The fourier series of continuous time periodic signals:-

Suppose that we are given a periodic signal $x(t)$ with time period T_p. We can represent the periodic signal by fourier series -

$$\boxed{x(t) = \sum_{k=-\alpha}^{\alpha} c_k e^{j2\pi k f_0 t} \quad \text{———} (1)} \quad ; \; F_0 = 1/T_p$$

To determine the expression for the coefficients $\{c_k\}$, we first multiply both sides of (1) by the complex exponential -

$$e^{-j2\pi l f_0 t}$$

where, l is an integer.

$$x(t) \cdot e^{-j2\pi l f_0 t} = \sum_{k=-\alpha}^{\alpha} c_k e^{j2\pi k f_0 t} \cdot e^{-j2\pi l f_0 t} \quad \text{———} (2)$$

We integrate both sides of (2) over a single period t_0 to $t_0 + T_p$ where t_0 is an arbitrary but mathematically convenient starting value.

$$\int_{t_0}^{t_0+T_p} x(t) e^{-j2\pi l f_0 k} \cdot dt = \int_{t_0}^{t_0+T_p} e^{-j2\pi l f_0 t} \cdot \left(\sum_{k=-\alpha}^{\alpha} c_k e^{j2\pi k f_0 t} \right) dt$$

$$= \sum_{k=-\alpha}^{\alpha} c_k \int_{t_0}^{t_0+T_p} e^{j2\pi f_0 t(k-l)} \cdot dt \quad \text{———} (3)$$

$$= \sum_{k=-\alpha}^{\alpha} c_k \left[\frac{e^{j2\pi f_0 t(k-l)}}{j2\pi f_0 (k-l)} \right]_{t_0}^{t_0+T_p} \quad \text{(4)}$$

for, $k \neq l$, the RHS of (4) yeilds zero.
On the other hand, if $k = l$, from (3) we get –

$$\int_{t_0}^{t_0+T_p} x(t)\, e^{-j2\pi f_0 kt} \cdot dt = \sum_{k=-\alpha}^{\alpha} c_k \int_{t_0}^{t_0+T_p} dt$$

$$= c_k T_p$$

$$\therefore c_k = 1/T_p \int_{t_0}^{t_0+T_p} x(t) \cdot e^{-j2\pi f_0 kt} \cdot dt$$

Since, t_0 is arbitrary –

$$\boxed{c_k = 1/T_p \int_{T_p} x(t)\, e^{-j2\pi f_0 kt} \cdot dt}$$

Dirichlet Conditions:-

* It gurantees that $\sum_{k=-\infty}^{\infty} c_k e^{j2\pi f_0 kt}$ ——— (1) is equal to

Periodic

$x(t)$, except at the values of t for which $x(t)$ is discontinuous. At these values of t, (1) converges to the midpoint of the discontinuity. The dirichlet conditions are-

(1) The signal $x(t)$ has a finite no. of discontinuities in any period.

(2) The signal $x(t)$ contains a finite no. of maxima and minima during any period.

(3) The signal $x(t)$ is absolutely integrable in any period, that is

$$\int_{T_p} |x(t)|\,dt < \infty$$

All periodic signals of practical interest satisfy these conditions.

* In summary, if $x(t)$ is periodic and satisfies the Dirichlet conditions, it can be represented in Fourier Series.

* Synthesis Equation $\rightarrow$ $\boxed{x(t) = \sum_{K=-\infty}^{\infty} c_K e^{j2\pi F_0 kt}}$

* Analysis Equation $\rightarrow$ $\boxed{c_K = 1/T_P \int_{T_P} x(t).e^{-j2\pi f_0 kt}.dt}$

* If the periodic signal is real, c_K and c_{-K} are complex conjugates.

$\therefore c_K = |c_K| e^{j\theta_K}$

$c_{-K} = |c_K| e^{-j\theta_K}$

$\therefore$ The fourier series can be represented as -

$$\boxed{x(t) = c_0 + 2\sum_{K=1}^{\infty} |c_K| \cos(2\pi k f_0 t + \theta_K)} \quad (2)$$

Where, c_0 is real valued when $x(t)$ is real.

