**Exercise 7.1**

**Question 1:**

sin 2*x*

Answer

The anti derivative of sin 2*x* is a function of *x* whose derivative is sin 2*x*. It is known that,

$$\frac{d}{dx}(\cos 2x) = -2\sin 2x$$

$$\Rightarrow \sin 2x = -\frac{1}{2}\frac{d}{dx}(\cos 2x)$$

$$\therefore \sin 2x = \frac{d}{dx}\left(-\frac{1}{2}\cos 2x\right)$$

Therefore, the anti derivative of $\sin 2x$ is $-\frac{1}{2}\cos 2x$

**Question 2:**

Cos 3*x*

Answer

The anti derivative of cos 3*x* is a function of *x* whose derivative is cos 3*x*.

It is known that,

$$\frac{d}{dx}(\sin 3x) = 3\cos 3x$$

$$\Rightarrow \cos 3x = \frac{1}{3}\frac{d}{dx}(\sin 3x)$$

$$\therefore \cos 3x = \frac{d}{dx}\left(\frac{1}{3}\sin 3x\right)$$

Therefore, the anti derivative of $\cos 3x$ is $\frac{1}{3}\sin 3x$ .

**Question 3:**

*e*$^{2x}$

Answer

The anti derivative of *e*$^{2x}$ is the function of *x* whose derivative is *e*$^{2x}$.

It is known that,

$$\frac{d}{dx}\left(e^{2x}\right) = 2e^{2x}$$

$$\Rightarrow e^{2x} = \frac{1}{2}\frac{d}{dx}\left(e^{2x}\right)$$

$$\therefore e^{2x} = \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right)$$

Therefore, the anti derivative of $e^{2x}$ is $\frac{1}{2}e^{2x}$.

**Question 4:**

$$\left(ax+b\right)^2$$

Answer

The anti derivative of $\left(ax+b\right)^2$ is the function of $x$ whose derivative is $\left(ax+b\right)^2$.

It is known that,

$$\frac{d}{dx}\left(ax+b\right)^3 = 3a\left(ax+b\right)^2$$

$$\Rightarrow \left(ax+b\right)^2 = \frac{1}{3a}\frac{d}{dx}\left(ax+b\right)^3$$

$$\therefore \left(ax+b\right)^2 = \frac{d}{dx}\left(\frac{1}{3a}\left(ax+b\right)^3\right)$$

Therefore, the anti derivative of $\left(ax+b\right)^2$ is $\frac{1}{3a}\left(ax+b\right)^3$.

**Question 5:**

$$\sin 2x - 4e^{3x}$$

Answer

The anti derivative of $\left(\sin 2x - 4e^{3x}\right)$ is the function of $x$ whose derivative is $\left(\sin 2x - 4e^{3x}\right)$.

It is known that,

$$\frac{d}{dx}\left(-\frac{1}{2}\cos 2x - \frac{4}{3}e^{3x}\right) = \sin 2x - 4e^{3x}$$

Therefore, the anti derivative of $\left(\sin 2x - 4e^{3x}\right)$ is $\left(-\frac{1}{2}\cos 2x - \frac{4}{3}e^{3x}\right)$.

