Assignment 2: Numerical Solution of the One-Dimensional Poisson's Equation With the Finite-Difference Method

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September 21, 2022

Consider the following boundary-value problem:

$$-\frac{d^2u_i}{dx^2} = f_i, x \in (0,3) \tag{1}$$

$$u_i(0) = 1, u_i(3) = 1, i = 1,2$$

with the source functions

$$f_1(x) = 3x - 2, f_2(x) = x^2 + 3x - 2, x \in [0,3]. \tag{2}$$

1. Find the exact solutions $u_1^{ex}(x)$ and $u_2^{ex}(x)$ of the problem (1) corresponding to $f_1(x)$ and $f_2(x)$ given in (2).

$$-u_i'' = f_i$$

$$-u_1'' = 3x - 2$$

$$-u_1' = \frac{3x^2}{2} - 2x + C$$

$$-u_1 = \frac{3x^3}{6} - \frac{2x^2}{2} + Cx + D$$

$$u_1(x) = \frac{-x^3}{2} + x^2 - Cx - D$$

Applying $u_1(0) = 1$

$$1 = (0) + (0) - (0) - D$$

$$D = -1$$

Applying $u_1(3) = 1$

$$1 = \frac{-27}{2} + 9 - 3C + 1$$

$$3C = \frac{18 - 27}{2}$$

$$C = \frac{-3}{2}$$

$$\therefore u_1(x) = \frac{-x^3}{2} + x^2 + \frac{3x}{2} + 1$$

$$-u_i'' = f_i$$

$$-u_2'' = x^2 + 3x - 2$$

$$-u_2' = \frac{x^3}{3} + \frac{3x^2}{2} - 2x + C$$

$$-u_2 = \frac{x^4}{12} + \frac{3x^3}{6} - \frac{2x^2}{2} + Cx + D$$

$$u_2(x) = -\frac{x^4}{12} + \frac{-x^3}{2} + x^2 - Cx - D$$

Applying $u_2(0) = 1$

$$1 = -(0) + (0) + (0) - (0) - D$$

$$D = -1$$

Applying $u_1(3) = 1$

$$1 = -\frac{81}{12} + \frac{-27}{2} + 9 - 3C + 1$$

$$3C = \frac{-27 - 18}{4}$$

$$C = \frac{-15}{4} +$$

$$\therefore u_2(x) = -\frac{x^4}{12} + \frac{-x^3}{2} + x^2 + \frac{15}{4}x + 1$$

2. Discretize the problem (1) using the Finite-Difference Method (FDM) on a uniform grid obtained by dividing the [0, 3] interval into n = 5 sub-intervals of equal length.

   (a) What is the step size h? How many internal and boundary points do you get? How many unknowns does your numerical problem have?

   $$h = \frac{3-0}{5} = \frac{3}{5}$$

   There are 4 internal points: $\frac{3}{5}, \frac{6}{5}, \frac{9}{5}, \frac{12}{5}$. There are 2 boundary points: $a = 0$ and $b = 3$. There are 4 unknowns corresponding to the 4 internal points.

   (b) Write down the $O(h^2)$ finite-difference (FD) approximation of the (negative) second derivative operator, including the explicit remainder term.
   For $u_1(x) = -\frac{x^3}{2} + x^2 + \frac{3x}{2} + 1$, Taking the taylor series of $u_1(x)$ around $x_1 = \frac{3}{5}$,

   $$u(x) = u(\frac{3}{5}) + u'(\frac{3}{5})(x - \frac{3}{5}) + \frac{u''(\frac{3}{5})}{2}(x - \frac{3}{5})^2 + \frac{u'''(\frac{3}{5})}{6}(x - \frac{3}{5})^3 + \ldots.$$

   $$u(x) = u_1 + u_1'(x - \frac{3}{5}) + \frac{u_1''}{2}(x - \frac{3}{5})^2 + \frac{u_1'''}{6}(x - \frac{3}{5})^3 + \ldots.$$

   Taking the nodes x_2 and x_0

   $$u_2 = u_1 + u_1'(\frac{6}{5} - \frac{3}{5}) + \frac{u_1''}{2}(\frac{6}{5} - \frac{3}{5})^2 + \frac{u_1'''}{6}(\frac{6}{5} - \frac{3}{5})^3 + \ldots.$$

$$u_0 = u_1 + u_1'(0-\frac{3}{5}) + \frac{u_1''}{2}(0-\frac{3}{5})^2 + \frac{u_1'''}{6}(0-\frac{3}{5})^3 + ...$$