* we can rewrite (2) in the form -

$$x(t) = c_0 + 2\sum_{K=1}^{\infty} |c_K| (\cos 2\pi f_0 kt \cos\theta_K - \sin 2\pi f_0 kt \sin\theta_K)$$

$$\therefore \boxed{x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos 2\pi k f_0 t - b_k \sin 2\pi f_0 k t\right)}$$

where, $a_0 = c_0$

$$a_k = 2|c_k| \cos\theta_k$$
$$b_k = 2|c_k| \sin\theta_k$$

Power Density Spectrum of Periodic Signal:-

* Power Signal $\longrightarrow P_x = \frac{1}{T_p}\int_{T_p} |x(t)|^2 \, dt$

* Energy Signal $\longrightarrow E_x = \int_{T_p} |x(t)|^2 \, dt$

* A periodic signal has infinite energy and a finite average power, which is given as-

$$P_x = \frac{1}{T_p}\int_{T_p} |x(t)|^2 \, dt$$

$$= \frac{1}{T_p}\int_{T_p} x(t) \cdot x(t)^* \, dt$$

$$= \frac{1}{T_p}\int_{T_p} x(t) . \sum_{k=-\infty}^{\infty} c_k^* \cancel{x(t)} e^{-j2\pi f_0 k t} . dt$$

$$\cancel{= \frac{1}{T_p}\sum_{k=-\infty}^{\infty} c_k^* \int}$$

$|z|^2 = z \times z^*$

$z = x + iy$

$z^* = x - iy$

$$= \sum_{k=-\infty}^{\infty} c_k^{*} \left[\frac{1}{T_P} \int_{T_P} x(t)\, e^{-j2\pi f_0 k t} \cdot dt \right]$$

$$= \sum_{k=-\infty}^{\infty} c_k^{*} \cdot c_k$$

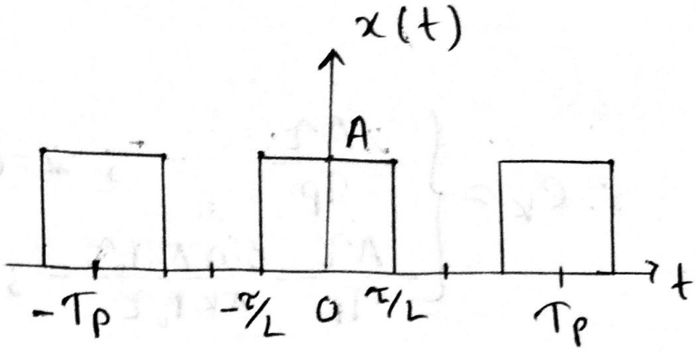
$$= \sum_{k=-\infty}^{\infty} |c_k|^2$$

Therefore, $\boxed{P_x = \frac{1}{T_P} \int_{T_P} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2}$ ——— (1)

Which is called <u>Parseval's Relation</u> of power signal.

* Periodic signal ⟶ Power signal
* Aperiodic signal ⟶ Energy signal

* **Ex- 4.1.1 :-**



Determine the fourier series and power density spectrum.

Soln :-

$$c_k = 1/T_p \int_{T_p} x(t)\, e^{-j2\pi f_0 tk}\, dt$$

$$= 1/T_p \int_{-T_p/2}^{T_p/2} x(t) \cdot e^{-j2\pi f_0 kt} \cdot dt$$

$$= 1/T_p \int_{-\tau/2}^{\tau/2} x(t)\, e^{-j2\pi f_0 kt} \cdot dt$$

At $k = 0$

$$c_0 = 1/T_p \int_{-\tau/2}^{\tau/2} x(0)\, dt$$

$$= 1/T_p \int_{-\tau/2}^{\tau/2} A\, dt$$

$$= \frac{A\tau}{T_p}$$ dc component

At $k \neq 0$

$$c_k = 1/T_p \int_{-\tau/2}^{\tau/2} A e^{-j2\pi f_0 kt}\, dt$$

$$= A/T_p \left[\frac{e^{-j2\pi f_0 kt}}{-j2\pi f_0 k} \right]_{-\tau/2}^{\tau/2}$$

$$= A/T_p \left(\frac{e^{-j\pi f_0 k\tau} - e^{j\pi f_0 k\tau}}{-j2\pi f_0 k} \right)$$

$$= \frac{A}{\pi f_0 k T_p} \left(\frac{e^{j\pi f_0 k\tau} - e^{-j\pi f_0 k\tau}}{2j} \right)$$

