Basic Queuing Network

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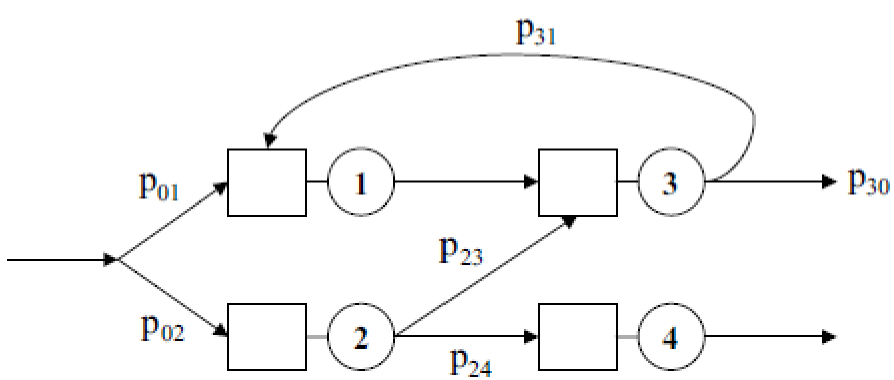
Queueing Networks

- A queueing network is a system composed of several interconnected stations, each with a queue.

- Customers, upon the completion of their service at a station, moves to another station for additional service or leave the system according some **routing rules** (deterministic or probabilistic).



Exponential Queues in Series

- In the queuing models that we have studied so far, a customer's entire service time is spent with a single server.
- In many situations the customer's service is not complete until the customer has been served by more than one server.

- 2-stage tandem network with Independent Service Times

- What the arrival process and rate at node 1 ?

2-stage tandem network - transition diagram

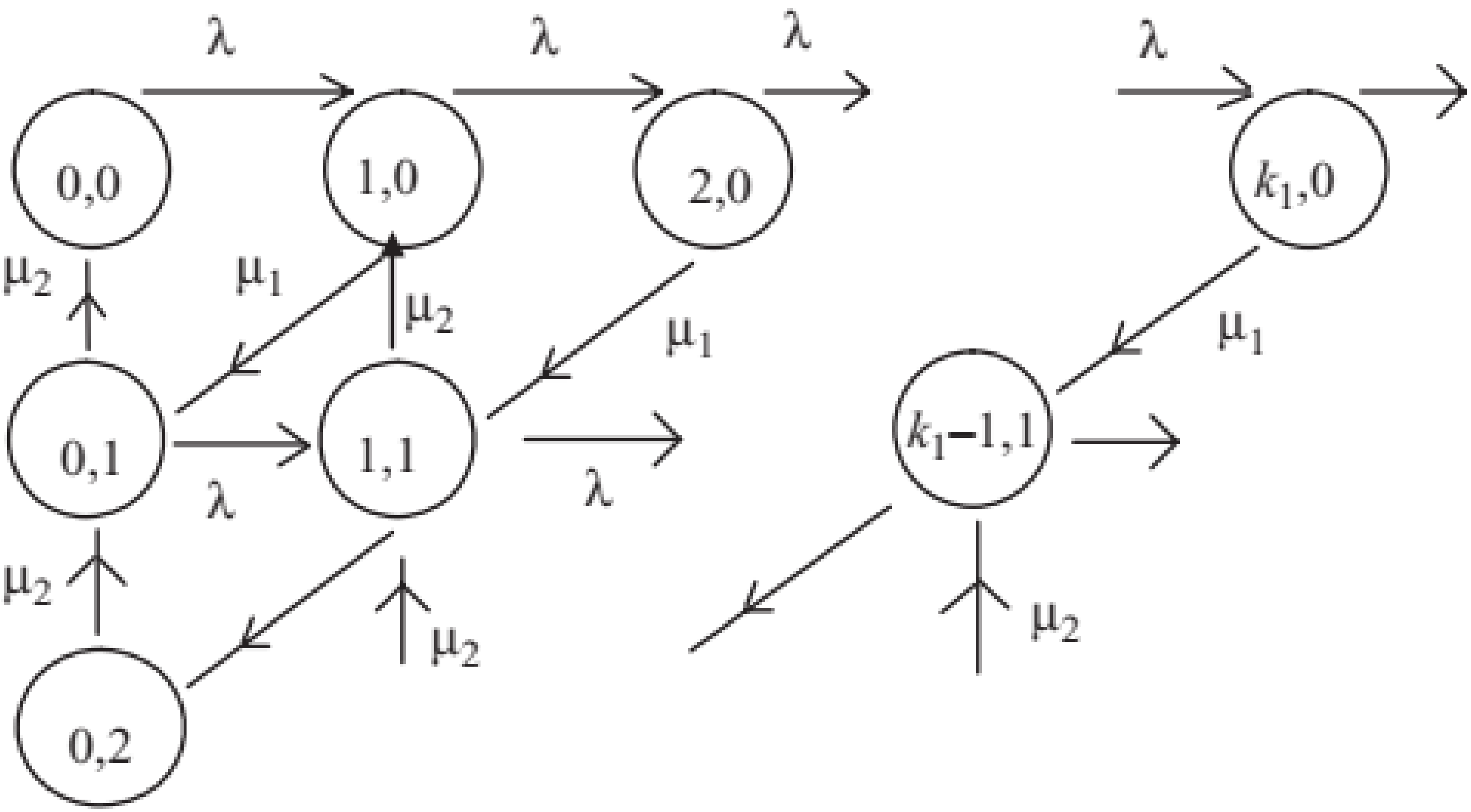
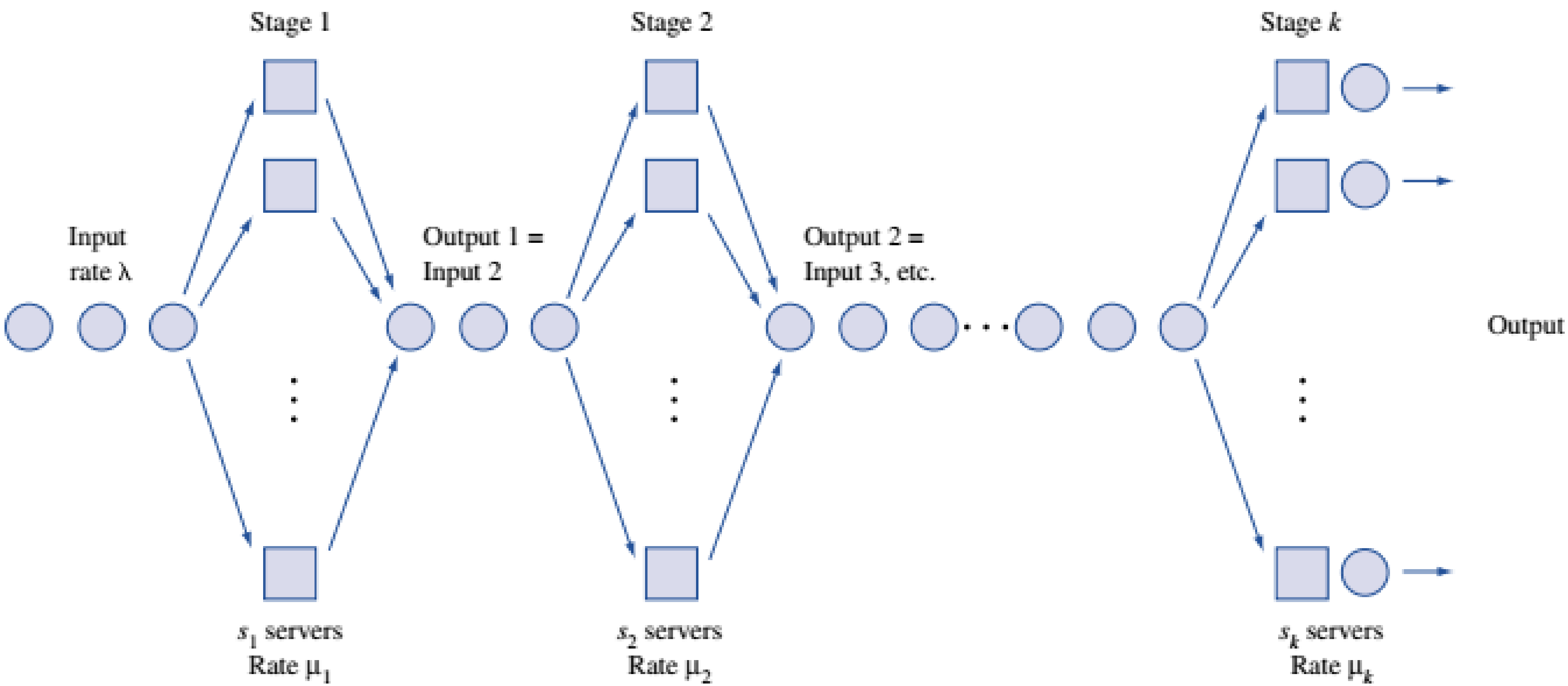


Figure 6.4 State transition diagram of the tandem queues

Exponential Queues in Series

- k-stage series queuing system



Exponential Queues in Series Networks

- **Theorem**
- **If**
  1) interarrival times for a series queuing system are exponential with rate λ,
  2) service times for each stage ***i*** server are exponential, and
  3) each stage has an infinite-capacity waiting room,
- **then** interarrival times for arrivals to each stage of the queuing system are exponential with rate λ.

Tandem network of M/M/1 queues

- M/M/1 queue, Poisson(λ) arrivals, exponential(μ) service
- Equilibrium distribution

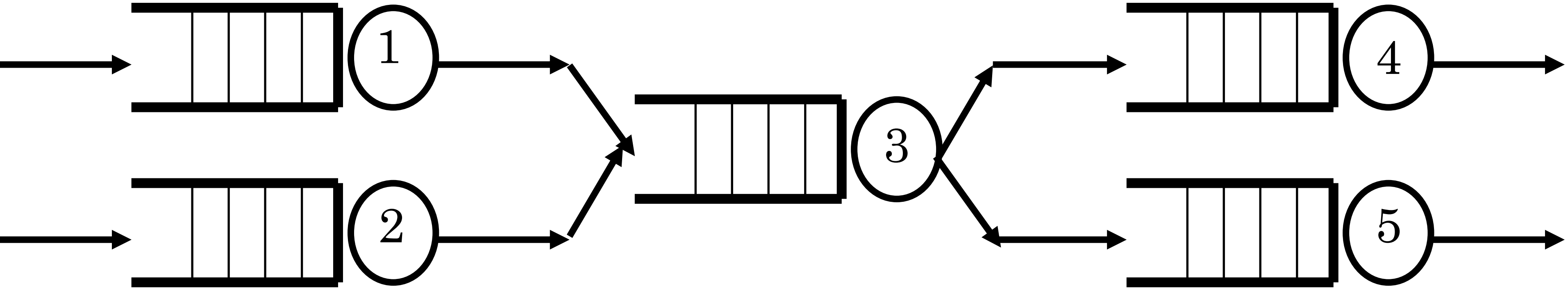
$$\pi_j = (1-\rho)\rho^j, \quad j = \{0,1,2,...\}, \rho = \lambda/\mu < 1$$

- For a k queue tandem system with Poisson arrival and expo. service times
- Jackson's theorem:

$$\pi(j_1,..., j_k) = \prod_{i=1}^{k} (1-\rho_i)\rho_i^{j_i}, \quad \rho_i = \lambda/\mu_i < 1$$

- Above formula is true when there are feedbacks among different queues
  - Each queue behaves as M/M/1 queue in isolation

