GRIFFITHS ELECTRODYNAMICS SOLUTIONS

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Problem 2.1
a) Twelve equal charges, q are situated at the corners of a regular 12-sided polygon (for instance, one on each numeral of a clock face.) What is the net force on a test charge Q at the center?
The force on Q must be 0 by symmetry. The x, y, and z components of the fields due to each charge cancel, so the force vanishes.
b) Suppose one of the 12 q's is removed (the one at "6 o'clock"). What is the force on Q? Explain your reasoning carefully.
Now it is like 10 of the charges cancel, the only one that doesn't cancel being the one directly opposite the missing charge. Thus, the force is

$$F = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2} \quad \square \tag{1}$$

pointing in the direction of the missing charge. Here, r is the distance from the center to each charge.
c) Now 13 equal charges, q, are placed at the corners of a regular 13-sided polygon. What is the force on a test charge Q at the center.
Again, by symmetry, the answer is still 0.
d) If one of the 13 q's is removed, what is the force on Q? Explain your reasoning.
The same answer as in part b, equation 1, and for the same reason.

Problem 2.2
a) Find the electric field (magnitude and direction) a distance z above the midpoint between two equal charges, q, a distance d apart. Check that your result is consistent with what you'd expect when $z \gg d$.
The force would not have components in the y or x direction, if we put the charges in the $x-z$ plane, there is no y-component of the field at the point in question, and the x-components cancel. The z-component of the field is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2q}{z^2+(d/2)^2}sin(\theta)\hat{\mathbf{z}} = \frac{1}{4\pi\epsilon_0}\frac{2qz}{(z^2+(d/2)^2)^{3/2}}\hat{\mathbf{z}} \quad \square \tag{2}$$

where the factor of 2 comes from adding up the two charges. When $z \gg d$, the electric field becomes

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2q}{z^2}\hat{\mathbf{z}} \tag{3}$$

which is what you would expect, since the system acts like a point charge of charge $2q$.
b) Repeat part (a), only this time make the right hand charge $-q$ instead of $+q$.
Now the z-components cancel and the x-components don't. Mathematically, this means replacing $sin(\theta)$ with $cos(\theta)$.

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2q}{z^2+(d/2)^2}cos(\theta)\hat{\mathbf{z}} = \frac{1}{4\pi\epsilon_0}\frac{qd}{(z^2+(d/2)^2)^{3/2}}\hat{\mathbf{x}} \quad \square \tag{4}$$

where the 2 has canceled out with the $d/2$ that comes from the cosine. If $z \gg d$, then from far away it looks like a dipole, and the field reduces to that of a dipole:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{qd}{z^3}\hat{\mathbf{x}} \tag{5}$$

Problem 2.3
Find the electric field a distance z above one end of a straight line segment of length L, which carries a uniform line charge λ. Check that your formula is consistent with what you would expect for the case $z \gg L$.
The electric field is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_0^L \frac{dq}{r^2} cos(\theta)\hat{\mathbf{x}} + \frac{1}{4\pi\epsilon_0}\int_0^L \frac{dq}{r^2} sin(\theta)\hat{\mathbf{z}} \tag{6}$$

replacing dq by λdx, r^2 by x^2+z^2, $cos(\theta)$ by $x/\sqrt{x^2+z^2}$, and $sin(\theta)$ by $z/\sqrt{x^2+z^2}$, we get

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\lambda\int_0^L \frac{xdx}{(x^2+z^2)^{3/2}}\hat{\mathbf{x}} + \frac{1}{4\pi\epsilon_0}\lambda\int_0^L \frac{zdx}{(x^2+z^2)^{3/2}}\hat{\mathbf{z}} \tag{7}$$

The first integral is a u-substitution, while the second can be evaluated using your favorite computer program.

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\lambda\{(\frac{-1}{z} + \frac{1}{\sqrt{L^2+z^2}})\hat{\mathbf{x}} + \frac{L}{z\sqrt{L^2+z^2}}\hat{\mathbf{z}}\} \quad \square \tag{8}$$

In the limit $z \gg L$, we obtain

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{\lambda L}{z^2}\hat{\mathbf{z}} \tag{9}$$

which is what you would expect from a point charge $q = \lambda L$.

Problem 2.5
Find the electric field a distance z above the center of a circular loop of radius r, which carries a uniform line charge λ.

$$\mathbf{E} = \int \frac{1}{4\pi\epsilon_0}\frac{dq}{R^2} \tag{10}$$

Now, $dq = \lambda dl$, where $dl = rd\theta$, while $R = \sqrt{r^2+z^2}$ is the distance from the edge of the loop to the point z. Furthermore, clearly the horizontal components will cancel out, leaving only a $\hat{\mathbf{z}}$ component of the electric field, meaning we need to include a factor of $cos(\theta) = z/\sqrt{r^2+z^2}$. Thus,

$$\mathbf{E} = \int_0^L \frac{\lambda}{4\pi\epsilon_0}\frac{dl}{(r^2+z^2)} cos(\theta)\hat{\mathbf{z}} = \int_0^{2\pi} \frac{1}{4\pi\epsilon_0}\lambda\frac{zrd\theta}{(r^2+z^2)^{3/2}}\hat{\mathbf{z}} \tag{11}$$

meaning the answer is

$$\mathbf{E} = \frac{\lambda rz}{2\epsilon_0(r^2+z^2)^{3/2}}\hat{\mathbf{z}} \quad \square \tag{12}$$

Problem 2.6
Find the electric field a distance z above the center of a flat circular dis ok radius R, which carries a uniform surface charge σ. What does your formula give in the limit $R \to \infty$? Also, check the case $z \gg R$.
The method is the same as in problem 2.5, except that now $dq = \sigma da = \sigma r dr d\theta$. Thus, the integral is

$$\mathbf{E} = \frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^R \frac{rzdrd\theta}{(r^2+z^2)^{3/2}}\hat{\mathbf{z}} = \frac{z\sigma}{2\epsilon_0}\frac{-1}{(r^2+z^2)^{1/2}}\hat{\mathbf{z}}\Big|_0^R = \frac{z\sigma}{2\epsilon_0}(\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}})\hat{\mathbf{z}} \quad \square \tag{13}$$

In the limit $R \to \infty$, we recover the field due to an infinite conducting plane, $E_z = \sigma/2\epsilon_0$, while in the limit that $z \gg R$, by Taylor expanding the square root, we recover the field between two point charges: $E = Q/4\pi\epsilon_0 z^2$, where $Q = \pi R^2\sigma$.

Problem 2.9
Suppose the electric field in some region is found to be $\mathbf{E} = kr^3\hat{\mathbf{r}}$, in spherical coordinates (k is some constant).
a) Find the charge density ρ.
This part is simple.

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E} = \epsilon_0 \frac{1}{r^2}\frac{\partial(r^2 E_r)}{\partial r} = \epsilon_0 k \frac{1}{r^2}\frac{\partial(r^5)}{\partial r} = 5\epsilon_0 k r^2 \tag{14}$$

b) Find the total charge contained in a sphere of radius R, centered at the origin. (Do it in two different ways).
Method 1:

$$Q_{enc} = \int_V \rho dV = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^R 5\epsilon_0 k r^2 r^2 dr = 4\pi\epsilon_0 k R^5 \quad \square \tag{15}$$

$$Q_{enc} = \epsilon_0 \oint \mathbf{E} \cdot d\mathbf{s} = \epsilon_0 k R^3 4\pi R^2 = 4\pi\epsilon_0 k R^5 \quad \square \tag{16}$$

Problem 2.10
A charge q sits at the back corner of a cube. What is the flux of E through one side?
We need to surround the charge symmetrically, since otherwise it is unclear what fraction of the flux is going through each face. Naively, you might think that the flux through one face is 1/6 of the total flux, but this is an assumption that we cannot justify. For instance, no flux goes through the three faces adjacent to the charge, since they are parallel to the electric field. However, if make a larger cube, one centered on the charge, we can determine the flux through each face by symmetry. If we center the cube on the charge, then we are effectively adding 7 cubes of the same dimensions as the original cube, yielding 8 total cubes. Now, the charge is centered in the cube, with the original face that we were examining being 1/4 of one face of this cube. Since the cube has six sides, the original face composes 1/24 of the cube. The flux is thus $q/24\epsilon_0$. $\square$

Problem 2.21
Find the potential inside and outside a uniformly charged solid sphere whose radius is R and whose total charge is q. Use infinity as your reference point. Compute the gradient of V in each region, and check that it yields the correct result.
Inside the sphere, the electric field

$$\int E \cdot da = \frac{1}{\epsilon_0}\int \rho dV \quad \Rightarrow \quad E 4\pi r^2 = \frac{3q}{4\pi\epsilon_0 R^3}\int dV = \frac{3q}{4\pi\epsilon_0 R^3}\frac{4\pi r^3}{3} \quad \Rightarrow \quad \mathbf{E} = \frac{qr}{4\pi\epsilon_0 R^3}\hat{\mathbf{r}} \tag{17}$$

whereas outside the electric field is just

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \tag{18}$$

Thus,

$$V_{out} = \int_r^{\infty} \mathbf{E} \cdot \mathbf{dl} = \frac{1}{4\pi\epsilon_0}\int_r^{\infty} \frac{q}{r'^2}\hat{\mathbf{r}} \cdot dr'\hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \quad \square \tag{19}$$

$$V_{in} = \int_r^{\infty} \mathbf{E} \cdot \mathbf{dl} = \int_r^R \frac{qr}{4\pi\epsilon_0 R^3}\hat{\mathbf{r}} \cdot dr\hat{\mathbf{r}} + \frac{1}{4\pi\epsilon_0}\int_R^{\infty} \frac{q}{r^2}\hat{\mathbf{r}} \cdot dr\hat{\mathbf{r}} = \frac{q}{8\pi\epsilon_0 R} - \frac{qr^2}{8\pi\epsilon_0 R^3} + \frac{1}{4\pi\epsilon_0}\frac{q}{R} \tag{20}$$

this can be rewritten as

$$V_{in} = \frac{3q}{8\pi\epsilon_0 R} - \frac{qr^2}{8\pi\epsilon_0 R^3} = \frac{q}{8\pi\epsilon_0 R}(3 - \frac{r^2}{R^2}) \quad \square \tag{21}$$

Taking the gradient of equations 19 and 21 trivially yields in equation 18 and 17, respectively.