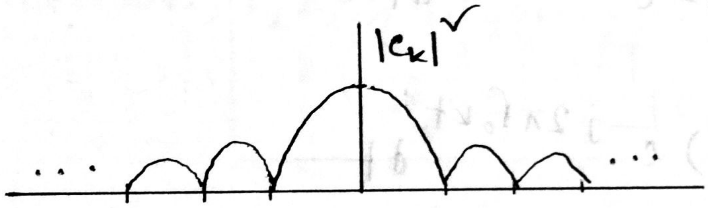
$$= A/T_p \cdot \frac{\sin \pi f_0 k\tau}{\pi f_0 k}$$

$$= \frac{A\tau}{T_p} \cdot \frac{\sin \pi f_0 k\tau}{\pi f_0 k\tau}$$

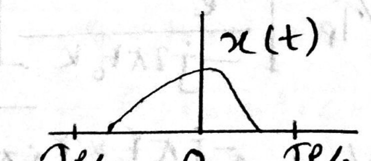
$$\therefore c_k = \begin{cases} \dfrac{A\tau}{T_p} & ; k=0 \\ \dfrac{A\tau}{T_p}\,\dfrac{\sin \pi k f_0 \tau}{\pi k f_0 \tau} & ; k=\pm 1, \pm 2 \cdots \end{cases}$$

$$\therefore |c_k|^2 = \begin{cases} \left(\dfrac{A\tau}{T_p}\right)^2 & ; k=0 \\ \left(\dfrac{A\tau}{T_p}\right)^2 \left(\dfrac{\sin \pi k f_0 \tau}{\pi k f_0 \tau}\right)^2 & ; k=\pm 1, \pm 2, \cdots \end{cases}$$

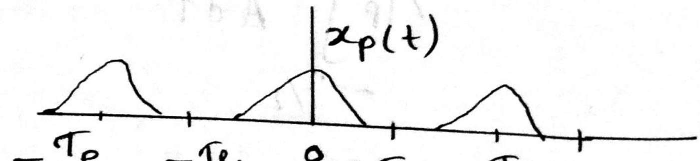
↳ Power density spectrum



☐ Fourier transform of continuous time aperiodic signal:-



↳ Aperiodic signal $x(t)$



↳ Periodic signal $x_p(t)$ constructed by repeating $x(t)$ with time period T_p

* $x(t) = \lim_{T_p \to \infty} x_p(t)$

* $f_0 = 1/T_p$

$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \quad \text{——— (1)}$$

where –

$$c_k = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x_p(t) e^{-j2\pi k f_0 t} \cdot dt$$

$\because x(t) = x_p(t)$ in $-T_p/2 \leq t \leq T_p/2$ –

$$c_k = 1/T_p \int_{-T_p/2}^{T_p/2} x(t) e^{-j2\pi f_0 k t} \cdot dt$$

It is also true that $x(t) = 0$ for $|t| > T_p/2$. So the limits can be replaced by $-\infty$ and ∞

$$c_k = 1/T_p \int_{-\infty}^{\infty} x(t) e^{-j2\pi f_0 k t} \cdot dt \quad \text{——— (2)}$$

Let,

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi F t} \quad \Big| \quad F = kf_0$$

$\therefore c_k = 1/T_p \, X(kf_0)$

$\Rightarrow c_k \cdot T_p = X(k/T_p)$

$\therefore$ (1) –

$$x_p(t) = 1/T_p \sum_{k=-\infty}^{\infty} X(k/T_p) \, e^{j2\pi f_0 k t} \quad \text{——— (3)}$$

Let $\Delta f = f_0 = 1/T_P$

(3) –

$$x_p(t) = \sum_{k=-\infty}^{\infty} \Delta f \, X(\Delta f . k) \, e^{j2\pi \Delta f k t} \quad \text{——— (4)}$$

$$x(t) = \lim_{T_p \to \infty} x_p(t)$$

$$= \lim_{\Delta f \to 0} x_p(t)$$

$$= \lim_{\Delta f \to 0} \sum_{k=-\infty}^{\infty} \cancel{\Delta f} \, X(\Delta f . k) \, e^{j2\pi \Delta f k t} . \Delta f$$

$$\therefore x(t) = \int_{-\infty}^{\infty} X(f) \, e^{j2\pi f t} \, df$$

---

* Synthesis equation

$$x(t) = \int_{-\infty}^{\infty} X(f) \, e^{j2\pi f t} \, df$$

* Analysis equation

$$X(f) = \int_{-\infty}^{\infty} x(t) \, e^{-j2\pi f t} \, dt$$

Energy density spectrum of aperiodic signals

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, dt \left[\int_{-\infty}^{\infty} X^*(F)\, e^{-j2\pi Ft} dF\right]$$

$$= \int_{-\infty}^{\infty} X^*(F)\, dF \left[\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi Ft} dt\right]$$

$$= \int_{-\infty}^{\infty} X^*(F) . X(F)\, dF$$

$$= \int_{-\infty}^{\infty} |X(F)|^2 dF$$

*** Ex- 4·1·2**

$$x(t) = \begin{cases} A, & |t| \le \tau/2 \\ 0, & |t| > \tau/2 \end{cases}$$

Determine the fourier transform and the energy density spectrum.

