Direct z-transform:-

↳ Transforms time domain signal $x(n)$ into its complex-plane representation $X(z)$

The inverse process is called inverse z-transform

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

$$X(z) \equiv z[x(n)]$$

$$\boxed{X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n}}$$

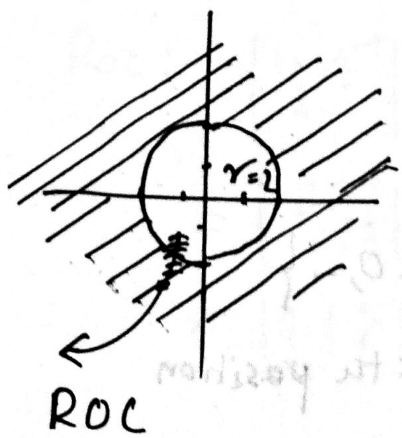
$$= x(\infty) z^{+\infty} + \ldots + x(-1) z^{1} + x(0) z^{0} + x(1) z^{-1} + \ldots + x(\infty) z^{-\infty}$$

The region of convergence (ROC)

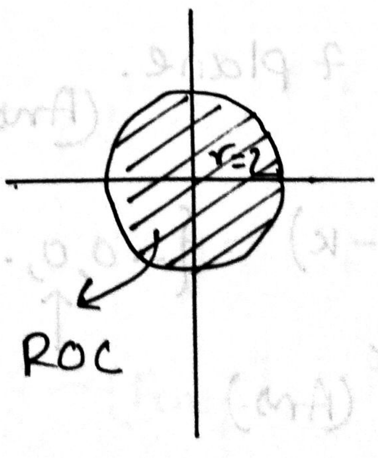
↳ set of all values of z for which $X(z)$ attains a finite value.

Ex -

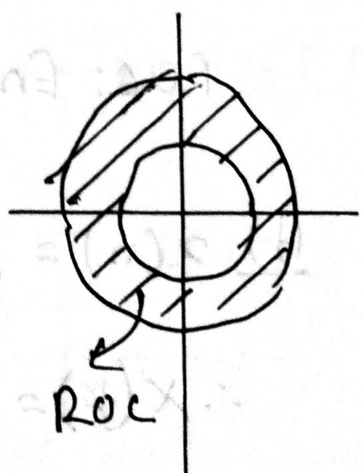
* ROC: $|z| > 2$



* ROC: $|z| < 2$



* ROC: $3 < r < 4$



* Ex – 3.1.1

(b) $x(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$

$\therefore X(z) = 1 \cdot z^{2} + 2 \cdot z + 5 + 7z^{-1} + z^{-3}$

ROC: All values of z, except $z = 0$ and $z = \infty$ $\longrightarrow$

(Ans.)

$X(z) = \infty$ আসা যাবেনা।
$z = 0$ হলে ঋণাংশগুলো ∞ হবে।
$z = \infty$ হলে অঋণাংশগুলো ∞ হবে।

(a) $x(n) = \{\underset{\uparrow}{1}, 2, 5, 7, 0, 1\}$

$\therefore X(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$

$= 1 + 2/z + 5/z^{2} + 7/z^{3} + 1/z^{5}$

ROC: All values of z except $z = 0$

(Ans.)

(c) $x(n) = \delta(n) = \{\ldots, 0, 0, \underset{\uparrow}{1}, 0, 0, \ldots\}$

$\therefore X(z) = 1 \cdot z^{0}$

$= 1$

ROC: Entire z plane.

(Ans.)

(d) $x(n) = \delta(n-k) = \{\ldots 0, \underset{\uparrow}{0}, \ldots, 1, \ldots 0, \ldots\}$

$\hookrightarrow$ kth position

$\therefore X(z) = z^{-k}$ (Ans.)

ROC: Entire z plane, except $z = 0$

(g) $x(n) = \delta(n+k) = \{..0, .., 1, 0, ... 0, ... 0, ..\}$

$\therefore X(z) = z^{k}$ $\quad$ $-k$ th position

ROC: Entire z plane except $z = \infty$

(Ans.)

* Ex-3.1.2 $\quad x(n) = (1/2)^n u(n)$

$= \{1, 1/2, (1/2)^2, (1/2)^3, ...\}$

$X(z) = 1 + (1/2) z^{-1} + (1/2)^2 z^{-2} + (1/2)^3 z^{-3} + \cdots$ $\quad | \quad 1 = (1/2)^0 z^0$

$$= \sum_{n=0}^{\infty} (1/2)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (1/2 z^{-1})^n$$

$$= \frac{1}{1 - (1/2 z^{-1})}$$

$$= \frac{2z}{2z - 1}$$

$1 + A + A^2 + \cdots = \frac{1}{1-A}$ if $|A| < 1$

Roc: $|1/2 z^{-1}| < 1$

$= |1/2z| < 1$

$\therefore |z| > 1/2$

(Ans.)

*Ex - 3.1.3

$$x(n) = a^n u(n)$$
$$= \{1, a, a^2, a^3, \dots\}$$
$$= \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$
$$= \sum_{n=0}^{\infty} (az^{-1})^n$$
$$= \frac{1}{1-az^{-1}}$$

$$\boxed{a^n u(n) \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}}$$

ROC : $|az^{-1}| < 1$

$\Rightarrow |1/z| < 1/a$

$\therefore |z| > a$

(Ans.)

* Ex - 3.1.4

$$x(n) = -a^n u(-n-1)$$

$u(n-1) = \{0, \underset{\uparrow}{0}, 1, 1, 1, 1, \dots\}$

$\therefore u(-n-1) = \{\dots 1, 1, 1, 1, \underset{\uparrow}{0}, 0, 0, 0 \dots\}$

$\therefore -a^n u(-n-1) = \{\dots, -a^{-4}, -a^{-3}, -a^{-2}, -a^{-1}, \underset{\uparrow}{0}, 0, 0, \dots\}$

$\therefore X(n) = \sum \{\dots, -a^{-4}z^4, -a^{-3}z^3, -a^{-2}z^2, -a^{-1}z, \dots$

$$\Rightarrow X(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} (az^{-1})^n$$

$$= -\sum_{l=\infty}^{1} (az^{-1})^{-l} \quad [\text{let, } l = -n]$$

$$= -\sum_{l=\infty}^{1} (a^{-1}z)^{l}$$

$$= \cancel{\sum_{l=1}^{\infty} (a^{-1}z)^{l}}$$

$$= -\frac{a^{-1}z}{1-a^{-1}z}$$

$$= \frac{1}{1-az^{-1}}$$

ROC: $|a^{-1}z| < 1$

$\therefore |z| < |a|$ (Ans.)