Problem 2.22

Find the potential a distance s from an infinitely long straight wire that carries a uniform line charge λ. Compute the gradient of your potential, and check that it yields the correct field.

Let's calculate the electric field for this wire:

$$\int E \cdot da = E2\pi sL = \frac{q}{\epsilon_0} = \frac{\lambda L}{\epsilon_0} \quad \Rightarrow \quad \mathbf{E} = \frac{\lambda}{2\pi s\epsilon_0}\hat{\mathbf{s}} \tag{22}$$

Now we integrate this over the path $ds\ \hat{\mathbf{s}}$ to find the potential. We don't integrate from s to ∞, since the integral will blow up. Instead we integrate from s to some point a.

$$V = \int_s^a E \cdot dl = \int_s^a \frac{\lambda}{2\pi s\epsilon_0} ds = \frac{\lambda}{2\pi\epsilon_0} ln\frac{a}{s} \quad \square \tag{23}$$

Taking the gradient of -V trivially reproduces equation 22.

Problem 2.31

a) Three charges are situated at the corners of a square (side a), as shown in Fig. 2.41. How much work does it take to bring in another charge $+q$, from far away and place it in the fourth corner?

The potential at the corner is

$$V = V_1 + V_2 + V_3 = \frac{1}{4\pi\epsilon_0}\frac{-q}{a} + \frac{1}{4\pi\epsilon_0}\frac{-q}{a} + \frac{1}{4\pi\epsilon_0}\frac{q}{a\sqrt{2}} = \frac{1}{4\pi\epsilon_0}\frac{q}{a}(-2 + \frac{1}{\sqrt{2}}) \tag{24}$$

So then the work $W = qV$ is, since the potential is zero at infinity,

$$W = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a}(-2 + \frac{1}{\sqrt{2}}) \quad \square \tag{25}$$

b) How much work does it take to assemble the whole configuration of four charges?

It takes no work to bring in the first charge. The second charge requires (supposing it is the negative charge) work $W = -qV = -q^2/4\pi\epsilon_0 a$, since, once again, the potential is zero at infinity. The third charge requires

$$W = \frac{1}{4\pi\epsilon_0}\frac{q(-q)}{a} + \frac{1}{4\pi\epsilon_0}\frac{-q(-q)}{a\sqrt{2}} = \frac{q^2}{4\pi\epsilon_0 a}(\frac{1}{\sqrt{2}} - 1) \tag{26}$$

and we have already calculated the work required for the fourth charge. Therefore, the work for the total configuration is:

$$W = 0 + \frac{1}{4\pi\epsilon_0}\frac{-q^2}{a} + \frac{q^2}{4\pi\epsilon_0 a}(\frac{1}{\sqrt{2}} - 1) + \frac{1}{4\pi\epsilon_0}\frac{q^2}{a}(-2 + \frac{1}{\sqrt{2}}) = \frac{q^2}{2\sqrt{2}\pi\epsilon_0 a} - \frac{q^2}{\pi\epsilon_0 a} \quad \square \tag{27}$$

Problem 2.35

A metal sphere of radius R, carrying charge q, is surrounded by a thick concentric metal shell (inner radius a, outer radius b, as in Fig. 2.48). The shell carries no net charge.

a) Find the surface charge density σ at R, at a, and at b.

All the charge has gone to the outer radius of the metal sphere, since it is a conductor, so at R, the surface charge density is $\sigma_R = q/A = q/4\pi R^2$. At a, there will be induced a charge -q, so the surface density is $\sigma_a = -q/4\pi a^2$. In order to keep the shell neutral, the remaining charge $+q$ will go to the outer edge of the shell, causing a surface charge density $\sigma_b = q/4\pi b^2$. $\square$

b) Find the potential at the center, using infinity as the reference point.

We'll use $V = \int_0^\infty E \cdot dl$.

$$V = \int_0^\infty E \cdot dl = \int_0^R (0)dr + \int_R^a \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr + \int_a^b (0)dr + \int_b^\infty \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr \tag{28}$$

where the (0) corresponds to regions of $E = 0$. Thus,

$$V = \frac{q}{4\pi\epsilon_0}(\frac{1}{b} + \frac{1}{R} - \frac{1}{a}) \quad \square \tag{29}$$

c) Now the outer surface is touched to a grounding wire, which lowers its potential to zero (same as at infinity). How do your answers to (a) and (b) change?
If the outer surface is grounded, the charge on the outer surface will go to ground, since it will go from a region of high potential to region of low potential. Thus, $\sigma_b = 0$. As a result, the potential at the center will have no contribution from the outer part of the shell.

$$V = \int_0^\infty E \cdot dl = \int_0^R (0)dr + \int_R^a \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr + \int_a^b (0)dr + \int_b^\infty (0)dr = \frac{q}{4\pi\epsilon_0}(\frac{1}{R} - \frac{1}{a}) \quad \square \tag{30}$$

Problem 2.36
Two spherical cavities, of radii a and b, are hollowed out from the interior of a (neutral) conducting sphere of radius R (Fig. 2.49). At the center of each cavity a point charge is placed – call these charges q_a and q_b.
a) Find the surface charges σ_a, σ_b, and σ_R.
The charge at the middle of each cavity will induce an equal and opposite charge density on the cavity surface. Let the radii of the cavities be a and b, respectively. Then

$$\sigma_a = -q_a/4\pi a^2, \quad \sigma_b = -q_b/4\pi b^2 \tag{31}$$

Now, since the sphere is overall neutral, the remaining charge must be on the outer surface R. Therefore,

$$\sigma_R = (q_a + q_b)/4\pi R^2 \tag{32}$$

b) What is the field outside the conductor?
The field outside a conductor is always $\mathbf{E} = \sigma/\epsilon_0\hat{\mathbf{n}}$. Therefore, outside of R:

$$\mathbf{E} = \frac{q_a + q_b}{4\pi\epsilon_0 R^2}\hat{\mathbf{r}} \tag{33}$$

Note that this is consistent with Gauss's Law: $\int E \cdot da = Q_{enc}/\epsilon_0$.
c) What is the field within each cavity?
Once again, by Gauss's law (or due to the boundary conditions, whichever you prefer),

$$\mathbf{E}_a = \frac{1}{4\pi\epsilon_0}\frac{q_a}{r^2}\hat{\mathbf{r}}_a \quad \forall\, r < a \tag{34}$$

and

$$\mathbf{E}_b = \frac{1}{4\pi\epsilon_0}\frac{q_b}{r^2}\hat{\mathbf{r}}_b \quad \forall\, r < b \tag{35}$$

Note that these unit vectors are, in general, different from each other and the unit vector $\hat{\mathbf{r}}$ in equation 33.
d) What is the force on q_a and q_b?
The electric field is due to the charges themselves, and the field due to charge a does not influence charge b, and vice versa. Therefore the force on each charge is 0. $\square$
e) Which of these answers would change if a third charge, q_c, were brought near the conductor?
σ_R would be different, since the new charge would induce some more charge on the outer surface. Furthermore, it would affect E_R, since the new charge contributes its own electric field. Note that it would be difficult to apply Gauss's law here, since we don't have the required symmetry.

Problem 3.18
The potential at the surface of a sphere (radius R) is given by $V_0 = kcos3\theta$, where k

is a constant. Find the potential inside and outside the sphere, as well as the surface charge density $\sigma(\theta)$ on the sphere. (Assume there's no charge inside or outside the sphere.)
This potential can be decomposed into a linear combination of Legendre polynomials:

$$V_0 = kcos3\theta = k[4cos^3\theta - 3cos\theta] = k[xP_3(cos\theta) + yP_1(cos\theta)] \tag{36}$$

I can tell immediately that it is some combination of P_3 and P_1 since the function is odd. So now I have the equality (if I replace P_3 and P_1 with the explicit forms of the polynomials):

$$4cos^3\theta - 3cos\theta = x\frac{5cos^3\theta - 3cos\theta}{2} + ycos(\theta) = \frac{5x}{2}cos^3\theta + \frac{2y-3x}{2}cos\theta \tag{37}$$

and therefore we have

$$\frac{5x}{2} = 4 \quad \frac{2y-3x}{2} = -3 \quad \Rightarrow \quad x = \frac{8}{5}, \quad y = \frac{-3}{5} \tag{38}$$

So our potential, written in terms of the Legendre polynomials, is

$$V_0 = \frac{k}{5}[8P_3(cos\theta) - 3P_1(cos\theta)] \tag{39}$$

Now the general solution to Laplace's equation (in spherical coordinates) is

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(cos\theta) + \frac{B_l}{r^{l+1}}P_l(cos\theta) \tag{40}$$

Inside the sphere, the B_l's must be 0, or we will have problems at the origin, while outside the sphere, the A_l's must be 0, since otherwise we will have problems at infinity:

$$V_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(cos\theta), \quad V_{out} = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}}P_l(cos\theta) \tag{41}$$

Clearly the only l's that contribute are $l = 1$ and $l = 3$:

$$V_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(cos\theta) = \frac{k}{5}[8P_3(cos\theta) - 3P_1(cos\theta)] \quad \Rightarrow \quad A_0 = 0, A_2 = 0, A_l = 0 \ \forall \ l > 3 \tag{42}$$

and so, at the surface of the sphere,

$$V_{in} = \frac{k}{5}[8P_3(cos\theta) - 3P_1(cos\theta)] = A_1 R P_1(cos\theta) + A_3 R^3 P_3(cos\theta) \ \Rightarrow \ A_1 = \frac{-3k}{5R}, \ A_3 = \frac{8k}{5R^3} \tag{43}$$

while equating this with outside the surface yields ($B_{l\neq 1,3} = 0$):

$$V_{out} = \frac{k}{5}[8P_3(cos\theta) - 3P_1(cos\theta)] = \frac{B_1}{R^2}P_1(cos\theta) + \frac{B_3}{R^4}P_3(cos\theta) \ \Rightarrow \ B_1 = \frac{-3kR^2}{5}, \ B_3 = \frac{8kR^4}{5} \tag{44}$$

