Optimization Methods **Fall 2023**

Homework 1

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Notice

- The submission email is: **optfall2023@163.com**.
- Please use the provided LaTeX file as a template.
- If you are not familiar with LaTeX, you can also use Word to generate a **PDF** file.

Problem 1: Norms

A function $f : \mathbb{R}^n \to \mathbb{R}$ with $\operatorname{dom} f = \mathbb{R}^n$ is called a norm if

- f is nonnegative: $f(x) \geq 0$ for all $x \in \mathbb{R}^n$
- f is definite: $f(x) = 0$ only if $x = 0$
- f is homogeneous: $f(tx) = |t|f(x)$, for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$
- f satisfies the triangle inequality: $f(x + y) \leq f(x) + f(y)$, for all $x, y \in \mathbb{R}^n$

We use the notation $f(x) = \|x\|$. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. The associated dual norm, denoted $\|\cdot\|_*$, is defined as

$$\|z\|_* = \sup\left\{z^{\mathrm{T}}x \mid \|x\| \leq 1\right\}$$

a) Prove that $\|\cdot\|_*$ is a valid norm.

b) Prove that the dual of the Euclidean norm (ℓ_2-norm) is the Euclidean norm, i.e., prove that

$$\|z\|_{2*} = \sup\left\{z^Tx \mid \|x\|_2 \leq 1\right\} = \|z\|_2$$

(*Hint*: Use Cauchy-Schwarz inequality.)

Solution. a) Because z^Tx is going to gain the maximum value, the direction of x is the same as z, and its l_2 norm is 1, i.e. $x = \frac{z}{||z||}$ Then $\angle(x, z) = 0$ holds, so the result of z^Tx is nonnegative, and is 0 only if $x = 0$.

$$f(tz) = ||tz||_* = \sup\{tz^Tx \mid ||x|| \leq 1\} = |t| \ \sup\{z^Tx \mid ||x|| \leq 1\} = |t|f(z)$$

$$f(x) + f(y) = \sup\{x^Tz_1 \mid ||z_1|| \leq 1\} + \sup\{y^Tz_2 \mid ||z_2|| \leq 1\} >= \sup\{x^Tz_1 \mid ||z_1|| \leq 1\} + y^Tz_1$$
$$\geq \sup\{(x^T + y^T)z_1 \mid ||z_1|| \leq 1\} = f(x + y)$$

b) $\|z\|_{2*} = \sup\left\{z^Tx \mid \|x\|_2 \leq 1\right\}$. Cauchy-Schwarz inequality implies that $||z^Tx||_2 \leq ||z||_2||x||_2 \leq ||z||_2$. So choosing $x = \frac{z}{||z||_2}$ achieves the upper bound.

□

Problem 2: Inequalities

Let $x \in \mathbb{R}^n, y \in \mathbb{R}^n$, where n is a positive integer. Let $\|\cdot\|$ denote the Euclidean norm.

a) Prove the triangle inequality $\|x + y\| \leq \|x\| + \|y\|$.

b) Prove $\|x + y\|^2 \leq (1 + \epsilon)\|x\|^2 + (1 + \frac{1}{\epsilon})\|y\|^2$ for any $\epsilon > 0$.

(*Hint*: You may need the Young's inequality for products, i.e. if a and b are nonnegative real numbers and p and q are real numbers greater than 1 such that $1/p + 1/q = 1$, then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.)

Solution. a) $||x+y|| = ((x+y)^T(x+y))^{\frac{1}{2}} = (\Sigma_{i=1}^n(x_i+y_i)^2)^{\frac{1}{2}}$
$= (\Sigma_{i=1}^n(x_i^2+y_i^2+2x_iy_i))^{\frac{1}{2}} \leq (\Sigma_{i=1}^n(x_i^2+y_i^2)+2(\Sigma_{i=1}^n x_i^2 \Sigma_{i=1}^n y_i^2)^{\frac{1}{2}})^{\frac{1}{2}}$
$= (\Sigma_{i=1}^n(x_i^2+y_i^2)+2(\Sigma_{i=1}^n x_i^2)^{\frac{1}{2}}(\Sigma_{i=1}^n y_i^2)^{\frac{1}{2}})^{\frac{1}{2}}$
$= (\Sigma_{i=1}^n x_i^2)^{\frac{1}{2}} + (\Sigma_{i=1}^n y_i^2)^{\frac{1}{2}} = ||x|| + ||y||$
b) $||x+y||^2 \leq (||x||+||y||)^2 = ||x||^2+||y||^2+2||x||||y|| \leq ||x||^2+||y||^2+\epsilon||x||^2+\frac{1}{\epsilon}||y||^2$