**Question 6:**

$$\int\left(4e^{3x} + 1\right)dx$$

Answer

$$\int\left(4e^{3x} + 1\right)dx$$

$$= 4\int e^{3x}dx + \int 1dx$$

$$= 4\left(\frac{e^{3x}}{3}\right) + x + C$$

$$= \frac{4}{3}e^{3x} + x + C$$

**Question 7:**

$$\int x^2\left(1 - \frac{1}{x^2}\right)dx$$

Answer

$$\int x^2\left(1 - \frac{1}{x^2}\right)dx$$

$$= \int\left(x^2 - 1\right)dx$$

$$= \int x^2dx - \int 1dx$$

$$= \frac{x^3}{3} - x + C$$

**Question 8:**

$$\int\left(ax^2 + bx + c\right)dx$$

Answer

$$\int (ax^2 + bx + c)\, dx$$

$$= a\int x^2 dx + b\int x\, dx + c\int 1.\, dx$$

$$= a\left(\frac{x^3}{3}\right) + b\left(\frac{x^2}{2}\right) + cx + C$$

$$= \frac{ax^3}{3} + \frac{bx^2}{2} + cx + C$$

**Question 9:**

$$\int (2x^2 + e^x)\, dx$$

Answer

$$\int (2x^2 + e^x)\, dx$$

$$= 2\int x^2 dx + \int e^x dx$$

$$= 2\left(\frac{x^3}{3}\right) + e^x + C$$

$$= \frac{2}{3}x^3 + e^x + C$$

**Question 10:**

$$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$$

Answer

$$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$$

$$= \int \left(x + \frac{1}{x} - 2\right) dx$$

$$= \int x\, dx + \int \frac{1}{x}\, dx - 2\int 1.\, dx$$

$$= \frac{x^2}{2} + \log|x| - 2x + C$$

**Question 11:**

$$\int \frac{x^3 + 5x^2 - 4}{x^2} dx$$

Answer

$$\int \frac{x^3 + 5x^2 - 4}{x^2} dx$$

$$= \int \left( x + 5 - 4x^{-2} \right) dx$$

$$= \int x\, dx + 5 \int 1.dx - 4 \int x^{-2} dx$$

$$= \frac{x^2}{2} + 5x - 4\left( \frac{x^{-1}}{-1} \right) + C$$

$$= \frac{x^2}{2} + 5x + \frac{4}{x} + C$$

**Question 12:**

$$\int \frac{x^3 + 3x + 4}{\sqrt{x}} dx$$

Answer

$$\int \frac{x^3 + 3x + 4}{\sqrt{x}} dx$$

$$= \int \left( x^{\frac{5}{2}} + 3x^{\frac{1}{2}} + 4x^{-\frac{1}{2}} \right) dx$$

$$= \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{3\left( x^{\frac{3}{2}} \right)}{\frac{3}{2}} + \frac{4\left( x^{\frac{1}{2}} \right)}{\frac{1}{2}} + C$$

$$= \frac{2}{7} x^{\frac{7}{2}} + 2x^{\frac{3}{2}} + 8x^{\frac{1}{2}} + C$$

$$= \frac{2}{7} x^{\frac{7}{2}} + 2x^{\frac{3}{2}} + 8\sqrt{x} + C$$

**Question 13:**

$$\int \frac{x^3 - x^2 + x - 1}{x - 1} dx$$

Answer

$$\int \frac{x^3 - x^2 + x - 1}{x - 1} dx$$

On dividing, we obtain

$$= \int (x^2 + 1) dx$$

$$= \int x^2 dx + \int 1 dx$$

$$= \frac{x^3}{3} + x + C$$

**Question 14:**

$$\int (1 - x)\sqrt{x} dx$$

Answer

$$\int (1 - x)\sqrt{x} dx$$

$$= \int \left( \sqrt{x} - x^{\frac{3}{2}} \right) dx$$

$$= \int x^{\frac{1}{2}} dx - \int x^{\frac{3}{2}} dx$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C$$

$$= \frac{2}{3} x^{\frac{3}{2}} - \frac{2}{5} x^{\frac{5}{2}} + C$$

**Question 15:**

$$\int \sqrt{x} \left( 3x^2 + 2x + 3 \right) dx$$

Answer

$$\int \sqrt{x}\left(3x^2 + 2x + 3\right)dx$$

$$= \int \left(3x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 3x^{\frac{1}{2}}\right)dx$$

$$= 3\int x^{\frac{5}{2}}dx + 2\int x^{\frac{3}{2}}dx + 3\int x^{\frac{1}{2}}dx$$

$$= 3\left(\frac{x^{\frac{7}{2}}}{\frac{7}{2}}\right) + 2\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right) + 3\frac{\left(x^{\frac{3}{2}}\right)}{\frac{3}{2}} + C$$

$$= \frac{6}{7}x^{\frac{7}{2}} + \frac{4}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + C$$

**Question 16:**

$$\int\left(2x - 3\cos x + e^x\right)dx$$

Answer

$$\int\left(2x - 3\cos x + e^x\right)dx$$

$$= 2\int x\,dx - 3\int \cos x\,dx + \int e^x dx$$

$$= \frac{2x^2}{2} - 3(\sin x) + e^x + C$$

$$= x^2 - 3\sin x + e^x + C$$

**Question 17:**

$$\int\left(2x^2 - 3\sin x + 5\sqrt{x}\right)dx$$

Answer

$$\int\left(2x^2 - 3\sin x + 5\sqrt{x}\right)dx$$

$$= 2\int x^2\,dx - 3\int \sin x\,dx + 5\int x^{\frac{1}{2}}\,dx$$

$$= \frac{2x^3}{3} - 3\left(-\cos x\right) + 5\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{2}{3}x^3 + 3\cos x + \frac{10}{3}x^{\frac{3}{2}} + C$$