Thus, outside, we have, (plugging our A_1, A_3, B_1, B_3 into the general solution, equation 40):

$$V_{in} = \frac{-3k}{5R} r P_1(cos\theta) + \frac{8k}{5R^3} r^3 P_3(cos\theta) \quad \square \tag{45}$$

$$V_{out} = \frac{-3kR^2}{5}\frac{1}{r^2}P_1(cos\theta) + \frac{8kR^4}{5}\frac{1}{r^4}P_3(cos\theta) \quad \square \tag{46}$$

To obtain σ, we use the fact that the discontinuity in the electric field at the boundary is equal to the surface charge density:

$$\frac{\sigma}{\epsilon_0} = -\left(\frac{\partial V}{\partial n}\Big|_{out} - \frac{\partial V}{\partial n}\Big|_{in}\right) = \frac{\partial V_{in}}{\partial r}\Big|_R - \frac{\partial V_{out}}{\partial R}\Big|_R \tag{47}$$

Cranking out the derivatives:

$$\frac{\partial V_{in}}{\partial r} = \frac{-3k}{5R}P_1(cos\theta) + \frac{24kr^2}{5R^3}P_3(cos\theta)\Big|_R = \frac{-3k}{5R}P_1(cos\theta) + \frac{24k}{5R}P_3(cos\theta) \quad (48)$$

$$\frac{\partial V_{out}}{\partial r} = \frac{6kR^2}{5r^3}P_1(cos\theta) - \frac{32kR^4}{5r^5}P_3(cos\theta)\Big|_R = \frac{6k}{5R}P_1(cos\theta) - \frac{32k}{5R}P_3(cos\theta) \quad (49)$$

Subtracting one from the other yields:

$$\sigma = \epsilon_0\frac{-9k}{5R}P_1(cos\theta) + \epsilon_0\frac{56k}{5R}P_3(cos\theta) \quad \square \quad (50)$$

Problem 3.25

Charge density $\sigma(\phi) = asin5\phi$ (where a is a constant) is glued over the surface of an infinite cylinder of radius R. Find the potential inside and outside the cylinder. Use the general solution to Laplace's equation with cylindrical symmetry:

$$V(s,\phi) = a_0 + b_0 ln(s) + \sum_{k=1}^{\infty}[s^k(a_k cos(k\phi) + b_k sin(k\phi)) + s^{-k}(c_k cos(k\phi) + d_k sin(k\phi)] \quad (51)$$

Inside the cylinder, we must have $b_0 = c_k = d_k = 0$, since otherwise the potential blows up at $s = 0$. Outside we require $b_0 = a_k = b_k = 0$, since otherwise the potential blows up at infinity. So,

$$V_{in}(s,\phi) = a_0 + \sum_{k=1}^{\infty} s^k(a_k cos(k\phi) + b_k sin(k\phi)) \quad (52)$$

$$V_{out}(s,\phi) = A_0 + \sum_{k=1}^{\infty} s^{-k}(c_k cos(k\phi) + d_k sin(k\phi)) \quad (53)$$

Using the definition

$$\frac{\sigma}{\epsilon_0} = -(\frac{\partial V}{\partial n}\Big|_{out} - \frac{\partial V}{\partial n}\Big|_{in}) = \frac{\partial V_{in}}{\partial s} - \frac{\partial V_{out}}{\partial s}\Big|_{s=R} \quad (54)$$

where σ is given by $\sigma(\phi) = asin5\phi$. Taking the derivatives of equations 52 and 53 and subtracting yields

$$\frac{asin5\phi}{\epsilon_0} = \sum_{k=1}^{\infty} kR^{k-1}(a_k cos(k\phi) + b_k sin(k\phi)) + kR^{-k-1}(c_k cos(k\phi) + d_k sin(k\phi)) \quad (55)$$

Clearly, $a_k = c_k = 0$, unless $k = 5$ to match the left hand side.

$$\frac{asin5\phi}{\epsilon_0} = 5R^4 b_5 sin(5\phi) + 5R^{-6} d_5 sin(5\phi) \quad (56)$$

In order to calculate b_5 and d_5, we need a second equation. The second equation is the continuity of the potential inside and outside the cylinder $V_{in}|_R = V_{out}|_R$:

$$a_0 + R^5 b_5 sin(5\phi) = A_0 + R^{-5} d_5 sin(5\phi) \quad \Rightarrow \quad a_0 = A_0, \quad b_5 = \frac{d_5}{R^{10}} \quad (57)$$

We can choose $a_0 = A_0 = 0$, and plugging this into equation 56 to solve for d_5

$$\frac{asin5\phi}{\epsilon_0} = 5R^4\frac{d_5}{R^{10}}sin(5\phi) + 5\frac{d_5}{R^6}sin(5\phi) \quad \Rightarrow \quad d_5 = \frac{aR^6}{10\epsilon_0} \quad \Rightarrow \quad b_5 = \frac{a}{10R^4\epsilon_0} \quad (58)$$

Now we have everything we need to plug in to the potentials inside (equation 52) and outside (equation 53). If we note that all the coefficients vanish except b_5 and d_5:

$$V_{in} = s^5\frac{a}{10R^4\epsilon_0}sin(5\phi) \quad \square \quad (59)$$

$$V_{out} = s^{-5}\frac{aR^6}{10\epsilon_0}sin(5\phi) \quad \square \tag{60}$$

Problem 3.26
A sphere of radius R, centered at the origin, carries charge density

$$\rho(r,\theta) = k\frac{R}{r^2}(R-2r)sin(\theta) \tag{61}$$

where k is a constant, and r, θ are the usual spherical coordinates. Find the approximate potential for points on the z axis, far from the sphere.
For reference, the exact form of the multipole expansion potential is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int r^n P_n(cos\theta)\rho(\mathbf{r})dV \tag{62}$$

The monopole term in the potential is

$$V_{mon}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \tag{63}$$

We need to find Q.

$$Q = \int \rho dV = 2\pi\int_0^\pi\int_0^R k\frac{R}{r^2}(R-2r)sin(\theta)r^2 sin(\theta)drd\theta = k\pi^2\int_0^R(R^2-2r)dr = 0 \tag{64}$$

Thus the total charge on the sphere is 0, so there is no monopole term, $V_{mon} = 0$.
The dipole term is

$$V_{dip}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int rcos(\theta)\rho dV = \frac{1}{4\pi\epsilon_0}\frac{2\pi}{r^2}\int_0^\pi\int_0^R rcos(\theta)k\frac{R}{r^2}(R-2r)sin(\theta)r^2 sin(\theta)drd\theta \tag{65}$$

and we can see immediately that the θ integral vanishes, so the dipole term is also zero, $V_{dip} = 0$.
The quadruple term won't be zero:

$$V_{quad} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int r^2\frac{3cos^2\theta-1}{2}\rho dV = \frac{2\pi kR}{8\pi\epsilon_0 r^3}\int_0^\pi\int_0^R r^2(3cos^2\theta-1)\frac{1}{r^2}(R-2r)sin(\theta)r^2 drd\theta \tag{66}$$

This is not a difficult integral, and it equals $k\pi^2R^5/48$. Thus, the potential at a point on the z-axis is

$$V(z) = \frac{1}{4\pi\epsilon_0}\frac{k\pi^2R^5}{48z^3} \quad \square \tag{67}$$

where I have let the original $r \to z$.

Problem 3.27
Four particles are placed as follows: charge q at $(0,0,-a)$, charges $-2q$ at $(0,a,0)$ and $(0,-a,0)$, and charge $3q$ at $(0,0,a)$. Find a simple approximate formula for the potential valid at points far from the origin. Express your answer in spherical coordinates.
To first order, (the monopole term) the potential vanishes, since the total charge is zero. Let's examine the dipole term:

$$V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \tag{68}$$

But $\mathbf{p} = \sum q_i d_i$, so the dipole moment is

$$\mathbf{p} = (3q)(a)\hat{\mathbf{z}} + (q)(-a)\hat{\mathbf{z}} + (-2q)(-a)\hat{\mathbf{y}} + (-2q)(a)\hat{\mathbf{y}} + 0\hat{\mathbf{x}} = 2qa\hat{\mathbf{z}} \tag{69}$$

So the dipole potential is

$$V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{2qa\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{2qacos\theta}{r^2} \quad \square \tag{70}$$

Problem 3.30

Two point charges, $3q$ and $-q$, are separated by a distance a. For each of the arrangements below, find (i) the monopole moment, (ii) the dipole moment, and (iii) the approximate potential (in spherical coordinates) at large r (include both the monopole and dipole contributions).

a) $3q$ at $(0, 0, a)$, $-q$ at origin.

The monopole moment is just the charge, $Q = 3q + (-q) = 2q$. The dipole moment is $\mathbf{p} = \sum q_i \mathbf{d}_i$

$$\mathbf{p} = (-q)(0) + (3q)(a)\hat{\mathbf{z}} = 3qa\hat{\mathbf{z}} \quad \Box \tag{71}$$

The approximate potential is the sum of the monopole and dipole terms:

$$V = \frac{1}{4\pi\epsilon_0}(\frac{Q}{r} + \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}) = \frac{1}{4\pi\epsilon_0}(\frac{2q}{r} + \frac{3qacos\theta}{r^2}) \quad \Box \tag{72}$$

where I have replaced $\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}$ with $cos\theta$.

b) $3q$ at origin, $-q$ at $(0, 0, -a)$.