Soln:-

$$X(F) = \int_{-\tau/2}^{\tau/2} A\, e^{-j2\pi Ft} . dt$$

$$= A.\frac{e^{-j\pi f\tau} - e^{j\pi f\tau}}{-j2\pi f}$$

$$= \frac{A\tau}{\pi f\tau} \cdot \frac{1}{2j}\left(e^{j\pi f\tau} - e^{-j\pi f\tau}\right)$$

$$= A\tau \frac{\sin \pi f\tau}{\pi f\tau}$$

$$C_k = \frac{1}{T_p} X(f_0 k)$$

$$= \frac{1}{T_p} . A\tau . \frac{\sin \pi f_0 k\tau}{\pi f_0 k\tau}$$

$$= \frac{A\tau}{T_p} . \frac{\sin \pi f_0 k\tau}{\pi f_0 k\tau}$$

$$\therefore E_x = (A\tau)^2 \left(\frac{\sin \pi f\tau}{\pi f\tau}\right)^2$$

□ <u>Fourier series for discrete time periodic signal</u>

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$

$$c_k = 1/N \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

* Ex- 4.2.1

Determine spectra of the signals.

(a) $x(n) = \cos\sqrt{2}\pi n$

$w = \sqrt{2}\pi$

$\Rightarrow 2\pi f = \sqrt{2}\pi$

$\therefore f = 1/\sqrt{2}$

$\therefore$ Not periodic. So, cannot be expanded using fourier series.

It's spectral content consists of the single frequency component at $w = \sqrt{2}\pi$

(b) $x(n) = \cos \pi n/3$

$f = 1/6$

$\therefore N = 6$

$$\cancel{c_k = 1/6 \sum_{n=0}^{5} x(n) e^{-j2\pi kn/6} ; k = 0,1,2,3,4,5}$$

$$\cancel{c_0 = 1/6 \sum \cos 0 . e^{-0}} =$$

$= 1/6$

$$\cancel{\cos \pi n/3 = \cos 2\pi n/6 = \frac{e^{j2\pi n/6} + e^{-j2\pi n/6}}{2}}$$

$\cos n\pi/3 = \cos 2n\pi/6$

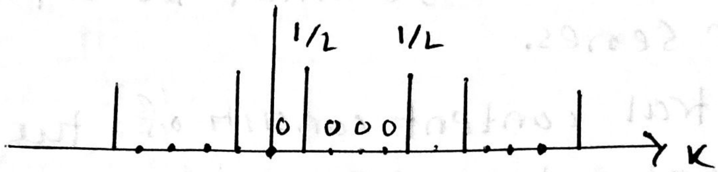
$= 1/2\, e^{j2\pi n/6} + 1/2\, e^{-j2\pi n/6}$

$= 1/2\, e^{j2\pi n/6} + 1/2\, e^{j2\pi 5n/6}$

$$= 0 \cdot e^{j2\pi 0n/6} + \tfrac{1}{2} e^{j2\pi n/6} + 0 \cdot e^{j2\pi 2n/6} + 0 \cdot e^{j2\pi 3n/6} + 0 \cdot e^{j2\pi 4n/6} + \tfrac{1}{2} e^{j2\pi 5n/6}$$