**Question 18:**

$$\int \sec x\left(\sec x + \tan x\right)dx$$

Answer

$$\int \sec x\left(\sec x + \tan x\right)dx$$

$$= \int \left(\sec^2 x + \sec x \tan x\right)dx$$

$$= \int \sec^2 x\,dx + \int \sec x \tan x\,dx$$

$$= \tan x + \sec x + C$$

**Question 19:**

$$\int \frac{\sec^2 x}{\cos ec^2 x}\,dx$$

Answer

$$\int \frac{\sec^2 x}{\cos ec^2 x}\,dx$$

$$= \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}} dx$$

$$= \int \frac{\sin^2 x}{\cos^2 x} dx$$

$$= \int \tan^2 x \, dx$$

$$= \int \left(\sec^2 x - 1\right) dx$$

$$= \int \sec^2 x \, dx - \int 1 \, dx$$

$$= \tan x - x + C$$

**Question 20:**

$$\int \frac{2 - 3\sin x}{\cos^2 x} dx$$

Answer

$$\int \frac{2 - 3\sin x}{\cos^2 x} dx$$

$$= \int \left(\frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x}\right) dx$$

$$= \int 2\sec^2 x \, dx - 3\int \tan x \sec x \, dx$$

$$= 2\tan x - 3\sec x + C$$

**Question 21:**

The anti derivative of $\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)$ equals

(A) $\dfrac{1}{3} x^{\frac{1}{3}} + 2x^{\frac{1}{2}} + C$

(B) $\dfrac{2}{3} x^{\frac{2}{3}} + \dfrac{1}{2} x^2 + C$

(C) $\dfrac{2}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$

(D) $\dfrac{3}{2} x^{\frac{3}{2}} + \dfrac{1}{2} x^{\frac{1}{2}} + C$

Answer

$$\left( \sqrt{x} + \frac{1}{\sqrt{x}} \right) dx$$

$$= \int x^{\frac{1}{2}} dx + \int x^{-\frac{1}{2}} dx$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \frac{2}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$$

Hence, the correct Answer is C.

**Question 22:**

If $\frac{d}{dx} f(x) = 4x^3 - \frac{3}{x^4}$ such that $f(2) = 0$, then $f(x)$ is

(A) $x^4 + \frac{1}{x^3} - \frac{129}{8}$   (B) $x^3 + \frac{1}{x^4} + \frac{129}{8}$

(C) $x^4 + \frac{1}{x^3} + \frac{129}{8}$   (D) $x^3 + \frac{1}{x^4} - \frac{129}{8}$

Answer

It is given that,

$$\frac{d}{dx} f(x) = 4x^3 - \frac{3}{x^4}$$

∴Anti derivative of $4x^3 - \frac{3}{x^4} = f(x)$

$$\therefore f(x) = \int 4x^3 - \frac{3}{x^4}\, dx$$

$$f(x) = 4\int x^3\, dx - 3\int (x^{-4})\, dx$$

$$f(x) = 4\left(\frac{x^4}{4}\right) - 3\left(\frac{x^{-3}}{-3}\right) + C$$

$$\therefore f(x) = x^4 + \frac{1}{x^3} + C$$

Also,

$$f(2) = 0$$

$$\therefore f(2) = (2)^4 + \frac{1}{(2)^3} + C = 0$$

$$\Rightarrow 16 + \frac{1}{8} + C = 0$$

$$\Rightarrow C = -\left(16 + \frac{1}{8}\right)$$

$$\Rightarrow C = \frac{-129}{8}$$

$$\therefore f(x) = x^4 + \frac{1}{x^3} - \frac{129}{8}$$

Hence, the correct Answer is A.

Exercise 7.2

**Question 1:**

$$\frac{2x}{1+x^2}$$

Answer

Let $1+x^2 = t$

$\therefore 2x\ dx = dt$

$$\Rightarrow \int \frac{2x}{1+x^2}\,dx = \int \frac{1}{t}\,dt$$
$$= \log|t| + C$$
$$= \log|1+x^2| + C$$
$$= \log(1+x^2) + C$$

**Question 2:**

$$\frac{(\log x)^2}{x}$$

Answer

Let $\log|x| = t$

$$\therefore \frac{1}{x}\,dx = dt$$

$$\Rightarrow \int \frac{\left(\log |x|\right)^2}{x} dx = \int t^2 dt$$

$$= \frac{t^3}{3} + C$$

$$= \frac{\left(\log |x|\right)^3}{3} + C$$

**Question 3:**

$$\frac{1}{x + x \log x}$$

Answer

$$\frac{1}{x + x \log x} = \frac{1}{x\left(1 + \log x\right)}$$

Let $1 + \log x = t$

$$\therefore \frac{1}{x} dx = dt$$

$$\Rightarrow \int \frac{1}{x\left(1 + \log x\right)} dx = \int \frac{1}{t} dt$$

$$= \log |t| + C$$
$$= \log |1 + \log x| + C$$

**Question 4:**

$\sin x \cdot \sin (\cos x)$

Answer

sin $x$ · sin (cos $x$)