$$A + A^2 + A^3 + \cdots = \frac{A}{1-A}$$

* Ex-3.1.5

$$x(n) = a^n u(n) + b^n u(-n-1)$$

$$X(n) = \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n}$$

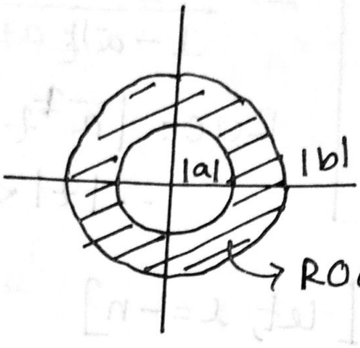
$$= \sum_{n=0}^{\infty} (az^{-1})^n + \sum_{l=1}^{\infty} (b^{-1}z)^n$$

$$= \frac{1}{1-az^{-1}} - \frac{1}{1-bz^{-1}}$$

$$= \frac{1}{a+b-z-abz^{-1}}$$

- For $a^n u(n)$, ROC: $|z| > |a|$
- for $b^n u(-n-1)$, ROC: $|z| < |b|$
- Now, if $|a| < |b$ ROC: $|a| < |z| < |$
- And, if $|a| > |b|$ ~~ROC: $|a| < |z|$~~

we do not find any value that satisfies both series. So $X(z)$ does not exists.

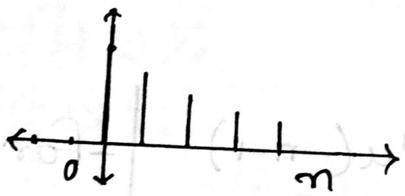


ROC when $|a| < |b|$

Characteristic families of signals with Their corresponding ROCs:-

* finite duration signals

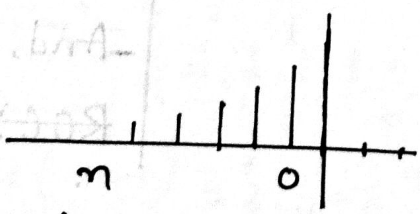
(1) Causal:-



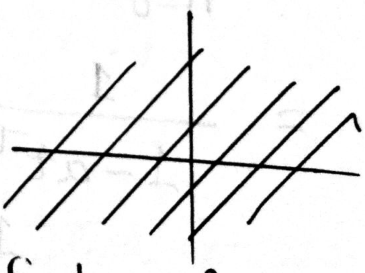
$x(n) = 0$ for $n < n_0 < 0$

ROC: Entire z-plane, except $z = 0$

(2) Anticausal:-

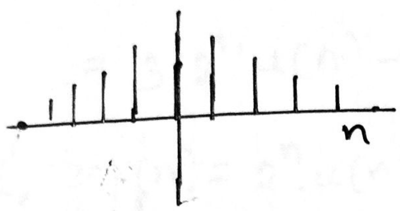


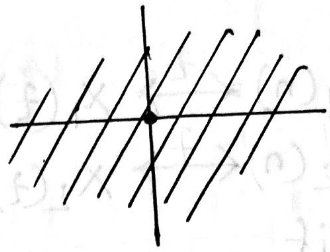
$x(n) = 0$ for $n > n_1 > 0$



Entire z plane except $z = \infty$

(3) Two sided:-

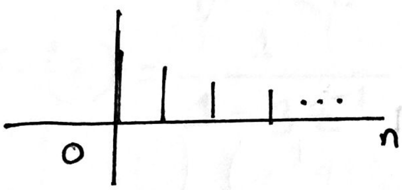


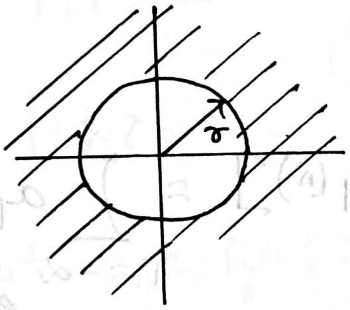


ROC: Entire z plane, except $z=0$ and $z=\infty$

* Infinite duration signals:

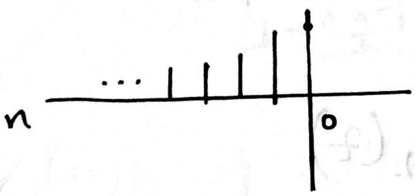
(1) Causal:-

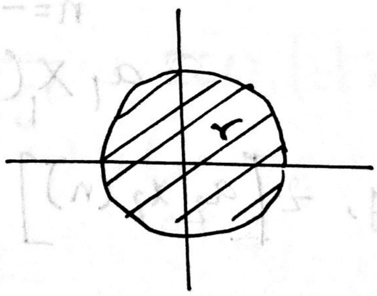




$|z| > r$

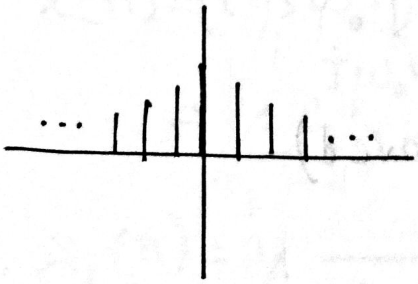
(2) Anticausal:-

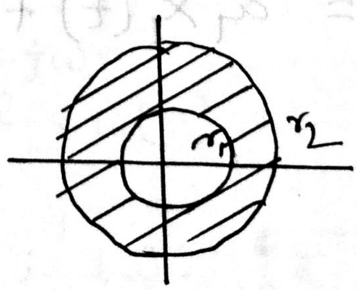




$|z| < r$

(3) Two sided:-





$r_1 < |z| < r_2$

Properties of z-transform:-

(1) Linearity:-

If $x_1(n) \xleftrightarrow{z} X_1(z)$
and, $x_2(n) \xleftrightarrow{z} X_2(z)$
then, if,

$$x(n) = a_1 x_1(n) + a_2 x_2(n)$$
$$X(z) = a_1 X_1(z) + a_2 X_2(z)$$

* Proof:-

$$z[a_1 x_1(n)] = \sum_{n=-\infty}^{\infty} a_1 x_1(n) z^{-n}$$
$$= a_1 \sum_{n=-\infty}^{\infty} x_1(n) z^{-n}$$
$$= a_1 X_1(z)$$

Similarly, $z[a_2 x_2(n)] = a_2 X_2(z)$

$$\therefore X(z) = z[a_1 x_1(n)] + z[a_2 x_2(n)]$$
$$= a_1 X_1(z) + a_2 X_2(z)$$

(Proved)

*Ex- 3.2.1

$x(n) = [3(2^n) - 4(3^n)]u(n)$

$= 3.2^n.u(n) - 4.3^n.u(n)$

Let, $x_1(n) = 2^n.u(n)$

$x_2(n) = 3^n.u(n)$

$\therefore x(n) = 3x_1(n) - 4x_2(n)$

$X_1(z) = \dfrac{1}{1-2z^{-1}}$; ROC: $|z| > 2$

$X_2(z) = \dfrac{1}{1-3z^{-1}}$; ROC: $|z| > 3$

$\because$ if $x(n) = a^n u(n)$

$X(z) = \dfrac{1}{1-az^{-1}}$

ROC: $|z| > |a|$

The intersection of both the ROC is $|z| > 3$

$\therefore X(z) = 3X_1(z) - 4X_2(z)$

$= \dfrac{3}{1-2z^{-1}} - \dfrac{4}{1-3z^{-1}}$; ROC: $|z| > 3$

(Ans.)

