\ \ \bar{e}_{kv}=e_{kv}. \tag{51}$$

Let $\bar{e}_x$ and $\bar{e}_v$ be the stack vector of $\bar{e}_{1x},\ldots,\bar{e}_{nx}$ and $\bar{e}_{1v},\ldots,\bar{e}_{nv}$ respectively, $\hat{\delta}=(\bar{e}_x^T,\bar{e}_v^T)^T$, and

$$\hat{L}^f=\begin{bmatrix}\bar{k}_1L_{11} & L_{12} & \cdots & L_{1r}\\ L_{21} & \bar{k}_2L_{22} & \cdots & L_{2r}\\ \vdots & \vdots & \ddots & \vdots\\ L_{r1} & L_{r2} & \cdots & \bar{k}_rL_{rr}\end{bmatrix}. \tag{52}$$

Then, one has

$$\dot{\hat{\delta}} = \begin{bmatrix} \breve{\psi}(t) I_n & \psi(t) I_n \\ -\psi(t)\hat{L}^f & -\psi(t)\hat{L}^f \end{bmatrix} \otimes I_m \hat{\delta} + \begin{bmatrix} 0 \\ F(t) \end{bmatrix}, \tag{53}$$

in which $F(t) = \bar{F}(t) - \tilde{G}(t)$, $\bar{F}(t)$ and $\tilde{G}(t)$ are the stack vectors composed of $g(t, x_1, v_1), \ldots, g(t, x_n, v_n)$ and $\tilde{G}_1(t), \ldots, \tilde{G}_r(t)$, respectively, and $\tilde{G}_{\hat{i}}(t) = \mathbf{1}_{s_{\hat{i}}-s_{\hat{i}-1}} \otimes g(t, x_{n+1}^{\tau_{\hat{i}}}, v_{n+1}^{\tau_{\hat{i}}})$ with $\hat{i} = 1, \ldots, r$.

Theorem 2. *Consider the MAS (2)–(3) with the proposed controller (48). Under Assumptions 1, 3–4, for a given finite time T, the prescribed-time cluster lag consensus for the MAS (2)–(3) can be achieved within time T, if for all $\hat{i} \in \mathcal{C}$, there exist constants $\bar{\Bbbk} \geq \frac{2}{p}$, $\bar{b} \geq 0$ and $\ell > 0$ which satisfy the following conditions:*

$$-\bar{k}_{\hat{i}}\,\lambda_{\min}(\Phi_{\hat{i}}) - \lambda_{\min}(\Pi) + \lambda_{\max}(R) < 0, \tag{54}$$

$$\bar{\Delta}_1\left(\bar{k}_{\hat{i}}\,\lambda_{\min}(\Phi_{\hat{i}}) + \lambda_{\min}(\Pi)\right) + \bar{\Delta}_2\lambda_{\max}(R) < 0, \tag{55}$$

$$-\bar{k}_{\hat{i}}\,\lambda_{\min}(\Phi_{\hat{i}}) - \lambda_{\min}(\Pi) + \bar{\Delta}_3\lambda_{\max}(R) < 0, \tag{56}$$

where

$$\begin{aligned}
\bar{\Delta}_1 &= -1 + \frac{2+\bar{b}T}{p} + \bar{\Bbbk}, \\
\bar{\Delta}_2 &= 2\bar{\Bbbk}\ell + \frac{2\ell}{p} + \frac{2T}{p}\left(\omega_1 + \bar{b}\ell\right), \\
\bar{\Delta}_3 &= 2 + \frac{1}{2p\ell} + \frac{\bar{\Bbbk}(1+2\ell)}{2\ell} + \frac{2T}{p}\left(1 + \omega_2 + \frac{\bar{b}}{2} + \frac{\bar{b}}{4\ell}\right), \\
\Phi_{\hat{i}} &= \widehat{W}_{\hat{i}} L_{\hat{i}\hat{i}} + L_{\hat{i}\hat{i}}^T \widehat{W}_{\hat{i}}, \\
\Pi &= R\hat{L}_h^f + \left(\hat{L}_h^f\right)^T R, \\
R &= \mathrm{diag}\left\{\widehat{W}_1, \widehat{W}_2, \ldots, \widehat{W}_r\right\}, \\
\hat{L}_h^f &= \hat{L}^f - \mathrm{diag}\left\{\bar{l}_1 L_{11}, \bar{l}_2 L_{22}, \ldots, \bar{l}_r L_{rr}\right\}
\end{aligned} \tag{57}$$

with $\omega_1 = \ell c_v + \frac{Tc_x}{p}(1+\ell)$ and $\omega_2 = c_v + \frac{Tc_x + pc_v}{4p\ell}$. Moreover, the input u_k is bounded.

Proof. Consider the following candidate Lyapunov function

$$V = \frac{1}{2}\hat{\delta}^T \left\{ \begin{bmatrix} R\hat{L}^f + (\hat{L}^f)^T R & R \\ R & R \end{bmatrix} \otimes I_m \right\} \hat{\delta}. \tag{58}$$

If the condition (54) is satisfied, we can get $R\hat{L}^f + \left(\hat{L}^f\right)^T R - R > 0$, and then $V > 0$ is guaranteed.

