Find the following limits:

(1) $\lim\limits_{x \to 1} \dfrac{\sqrt{x+7} - \sqrt{3x+5}}{\sqrt{3x+5} - \sqrt{5x+3}}$ [Ans: 1]	(2) $\lim\limits_{x \to 1^+} \dfrac{\sqrt{x^2 - 1}}{\sqrt{x^3 - 1} + \sqrt{x^4 - 1}}$ [Ans: $\sqrt{2}\,(2 - \sqrt{3})$]
(3) $\lim\limits_{x \to 0} \dfrac{\sin ax - \sin bx}{\sin cx - \sin dx}$ $\left[\text{Ans}: \dfrac{a-b}{c-d} \right]$	(4) $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{\tan x - 1}{x - \frac{\pi}{4}}$ [Ans: 2]
(5) $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{\sin x - \cos x}{\frac{\pi}{4} - x}$ [Ans: $-\sqrt{2}$]	(6) $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{\sin 3x + \cos 3x}{x - \frac{\pi}{4}}$ [Ans: $-3\sqrt{2}$]
(7) $\lim\limits_{x \to \sqrt{3}} \dfrac{\tan^{-1} x - \frac{\pi}{3}}{x - \sqrt{3}}$ $\left[\text{Ans}: \dfrac{1}{4} \right]$	(8) $\lim\limits_{x \to \frac{\pi}{6}} \dfrac{2 \sin x - 1}{\sqrt{3} \tan x - 1}$ $\left[\text{Ans}: \dfrac{3}{4} \right]$
(9) $\lim\limits_{x \to -1^+} \dfrac{\sqrt{x^3 + 1} + \sqrt{x^5 + 1}}{\sqrt{x^{[illegible]} + 1}}$ $\left[\text{Ans}: \dfrac{1}{7} (\sqrt{35} + \sqrt{21}) \right]$	(10) $\lim\limits_{x \to 1} \dfrac{x + x^2 + x^3 + \ldots + x^n - n}{x - 1}$ $\left[\text{Ans}: \dfrac{n(n+1)}{2} \right]$
(11) $\lim\limits_{x \to 3} \dfrac{x^4 - 8x^2 - x - 6}{2x^3 - 15x - 9}$ $\left[\text{Ans}: \dfrac{59}{39} \right]$	(12) $\lim\limits_{x \to 2} \dfrac{x^6 - 64}{x^{\frac{1}{3}} - 2^{\frac{1}{3}}}$ $\left[\text{Ans}: 9 \times 2^{\frac{20}{3}} \right]$
(13) $\lim\limits_{x \to 2} \dfrac{(3x+3)^{\frac{1}{4}} - (4x+1)^{\frac{1}{4}}}{x^3 - 8}$ $\left[\text{Ans}: -\dfrac{1}{144\sqrt{3}} \right]$	(14) $\lim\limits_{x \to 2} \dfrac{\sqrt[3]{x+6} - \sqrt[3]{2x+4}}{x^2 - 4}$ $\left[\text{Ans}: -\dfrac{1}{48} \right]$

Find the following limits:

(15) $\lim\limits_{x \to \frac{1}{2}} \dfrac{\sin^{-1} x - \frac{\pi}{6}}{2x - 1}$ $\left[\text{Ans : } \dfrac{1}{\sqrt{3}} \right]$

(16) $\lim\limits_{x \to a} \dfrac{\tan x - \tan a}{\tan^{-1} x - \tan^{-1} a}$ $\left[\text{Ans : } (1 + a^2) \sec^2 a \right]$

(17) $\lim\limits_{x \to 0} \dfrac{\cos ax - \cos bx}{\cos cx - \cos dx}$ $\left[\text{Ans : } \dfrac{a^2 - b^2}{c^2 - d^2} \right]$

(18) $\lim\limits_{x \to a} \dfrac{(x^n - a^n) - na^{n-1}(x - a)}{(x - a)^2}$ $\left[\text{Ans : } \dfrac{n(n-1)}{2} a^{n-2} \right]$