*Ex-3.2.2

(a) $x(n) = (\cos w_0 n)\, u(n)$

$= \frac{1}{2}e^{jw_0 n}.u(n) + \frac{1}{2}e^{-jw_0 n}.u(n)$

$\therefore X_1(n) = \frac{1}{2}\dfrac{1}{1-e^{jw_0}.z^{-1}}$; ROC: $|z| > |e^{jw_0}|$

$\therefore |z| > 1$

$$\therefore X_2(z) = \frac{1}{2}\frac{1}{1 - e^{-j\omega_0} \cdot z^{-1}} \quad ; ROC : |z| > |e^{-j\omega_0}|$$

$$\therefore |z| > 1$$

$$\therefore X(z) = \frac{1}{2}\frac{1}{1 - e^{j\omega_0} z^{-1}} + \frac{1}{2}\frac{1}{1 - e^{-j\omega_0} z^{-1}}$$

$$= \frac{1}{2}\frac{1 - e^{-j\omega_0} \cdot z^{-1} + 1 - e^{j\omega_0} \cdot z^{-1}}{1 - e^{j\omega_0} \cdot z^{-1} - e^{-j\omega_0} \cdot z^{-1} + z^{-2}}$$

$$= \frac{1}{2}\frac{2 - 2z^{-1}\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right)}{1 - 2z^{-1}\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right) + z^{-2}}$$

$$= \frac{1 - z^{-1}\cos\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}$$

$$|e^{\pm j\omega_0}|$$

$$= |\sin(\pm\omega_0) + j\cos(\pm\omega_0)|$$

$$= \sqrt{\sin^2\omega_0 + \cos^2\omega_0}$$

$$= 1$$

ROC: $|z| > 1$ (Ans.)

<u>(b)</u>

$$x(n) = (\sin\omega_0 n)\, u(n)$$

$$= \frac{1}{2j}\left(e^{j\omega_0 n} - e^{-j\omega_0 n}\right) u(n)$$

$$= \frac{1}{2j}\left[e^{j\omega_0 n} \cdot u(n) - e^{-j\omega_0 n} \cdot u(n)\right]$$

$$X(z) = \frac{1}{2j}\frac{1}{1 - e^{j\omega_0} z^{-1}} - \frac{1}{2j}\frac{1}{1 - e^{-j\omega_0} \cdot z^{-1}}$$

$$= \frac{z^{-1}\sin\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}$$

ROC: $|z| > 1$ (Ans.)

(2) Shifting:-

If $x(n) \overset{z}{\longleftrightarrow} X(z)$

Then, $x(n-k) \overset{z}{\longleftrightarrow} z^{-k}X(z)$

*Proof:-

$$z[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$\therefore z[x(n-k)] = \sum_{n=-\infty}^{\infty} x(n-k)\, z^{-n}$$

$$= \sum_{l=-\infty}^{\infty} x(l)\, z^{-(l+k)} \quad [\text{Let, } l = n-k]$$

$$= z^{-k} \sum_{l=-\infty}^{\infty} x(l)\, z^{-l} \cancel{= z^{-k} \sum_{n=\infty}^{\infty} x(n)\, z^{-n}}$$

$$= z^{-k} X(z)$$

ROC: If $k>0$, same as $X(z)$ except $z=0$

If $k<0$, same as $X(z)$ except $z=\infty$

* Ex-3.2.3

$x_1(n) = \{\underset{\uparrow}{1}, 2, 5, 7, 0, 1\}$

$x_2(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$

$x_3(n) = \{\underset{\uparrow}{0}, 0, 1, 2, 5, 7, 0, 1\}$

$\therefore\ x_2(n) = x_1(n-2)$

$x_3(n) = x_1(n+2)$

$$\therefore\ X_2(z) = z^2 X_1(z)$$
$$= z^2 + 2z + 5 + 7z^{-1} + z^{-3}$$

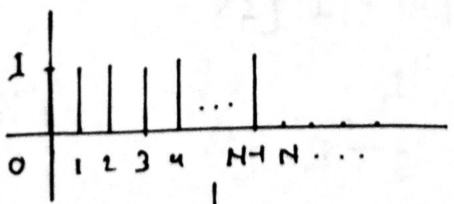
$$\therefore\ X_3(z) = z^{-2} X_1(z)$$
$$= z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}$$

(Ans.)

* Ex-3.2.4

$$x(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$$

↓



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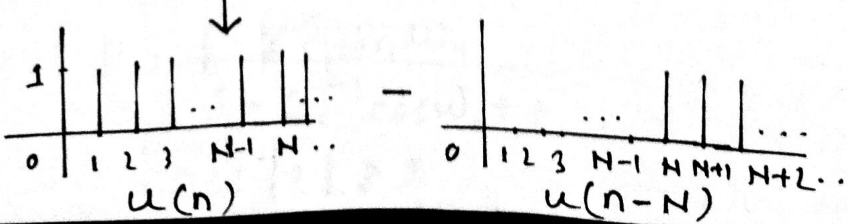


$\therefore\ x(n) = u(n) - u(n-N)$

$\therefore\ X(z) = Z[u(n)] - Z[u(n-N)]$

$= Z[u(n)] - z^{-N} Z[u(n)]$

$= (1 - z^{-N}) Z[u(n)]$

we know, $a^n u(n) \xleftrightarrow{Z} \frac{1}{1 - az^{-1}}$ and $|z| > |a|$

Here, $a = 1$

$\therefore\ X(z) = (1 - z^{-N}) \cdot \frac{1}{1 - z^{-1}}$

$= \frac{1 - z^{-N}}{1 - z^{-1}}$; ROC: $|z| > 1$

(3) Scaling:-

If -

$$x(n) \xleftrightarrow{z} X(z) \qquad ROC: r_1 < |z| < r_2$$

then -

$$a^n x(n) \xleftrightarrow{z} X(a^{-1}z) \qquad ROC: |a|r_1 < |z| < |a|r_2$$

for any constant a — real or complex.