For $t \in [t_0, t_0 + T)$, calculating the derivative of V along (53) yields

$$\begin{aligned}
\dot{V} = {}& \psi(t)\,\hat{\delta}^T \left\{ \begin{bmatrix} R\hat{L}^f + (\hat{L}^f)^T R & R \\ R & R \end{bmatrix} \right. \\
& \left. \begin{bmatrix} \frac{1}{p} I_n & I_n \\ -\hat{L}^f & -\hat{L}^f \end{bmatrix} \otimes I_m \right\} \hat{\delta} \\
& + \hat{\delta}^T \left\{ \begin{bmatrix} R\hat{L}^f + (\hat{L}^f)^T R & R \\ R & R \end{bmatrix} \otimes I_m \right\} \begin{bmatrix} 0 \\ F(t) \end{bmatrix}.
\end{aligned} \tag{59}$$

Under Assumption 1, by applying Lemma 1 and the fact $\frac{1}{\psi(t)} \leq \frac{T}{p}$, one has

$$\begin{aligned}
&\left(\bar{e}^T_{(s_{\hat{i}-1}+1)x}, \ \bar{e}^T_{(s_{\hat{i}-2}+1)x}, \ldots, \ \bar{e}^T_{s_{\hat{i}}x} \right) \left(\widehat{W}_{\hat{i}} \otimes I_{n_{\hat{i}}} \right) F_{\hat{i}}(t) \\
&= \sum_{l=1}^{n_{\hat{i}}} \frac{1}{w_{\hat{i}l}} \bar{e}^T_{(s_{\hat{i}-1}+l)x} \left(g(t, x_{(s_{\hat{i}-1}+l)}, v_{(s_{\hat{i}-1}+l)}) \right. \\
&\quad \left. - g(t, x^{\tau_{\hat{i}}}_{n+1}, v^{\tau_{\hat{i}}}_{n+1}) \right) \\
&\leq \left(\frac{Tc_x}{p} + \ell c_v \right) \sum_{l=1}^{n_{\hat{i}}} \frac{1}{w_{\hat{i}l}} \left\| \bar{e}^T_{(s_{\hat{i}-1}+l)x} \right\|^2 \\
&\quad + \frac{c_v}{4\ell} \sum_{l=1}^{n_{\hat{i}}} \frac{1}{w_{\hat{i}l}} \left\| \bar{e}^T_{(s_{\hat{i}-1}+l)v} \right\|^2
\end{aligned} \tag{60}$$

and

$$\begin{aligned}
&\left(\bar{e}^T_{(s_{\hat{i}-1}+1)v}, \ \bar{e}^T_{(s_{\hat{i}-2}+1)v}, \ldots, \ \bar{e}^T_{s_{\hat{i}}v} \right) \left(\widehat{W}_{\hat{i}} \otimes I_{n_{\hat{i}}} \right) F_{\hat{i}}(t) \\
&\quad \leq \frac{T\ell c_x}{p} \sum_{l=1}^{n_{\hat{i}}} \frac{1}{w_{\hat{i}l}} \left\| \bar{e}^T_{(s_{\hat{i}-1}+l)x} \right\|^2 \\
&\qquad + \left(\frac{Tc_x}{4p\ell} + c_v \right) \sum_{l=1}^{n_{\hat{i}}} \frac{1}{w_{\hat{i}l}} \left\| \bar{e}^T_{(s_{\hat{i}-1}+l)v} \right\|^2.
\end{aligned} \tag{61}$$

Hence, we have

$$\begin{aligned}
&\hat{\delta}^T \left\{ \begin{bmatrix} R\hat{L}^f + (\hat{L}^f)^T R & R \\ R & R \end{bmatrix} \otimes I_m \right\} \begin{bmatrix} 0 \\ F(t) \end{bmatrix} \\
&\leq \hat{\delta}^T \left\{ \begin{bmatrix} \omega_1 R & 0 \\ 0 & \omega_2 R \end{bmatrix} \otimes I_m \right\} \hat{\delta}.
\end{aligned} \tag{62}$$

Let $\bar{H}(t) = \dot{V} + (\bar{\Bbbk}\psi(t) + \bar{b})V$. With some calculations, one has

$$\bar{H}(t) \leq \frac{1}{2}\psi(t)\,\hat{\delta}^T \left\{ \begin{bmatrix} \Upsilon_1 & 0 \\ 0 & \Upsilon_2 \end{bmatrix} \otimes I_m \right\} \hat{\delta}, \tag{63}$$

where

$$\begin{aligned}
\Upsilon_1 = {}& \left(-1 + \frac{2+\bar{b}T}{p} + \bar{\Bbbk} \right) \left(R\hat{L}^f + (\hat{L}^f)^T R \right) \\
& + \left[2\bar{\Bbbk}\ell + \frac{2\ell}{p} + \frac{2T}{p}\left(\omega_1 + \bar{b}\ell\right) \right] R, \\
\Upsilon_2 = {}& -\left(R\hat{L}^f + (\hat{L}^f)^T R \right) + \left[2 + \frac{1}{2p\ell} \right. \\
& \left. + \frac{\bar{\Bbbk}(1+2\ell)}{2\ell} + \frac{2T}{p}\left(1 + \omega_2 + \frac{\bar{b}}{2} + \frac{\bar{b}}{4\ell}\right) \right] R.
\end{aligned} \tag{64}$$