Let cos $x = t$

∴ −sin $x$ $dx = dt$

$$\Rightarrow \int \sin x \cdot \sin(\cos x)\,dx = -\int \sin t\,dt$$
$$= -\left[-\cos t\right] + C$$
$$= \cos t + C$$
$$= \cos(\cos x) + C$$

**Question 5:**

$$\sin(ax+b)\cos(ax+b)$$

Answer

$$\sin(ax+b)\cos(ax+b) = \frac{2\sin(ax+b)\cos(ax+b)}{2} = \frac{\sin 2(ax+b)}{2}$$

Let $2(ax+b) = t$

∴ 2$adx = dt$

$$\Rightarrow \int \frac{\sin 2(ax+b)}{2}dx = \frac{1}{2}\int \frac{\sin t \ dt}{2a}$$

$$= \frac{1}{4a}[-\cos t] + C$$

$$= \frac{-1}{4a}\cos 2(ax+b) + C$$

**Question 6:**

$$\sqrt{ax+b}$$

Answer

Let $ax + b = t$

$\Rightarrow adx = dt$

$$\therefore dx = \frac{1}{a}dt$$

$$\Rightarrow \int (ax+b)^{\frac{1}{2}} \ dx = \frac{1}{a}\int t^{\frac{1}{2}}dt$$

$$= \frac{1}{a}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{2}{3a}(ax+b)^{\frac{3}{2}} + C$$

**Question 7:**

$$x\sqrt{x+2}$$

Answer

Let $(x+2) = t$

$\therefore dx = dt$

$$\Rightarrow \int x\sqrt{x+2}\,dx = \int(t-2)\sqrt{t}\,dt$$

$$= \int\left(t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\right)dt$$

$$= \int t^{\frac{3}{2}}dt - 2\int t^{\frac{1}{2}}dt$$

$$= \frac{t^{\frac{5}{2}}}{\frac{5}{2}} - 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}} + C$$

$$= \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + C$$

**Question 8:**

$$x\sqrt{1+2x^2}$$

Answer

Let $1 + 2x^2 = t$

$\therefore 4x\,dx = dt$

$$\Rightarrow \int x\sqrt{1+2x^2}\,dx = \int \frac{\sqrt{t}\,dt}{4}$$

$$= \frac{1}{4}\int t^{\frac{1}{2}}\,dt$$

$$= \frac{1}{4}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{1}{6}\left(1+2x^2\right)^{\frac{3}{2}} + C$$

**Question 9:**

$$(4x+2)\sqrt{x^2+x+1}$$

Answer

Let $x^2+x+1=t$

$\therefore (2x+1)dx = dt$

$$\int (4x+2)\sqrt{x^2+x+1}\ dx$$

$$= \int 2\sqrt{t}\ dt$$

$$= 2\int \sqrt{t}\ dt$$

$$= 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{4}{3}\left(x^2+x+1\right)^{\frac{3}{2}} + C$$

**Question 10:**

$$\frac{1}{x-\sqrt{x}}$$

Answer

$$\frac{1}{x-\sqrt{x}} = \frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}$$

Let $\left(\sqrt{x}-1\right) = t$

$$\therefore \frac{1}{2\sqrt{x}}dx = dt$$

$$\Rightarrow \int \frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}dx = \int \frac{2}{t}dt$$

$$= 2\log|t| + C$$

$$= 2\log\left|\sqrt{x}-1\right| + C$$

**Question 11:**

$$\frac{x}{\sqrt{x+4}}, x > 0$$

Answer

Let $x+4 = t$

$\therefore dx = dt$

$$\int \frac{x}{\sqrt{x+4}}\, dx = \int \frac{(t-4)}{\sqrt{t}}\, dt$$

$$= \int \left( \sqrt{t} - \frac{4}{\sqrt{t}} \right) dt$$

$$= \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - 4\left( \frac{t^{\frac{1}{2}}}{\frac{1}{2}} \right) + C$$