* **Proof:-**

$$z[a^n x(n)] = \sum_{n=\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)(a^{-1}z)^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot p^{-n} \quad , \text{let } p = a^{-1}z$$

$$= X(p)$$

$$= X(za^{-1})$$

$$ROC: r_1 < |za^{-1}| < r_2$$

$$\therefore |a|r_1 < |z| < |a|r_2$$

* **Ex - 3.2.5**

(a)

$$x(n) = a^n (\cos \omega_0 n) u(n)$$

$$= a^n x_1(n) \quad , \text{ let } x_1(n) = (\cos \omega_0 n) u(n)$$

$$\therefore X(z) = X_1(a^{-1} z)$$

$$= \frac{1 - a z^{-1} \cos \omega_0}{1 - 2 a z^{-1} \cos \omega_0 + a^2 z^{-2}} \quad [\text{Ex} - 3.2.2]$$

ROC: $|z| > |a|$ (Ans.)

(b)

$$x(n) = a^n (\sin \omega_0 n) u(n)$$

$$= a^n x_1(n)$$

$$\therefore X(z) = X_1(a^{-1} z)$$

$$= \frac{a z^{-1} \sin \omega_0}{1 - 2 a z^{-1} \cos \omega_0 + a^2 z^{-2}}$$

ROC: $|z| > |a|$ (Ans.)

(4) Time Reversal :-

If -

$$x(n) \xleftrightarrow{z} X(n) \qquad \text{ROC: } r_1 < |z| < r_2$$

Then,

$$x(-n) \xleftrightarrow{z} X(z^{-1}) \qquad \text{ROC: } \frac{1}{r_2} < |z| < 1/r_1$$

* **Proof:-**

$$\sum_{n=-\infty}^{\infty} \text{ and } \sum_{n=\infty}^{-\infty}$$

will result in same

$$Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

$$= \sum_{l=-\infty}^{\infty} x(l) z^{l} \quad ; \text{let } l=-n$$

$$= \sum_{l=-\infty}^{\infty} x(l) (z^{-1})^{-l}$$

$$= X(z^{-1})$$

ROC: $r_1 < |z^{-1}| < r_2$

$\therefore \quad 1/r_2 < |z| < 1/r_1$

* **Ex-3.2.6**

$x(n) = u(-n)$

~~$\therefore X(z) =$~~

We know,

$$u(n) \xleftrightarrow{z} \frac{1}{1-z^{-1}}$$

ROC: $|z| > 1$

$$u(-n) \xleftrightarrow{z} \frac{1}{1-z}$$

ROC: $|z| < 1$

(Ans.)

(5) Differentiation in z-domain

If,

$$x(n) \xleftrightarrow{z} X(n)$$

Then,

$$n\,x(n) \xleftrightarrow{z} -z\,\frac{dX(z)}{dz}$$

* **Proof:-**

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$\therefore \frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x(n).(-n)\, z^{-n-1}$$

$$= -z^{-1} \sum_{n=\infty}^{\infty} n\,x(n)\, z^{-n}$$

$$= -z^{-1}\, Z\left[n\,x(n)\right]$$

$$\therefore Z\left[n\,x(n)\right] = -z\,\frac{dX(z)}{dt}$$

ROC will be same.

* Ex - 3.2.7

$$x(n) = n a^n u(n)$$
$$= n x_1(n) \quad , \text{let } x_1(n) = a^n u(n)$$

$$\therefore X(z) = -z \frac{dX_1(z)}{dz}$$

$$= -z \frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right) \quad \Bigg| \text{ROC: } |z| > |a|$$

$$= -z \cdot \frac{d}{dz} (1-az^{-1})^{-1}$$

$$= -z \left[-1 (1-az^{-1})^{-2} \cdot (-a)(-1) z^{-2} \right]$$

$$= az^{-1} (1-az^{-1})^{-2}$$

$$= \frac{az^{-1}}{(1-az^{-1})^2}$$

ROC: $|z| > |a|$ (Ans.)

(6) Convolution of two sequences

If, $x_1(n) \xleftrightarrow{z} X_1(n)$

$x_2(n) \xleftrightarrow{z} X_2(n)$

Then, $x_1(n) * x_2(n) \xleftrightarrow{z} X_1(z) X_2(z)$

* **Proof:-**

$$x(n) = x_1(n) * x_2(n)$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \cancel{\sum_{n=-\infty}^{\infty} \sum x_1 t} \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) \right] z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \left[\sum_{n=-\infty}^{\infty} x_2(n-k)\, z^{-n} \right]$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \,.\, z^{-k} \,.\, X_2(z)$$

$$= X_2(z) \sum_{k=-\infty}^{\infty} x_1(k) \,.\, z^{-k}$$

$$\therefore X(z) = X_2(z) \,.\, X_1(z)$$

ROC of $X(z)$ is the intersection of ROC of $X_1(z)$ and $X_2(z)$.

* Ex - 3.2.9

$x_1(n) = \{\underset{\uparrow}{1}, -2, 1\}$

$x_2(n) = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & \text{elsewhere} \end{cases}$

$= \{\underset{\uparrow}{1}, 1, 1, 1, 1, 1\}$

$X_1(z) = 1 - 2z^{-1} + z^{-2}$

$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$

$\therefore$ ~~x(z)~~ $X_1(z) . X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$
$- 2z^{-1} - 2z^{-2} - 2z^{-3} - 2z^{-4} - 2z^{-5} - 2z^{-6}$
$+ z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6} + z^{-7}$

$= 1 - z^{-1} - z^{-6} + z^{-7}$

$\therefore x(n) = \{\underset{\uparrow}{1}, -1, 0, 0, 0, 0, -1, 1\}$

(Ans.)

(7) Correlation of two sequences:-

If, $x_1(n) \xleftrightarrow{z} X_1(z)$

$x_2(n) \xleftrightarrow{z} X_2(z)$

$$r_{x_1x_2}(l) = \underbrace{\sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n-l)}_{\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(k-n)} \xleftrightarrow{z} R_{x_1x_2}(z) = X_1(z) X_2(z^{-1})$$

*Proof:-

$$x(n) = \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(k-n)$$

$$\therefore X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(k-n)\, z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \left[\sum_{n=-\infty}^{\infty} x_2(k-n)\, z^{-n} \right]$$

let,
$l = k-n$
$\therefore -n = l-k$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \left[\sum_{l=-\infty}^{\infty} x_2(l)\, z^{-(l-k)} \right]$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \left[\sum_{l=-\infty}^{\infty} x_2(l) (z^{-1})^{-l} \cdot z^{-k} \right]$$

$$= X_2(z^{-1}) \sum_{k=-\infty}^{\infty} x_1(k)\, z^{-k}$$

$$= X_1(z)\, X_2(z^{-1})$$

(Ans.)