Note that $R\hat{L}^f + (\hat{L}^f)^T R = \mathrm{diag}\left\{ \bar{k}_1\left(\widehat{W}_1 L_{11} + L_{11}^T \widehat{W}_1\right), \ldots, \bar{k}_r\left(\widehat{W}_r L_{rr} + L_{rr}^T \widehat{W}_r\right) \right\} + R\hat{L}_h^f + \left(\hat{L}_h^f\right)^T R$, then according to Lemma 5, it yields

$$\begin{aligned}
&\lambda_{\min}\left(R\hat{L}^f + (\hat{L}^f)^T R \right) \\
&\quad \geq \bar{k}_{\hat{i}}\,\lambda_{\min}\left(\widehat{W}_{\hat{i}} L_{\hat{i}\hat{i}} + L_{\hat{i}\hat{i}}^T \widehat{W}_{\hat{i}} \right) + \lambda_{\min}\left(R\hat{L}_h^f + (\hat{L}_h^f)^T R \right), \\
&\lambda_{\max}\left(R\hat{L}^f + (\hat{L}^f)^T R \right) \\
&\quad \leq \bar{k}_{\hat{i}}\,\lambda_{\max}\left(\widehat{W}_{\hat{i}} L_{\hat{i}\hat{i}} + L_{\hat{i}\hat{i}}^T \widehat{W}_{\hat{i}} \right) + \lambda_{\max}\left(R\hat{L}_h^f + (\hat{L}_h^f)^T R \right).
\end{aligned} \tag{65}$$

Based on the above inequalities, it can be verified that if conditions (55) and (56) are satisfied, then we have $\Upsilon_1 < 0$ and

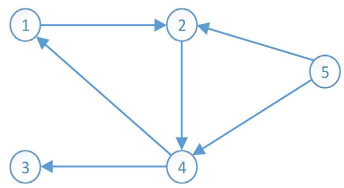


Fig. 1. Communication topology for Example 1.

$\Upsilon_2 < 0$ which implies that $\bar{H}(t) < 0$. Next, by adopting similar analyses as in Theorem 1, we can prove that (45) and (46) hold, which implies that prescribed-time cluster lag consensus for MAS (2)–(3) is achieved according to Definition 2.

Finally, we can prove that the input signal for each follower is bounded. As the proof process for this conclusion is similar to that in Theorem 1, we no longer show it here.

A procedure for determining the control gain $\bar{k}_{\hat{i}}$ in controller (48) is now presented.

Algorithm 1. For a given MAS (2)–(3) with the corresponding Laplacian matrix L in which L^f is given in the form of (44), we can follow the steps below to determine $\bar{k}_{\hat{i}}$.

(1) Determine c_x and c_v according to Assumption 1.

(2) Choose the matrices $\widehat{W}_1, \ldots, \widehat{W}_r$ on the basis of Lemma 4, then compute the values of $\lambda_{\min}(\Phi_{\hat{i}})$, $\lambda_{\min}(\Pi)$ and $\lambda_{\max}(R)$.

(3) Select $\bar{\Bbbk}$, p, $\bar{b}$ satisfying $\bar{b} \geq 0$, $p > 1$, $\bar{\Bbbk} \geq \frac{2}{p}$ and $\bar{\Delta}_1 < 0$.

(4) Choose $\ell > 0$ and $\bar{k}_{\hat{i}} > 0$ such that (54)–(56) hold. In fact, according to conditions (54)–(56), $\bar{k}_{\hat{i}}$ can be chosen to satisfy the following inequality

$$\bar{k}_{\hat{i}} > \max\left\{\frac{\lambda_{\max}(R) - \lambda_{\min}(\Pi)}{\lambda_{\min}(\Phi_{\hat{i}})}, \frac{-\bar{\Delta}_2\lambda_{\max}(R) - \bar{\Delta}_1\lambda_{\min}(\Pi)}{\bar{\Delta}_1\lambda_{\min}(\Phi_{\hat{i}})}, \frac{\bar{\Delta}_3\lambda_{\max}(R) - \lambda_{\min}(\Pi)}{\lambda_{\min}(\Phi_{\hat{i}})}\right\}. \tag{66}$$