Rearranging

$$-\frac{u_1''}{2}(\frac{6}{5}-\frac{3}{5})^2 = u_1 + u_1'(\frac{6}{5}-\frac{3}{5}) - u_2 + \frac{u_1'''}{6}(\frac{6}{5}-\frac{3}{5})^3 + ...$$

$$-\frac{u_1''}{2}(0-\frac{3}{5})^2 = u_1 + u_1'(0-\frac{3}{5}) - u_0 + \frac{u_1'''}{6}(0-\frac{3}{5})^3 + ...$$

Dividing both equations by h^2,

$$-\frac{u_1''}{2} = \frac{-u_2+u_1}{(\frac{6}{5}-\frac{3}{5})^2} + \frac{u_1'}{(\frac{6}{5}-\frac{3}{5})} + \frac{u_1'''}{6}(\frac{6}{5}-\frac{3}{5}) + ...$$

$$-\frac{u_1''}{2} = \frac{-u_0+u_1}{(-\frac{3}{5})^2} + \frac{u_1'}{(-\frac{3}{5})} + \frac{u_1'''}{6}(-\frac{3}{5}) + ...$$

Adding both equations,

$$-u_1'' = \frac{-u_0-u_2+2u_1}{(\frac{3}{5})^2} + 2u_1''''\frac{h^2}{24}$$

The explicit remainder term is

$$+2u_1''''\frac{h^2}{24} = \frac{3u_1''''}{100}$$

$$\therefore u_1'' = -\frac{-u_0-u_2+2u_1}{(\frac{3}{5})^2} - \frac{3u_1''''}{100}$$

Similarly,

$$\therefore u_2'' = -\frac{-u_1-u_3+2u_2}{(\frac{3}{5})^2} - \frac{3u_2''''}{100}$$

(c) Write down all discrete FD equations for your problem, explicitly computing numerical constants.

Continuous problem:

$$-u'' = f, x \in \Omega = [0,3];$$

$$u(a) = u(0) = g_0 = 1, u(b) = u(3) = g_1 = 1$$

Discrete FD equations:

$u_0 = g_0 = 1$ left BC

$$-\frac{1}{h^2}(1-2u_1+u_2) = f_1$$

$$-\frac{1}{h^2}(u_1-2u_2+u_3) = f_2$$

$$-\frac{1}{h^2}(u_2-2u_3+u_4) = f_3$$

$$-\frac{1}{h^2}(u_3-2u_4+1) = f_4$$

$u_5 = g_1 = 1$ right BC

Corresponding to u_1, these values are:

$$f_1 = \frac{-1}{5}$$

$$f_2 = \frac{8}{5}$$
$$f_3 = \frac{17}{5}$$
$$f_4 = \frac{26}{5}$$

Corresponding to u_2, these values are:

$$f_1 = \frac{4}{25}$$
$$f_2 = \frac{76}{25}$$
$$f_3 = \frac{166}{25}$$
$$f_4 = \frac{274}{25}$$

(d) Write out the system matrix A and the right-hand-side vectors f_1 and f_2, for the source functions (2)

$$-\frac{1}{h^2}(-2u_1 + u_2) = f_1 + \frac{1}{h^2}$$
$$-\frac{1}{h^2}(u_1 - 2u_2 + u_3) = f_2$$
$$-\frac{1}{h^2}(u_2 - 2u_3 + u_4) = f_3$$
$$-\frac{1}{h^2}(u_3 - 2u_4) = f_4 + \frac{1}{h^2}$$

The system matrix and RHS vector is as follows

$$\frac{-1}{h^2}\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f_1 + \frac{1}{h^2} \\ f_2 \\ f_3 \\ f_4 + \frac{1}{h^2} \end{bmatrix}$$

System matrix,

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

RHS vector= $\begin{bmatrix} f_1 + \frac{1}{h^2} \\ f_2 \\ f_3 \\ f_4 + \frac{1}{h^2} \end{bmatrix}$ For u_1 this is

$$\begin{bmatrix} 2.57777778 \\ 1.6 \\ 3.4 \\ 7.97777778 \end{bmatrix}$$

For u_2 this is

$$\begin{bmatrix} 1.16 \\ 3.04 \\ 6.64 \\ 33.2222222 \end{bmatrix}$$

(e) Compute the theoretical eigenvalues of A and present them as a column of a LaTeX table in your report

$$\lambda_i = \frac{4}{h^2}\sin^2(\pi\frac{i}{2N})$$

$$h = \frac{3}{5}, N = 5$$

Eigenvalues of A	
i	λ_i
1	1.061016697916959
2	3.838794475694737
3	7.272316635416375
4	10.05009441319415
5	11.11111111111111

(f) Compute the first 4 (smallest) eigenvalues of the negative second derivative operator and present them as the second column in your table.