$$= \frac{2}{3}(t)^{\frac{3}{2}} - 8(t)^{\frac{1}{2}} + C$$

$$= \frac{2}{3} t \cdot t^{\frac{1}{2}} - 8 t^{\frac{1}{2}} + C$$

$$= \frac{2}{3} t^{\frac{1}{2}} (t - 12) + C$$

$$= \frac{2}{3}(x+4)^{\frac{1}{2}} (x + 4 - 12) + C$$

$$= \frac{2}{3}\sqrt{x+4}\,(x-8) + C$$

**Question 12:**

$$\left( x^3 - 1 \right)^{\frac{1}{3}} x^5$$

Answer

Let $x^3 - 1 = t$

$\therefore 3x^2 dx = dt$

$$\Rightarrow \int \left(x^3-1\right)^{\frac{1}{3}} x^5 \, dx = \int \left(x^3-1\right)^{\frac{1}{3}} x^3 \cdot x^2 \, dx$$

$$= \int t^{\frac{1}{3}} \left(t+1\right) \frac{dt}{3}$$

$$= \frac{1}{3} \int \left(t^{\frac{4}{3}} + t^{\frac{1}{3}}\right) dt$$

$$= \frac{1}{3} \left[\frac{t^{\frac{7}{3}}}{\frac{7}{3}} + \frac{t^{\frac{4}{3}}}{\frac{4}{3}}\right] + C$$

$$= \frac{1}{3} \left[\frac{3}{7} t^{\frac{7}{3}} + \frac{3}{4} t^{\frac{4}{3}}\right] + C$$

$$= \frac{1}{7} \left(x^3-1\right)^{\frac{7}{3}} + \frac{1}{4} \left(x^3-1\right)^{\frac{4}{3}} + C$$

**Question 13:**

$$\frac{x^2}{\left(2+3x^3\right)^3}$$

Answer

Let $2+3x^3 = t$

$\therefore 9x^2 \, dx = dt$

$$\Rightarrow \int \frac{x^2}{\left(2+3x^3\right)^3}dx = \frac{1}{9}\int \frac{dt}{\left(t\right)^3}$$

$$= \frac{1}{9}\left[\frac{t^{-2}}{-2}\right]+C$$

$$= \frac{-1}{18}\left(\frac{1}{t^2}\right)+C$$

$$= \frac{-1}{18\left(2+3x^3\right)^2}+C$$

**Question 14:**

$$\frac{1}{x\left(\log x\right)^m} \ , \ x>0$$

Answer

Let log $x = t$

$$\therefore \ \frac{1}{x}dx = dt$$

$$\Rightarrow \int \frac{1}{x\left(\log x\right)^m}dx = \int \frac{dt}{\left(t\right)^m}$$

$$= \left(\frac{t^{-m+1}}{1-m}\right)+C$$

$$= \frac{\left(\log x\right)^{1-m}}{\left(1-m\right)}+C$$

**Question 15:**

$$\frac{x}{9-4x^2}$$

Answer

Let $9 - 4x^2 = t$

$\therefore -8x \, dx = dt$

$$\Rightarrow \int \frac{x}{9 - 4x^2} dx = \frac{-1}{8} \int \frac{1}{t} dt$$
$$= \frac{-1}{8} \log |t| + C$$
$$= \frac{-1}{8} \log \left| 9 - 4x^2 \right| + C$$

**Question 16:**

$e^{2x+3}$

Answer

Let $2x + 3 = t$

$\therefore 2dx = dt$

$$\Rightarrow \int e^{2x+3} dx = \frac{1}{2} \int e^t dt$$
$$= \frac{1}{2} \left( e^t \right) + C$$
$$= \frac{1}{2} e^{(2x+3)} + C$$

**Question 17:**

$$\frac{x}{e^{x^2}}$$

Answer

Let $x^2 = t$

$\therefore 2x dx = dt$

$$\Rightarrow \int \frac{x}{e^{x^2}} dx = \frac{1}{2} \int \frac{1}{e^t} dt$$

$$= \frac{1}{2} \int e^{-t} dt$$

$$= \frac{1}{2} \left( \frac{e^{-t}}{-1} \right) + C$$

$$= -\frac{1}{2} e^{-x^2} + C$$

$$= \frac{-1}{2e^{x^2}} + C$$

**Question 18:**

$$\frac{e^{\tan^{-1} x}}{1 + x^2}$$

Answer

Let $\tan^{-1} x = t$

$\therefore \frac{1}{1 + x^2} dx = dt$

$$\Rightarrow \int \frac{e^{\tan^{-1} x}}{1+x^2}\,dx = \int e^t\,dt$$

$$= e^t + C$$

$$= e^{\tan^{-1} x} + C$$

**Question 19:**

$$\frac{e^{2x}-1}{e^{2x}+1}$$

Answer

$$\frac{e^{2x}-1}{e^{2x}+1}$$

Dividing numerator and denominator by $e^x$, we obtain

$$\frac{\dfrac{\left(e^{2x}-1\right)}{e^x}}{\dfrac{\left(e^{2x}+1\right)}{e^x}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Let $e^x + e^{-x} = t$