The monopole moment is that total charge, which is again $2q$. The dipole moment is

$$\mathbf{p} = (3q)(0) + (-q)(-a)\hat{\mathbf{z}} = qa\hat{\mathbf{z}} \quad \Box \tag{73}$$

The approximate potential is

$$V = \frac{1}{4\pi\epsilon_0}(\frac{Q}{r} + \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}) = \frac{1}{4\pi\epsilon_0}(\frac{2q}{r} + \frac{qacos\theta}{r^2}) \quad \Box \tag{74}$$

c) $3q$ at $(0, a, 0)$, $-q$ at origin.

The monopole moment is the total charge $2q$, the dipole moment is

$$\mathbf{p} = (3q)(a) + (-q)(0)\hat{\mathbf{y}} = 3qa\hat{\mathbf{y}} \quad \Box \tag{75}$$

meaning that the potent is

$$V = \frac{1}{4\pi\epsilon_0}(\frac{Q}{r} + \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}) = \frac{1}{4\pi\epsilon_0}(\frac{2q}{r} + \frac{3qasin\theta sin\phi}{r^2}) \quad \Box \tag{76}$$

since $\hat{\mathbf{y}}\cdot\hat{\mathbf{r}} = sin\theta sin\phi$.

Problem 3.31

A "pure" dipole p is situated at the origin, pointing in the z direction.

a) What is the force on a point charge q at $(a,\ 0,\ 0)$ (Cartesian coordinates)?

The potential on the x-axis due to a dipole is

$$V = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{pcos\theta}{r^2} \tag{77}$$

So the force is

$$\mathbf{F} = -q\nabla V = -q\frac{\partial V}{\partial r}\hat{\mathbf{r}} - \frac{q}{r}\frac{\partial V}{\partial\theta}\hat{\theta} = \frac{q}{4\pi\epsilon_0}\frac{2pcos\theta}{r^3}\hat{\mathbf{r}} + \frac{1}{4\pi\epsilon_0}\frac{qpsin\theta}{r^3}\hat{\theta}\Big|_{r=a,\theta=\pi/2} = -\frac{pq}{4\pi\epsilon_0 a^3}\hat{\mathbf{z}} \tag{78}$$

since for $\theta = \pi/2$, the unit vector $\hat{\theta}$ is in the -z direction. By the way, $\theta = \pi/2$ since the point we are looking at is on the x axis, while the dipole moment is in the z direction.

b) What is the force on q at $(0,\ 0,\ a)$?

This time, $\theta = 0$, so

$$\mathbf{F} = \frac{pq}{2\pi\epsilon_0 a^3}\hat{\mathbf{z}} \tag{79}$$

Since the $sin\theta$ kills the second term in equation 78 above, and the $\hat{\mathbf{r}}$ direction is in $\hat{\mathbf{z}}$.

c) How much work does it take to move q from $(a,\ 0,\ 0)$ to $(0,\ 0,\ a)$?

This is just the difference between the answers to parts a and b, multiplied by q, since $W = qV$. Thus,

$$W = \frac{pq}{4\pi\epsilon_0 a^2} \tag{80}$$

Problem 3.32

Three point charges are located as shown in Fig. 3.38, each a distance a from the origin. Find the approximate electric field at points far from the origin. Express your answer in spherical coordinates, and include the two lowest orders in the multipole expansion.

The first order term in the potential is $V = -q/4\pi\epsilon_0 r^2$, since the net charge is -q. The dipole term is

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\int \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \tag{81}$$

where the dipole moment is

$$\mathbf{p} = \int \rho(\mathbf{r})\mathbf{r}dV = \int -q\delta(y+a) + q\delta(z-a) - q\delta(y-a)(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})dxdydz = \tag{82}$$

multiplying this out:

$$\mathbf{p} = \int q\delta(z-a)zdz\hat{\mathbf{z}} - q(\delta(y-a) + \delta(y+a))ydy\hat{\mathbf{y}} = qa\hat{\mathbf{z}} - qa\hat{\mathbf{y}} - q(-a)\hat{\mathbf{y}} = qa\hat{\mathbf{z}} \tag{83}$$

Thus, the dipole potential is

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{qa\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{qa}{r^2}cos\theta \tag{84}$$

Thus the total potential, monopole and dipole terms included, is:

$$V = -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2} + \frac{1}{4\pi\epsilon_0}\frac{qa}{r^2}cos\theta \tag{85}$$

The electric field, therefore, is

$$\mathbf{E} = -\nabla V = \frac{q}{4\pi\epsilon_0}[\frac{-1}{r^2}\hat{\mathbf{r}} + \frac{a}{r^3}(2cos\theta\hat{\mathbf{r}} + sin\theta\hat{\theta})] \quad \square \tag{86}$$

Problem 4.20

A sphere of linear dielectric material has embedded in it a uniform free charge density ρ. Find the potential at the center of the sphere (relative to infinity), if its radius is R and its dielectric constant is ϵ_r.

We need to find the electric field everywhere first. Inside the sphere,

$$\int D\cdot da = D4\pi r^2 = Q_f = \int \rho dV = \rho\frac{4}{3}\pi r^3 \quad \Rightarrow \quad \mathbf{D}_{in} = \frac{1}{3}\rho r\hat{\mathbf{r}} \tag{87}$$

while outside the sphere,

$$\int D\cdot da = D4\pi r^2 = Q_f = \int \rho dV = \rho\frac{4}{3}\pi R^3 \quad \Rightarrow \quad \mathbf{D}_{out} = \frac{1}{3r^2}\rho R^3\hat{\mathbf{r}} \tag{88}$$

Therefore, inside the sphere,

$$\mathbf{E}_{in} = \frac{\mathbf{D}}{\epsilon} = \frac{\mathbf{D}}{\epsilon_0\epsilon_r} = \frac{\rho r}{3\epsilon_0\epsilon_r}\hat{\mathbf{r}} \tag{89}$$

Outside the sphere, the electric field is

$$\mathbf{E}_{out} = \frac{\mathbf{D}}{\epsilon} = \frac{\mathbf{D}}{\epsilon_0} = \frac{\rho R^3}{3\epsilon_0 r^2}\hat{\mathbf{r}} \tag{90}$$

From which we can find the potential:

$$V = \int_0^\infty \mathbf{E}\cdot d\mathbf{l} = \int_0^R \mathbf{E}_{in}\cdot d\mathbf{l} + \int_R^\infty \mathbf{E}_{out}\cdot d\mathbf{l} = \int_0^R \frac{\rho r}{3\epsilon_0\epsilon_r}dr + \int_R^\infty \frac{\rho R^3}{3\epsilon_0 r^2}dr \tag{91}$$

$$= \frac{\rho R^2}{6\epsilon_0\epsilon_r} + \frac{\rho R^2}{3\epsilon_0} = \frac{\rho R^2}{3\epsilon_0}(1+\frac{1}{2\epsilon_r}) \quad \square \tag{92}$$

Problem 4.23

Find the field inside a sphere of linear dielectric material in an otherwise uniform electric field $\mathbf{E}_0$ by the following method of successive approximations: First pretend the field inside is just $\mathbf{E}_0$, and use Eq. 4.30 to write down the resulting polarization $\mathbf{P}_0$. This polarization generates a field of its own, $\mathbf{E}_1$ (Ex. 4.2), which in turn modifies the polarization by an amount $\mathbf{P}_1$, which further changes the field by an amount $\mathbf{E}_2$, and so on. The resulting field is $\mathbf{E}_0 + \mathbf{E}_1 + \mathbf{E}_2 + ...$ Sum the series, and compare your answer with Eq. 4.49.

The polarization due to the field $\mathbf{E}_0$ is $\mathbf{P}_0 = \epsilon_0\chi_e\mathbf{E}_0$. The field resulting from this polarization is given by equation 4.18: $\mathbf{E} = -\mathbf{P}/3\epsilon_0$. Thus,

$$\mathbf{E}_1 = -\frac{\mathbf{P}_0}{3\epsilon_0} = -\frac{\epsilon_0\chi_e\mathbf{E}_0}{3\epsilon_0} = -\frac{\chi_e\mathbf{E}_0}{3} \tag{93}$$

Now the polarization due to this field is $\mathbf{P}_1 = \epsilon_0\chi_e\mathbf{E}_1$, which results in the field $\mathbf{E}_2 = -\mathbf{P}_1/3\epsilon_0$:

$$\mathbf{E}_2 = -\frac{\mathbf{P}_1}{3\epsilon_0} = -\frac{\epsilon_0\chi_e\mathbf{E}_1}{3\epsilon_0} = -\frac{\chi_e}{3}(-\frac{\chi_e\mathbf{E}_0}{3}) = \frac{\chi_e^2}{9}\mathbf{E}_0 \tag{94}$$

Thus we obtain an infinite geometric series

$$\mathbf{E}_{tot} = \mathbf{E}_0\sum_{n=0}^{\infty}(-\frac{\chi_e}{3})^n = \frac{\mathbf{E}_0}{1+\chi_e/3} \tag{95}$$

Using $\chi_e = \epsilon_r - 1$, we obtain:

$$\mathbf{E}_{tot} = \frac{\mathbf{E}_0}{2/3+\epsilon_r/3} = \frac{3}{\epsilon_r+2}\mathbf{E}_0 \quad \square \tag{96}$$

Problem 4.26

A spherical conductor, of radius a, carries a charge Q. It is surrounded by a linear dielectric material of susceptibility χ_e, out to radius b. Find the energy of this configuration.