$\therefore C_0 = C_2 = C_3 = C_4 = 0$

$C_1 = C_5 = 1/2$



(c) $x(n) = \{\underset{\uparrow}{1}, 1, 0, 0\}$

$N = 4$

$$c_k = \frac{1}{4} \sum_{n=0}^{3} x(n) e^{-j2\pi kn/4}$$

$$= \frac{1}{4}\left[e^{-j2\pi \cdot k \cdot 0/4} + e^{-j2\pi k \cdot 1/4} + 0 + 0\right]$$

$$\therefore c_k = \frac{1}{4}\left(1 + e^{-j\pi k/2}\right)$$

$$c_0 = \frac{1}{4}\left(1 + e^{-0}\right) = \frac{1}{2}$$

$$c_1 = \frac{1}{4}\left(1 + e^{-j\pi/2}\right)$$

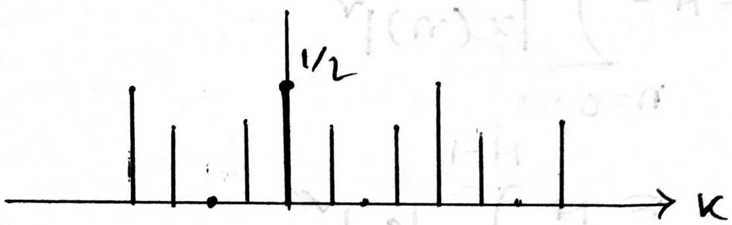
$$= \frac{1}{4}\left(1 + \cos \pi/2 + j\sin(-\pi/2)\right)$$

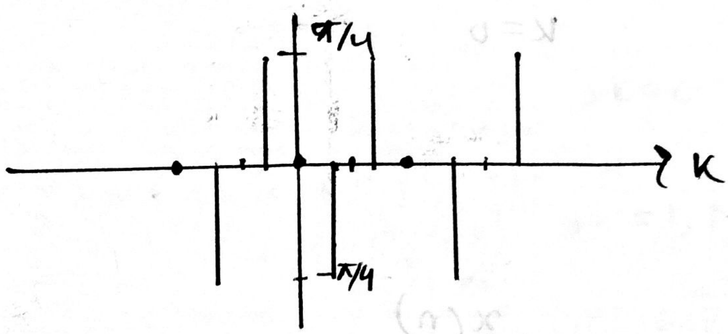
$$= \frac{1}{4}(1 - j)$$

$c_2 = 0$

$c_3 = \frac{1}{4}(1 + j)$

$\angle c_0 = 0$	$\lvert c_0 \rvert = \frac{1}{2}$
$\angle c_1 = -\pi/4$	$\lvert c_1 \rvert = \sqrt{2}/4$
$\angle c_2 =$ undefined	$\lvert c_2 \rvert = 0$
$\angle c_3 = \pi/4$	$\lvert c_4 \rvert = \sqrt{2}/4$





□ Power density spectrum of periodic signals

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2$$

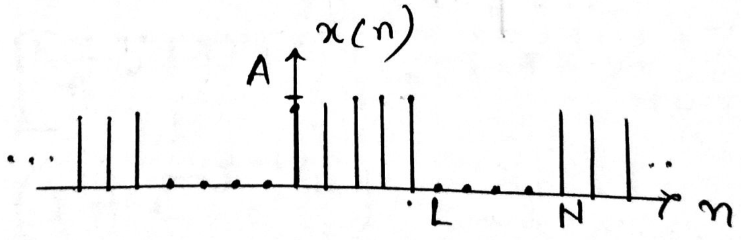
$$= \sum_{k=0}^{N-1} |c_k|^2$$

* If we are interested in the energy of the sequence $x(n)$ over a single period :-

$$E_N = \sum_{n=0}^{N-1} |x(n)|^2$$

$$= N \sum_{k=0}^{N-1} |c_k|^2$$

* Ex - 4.2.2



Determine fourier series coefficients and the power density spectrum.

Soln :-

$$c_k = \frac{1}{N}\sum_{n=0}^{L-1} A e^{-j2\pi kn/N} = \frac{A}{N}\sum_{n=0}^{L-1} e^{-j2\pi kn/N} \quad \text{———} (1)$$

$$= A/N \cdot \frac{1 - e^{-j2\pi KL/N}}{1 - e^{-j2\pi K/N}}$$

$$= A/N \cdot \frac{e^{-j\pi KL/N} \cdot e^{-j\pi KL/N} - e^{-j2\pi KL/N}}{e^{j\pi K/N} \cdot e^{-j\pi K/N} - e^{-j2\pi K/N}}$$

$$= A/N \cdot \frac{e^{-j\pi KL/N}}{e^{-j\pi K/N}} \cdot \frac{e^{j\pi KL/N} - e^{-j\pi KL/N}}{e^{j\pi K/N} - e^{-j\pi K/N}}$$

$$\therefore C_K = A/N \frac{\sin(\pi KL/N)}{\sin(\pi K/N)} \cdot e^{-j\pi K(L-1)/N}$$

$$\therefore C_k = \begin{cases} AL/N & ; K = 0 \\ A/N \cdot e^{-j\pi k(L-1)/N} \cdot \frac{\sin(\pi kL/N)}{\sin(\pi k/N)} & ; k = 1, 2, 3 .., N-1 \end{cases}$$

