List 2 Exercise 1 PDF for UMSI

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June 2024

1 Task Description

Let $X \in \{0,1\}$ be an input drawn from a Bernoulli distribution $B(p)$ $p < \frac{1}{2}$, and let $Y \in \{0,1\}$ be the output obtained as follows:

$$Y = \begin{cases} X & \text{with probability } 1-p, \\ 1-X & \text{with probability } p \end{cases}$$

Determine the optimal Bayes classifier and its risk.

2 Solution

Bayes Classifier

$h(y)$ chooses x that maximizes possibility of hitting right value.

$$h(y) = \arg\max_{x \in 0,1} P(X = x|Y = y)$$

It means that

$$h(0) = 0 \iff P(X = 0|Y = 0) > P(X = 1|Y = 0)$$

$$h(1) = 0 \iff P(X = 0|Y = 1) > P(X = 1|Y = 1)$$

So we need to calculate these probabilities and a lot of others.

- From Task Description
  $P(X = 0) = 1 - p$
  $P(X = 1) = p$
  $P(Y = 0|X = 0) = 1 - p$
  $P(Y = 0|X = 1) = p$
  $P(Y = 1|X = 0) = p$
  $P(Y = 1|X = 1) = 1 - p$
- Marginal Probabilities
  $P(Y = 0) = P(Y = 0 \wedge X = 0) + P(Y = 0 \wedge X = 1) = (1-p)^2 + p^2$
  $P(Y = 1) = P(Y = 1 \wedge X = 0) + P(Y = 1 \wedge X = 1) = (1-p) \cdot p + (1-p) \cdot p$
- Posterior Probabilities from Bayes
  $P(X = 0|Y = 0) = \frac{(1-p)^2}{(1-p)^2+p^2}$
  $P(X = 1|Y = 0) = \frac{p^2}{(1-p)^2+p^2}$
  $P(X = 0|Y = 1) = \frac{p \cdot (1-p)}{2p \cdot (1-p)}$
  $P(X = 1|Y = 1) = \frac{p \cdot (1-p)}{2p \cdot (1-p)}$

From above we can say that $h(0) = 0$, because for $p < \frac{1}{2}$, we have $p^2 < (1-p)^2$.
On the other hand, as $P(X = 0|Y = 1) = P(X = 1|Y = 1)$, we can choose any value for $h(1)$ so for symmetry we choose $h(1) = 1$

Risk

$$L^* = P(X=0) \min P(Y=1|X=0), P(Y=0|X=0) + P(X=1) \min P(Y=1|X=1), P(Y=0|X=1)$$

$$L^* = (1-p) \cdot (p) + p^2 = p$$