(19) $\lim\limits_{x \to 0} \dfrac{(1 + mx)^n - (1 + nx)^m}{x^2}$ $\left[\text{Ans : } \dfrac{mn(n - m)}{2} \right]$

(20) $\lim\limits_{x \to 1} \left(\dfrac{m}{1 - x^m} - \dfrac{n}{1 - x^n} \right)$, $m, n \in N$ $\left[\text{Ans : } \dfrac{m - n}{2} \right]$

(21) $\lim\limits_{x \to 1} \dfrac{x^{n+1} - (n + 1)x + n}{(x - 1)^2}$, $n \in N$ $\left[\text{Ans : } \dfrac{n(n + 1)}{2} \right]$

(22) $\lim\limits_{n \to \infty} \dfrac{1^2 + 3^2 + 5^2 + \dots + (2n - 1)^2}{[1 + 3 + 5 + \dots + (2n - 1)]\, n}$ $\left[\text{Ans : } \dfrac{4}{3} \right]$

(23) $\lim\limits_{x \to 0} \dfrac{2^{5x} - 2^{3x}}{\sin x}$ $\left[\text{Ans : } 2 \log 2 \right]$

(24) $\lim\limits_{x \to 0} \left(\dfrac{1 - x}{1 + x} \right)^{\frac{1}{x}}$ $\left[\text{Ans: } e^{-2} \right]$

(25) $\lim\limits_{x \to e} \dfrac{\log x - 1}{x - e}$ $\left[\text{Ans : } e^{-1} \right]$

(26) $\lim\limits_{n \to \infty} \sum\limits_{r=1}^{n} \dfrac{1}{n} 3^{\frac{r}{n}}$ $\left[\text{Ans : } 2 \log_3 e \right]$

(27) $\lim\limits_{x \to \infty} \sum \dfrac{1}{n(n + 1)}$, $n \in N$ $[\text{Ans: } 1]$

(28) $\lim\limits_{x \to \infty} \dfrac{\sum n^3}{n \sum n^2}$, $n \in N$ $\left[\text{Ans : } \dfrac{3}{4} \right]$

(29) $\lim\limits_{n \to \infty} \left(\sqrt{n^2 + n + 1} - n \right)$ $\left[\text{Ans : } \dfrac{1}{2} \right]$

(30) $\lim\limits_{x \to 0} \dfrac{x \tan x}{e^x - 2 + e^{-x}}$ $[\text{Ans: } 1]$

Find the following limits:

(31) $\displaystyle\lim_{x\to 0} \frac{e^{3x} - 2e^{2x} + 2^x}{x}$ $\left[\text{Ans}: \log\frac{2}{e}\right]$	(32) $\displaystyle\lim_{x\to 0} \frac{6^x - 3^x - 2^x + 1}{x^2}$ $[\text{Ans}: (\log 3)(\log 2)]$
(33) $\displaystyle\lim_{x\to 0} \frac{\sqrt[4]{x^4 + 1} - \sqrt{x^2 + 1}}{x^2}$ $\left[\text{Ans}: -\frac{1}{2}\right]$	(34) $\displaystyle\lim_{x\to 1} \frac{\sqrt[3]{7 + x^3} - \sqrt{3 + x^2}}{x - 1}$ $\left[\text{Ans}: -\frac{1}{4}\right]$
(35) $\displaystyle\lim_{x\to \frac{\pi}{3}} \frac{\tan^3 x - 3\tan x}{\cos\left(x + \frac{\pi}{6}\right)}$ $[\text{Ans}: -24]$	(36) $\displaystyle\lim_{x\to 0} \frac{a^{x+1} - 2a^x - a + 2}{a^{x+1} - 3a^x - a + 3}$ $\left[\text{Ans}: \frac{a-2}{a-3}\right]$
(37) $\displaystyle\lim_{x\to a^+} \frac{\sqrt{x} - \sqrt{a} + \sqrt{x - a}}{\sqrt{x^2 - a^2}}$ $\left[\text{Ans}: \frac{1}{\sqrt{2a}}\right]$	(38) $\displaystyle\lim_{x\to a^-} \frac{\sqrt{a^2 - x^2} + (a - x)}{\sqrt{a^3 - x^3} + \sqrt{a - x}}$, $a > 0$ $\left[\text{Ans}: \frac{\sqrt{2a}}{a\sqrt{3} + 1}\right]$
(39) $\displaystyle\lim_{x\to -8} \frac{\sqrt{1 - x} - 3}{2 + \sqrt[3]{x}}$ $[\text{Ans}: -2]$	(40) $\displaystyle\lim_{x\to \sqrt{2}} \frac{x^4 - 4}{x^2 + 3x\sqrt{2} - 8}$ $\left[\text{Ans}: \frac{8}{5}\right]$
(41) $\displaystyle\lim_{x\to 0} \frac{\sin x^\circ}{x}$ $\left[\text{Ans}: \frac{\pi}{180}\right]$	(42) $\displaystyle\lim_{x\to 0} \frac{(\sqrt{1 + x^2} - 1)\sin^{-1}x}{(\tan^{-1}x)^3}$ $\left[\text{Ans}: \frac{1}{2}\right]$
(43) $\displaystyle\lim_{x\to 0} \frac{(\sin^{-1}x)^2}{1 - \sqrt{1 - x^2}}$ $[\text{Ans}: 2]$	(44) $\displaystyle\lim_{h\to 0} \frac{e^{\tan^{-1}h} - e^{\sin^{-1}h}}{\tan^{-1}h - \sin^{-1}h}$ $[\text{Ans}: 1]$
(45) $\displaystyle\lim_{x\to \pi} \frac{\sqrt{5 + \cos x} - 2}{(\pi - x)^2}$ $\left[\text{Ans}: \frac{1}{8}\right]$	(46) $\displaystyle\lim_{x\to \infty} \frac{2\sqrt{x} + 3\sqrt[3]{x} + 5\sqrt[5]{x}}{\sqrt{3x - 2} + \sqrt[3]{2x - 3}}$ $\left[\text{Ans}: \frac{2}{\sqrt{3}}\right]$

Find the following limits:

(47) $\lim_{x \to 3} \dfrac{(x^3 + 27)\log(x - 2)}{x^2 - 9}$ [Ans : 9]	(48) $\lim_{x \to 0} (\cos x)^{\cot^2 x}$ [Ans : $e^{-\frac{1}{2}}$]
(49) $\lim_{x \to \frac{\pi}{4}} (\sin 2x)^{\tan^2 2x}$ [Ans : $e^{-\frac{1}{2}}$]	(50) $\lim_{x \to \infty} \left(\dfrac{3x - 2}{3x + 2}\right)^{3x}$ [Ans : e^{-4}]
(51) $\lim_{x \to 0} (1 - 4x)^{\frac{1 - x}{x}}$ [Ans : e^{-4}]	(52) $\lim_{x \to 0} \left(\dfrac{\sin x}{x}\right)^{\frac{\sin x}{x - \sin x}}$ [Ans : e^{-1}]
(53) $\lim_{n \to \infty} \dfrac{(n + 1) + (n + 2) + \ldots + (n + n)}{n^2}$ [Ans : $\frac{3}{2}$]	(54) $\lim_{n \to 0} \dfrac{8^n - 4^n - 2^n + 1}{\sqrt{5} - \sqrt{4 + \cos n}}$ [Ans : $8\sqrt{5}\,(\log_e 2)^2$]
(55) $\lim_{n \to \infty} \sum_{r=1}^{n} \dfrac{1}{4r^2 - 1}$ [Ans : $\frac{1}{2}$]	(56) $\lim_{n \to \infty} \dfrac{3^{n+1} - 2^n}{3^n + 2^{n+1}}$ [Ans: 3]
(57) $\lim_{x \to \infty} \left[\sqrt{x + \sqrt{x + x}} - \sqrt{x}\right]$ [Ans : $\frac{1}{2}$]	(58) $\lim_{x \to a} \dfrac{xe^{-x} - ae^{-a}}{x - a}$ [Ans : $(1 - a)e^{-a}$]
(59) $\lim_{n \to \infty} \cos\frac{x}{2} \cdot \cos\frac{x}{2^2} \cdot \ldots \cdot \cos\frac{x}{2^n}$ [Ans : $\frac{\sin x}{x}$]	(60) $\lim_{\theta \to 0} \dfrac{1 - \cos\{1 - \cos(1 - \cos\theta)\}}{\theta^8}$ [Ans : $\frac{1}{128}$]
(61) $\lim_{x \to 0} \dfrac{35^x - 2 \cdot 7^x + 1}{x}$ [Ans : $\frac{35}{49}$]	(62) $\lim_{x \to 0} \dfrac{e^{\alpha x} - e^{\beta x}}{\sin \alpha x - \sin \beta x}$ [Ans: 1]