$$[\mathbf{v}_i]_j = \sin(\pi\frac{ij}{N})$$

i,j=1,2,....N-1

Eigenvalues and Eigenvectors of A		
i	λ_i	$\mathbf{v}_i$
1	1.061016697916959	$\begin{bmatrix} 0.587785252292473 \\ 0.951056516295154 \\ 0.951056516295154 \\ 0.587785252292473 \\ 0 \end{bmatrix}$
2	3.838794475694737	$\begin{bmatrix} 0.951056516295154 \\ 0.587785252292473 \\ -0.587785252292473 \\ -0.951056516295154 \\ 0 \end{bmatrix}$
3	7.272316635416375	$\begin{bmatrix} 0.951056516295154 \\ -0.587785252292473 \\ -0.587785252292473 \\ 0.951056516295154 \\ 0 \end{bmatrix}$
4	10.05009441319415	$\begin{bmatrix} 0.587785252292473 \\ -0.951056516295154 \\ 0.951056516295154 \\ -0.587785252292473 \\ 0 \end{bmatrix}$
5	11.11111111111111	

3. Check your Python environment. Enter the following lines in an ASCII text file called assignment-2.py. Run this code in python3 or ipython3. If it does not produce any errors, all modules have been installed correctly.
   Done in Jupyter Notebook. All modules have been installed correctly.

4. Construct a uniform grid and display the source functions and the exact solutions.

   (a) Study np.linspace() function and use it to construct a numpy array xgrid containing the uniform grid obtained by dividing the interval [0, 3] into n equal sub-intervals. This will add the following lines of the code:
   Done in Jupyter Notebook.

   (b) Use the def operator to define two python functions, func1() and func2(), that return the values of f1(x) and f2(x) for any x. This will add the following lines to your code:
   Done in Jupyter Notebook.

   (c) Use your functions to compute the grid values of $f_1(x)$ and $f_2(x)$ as follows:
   Done in Jupyter Notebook.

   (d) Study plt.figure(), plt.plot(), plt.xlabel(), plt.ylabel(), plt.title(), plt.legend() functions and make a plot of the right-hand-side functions $f_1(x)$ and $f2(x)$ on the grid with n = 5. Use solid lines of blue and red color with circular markers to show both functions in the same plot. The plot legend should identify each function by its LaTeX symbol, i.e., as f_1 and f_2. This will add the following lines to your code:
   Done in Jupyter Notebook.

(e) Make sure that your code, including the plots, works for any n.
Done in Jupyter Notebook.

(f) Export the pdf image of your Python figure for n = 5 and insert it in your LaTeX report.

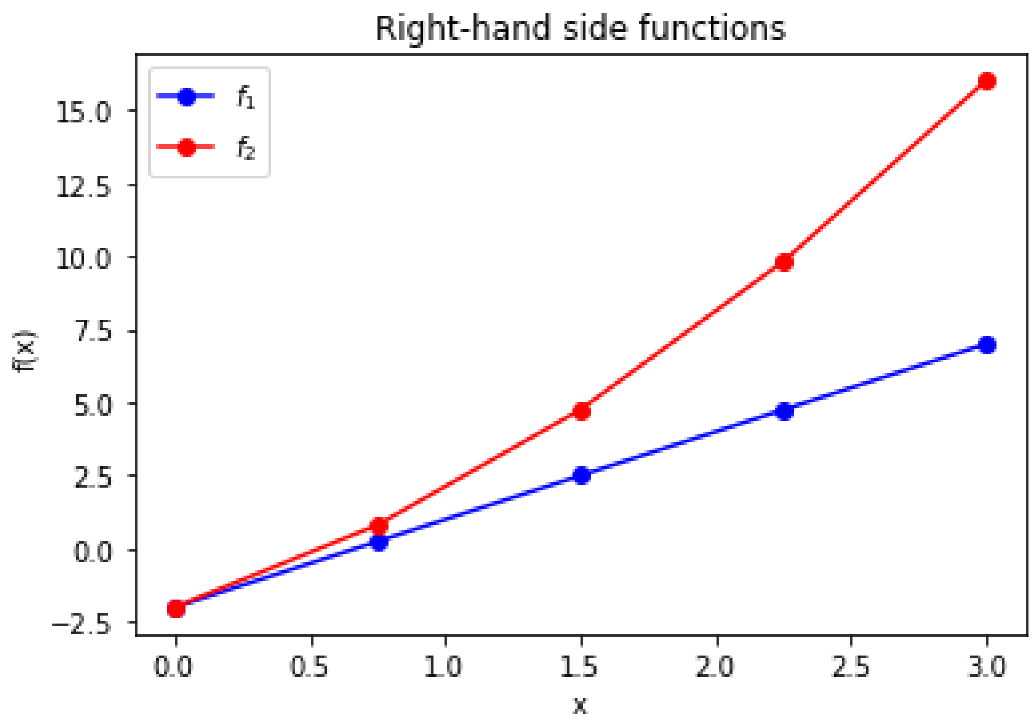

Figure 1: Plot from Matplotlib for 4. (d)

(g) Repeat steps 4 (b)–(e) with the exact solutions $u_1(x)$ and $u_2(x)$ found in step 1. Create arrays u1ex and u2ex to store the grid values of the exact solutions.
Done in Jupyter Notebook.