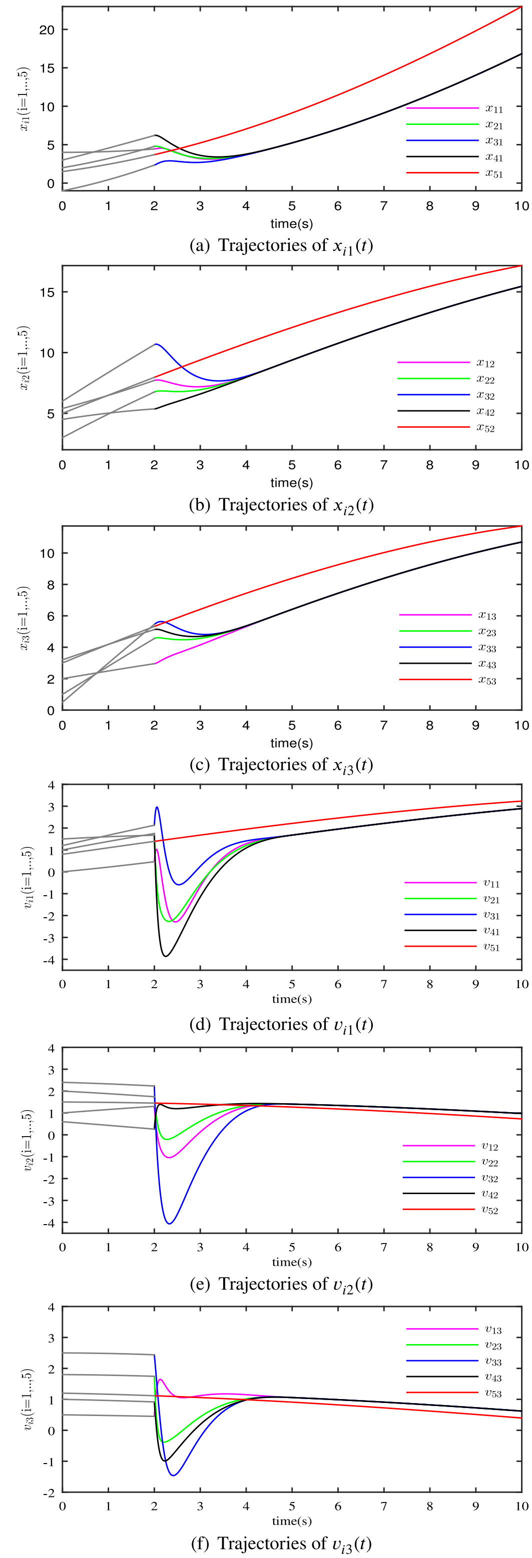
Remark 6. Cluster synchronization problems of complex network systems and cluster consensus problems of MASs have been widely studied (see [29,38,39] and references therein). In contrast, there are relatively few studies on cluster lag consensus of MASs. As mentioned in the introduction part, in practical applications, cluster lag consensus control can avoid congestion. Moreover, it will be more in line with actual application requirements if the cluster lag consensus could be achieved within a user-specified time. Therefore, compared with [29–31], the research on prescribed-time lag consensus in this section is of great practical significance.

4. Simulation examples

Below we show the simulation results of MAS (2)–(3) under controllers (15) (designed for lag consensus) and (48) (designed for cluster lag consensus) to verify the effectiveness of these two control laws.

Example 1 (*Lag Consensus*)**.** Consider the spacecraft formation flying model in [40]. We take the same non-linear function as in paper [40], that is, for agent k, its expression is as follow:

$$g(t, x_k, v_k) = \begin{bmatrix} * & * & * \\ * & 3\varrho^2 & * \\ * & * & -\varrho^2 \end{bmatrix} x_k + \begin{bmatrix} * & 2\varrho & * \\ -2\varrho & * & * \\ * & * & * \end{bmatrix} v_k, \tag{67}$$



(a) Trajectories of $x_{i1}(t)$

(b) Trajectories of $x_{i2}(t)$

(c) Trajectories of $x_{i3}(t)$

(d) Trajectories of $v_{i1}(t)$

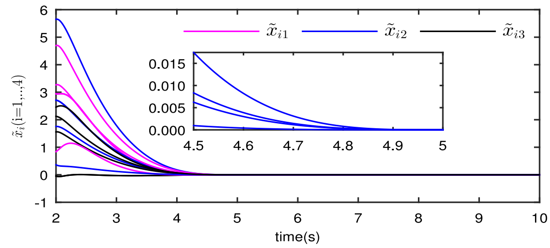
(e) Trajectories of $v_{i2}(t)$

(f) Trajectories of $v_{i3}(t)$

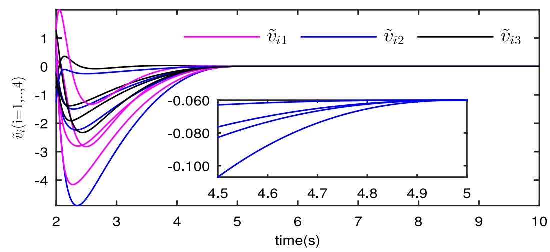
Fig. 2. System trajectories of lag consensus with $t_0 = 2$ s, $T = 3$ s and $\tau = 2$ s.

where the asterisks in these matrices represent 0 and ϱ represents the angular rate. Set $\varrho = 0.1$. Through some simple calculations, we can verify that Assumption 1 is satisfied with $c_x = 0.03$ and $c_v = 0.2$.

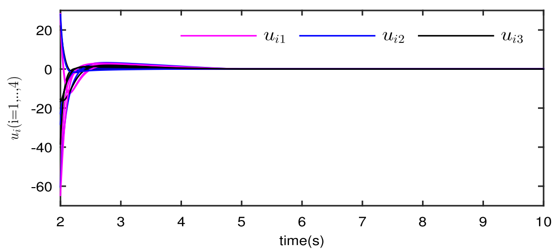
Suppose there are four followers and one leader (indexed as 5). Fig. 1 depicts the communication topology, and the Laplacian



