Modelling the stock price

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1 Problem

Consider the system of (Itô) SDEs for the price S_t of a stock, the stochastic and past-dependent volatility σ_t and the long-term averaged volatility ξ_t:

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma_t S_t dw_t \\ d\sigma_t &= -(\sigma_t - \xi_t)dt + p\sigma_t dw_t \\ d\xi_t &= \frac{1}{\alpha}(\sigma_t - \xi_t)dt \end{aligned} \tag{1}$$

with $S_0 = 50€$, $\sigma_0 = 0.20$, $\xi_0 = 0.20$ and $\mu_0 = 0.10$.
For $p = 0$ and $\alpha \neq 0$, the well-known Black-Scholes model is obtained:

$$dS_t = \mu S_t dt + \sigma_0 S_t dw_t \tag{2}$$

1.1 Implementation of the Euler and Milstein Scheme for the Black-Scholes Model

Implement the Euler and the Milstein strong order 1.0 scheme and generate numerical tracks for $0 \leq t \leq 1$ year for different values of p and α.

The Euler scheme for the above system of equations can be given as:

$$S_{t+dt} = S_t + \mu * S_t * dt + \sigma_t * S_t * dw_t^1 \tag{3}$$

$$\sigma_{t+dt} = \sigma_t - (\sigma_t - \xi_t) * dt + p * \sigma_t * dw_t^2 \tag{4}$$

$$\xi_{t+dt} = \xi_t + (1/\alpha) * (\sigma_t - \xi_t) * dt \tag{5}$$

The Milstein scheme is derived from the Taylor expansion of the solution to the SDEs. The general form of the Milstein scheme is given by the following equations:

$$S_{t+dt} = S_t + \mu * S_t * dt + \sigma_t * S_t * dw_t^1 + (0.5 * \sigma_t * S_t) * \sigma_t * ((dw_t^1)^2 - dt) \tag{6}$$

$$\sigma_{t+dt} = \sigma_t - (\sigma_t - \xi_t) * dt + p * \sigma_t * dw_t^2 + (0.5 * \sigma_t * p) * p * ((dw_t^2)^2 - dt) \tag{7}$$

$$\xi_{t+dt} = \xi_t + (1/\alpha) * (\sigma_t - \xi_t) * dt \tag{8}$$

This schemes have been implemented specifically in Python. This implementation produced the following plots for various values of p and α.
For each combination of α and p, there are three plots representing the stock price S_t, the volatility σ_t and the long-term averaged volatility ξ_t, each of them obtained in two ways: using the Euler scheme and using the Milstein scheme.