$$\therefore \left(e^x - e^{-x}\right)dx = dt$$

$$\Rightarrow \int \frac{e^{2x}-1}{e^{2x}+1}\,dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$$

$$= \int \frac{dt}{t}$$

$$= \log|t| + C$$

$$= \log\left|e^x + e^{-x}\right| + C$$

**Question 20:**

$$\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$$

Answer

Let $e^{2x} + e^{-2x} = t$

$$\therefore \left(2e^{2x} - 2e^{-2x}\right)dx = dt$$

$$\Rightarrow 2\left(e^{2x} - e^{-2x}\right)dx = dt$$

$$\Rightarrow \int\left(\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\right)dx = \int\frac{dt}{2t}$$

$$= \frac{1}{2}\int\frac{1}{t}dt$$

$$= \frac{1}{2}\log|t| + C$$

$$= \frac{1}{2}\log\left|e^{2x} + e^{-2x}\right| + C$$

**Question 21:**

$$\tan^2(2x - 3)$$

Answer

$$\tan^2(2x - 3) = \sec^2(2x - 3) - 1$$

Let $2x - 3 = t$

$$\therefore 2dx = dt$$

$$\Rightarrow \int \tan^2(2x-3)\,dx = \int \left[\left(\sec^2(2x-3)\right)-1\right]dx$$

$$= \frac{1}{2} \int \left(\sec^2 t\right)dt - \int 1\,dx$$

$$= \frac{1}{2} \int \sec^2 t\,dt - \int 1\,dx$$

$$= \frac{1}{2} \tan t - x + C$$

$$= \frac{1}{2} \tan(2x-3) - x + C$$

**Question 22:**

$$\sec^2(7-4x)$$

Answer

Let $7 - 4x = t$

$\therefore -4dx = dt$

$$\therefore \int \sec^2(7-4x)\,dx = \frac{-1}{4} \int \sec^2 t\,dt$$

$$= \frac{-1}{4} \left(\tan t\right) + C$$

$$= \frac{-1}{4} \tan(7-4x) + C$$

**Question 23:**

$$\frac{\sin^{-1} x}{\sqrt{1-x^2}}$$

Answer

Let $\sin^{-1} x = t$

$$\therefore \frac{1}{\sqrt{1-x^2}}\,dx = dt$$

$$\Rightarrow \int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx = \int t\,dt$$

$$= \frac{t^2}{2} + C$$

$$= \frac{\left(\sin^{-1} x\right)^2}{2} + C$$

**Question 24:**

$$\frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$$

Answer

$$\frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} = \frac{2\cos x - 3\sin x}{2\left(3\cos x + 2\sin x\right)}$$

Let $3\cos x + 2\sin x = t$

$$\therefore \left(-3\sin x + 2\cos x\right)dx = dt$$

$$\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}\,dx = \int \frac{dt}{2t}$$

$$= \frac{1}{2}\int \frac{1}{t}\,dt$$

$$= \frac{1}{2}\log|t| + C$$

$$= \frac{1}{2}\log|2\sin x + 3\cos x| + C$$

**Question 25:**

$$\frac{1}{\cos^2 x (1 - \tan x)^2}$$

Answer

$$\frac{1}{\cos^2 x (1 - \tan x)^2} = \frac{\sec^2 x}{(1 - \tan x)^2}$$

Let $(1 - \tan x) = t$

$$\therefore \; -\sec^2 x\, dx = dt$$

$$\Rightarrow \int \frac{\sec^2 x}{(1 - \tan x)^2}\, dx = \int \frac{-dt}{t^2}$$

$$= -\int t^{-2}\, dt$$

$$= +\frac{1}{t} + C$$

$$= \frac{1}{(1 - \tan x)} + C$$

**Question 26:**

$$\frac{\cos \sqrt{x}}{\sqrt{x}}$$

Answer

Let $\sqrt{x} = t$

$$\therefore \; \frac{1}{2\sqrt{x}}\, dx = dt$$

$$\Rightarrow \int \frac{\cos \sqrt{x}}{\sqrt{x}}\, dx = 2 \int \cos t\, dt$$

$$= 2 \sin t + C$$

$$= 2 \sin \sqrt{x} + C$$

**Question 27:**

$$\sqrt{\sin 2x}\, \cos 2x$$

Answer

Let sin 2x = t

$$\therefore \quad 2 \cos 2x\, dx = dt$$

$$\Rightarrow \int \sqrt{\sin 2x}\, \cos 2x\, dx = \frac{1}{2} \int \sqrt{t}\, dt$$