Find the following limits:

(63) $\lim\limits_{y \to a} \sin \dfrac{y - a}{2} \tan \dfrac{\pi y}{2a}$

$\left[\text{Ans: } -\dfrac{a}{\pi} \right]$

(64) $\lim\limits_{x \to a} \dfrac{\sin^{-1} x - \sin^{-1} a}{x - a}$

$\left[\text{Ans: } \dfrac{1}{\sqrt{1 - a^2}} \right]$

(65) $\lim\limits_{x \to 0} \dfrac{1 - \cos(1 - \cos x)}{x^4}$ $\left[\text{Ans: } \dfrac{1}{8} \right]$

(66) $\lim\limits_{x \to \sqrt{2}} \dfrac{4 \sec^{-1} x - \pi}{x - \sqrt{2}}$ $[\text{Ans: } 2\sqrt{2}]$

(67) $\lim\limits_{h \to 0} \dfrac{(a + h)^2 \sin(a + h) - a^2 \sin a}{h}$

$\left| \text{Ans: } a^2 \cos a + 2a \sin a \right|$

(68) Prove that $\lim\limits_{x \to 2} (3x + 2) \neq 5$

(69) Prove that $\lim\limits_{x \to 0} x^2 \sin \dfrac{1}{x} = 0$

Solve the following problems:

(70) Express $\left\{ x \mid \dfrac{1}{|2x - 3|} \leq \dfrac{1}{5},\ x \in R - \left\{ \dfrac{3}{2} \right\} \right\}$ as the complement of an interval.

$[\text{Ans: } R - (-1, 4)]$

(71) Express 1 – neighbourhood of 0.1 in the interval and modulus forms.

$[\text{Ans: } (-0.9, 1.1),\ |x - 0.1| < 1]$

(72) Express the following set in the form of an interval and also in the form $N(p, \delta)$, $p \in R$, $\delta > 0$. $\{ x \mid 0 \leq |3x + 1| < 2,\ x \in R \}$.

$\left[\text{Ans: } \left(-1, \dfrac{1}{3} \right),\ N\left(-\dfrac{1}{3}, \dfrac{2}{3} \right) \right]$

(73) Express $\left\{ x \mid \dfrac{1}{|3x + 2|} \leq \dfrac{1}{5},\ x \in R - \left\{ -\dfrac{2}{3} \right\} \right\}$ as the complement of an interval.

$\left[\text{Ans: } R - \left(-\dfrac{7}{3}, 1 \right) \right]$

(74) Prove that the intersection of two δ - neighbourhoods of p is a δ - neighbourhood of p.

(75) Assuming that the intersection of two open intervals is either empty or it is an open interval, prove that the intersection of two neighbourhoods of $p \in R$ is a neighbourhood of p.

(76) If
$$f(x) = \begin{cases} x, & x < -1 \\ 0, & x = 0 \\ x^2, & x > 1, \end{cases}$$
then can we find $\lim\limits_{x \to 0} f(x)$?