If, $k = 0$, from(1)

$$C_k = A/N \sum_{n=0}^{L-1} 1 = AL/N$$

$$\therefore |C_k|^2 = \begin{cases} (AL/N)^2 & ; k = 0 \\ (A/N)^2 \frac{\sin^2(AKL/N)}{\sin^2(AK/N)} & ; k = 1, 2, .., N-1 \end{cases}$$

□ Fourier Transform of discrete time aperiodic Signal:-

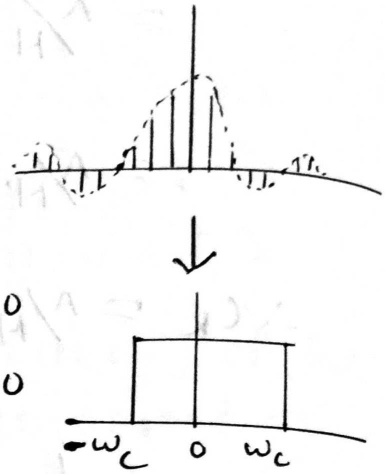
$$* \; x(n) = 1/2\pi \int_{2\pi} X(\omega) e^{j\omega n} d\omega$$

$$* \; X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

Gibbs Phenomenon

Suppose,

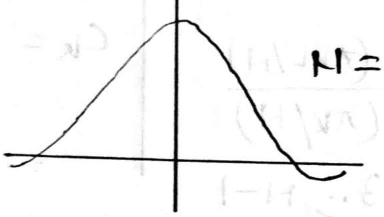
$$X(\omega)=\begin{cases}1, & |\omega|\leq\omega_c\\ 0, & \omega_c<|\omega|\leq\pi\end{cases}$$



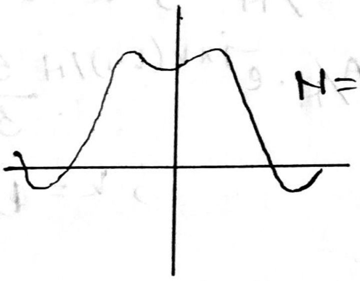
By calculating –

$$x(n)=\begin{cases}\omega_c/\pi, & n=0\\ \omega_c/\pi\cdot\dfrac{\sin\omega_c n}{\omega_c n}, & n\neq 0\end{cases}$$

$$\boxed{X_N(\omega)=\sum_{n=-N}^{N}\frac{\sin\omega n}{\pi n}e^{-j\omega n}}$$

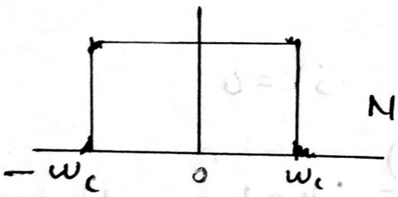


N = 1



N = 3

. . .



N = 70

We notice that there is a significant oscillatory overshoot at $\omega=\omega_c$. It is independent of the value of N. As N increases the oscilations become more rapid but the size of the ripple remains the same. One can show that as $N\to\infty$, the oscillation converges to the point of discontinuity but their amplitude does not go to zero.

↳ The oscillatory behaviour of the approximation $X_N(\omega)$ to the f^n $X(\omega)$ at a point of discontinuity of $X(\omega)$ is called the Gibbs phenomenon.

Energy density spectrum of aperiodic signals

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

Here, E_x is complex valued because of $X(\omega)$ is complex valued

$$X(\omega) = |X(\omega)| e^{j\theta(\omega)}$$

$$\theta(\omega) = \angle X(\omega)$$

$\therefore$ Spectrum :-

$$S_{xx}(\omega) = |X(\omega)|^2$$

* **Ex :- 4.2.3**

$x(n) = a^n u(n)$; $-1 < a < 1$

Soln :-

$$\sum_{n=-\infty}^{\infty} |x(n)| = \sum_{n=0}^{\infty} |a|^n = \frac{1}{1-|a|} < \infty$$

$\therefore$ Fourier transform exists.

$$X(\omega) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \frac{1}{1-ae^{-j\omega}}$$

$$X^*(\omega) = \frac{1}{1-ae^{j\omega}}$$

$$S_{xx} = |X(\omega)|^2$$
$$= X(\omega)X^*(\omega)$$
$$= \frac{1}{(1-ae^{-j\omega})} \cdot \frac{1}{(1-ae^{j\omega})}$$
$$= \frac{1}{(1-ae^{-j\omega})(1-ae^{j\omega})}$$

(Ans.)