Example: feed forward network of M/M/1 queues



arrival rate $\lambda_i, i = 1,2$

service rate $\mu_i, i = 1,\ldots,5$

routing probabilit ies p_{34}, p_{35}

$$\lambda_3 = \lambda_1 + \lambda_2$$

$$\lambda_4 = \lambda_3 p_{34}$$

$$\lambda_5 = \lambda_3 p_{35}$$

$$\pi(j_1,\ldots,j_5) = \prod_{i=1}^{5} (1-\rho_i)\rho_i^{\,j_i}, \quad \rho_i = \lambda_i / \mu_i < 1$$

Exponential Queues in Series Networks

- The last two works in a car manufacturing process are installing the engine and putting on the tires. An average of 54 cars per hour arrive requiring these two tasks.
- One worker is available to install the engine and can service an average of 60 cars per hour.
- After the engine is installed, the car goes to the tire station and waits for its tires to be attached. Three workers serve at the tire station. Each works on one car at a time and can put tires on a car in an average of 3 minutes.
- Both interarrival times and service times are exponential.
  - Determine the mean queue length at each work station.
  - Determine the total expected time that a car spends waiting for service

Exponential Queues in Series Networks

- This is a series queuing system with
  - λ=54 cars per hour, s_1 = 1, μ_1 = 60 cars per hour,
  - s_2 = 3, and μ_2 = 20 cars per hour
  - Since $\lambda < \mu_1$ and $\lambda < 3\mu_2$, neither queue will “blow up,”
- For stage 1 (engine), ρ = 54/60 = 0.9

$$L_q \text{ (for engine)} = \left(\frac{\rho^2}{1-\rho}\right) = \left[\frac{(.90)^2}{1-.90}\right] = 8.1 \text{ cars}$$

$$W_q \text{ (for engine)} = \frac{L_q}{\lambda} = \frac{8.1}{54} = 0.15 \text{ hour}$$

Exponential Queues in Series Networks

- For stage 2 (Tires), ρ = 54/(3*20) = 0.9

$$L_q = \frac{(s\rho)^s}{s!}\pi_0 \frac{\rho}{(1-\rho)^2} \qquad L_q = \frac{P(j \geq s)\rho}{1-\rho} \qquad P(j \geq s) = \frac{(s\rho)^s \pi_0}{s!(1-\rho)}$$

$$\pi_0 = \left(\sum_{j=0}^{S-1} \frac{(s\rho)^j}{j!} + \frac{(s\rho)^s}{s!(1-\rho)} \right)^{-1}$$

$$= (1 + 2.7 + 3.645 + 32.805)^{-1} = 0.025$$

$$P(j \geq s) = 0.82$$

$$L_q = \frac{0.82 * 0.9}{1-0.9} = 7.4 \quad cars$$

total expected waiting time is

0.15 +0.137 = 0.287 hour

$$W_q = \frac{7.4}{54} = 0.137\, hrs$$

Open Queuing Network

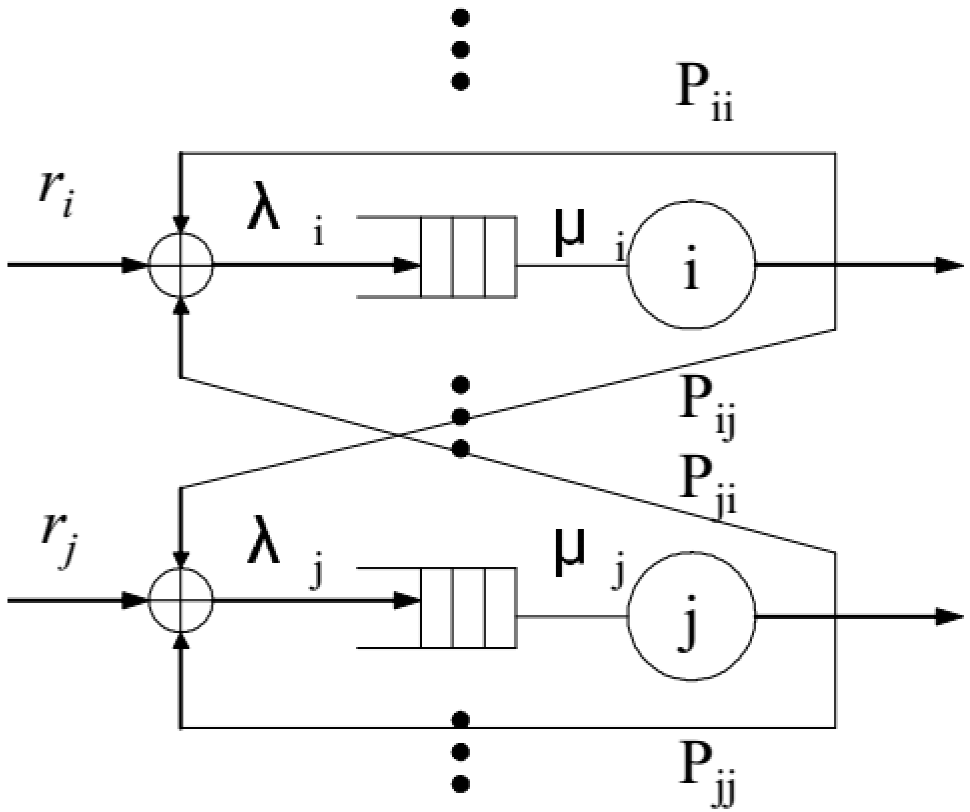
- Jobs arrive from external sources, circulate, and eventually depart

Jackson Network Definition

1. All outside arrivals to each queuing station in the network must follow a Poisson process.
2. All service times must be exponentially distributed.
3. All queues must have unlimited capacity.
4. When a job leaves one station, the probability that it will go to another station is independent of its past history and is independent of the location of any other job.

In essence, a Jackson network is a collection of connected $M/M/s$ queues with known parameters.

Jackson Network Definition



In essence, a Jackson network is a collection of connected $M/M/s$ queues with known parameters.

Jackson's Theorem

1. Each node is an independent queuing system with Poisson input determined by partitioning, merging and tandem queuing example.

2. Each node can be analyzed separately using $M/M/1$ or $M/M/s$ model.

3. Mean delays at each node can be added to determine mean system (network) delays.

Computation of Input Rate

Let r_i = external arrival rate to station $i = 1, \ldots, k$

P_{ij} = probability of going from station i to j in network

λ_i = total input to station i

In steady state there must be flow balance at each station.

$$\lambda_i = r_i + \sum_{m=1}^{k} P_{mi} \lambda_m, \quad i = 1, \ldots, k$$

$$\lambda_i = r_i + \sum_{m=1}^{k} P_{mi} \times \lambda_m \qquad k = 5$$

$$\lambda_i = r_i + \boxed{P_{1i} \lambda_1 + P_{2i} \lambda_2 + \cdots + P_{ki} \lambda_k}$$

$$\lambda_1 = r_1 + P_{11} \lambda_1 + P_{21} \lambda_2 + \cdots P_{51} \lambda_5$$

$$\lambda_2 = r_2 + P_{12} \lambda_1 + P_{22} \lambda_2 + \cdots + P_{52} \lambda_5$$

$$\vdots$$

$$\lambda_5 = r_5 + P_{15} \lambda_1 + P_{25} \lambda_2 + \cdots + P_{55} \lambda_5$$

$$(1-P_{11})\lambda_1 - P_{21}\lambda_2 - P_{31}\lambda_3 - \cdots - P_{51}\lambda_5 = r_1$$

$$-P_{12}\lambda_1 + (1-P_{22})\lambda_2 - P_{32}\lambda_3 - \cdots - P_{52}\lambda_5 = r_2$$

$$\vdots$$

$$-P_{15}\lambda_1 - P_{25}\lambda_2 - \cdots + (1-P_{55})\lambda_5 = r_5$$

$$\left\{\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}\right. \qquad \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} - & - & - & - \\ - & - & - & - \\ \\ - & - & - & \\ \\ \end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots \\ x_5\end{bmatrix}$$

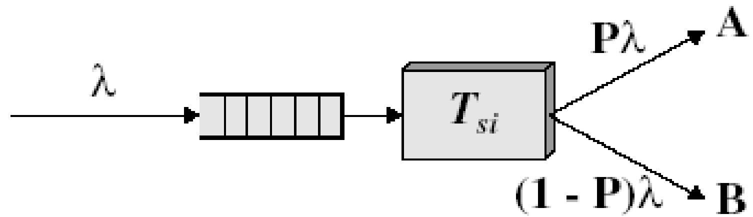
$$P = \begin{bmatrix}1-P_{11} & -P_{21} & \cdots & -P_{51}\\ -P_{12} & 1-P_{22} & \cdots & -P_{25}\\ \vdots & & & \\ P_{15} & - & - & 1-P_{55}\end{bmatrix}$$

$$\frac{I-P}{} =$$

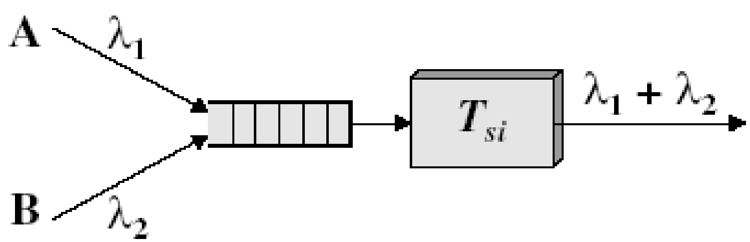
$$r_1 - r_2$$

$$[I-P]\,x = r$$

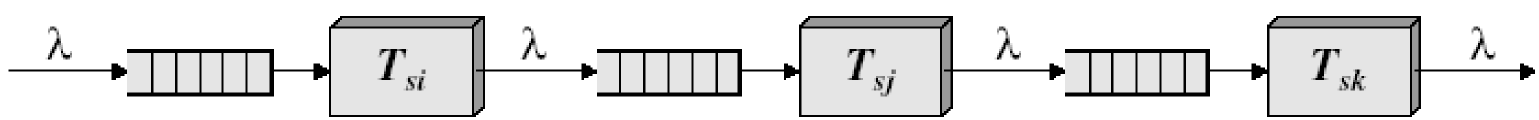
Element of a Queuing Network



(a) Traffic partitioning



(b) Traffic merging



(c) Simple tandem queue

Matrix Form of Computations

Property 1: Let **P** be the $k \times k$ probability matrix that describes the routing of units within a Jackson network, and let r_i denote the mean arrival rate of units going directly to station i from outside the system. Then

$$\boldsymbol{\lambda} = \mathbf{r}(\mathbf{I} - \mathbf{P})^{-1}$$

$(I - P)\lambda = r$

where **r** $= (r_1, \ldots, r_k)$ give the external arrival rates into the various station; and **I** is the identity matrix,

λ_i is the net arrival rate into station i.

$\boxed{AX = B}$

Note: Unlike the state-transition matrix used for Markov chains, the rows of the **P** matrix here need not sum to one; that is $\Sigma_j P_{ij} \le 1$

Simplification of Network

After the net rate into each node is known, the network can be decomposed and each node treated as if it were an independent queuing system with Poisson input.

Property 2: Consider a Jackson network comprising k nodes. Let N_i denote a random variable indicating the number of jobs at node i (the number in the queue plus the number in service). Then,

$$\Pr\{N_1 = n_1, \ldots, N_m = n_m\} = \Pr\{N_1 = n_1\} \times \cdots \times \Pr\{N_m = n_m\}$$

and

$\Pr\{N_i = n_i\}$ for all $n_i = 0, 1, \ldots$ can be calculated using the equations for independent $M/M/s$ seen previously.