We will use equation 4.58 in Griffiths:

$$W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}d\tau \tag{97}$$

It is evident that for $r < a$, both $\mathbf{D}$ and $\mathbf{E}$ are 0. Outside the conductor, using the fact that $\epsilon = \epsilon_0(1+\chi_e)$:

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}} \quad r > a \tag{98}$$

$$\mathbf{E} = \frac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}} \quad a < r < b \tag{99}$$

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \quad r > b \tag{100}$$

Thus, the energy is

$$W = \frac{1}{2}\int_a^b \frac{Q}{4\pi r^2}\hat{\mathbf{r}}\cdot\frac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}4\pi r^2 dr + \frac{1}{2}\int_b^\infty \frac{Q}{4\pi r^2}\hat{\mathbf{r}}\cdot\frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}4\pi r^2 dr \tag{101}$$

where the integral from 0 to a contributes no energy to the system. Thus,

$$W = \frac{Q^2}{8\pi\epsilon}\int_a^b \frac{dr}{r^2} + \frac{Q^2}{8\pi\epsilon_0}\int_b^\infty \frac{dr}{r^2} = \frac{Q^2}{8\pi\epsilon}(\frac{1}{a}+\frac{1}{b}) - \frac{Q^2}{8\pi\epsilon_0 b} \tag{102}$$

Or, writing the answer in terms of the susceptibility,

$$W = \frac{Q^2}{8\pi\epsilon_0(1+\chi_e)}(\frac{1}{a}+\frac{1}{b}) - \frac{Q^2}{8\pi\epsilon_0 b} \quad \square \tag{103}$$

Problem 4.31
A dielectric cube of side a, centered at the origin, carries a "frozen-in" polarization $\mathbf{P} = k\mathbf{r}$, where k is a constant. Find all the bound charges, and check that they add up to zero.
The polarization can be rewritten in Cartesian coordinates,

$$\mathbf{P} = k(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) \tag{104}$$

We also know that $-\nabla \cdot \mathbf{P} = \rho_b$ and $\mathbf{P}\cdot\hat{\mathbf{n}} = \sigma_b$. Let's find the bound volume charge first:

$$\rho_b = -\nabla\cdot\mathbf{P} = -k(1+1+1) = -3k \tag{105}$$

Meanwhile, the bound surface charge on *one side* is

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = k(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})\cdot\hat{\mathbf{z}}\Big|_{z=a/2} = \frac{ka}{2} \tag{106}$$

Thus, since there are six sides to a cube, the total bound surface charge is $\sigma_b = 3ka$. Now we need to find the total charge:

$$Q_{tot} = \int \rho_b dV + \int \sigma_b da = -3k(a^3) + 3ka(a^2) = 0 \quad \square \tag{107}$$

Problem 4.32
A point charge q is imbedded at the center of a sphere of linear dielectric material (with susceptibility χ_e and radius R). Find the electric field, the polarization, and the bound charge densities ρ_b and σ_b. What is the total bound charge on the surface? Where is the compensating negative bound charge located?
The displacement field $\mathbf{D}$ is easily found to be $\mathbf{D} = q/4\pi r^2\hat{\mathbf{r}}$. The electric field, written in terms of the susceptibility, defined such that $\epsilon = \epsilon_0(1+\chi_e)$, is

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0(1+\chi_e)r^2}\hat{\mathbf{r}} \tag{108}$$

The polarization is $\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$, so

$$\mathbf{P} = \frac{\chi_e q}{4\pi(1+\chi_e)r^2}\hat{\mathbf{r}} \tag{109}$$

The bound volume charge is

$$\rho_b = -\nabla\cdot\mathbf{P} = -\frac{\chi_e q}{4\pi(1+\chi_e)}\nabla\cdot\frac{\hat{\mathbf{r}}}{r^2} = -\frac{\chi_e q}{(1+\chi_e)}\delta^3(r) \tag{110}$$

where I have used the fact that $\nabla\cdot(\hat{\mathbf{r}}/r^2) = 4\pi\delta^3(r)$. Meanwhile the bound surface charge is

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}\Big|_R = \frac{\chi_e q}{4\pi(1+\chi_e)R^2} \tag{111}$$

Thus the total bound charge on the surface is

$$Q_{surface} = \int \sigma_b da = \frac{\chi_e q}{4\pi(1+\chi_e)R^2}4\pi R^2 = \frac{\chi_e}{1+\chi_e}q \tag{112}$$

The compensating negative charge is located at the origin (in the ρ_b term):

$$Q_{comp} = \int \rho_b dV = -\frac{\chi_e q}{(1+\chi_e)} \int \delta^3(r) = -\frac{\chi_e q}{(1+\chi_e)} \tag{113}$$

which exactly cancels the surface charge in equation 112.

Problem 5.4

Suppose that the magnetic field in some region has the form $\mathbf{B} = kz\hat{\mathbf{x}}$. (where k is a constant). Find the force on a square loop (side a), lying in the yz plane and centered at the origin, if it carries a current I, flowing counterclockwise, when you look down the x axis.

The force is $F = Idl \times B$. The force on the two horizontal segments in $\hat{\mathbf{z}}$ will cancel out, since the field flips sign.

$$\mathbf{F} = I \int \mathbf{dl} \times \mathbf{B} = I \int_0^a dz\hat{\mathbf{y}} \times kz\hat{\mathbf{x}} = \frac{1}{2} Ika^2 \hat{\mathbf{z}} \tag{114}$$

But that was only for one edge of the loop. The force on both edges will conspire, so the total force is

$$\mathbf{F} = Ika^2\hat{\mathbf{z}} \quad \square \tag{115}$$

Problem 5.5

A current I flows down a wire of radius a.

a) If it is uniformly distributed over the surface, what is the surface current density K?

$\mathbf{K} = d\mathbf{I}/dl_\perp$, so $I = \int K dl_\perp = K2\pi a$, since the length perpendicular to the flow is the circumference (not the surface area, which has a component parallel to the flow). Thus, $K = I/2\pi a \quad \square$

b) If it is distributed in such a way that the volume current density is inversely proportional to the distance from the axis, what is J?

We have that $J \propto 1/s$, or more precisely $J = \alpha/s$, with α some constant. Thus,

$$I = \int J dS = \alpha \int_0^a \frac{2\pi s}{s} ds = 2\pi a \alpha \quad \Rightarrow \quad \alpha = \frac{I}{2\pi a} \tag{116}$$

and therefore

$$J = \frac{I}{2\pi s a} \quad \square \tag{117}$$

Problem 5.9

Find the magnetic field at point P for each of the steady current configurations shown in Fig. 5.23

We will try to use the Biot Savart law:

a)

$$B = \frac{\mu_0}{4\pi} \int \frac{Idl \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi} \int_0^{\pi/2} \frac{ad\phi\hat{\phi} \times \hat{\mathbf{r}}}{a^2} + \frac{\mu_0 I}{4\pi} \int_{\pi/2}^0 \frac{bd\phi\hat{\phi} \times \hat{\mathbf{r}}}{b^2} \tag{118}$$

where I have used that $\mathbf{dl} = rd\phi\hat{\phi}$, for $r = a$ and $r = b$. There is clearly no contribution from the straight sides, since here $dl \times \hat{\mathbf{r}} = 0$. Thus,

$$\mathbf{B} = \frac{\mu_0 I}{4\pi a}(\hat{\mathbf{z}})\frac{\pi}{2} + \frac{\mu_0 I}{4\pi b}(\hat{\mathbf{z}})(-\frac{\pi}{2}) = \frac{\mu_0 I}{8}(\frac{1}{a} - \frac{1}{b})\hat{\mathbf{z}} \quad \square \tag{119}$$

b)

Here, the Biot Savart integral becomes (for the semicircular side):

$$B = \frac{\mu_0 I}{4\pi} \int_{-\pi/2}^{\pi/2} \frac{Rd\phi(-\hat{\phi}) \times \hat{\mathbf{r}}}{R^2} = -\frac{\mu_0 I}{4R}\hat{\mathbf{z}} \tag{120}$$

For the two straight lines, the Biot Savart integral is awkward and difficult to compute, so we will refer to Griffiths example 5, in which he states that the magnetic field between these two wires is going to be $B = \mu_0 I/2\pi s$, so the total field at $s = R$ is:

$$\mathbf{B} = (-\frac{\mu_0 I}{4R} + \frac{\mu_0 I}{2\pi R})\hat{\mathbf{z}} \tag{121}$$

Problem 5.12

Suppose you have two infinite straight line charges λ, a distance d apart, moving along at a constant speed v. How great would v have to be in order for the magnetic attraction to balance the electric repulsion?

A line charge λ traveling at velocity v constitutes a current $I = \lambda v$. The magnetic field at a distance d due to the first (second) wire is

$$B = \frac{\mu_0}{2\pi d} I_1(I_2) \tag{122}$$

There is an attractive force between the wires, since the current is in the same direction. The magnitude of the force is $F = ILB$, where L is the length of the wire. Now, the total force is infinite, but the force per unit length is $f = IB$, so the force per unit length between these two wires is, substituting $I = \lambda v$,

$$f_B = \frac{\mu_0 I_1 I_2}{2\pi d} = \frac{\mu_0 \lambda^2 v^2}{2\pi d} \tag{123}$$

Meanwhile, we need to find the electric field due to a wire of charge density λ:

$$\int \mathbf{E} \cdot d\mathbf{l} = E 2\pi d = \frac{\lambda L}{\epsilon_0} \quad \Rightarrow \quad E = \frac{\lambda L}{2\pi\epsilon_0 d}, \tag{124}$$

meaning that the force per unit length between the wires is

$$f_e = \frac{\lambda E}{L} = \frac{\lambda^2}{2\pi\epsilon_0 d} \tag{125}$$

Thus the electric force per unit length $f_e \equiv F_e/L$ can be set equal to the magnetic force per unit length:

$$f_B = f_e \quad \Rightarrow \quad \frac{\mu_0 \lambda^2 v^2}{2\pi d} = \frac{\lambda^2}{2\pi\epsilon_0 d} \quad \Rightarrow \quad v = \sqrt{\frac{1}{\epsilon_0 \mu_0}} = c \quad \square \tag{126}$$

Thus you could never get the wires moving fast enough – the electric force will always dominate over the magnetic force, and the wires will repel.

Problem 5.15

Two long coaxial solenoids each carry current I, but in opposite directions, as shown in Fig. 5.42. The inner solenoid (radius a) has n_1 turns per unit length, and the outer one (radius b) has n_2. Find B in each of the three regions: (i) inside the inner solenoid, (ii) between them, and (iii) outside both.