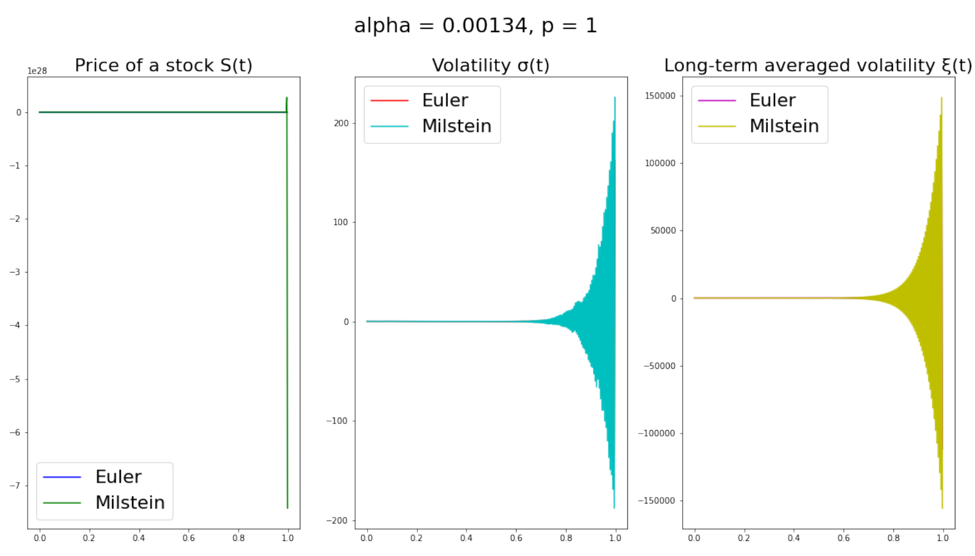

Figure 1

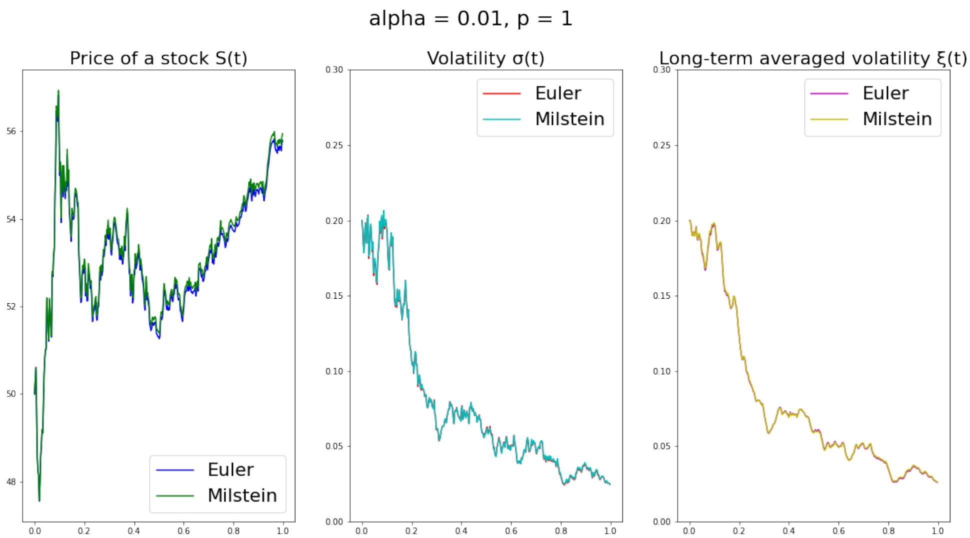

Figure 2

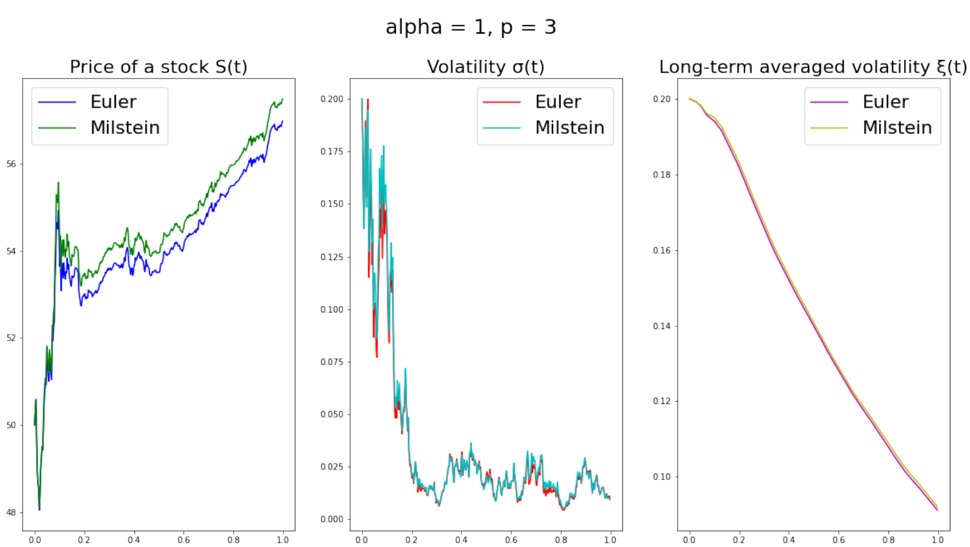

Figure 3

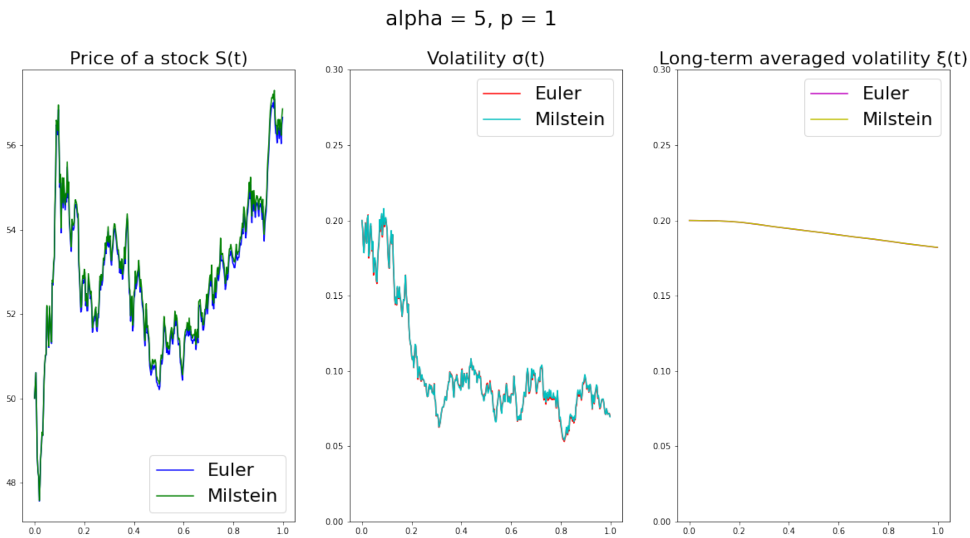

Figure 4

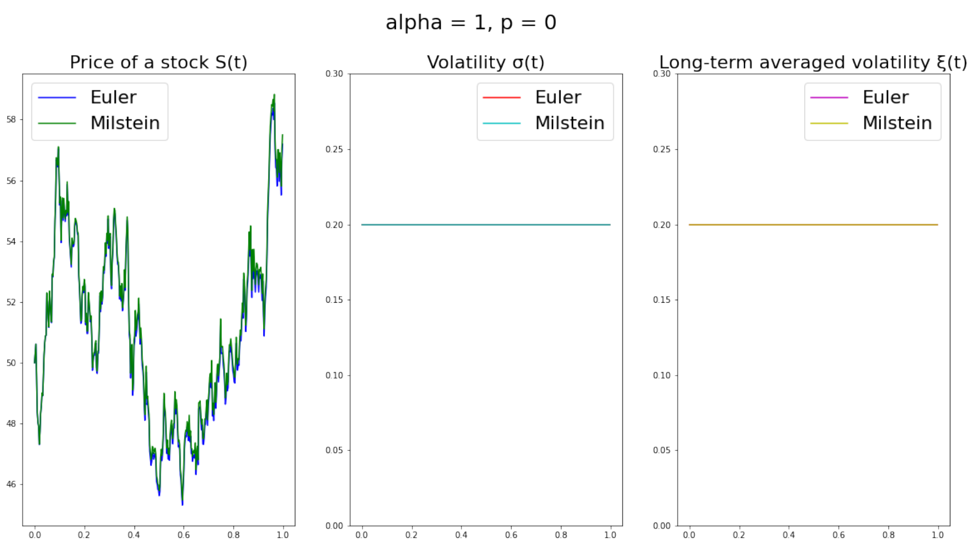

Figure 5

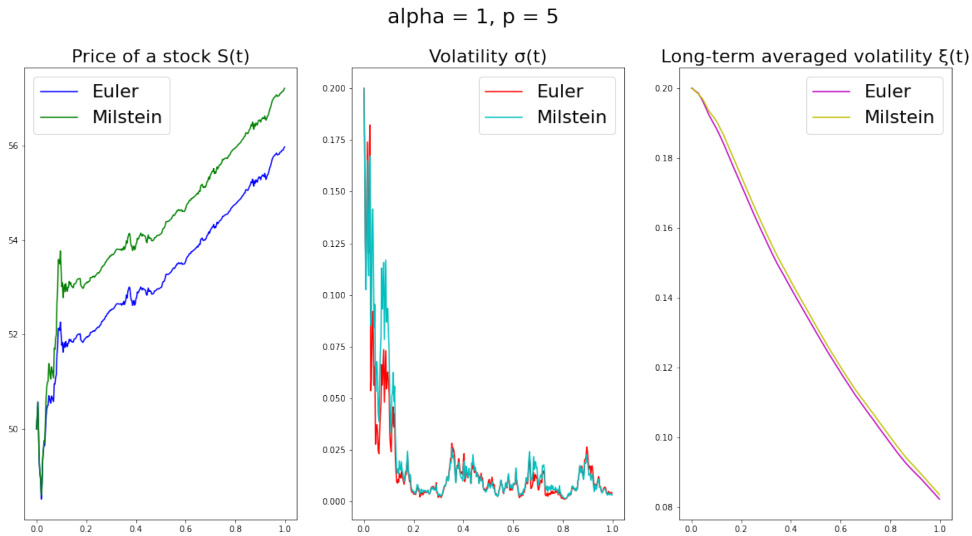

Figure 6

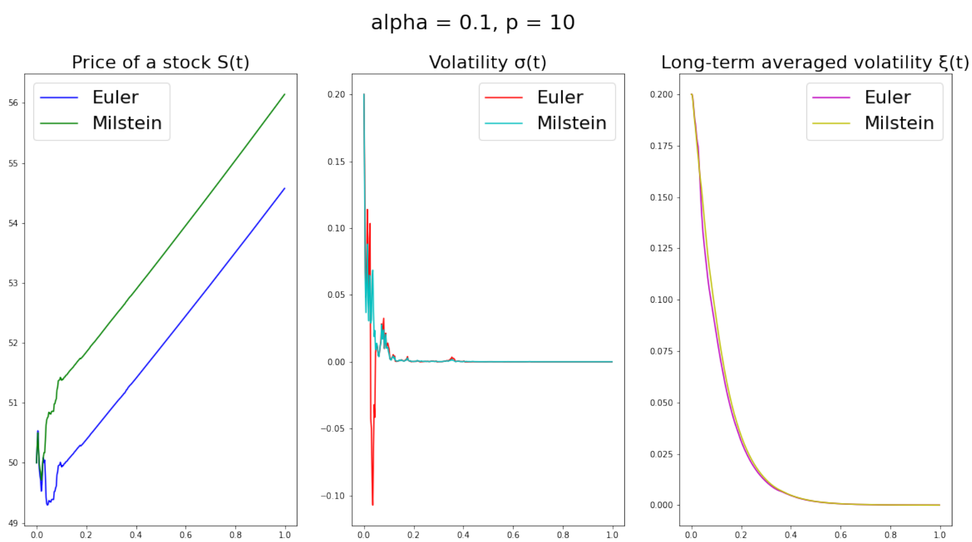

Figure 7

1.2 Influence of parameters p and α on the behaviour of S_t

Study the influence of parameters p and α on the behaviour of S_t.

Testing methodology With these plots the goal was to study the behaviour of S_t, σ_t and ξ_t, and the effects that varying α and p have on their trend. On one side we want to bypass the non-deterministic nature of the values of the stock price and to not let it pollute the conclusions we want to draw about the stock price, so it is preferable to have multiple samples represented in the same graph; on the other hand, it is necessary to have clarity in the plot and to be able to distinguish easily between the different samples. Compromising between these two necessities, it has been decided to use five samples per plot.