$\square$

Problem 3: Definition of convexity

Which of the following sets are convex? Please provide explanations for your choices.

a) a set of the form $\left\{x \in \mathbf{R}^n \mid \alpha \leq a^T x \leq \beta\right\}$.

b) a set of the form $\{x \in \mathbf{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = 1, 2, ..., n\}$.

c) a set of the form $\left\{x \in \mathbf{R}^n \mid a_1^T x \leq b_1, a_2^T x \leq b_2\right\}$.

d) The set of points closer to a given point than a given set, *i.e.*,

$$\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\}$$

where $S \subseteq \mathbf{R}^n$.

e) The set of points closer to one set than another, *i.e.*,

$$\{x \mid \mathbf{dist}(x, S) \leq \mathbf{dist}(x, T)\},$$

where $S, T \subseteq \mathbf{R}^n$, and

$$\mathbf{dist}(x, S) = \inf\left\{\|x - z\|_2 \mid z \in S\right\}.$$

f) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, *i.e.*, the set $\{x \mid \|x - a\|_2 \leq \theta\|x - b\|_2\}$. You can assume $a \neq b$ and $0 \leq \theta \leq 1$.

Solution. a) Yes. This describes two hyperplane and a space in the mid of them. Then obviously it is convex.
b) Yes. Due to a), this describes the intersection of finite spaces, or a rectangle, so it is still convex.
c) Yes. This describes the intersection of two arbitrary spaces, so it is convex.
d) Yes. This can be shown as $\cap_{y\in S}\{x \mid ||x-x_0||_2 \leq ||x-y||_2\}$, or the intersection of some halfspaces, so is convex.
e) No. Counterexample: In R^2 , S = (0,-2) , (0,2) , T = (0,0), then $\{x \mid \mathbf{dist}(x,S) \leq \mathbf{dist}(x,T)\} = \{x \leq -1 \cup x \geq 1\}$, and it is not convex.
f) Yes. $\{x \mid \|x-a\|_2 \leq \theta\|x-b\|_2\} = \left\{x \mid \|x-a\|_2^2 \leq \theta^2\|x-b\|_2^2\right\} = \{x \mid x^Tx - 2a^Tx + a^Ta \leq \theta^2(x^Tx - 2b^Tx + b^Tb)\} = \{x \mid (1-\theta^2)x^Tx - 2(a-\theta^2 b)^T x + (a^Ta - \theta^2 b^T b) \leq 0\}$, and it is a ball.

$\square$

Problem 4: Examples

Let $C \subseteq \mathbb{R}^n$ be the solution set of a quadratic inequality,

$$C = \{x \in \mathbb{R}^n | x^T A x + b^T x + c \leqslant 0\}$$

with $A \in \mathbb{S}^n$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

a) Show that C is convex if $A \succeq 0$.

b) Is the following statement true? The intersection of C and the hyperplane defined by $g^T x + h = 0$ is convex if $A + \lambda g g^T \succeq 0$ for some $\lambda \in \mathbb{R}$.

Solution. a) We use the trick that the a set is convex if and only if its intersection with an arbitrary line $\{x + tv \mid t \in R\}$ is convex. If we substitute the line equation in the quadratic formula we end up with:$\{x + tv \mid \alpha t^2 + \beta t + \gamma \leq 0\}$ with $\alpha = v^T Av, \beta = b^T v + x^T Av + v^T Ax$ *and* $\gamma = c + b^T x + x^T Ax.$
If $A \succeq 0$, then $\alpha = v^T Av \geq 0$, then the solution to $\alpha t^2 + \beta t + \gamma \leq 0$ is a bounded interval. Since the arbitrary value of v, each line intersecting with the set is convex, so the set is convex.
b) Let $B = A + \lambda gg^T \succeq 0$, then $x^T Ax + b^T x + c = x^T (B - \lambda gg^T)x + b^T x + c = x^T Ax - \lambda(g^T x)^T g^T x + b^T x + c = x^T Ax + b^T x + c - \lambda h^2$. Replace the c in the question a) by $c - \lambda h^2$ and can get $B \succeq 0$.

$\square$

Problem 5: Dual cones

Describe the dual cone for each of the following cones.

a) $K = \{0\}$.

b) $K = \mathbf{R}^2$.

c) $K = \{(x_1, x_2) \,||x_1 \,| \leq x_2\}$.

d) $K = \{(x_1, x_2) \mid x_1 + x_2 = 0\}$.

Solution. a) $K^* = R^n$
b) $K^* = (R^2)^\perp$. If K is in R^2, then $K^* = \{0\}$
c) $K^* = K$
d) $K^* = 0$

$\square$