(a) Tracking errors of the position states of the followers

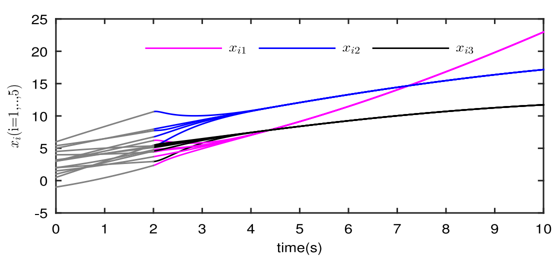


(b) Tracking errors of the velocity states of the followers

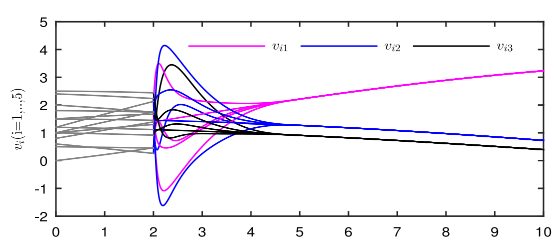


(c) Values of input signals of the followers

Fig. 3. Tracking errors and inputs of the followers of lag consensus with $t_0 = 2$ s, $T = 3$ s and $\tau = 2$ s.



(a) Variables of position states of all the agents

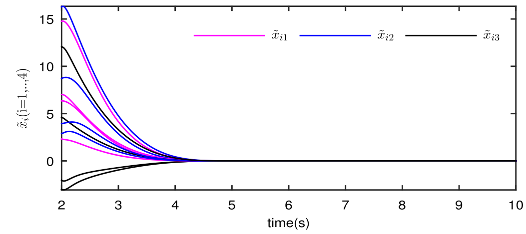


(b) Variables of velocity states of all the agents

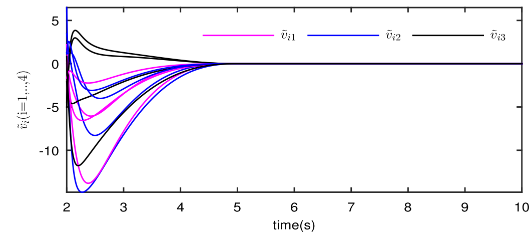
Fig. 4. System trajectories of MAS (2)–(3) with $t_0 = 2$ s, $T = 3$ s and $\tau = 0$ s.

matrix L is given by:

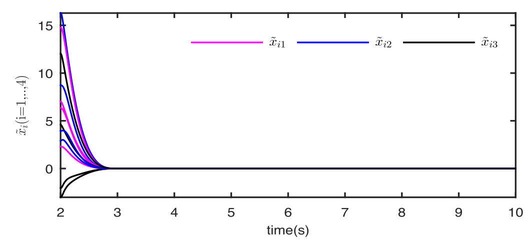
$$L = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ -0.5 & 2 & 0 & 0 & -1.5 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & -0.8 & 0 & 2 & -1.2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{68}$$



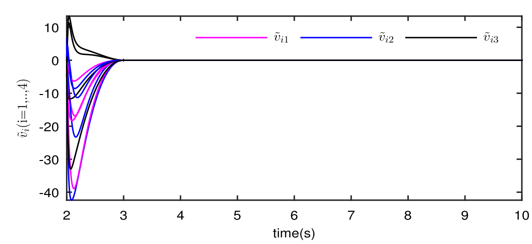
(a) Tracking errors of the position states with $T = 3s$



(b) Tracking errors of the velocity states with $T = 3s$



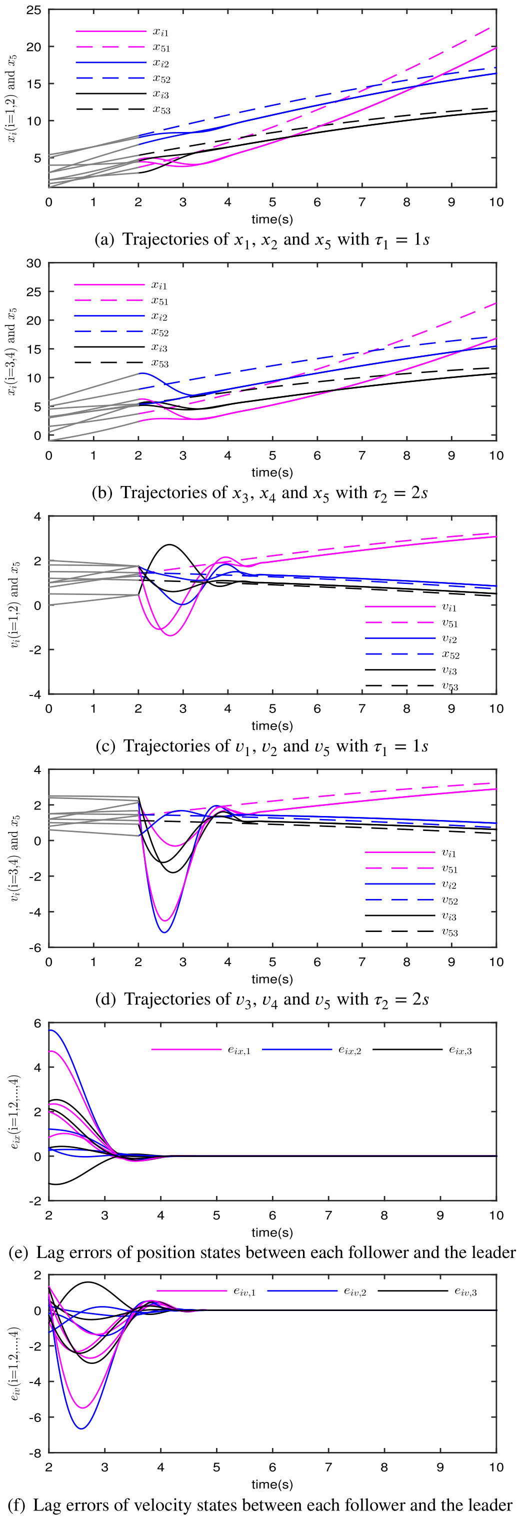
(c) Tracking errors of the position states with $T = 1s$