The field inside a solenoid is $\mathbf{B} = \mu_0 n I$. Outside, the field is zero. Thus, in region (i), we have contributions from both the larger and the smaller solenoid. The field in region (i) is

$$\mathbf{B}_{(i)} = \mu_0 n_1 I \hat{\mathbf{z}} + \mu_0 n_2 I(-\hat{\mathbf{z}}) = \mu_0 I(n_1 - n_2)\hat{\mathbf{z}} \quad \square \tag{127}$$

where the minus sign comes from the fact that the current is being carried in the opposite direction in the larger solenoid. In region (ii), the only contribution to the magnetic field is from the larger solenoid. The region is outside the smaller solenoid, so the magnetic field due to the smaller one is zero. Thus, the field in region (ii) is

$$\mathbf{B}_{(ii)} = \mu_0 n_2 I(-\hat{\mathbf{z}}) = -\mu_0 n_2 I \hat{\mathbf{z}} \quad \square \tag{128}$$

Outside both solenoids, the field is zero. Thus, in region (iii),

$$\mathbf{B}_{(iii)} = 0 \quad \square \tag{129}$$

Problem 5.16

A large parallel-plate capacitor with uniform surface charge σ on the upper plate and $-\sigma$ on the lower is moving with a constant speed v, as shown in Fig. 5.43.

a) Find the magnetic field between the plates and also above and below them.

We consider each moving plate to be a surface current K, where $K = \sigma v$. According to example 5.8, the magnetic field due to such a surface current is $B = \pm\mu_0 K/2$, where the $+$ is below the plane and the $-$ is above the plane. How is this derived? Well, use Ampere's law: $\int B \cdot dl = 2BL = \mu_0 I_{enc} = \mu_0 KL \Rightarrow B = \mu_0 K/2$. Thus, above and below the capacitor, the field is zero, since the surface current K is pointed in opposite direction due to the opposite charge on the plates. In between the capacitor plates, the field adds:

$$B_{between} = \frac{\mu_0}{2}\sigma v - \frac{\mu_0}{2}(-\sigma v) = \mu_0 \sigma v\ (in) \quad \square \tag{130}$$

b) Find the magnetic force per unit area on the upper plate, including its direction.

The force per unit area is $\mathbf{f} = \mathbf{K} \times \mathbf{B}$, so we obtain, for $the upper plate$

$$f = \sigma v \frac{\mu_0}{2}\sigma v = \frac{\mu_0 \sigma^2 v^2}{2} \tag{131}$$

Doing the right hand rule will convince you that the force is up.

c) At what speed v would the magnetic force balance the electrical force?

The electric field of the bottom plate is $E = \sigma/2\epsilon_0$, so the force on the upper plate is $F = \sigma^2 A/2\epsilon_0$. The force per unit area is, thus, $f = \sigma^2/2\epsilon_0$. Equating this force with the force per unit area in part (b) yields:

$$\frac{\mu_0 \sigma^2 v^2}{2} = \frac{\sigma^2}{2\epsilon_0} \quad \Rightarrow \quad v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \quad \square \tag{132}$$

So the plates would have to move at the speed of light. Thus, the forces would never balance.

Problem 5.22

Find the magnetic vector potential of a finite segment of a straight wire, carrying a current I. [Put the wire on the z axis, from z_1 to z_2, and use equation 5.64:

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}dl}{|\mathbf{r} - \mathbf{r}'|} \tag{133}$$

Check that your answer is consistent with equation 5.35:

$$\mathbf{B} = \frac{\mu_0 I}{4\pi s}(sin\theta_2 - sin\theta_1)\hat{\phi} \tag{134}$$

The distance between the wire and any point s off the wire is $\sqrt{z^2 + s^2}$, and, letting $\mathbf{I} = I\hat{\mathbf{z}}$, and $dl = dz$,

$$\mathbf{A} = \frac{\mu_0 I}{4\pi}\int_{z_1}^{z_2} \frac{\hat{\mathbf{z}}dz}{\sqrt{z^2 + s^2}} = \frac{\mu_0 I}{4\pi} ln \frac{\sqrt{s^2 + z_2^2} + z_2}{\sqrt{s^2 + z_1^2} + z_1}\hat{\mathbf{z}} \quad \square \tag{135}$$

To confirm whether this matches equation 5.35, we take the curl of $\mathbf{A}$:

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial s}\hat{\phi} \tag{136}$$

If you take this derivative, you get

$$\mathbf{B} = \hat{\phi}\frac{\mu_0 I}{4\pi s}\left[\frac{z_2}{\sqrt{z_2^2+s^2}} - \frac{z_1^2}{\sqrt{z_1^2+s^2}}\right] \tag{137}$$

But recall that $sin\theta_j = z_j/\sqrt{z_j^2+s^2}$, so

$$\mathbf{B} = \hat{\phi}\frac{\mu_0 I}{4\pi s}(sin\theta_2 - sin\theta_1) \quad \square \tag{138}$$

Problem 6.1
Calculate the torque exerted on the square loop shown in Fig. 6.6, due to the circular loop (assume r is much larger than a or b. If the square loop is free to rotate, what will its equilibrium orientation be?
Torque is given by $\mathbf{N} = \mathbf{m} \times \mathbf{B}$. The dipole moment of the circular loop is $\mathbf{m} = I\pi a^2\hat{\mathbf{z}}$. The magnetic field due to the circular loop is given by everyone's favorite formula for the magnetic field due to a dipole:

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] = -\frac{\mu_0}{4\pi}\frac{I\pi a^2}{r^3}\hat{\mathbf{z}} \tag{139}$$

since $\mathbf{m}\cdot\hat{\mathbf{r}} = 0$. The $\mathbf{m}$ in the torque equation, however, refers to the dipole moment of the *square* loop, which is $\mathbf{m} = Ib^2\hat{\mathbf{x}}$. Thus,

$$\mathbf{N} = -Ib^2\hat{\mathbf{x}} \times \frac{\mu_0}{4\pi}\frac{I\pi a^2}{r^3}\hat{\mathbf{z}} = -\frac{I^2a^2b^2\mu_0}{4r^3}\hat{\mathbf{y}} \tag{140}$$

The final orientation of the square loop will be when there is no torque on it. Since the magnetic field is in the $\hat{\mathbf{z}}$ direction, the loop will rotate until it points in -$\hat{\mathbf{z}}$. Why minus? Since the torque is in -$\hat{\mathbf{y}}$, the loop is spinning in the direction of negative z, so it will anti align with the other loop.

Problem 6.7
An infinitely long circular cylinder carries a uniform magnetization M parallel to its axis. Find the magnetic field (due to M) inside and outside the cylinder.
If the magnetization is parallel to the axis then we can write $\mathbf{M} = M\hat{\mathbf{z}}$. Then,

$$\mathbf{J}_b = \nabla \times \mathbf{M} = 0, \quad \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = M\hat{\mathbf{z}} \times \hat{\mathbf{s}} = M\hat{\phi} \tag{141}$$

So the cylinder acts like a solenoid, since it carries a surface current in the azimuthal direction. When that happens, the magnetic field is $B = \mu_0 nI$, where n is the number of turns per unit length. But $n = N/L$, and $I = K_b L$, thus the product $nI = NK_b$, and, since $N = 1$ for this cylinder,

$$\mathbf{B} = \mu_0 K_b\hat{\mathbf{z}} = \mu_0 M\hat{\mathbf{z}} = \mu_0\mathbf{M} \quad \square \tag{142}$$

Outside the cylinder, where the magnetization is zero, the field is also zero.

Problem 6.8
A long circular cylinder of radius R carries a magnetization $\mathbf{M} = ks^2\hat{\phi}$, where k is a constant, s is the distance from the axis, and $\hat{\phi}$ is the usual azimuthal unit vector (Fig. 6.13). Find the magnetic field due to M, for points inside and outside the cylinder.
The easiest way to do this is to calculate the bound currents:

$$\mathbf{J}_b = \nabla \times \mathbf{M} = \frac{1}{s}\frac{\partial(sM_\phi)}{\partial s}\hat{\mathbf{z}} = 3ks\hat{\mathbf{z}}, \quad \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = -ks^2\hat{\mathbf{z}}\Big|_{s=R} = -kR^2\hat{\mathbf{z}} \tag{143}$$

Now we can get the magnetic field inside the cylinder by using Ampere's law. Since the surface current is... on the surface, the only contribution to I_{enc} is from $\mathbf{J}_b$:

$$\int B\cdot dl = B2\pi s = \mu_0 I_{enc} = \mu_0\int J_b da = \mu_0\int_0^s 3ks\hat{\mathbf{z}}\cdot sd\phi\hat{\mathbf{z}} = 2\pi\mu_0 ks^3 \quad \Rightarrow \quad \mathbf{B} = \mu_0 ks^2\hat{\phi} \tag{144}$$

or, in terms of the magnetization,

$$\mathbf{B}_{in} = \mu_0 k s^2 \hat{\phi} = \mu_0 \mathbf{M} \quad \Box \tag{145}$$

Inside the cylinder, the magnetic field is zero since the total current enclosed by an Amperian loop is zero. Let's check that:

$$I_{enc} = \int J_b da + \int K_b dl = \int_0^R 3ks\hat{\mathbf{z}} \cdot s d\phi \hat{\mathbf{z}} - kR^2 \int dl = 2\pi k R^3 - kR^2 2\pi R = 0 \tag{146}$$

Problem 6.12

An infinitely long cylinder, of radius R, carries a frozen-in magnetization, parallel to the axis, $\mathbf{M} = ks\hat{\mathbf{z}}$, where k is constant and s is the distance from the axis; there is no free current anywhere. Find the magnetic field inside and outside the cylinder by two different methods:

a) As in Sect. 6.2, locate all the bound currents. and calculate the field they produce.