$$= \frac{1}{2} \left( \frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + C$$

$$= \frac{1}{3} t^{\frac{3}{2}} + C$$

$$= \frac{1}{3} (\sin 2x)^{\frac{3}{2}} + C$$

**Question 28:**

$$\frac{\cos x}{\sqrt{1 + \sin x}}$$

Answer

Let $1 + \sin x = t$

$\therefore \cos x \, dx = dt$

$$\Rightarrow \int \frac{\cos x}{\sqrt{1+\sin x}} dx = \int \frac{dt}{\sqrt{t}}$$

$$= \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= 2\sqrt{t} + C$$

$$= 2\sqrt{1+\sin x} + C$$

**Question 29:**

cot *x* log sin *x*

Answer

Let log sin *x* = *t*

$$\Rightarrow \frac{1}{\sin x} \cdot \cos x \, dx = dt$$

$$\therefore \cot x \, dx = dt$$

$$\Rightarrow \int \cot x \, \log \sin x \, dx = \int t \, dt$$

$$= \frac{t^2}{2} + C$$

$$= \frac{1}{2} \left( \log \sin x \right)^2 + C$$

**Question 30:**

$$\frac{\sin x}{1+\cos x}$$

Answer

Let 1 + cos *x* = *t*

$\therefore -\sin x\ dx = dt$

$$\Rightarrow \int \frac{\sin x}{1+\cos x}\ dx = \int -\frac{dt}{t}$$
$$= -\log|t| + C$$
$$= -\log|1+\cos x| + C$$

**Question 31:**

$$\frac{\sin x}{(1+\cos x)^2}$$

Answer

Let $1 + \cos x = t$

$\therefore -\sin x\ dx = dt$

$$\Rightarrow \int \frac{\sin x}{(1+\cos x)^2}\ dx = \int -\frac{dt}{t^2}$$
$$= -\int t^{-2}dt$$
$$= \frac{1}{t} + C$$
$$= \frac{1}{1+\cos x} + C$$

**Question 32:**

$$\frac{1}{1+\cot x}$$

Answer

Let $I = \int \dfrac{1}{1+\cot x}dx$

$= \int \dfrac{1}{1+\dfrac{\cos x}{\sin x}}dx$

$= \int \dfrac{\sin x}{\sin x + \cos x}dx$

$= \dfrac{1}{2}\int \dfrac{2\sin x}{\sin x + \cos x}dx$

$= \dfrac{1}{2}\int \dfrac{(\sin x + \cos x) + (\sin x - \cos x)}{(\sin x + \cos x)}dx$

$= \dfrac{1}{2}\int 1\,dx + \dfrac{1}{2}\int \dfrac{\sin x - \cos x}{\sin x + \cos x}dx$

$= \dfrac{1}{2}(x) + \dfrac{1}{2}\int \dfrac{\sin x - \cos x}{\sin x + \cos x}dx$

Let sin $x$ + cos $x$ = $t$ ⇒ (cos $x$ − sin $x$) $dx$ = $dt$

$\therefore I = \dfrac{x}{2} + \dfrac{1}{2}\int \dfrac{-(dt)}{t}$

$= \dfrac{x}{2} - \dfrac{1}{2}\log|t| + C$

$= \dfrac{x}{2} - \dfrac{1}{2}\log|\sin x + \cos x| + C$

**Question 33:**

$\dfrac{1}{1-\tan x}$

Answer

Let $I = \int \dfrac{1}{1 - \tan x} dx$

$= \int \dfrac{1}{1 - \dfrac{\sin x}{\cos x}} dx$

$= \int \dfrac{\cos x}{\cos x - \sin x} dx$

$= \dfrac{1}{2} \int \dfrac{2 \cos x}{\cos x - \sin x} dx$

$= \dfrac{1}{2} \int \dfrac{\left(\cos x - \sin x\right) + \left(\cos x + \sin x\right)}{\left(\cos x - \sin x\right)} dx$

$= \dfrac{1}{2} \int 1 \, dx + \dfrac{1}{2} \int \dfrac{\cos x + \sin x}{\cos x - \sin x} dx$

$= \dfrac{x}{2} + \dfrac{1}{2} \int \dfrac{\cos x + \sin x}{\cos x - \sin x} dx$

Put $\cos x - \sin x = t \Rightarrow (-\sin x - \cos x)\, dx = dt$

$\therefore I = \dfrac{x}{2} + \dfrac{1}{2} \int \dfrac{-(dt)}{t}$

$= \dfrac{x}{2} - \dfrac{1}{2} \log |t| + C$

$= \dfrac{x}{2} - \dfrac{1}{2} \log |\cos x - \sin x| + C$

**Question 34:**

$\dfrac{\sqrt{\tan x}}{\sin x \cos x}$

Answer

Let $I = \int \dfrac{\sqrt{\tan x}}{\sin x \cos x} dx$

$\qquad = \int \dfrac{\sqrt{\tan x} \times \cos x}{\sin x \cos x \times \cos x} dx$