- λ_N = total average load on network = total throughput

  $= \sum_{i=1}^{m} r_i$
- Consider also $\mathbf{v_i}$ the **average number of visits** to station i for each arriving customer:

$$v_i = \lambda_i / \lambda_N$$

- L_N = Average number of customers in network

$$= \sum_{i=1}^{m} L_i = \sum_{i=1}^{m} \frac{\rho_i}{1-\rho_i}$$

- W_N = Average delay through network

  = mean sojourn time

$$= \frac{L_N}{\lambda_N} = \sum_{i=1}^{m} v_i W_i = \frac{1}{\lambda_N} \sum \lambda_i W_i$$

$For \quad M/M/1$

$$L_i = \frac{\rho_i}{1-\rho_i}$$

$$W_i = \frac{L_i}{\lambda_i} = \frac{1}{\mu_i (1-\rho_i)}$$

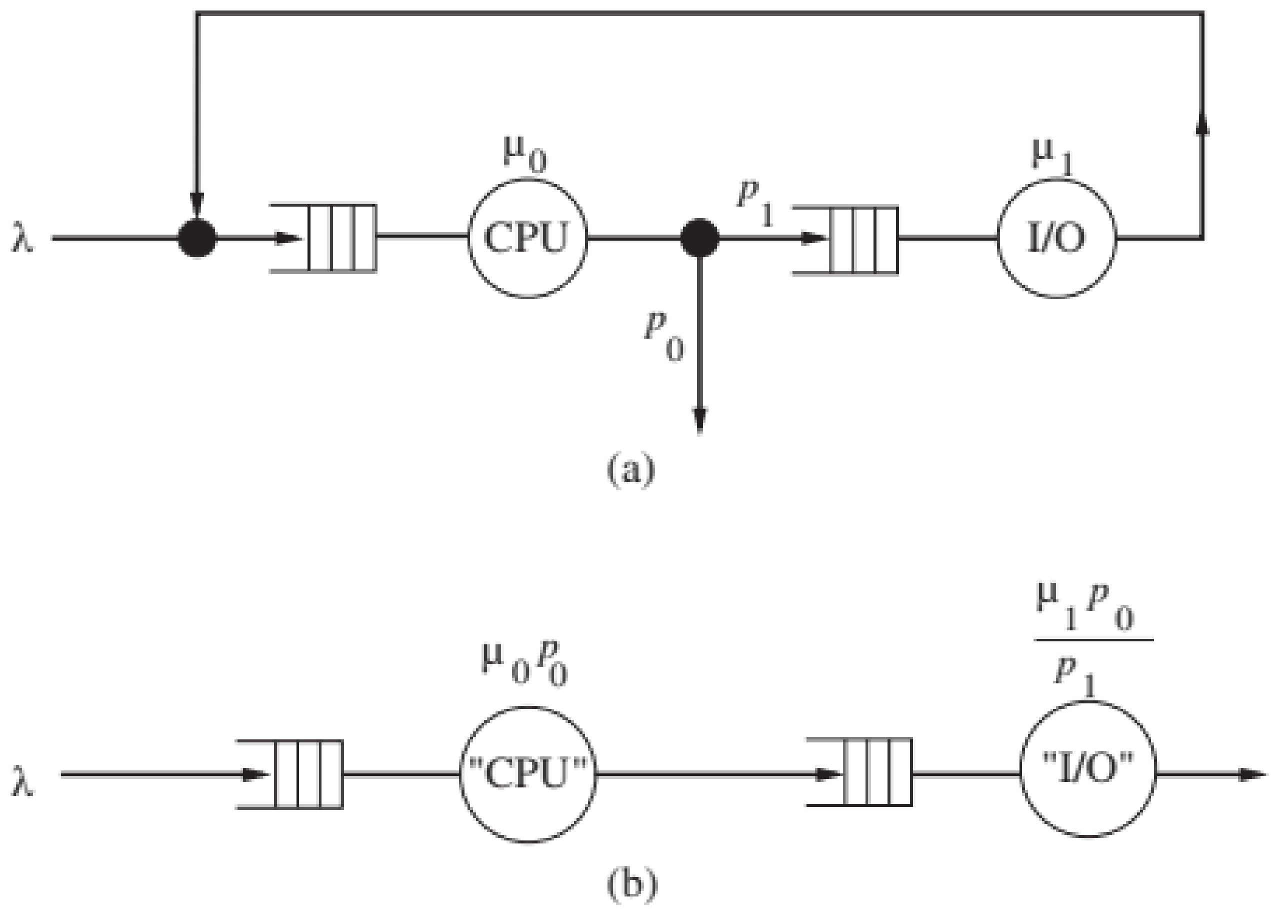


Figure 9.4. (a) An open network with feedback; (b) an "equivalent" network without feedback

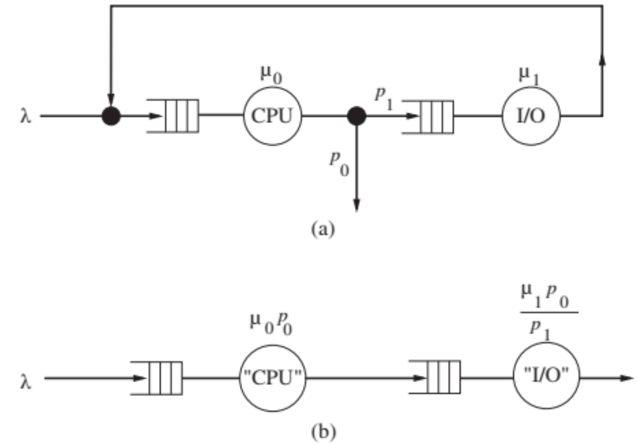


Figure 9.4. (a) An open network with feedback; (b) an "equivalent" network without feedback

$$p(k_0, k_1) = (1-\rho_0)\rho_0^{k_0}(1-\rho_1)\rho_1^{k_1}$$

where $\lambda_0/\mu_0 = \rho_0$ and $\lambda_1/\mu_1 = \rho_1$.

$$\rho_0, \rho_1 < 1$$

$$\lambda_0 = \lambda + \lambda_1.$$

$$\lambda_1 = \lambda_0 p_1$$

$$\lambda_0 = \frac{\lambda}{1-p_1} = \frac{\lambda}{p_0}$$

$$\lambda_1 = \frac{p_1 \lambda}{p_0}.$$

$$\rho_0 = \frac{\lambda}{p_0 \mu_0} \quad \text{and} \quad \rho_1 = \frac{p_1 \lambda}{p_0 \mu_1}.$$

$$W = E[R] = \left(\frac{\rho_0}{1-\rho_0} + \frac{\rho_1}{1-\rho_1}\right)\frac{1}{\lambda}$$

$$= \frac{1}{p_0 \mu_0 - \lambda} + \frac{1}{\frac{p_0 \mu_1}{p_1} - \lambda}$$

$$W_{s0} = 1/(p_0 \mu_0) \qquad W_{s1} = p_1/(p_0 \mu_1)$$

For, M/M/1

$$L_s = 0\pi_0 + 1(\pi_1 + \pi_2 + \cdots) = 1 - \pi_0$$

$$= 1 - (1-\rho) = \rho$$

$$W_s = \frac{L_s}{\lambda} = \frac{\rho}{\lambda} = \frac{1}{\mu}$$

Computation Center Example

- A high performance computation center is composed of 3 work stations comprising: (1) input processors, (2) central computers, and (3) a print center.
- All jobs submitted must first pass through an input processor for error checking before moving on to a central processor → 80% go through and 20% are rejected.
- Of the jobs that pass through the central processor, 40% are routed to a printer.
- Jobs arrive randomly at the computation center at an average rate of 10/min. To handle the load, each station may have several parallel processors.

Data for the Computation Center

We know from previous statistics that the time for the three steps have exponential distributions with means as follows:

10 seconds for an input processor

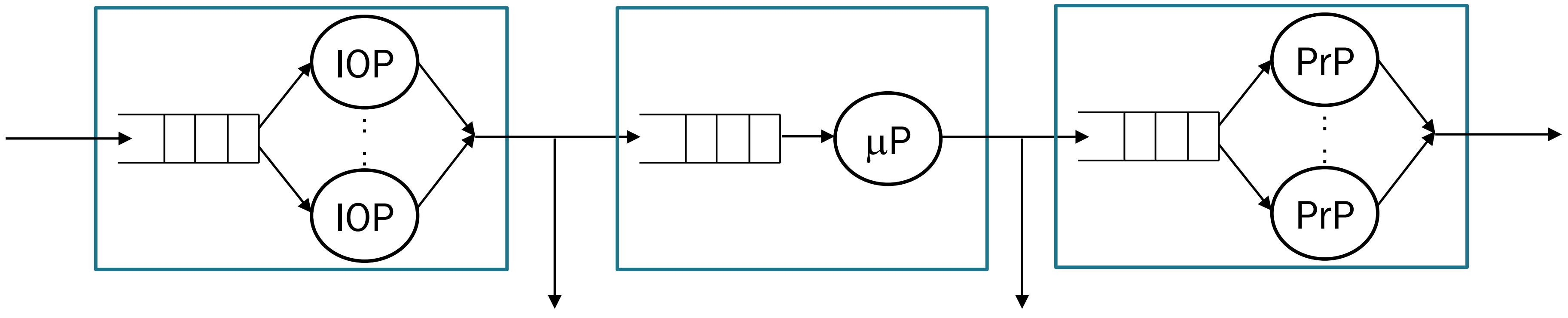
3 seconds for a central processor

70 seconds for a print processor

All queues are assumed to have unlimited capacity.

Goal

Model system as a Jackson network. Find the minimum number of processors of each type and compute the average time require for a job to pass through the system.


IOP
IOP
μP
PrP
PrP

Arrival Rate Computations

Using general equation:

With k = 3, r_1 = 10, p_{12} = 0.8, p_{23} = 0.4

we get:

$$\lambda_1 = 10$$

$$\lambda_2 = 0.8\lambda_1 = 8$$

$$\lambda_3 = 0.4\lambda_2 = 3.2$$

I/O Data for the Computation Center

System measure	Input processor	Central processor	Printer
External arrival rate, r_i	10/min	0	0
Total arrival rate, λ_i	10/min	8/min	3.2/min
Service rate, μ_i	6/min	20/min	0.857/min
Minimum channels, s_i	2	1	4
Traffic intensity, ρ_i	0.833	0.400	0.933

Results for Computation Center

$$for\ \ M/M/1 \quad \pi_0 = 1-\rho \qquad\qquad \rho = \lambda/\mu$$

$$for\ \ M/M/s \quad \pi_0 = \left(\sum_{j=0}^{S-1} \frac{(s\rho)^j}{j!} + \frac{(s\rho)^s}{s!(1-\rho)} \right)^{-1} \qquad\qquad \rho = \lambda/s\mu$$

	ρ	π_0
Input processor (M/M/2)	0.833	$10.976^{-1} = 0.09$
Central Processor (M/M/1)	0.40	0.6
Printer (M/M/4)	0.933	$\approx 141^{-1}$