(d) Tracking errors of the velocity states with $T = 1s$

Fig. 5. Comparison results for lag consensus with $t_0 = 2$ s, $\tau = 1$ s under different values of T.

Set $t_0 = 2$ s, $T = 3$ s and $\tau = 2$ s, then (22)–(24) in Theorem 1 are satisfied when we choose $p = 5$, $k_1 = 4.74$, $k_2 = 5.58$, $\Bbbk = 2/p = 0.4$, $b = 0$, $\beta = 0.92$, $\gamma = 0.9$ and $\ell = 0.3$. It follows from Theorem 1 that under the designed controller (15), the prescribed-time lag consensus will be achieved for the leader-following MAS (2)–(3). In the simulation, we consider three-dimensional state vectors, i.e., $m = 3$. The initial state values are given by $x(0) = [4, 5.4, 2, 2, 3, 1, -1, 6, 0.5, 3, 4.5, 3.2, 1.5, 5, 3]^T$, $v(0) = [0, 1, 0.5, 1, 2, 1.8, 1.2, 2.4, 2.5, 1.5, 0.6, 1, 0.8, 1.5, 1.2]^T$. Figs. 2–3 show the simulation results in this case. The trajectories of state variables of all the five agents are shown in detail in Fig. 2, and the tracking errors of all the followers and the input signal for each follower are presented in Fig. 3. It can be seen from Fig. 3(a)–(b) that the tracking errors of position states and velocity states of each follower are converge to zero within $T = 3$ s, which means that the prescribed-time lag consensus is achieved at the lag time $\tau = 2$ s with the UBST T. Furthermore, the results in Fig. 3(c) show that the inputs are uniformly bounded. The prescribed-time consensus is a special case of the corresponding prescribed-time lag consensus with $\tau = 0$ s, and the simulation results in this case where all parameter values



(a) Trajectories of x_1, x_2 and x_5 with $\tau_1 = 1s$

(b) Trajectories of x_3, x_4 and x_5 with $\tau_2 = 2s$

(c) Trajectories of v_1, v_2 and v_5 with $\tau_1 = 1s$

(d) Trajectories of v_3, v_4 and v_5 with $\tau_2 = 2s$

(e) Lag errors of position states between each follower and the leader

(f) Lag errors of velocity states between each follower and the leader

Fig. 6. System trajectories and tracking errors of cluster lag consensus with $t_0 = 2$ s, $T = 3$ s, $\tau_1 = 1$ s and $\tau_2 = 2$ s.

remain unchanged except τ are presented in Fig. 4, from which it can be concluded that the leader-following consensus is achieved within $T = 3$ s.

In order to further illustrate that the USBT can be assigned arbitrarily, we give two different T values: $T = 3$ s and $T = 1$ s for comparison, and change the initial state values of the five agents as $x_1(0) = [2, 5, -1]^T$, $x_2(0) = [4, 7, 1.7]^T$, $x_3(0) = [6, 3.5, 1]^T$, $x_4(0) = [1, 4.2, 3.2]^T$, $x_5(0) = [-2, 3, 0]^T$, $v_1(0) = [2, 3.5, 2.5]^T$, $v_2(0) = [0.8, 6, 5]^T$, $v_3(0) = [4.5, 2.4, 2]^T$, $v_4(0) = [1, 0.8, 8]^T$, $v_5(0) = [3, 0, 7]^T$. Under the above-mentioned initial condition, the tracking errors of the MAS under the two cases of $T = 3$ s and $T = 1$ s are shown in Fig. 5. We can see from Fig. 5(a)–(b) that the lag consensus of the leader-following MAS (2)–(3) is achieved within $T = 3$ s, and the results in Fig. 5(c)–(d) show that the lag consensus can also be achieved within $T = 1$ s. In addition, by comparing the results in Fig. 5(a)–(b) with those in Fig. 2, it can be concluded that no matter what initial value the MAS takes, the designed controller (15) can drive the MAS (2)–(3) to achieve lag consensus within the specified time.