[Ans: No. $\because$ f is not defined in any neighbourhood of zero.]

(77) Examine the continuity of f at x = 0 where $f(x) = \dfrac{|x|}{x}$, for $x \neq 0$, $f(0) = 1$.

[Ans: discontinuous]

(78) Examine the continuity of $f(x) = [x]$, $x \in R$ at $n \in Z$.

[Ans: discontinuous $\forall\ n \in Z$]

(79) If for each x, $a \in D_f$, we have $|f(x) - f(a)| \leq |x - a|$, then prove that f is continuous at a.

(80) For $f(x) = x - [x]$ find $\lim\limits_{x \to n^-} f(x)$ and $\lim\limits_{x \to n^+} f(x)$ where $n \in Z$ and hence examine the continuity of f at n. [Ans: discontinuous]

(81) If $f(x) = \dfrac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}}$, $x \neq 0$, then prove that $\lim\limits_{x \to 0} f(x)$ does not exist.

(82) Is it true that $\lim\limits_{n \to 0} \dfrac{\binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}}{n} = \log_e 2$?

$\left[\text{Ans: No. In } \binom{n}{r},\ n \in N.\ \therefore\ n \text{ is not defined everywhere in any neighbourhood of } 0. \right]$

(83) Express 2.123123123..... as a vulgar fraction $\left[\text{Ans: } \dfrac{707}{333} \right]$

(84) If $f(x) = \dfrac{x}{e^{\frac{1}{x}} - 1}$, $x \neq 0$, find $\lim\limits_{x \to 0} f(x)$. [Ans: 0]

(85) Find limit of the sequence $\dfrac{1 \cdot 5}{9 \cdot 13}, \dfrac{2 \cdot 7}{12 \cdot 17}, \dfrac{3 \cdot 9}{15 \cdot 21}, \ldots$ $\left[\text{Ans} : \dfrac{1}{6} \right]$

(86) Find limit of the sequence $\dfrac{1^3}{1 \cdot 2 \cdot 3}, \dfrac{2^3}{4 \cdot 5 \cdot 5}, \dfrac{3^3}{7 \cdot 8 \cdot 7}, \ldots$ $\left[\text{Ans} : \dfrac{1}{18} \right]$

(87) Obtain the maximum $\delta > 0$ such that $\forall x$, $x \in N^*(2, \delta) \Rightarrow f(x) \in N(13, 0.001)$, where $f(x) = 4x + 5$, $x \in R$.

[Ans: 0.00025]

(88) For $f(x) = 2x$, $x \in \left\{ 1 + \dfrac{1}{n},\ n \in N \right\}$, show that

$\forall x \in N^*(1, \delta)$, $x \in D_f \Rightarrow | f(x) - 2 | < \varepsilon$. What can be said about $\lim\limits_{x \to 1} f(x)$?

[Ans: limit does not exist as f is not defined everywhere in any neighbourhood of 1]

(89) Express $\left\{ x \mid \left| x + \dfrac{1}{x} \right| < 4,\ x \in R,\ x \neq 0 \right\}$ as the union of the intervals.

$\left[\text{Ans} : (-2, -\sqrt{3}, -2 + \sqrt{3}) \cup (2 - \sqrt{3}, 2 + \sqrt{3}) \right]$

(90) Express $\left\{ x \mid \left| 1 + \dfrac{3}{x} \right| > 2,\ x \in R - \{ 0 \} \right\}$ as the union of the intervals.

[Ans: $(-1, 0) \cup (0, 3)$]

(91) Express $\left\{ x \mid \left| x^2 - \dfrac{13}{2} \right| < \dfrac{5}{2},\ x \in R \right\}$ as the union of the intervals.

[Ans: $(-3, -2) \cup (2, 3)$]

(92) Prove that $\lim\limits_{x \to 4} f(x)$ where $f(x) = \sqrt{x} - [\sqrt{x}]$ does not exist.