$for \quad M/M/1$

$$L_q = L - L_s = \frac{\rho}{1-\rho} - \rho = \frac{\rho^2}{1-\rho}$$

$for \quad M/M/s$

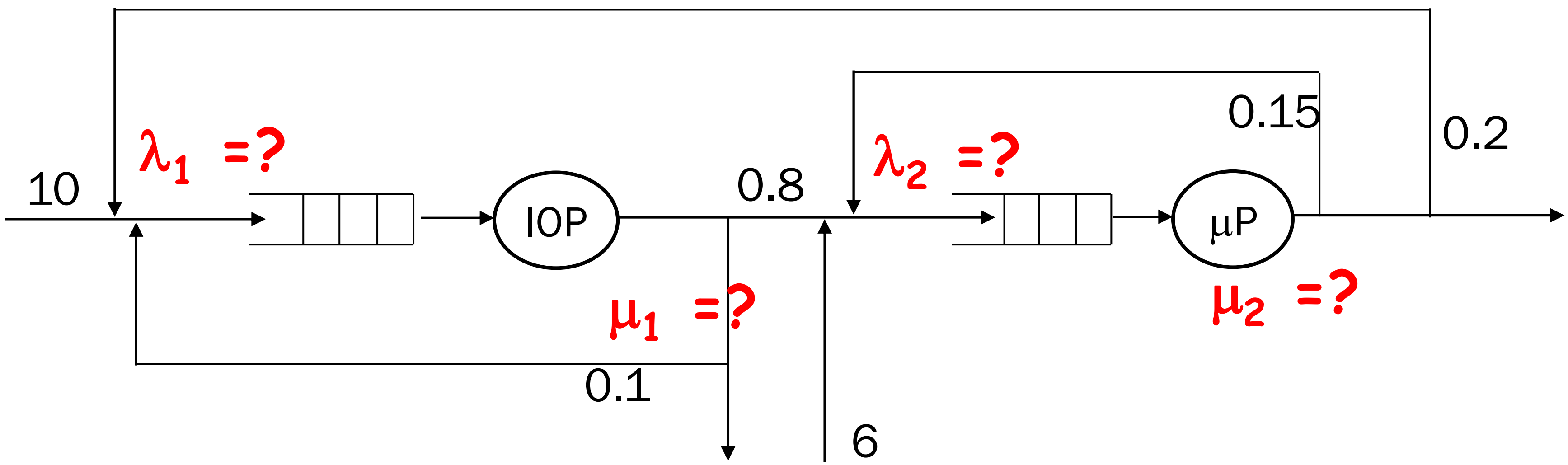
$$L_q = \frac{(s\rho)^s}{s!}\pi_0 \frac{\rho}{(1-\rho)^2} \qquad L_q = \frac{P(j \geq s)\rho}{1-\rho} \qquad P(j \geq s) = \frac{(s\rho)^s \pi_0}{s!(1-\rho)}$$

$$L = L_q + \frac{\lambda}{\mu} \qquad L_s = \frac{\lambda}{\mu} \qquad \text{Since } W_s = \frac{1}{\mu}$$

Results for Computation Center

Measure	Input processor	Central processor	Printer station	Total
Model	$M/M/2$	$M/M/1$	$M/M/4$	
L_q	3.788	0.267	12.023	16.077
W_q	0.379	0.033	3.757	4.169
L_s	1.667	0.400	3.734	5.801
W_s	0.167	0.050	1.167	1.384

Problem



$$\lambda_1 = 10 + 0.1\lambda_1 + 0.2\lambda_2$$

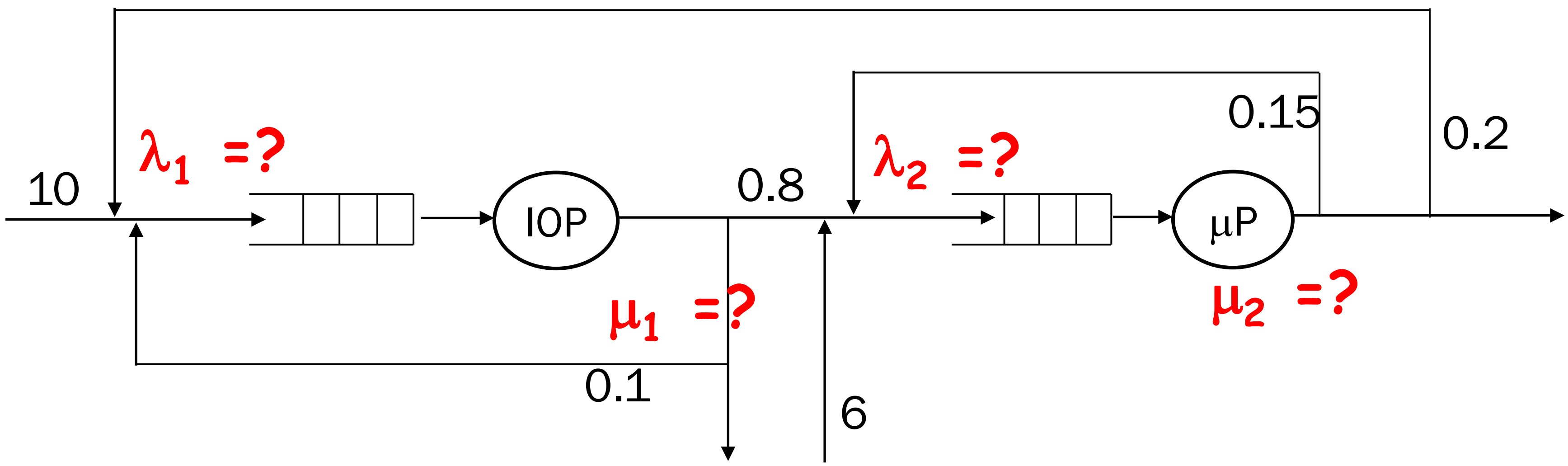
$$\lambda_2 = 6 + 0.8\lambda_1 + 0.15\lambda_2$$

Solves to

$$\lambda_1 \approx \mathbf{15.21}$$

$$\lambda_2 \approx \mathbf{22.15}$$

Problem



$$\mathbf{r} = [r_1 \quad r_2] = [10 \quad 6]$$

$$\mathbf{P} = \begin{bmatrix} 0.1 & 0.8 \\ 0.2 & 0.15 \end{bmatrix}$$

$$\mathrm{I} - \mathbf{P} = \begin{bmatrix} 0.9 & -0.8 \\ -0.2 & 0.85 \end{bmatrix}$$

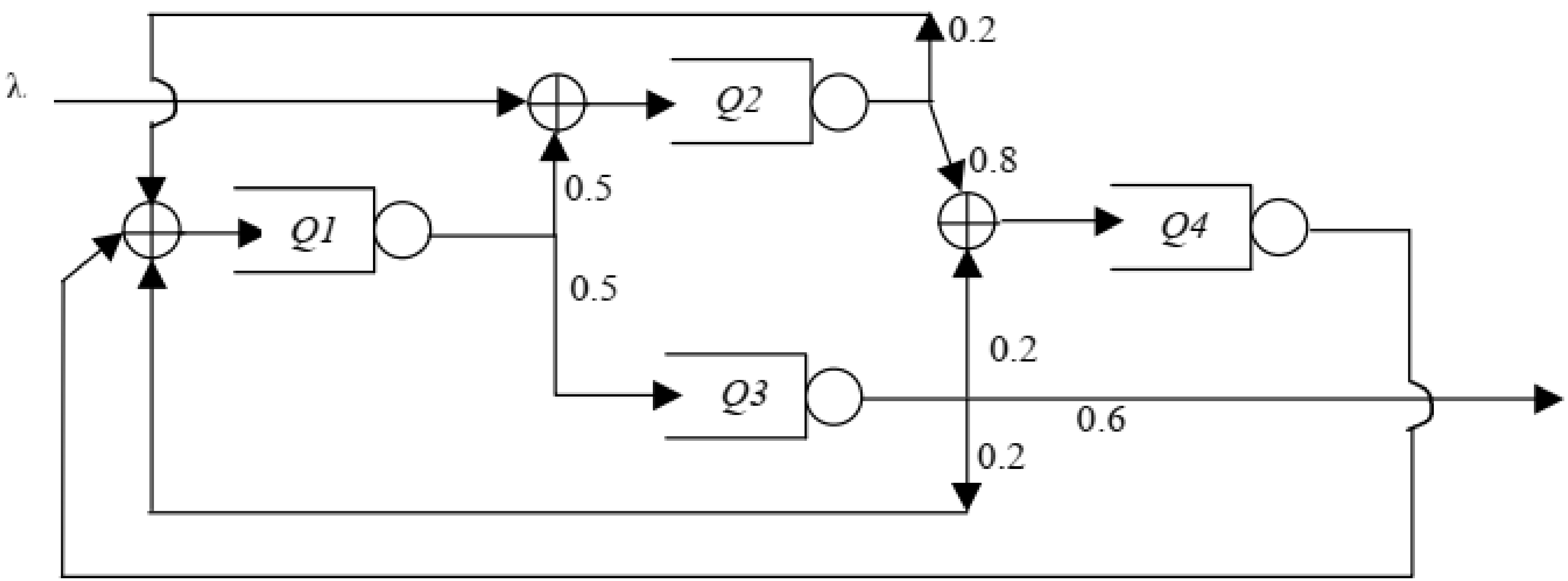
$$\left(\mathrm{I} - \mathbf{P}\right)^{-1} = \frac{1}{0.765 - 0.16} \begin{bmatrix} 0.85 & 0.8 \\ 0.2 & 0.9 \end{bmatrix}$$

$$[\lambda_1 \quad \lambda_2] = \mathbf{r}(\mathbf{I} - \mathbf{P})^{-1}$$

$$= [15.21 \quad 22.15]$$

Problem - DIY

Consider the open network of single-server, FCFS, exponential service queues shown in the figure.



The external arrivals are at Q2 from a Poisson process with average arrival rate λ. The mean service rates are $\mu_1 = \mu_3 = \mu$ and $\mu_2 = \mu_4 = 0.5\mu$.

(a) What will be the maximum value of λ for which the system will be stable? [3]

Do the following for λ=0.1 and μ=1

(b) Give the state distribution of the system. [2]

(c) Give the mean number in each queue. [3]

(d) Give the mean time spent in system by a customer entering the system. [2]

Problem - DIY

1. We can write the flow balance equations and solve them. In this case, it is easy to get the following by inspection -

$$0.6\lambda_3 = \lambda \quad \Rightarrow \quad \lambda_3 = 1.6667\lambda$$

$$0.5\lambda_1 = \lambda_3 \quad \Rightarrow \quad \lambda_1 = 3.3333\lambda$$

$$0.5\lambda_1 + \lambda = \lambda_2 \quad \Rightarrow \quad \lambda_2 = 2.6667\lambda$$

$$0.8\lambda_2 + 0.2\lambda_3 = \lambda_4 \quad \Rightarrow \quad \lambda_4 = 2.4667$$

Therefore $\quad \tilde{\lambda} = (3.3333\lambda, 2.6667\lambda, 1.6667\lambda, 2.4667\lambda)$

and $\quad \tilde{\rho} = (3.3333\rho, 5.3334\rho, 1.6667\rho, 4.9334\rho)$ with $\rho = \dfrac{\lambda}{\mu}$