The bound volume current is $\mathbf{J}_b = \nabla \times \mathbf{M}$ and the bound surface current is $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$. Let's calculate these:

$$\mathbf{J}_b = -\frac{dM_z}{ds}\hat{\phi} = -k\hat{\phi}, \quad \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{s}}\Big|_{s=R} = kR\hat{\phi} \tag{147}$$

Thus, the total current through a square Amperian loop of side length l, placed a distance $R - s$ into the cylinder is

$$I_{enc} = \int J_b da + \int K_b dl = -k(R-s)l + kR(l) = ksl \tag{148}$$

and therefore the magnetic field is

$$\int \mathbf{B} \cdot \mathbf{dl} = Bl = \mu_0 I_{enc} = \mu_0 ksl \quad \Rightarrow \quad B_{in} = \mu_0 ks\hat{\mathbf{z}} \quad \Box \tag{149}$$

It is in the $\hat{\mathbf{z}}$ direction, since the magnetization is in that direction and the surface and volume currents are in the $\hat{\phi}$ directions. Meanwhile, outside, there is no current through the Amperian loop outside, so $\mathbf{B}_{out} = 0$.

b) Use Ampere's law (in the form of Eq. 6.20) to find H, and the get B from Eq. 6.18. (Notice that the second method is much faster, and avoids any explicit reference to the bound currents.)

Ampere's law is $\int \mathbf{H} \cdot \mathbf{dl} = I_{free,enc}$. Since there is no free current anywhere, $\mathbf{H} = 0$. Therefore, since $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$, we can immediately find the magnetic field. Outside, where $\mathbf{M} = 0$, $\mathbf{B} = 0$ also. Inside, $\mathbf{B} = \mu_0 \mathbf{M}$, so

$$\mathbf{B} = \mu_0 ks\hat{\mathbf{z}} \quad \Box \tag{150}$$

Problem 6.16

A coaxial cable consists of two very long cylindrical tubes, separated by linear insulating material of magnetic susceptibility chi_m. A current I flows down the inner conductor and returns along the outer one; in each case the current distributes itself uniformly over the surface (Fig. 6.24). Find the magnetic field in the region between the tubes. As a check, calculate the magnetization and the bound currents, and confirm that (together, of course, with the free currents) they generate the correct field.

Using Ampere's law: $\int H \cdot dl = I_{free} \Rightarrow H = I_free/2\pi s$. Therefore, since $B = \mu H = \mu_0(1+\chi_m)H$, we obtain

$$\mathbf{B} = \mu_0(1+\chi_m)\frac{I}{2\pi s}\hat{\phi} \tag{151}$$

Magnetization is given by $\mathbf{M} = \chi_m \mathbf{H}$, so

$$\mathbf{M} = \frac{I\chi_m}{2\pi s}\hat{\phi} \tag{152}$$

Therefore,

$$\mathbf{J} = \nabla \times \mathbf{M} = \frac{1}{s}\frac{\partial(sM_\phi)}{\partial s}\hat{\mathbf{z}} = 0 \tag{153}$$

$$\mathbf{K}_a = \mathbf{M} \times \hat{\mathbf{n}}\Big|_{s=a} = \frac{I\chi_m}{2\pi s}\hat{\phi} \times \hat{\mathbf{s}}\Big|_{s=a} = \frac{I\chi_m}{2\pi a}\hat{\mathbf{z}} \tag{154}$$

and

$$\mathbf{K}_a = \mathbf{M} \times \hat{\mathbf{n}}\Big|_{s=b} = \frac{I\chi_m}{2\pi s}\hat{\phi} \times \hat{\mathbf{s}}\Big|_{s=b} = -\frac{I\chi_m}{2\pi b}\hat{\mathbf{z}} \tag{155}$$

Thus, the total current is

$$I_{tot} = I + \int K dl = I + \int \frac{I\chi_m}{2\pi a}\hat{\mathbf{z}} \cdot dl\hat{\mathbf{z}} = I + \frac{I\chi_m}{2\pi a}2\pi a = I(1+\chi_m) \tag{156}$$

Thus,

$$\int B \cdot dl = \mu_0 I_{enc} \quad \Rightarrow \quad B 2\pi s = \mu_0 I(1+\chi_m) \quad \Rightarrow \quad \mathbf{B} = \frac{\mu_0 I(1+\chi_m)}{2\pi s}\hat{\phi} \quad \square \tag{157}$$

Problem 6.17

A current I flows down a long straight wire of radius a. If the wire is made of linear material (copper, say, or aluminum) with susceptibility χ_m, and the current is distributed uniformly, what is the magnetic field a distance s from the axis? Find all the bound currents. What is the net bound current flowing down the wire.

At a distance $s < a$ from the center of the wire, $\int H \cdot dl = 2\pi s H = I_{free}$. Since the current is distributed uniformly, the current at that distance is Is^2/a^2. Thus,

$$H_{in} = I\frac{s}{2\pi a^2}\hat{\phi} \tag{158}$$

Meanwhile, outside the wire, the enclosed current is just I. Thus,

$$H_{out} = \frac{I}{2\pi s}\hat{\phi} \tag{159}$$

Now, $B = \mu H = \mu_0(1+\chi_m)H$. Thus,

$$B_{in} = I\frac{\mu_0(1+\chi_m)s}{2\pi a^2}\hat{\phi} \tag{160}$$

and

$$B_{out} = \frac{I\mu_0}{2\pi s}\hat{\phi} \tag{161}$$

since the susceptibility of free space vanishes. The free current is $J_f = I/\pi a^2$ and $J_b = \chi_m J_f$, so

$$J_b = \frac{I}{\pi a^2}\chi_m\hat{\mathbf{z}} \tag{162}$$

while

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = \chi_m \mathbf{H} \times \hat{\mathbf{n}}\Big|_{s=a} = \frac{\chi_m I}{2\pi a}(-\hat{\mathbf{z}}) \tag{163}$$

The net bound current is

$$I_b = J_b \pi a^2 K_b 2\pi a = I\chi_m - I\chi_m = 0 \quad \square \tag{164}$$

Problem 7.7

A metal bar of mass m slides frictionlessly on two parallel conducting rails a distance l apart. A resistor R is connected across the rails and a uniform magnetic field B, pointing into the page, fills the entire region.

a) If the bar moves to the right at speed v, what is the current in the resistor? In

what direction does it flow?

If the bar is moving to the right, that means that the flux through the loop is increasing. The area of the loop is lvt, so that the flux (and it's time derivative) through the loop is

$$\phi_B = lvtB \quad \Rightarrow \quad \frac{d\phi_B}{dt} = Blv \tag{165}$$

By Faraday's law, this produces an EMF in the loop, whose magnitude is

$$EMF = -\frac{d\phi_B}{dt} = -Blv = IR \tag{166}$$

meaning that the current is

$$I = -\frac{Blv}{R} \quad \Box \tag{167}$$

The negative sign means that the current is flowing counterclockwise (i.e. downward through the resistor), since the positive sense was defined by the direction of the magnetic field.

b) What is the magnetic force on the bar? In what direction?

The magnetic force is $\mathbf{F} = I\mathbf{dl} \times \mathbf{B}$. Since the current is up through the bar and the magnetic field is into the page, the force is to the left, as it must be. Otherwise, the bar would gain more and more energy as it sped up, without a source of that energy. The bar must slow down. The force is

$$F = IlB = -\frac{B^2l^2v}{R} \tag{168}$$

where the minus sign means to the left. $\Box$.

c) If the bar starts out with speed v_0 at time $t = 0$, and is left to slide, what is its speed at a later time t?

$$m\frac{dv}{dt} = -\frac{B^2l^2v}{R} \quad \Rightarrow \quad v(t) = v_0e^{-B^2l^2t/Rm} \quad \Box \tag{169}$$

d) The initial kinetic energy of the bar was, of course, $mv_0^2/2$. Check the energy delivered to the resistor is exactly $mv_0^2/2$.

The power delivered to the resistor is $P = I^2R$, and power is the time derivative of energy: $P \equiv dW/dt$:

$$P = \frac{dW}{dt} = I^2R = \frac{B^2l^2v^2}{R} \quad \Rightarrow \quad dW = \frac{B^2l^2v^2}{R}dt \tag{170}$$

Plugging in my result for v^2 from equation 169:

$$W = \frac{B^2l^2}{R}\int_0^\infty v_0^2e^{-2B^2l^2t/Rm}dt = \frac{B^2l^2v_0^2}{R}\frac{Rm}{-2B^2l^2}(e^{-2B^2l^2t/Rm})\Big|_0^\infty = \frac{1}{2}mv_0^2 \quad \Box \tag{171}$$

Problem 7.12

A long solenoid, of radius a, is driven by an alternating current, so that the field inside is sinusoidal: $\mathbf{B}(t) = B_0cos(\omega t)\hat{\mathbf{z}}$. A circular loop of wire, or radius $a/2$ and resistance R, is placed inside the solenoid, and coaxial with it. Find the current induced in the loop, as a function of time.