5. Assemble the Finite-Difference (negative) Laplacian matrix.

$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

(a) Use 2 (e) to understand the structure of the system matrix A. Pay attention to the negative sign in front of the second derivative in (1), it should be preserved in the system matrix as well. What are the dimensions of A in terms of n?
n-1 X n-1

(b) Use np.diag() to construct A from its three diagonals for any given n.
Done in Jupyter Notebook.

(c) Use plt.spy() to visualize the structure of A with green or red round markers. Insert the corresponding plot in your LaTeX report.

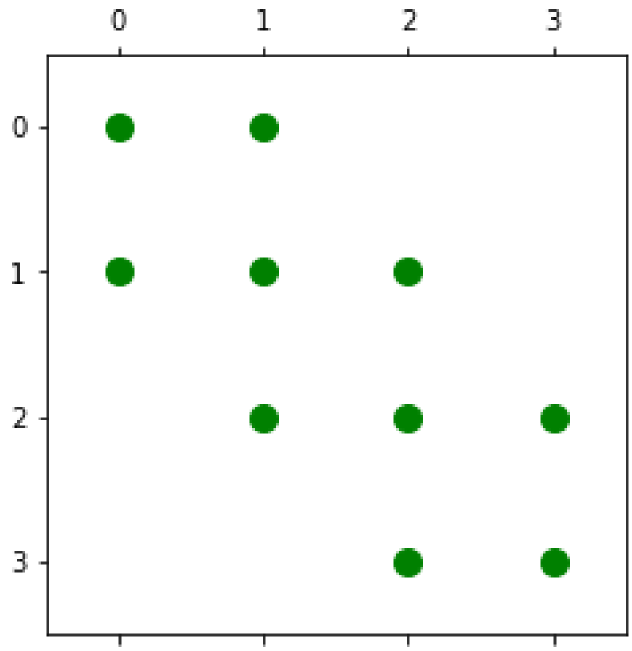

Figure 2: Plot from Matplotlib for 4. (d)

(d) Use np.linalg.eig() to compute numerically the eigenvalues of A for n = 5. Add your results as the third column to the table you have created in 2 (g), (h). Sometimes, numerically computed eigenvalues need to be sorted first.

Eigenvalues and Eigenvectors of A, and numerically calculated eigenvalues			
i	λ_i	$\mathbf{v}_i$	Numerically calculated eigenvalues
1	1.061016697916959	$\begin{bmatrix} 0.587785252292473 \\ 0.951056516295154 \\ 0.951056516295154 \\ 0.587785252292473 \\ 0 \end{bmatrix}$	-1.0610167
2	3.838794475694737	$\begin{bmatrix} 0.951056516295154 \\ 0.587785252292473 \\ -0.587785252292473 \\ -0.951056516295154 \\ 0 \end{bmatrix}$	-3.83879448
3	7.272316635416375	$\begin{bmatrix} 0.951056516295154 \\ -0.587785252292473 \\ -0.587785252292473 \\ 0.951056516295154 \\ 0 \end{bmatrix}$	-10.05009441
4	10.05009441319415	$\begin{bmatrix} 0.587785252292473 \\ -0.951056516295154 \\ 0.951056516295154 \\ -0.587785252292473 \\ 0 \end{bmatrix}$	-7.27231664
5	11.11111111111111		

6. Solve the linear algebraic problem.

   (a) Use your source-function arrays f1 and f2 to create two RHS arrays f1rhs and f2rhs with the same entries as f1 and f2 in 2 (f).
   Done in Jupyter Notebook.

   (b) Use np.linalg.solve() to find the solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ of the linear algebraic problems $A\mathbf{u}_1 = \mathbf{f}_1$ and $A\mathbf{u}_2 = \mathbf{f}_2$.
   Done in Jupyter Notebook.

   (c) Append boundary values to the obtained solution arrays, storing the results as the arrays u1 and u2, and display them in the same plot together with the two exact solution arrays u1ex and u2ex. Use colors corresponding to the colors you used to display $f_1(x)$ and $f_2(x)$. Set lines for the exact solution to solid and for the FDM approximations to dashed. Include a legend identifying the numerical solutions as u_1 and u_2 and exact solutions as u_1^{ex} and u_2^{ex} 2 . Add this figure to your report.

   Done in Jupyter Notebook.

7. Analyze your results.

   (a) Compute the global error between the numerical and the exact solutions for n = 5. Explain your results.

(b) Study the rate of convergence of the Finite-Difference Method by changing n. Display the logarithm of the global error as a function of n. Explain your results.