Example 2 (*Cluster Lag Consensus*)**.** Consider a SOMAS with five agents, where the four followers are divided into two groups with $q_1 = \{1, 2\}$ and $q_2 = \{3, 4\}$. Let $g(\cdot, \cdot, \cdot)$ be the same as in Example 1 and

$$L = \begin{bmatrix} 1 & 0 & -0.2 & 0.2 & -1 \\ -0.1 & 1 & 0 & 0 & -0.9 \\ 0.5 & -0.5 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Set $t_0 = 2$ s, $T = 3$ s, $\tau_1 = 1$ s, and $\tau_2 = 2$ s. When we choose $p = 5$, $\bar{k}_1 = 2.9$, $\bar{k}_2 = 2.6$, $\bar{b} = 0.01$, $\ell = 0.51$ and $\Bbbk = 0.4$, (54)–(56) in Theorem 2 are satisfied. Then, under the proposed controller (48), for each follower in group i, $i = 1, 2$, it is observed from Theorem 2 that it can achieve lag consensus at the lag time τ_i with the leader within $T = 3$ s. The performance of the MAS under the same initial condition as in Fig. 2 are shown in Fig. 6. Specifically, in Fig. 6(a)–(d), we separately show the trajectories of position state variables and velocity state variables of followers in different clusters. At the same time, for clear visibility, the corresponding trajectories of the leader are also presented in these four subgraphs. From Fig. 6(a)–(d), we can see that within time $T = 3$ s, follower nodes 1, 2 in cluster q_1 achieve lag consensus with leader node 5 at the lag time $\tau_1 = 1$ s, and followers 3, 4 in cluster q_2 achieve lag consensus at lag time $\tau_2 = 2$ s. The results in Fig. 6(e)–(f) further show that both the position lag errors and velocity lag errors eventually converge to zero at a time $t < t_0 + T = 5$ s, which indicates that the prescribed-time cluster lag consensus for MAS (2)–(3) is achieved under the controller (48).

Remark 7. Although we draw upon a spacecraft formation flying model for simulation details, we choose simpler parameter values for clarity, which may not be suitable for practical problems. Therefore, the units of all ordinates in the simulation figures are not specified.