Study α Let's first variate α. For values < 0.00134 (see Figure 8), the volatilities start exploding and oscillate between positive and negative values of magnitude 10^3 to 10^6, and we conclude that the stock price model is broken, as its magnitude jumps to values up to 10^{200}. Taking values of α between 0.0015 and 5, the stock price value seems to oscillate without particular sudden changes in a range of values between 30 and 90 €.

In general we can't see a marked effect of varying α on the behaviour of the stock price; if anything, the stock price seems to have slightly bigger oscillations with higher values of α; the same holds for the volatility. The case is different for the long-term averaged volatility: there α has a significant impact, in fact it appears that the higher α, the smoother and the smaller the decrease in the long-term volatility ξ_t. This can be maybe explained with the fact that the differences in ξ_t will have an impact in the long term, so after t=1. Looking at specific values of α, for low values ($\alpha \leq 0.01$, see Figures 9, 10), ξ_t and σ_t are roughly the same; for higher values ($\alpha \geq 1$, see Figures 11, 12), while the volatility σ_t doesn't change, ξ_t has a quasi-linear trend, and for $\alpha = 5$, ξ_t is almost a horizontal line. Note that this doesn't hold for the higher values of p (≥ 3); there ξ_t and σ_t settle on zero (see Figures 14 and 15).