*** Ex - 4.2.4**

$$x(n) = \begin{cases} A, & 0 \leq n \leq L-1 \\ 0, & \text{otherwise} \end{cases}$$

Soln:-

$$\sum_{n=-\infty}^{\infty} x(n) = \sum_{n=0}^{L-1} A = A|L| < \infty$$

$\therefore$ Fourier transform exists

$$X(\omega) = \sum_{n=0}^{L-1} A e^{-j\omega n}$$
$$= A \frac{1-e^{-j\omega L}}{1-e^{-j\omega}}$$
$$= A \cdot \frac{e^{-j\omega L/2}}{e^{-j\omega/2}} \cdot \frac{\sin \omega L/2}{\sin \omega/2}$$
$$= A \cdot e^{-j\omega(L-1)/2} \cdot \frac{\sin \omega L/2}{\sin \omega/2}$$

$$S_{xx} = \begin{cases} |A|^2 L^2, & \omega = 0 \\ |A|^2 \left(\frac{\sin \omega L/2}{\sin \omega/2}\right)^2, & \text{otherwise} \end{cases}$$

□ Relationship between fourier and z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \qquad \text{ROC: } r_2 < |z| < r_1$$

$$= \sum_{n=-\infty}^{\infty} [x(n) r^{-n}] e^{-j\omega n} \qquad \begin{array}{|l} z = re^{j\omega} \\ r = |z| \end{array}$$

If $|z| = 1$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= X(\omega)$$

∴ fourier transform is the z-transform of the sequence evaluated on the unit circle.

- If $X(z)$ does not converge in $|z|=1$, the fourier transform of $X(\omega)$ does not exist.
- Note that the existence of z-transform requires that the sequence $\{x(n) r^{-n}\}$ be absolutely summable

$$\sum_{n=-\infty}^{\infty} |x(n) r^{-n}| < \infty$$

□ Symmetry properties of the fourier transform

↳ Book - 4.4.1

Fourier Transform Theorem and properties

* Linearity:-

If $x_1(n) \xleftrightarrow{f} X_1(\omega)$

And $x_2(n) \xleftrightarrow{f} X_2(\omega)$

Then $a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{f} a_1 X_1(\omega) + a_2 X_2(\omega)$

Ex- 4.4.3

Determine fourier transform -

$$x(n) = a^{|n|} \quad , -1 < a < 1$$

Soln:-

$$x(n) = x_1(n) + x_2(n)$$

$$x_1(n) = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$x_2(n) = \begin{cases} a^{-n}, & n < 0 \\ 0, & n \geq 0 \end{cases}$$

$$X_1(\omega) = \sum_{n=0}^{\alpha} a^n e^{-j\omega n}$$

$$= \frac{1}{1 - ae^{-j\omega}}$$

$|a| < 1$

$$X_2(\omega) = \sum_{n=-\alpha}^{-1} a^{-n} e^{-j\omega n}$$

$$= \sum_{n=-\alpha} (ae^{j\omega})^{-n}$$

$$= \sum_{k=\alpha}^{1} (ae^{j\omega})^k$$

$$= \frac{ae^{j\omega}}{1 - ae^{j\omega}}$$

$|a| < 1$

$$\therefore X(\omega) = \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}}$$

(Ans.)

* Time shifting:-

If, $x(n) \xleftrightarrow{f} X(\omega)$

Then, $x(n-k) \longleftrightarrow e^{-j\omega k} X(\omega)$

* Time reversal:-

If, $x(n) \xleftrightarrow{f} X(\omega)$

Then, $x(-n) \xleftrightarrow{f} X(-\omega)$

* Convolution theorem:-

If $x_1(n) \xleftrightarrow{f} X_1(\omega)$

And $x_2(n) \xleftrightarrow{f} X_2(\omega)$

Then $x(n) = x_1(n) * x_2(n) \xleftrightarrow{f} X(\omega) = X_1(\omega) . X_2(\omega$

Example-4.44.