5. Conclusions

The research on prescribed-time consensus problem aims to provide a solution that enables the MAS to achieve consensus within any pre-specified time. Compared with studies on finite-time/fixed-time consensus problems, in which the UBST is estimated online and depends on the initial conditions or designed parameters of the system, the study on prescribed-time consensus problems is a more challenging issue. In this work, the prescribed-time consensus for SOMASs with a leader-following configuration has been investigated. When all the followers are in one group, we have proved that, under the designed distributed control law, lag consensus is reached within a user-preassigned time. The cluster lag consensus problem when followers are divided into several groups has also been discussed. We have theoretically proved the effectiveness of the designed controllers and some sufficient conditions for achieving lag consensus and cluster lag consensus have been derived, respectively. Prescribed-time consensus problems of MASs with bounded disturbances under time-varying topologies will be discussed in our follow-up works.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] Jadbabaie A, Lin J, Morse AS. Coordination of groups of mobile autonomous agents using nearest neighbor rules. IEEE Trans Automat Control 2003;48(6):988–1001.
[2] Kolaric P, Chen C, Dalal A, Lewis FL. Consensus controller for multi-UAV navigation. Control Theory Technol 2018;16(2):110–21.
[3] Chen Q, Yin C, Zhou J, Wang Y, Wang X, Chen C. Hybrid consensus-based cubature Kalman filtering for distributed state estimation in sensor networks. IEEE Sens J 2018;18(11):4561–9.
[4] Schiffer J, Seel T, Raisch J, Sezi T. Voltage stability and reactive power sharing in inverter-based microgrids with consensus-based distributed voltage control. IEEE Trans Control Syst Technol 2016;24(1):96–109.
[5] Liu W, Zhou S, Qi Y, Wu X. Leaderless consensus of multi-agent systems with Lipschitz nonlinear dynamics and switching topologies. Neurocomputing 2016;173:1322–9.
[6] Song Q, Cao J, Yu W. Second-order leader-following consensus of nonlinear multi-agent systems via pinning control. Systems Control Lett 2010;59(9):553–62.
[7] Cai Y, Zhang H, Zhang K, Liang Y. Distributed leader-following consensus of heterogeneous second-order time-varying nonlinear multi-agent systems under directed switching topology. Neurocomputing 2019;325:31–47.
[8] Wen G, Hu G, Yu W, Cao J, Chen G. Consensus tracking for higher-order multi-agent systems with switching directed topologies and occasionally missing control inputs. Systems Control Lett 2013;62(12):1151–8.
[9] Lin P, Jia Y. Average consensus in networks of multi-agents with both switching topology and coupling time-dela. Physica A 2008;387(1):303–13.
[10] Wang Y, Ma Z. Lag consensus of the second-order leader-following multi-agent systems with nonlinear dynamics. Neurocomputing 2016;171:82–8.
[11] Chen Y, Lü J, Han F, Yu X. On the cluster consensus of discrete-time multi-agent systems. Systems Control Lett 2011;60(7):517–23.
[12] Liu Y, Jia Y. H_∞ consensus control for multi-agent systems with linear coupling dynamics and communication delays. Internat J Systems Sci 2012;43(1):50–62.
[13] Bhat SP, Bernstein DS. Finite-time stability of continuous autonomous systems. SIAM J Control Optim 2000;38(3):751–66.
[14] Huang J, Wen C, Wang W, Song YD. Adaptive finite-time consensus control of a group of uncertain nonlinear mechanical systems. Automatica 2015;51:292–301.
[15] Yu S, Long X. Finite-time consensus for second-order multi-agent systems with disturbances by integral sliding mode. Automatica 2015;54:158–65.
[16] Wang L, Ge MF, Zeng Z, Hu J. Finite-time robust consensus of nonlinear disturbed multiagent systems via two-layer event-triggered control. Information Science 2018;466:270–83.
[17] Aghababa MP. Finite time control of a class of nonlinear switched systems in spite of unknown parameters and input saturation. Nonlinear Anal Hybrid Syst 2019;31:220–32.
[18] Huang Y, Jia Y. Fixed-time consensus tracking control for second-order multi-agent systems with bounded input uncertainties via NFFTSM. IET Control Theory Appl 2017;11(16):2900–9.
[19] Zuo Z, Han QL, Ning B, Ge X, Zhang XM. An overview of recent advances in fixed-time cooperative control of multiagent systems. IEEE Trans Ind Inf 2018;14(6):2322–34.
[20] Gao F, Wu Y, Zhang Z, Liu Y. Global fixed-time stabilization for a class of switched nonlinear systems with general powers and its application. Nonlinear Anal Hybrid Syst 2019;31:56–68.
[21] Wang Y, Song Y. Leader-following control of high-order multi-agent systems under directed graphs: Pre-specified finite time approach. Automatica 2018;87:113–20.
[22] Wang Y, Song Y, Hill DJ, Krstic M. Prescribed-time consensus and containment control of networked multiagent systems. IEEE Trans Cybern 2018;99:1–10.
[23] Liu Y, Zhao Y, Ren W, Chen G. Appointed-time consensus: Accurate and practical designs. Automatica 2018;89:425–9.
[24] Wang B, Wang J, Zhang B, Chen B, Zhang Z. Leader–follower consensus of multivehicle wirelessly networked uncertain systems subject to nonlinear dynamics and actuator fault. IEEE Trans Autom Sci Eng 2018;15(2):492–505.
[25] Yang X, Zhu Q, Huang C. Generalized lag-synchronization of chaotic mix-delayed systems with uncertain parameters and unknown perturbations. Nonlinear Anal RWA 2011;12(1):93–105.
[26] Li K, Yu W, Ding Y. Successive lag synchronization on nonlinear dynamical networks via linear feedback control. Nonlinear Dynam 2015;80(1–2):421–30.
[27] Wang Y, Ma Z, Zheng S, Chen G. Pinning control of lag-consensus for second-order nonlinear multiagent systems. IEEE Trans Cybern 2016;47(8):2203–11.
[28] Zhou Y, Zhang G. Distributed finite-time lag-consensus for second-order nonlinear multi-agent systems with disturbances. In: 2018 13th World Congress on Intelligent Control and Automation (WCICA). IEEE; 2018, p. 1594–9.
[29] Wang Y, Ma Z, Chen G. Avoiding congestion in cluster consensus of the second-order nonlinear multiagent systems. IEEE Trans Neural Netw Learn Syst 2017;29(8):3490–8.
[30] Wang Y, Li Y, Ma Z, Cai G, Chen G. Cluster lag consensus for second-order multiagent systems with nonlinear dynamics and switching topologies. IEEE Trans Syst, Man, Cybern: Syst 2018;99:1–8.
[31] Zhang Y, Wen G, Peng Z, Yu Y, Rahmani A. Group multiple lags consensus of fractional-order nonlinear leader-following multi-agent systems via adaptive control. Trans Inst Meas Control 2019;41(5):1313–22.
[32] Sánchez-Torres JD, Muñoz Vázquez AJ, Defoort M, Jiménez-Rodríguez E, Loukianov AG. A class of predefined-time controllers for uncertain second-order systems. Eur J Control 2020;53:52–8.
[33] Ni J, Liu L, Tang Y, Liu C. Predefined-time consensus tracking of second-order multiagent systems. IEEE Trans Syst, Man, Cybern: Syst 2019. http://dx.doi.org/10.1109/TSMC.2019.2916257.
[34] Qin J, Zhang G, Zheng W, Kang Y. Adaptive sliding mode consensus tracking for second-order nonlinear multiagent systems with actuator faults. IEEE Trans Cybern 2019;49(5):1605–15.
[35] Qu Z. Cooperative Control of Dynamical Systems: Applications to Autonomous Vehicles. Springer Science & Business Media; 2009.
[36] Wu W, Zhou W, Chen T. Cluster synchronization of linearly coupled complex networks under pinning control. IEEE Trans Circuits Syst I Regul Pap 2009;56(4):829–39.
[37] Xia W, Cao M. Clustering in diffusively coupled networks. Automatica 2011;47(11):2395–405.
[38] Liu L, Zhou W, Li X, Sun Y. Dynamic event-triggered approach for cluster synchronization of complex dynamical networks with switching via pinning control. Neurocomputing 2019;340:32–41.
[39] Sun Y, Zhang Y, Zhou W, Zhang X, Wang X. New criteria on event-triggered cluster synchronization of neutral-type neural networks with lévy noise and non-Lipschitz condition. Neurocomputing 2020;384:156–69.
[40] Li Z, Duan Z, Chen G, Huang L. Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint. IEEE Trans Circuits Syst I Regul Pap 2010;57(1):213–24.

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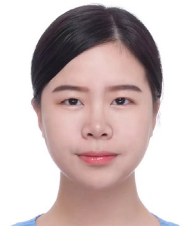

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