信息与计算科学导论
Graph Theory
221502023 沈硕 第四次作业

Problem 5.4: Order the vertices of a graph G according to their degrees, so that $V(G) = \{x_1, x_2, \cdots, x_n\}$ and $d(x_1) \geq d(x_2) \geq \cdots$. Show that in this order the greedy algorithm uses at most $\max_i \min\{d(x_i) + 1, i\}$ colours, and so if k is the maximal natural number for which $k \leq d(x_k) + 1$ then $\chi(G) \leq k$.

Solution: Color G by following this non-increasing degree sequence. Each time when we color the i^{th} vertex v_i, it has at most $\min\{d_i, i-1\}$ colored neighbours. So at this very moment we use at most $1 + \min\{d_i, i-1\}$, or , $\min\{d(x_i) + 1, i\}$ colors. Since this holds for all vertices, maximizing i can get the upper bound.
If k is the maximal natural number for which $k \leq d(x_k) + 1$, then coloring $v_1, v_2, \cdots, v_k$ uses at most k colors. $\forall l > k$, $l > d(x_l) + 1$, then for $v_{k+1}, v_{k+2}, \cdots, v_n$, when we come to color each of them, we will use at most $d(x_l) + 1$ colors. As the degree sequence is non-increasing, $d(x_l) + 1 \leq k$ holds. So $\chi(G) \leq k$.

Problem 5.5: Deduce from Exercise 4 that if G has n vertices then

$$\chi(G) + \chi(\overline{G}) \leq n + 1$$

Solution: We have non-increasing degree sequence for G:
$V(G) = \{x_1, x_2, \cdots, x_n\}$ and $d(x_1) \geq d(x_2) \geq \cdots \geq d(x_n)$.

Correpondingly, there is a non-increasing sequence for $\overline{G}$:
$d(x_n) \geq d(x_{n-1}) \geq \cdots \geq d(x_1)$.
Let k be the maximal natural number for which $k \leq d(x_k) + 1$ then
$\chi(G) \leq k$. Then for $\overline{G}$ this must be below $n + 1 - k$. Summing up these two can solve the problem. And the equal situation is G or $\overline{G}$ is a complete gragh.

Problem 5.6: Show that $\chi(G) + \chi(\overline{G}) \geq 2\sqrt{n}$.

Solution: Since k is at least 1 and at most n, we have $\chi(G)\chi(\overline{G}) \geq n$. So $(\chi(G) + \chi(\overline{G}))^2 \geq 4\chi(G)\chi(\overline{G})$. Then we are done.

Problem 5.29: Find the edge chromatic number of K_n.

Solution: If n is odd, then $\chi^{'}(G) = n\ (= \Delta + 1)$; if n is even, then
$\chi^{'}(G) = n - 1\ (= \Delta)$.
Case even: G must contain n-1 different perfect matchings. Then for each perfect matching, we apply a different color. So their will be
$\chi^{'}(G) = n - 1\ (= \Delta)$.
Case odd: n can be shown as 2k+1, so $\mid E \mid = (2k + 1)k$. Since each color can color k edges, we need at least $\lceil \frac{|E|}{k} \rceil = 2k + 1 = n$.