* Ex-3.2.10

$x(n) = a^n u(n)$, $-1 < a < 1$

Autocorrelation, $R_{xx}(z) = X(z)X(z^{-1})$

$$= \frac{1}{1-az^{-1}} \cdot \frac{1}{1-az}$$

$$= \frac{1}{1-a(z+z^{-1})+a^2}$$

for $X(z)$, ROC: $|z| > |a|$

for $X(z^{-1})$, ROC: $|z| < |1/a|$

$\therefore$ ROC: $|a| < |z| < |1/a|$

(Ans.)

Table - 3.1

- 3.2

- 3.3

(4) Poles and zeros:-

Poles

↳ The poles of a z-transform are the values of z for which $X(z) = \infty$.

Zeros

↳ The zeros of a z-transform are the values of z for which $X(z) = 0$

***Ex - 3.3.1**

$x(n) = a^n u(n)$

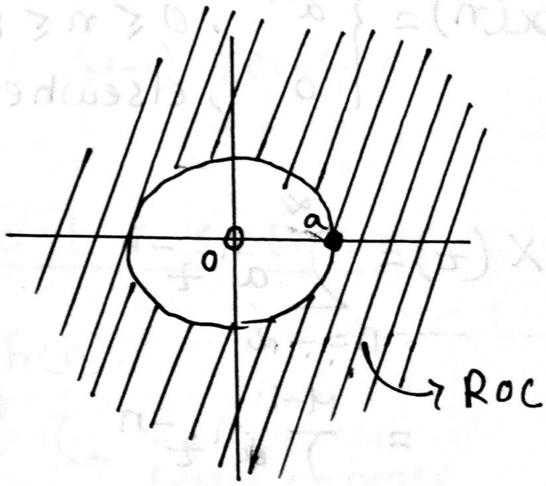
$X(z) = \frac{1}{1 - az^{-1}}$

$= \frac{z}{z - a}$

ROC: $|z| > a$

$X(z)$ will be zero at $z = 0$

$X(z)$ " " ∞ " $z = a$



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(1) ~~finite series sum~~

$$\cancel{a + ar + ar^2 + ar^3 + \dots + ar^n = \frac{a(1 - r^{n+1})}{1 - r} ; \text{ if } r \neq 1}$$

(1) Finite series sum:-

$$* \quad * \sum_{n=0}^{N} ar^n = \frac{a(1-r^{N+1})}{1-r} \quad ; r \neq 1$$

$$* \sum_{n=1}^{N} ar^{n-1} = \frac{a(1-r^N)}{1-r} \quad ; r \neq 1$$

(2) Infinite series sum:-

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad ; r < 1$$

---

*Ex-3.3.2 (***)

$$x(n) = \begin{cases} a^n & , 0 \leq n \leq M-1 \\ 0 & , \text{elsewhere} \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{M-1} a^n z^{-n} \qquad \left| \begin{array}{l} A = 1 \\ R = az^{-1} \end{array} \right.$$

$$= \frac{1[1-(az^{-1})^M]}{1-az^{-1}}$$

$$= \frac{1-a^M z^{-M}}{1-az^{-1}} \quad ; |az^{-1}| < 1$$

$$= \frac{z^M - a^M}{z^{M-1}(z-a)} \qquad \therefore |z| > |a|$$

Poles:

$$z^{M-1} = 0 \quad \text{or} \quad z - a = 0$$

$\therefore z = 0$ → M−1 গুলো | $z = a$

zeros:

$$z^M - a^M = 0$$
$$\Rightarrow z^M = a^M$$

$\because a > 0$, $z^M = a^M$ has M roots at:—

$$z_k = a e^{j 2\pi k / M} \; ; \; k = 0, 1, \ldots M-1$$

Now, the zero at $k = 0$, $z_0 = a$, cancels the pole at $z = a$.

$\therefore X(z)$ has $(M-1)$ poles and $(M-1)$ zeros.

The System Function of a LTI System:—

$$y(n) = x(n) . h(n)$$

Output sequence — $y(n)$; Input sequence — $x(n)$; Unit sample response — $h(n)$

$$\therefore Y(z) = X(z) . H(z)$$
$$\Rightarrow H(z) = \frac{Y(z)}{X(z)}$$

Note:—
* Partial, complete soln আর z-transformation দিয়ে diff. eqn solve করতে পারি।

* Ex - 3.3.4

$$y(n) = \tfrac{1}{2} y(n-1) + 2x(n)$$

$$\Rightarrow Y(z) = \tfrac{1}{2} z^{-1} Y(z) + 2 . X(z)$$

$$\Rightarrow (1 - \tfrac{1}{2} z^{-1}) . Y(z) = 2 . X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{2}{1 - \tfrac{1}{2} z^{-1}}$$

$$\Rightarrow H(z) = 2 . \frac{1}{1 - \tfrac{1}{2} . z^{-1}}$$

$$\therefore h(n) = 2 . (\tfrac{1}{2})^n u(n)$$

(Ans.)

Inversion of z-transform:-

(1) By contour integration ⟶ দেখবো না
(2) By expansion of series
(3) Partial fraction expansion

* Ex - 3.4.2

$$X(z) = \frac{1}{1 - 1.5 z^{-1} + 0.5 z^{-2}}$$

Determine inverse z-transform when-

(a) ROC: $|z| > 1$
(b) ROC: $|z| < 0.5$

(a) Since, ROC is the exterior of a circle, we expect $x(n)$ to be a causal signal. Thus we seek a power series expansion in negative powers of z.

$$1 - \tfrac{3}{2}z^{-1} + \tfrac{1}{2}z^{-2} \,\big)\, 1 \,\big(\, 1 + \tfrac{3}{2}z^{-1} + \tfrac{7}{4}z^{-2} + \tfrac{15}{8}z^{-3}$$

$$\underline{\overset{(-)}{1} \overset{(+)}{-} \tfrac{3}{2}z^{-1} \overset{(-)}{+} \tfrac{1}{2}z^{-2}}$$

$$\tfrac{3}{2}z^{-1} - \tfrac{1}{2}z^{-2}$$

$$\underline{\overset{(-)}{\tfrac{3}{2}}z^{-1} \overset{(+)}{-} \tfrac{9}{4}z^{-2} \overset{(-)}{+} \tfrac{3}{4}z^{-3}}$$

$$\tfrac{7}{4}z^{-2} - \tfrac{3}{4}z^{-3}$$

$$\underline{\overset{(-)}{\tfrac{7}{4}}z^{-2} \overset{(+)}{-} \tfrac{21}{8}z^{-3} \overset{(-)}{+} \tfrac{7}{8}z^{-4}}$$

$$\tfrac{15}{8}z^{-3} - \tfrac{7}{8}z^{-4}$$

$$\underline{\overset{(-)}{\tfrac{15}{8}}z^{-3} \overset{(+)}{-} \tfrac{45}{16}z^{-4} \overset{(-)}{+} \tfrac{15}{16}z^{-5}}$$

$$\tfrac{31}{16}z^{-4} - \tfrac{15}{16}z^{-5}$$

$$\therefore X(z) = 1 + \tfrac{3}{2}z^{-1} + \tfrac{7}{4}z^{-2} + \tfrac{15}{8}z^{-3} + \cdots$$

$$\therefore x(n) = \{\underset{\uparrow}{1}, \tfrac{3}{2}, \tfrac{7}{4}, \tfrac{15}{8}, \cdots\}$$ (Ans.)