Maximum Value of λ for which the queuing network will be stable = 0.1875μ

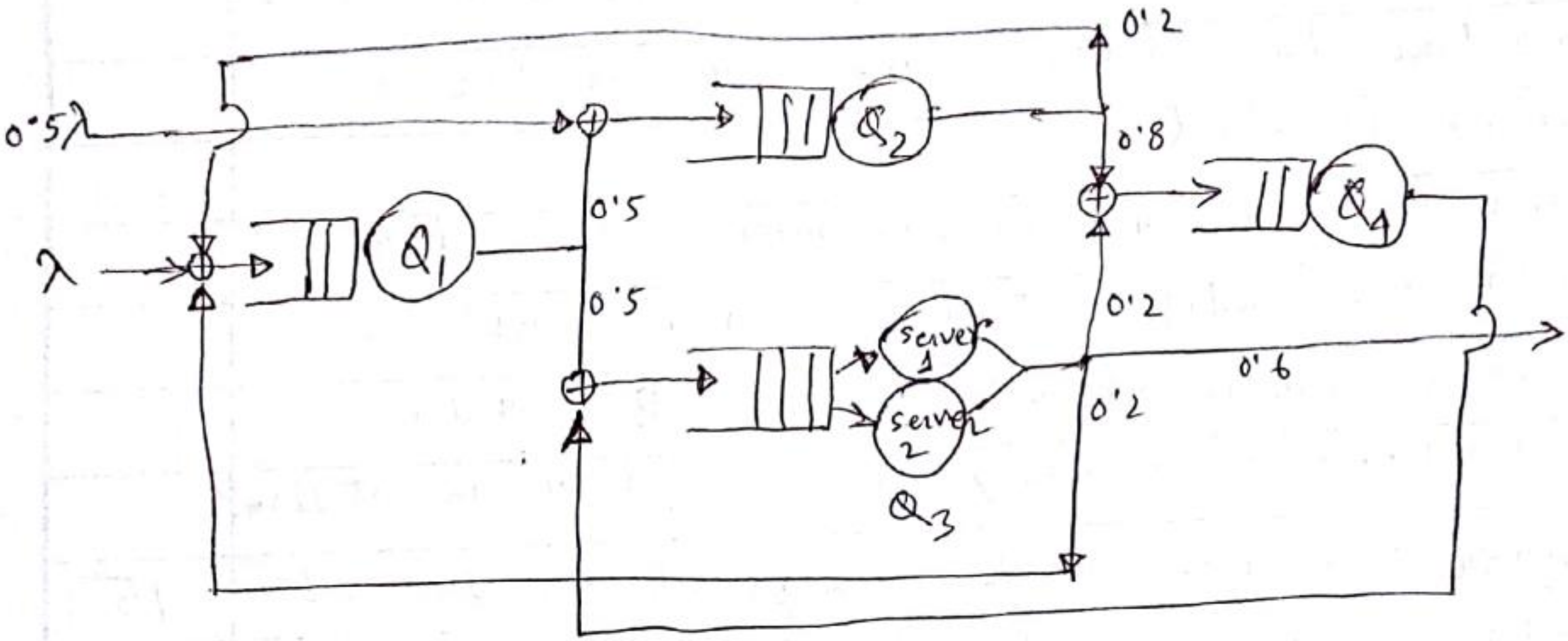
For $\lambda=0.1$, $\mu=1$, we get $\rho=0.1$

State Distribution is -

$$P(\tilde{n}) = (0.66667)(0.46666)(0.83333)(0.50666)(0.33333)^{n_1}(0.53334)^{n_2}(0.16667)^{n_3}(0.49334)^{n_4}$$

or $\quad P(\tilde{n}) = (0.13135)(0.33333)^{n_1}(0.53334)^{n_2}(0.16667)^{n_3}(0.49334)^{n_4}$

Mean Numbers in the various queues are $\quad$ *(0.5, 1.14286, 0.2, 0.973684)*
Mean of Total Number in Network = *2.81654*
Mean Time Spent in System by a customer = *2.81654/0.1=28.1654*

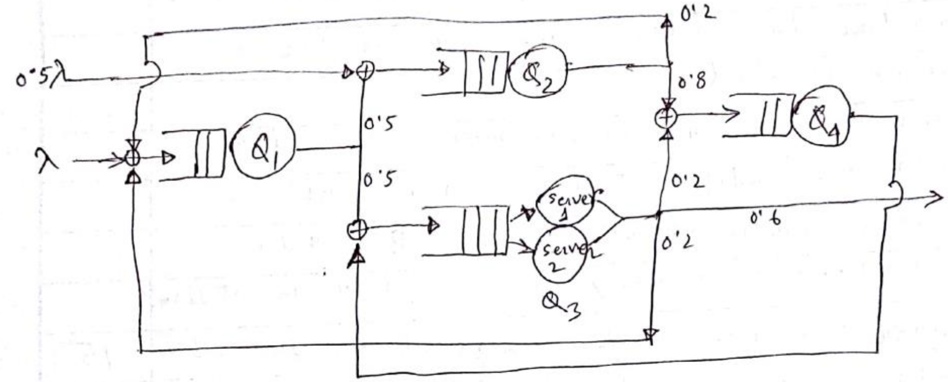


$$\lambda_1 = \lambda + 0.2\,\lambda_2 + 0.2\,\lambda_3$$

$$\lambda_2 = 0.5\,\lambda + 0.5\,\lambda_1$$

$$\lambda_3 = 0.5\,\lambda_1 + \lambda_4$$

$$\lambda_4 = 0.8\,\lambda_2 + 0.2\,\lambda_3$$



$$\lambda_1 = \lambda + 0'2\lambda_2 + 0'2\lambda_3$$

$$\lambda_2 = 0'5\lambda + 0'5\lambda_1$$

$$\lambda_3 = 0'5\lambda_1 + \lambda_4$$

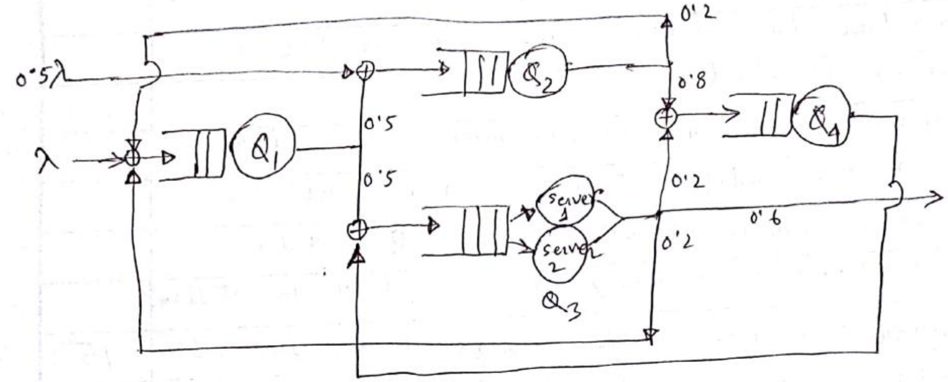
$$\lambda_4 = 0'8\lambda_2 + 0'2\lambda_3$$

$$1\cdot\lambda_1 - 0'2\lambda_2 - 0'2\lambda_3 + 0\lambda_4 = \lambda$$

$$-0'5\lambda_1 + 1\cdot\lambda_2 + 0 + 0 = 0'5\lambda$$

$$-0'5\lambda_1 + 0 + \lambda_3 - 1\lambda_4 = 0$$

$$0 - 0'8\lambda_2 - 0'2\lambda_3 + 1\cdot\lambda_4 = 0$$

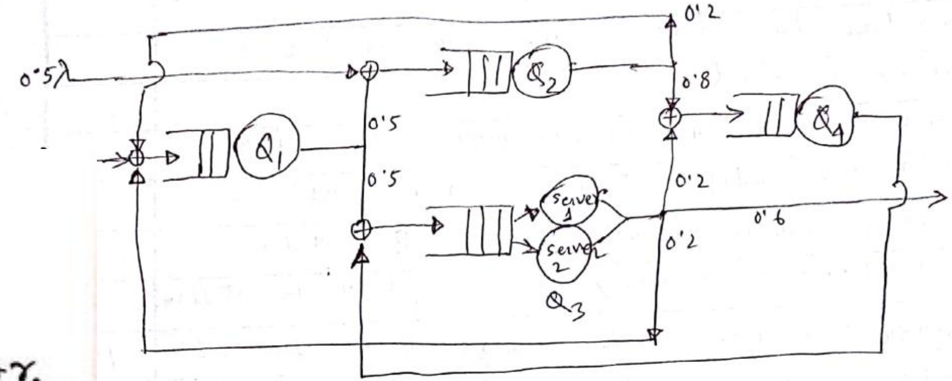


$$\lambda_1 = \lambda + 0'2\lambda_2 + 0'2\lambda_3$$
$$\lambda_2 = 0'5\lambda + 0'5\lambda_1$$
$$\lambda_3 = 0'5\lambda_1 + \lambda_4$$
$$\lambda_4 = 0'8\lambda_2 + 0'2\lambda_3$$

$$1\cdot\lambda_1 - 0'2\lambda_2 - 0'2\lambda_3 + 0\lambda_4 = \lambda$$
$$-0'5\lambda_1 + 1\cdot\lambda_2 + 0 + 0 = 0'5\lambda$$
$$-0'5\lambda_1 + 0 + \lambda_3 - 1\lambda_4 = 0$$
$$0 \quad -0'8\lambda_2 - 0'2\lambda_3 + 1\cdot\lambda_4 = 0$$

$$\left[\begin{array}{cccc|c} 1 & -0'2 & -0'2 & 0 & 1 \\ -0'5 & 1 & 0 & 0 & 0'5 \\ -0'5 & 0 & 1 & -1 & 0 \\ 0 & -0'8 & -0'2 & +1 & 0 \end{array}\right] \begin{array}{l} \\ \to 2r_2 + r_1 \\ \to 2r_3 + r_1 \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -0'2 & -0'2 & 0 & 1 \\ 0 & 1'8 & -0'2 & 0 & 2 \\ 0 & -0'2 & 1'8 & -2 & 1 \\ 0 & -0'8 & -0'2 & 1 & 0 \end{array}\right] \begin{array}{l} \\ \to 2r_2 + r_1 \\ \to 2r_3 + r_1 \\ \\ \end{array}$$



$$\left[\begin{array}{cccc|c}
1 & -0.2 & -0.2 & 0 & 1 \\
0 & 1.8 & -0.2 & 0 & 2 \\
0 & 0 & 16 & -18 & 11 \\
0 & 0 & -0.65 & 2.25 & 2 \\
0 & 0 & 0 & 37.38 & 60.23
\end{array}\right]
\begin{array}{l}
\\
\\
\rightarrow 9r_3 + r_2 \\
\rightarrow 2.25 r_4 + r_2 \\
\\
\end{array}$$

$$\therefore \lambda_4 = \frac{60.23}{37.38} \simeq 1.61\,\lambda$$

$$\lambda_3 = \frac{11 + 18 \times 1.61}{16} \approx 2.5\,\lambda$$

$$\lambda_2 = \frac{2 + 0.2 \times 2.5}{1.8} = 1.388\,\lambda$$

$$\lambda_1 = 1 + 0.2(1.388 + 2.5)$$
$$= 1.77\,\lambda$$

$$0.6\, x_3 = 1.5\lambda \Rightarrow x_3 = 2.5\,\lambda$$

Example 6.4 Hock n Hee

Figure 6.10 shows a queueing model of a multi-programming computer system. Computer jobs arrive at the CPU according to a Poisson process with rate γ. A job gets executed in the CPU for an exponential time with mean μ_1^{-1} and then requests service from either I/O 1 or I/O 2 with equal probability. The processing times of these two I/O devices are exponentially distributed with means μ_2^{-1} and μ_3^{-1}, respectively. After going through either one of the I/O devices, the job may return to the CPU or leave the system according to the probabilities shown on the diagram. Calculate:

(i) the joint probability mass function of the whole network;
(ii) if $\mu_1 = 8\gamma$ and $\mu_2 = \mu_3 = 4\gamma$, find the mean number of jobs at each queue;
(iii) the mean number of jobs in the network, and the time a job spends in the network.