(93) Find the limit of the sequence $\sqrt{5}$, $\sqrt{5\cdot\sqrt{5}}$, $\sqrt{5\cdot\sqrt{5\cdot\sqrt{5}}}$, ... [Ans: 5]

(94) Find minimum $m \in N$ such that $n \geq m \Rightarrow f(n) \in (1, 0.01)$, $f(n) = \frac{3^n - 1}{3^n}$

[Ans: 5]

(95) If $f(x + y) = f(x) + f(y)$ $\forall$ $x, y \in R$ and if f is continuous at $x = 0$, then prove that f is continuous at every $x \in R$.

(96) If $f(x) = \frac{(3^{\sin x} - 1)^2}{x \log(1 + x)}$, $x \neq 0$ is continuous at $x = 0$, then find $f(0)$.

$\left[\text{Ans: } (\log 3)^2\right]$

(97) If

$$f(x) = \frac{\sin(a+1)x + \sin x}{x}, \quad x < 0$$
$$= c, \quad x = 0$$
$$= \frac{(x + bx^2)^{\frac{1}{2}} - x^{\frac{1}{2}}}{bx^{\frac{3}{2}}}, \quad x > 0$$

is continuous at $x = 0$, then find a, b, c.

$\left[\text{Ans: } a = -\frac{3}{2},\ b \neq 0,\ c = \frac{1}{2}\right]$

(98) If

$$f(x) = -2\sin x, \quad x \leq -\frac{\pi}{2}$$
$$= A\sin x + B, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$
$$= \cos x, \quad x \geq \frac{\pi}{2}$$

is continuous on R, then find A and B.

[Ans: A = - 1, B = 1]

Solve the following problems of infinite series:

(99) Find $\sum_{n=1}^{\infty} \frac{n^3 + n^2}{n!} x^n$ and hence obtain $\sum_{n=1}^{\infty} \frac{n^3 + n^2}{n!}$

[Ans: $(x^3 + 4x^2 + 2x)e^x$, $7e$]

Solve the following problems of infinite series:

(100) If $S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2$, then find $\sum_{n=1}^{\infty} \frac{S_n x^n}{n!}$.

$\left[\text{Ans: } \frac{1}{6}(2x^3 + 9x^2 + 6x)e^x \right]$

(101) Prove that $1 + \lim_{x \to 1} \left[\frac{1+x}{2!} + \frac{1+x+x^2}{3!} + \ldots \infty \right] = e$

(102) Prove that $n! > \left(\frac{n}{e}\right)^n$

(103) Prove that $\frac{1 + \frac{1}{2!} + \frac{1}{4!} + \ldots \infty}{1 + \frac{1}{3!} + \frac{1}{5!} + \ldots \infty} = 1 + \frac{2}{e^2 - 1}$

(104) Find $\sum_{n=1}^{\infty} \frac{2(n+1)}{(2n+1)!}$ [Ans: $e - 2$]

(105) Find $\frac{7}{3 \cdot 4 \cdot 5} + \frac{9}{5 \cdot 6 \cdot 7} + \frac{11}{7 \cdot 8 \cdot 9} + \ldots \infty$ $\left[\text{Ans: } 3\log 2 - \frac{11}{6} \right]$

(106) Prove that $\sum_{1}^{\infty} \frac{(\log x)^{2n-1}}{(2n-1)!} = \frac{x^2 - 1}{2x}$

(107) Find $\sum_{1}^{\infty} \frac{1}{n} \frac{x^n - 1}{(x+1)^n}$ [Ans: $\log x$]

(108) Prove that $\sum_{1}^{\infty} \frac{1}{(2n-1)(2x+1)^{(2n-1)}} = \log \sqrt{\frac{1+x}{x}}$

(109) Find approximate value of $(131)^{\frac{1}{3}}$ correct to four decimal places. [Ans: 5.0788]

(110) If n is very large, prove that $\left(1 + \frac{1}{n}\right)^n = e\left(1 - \frac{1}{2n} + \frac{11}{24n^2}\right)$