(b)

Since ...

$$\tfrac{1}{2}z^{-2} - \tfrac{3}{2}z^{-1} + 1 \,\big)\, 1 \,\big(\, 2z^2 + 6z^3 + 14z^4 + 30z^5 + \cdots$$

$$\underline{\overset{(-)}{1} \overset{(+)}{-} 3z \overset{(-)}{+} 2z^2}$$

$$3z - 2z^2$$

$$\underline{\overset{(-)}{3z} \overset{(+)}{-} 9z^2 \overset{(-)}{+} 6z^3}$$

$$7z^2 - 6z^3$$

$$\underline{\overset{(-)}{7z^2} \overset{(+)}{-} 21z^3 \overset{(-)}{+} 14z^4}$$

$$15z^3 - 14z^4$$

$$\underline{\overset{(-)}{15z^3} \overset{(+)}{-} 45z^4 \overset{(-)}{+} 30z^5}$$

$$31z^4 - 30z^5$$

$$\therefore X(z) = 2z^2 + 6z^3 + 14z^4 + 30z^5 + \cdots$$

Here, $x(n) = 0$ for $n \geq 0$

$$\therefore x(n) = \{\cdots, 30, 14, 6, 2, 0, \underset{\uparrow}{0}\}$$ (Ans.)

* Ex- 3.4.5

Determine partial fraction expansion.

$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

$$= \cancel{2R} \; \frac{z^2}{z^2-1.5z+0.5}$$

$$= \frac{z^2}{(z-0.5)(z-1)}$$

$$\Rightarrow \frac{X(z)}{z} = \frac{z}{(z-0.5)(z-1)}$$

Let, $\frac{z}{(z-0.5)(z-1)} = \frac{A}{(z-0.5)} + \frac{B}{(z-1)}$

$\Rightarrow z = A(z-1) + B(z-0.5)$ ——— (1)

Putting, $z=1$ in (1)

$1 = B \cdot 0.5$

$\therefore B = 2$

Putting $z = 0.5$ in (1)

$0.5 = -0.5A$

$\therefore A = -1$

$$\therefore \frac{X(z)}{z} = \frac{2}{z-1} - \frac{\cancel{0.5}\,1}{z-0.5}$$ (Ans.)

*Ex-3.4.6

$$X(z) = \frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}$$

$$= \frac{(z+1)z}{z^2-z+0.5}$$

$$= \frac{(z+1)z}{(z-P_1)(z-P_2)}$$; where, $P_1 = \frac{1+j}{2}$

$P_2 = \frac{1-j}{2}$

By solving $z^2 - z + 0.5 = 0$

$$\Rightarrow \frac{X(z)}{z} = \frac{z+1}{(z-P_1)(z-P_2)}$$

Let, $$\frac{z+1}{(z-P_1)(z-P_2)} = \frac{A}{z-P_1} + \frac{B}{z-P_2}$$

$$\Rightarrow z+1 = A(z-P_2) + B(z-P_1) \quad \text{——(1)}$$

Putting $z = P_1$ in (1)

$P_1 + 1 = A(P_1 - P_2)$

$\therefore A = 1/2 - j\,3/2$

Putting $z = P_2$ in (1)

$\therefore B = 1/2 + j\,3/2$

$$\therefore \frac{X(z)}{z} = \frac{1/2 - 3/2\,j}{z - 1/2 - 1/2\,j} + \frac{1/2 + 3/2\,j}{z + 1/2 + 1/2\,j}$$

(Ans.)

* Ex – 3.4.7

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}$$

$$= \frac{z^3}{(z+1)(z-1)^2}$$

$$\therefore \frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2}$$

Let, $$\frac{z^2}{(z+1)(z-1)^2} = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$\therefore z^2 = A(z-1)^2 + B(z-1)(z+1) + C(z+1) \quad \text{—(1)}$$

Putting $z = 1$ in (1)

$$1 = 2.C$$

$$\therefore C = 1/2$$

Putting $z = -1$ in (1)

$$1 = 4.A$$

$$\therefore A = 1/4$$

Putting $C = 1/2$ and $A = 1/4$ in (1)

$$z^2 = 1/4(z^2 - 2z + 1) + B(z^2 - 1) + 1/2(z+1)$$

$$= 1/4 z^2 - 1/2 z + 1/4 + Bz^2 - B + 1/2 z + 1/2$$

$$= z^2(1/4 + B) + (3/4 - B) \quad \text{—(2)}$$

$$\Rightarrow 0 = z^2(1/4 + B - 1) + (3/4 - B) \quad \text{—(2)}$$

Comparing coefficients

$$3/4 - B = 0 \qquad \therefore B = 3/4$$

$$\therefore \frac{X(z)}{z} = \frac{1/4}{z+1} + \frac{3/4}{z-1} + \frac{1/2}{(z-1)^2} \quad \text{(Ans.)}$$

$\boxed{\text{Poles are distinct}}$

$$* \; Z^{-1}\left\{\frac{1}{1-P_k z^{-1}}\right\} = \begin{cases} (P_k)^n u(n) & ; \text{if ROC: } |z| > |P_k| \\ -(P_k)^n u(-n-1) & ; \text{if ROC: } |z| < |P_k| \end{cases}$$

* Ex – 3.4.8

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

from 3.4.5 → $\boxed{\text{Can't skip at exam}}$

$$X(z) = \frac{2}{1 - z^{-1}} - \frac{1}{1 - 0.5z^{-1}}$$

(a) ROC: $|z| > 1$

In this case, $x(n)$ is causal and both terms are in causal terms.

$$\therefore x(n) = 2.(1)^n u(n) - (0.5)^n u(n)$$
$$= [2 - (0.5)^n] u(n) \quad \text{(Ans.)}$$

(b) ROC: $|z| < 0.5$

In this case, $x(n)$ is anticausal.

$$x(n) = -2.u(-n-1) + 0.5^n u(-n-1)$$
$$= (-2 + 0.5^n) u(-n-1) \quad \text{(Ans.)}$$

(c) ROC: $0.5 < |z| < 1$

In this case, $x(n)$ is two sided. The given ROC is the overlapping of $|z| > 0.5$ and $|z| < 1$. Hence, the pole $P_2 = 0.5$ provides the causal part and $P_1 = 1$ provides the anticausal part.

$$x(n) = -2 \cdot (1)^n \cdot u(-n-1) - (0.5)^n u(n)$$

(Ans.)