The flux through the loop is

$$\phi_B = \pi(\frac{a}{2})^2B_0cos(\omega t) = \frac{\pi}{4}a^2B_0cos(\omega t) \tag{172}$$

so the induced EMF is

$$EMF = -\frac{d\phi_B}{dt} = \frac{\pi a^2B_0\omega}{4}sin(\omega t) \tag{173}$$

meaning that the current $I = EMF/R$ is

$$I = \frac{\pi a^2 B_0 \omega}{4R} sin(\omega t) \tag{174}$$

Problem 7.15
A long solenoid with radius a and n turns per unit length carries a time-dependent current $I(t)$ in the $\hat{\phi}$ direction. Find the electric field (magnitude and direction) at a distance s from the axis (both inside and outside the solenoid), in the quasi static approximation.
The magnetic field inside a solenoid is $B = \mu_0 nI$, and outside is 0. Thus, the flux through an Amperian loop of radius s inside the solenoid is

$$\phi_B = \mu_0 nI(t)\pi s^2 \tag{175}$$

The induced EMF is

$$EMF = -\mu_0 n\pi s^2 \frac{dI}{dt} \tag{176}$$

and the electric field can be calculated via

$$EMF = \int \mathbf{E} \cdot \mathbf{dl} = E2\pi s \quad \Rightarrow \quad E_{in} = -\frac{\mu_0 ns}{2}\frac{dI}{dt} \quad \square \tag{177}$$

The electric field circles around the solenoid, since the magnetic field is in the $\hat{\mathbf{z}}$ direction through the solenoid. Meanwhile, the flux through an Amperian loop *outside* the solenoid is

$$\phi_B = \mu_0 nI(t)\pi a^2 \tag{178}$$

since the magnetic field outside the solenoid is 0, the contribution to $\int \mathbf{B} \cdot \mathbf{da}$ vanishes outside. Thus, the electric field can be found through

$$\int \mathbf{E} \cdot \mathbf{dl} = -\frac{d}{dt}\mu_0 nI(t)\pi a^2 \hat{\phi} \tag{179}$$

Since the loop of integration is taken around the point *outside* the solenoid, the integral becomes $2\pi sE$.

$$E2\pi s = -\mu_0 n\pi a^2 \frac{dI}{dt} \quad \Rightarrow \quad E = -\frac{\mu_0 na^2}{2s}\frac{dI}{dt}\hat{\phi} \tag{180}$$

Problem 7.31
A fat wire, radius a, carries a constant current I, uniformly distributed over its cross section. A narrow gap in the wire, of width $w \ll a$, forms a parallel-plate capacitor, as shown in Fig. 7.43. Find the magnetic field in the gap, at a distance $s < a$ from the axis.
This is a displacement current problem. The displacement current is given by the change of electric flux. The electric field due to the charge density on the edge of the wire is $E = \sigma/\epsilon_0$. Thus, the flux through an Amperian loop of radius s is

$$\phi_E = \frac{\sigma}{\epsilon_0}\pi s^2 \quad \Rightarrow \quad \frac{d\phi_E}{dt} = \frac{\pi s^2}{\epsilon_0}\frac{d\sigma}{dt} = \frac{\pi s^2}{\epsilon_0}\frac{1}{\pi a^2}\frac{dq}{dt} = \frac{Is^2}{a^2\epsilon_0} \tag{181}$$

Thus, we have

$$\int B \cdot dl = \epsilon_0\mu_0 \frac{\partial \phi_E}{\partial t} = \frac{\epsilon_0\mu_0 Is^2}{a^2\epsilon_0} \quad \Rightarrow \quad \mathbf{B} = \frac{\mu_0 Is^2}{a^2}\frac{1}{2\pi s}\hat{\phi} = \frac{\mu_0 Is}{2\pi a^2}\hat{\phi} \quad \square \tag{182}$$

Problem 7.32
The preceding problem was an artificial model for the charging capacitor, designed to avoid complications associated with the current spreading out over the surface of the

plates. For a more realistic model, imagine *thin* wires that connect to the centers of the plates (Fig. 7.44a). Again, the current I is constant, the radius of the capacitor is a, and the separation of the plates is $w \ll a$. Assume that the current flows out over the plates in such a way that the surface charge is uniform, at any given time, and is zero at $t = 0$.

a) Find the electric field between the plates, as a function of t.

Between the plates, we have the usual electric field in a capacitor, $E = \sigma/\epsilon_0 = q/\pi\epsilon_0 a^2$. But $q/t = I$, so the electric field as a function of time is

$$\mathbf{E} = \frac{It}{\pi\epsilon_0 a^2}\hat{\mathbf{z}} \quad \square \tag{183}$$

b) Find the displacement current through a circle of radius s in the plane midway between the plates. Using this circle as your "Amperian loop," and the flat surface that spans it, find the magnetic field at a distance s from the axis.

The flux through this circle is this electric field multiplied by the area of the circle, πs^2:

$$\phi_E = \frac{It}{\pi\epsilon_0 a^2}\pi s^2 \tag{184}$$

Therefore, the displacement current is

$$I_d = \epsilon_0 \frac{d\phi_E}{dt} = \epsilon_0 \frac{I}{\pi\epsilon_0 a^2}\pi s^2 = \frac{Is^2}{a^2} \quad \square \tag{185}$$

From this we can obtain the magnetic field

$$\int B \cdot dl = \mu_0 I_d \quad \Rightarrow \quad B 2\pi s = \frac{\mu_0 I s^2}{a^2} \quad \Rightarrow \quad \mathbf{B} = \frac{\mu_0 I s}{2\pi a^2}\hat{\phi} \quad \square \tag{186}$$

c) Repeat part (b), but this time use the cylindrical surface in Fig. 7.44b, which is open at the right end and extends to the left through the plate and terminates outside the capacitor. Notice that the displacement current through this surface is zero, and there are two contributions to I_{enc}. We know have a total enclosed current of (i) the current in the wire with radius s, and (ii) the displacement current from the donut region around the wire. The displacement current will be the old displacement current in equation 185, minus the current in the wire. So

$$I_{d,new} = I_{disp,old} - I_{wire} = I_{wire}\frac{s^2}{a^2} - I_{wire} \tag{187}$$

To obtain the total current to be enclosed by our Amperian loop, we need to add the real current flowing in the wire, I:

$$I_{tot} = I_{d,new} + I_{wire} = I_{wire}\frac{s^2}{a^2} - I_{wire} + I_{wire} = I_{wire}\frac{s^2}{a^2} \tag{188}$$

Thus, we again find, as in equation 186, that

$$\mathbf{B} = \frac{\mu_0 I s}{2\pi a^2}\hat{\phi} \quad \square \tag{189}$$

Problem 8.2

Consider the charging capacitor in Prob. 7.31.

a) Find the electric and magnetic fields in the gap, as functions of the distance s from the axis and the time t. (Assume the charge is zero at $t = 0$.)

We already found the magnetic field in problem 7.31 by using Maxwell's law $\int B \cdot dl = \epsilon_0\mu_0\dot{\phi}_E$. We found it to be

$$\mathbf{B} = \frac{\mu_0 I s}{2\pi a^2}\hat{\phi} \quad \square \tag{190}$$

Meanwhile, the electric field in the gap is given by $\mathbf{E} = \sigma/\epsilon_0\hat{\mathbf{z}}$. Using $\sigma = q/A = It/A$, we obtain

$$\mathbf{E} = \frac{It}{\epsilon_0 \pi a^2}\hat{\mathbf{z}} \quad \square \tag{191}$$

We could have also found this by using Maxwell's law in differential form:

$$\nabla \times \mathbf{B} = \frac{1}{s}\frac{\partial(sB_\phi)}{\partial s}\hat{\mathbf{z}} = \frac{\mu_0 I}{\pi a^2}\hat{\mathbf{z}} = \epsilon_0\mu_0\frac{\partial \mathbf{E}}{\partial t} \quad \Rightarrow \quad \mathbf{E} = \frac{It}{\epsilon_0\pi a^2}\hat{\mathbf{z}} \tag{192}$$

b) Find the energy density u_{em} and the Poynting vector S in the gap. Note especially the direction of S. Check that Eq. 8.14 is satisfied.
The energy density is

$$u_{em} = \frac{\epsilon_0}{2}E^2 + \frac{B^2}{2\mu_0} = \frac{1}{2}\frac{I^2t^2}{\epsilon_0\pi^2a^4} + \frac{1}{2}\frac{\mu_0 I^2 s^2}{4\pi^2 a^4} = \frac{\mu_0 I^2}{2\pi^2 a^4}(c^2t^2 + \frac{s^2}{4}) \tag{193}$$

Meanwhile, the Poynting vector is trivially computed:

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = -\frac{I^2 st}{2\pi^2\epsilon_0 a^4}\hat{\mathbf{s}} \tag{194}$$

Eq. 8.14 is

$$\frac{\partial}{\partial t}(u_{mech} + u_{em}) = -\nabla\cdot\mathbf{S} \tag{195}$$

$u_{mech} = 0$ here, so it remains to take the time derivative of equation 193 and compare it to the divergence of equation 194:

$$\frac{\partial u_{em}}{\partial t} = \frac{\mu_0 I^2}{\pi^2 a^4}c^2 t = -\nabla\cdot\mathbf{S} = \frac{I^2 t}{2\pi^2\epsilon_0 a^4}\nabla\cdot(s\hat{\mathbf{s}}) = \frac{I^2t}{\pi^2\epsilon_0 a^4} \tag{196}$$

(Recall that the divergence is taken in cylindrical coordinates). Since $c^2 = 1/\mu_0\epsilon_0$, equation 8.14 is confirmed.
c) Determine the total energy in the gap, as a function of time. Calculate the total power flowing into the gap, by integrating the Poynting vector of the appropriate surface. Check that the power input is equal to the rate of increase of energy in the gap (Eq. 8.9 – in this case $W = 0$, because there is no charge in the gap).
The total energy is given by

$$U_{em} = \int u_{em}dV = \int_0^b u_{em}w2\pi s ds = \frac{\mu_0 w I^2 b^2}{2\pi a^4}[c^2t^2 + \frac{b^2}{16}] \tag{197}$$

for an arbitrary surface of radius b. w is the width of the gap. The total power is

$$P = \int \mathbf{S}\cdot d\mathbf{a} = \frac{I^2 t}{2\pi^2\epsilon_0 a^4}[b\hat{\mathbf{s}}\cdot(2\pi bw\hat{\mathbf{s}}] = \frac{I^2 wtb^2}{\pi\epsilon_0 a^4} \tag{198}$$

Problem 10.3
Find the fields, and the charge and current distributions, corresponding to $V(\mathbf{r}, t) = 0$, $\mathbf{A}(\mathbf{r}, t) = -qt/4\pi\epsilon_0 r^2\hat{\mathbf{r}}$.
This is a rather simple exercise:

$$\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \tag{199}$$

$$\mathbf{B} = \nabla\times\mathbf{A} = 0 \tag{200}$$

It looks like this corresponds to a point charge at the origin:

$$\nabla\cdot\mathbf{E} = \frac{q}{4\pi\epsilon_0}\nabla\cdot(\frac{\hat{\mathbf{r}}}{r^2}) = \frac{q}{\epsilon_0}\delta^3(\mathbf{r}) \quad \Rightarrow \quad \rho = q\delta^3(\mathbf{r}) \tag{201}$$

$$\nabla\times\mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{J} = 0 \tag{202}$$