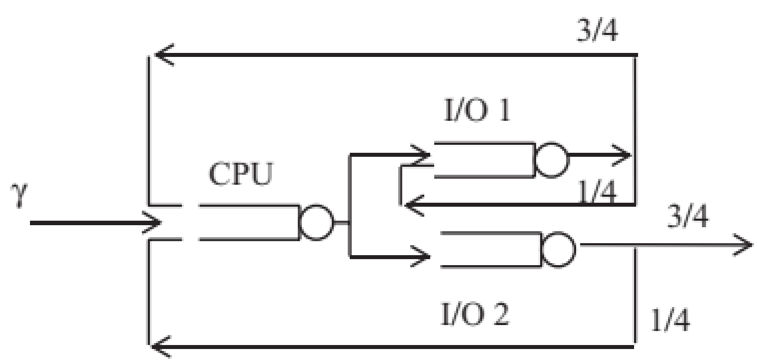


Figure 6.10 A multi-programming computer

Solution

$$P(N_1 = n_1, N_2 = n_2, \ldots, N_k = n_k) = \prod_{i=1}^{k}(1-\rho_i)\rho_i^{n_i},$$

- Let λ_1, λ_2, λ_3 be the effective arrival rates to the CPU , I/O 1 and I/O 2, respectively. By the flow conservation principle, we have

$$\lambda_1 = \gamma + \frac{3}{4}\lambda_2 + \frac{1}{4}\lambda_3$$

$$\lambda_2 = \frac{1}{2}\lambda_1 + \frac{1}{4}\lambda_2$$

$$\lambda_3 = \frac{1}{2}\lambda_1$$

Solving them, we obtain

$$\lambda_1 = \frac{8}{3}\gamma, \quad \lambda_2 = \frac{16}{9}\gamma, \quad \lambda_3 = \frac{4}{3}\gamma$$

(i) The joint probability mass function is then given by Jackson's theorem:

$$P(n_1, n_2, n_3) = \left(1-\frac{8\gamma}{3\mu_1}\right)\left(1-\frac{16\gamma}{9\mu_2}\right)\left(1-\frac{4\gamma}{3\mu_3}\right)\left(\frac{8\gamma}{3\mu_1}\right)^{n_1}\left(\frac{16\gamma}{9\mu_2}\right)^{n_2}\left(\frac{4\gamma}{3\mu_3}\right)^{n_3}$$

(ii) $\rho_1 = \frac{\lambda_1}{\mu_1} = \frac{1}{3}, \quad \rho_2 = \frac{4}{9}, \quad \rho_3 = \frac{1}{3}$

$$N_{CPU} = \frac{\rho_1}{1-\rho_1} = \frac{1/3}{1-(1/3)} = \frac{1}{2}$$

$$N_{I/O1} = \frac{4}{5} \quad \text{and} \quad N_{I/O2} = \frac{1}{2}$$

(iii) $N = N_{CPu} + N_{I/O1} + N_{I/O2} = \frac{9}{5}$

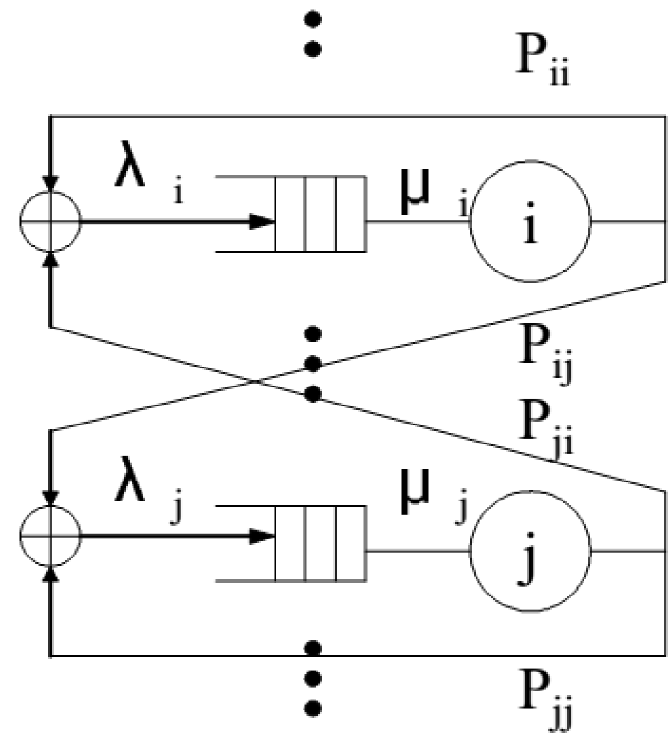
$$T = \frac{9/5}{\gamma} = \frac{9}{5\gamma}$$

Closed Queuing Network

Closed Queuing Network

- Fixed population of jobs circulate continuously and never leave
  - No arrivals from outside and no departures from the network
- Example: CPU job scheduling problem
- Since the number of jobs in the system is always constant, the distribution of jobs at different servers cannot be independent.
- Simplest case, *K* customers circulating among *m* queues
  - Each queue *i* has a server with exponentially distributed service time μ_i
  - P_{ij} be the routing probability from Q_i to Q_m

$$\sum_{j=1}^{m} P_{ij} = 1 \ ; \quad \forall\, i = 1, \ldots, m$$



Closed Queuing Network

- State of network at time t defined by

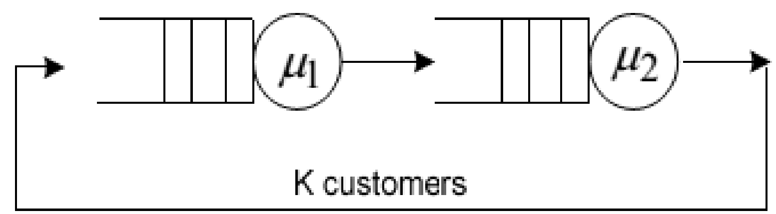
$$\mathbf{n} = \left(\tilde{n}_1(t), \tilde{n}_2(t), \ldots, \tilde{n}_m(t)\right)$$

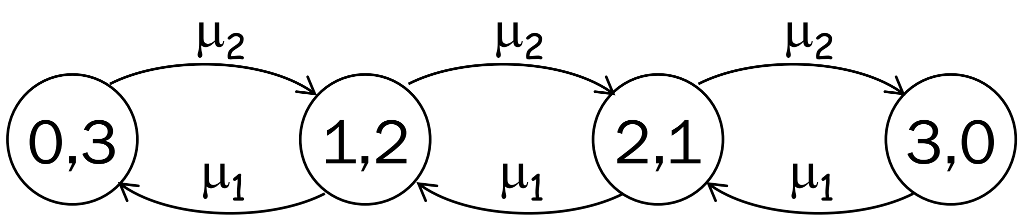
or simply, $= (n_1, n_2, \ldots, n_m)$

- which is m dimensional Markov process.
- The state space S is determined by

$$S = \left\{(n_1, n_2, \ldots, n_m) : 0 \le n_i \le K \;\; \forall i ; \sum_{i=1}^{m} n_i = K \right\}$$

- For example, $M = 2$, $K = 3$
- (n1, n2) state diagram





Closed Queuing Network

- Gordon and Newell (1967) showed that any arbitrary closed networks of m-server queues with exponentially distributed service times also have a product form solution
- The solution of the flow balance equation

$$\pi(\mathbf{n}) = \frac{1}{G(K,m)} \prod_{i=1}^{m} \rho_i^{n_i} \text{ where } \qquad \rho_i = \frac{\lambda_i}{\mu_i}$$

- G(K, m) is a normalization constant so that $\sum_{\mathbf{n} \in S} \pi(\boldsymbol{n}) = 1$ given by

$$G(K,m) = \sum_{\mathbf{n} \in S} \prod_{i=1}^{m} \rho_i^{n_i}$$

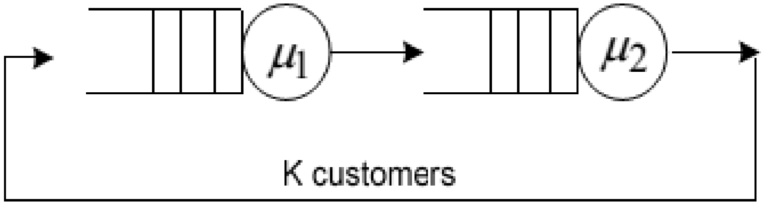
Closed Queuing Network

- In order to determine $G(K, m)$ we need λ_i ; $\forall i$
- Flow conservation equation is $$\lambda_i = \sum_{j=1}^{m} P_{ji}\lambda_j \quad ; \quad \forall\, i = 1,\ldots, m$$
  - same as open network case without external arrivals or departures.
  - This equation have no unique solution. Fortunately, it turns out that we can use any solution to help us get steady-state probabilities
  - arrival rates are found relative to each other,

    set λ_1= 1 or set λ_1 = μ_1 => $\rho1$ = 1

Closed Queuing Network

- For example, consider the tandem queue model with K=3.
  with $\mu_1 = 1$ and $\mu_2 = 2$



- From the diagram $P_{12} = P_{21} = 1$
- *i.e.* $\lambda_1 = \lambda_2$

- State space S = { (0,3), (1,2), (2,1), (3,0) }

- $G(K, m) = G(3,2) = \sum_{\mathbf{n}\in S} \prod_{i=1}^{m} \rho_i^{n_i} = \rho_2^3 + \rho_1 \rho_2^2 + \rho_1^2 \rho_2 + \rho_1^3$

- choosing $\lambda_1 = 1$ $\lambda_2 = 1$ $\rho_1 = 1$, $\rho_2 = 0.5$, $G(3,2) = 1.875$

- $$\pi(0,3) = \frac{\rho_1^0 \rho_2^3}{G(K,m)} = 0.0667 \qquad \pi(1,2) = \frac{\rho_1 \rho_2^2}{G(K,m)} = 0.1333$$

  $$\pi(2,1) = \frac{\rho_1^2 \rho_2^1}{G(K,m)} = 0.2667 \qquad \pi(3,0) = \frac{\rho_1^3 \rho_2^0}{G(K,m)} = 0.5333$$

Closed Queuing Network

- The computation of *G*(*K*, *m*) is difficult when the state space become large.
- For a closed network of *m* queues with *K* customers the number of states is given by

  Number of states = $^{K+m-1}C_{m-1} = ^{K+m-1}C_{K}$

- For even small networks, this is large.
  - For example *K* = 9, *m* = 3 = 55 states
  - And, direct computation of G(K,m) is very tedious

- One popular technique to determine is **Buzen's algorithm** (also called the **convolution algorithm**)
  $G(K, m) = G(K, m-1) + \rho_m G(K-1, m)$
- With initial condition

$$G(0, m) = 1 \qquad m = 1, 2, \ldots, M$$

$$G(k, 1) = \rho_1^k \qquad k = 1, 2, \ldots, K$$

Closed Queuing Network

- This can be computed in a simple tabular form

$$\begin{array}{c|c}
 & \begin{array}{ccccc} \rho_1 & \rho_2 & \rho_3 & & \rho_M \\ 1 & 2 & 3 & \cdots & M \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ K \end{array} &
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
\rho_1 & \rho_1 + \rho_2 & \rho_1 + \rho_2 + \rho_3 & \cdots & \\
\rho_1^2 & \rho_1^2 + \rho_2(\rho_1 + \rho_2) & \cdots & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \\
\rho_1^K & \cdots & \cdots & \cdots &
\end{bmatrix}
\end{array}$$

- The *i j* element in the table is computed by taking the *i*,(*j*-1) element and adding ρ_j . (*i* -1, *j*) *element*
- For the two queue example.

$$\lambda_1 = 0.5 \Rightarrow \lambda_2 = 0.5 \Rightarrow \rho_1 = 0.5\ ,\quad \rho_2 = 0.25$$