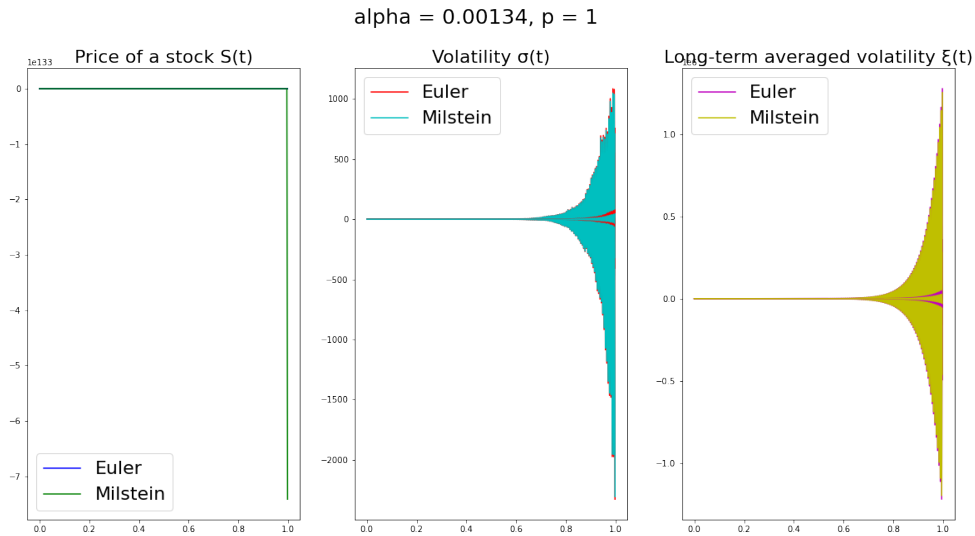

Figure 8

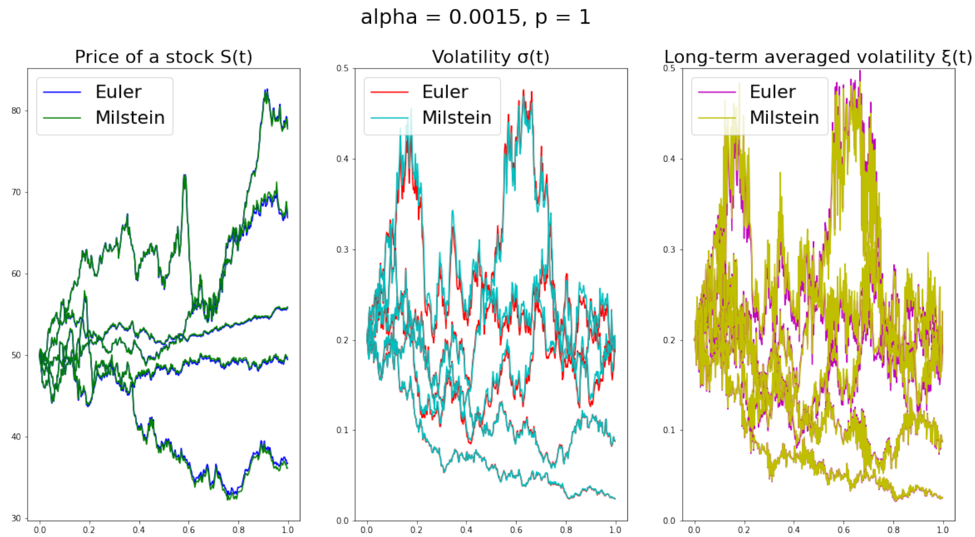

Figure 9

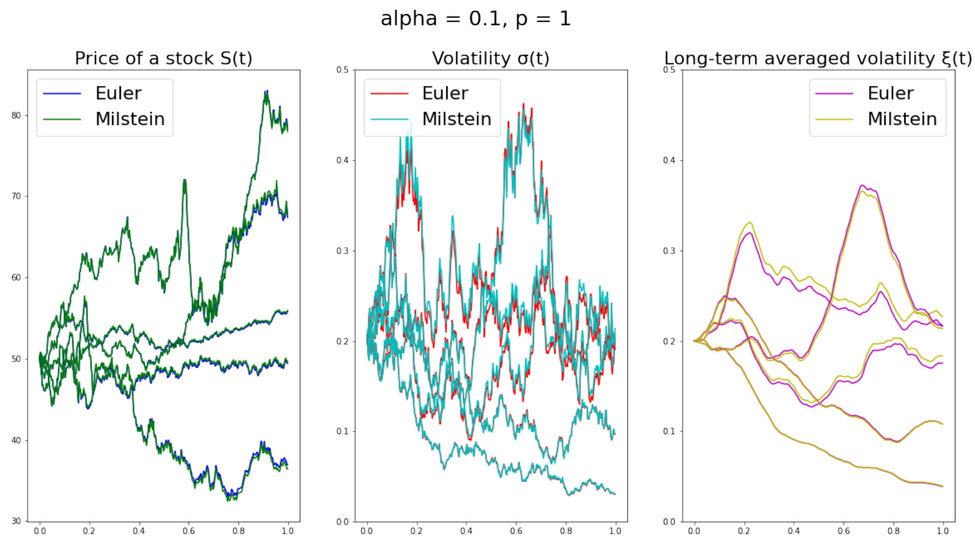

Figure 10

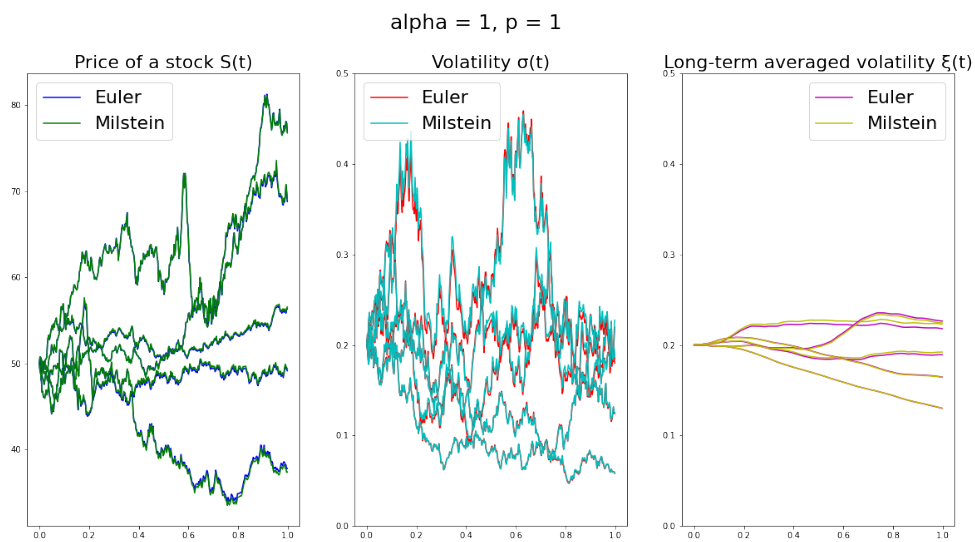

Figure 11

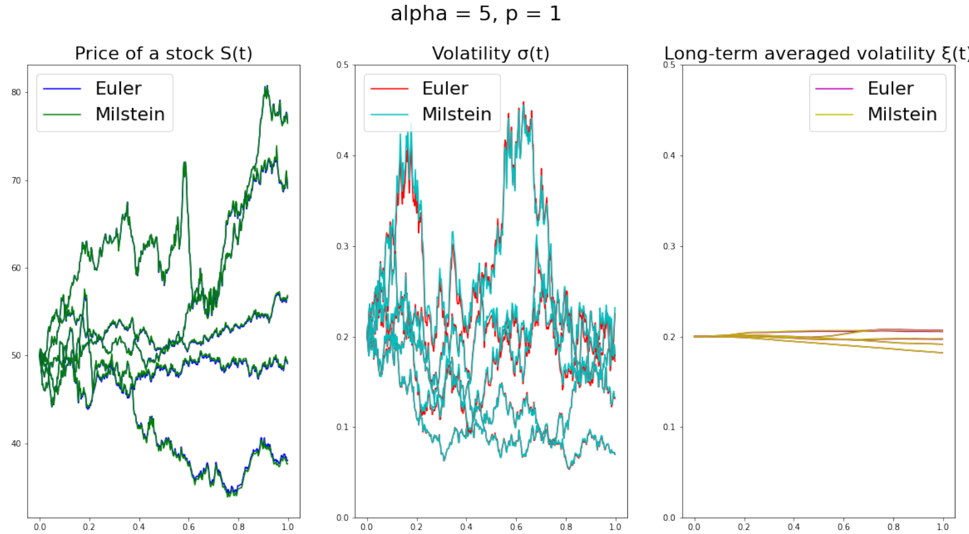

Figure 12

Study p Let's now look at p. For p=0, the volatilities don't depend on time and so remain constant at 0.20 (it is already clear from the one-sample plots in Figure 5). The Euler and Milstein scheme return almost identical results. For $p \leq 2$ (see 13), there aren't big differences visible in the behaviour of the volatilities, nor of S_t. The stock price has an oscillatory behaviour that stays in the range of 40-90 €. When $p = 3$ (see Figure 14), around after t=0.5 the stock price starts resembling roughly a linear growth, and the volatilities settle on 0 (with outliers occasionally occurring). For $p \geq 5$ (see Figure 15), the range of values that the stock price assumes gets smaller as p increases. The trend of the stock price, after showing sudden oscillations in a first interval of time, gets closer and closer to a linear growth as p increases.
Another interesting fact to note about varying p is how it affects the results of the Euler and Milstein scheme and the difference between the two. For $p < 3$ the two methods' stock price almost overlap, from $p \geq 3$ a gap between them appears. For $p \geq 5$ the approximations of the two methods starts to differ by a lot.

