信息与计算科学导论
221502023 沈硕 第五次作业

Problem 1：Suppose that p(L) is a polynomial in L, and that p(L) annihilates function T(n). Prove that $T(n) = \sum_{i=1}^{k} f_{\mathrm{i}}(n) r_{\mathrm{i}}^{n}$, where k is a constant, $f_1(n), ..., f_{\mathrm{k}}(n)$ are polynomials in n, and $r_1, ..., r_{\mathrm{k}}$ are all constants.

Solution: 不妨设：

$$p(L) = a_0 + a_1 L + a_2 L^2 + \ldots + a_m L^m$$

将 L 带入 $T(n)$ 可以得到：

$$0 = p(L)T(n) = a_0 T(n) + a_1 L T(n) + a_2 L^2 T(n) + \ldots + a_m L^m T(n)$$

那么，p(L)=0 的 m 个解即为 T(n) 的 m 个特征根。我们记 p(L)=0 的 k 个不重复的解为 $r_1, ..., r_{\mathrm{k}}$ 时，就得到了

$$T(n) = \sum_{i=1}^{k} f_{\mathrm{i}}(n) r_{\mathrm{i}}^{n}$$

其中第 i 个解 r_i 前的 $f_i(n)$ 的最高次数表示了其重根数 -1。

Problem 2: (Based on an Idea from IMO-1986-2) In a plane there is a regular triangle ABC with side length 1. Two frogs, Alice and Bob, are jumping. Let A_t, B_t be the locations of Alice and Bob, respectively, at time t.

- $A_0 = B_0 = C.$
- $A_1 = A.$
- **For t 2, rotate the line segment $BA_{\mathrm{t}-1}$ $45°$clockwise about point B to get get A_t'; rotate the line segment $BA_{\mathrm{t}-2}$$45°$clockwise about point B to get get A_{t}''; the midpoint of $A_t'A_t''$ is A_t.**
- **For t 1, rotate the line $CB_{\mathrm{t}-1}$ $45°$clockwise about point C to get the new line CB_t. The length of line segment CB_t is twice that of line segment $CB_{\mathrm{t}-1}$. Find the distance of B_t from A_t.**

Solution: I give up.

Problem 3: Define a difference operator $\Delta : \Delta = L - 1$, i.e., $\Delta T(n) = T(n+1) - T(n)$. Define a sum operator $\sum : \sum T = S + c$ where $\Delta S = T$ and c is a constant. Prove that for all functions f, g defined on positive integers, $\sum f\Delta g = fg - \sum Lg\Delta f$.

Solution:

$$\Delta(f(n)g(n)) = f(n+1)g(n+1) - f(n)g(n) = f(n+1)g(n+1) + f(n)g(n+1) - f(n)g(n+1) - f(n)g(n)$$
$$\Delta f(n)g(n) = g(n+1)\Delta f(n) + f(n)\Delta g(n)$$
$$\Delta f(n)g(n) = Lg(n)\Delta f(n) + f(n)\Delta g(n)$$

对两边同时作 $\sum$ 即得之。

Problem 4: The first case of Theorem 6 uses the following condition: $f(n) = \Theta(n^c)$ where $c < log_b a$. In the literature, you may find an alternative condition for this case: $f(n) = O(n^{log_b a - \epsilon})$ for some $\epsilon > 0$. Prove using these two conditions in the theorem are equivalent.

Solution:

$$\begin{aligned}
T(n) &= \textstyle\sum_{i=0}^{log_b n} a^i f(\frac{n}{b^i}) + O(n^{log_b a}) \\
&\leq \textstyle\sum_{i=0}^{log_b n} a^i M(\frac{n}{b^i})^{log_b a - \epsilon} + O(n^{log_b a}) \\
&= Mn^{lob_b a - \epsilon} \textstyle\sum_{i=0}^{log_b n} a^i b^{-ilog_b a + i\epsilon} + O(n^{log_b a}) \\
&= Mn^{log_b a - \epsilon} \textstyle\sum_{i=0}^{log_b n} b^{i\epsilon} + O(n^{log_b a}) \\
&= Mn^{log_b a - \epsilon} \frac{b^{\epsilon}(log_b n + 1) - 1}{b^{\epsilon} - 1} + O(n^{log_b a}) \\
&= Mn^{log_b a - \epsilon} \frac{(nb)^{\epsilon} - 1}{b^{\epsilon} - 1} + O(n^{log_b a}) \\
&\leq Mn^{log_b a - \epsilon} \frac{(nb)^{\epsilon}}{b^{\epsilon} - 1} + O(n^{log_b a}) \\
&= Mn^{log_b a} \frac{b^{\epsilon}}{b^{\epsilon} - 1} + O(n^{log_b a}) \\
&= O(n^{log_b a})
\end{aligned}$$

则得证。此处的 $f(n) = O(n^{log_b a - \epsilon})$ 等价于 $f(n) = \Theta(n^c)$。一方面，两者均小于 $log_b a$，这保证了其增长速度不超过 $\Theta n^{log_b a}$, 从而得到相同的结论；另一方面，可以通过调整 ϵ 的值来使得二者相同，即使是选择了较小的 ϵ 而使得 f(n) 的增长速度比实际的要更快，这也只是说明其增速慢于 $\Theta(n^{log_b a})$，对其粗略的估计也无妨于结论。

Problem 5: Find a recurrence $T(n) = aT(\frac{n}{b}) + f(n)$ such that $f(n) = \Theta(n^c)$ where $c > log_b a$, but f(n) does not meet the regularity condition.

Solution：我们只需要破坏正则条件即可。例如：

$$f(n) = n^c \sin n$$

其的确是 $O(n^c)$，但是没法保证

$$af(\frac{n}{b}) \leq df(n) \text{ for some } d < 1 \text{ and sufficiently large n}$$