	ρ_1	ρ_2
	1	2
0	1	1
1	0.5	0.75
2	0.25	0.4315
3	0.125	0.2344

Closed Queuing Network

- The performance measures can be written in terms of $G(K, M)$

$$L_i = \frac{1}{G(K,M)} \sum_{k=1}^{K} \rho_i^k G(K-k,M)$$

$$P(n_i \geq k) = \rho_i^k \frac{G(K-k,M)}{G(K,M)}$$

$$e_i = \lambda_i \frac{G(K-1,M)}{G(K,M)}$$

$$W_i = \frac{L_i}{e_i}$$

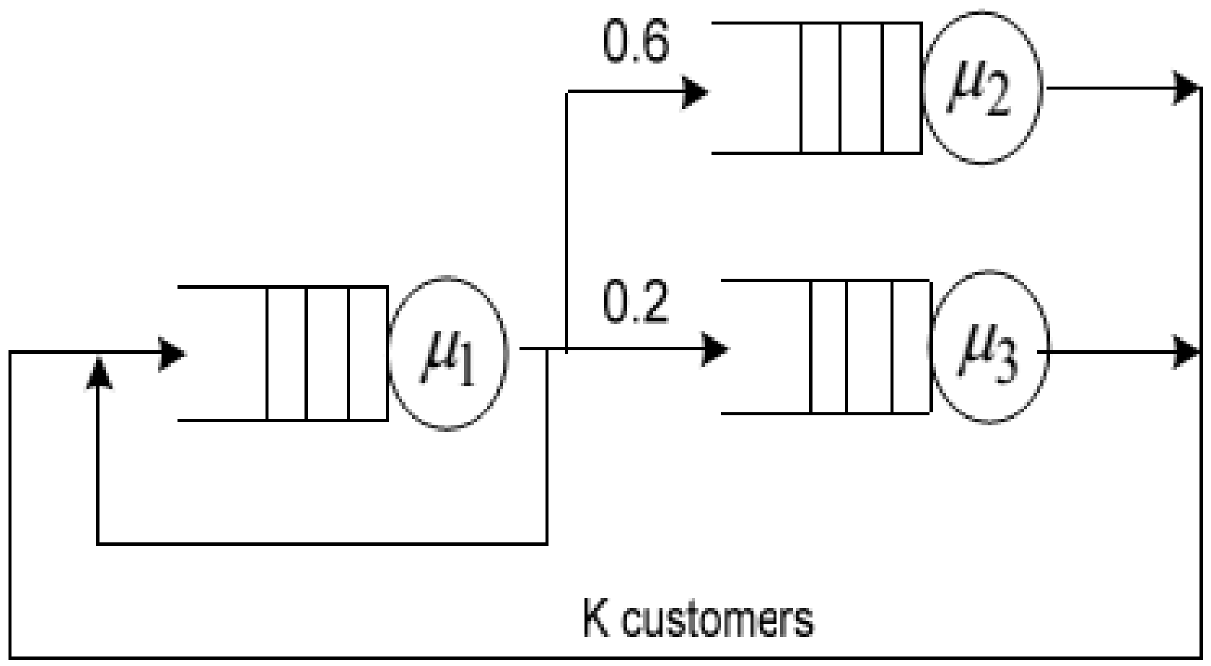
The effective server utilization $\rho_{e_i} = \frac{e_i}{\mu_i}$

- L_i = average customer at queue i, e_i = effective arrival rate

Example

Consider the simple model of a computer system shown below, queue 1– the CPU, queue 2–disk drive, and queue 3–I/O.

Given $\mu_1 = 10$, $\mu_2 = 5$, $\mu_3 = 1$, K = 4 jobs

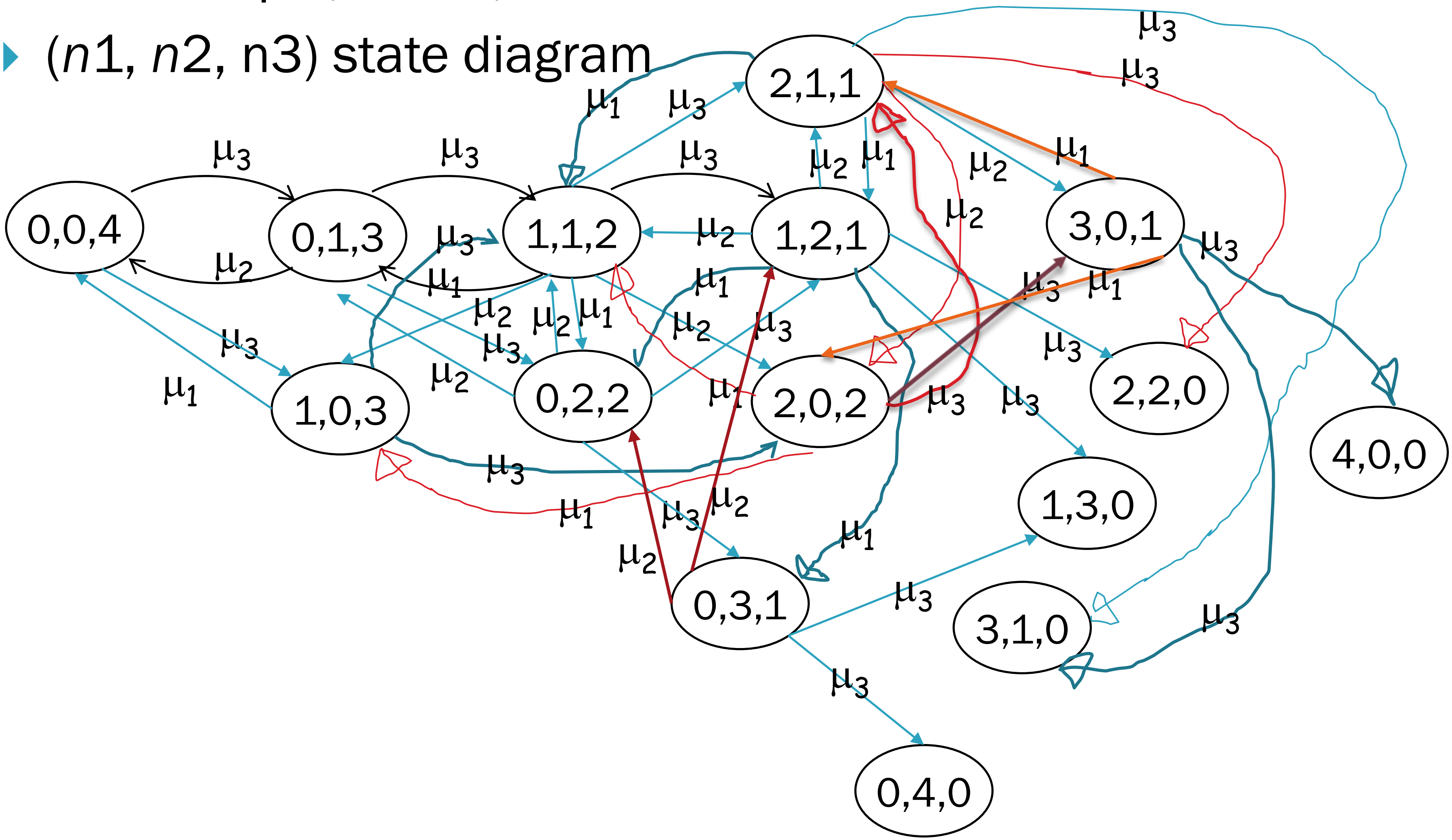


From the diagram

$r_{11} = 0.2$, $r_{12} = 0.6$, $r_{13} = 0.2$, $r_{21} = r_{31} = 1$,

Closed Queuing Network

- For example, $M = 3$, $K = 4$
- ($n1$, $n2$, n3) state diagram



Example

- Choosing $\lambda_1 = 10 \Rightarrow \lambda_2 = 6$, $\lambda_3 = 2$, and $\rho_1 = 1$, $\rho_2 = 1.2$, $\rho_3 = 2$
- Computing $G(4,3)$

	$\rho_1 = 1$	$\rho_2 = 1.2$	$\rho_3 = 2$
	1	2	3
0	1	1	1
1	1	2.2	4.2
2	1	3.64	12.04
3	1	5.368	29.448
4	1	7.4416	66.3376

Example

- Computing the effective arrival rates

$$e_1 = \lambda_1 \frac{G(3,3)}{G(4,3)} = 10 \times \frac{29.448}{66.3376} = 4.4391 \quad , e_2 = 2.6635 \quad , \quad e_3 = 0.8878$$

- The mean number in system at each queue

$$L_1 = \frac{1}{G(4,3)} \sum_{k=1}^{4} \rho_1^k G(4-k,3) = \frac{1}{G(4,3)} \left[\rho_1 G(3,3) + \rho_1^2 G(2,3) + \rho_1^3 G(1,3) + \rho_1^4 G(0,3) \right]$$

$$L_1 = 0.7038 \quad , \qquad L_2 = 0.9347 \ , \qquad L_3 = 2.3615$$

$$W_1 = L_1 / e_1 = 0.1585 \qquad W_2 = 0.3509 \qquad W_3 = 2.6599$$

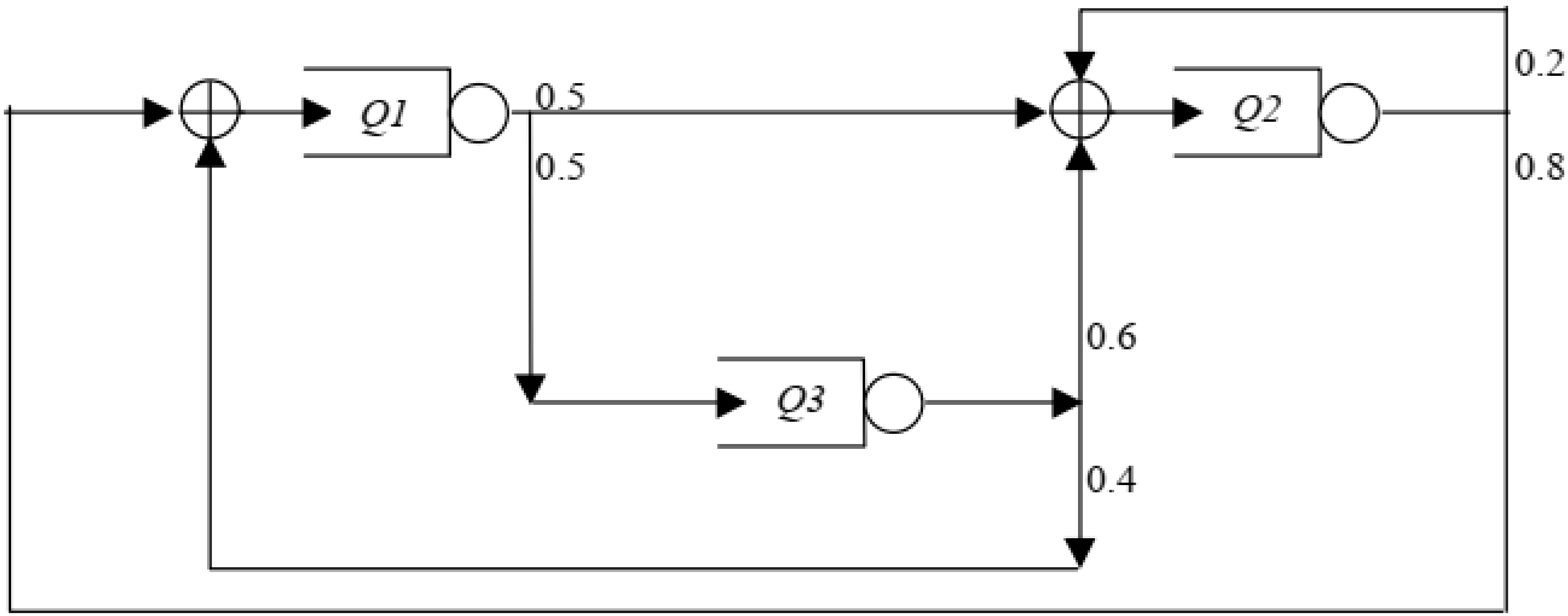
Example

P(q1 idle) = P(0,0,4) + P(0,1,3) + P(0,2,2) + P(0,3,1) + P(0,4,0)