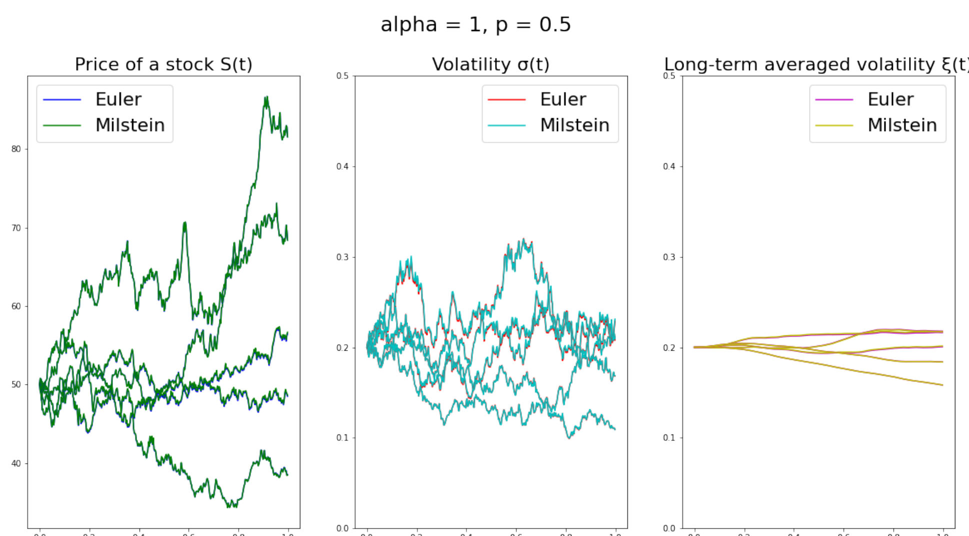

Figure 13

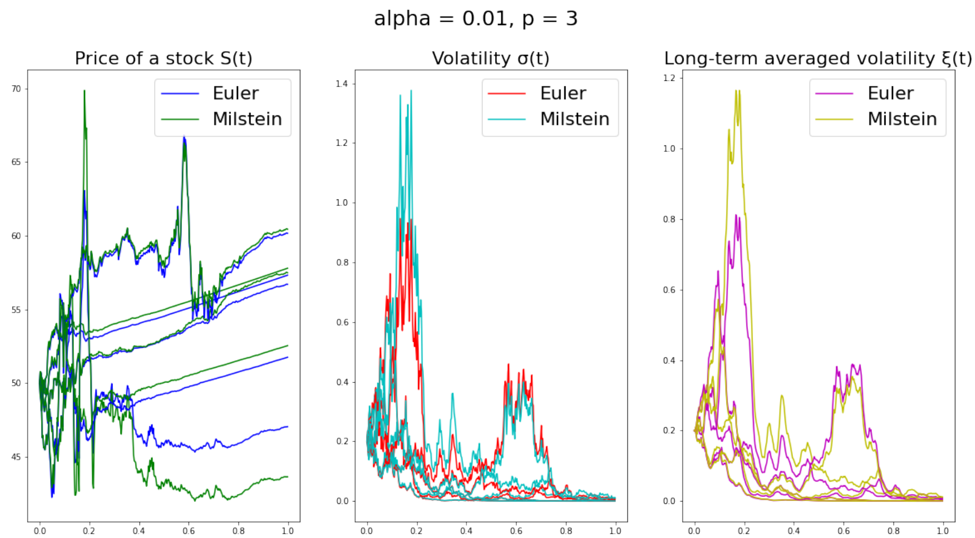

Figure 14

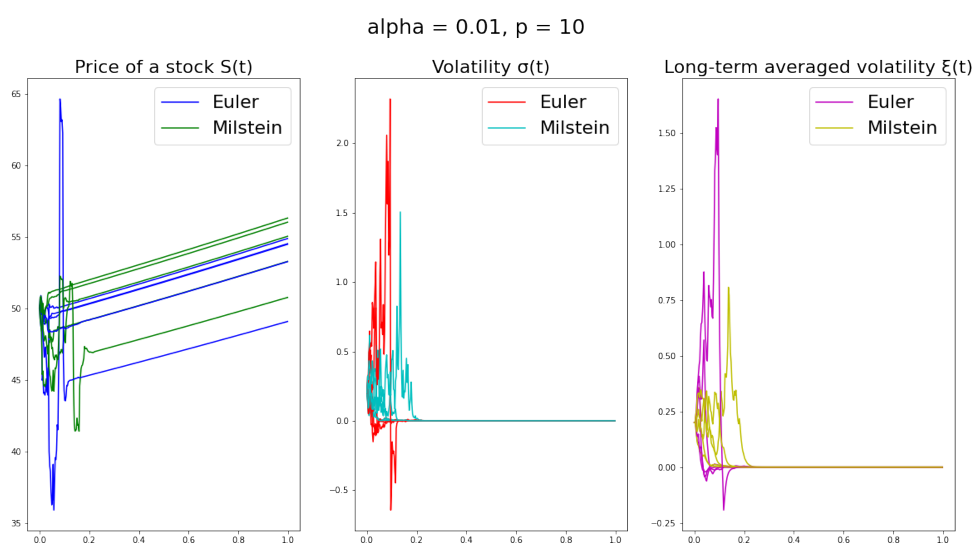

Figure 15

1.3 Convergence behaviour of the schemes for $\Delta t \to 0$ in the strong sense for the Black-Scholes model

Study by means of simulations the convergence behaviour of the schemes for $\Delta t \to 0$ in the strong sense for the Black-Scholes model and discuss the results.

The Black-Scholes model states:

$$dS_t = \mu S_t dt + \sigma_0 S_t dw_t \tag{9}$$

The aim is now to experimentally study the convergence behaviour of the Euler and of the Milstein schemes in the strong sense. Recall that the strong order of convergence is j if $\exists$ K, $\Delta > 0$ such that for fixed $T = N\Delta T$:

$$E\{|X_T - X_N|\} \leq K(\Delta t)^j, \forall\ 0 < \Delta t < \Delta. \tag{10}$$

It is known that the exact solution for the Black-Scholes model is:

$$S_t = S_0 e^{\mu - \frac{\sigma^2}{2} t + \sigma W(t)}. \tag{11}$$

A range of time steps sizes are considered. For this and the next section, the time step sizes considered were $[\frac{1}{365 \cdot 8}, \frac{2}{365 \cdot 8}, ..., \frac{24}{365 \cdot 8}]$. For each of them, 100 samples were generated. For each sample, the Euler and the Milstein schemes were used to compute numerical approximations of S_t for various time steps, and the difference of the approximation and the exact solution was taken. Only after that these error terms were computed for each individual sample, the average over all the samples was taken. Figure 16 plots the strong error terms for both the approximations against the various Δt values.
From the theory it is known that the strong order of convergence of the Euler method should be $\mathcal{O}(\Delta t^{\frac{1}{2}})$ while that of the Milstein method should be $\mathcal{O}(\Delta t)$; therefore in addition to the strong errors of the two methods, two reference functions $e_1(\Delta t) = \Delta t$ and $e_2(\Delta t) = \Delta t^{\frac{1}{2}}$ have been plotted so that the reader can easily see the similarities in the trends of the functions.

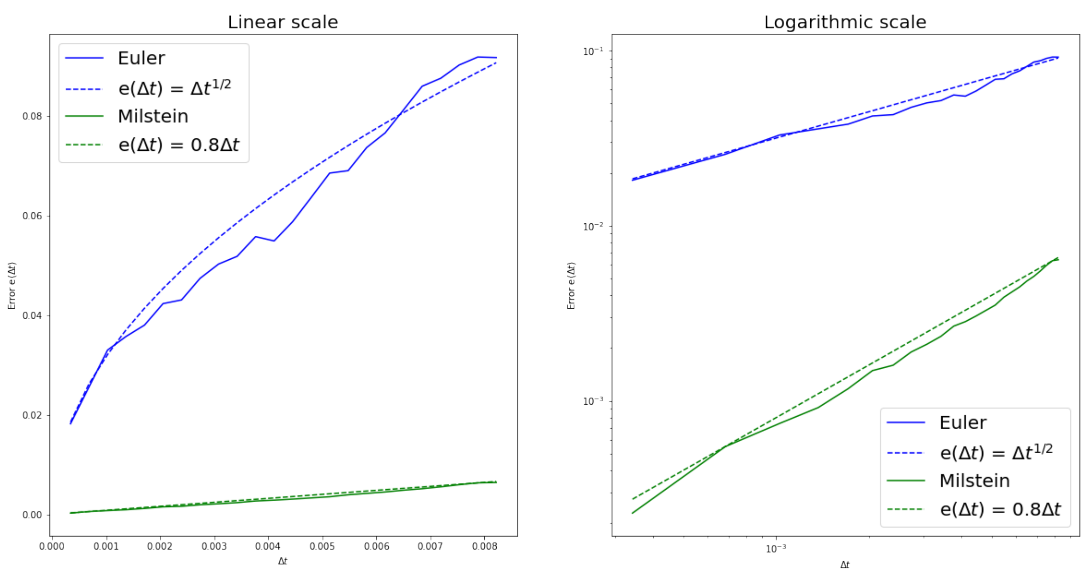

Figure 16: Error convergence in the strong sense of the Euler and Milstein schemes for the stock price using 100 samples

