Assignment 1: Derivation and Analysis of PDEs

Shraddha Raghuram

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Consider a finite, simply-connected open domain $\Omega \subset R2$ with the smooth boundary $\partial\Omega$. A substance is produced in Ω at the local rate f (x, y, t) > 0. The produced substance diffuses inside Ω. The diffusion coefficient is k(x, y) > 0, (x, y) $\in \Omega = \Omega \cup \partial\Omega$.

1. Denoting by c(x, y, t) [m^{-2}] the concentration of the substance and by J(x, y, t) its flux density, write down the global conservation law. What are the physical units of f and J?

$$[c(x, y, t)] = [m^{-2}]$$

According to the global form of conservation laws, if Q(t) is the total amount of some quantity inside the domain Ω at time t, Q changes with time due to the outward flux ϕ, and the creation rate R(t) inside Ω. The rate of change of Q inside Ω,

$$\frac{dQ}{dt} = -\phi + R \tag{1}$$

$$Q = \iint_A c(x, y, t)\, dx\, dy$$

$$\phi = \int J(x, y, t)\, d\partial\Omega$$

$$R = \iint_A f(x, y, t)\, dx\, dy$$

Substituting the above in equation (1),

$$\therefore \iint_A c(x, y, t)\, dx\, dy = -\int J(x, y, t)\, d\partial\Omega + \iint_A f(x, y, t)\, dx\, dy$$

Using dimensional analysis,

$$\left[\int J(x, y, t)\, d\partial\Omega\right] = \left[\iint_A c(x, y, t)\, dx\, dy\right] = [m^{-2}s^{-1}m^2]$$

$$\therefore [J] = [s^{-1}m^{-1}]$$

$$\left[\iint_A f(x, y, t)\, dx\, dy\right] = [m^{-2}s^{-1}m^2]$$

$$\therefore [f] = [s^{-1}m^{-2}]$$

2. Derive the local conservation law for (x, y) ∈ Ω.
Let c(x,y,t) be the density.

$$Q(t) = \iint_{\Omega} c(x, y, t)\, dx\, dy$$

$$R(t) = \iint_{\Omega} f(x, y, t)\, dx\, dy$$

Substituting the above in equation 1,

$$\frac{d \iint_{\Omega} c(x, y, t)\, dx\, dy}{dt} = -\int J(x, y, t)\, d\partial\Omega + \iint_{\Omega} f(x, y, t)\, dx\, dy = -\iint_{\Omega} \nabla \cdot \vec{J}\, dx\, dy + \iint_{\Omega} f(x, y, t)\, dx\, dy$$

$$\therefore \frac{\partial c}{\partial t} + \nabla \cdot \vec{J} = f \tag{2}$$

3. Use the Fick's law of diffusion to formulate the partial differential equation describing the local transport of c for (x, y) ∈ Ω. What are the physical units of k?
According to Fick's law of diffusion,

$$J = -k(x, y) \cdot \nabla c(x, y, t)$$

Substituting this in equation (2),

$$\frac{\partial c}{\partial t} - \nabla \cdot k(x, y) \cdot \nabla c(x, y, t) = f$$

$$[\frac{\partial c}{\partial t}] = [\nabla \cdot k(x, y) \cdot \nabla c(x, y, t)]$$

$$[m^{-2} s^{-1}] = [k * m^{-4}]$$

$$\therefore [k] = [m^{2} s^{-1}]$$

4. Assuming that no substance was present in $\overline{\Omega}$ at t = 0, write down the initial condition on c.
There is no substance in $\overline{\Omega}$ at t=0

$$\overline{\Omega} = \Omega \cup \partial\Omega$$

$$\therefore c(x, y, t = 0) = 0$$

5. Assuming that the substance flows (outwards) through the boundary $\partial\Omega$ at a local rate (normal flux density) proportional to the local concentration of the substance at $\partial\Omega$, with the coefficient of proportionality $\alpha(x, y) > 0$, write down the boundary condition on c. What are the physical units of α?
The rate of flow of the substance through the boundary is directly proportional to the local concentration of the substance at the boundary. So,

$$J(x, y, t) \cdot n \propto c(x, y, t)$$

$$J(x, y, t) \cdot n = -\alpha(x, y) * c(x, y, t)$$

Since the flow is outward,

$$J(x, y, t) \cdot n = -\alpha(x, y) * c(x, y, t)$$

Using dimensional analysis,

$$[J(x,y,t)] = [\alpha(x,y)][c(x,y,t)]$$
$$[m^{-1}s^{-1}] = [\alpha][m^{-2}]$$
$$\therefore [\alpha] = [ms^{-1}]$$

6. Formulate the steady-state boundary-value problem and analyze it using the definitions and theorems from the Chapters 1, 2 of the book and the slides of Lectures 1, 2. Specifically, address the following points:

   (a) What is the class/type of the PDE? What is the type of the boundary condition?
   The steady-state boundary-value problem is as follows:

$$\frac{\partial c}{\partial t} - \nabla \cdot k * \nabla c(x,y,t) = f$$

$$c(x,y,t=0) = 0$$

$$\frac{\partial c}{\partial t} = -\alpha(x,y) \cdot c(x,y,t)$$

$$-\nabla \cdot k * \nabla c(x,y,t) = -\nabla \cdot k \begin{bmatrix} \frac{\partial c}{\partial x} \\ \frac{\partial c}{\partial y} \end{bmatrix}$$

$$= -\frac{\partial k}{\partial x}\frac{\partial c}{\partial x} - k\frac{\partial^2 c}{\partial x^2} - \frac{\partial k}{\partial y}\frac{\partial c}{\partial y} - k\frac{\partial^2 c}{\partial y^2}$$

   $a_1 = -k, a_{12} = 0, a_{22} = -k$

$$a_{12}^2 - a_{11} * a_{22} = 0 - (-k) * (-k) = -k < 0$$

   Therefore it is an elliptic PDE. Since the boundary condition is of the form

$$k(\mathbf{x})D_{\mathbf{n}}u(\mathbf{x}) + \sigma u(\mathbf{x}) = g_2(\mathbf{x})$$

   , the type of boundary condition is a Robin boundary condition.

   (b) Let k = 1 and suppose that there is a solution c of the steady- state boundary-value problem satisfying $c \in C^2(\Omega) \cap C(\overline{\Omega})$. Is it unique? To establish this fact, you also need to prove the corresponding Theorem.
   Since the Robin boundary condition exists, and $c \in C^2(\Omega) \cap C(\overline{\Omega})$. This means that $c \in C^2(\Omega) \cap C(\partial\Omega)$ satisfies

$$-\Delta c = f(\mathbf{x}), \mathbf{x} \in \Omega$$

$$\sigma c + D_{\mathbf{c}}u = g(\mathbf{x}), \in \partial\Omega$$

   with $\sigma > 0$.
   Let us assume that the solution for this boundary-value problem is non unique. Let these non unique solutions be $c_1 and c_2$.
   This means that they satisfy

$$-\Delta \mathbf{c}_1 = f(\mathbf{x}), \mathbf{x} \in \Omega \tag{3}$$

$$\sigma \mathbf{c}_1 + D_{\mathbf{n}}\mathbf{c}_1 = g(\mathbf{x}), \mathbf{x} \in \partial\Omega \tag{4}$$

$$-\Delta \mathbf{c}_2 = f(\mathbf{x}), \mathbf{x} \in \Omega \tag{5}$$

$$\sigma \mathbf{c}_2 + D_{\mathbf{n}} \mathbf{c}_2 = g(\mathbf{x}), \mathbf{x} \in \partial\Omega \tag{6}$$

(3)-(5)

$$\Delta(\mathbf{c}_2 - \mathbf{c}_1) = f - f = 0$$

Let $\mathbf{c} = \mathbf{c}_1 - \mathbf{c}_2, c \neq 0$.

$$-\Delta c = 0 \tag{7}$$

(4)-(6)

$$\sigma(\mathbf{c}_1 - \mathbf{c}_2) + D_{\mathbf{n}}(\mathbf{c}_1) - D_{\mathbf{n}}(\mathbf{c}_2) = g - g = 0$$

$$\sigma \mathbf{c} + \nabla \mathbf{c}_1 \cdot \mathbf{n} - \nabla \mathbf{c}_2 \cdot \mathbf{n} = 0$$

$$\sigma \mathbf{c} + \nabla(\mathbf{c}_1 - \mathbf{c}_2) \cdot \mathbf{n} = 0$$

$$\sigma \mathbf{c} + \nabla \mathbf{c} \cdot \mathbf{n} = 0 \tag{8}$$

$$\iint_\Omega \nabla \cdot (\mathbf{c}\nabla\mathbf{c})\, dA = \iint_\Omega (\mathbf{c}\nabla\mathbf{c})\, dA + \iint_\Omega (\nabla\mathbf{c})^2\, dA = \iint_\Omega (\nabla\mathbf{c})^2\, dA \geq 0$$

Using the divergence theorem,

$$\iint_\Omega \nabla \cdot (\mathbf{c}\nabla\mathbf{c})\, dA = \int_{\partial\Omega} \mathbf{c}\nabla\mathbf{c}\, \mathbf{dS} = \int_{\partial\Omega} \mathbf{c}\nabla\mathbf{c} \cdot \mathbf{n}\, dS = \int_{\partial\Omega} -\sigma\mathbf{c}^2 dS \leq 0$$

$$\therefore \iint_\Omega (\nabla\mathbf{c})^2\, dA = 0$$

$$\therefore \nabla\mathbf{c} = 0$$

From (8),

$$\sigma\mathbf{c} = -\nabla\mathbf{c} \cdot \mathbf{n}$$

$$\sigma c = 0 * n = 0$$

Since $\sigma > 0$, c should be 0. But this would mean $c_1 - c_2 = 0$ and thus $c_1 = c_2$, and this implies the existence of a unique solution. Therefore c is the unique solution for the steady-state boundary-value problem satisfying the given questions.