Determine the convolution of the sequence-

$x_1(n) = x_2(n) = \{1, \underset{\uparrow}{1}, 1\}$

Soln:-

$X(\omega) = X_1(\omega) X_2(\omega)$

$$X_1(\omega) = X_2(\omega) = \sum_{n=-1}^{1} e^{-j\omega n}$$

$$= e^{j\omega} + e^{0} + e^{-j\omega}$$

$$= 1 + 2 \frac{e^{j\omega} + e^{-j\omega}}{2}$$

$$= 1 + 2\cos\omega$$

$X(\omega)$

$= (1 + 2\cos\omega)$

$= 1 + 4\cos\omega + 4\cos^2\omega$

$= 1 + 4\cos\omega + 2(1 + \cos 2$

$= 1 + 4\cos\omega + 2 + 2\cos 2A$

$= 3 + 4\cos\omega + 2\cos 2A$

$= 3 + 2e^{j\omega} + 2e^{-j\omega} + e^{2j\omega} + e^{-2j\omega}$

$\therefore x(\omega) = e^{-2j\omega} + 2e^{-j\omega} + 3e^{0j\omega} + 2e^{j\omega} + e^{2j\omega}$

$\therefore$ The sequence $= \{1, 2, \underset{\uparrow}{3}, 2, 1\}$

Response to complex exponentials and sinusoidals

* Convolution-

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

If $x = Ae^{j\omega n}$ complex

$$y(n) = \sum_{k=-\infty}^{\infty} h(k).\, Ae^{j\omega(n-k)}$$

$$= Ae^{j\omega n} \sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega k}$$

$$= x(n).\, H(\omega)$$

Ex - 5.1.1

Impulse response, $h(n) = (1/2)^n u(n)$

Input, $x(n) = Ae^{j\pi n/2}$

Determine output of the system.

Soln.-

$$H(\omega) = \sum_{n=-\infty}^{\infty} (1/2)^n e^{-j\omega n} = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

$\therefore H(\pi/2)$

$$= \frac{1}{1 + j/2}$$

$$= 2/\sqrt{5} \cdot e^{-j26.6^\circ}$$

$$\therefore y(n) = x(n).\, H(\omega)$$

$$= x(n).\, H(\pi/2)$$

$$= A.e^{j\pi n/2} \cdot 2/\sqrt{5} \cdot e^{-j26.6^\circ}$$

$$\therefore y(n) = 2A/\sqrt{5} \cdot e^{j(\pi n/2 - 26.6^\circ)}$$

(Ans.)

Ex-5.1.3
↳ Book

Response to aperiodic input signals

$$Y(\omega) = X(\omega) H(\omega)$$

$$\therefore y(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(\omega) e^{j\omega n} d\omega$$

$$|Y(\omega)| = |X(\omega)| |H(\omega)|$$

$$\therefore S_{yy}(\omega) = |Y(\omega)|^2$$

$$= |H(\omega)|^2 |X(\omega)|^2$$

$$= |H(\omega)|^2 S_{xx}(\omega)$$

$$\therefore E_y = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{yy}(\omega) d\omega$$

*Ex-5.1.5

$h(n) = (1/2)^n u(n)$

$x(n) = (1/4)^n u(n)$

Determine spectrum and energy density spectrum.

Soln:-

$$H(\omega) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$$

$$X(\omega) = \frac{1}{1 - \frac{1}{4} e^{-j\omega}}$$

$$Y(\omega) = H(\omega).X(\omega)$$

$$= \frac{1}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$$ → Spectrum

$$|H(\omega)|^2 = H(\omega).H^*(\omega)$$

$$= \frac{1}{1-\frac{1}{2}e^{-j\omega}} \cdot \frac{1}{1-\frac{1}{2}e^{j\omega}}$$

$$= \frac{1}{1-\frac{1}{2}e^{j\omega}-\frac{1}{2}e^{-j\omega}+\frac{1}{4}}$$

$$= \frac{1}{\frac{5}{4}-\cos\omega}$$

Similarly, $|X(\omega)|^2 = X(\omega).X^*(\omega)$

$$= \frac{1}{\frac{17}{16}-\frac{1}{2}\cos\omega}$$

$$\therefore \; S_{yy}(\omega) = \frac{1}{(\frac{5}{4}-\cos\omega)(\frac{17}{16}-\frac{1}{2}\cos\omega)}$$ → Energy density spectrum.

5.4.3 Digital Resonators
5.4.4 Notch filters
5.4.5 Comb filters

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