It is clear from Figure 16 that the Euler scheme is indeed $\mathcal{O}(\Delta t^{\frac{1}{2}})$ in the strong sense, as its error roughly follows the trend of the function $e_1(\Delta t) = \Delta t^{\frac{1}{2}}$; in a similar fashion, it can be concluded that the order of convergence of the Milstein scheme is $\mathcal{O}(\Delta t)$ in the strong sense as the trend of its error follows closely that of the function $e_2(\Delta t) = \Delta t^{\frac{1}{2}}$.

1.4

Verify experimentally the weak order of convergence for the Black-Scholes model and discuss the results.

The weak order of convergence of a solution is j if there exists a positive constant K and a positive constant Δ such that for fixed $T = N\Delta t$, the error of expectations is minimized by the formula below:

$$|E\{h(X_T, T)\} - E\{h(X_N, N)\}| <= K(\Delta t)^j \tag{12}$$

for all $0 < \Delta t < \Delta$ and for all functions h with polynomial growth, where N denotes the total number of time steps, T denotes the time interval and Δt describes the duration of each time step.
The strong order of convergence is calculated from the mean of all errors, whereas the weak error term simply

computes the error between the expected values of the two exact and the approximate solutions at a given point. So basically the weak convergence captures the average behavior of the various samples.
Convergence in the strong sense follows the approach of approximating the exact track S_t as accurately as possible by a numerical track S_n, so it's required if one is interested in the closeness of each sample to the exact solution. When this is not necessary and one is more interested in the expected value of the SDE, the weak order is preferred.

Figure 17 plots the weak error terms for both the approximations against the various Δt values.
From our simulations we see that both the Euler and Milstein schemes converge in the weak sense at $\mathcal{O}(\Delta t)$. This can be demonstrated by the trends followed by their errors. For the Euler scheme the error approximately scales with $2.1\Delta t$ and for the Milstein scheme the error approximately scales with $0.25\Delta t$. So weak convergence captures the average behavior of the simulated approximations.

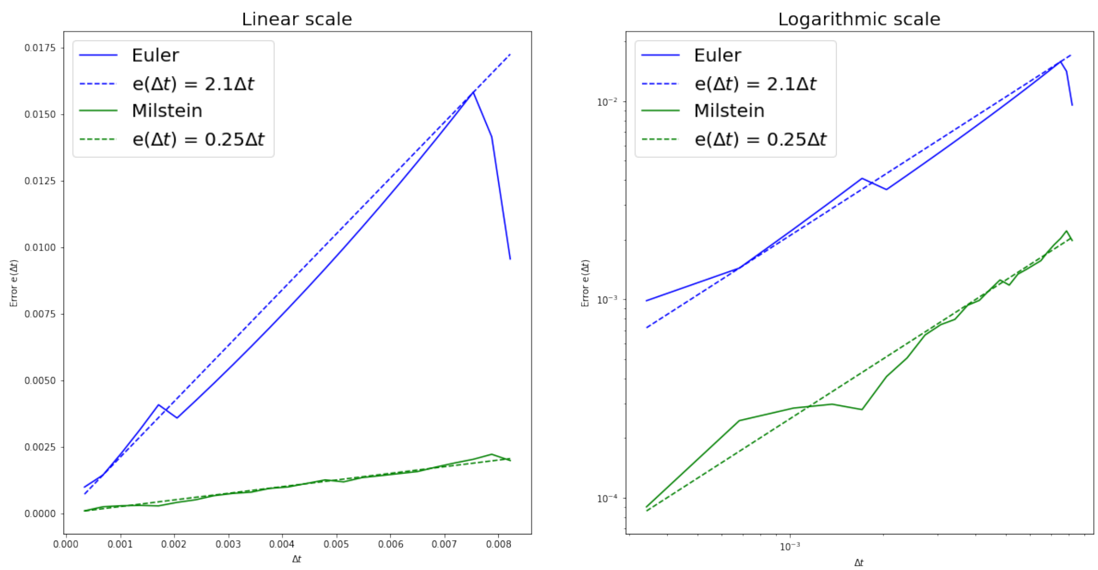

Figure 17: Error convergence in the weak sense of the Euler and Milstein schemes for the stock price using 100 samples

1.5

Repeat the experiments in section 1.3 and section 1.4 for the full model in eq. (1) and discuss the results.

The goal of this section was to repeat the convergence tests for the error both in the strong and weak sense for the full model in eq. (1). The main challenge of this compared to the Black-Scholes model was that this model doesn't have a known analytical solution. Therefore in the computation of the error in both eq. (10) and eq. (12), X_N cannot be taken equal to an exact solution, as it doesn't exist. Instead, our approach was the following: given a certain Δt_1, the computation of the error terms of the Euler approximation for that time step was performed using as X_N the Euler numerical approximation obtained using a twice as refined time step, namely $\Delta t_2 = \frac{\Delta t_1}{2}$.The approach can be intuitively explained stating that the error terms were computed taking the difference of the numerical approximation on a coarse time grid and the numerical approximation on a fine time grid; the same approach was used for the computation of the error terms of the Milstein method.