	ρ1	ρ2	ρ3		n1	n2	n3	ρ1^n	ρ2^n2	ρ3^n3	$\Pi\rho^i$	P(n1,n2,n3)	
	1	1.2	2		0	0	4	1	1	16	16	0.241	
					0	1	3	1	1.2	8	9.6	0.145	
	1	2	3		0	2	2	1	1.44	4	5.76	0.087	P(q1 idle)
0	1	1	1		0	3	1	1	1.73	2	3.46	0.052	
1	1	2.2	4.2		0	4	0	1	2.07	1	2.07	0.031	0.56
2	1	3.64	12		1	3	0	1	1.73	1	1.73	0.026	
3	1	5.37	29.4		2	2	0	1	1.44	1	1.44	0.022	
4	1	7.44	66.3		3	1	0	1	1.2	1	1.2	0.018	
			G(4,3)		4	0	0	1	1	1	1	0.015	
					3	0	1	1	1	2	2	0.03	
	λ1	λ2	λ3		2	0	2	1	1	4	4	0.06	
	10	6	2		1	0	3	1	1	8	8	0.121	
					1	1	2	1	1.2	4	4.8	0.072	
	e1	e2	e3		1	2	1	1	1.44	2	2.88	0.043	
	4.4	2.66	0.89		2	1	1	1	1.2	2	2.4	0.036	
										G(4,3)	66.3	1	

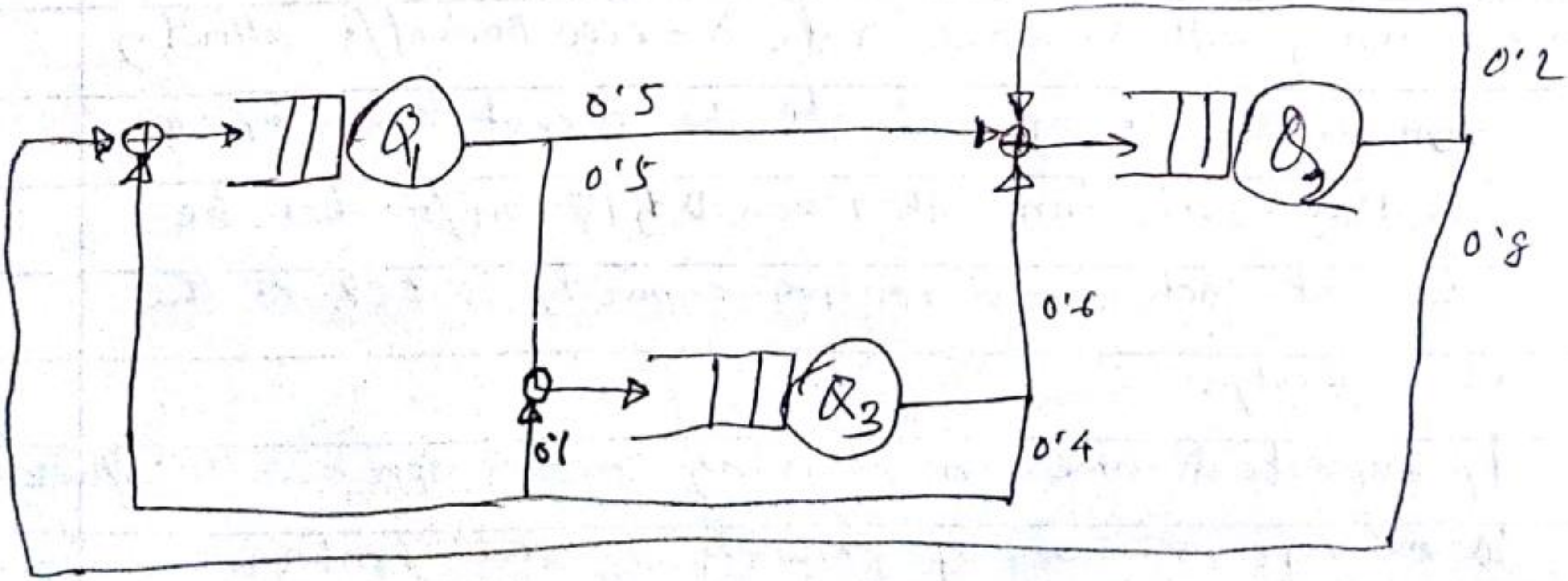
DIY

Consider the closed queuing network of single server queues with exponentially distributed service times, as shown in the figure below.



The average service rates of Q1, Q2 and Q3 are respectively $\mu_1 = 1$, $\mu_2 = 0.5$, and $\mu_3 = 0.5$. The system has a total user population of 4. **Using the convolution approach,** obtain the following –

(a) The Normalization Constant [5]
(b) The state probability distribution for the queuing network [2]
(c) The actual throughput of each queue [3]
(d) The mean number in each queue [5]



The average service rates of Q_1, Q_2 and Q_3 are respectively $\mu_1 = 1$, $\mu_2 = \mu_3 = 0'5$. The system has a total user population of 4. Using the convolution approach, obtain the following:-

i) The normalization constant,

ii) The state probability distribution for the network.

iii) The actual throughput of each queue.

iv) The average waiting time of each queue.

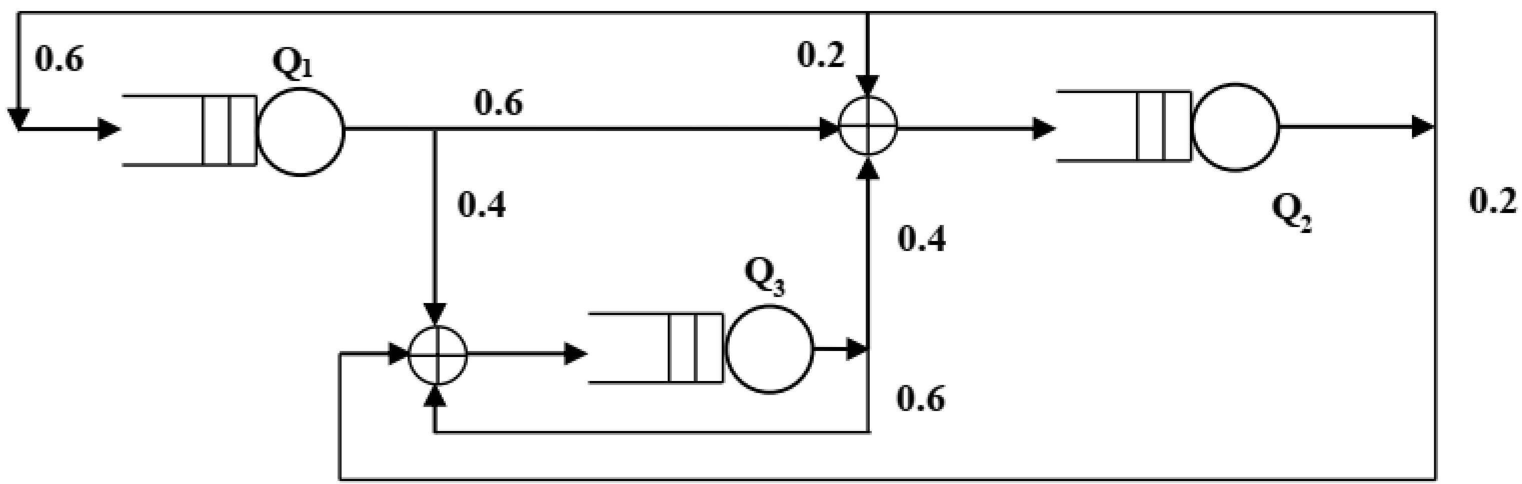
	lmd1	lmd2	lmd3		lmd1	lmd2	lmd3		lmd1	lmd2	lmd3		
lmd1=	0	0.8	0.3		-1	0.8	0.3		1				
lmd2=	0.5	0.2	0.6		0.5	-0.8	0.6			-0.8	0.6	0.5	
lmd3=	0.5	0	0.1		0.5	0	-0.9			0	-0.9	0.5	
let, lmd1=1		1		mu1	1			lmd2	lmd3				
lmd2 =		1.041667		mu2	0.5			1	0.75	0.625			
lmd3=		0.555556		mu3	0.5			0	-0.9	-0.5			
				G(0,m) = 1									
	r1	r2	r3	G(k,1) = r1^k				eff1	0.436275	L1	0.68845	W1	1.5
	1	2.083333	1.111111	G(k,m) = G(k, m-1) +rmG(k-1,m)				eff2	0.454453	L2	2.502848	W2	5.50
		machine						eff3	0.242375	L3	0.808702	W3	3.33
Job	**1**	**2**	**3**		**1**	**2**	**3**						
0	1	1	1							u1	0.436275		
1	1	3.083333	4.194444		29.89264	62.27633	33.21404			u2	0.908906		
2	1	7.423611	12.0841		12.0841	52.44837	14.91865			u3	0.48475		
3	1	16.46586	29.89264		4.194444	37.9272	5.753696						
4	1	35.30387	68.51791		1	18.83801	1.524158						
				sum	47.17119	171.4899	55.41055						

- Consider the closed queueing network of single server queues with exponentially distributed service times, as shown in the figure below. The average service rates of Q1, Q2, and Q3 are respectively, $\mu 1 = 0.5$, $\mu 2 = 1$, and $\mu 3 = 0.5$. If the system has a total user population of 4,calculate the followings using Buzen's algorithm.

  i) The state probability distribution of the network.

  ii) The actual throughput of each queue.

  iii) The average waiting time of each queue.



	lmd1	lmd2	lmd3		lmd1	lmd2	lmd3		lmd1	lmd2	lmd3		
lmd1=	0	0.6	0		-1	0.6	0		0.5				
lmd2=	0.6	0.2	0.4		0.6	-0.8	0.4			-0.8	0.4	0.3	
lmd3=	0.4	0.2	0.6		0.4	0.2	-0.4			0.2	-0.4	0.2	
let, lmd1=1		0.5		mu1	0.5		lmd2	lmd3			lmd2	lmd3	
lmd2 =		0.8333		mu2	1		1	-0.5	-0.375		1	0.5	0.375
lmd3=		0.9167		mu3	0.5		1	-2	1		0	1.5	1.375
			eff. Arrival rate		$e_i = \lambda_i \frac{G(K-1,M)}{G(K,M)}$			Exp. Cust. In a Q		$L_i = \frac{1}{G(K,M)} \sum_{k=1}^{K} \rho_i^k G(K-k,M)$			
				G(0,m) = 1									
	r1	r2	r3	G(k,1) = r1^k				eff1	0.2485	L1	0.8417	W1	3.387
	1	0.8333	1.8333	G(k,m) = G(k, m-1) +rmG(k-1				eff2	0.4141	L2	0.6377	W2	1.54
		machine						eff3	0.4556	L3	2.5206	W3	5.533
Job	**1**	**2**	**3**		**1**	**2**	**3**						
0	1	1	1							u1	0.497		
1	1	1.8333	3.6667		20.065	16.721	36.79			u2	0.4141		
2	1	2.5278	9.25		9.25	6.4236	31.09			u3	0.9111		
3	1	3.1065	20.065		3.6667	2.1219	22.59						
4	1	3.5887	40.374		1	0.4823	11.3						
				sum	33.981	25.748	101.8						
n1	n2	n3		ρ1^r	ρ2^n2	ρ3^n3			P(n1,n2,n3)				
0	0	4		1	1	11.3		11.3		0.28			
0	1	3		1	0.83	6.16		5.14		0.13			
0	2	2		1	0.69	3.36		2.33		0.06	P(q1		
0	3	1		1	0.58	1.83		1.06		0.03	idle)	busy	
0	4	0		1	0.48	1		0.48		0.01	0.5	0.497	