* Ex- 3.4.9

$$X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}}$$; Determine the causal signal $x(n)$.

from Ex- 3.4.6 —

$$X(z) = \frac{A_1}{1 - P_1 z^{-1}} + \frac{A_2}{1 - P_2 z^{-1}}$$

where, $A_1 = \frac{1}{2} - \frac{3}{2} j$

$A_2 = \frac{1}{2} + \frac{3}{2} j$

And, $P_1 = \frac{1}{2} + \frac{1}{2} j$

$P_2 = \frac{1}{2} - \frac{1}{2} j$

$A_1 = \frac{\sqrt{10}}{2} e^{-j\, 71.565}$

$P_1 = \frac{1}{\sqrt{2}} e^{j\pi/4}$

$$\therefore\ x(n) = 2 \cdot \frac{\sqrt{10}}{2} \cdot \left(\frac{1}{\sqrt{2}}\right)^n \cdot \cos\left(\frac{\pi}{4} \cdot n - 71.565\right) \cdot u(n)$$

(Ans.)

* when, $x(n) = [A_K (P_K)^n + A_K^* (P_K^*)^n]\, u(n)$

$\hookrightarrow A_K = |A_K|\, e^{j\alpha_K}$

$P_K = \underbrace{|P_K|}_{r_K}\, e^{j\beta_K}$

Then, $x(n) = 2\,|A_K|\,(r_K)^n \cos(\beta_K n + \alpha_K)\, u(n)$

* Ex – 3.4.10

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2} \longrightarrow \text{Determine the causal signal } x(n).$$

$$\Rightarrow X(z) = \frac{1/4}{1+z^{-1}} + \frac{3/4}{1-z^{-1}} + \frac{1/2\, z^{-1}}{(1-z^{-1})^2}$$

$$\therefore\ x(n) = \tfrac{1}{4} \cdot (-1)^n u(n) + \tfrac{3}{4} \cdot u(n) + \tfrac{1}{2}\, n\, u(n)$$

(Ans.)

$$* \; y_n = \underbrace{\sum_{K=1}^{N} A_K (P_K)^n u(n)}_{\substack{\text{Natural} \\ \text{Response} \\ y_{nr}(n)}} + \underbrace{\sum_{K=1}^{N} Q_K (q_K)^n u(n)}_{\substack{\text{Forced} \\ \text{Response} \\ y_{fr}(n)}}$$

y_n : Output signal

Transient Response $\rightarrow$ If $|P_K| < 1$ for all K, then $y_{nr}(n)$ decays to zero as n approaches infinity. In such case, we refer to the natural response of the system as transient response.

Steady State Response $\rightarrow$ when the causal input signal is a sinusoid, the poles fall on the unit circle and consequently, the forced response is also a sinusoid that persist for all $n \geq 0$. In this case, the forced response is called the steady-state response.

* Ex – 3.6.2

$$y(n) = 0.5y(n-1) + x(n)$$

$$\Rightarrow Y(z) = 0.5 z^{-1} Y(z) + X(z)$$

$$\Rightarrow Y(z)(1 - 0.5z^{-1}) = X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1}{1 - 0.5z^{-1}}$$

$$x(n) = 10\cos(\pi n/4)\,u(n)$$

$$\therefore X(z) = \frac{10\{1 - \cos(\pi/4) z^{-1}\}}{1 - 2 z^{-1}\cos\pi/4 + z^{-2}}$$

$$= \frac{10(1 - 1/\sqrt{2}\, z^{-1})}{1 - \sqrt{2} z^{-1} + z^{-2}}$$

$$\therefore H(z) = \frac{1}{1-0.5z^{-1}}$$

$$= \frac{10z^2(1-1/\sqrt{2}\,z^{-1})}{z^2-\sqrt{2}z+1}$$

$$= \frac{10z^2(1-1/\sqrt{2}\,z^{-1})}{(z-e^{j\pi/4})(z-e^{-j\pi/4})}$$

$$\therefore X(z) = \frac{10z^2(1-1/\sqrt{2}\,z^{-1})}{(1-e^{j\pi/4}z^{-1})(1-e^{-j\pi/4}z^{-1})}$$

$X(z)\,H(z)$

↑

$$\therefore Y(z) = \frac{10(1-1/\sqrt{2}\,z^{-1})}{(1-0.5z^{-1})(1-e^{j\pi/4}z^{-1})(1-e^{-j\pi/4}z^{-1})}$$

$$= \frac{\dfrac{10(z-1/\sqrt{2})}{z}}{\dfrac{(z-0.5)(z-e^{j\pi/4})(z-e^{-j\pi/4})}{z^3}}$$

$$= \frac{10z^2(z-1/\sqrt{2})}{(z-0.5)(z-e^{j\pi/4})(z-e^{-j\pi/4})}$$

$$\Rightarrow \frac{Y(z)}{z} = \frac{10z(z-1/\sqrt{2})}{(z-0.5)(z-e^{j\pi/4})(z-e^{-j\pi/4})}$$

$$\Rightarrow \text{let, } \frac{10z(z-1/\sqrt{2})}{(z-0.5)(z-e^{j\pi/4})(z-e^{-j\pi/4})} = \frac{A}{z-0.5} + \frac{B}{z-e^{j\pi/4}} + \frac{C}{z-e^{-j\pi/4}}$$

$$\Rightarrow 10(z)(z-1/\sqrt{2}) = A(z-e^{j\pi/4})(z-e^{-j\pi/4}) + B(z-0.5)(z-e^{-j\pi/4}) + C(z-0.5)(z-e^{j\pi/4}) \quad \text{—— (1)}$$

Putting, $z = 0.5 \Rightarrow A = -1.91$

$z = e^{j\pi/4} \Rightarrow B = 6.78 e^{-j28.7}$

$z = e^{-j\pi/4} \Rightarrow C = 6.78 e^{j28.7}$

$$\therefore \frac{Y(z)}{z} = \frac{-1.91}{z-0.5} + \frac{6.78 e^{-j28.7}}{z - e^{j\pi/4}} + \frac{6.78 e^{j28.7}}{z - e^{-j\pi/4}}$$

$$\Rightarrow Y(z) = \frac{-1.91}{1-0.5z^{-1}} + \frac{6.78 e^{-j28.7}}{1-e^{j\pi/4} z^{-1}} + \frac{6.78 e^{j28.7}}{1-e^{-j\pi/4} z^{-1}}$$

$\therefore$ The transient/natural response

$$y_{nr}(n) = (0.5)^n u(n) \times -1.91$$
$$= (-1.91)(0.5)^n u(n)$$

$\therefore$ The forced/steady state response

$$y_{fr}(n) = \left[6.78 e^{-j28.7} \cdot (e^{j\pi n/4}) + 6.78 e^{j28.7} \cdot (e^{-j\pi n/4})\right] u(n)$$

$$= 2 \cdot (6.78) \cdot (1)^n \cos(\pi n/4 - 28.7)\, u(n)$$

$$= 13.56 \cos(\pi n/4 - 28.7) \cdot u(n)$$

(Ans.)