After implementing the above described computation of the error terms, we plot the error convergence in the strong and weak sense for both the approximations against the various Δt values.
As stated in the previous two sections, from the theory the Euler method was expected to be $\mathcal{O}(\Delta t^{\frac{1}{2}})$ in the strong sense and $\mathcal{O}(\Delta t)$ in the weak sense, while the Milstein method to be $\mathcal{O}(\Delta t)$ for both the strong and weak sense. This is the first way to assess whether our implementation was providing correct results. Moreover, note that if the constant p in the full model is taken equal to zero, eq. (1) would just reduce to the Black-Scholes model of eq. (2).

Therefore the second and more accurate way to assess the correctness of the implementation was to plot the strong and weak error convergence of the full model using $p = 0$, and study whether the results were the same as those shown in Figure 16 and Figure 17. This was done in Figure 18 and Figure 19.

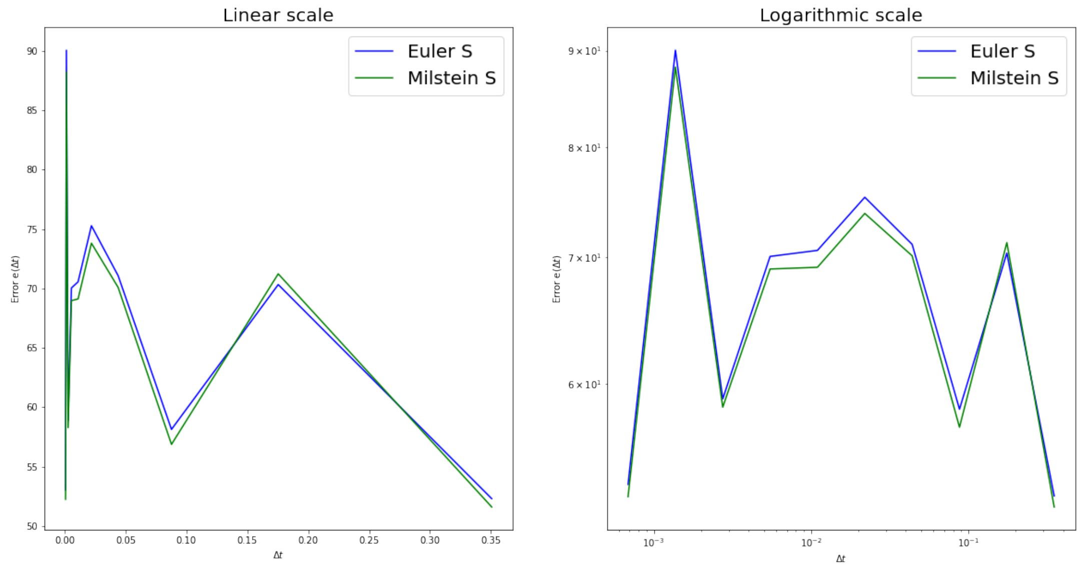

Figure 18: Error convergence in the strong sense of the Euler and Milstein schemes for the general model using 100 samples

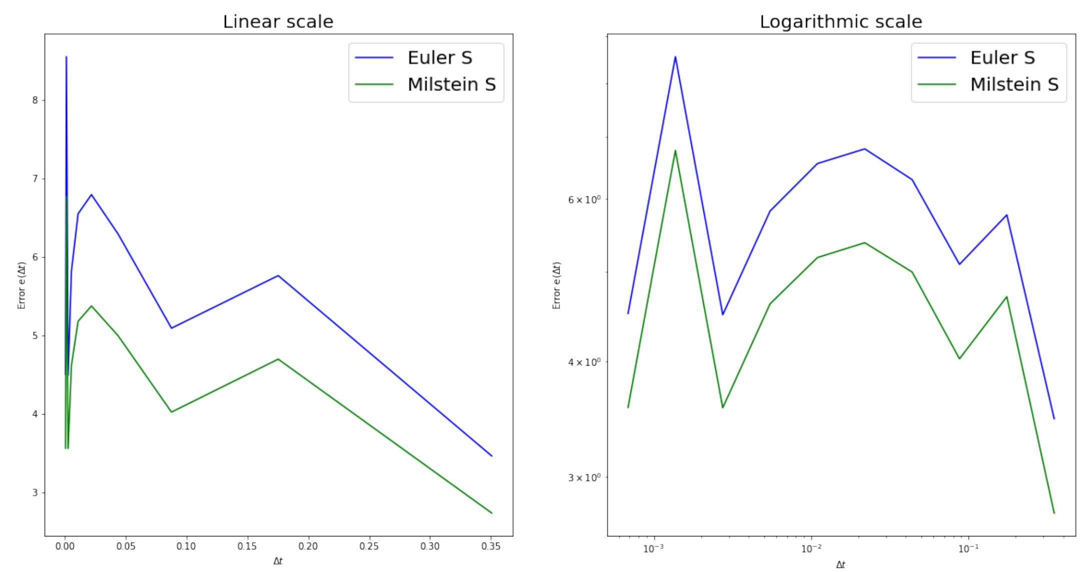

Figure 19: Error convergence in the weak sense of the Euler and Milstein schemes for the general model using 100 samples

The results shown in Figure 18 and Figure 19 greatly differ from those in Figure 16 and Figure 17; the magnitude of both the errors is way bigger and the trend of the functions is nothing close to Δt (or $\Delta t^{\frac{1}{2}}$ for the strong error of the Euler method). On the contrary, the errors get smaller as the time step size used is bigger; this is counter-intuitive and clearly indicates something is wrong with the implementation of the computation of the error terms.

Unfortunately, after several hours of inspection, we were unable to detect and fix the bug in our code which led to such unexpected results. However some speculations were made about what might have gone wrong. One hypothesis was that the definition of X_N was incorrect, and instead of what explained above, X_N should have been fixed for every different time step size, and taken equal to the numerical approximation (let it be Euler or Milstein) obtained using a fixed very small Δt that was considered then an "apparent" exact solution.