Stable/Unstable $\rightarrow$ A LTI system is BIBO (Bounded Input B. output) stable if the ROC of $H(z)$ contains the unit circle ($|z|=1$) and vice versa.

$\star\star\star$

* Ex - 3.6.3

$$H(z) = \frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}}$$

$$= \frac{(3-4z^{-1})z^2}{(z-3)(z-1/2)}$$

$$= \frac{3z^2-4z}{(z-3)(z-1/2)}$$

$$= \quad \vdots$$

$$= \frac{1}{1-1/2 z^{-1}} + \frac{2}{1-3z^{-1}}$$

The system has two poles $= 1/2, 3$

(a) The system is stable:

For causal, the ROC will be $|z| > 3$, which will not contain $|z|=1$.

For anticausal, the ROC will be $|z| < 1/2$, which will not contain $|z|=1$

For two sided, the ROC will be $1/2 < |z| < 3$ which contains $|z|=1$. In this case, the system will be stable.

$\therefore h(n) = (1/2)^n u(n) - 2(3)^n u(-n-1)$ (Ans.)

(b) The system is causal:

For causal, ROC: $|z| > 3$, and unstable

$\therefore h(n) = [(1/2)^n + 2(3)^n] u(n)$

(c) The system is anticausal:-

For anticausal, ROC: $|z| < 1/2$, and unstable.

$$\therefore h(n) = -[(1/2)^n + 2(3)^n]\,u(-n-1)$$

(Ans)

Pole-zero Cancellation → When a z-transform has a pole that is at the same location as a zero, the term containing that pole in the inverse z-transform vanishes.

* Ex - 3.6.4 → find unit sample response signal

$$y(n) = 2.5\,y(n-1) - y(n-2) + x(n) - 5x(n-1) + 6x(n-2)$$

$$\Rightarrow Y(z) = 2.5\,z^{-1}Y(z) - z^{-2}Y(z) + X(z) - 5z^{-1}X(z) + 6z^{-2}X(z)$$

$$\Rightarrow (1 - 2.5z^{-1} + z^{-2})\,Y(z) = (1 - 5z^{-1} + 6z^{-2})\,X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1 - 5z^{-1} + 6z^{-2}}{1 - 2.5z^{-1} + z^{-2}}$$

$$\Rightarrow H(z) = \frac{1 - 5z^{-1} + 6z^{-2}}{1 - 2.5z^{-1} + z^{-2}}$$

$$\therefore H(z) = \frac{1 - 5z^{-1} + 6z^{-2}}{(1 - 1/2\,z^{-1})(1 - 2z^{-1})} \rightarrow H(z) \text{ system } f^n$$

The system has poles at $P_1 = 2$ and $P_2 = 1/2$.

$\therefore$ Unit sample response –

$$Y(z) = H(z)\,X(z)$$
$$= H(z)$$
$$= \frac{1 - 5z^{-1} + 6z^{-2}}{(1 - 1/2\,z^{-1})(1 - 2z^{-1})}$$
$$= \frac{z^2(1 - 5z^{-1} + 6z^{-2})}{(z - 1/2)(z - 2)}$$

$$\Rightarrow \frac{Y(z)}{z} = \frac{z(1 - 5z^{-1} + 6z^{-2})}{(z - 1/2)(z - 2)}$$

Let, $\dfrac{z(1 - 5z^{-1} + 6z^{-2})}{(z - 1/2)(z - 2)} = \dfrac{A}{z - 1/2} + \dfrac{B}{z - 2}$

$$\Rightarrow z(1 - 5z^{-1} + 6z^{-2}) = A(z - 2) + B(z - 1/2) \quad \text{—— (1)}$$

Putting $z = 2 \Rightarrow B = 0$

$z = 1/2 \Rightarrow A = \cancel{5/2}\; -5$

$B = 0$ indicates that there exists a zero at $z = 2$ which cancels the pole at $z = 2$.

By cancelling common poles and zeros

$$H(z) = \frac{1 - 5z^{-1} + 6z^{-2}}{(1 - 1/2\,z^{-1})(1 - 2z^{-1})}$$
$$= \frac{z^2 - 5z + 6}{(z - 1/2)(z - 2)}$$
$$= \frac{(z - 3)(z - 2)}{(z - 1/2)(z - 2)}$$
$$= \frac{z - 3}{z - 1/2}$$
$$= \frac{z - 1/2 + 2.5}{z - 1/2}$$
$$= 1 - \frac{2.5}{z - 1/2}$$
$$\therefore H(z) = 1 - \frac{2.5z^{-1}}{1 - 1/2\,z^{-1}}$$

$\therefore h(n) = \delta(n) - 2.5 \cdot (1/2)^n u(n-1)$ (Answer)

Now, $H(z) = \frac{1-3z^{-1}}{1-1/2 z^{-1}}$

$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1-3z^{-1}}{1-1/2 z^{-1}}$

$\Rightarrow Y(z) - 1/2 z^{-1} Y(z) = X(z) - 3z^{-1} X(z)$

$\therefore y(n) = 1/2\, y(n-1) + x(n) - 3x(n-1)$ (Ans.)

* Ex - 3.4.4

Determine the response of the system -

$y(n) = 5/6\, y(n-1) - 1/6\, y(n-2) + x(n)$

$x(n) = \delta(n) - 1/3\, \delta(n-1)$

$Y(z) = 5/6 \cdot z^{-1} Y(z) - 1/6 z^{-2} Y(z) + X(z)$

$\Rightarrow (1 - 5/6 z^{-1} + 1/6 z^{-2}) Y(z) = X(z)$

$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1}{1 - 5/6 z^{-1} + 1/6 z^{-2}}$

$\Rightarrow H(z) = \frac{1}{1 - 5/6 z^{-1} + 1/6 z^{-2}}$

$\therefore H(z) = \frac{1}{(1-1/2 z^{-1})(1-1/3 z^{-1})}$

$X(z) = 1 - 1/3 z^{-1}$

$\therefore Y(z) = \frac{1}{1-1/2 z^{-1}}$ $\quad \therefore y(n) = (1/2)^n u(n)$

(Ans.)

System has two poles at $z = 1/3, 1/2$ and input signal has zero at $z = 1/3$ which cancels the pole.