$\qquad = \int \dfrac{\sqrt{\tan x}}{\tan x \cos^2 x} dx$

$\qquad = \int \dfrac{\sec^2 x \, dx}{\sqrt{\tan x}}$

Let $\tan x = t \implies \sec^2 x \, dx = dt$

$\therefore I = \int \dfrac{dt}{\sqrt{t}}$

$\qquad = 2\sqrt{t} + C$

$\qquad = 2\sqrt{\tan x} + C$

**Question 35:**

$\dfrac{(1 + \log x)^2}{x}$

Answer

Let $1 + \log x = t$

$\therefore \dfrac{1}{x} dx = dt$

$\Rightarrow \int \dfrac{(1 + \log x)^2}{x} dx = \int t^2 dt$

$\qquad\qquad\qquad = \dfrac{t^3}{3} + C$

$\qquad\qquad\qquad = \dfrac{(1 + \log x)^3}{3} + C$

**Question 36:**

$$\frac{(x+1)(x+\log x)^2}{x}$$

Answer

$$\frac{(x+1)(x+\log x)^2}{x} = \left(\frac{x+1}{x}\right)(x+\log x)^2 = \left(1+\frac{1}{x}\right)(x+\log x)^2$$

Let $(x+\log x) = t$

$\therefore$ $\left(1+\frac{1}{x}\right)dx = dt$

$$\Rightarrow \int\left(1+\frac{1}{x}\right)(x+\log x)^2 \, dx = \int t^2 dt$$

$$= \frac{t^3}{3}+C$$

$$= \frac{1}{3}(x+\log x)^3 +C$$

**Question 37:**

$$\frac{x^3 \sin\left(\tan^{-1} x^4\right)}{1+x^8}$$

Answer

Let $x^4 = t$

$\therefore 4x^3 \, dx = dt$

$$\Rightarrow \int \frac{x^3 \sin\left(\tan^{-1} x^4\right)}{1+x^8} dx = \frac{1}{4} \int \frac{\sin\left(\tan^{-1} t\right)}{1+t^2} dt \qquad \ldots(1)$$

Let $\tan^{-1} t = u$

$$\therefore \frac{1}{1+t^2} dt = du$$

From (1), we obtain

$$\int \frac{x^3 \sin\left(\tan^{-1} x^4\right) dx}{1+x^8} = \frac{1}{4} \int \sin u \, du$$

$$= \frac{1}{4}\left(-\cos u\right) + C$$

$$= \frac{-1}{4} \cos\left(\tan^{-1} t\right) + C$$

$$= \frac{-1}{4} \cos\left(\tan^{-1} x^4\right) + C$$

**Question 38:**

$$\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx \text{ equals}$$

(A)    $10^x - x^{10} + C$        (B)    $10^x + x^{10} + C$

(C)    $\left(10^x - x^{10}\right)^{-1} + C$       (D)    $\log\left(10^x + x^{10}\right) + C$

Answer

Let $x^{10} + 10^x = t$

$$\therefore \left(10x^9 + 10^x \log_e 10\right) dx = dt$$

$$\Rightarrow \int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx = \int \frac{dt}{t}$$

$$= \log t + C$$

$$= \log\left(10^x + x^{10}\right) + C$$

Hence, the correct Answer is D.

**Question 39:**

$$\int \frac{dx}{\sin^2 x \cos^2 x} \text{ equals}$$

A. $\tan x + \cot x + C$

B. $\tan x - \cot x + C$

C. $\tan x \cot x + C$

D. $\tan x - \cot 2x + C$

Answer

$$\text{Let } I = \int \frac{dx}{\sin^2 x \cos^2 x}$$

$$= \int \frac{1}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \sec^2 x\, dx + \int \mathrm{cosec}^2 x\, dx$$

$$= \tan x - \cot x + C$$

Hence, the correct Answer is B.

**Exercise 7.3**

**Question 1:**

$$\sin^2(2x+5)$$

Answer

$$\sin^2(2x+5) = \frac{1-\cos 2(2x+5)}{2} = \frac{1-\cos(4x+10)}{2}$$

$$\Rightarrow \int \sin^2(2x+5)\,dx = \int \frac{1-\cos(4x+10)}{2}\,dx$$

$$= \frac{1}{2}\int 1\,dx - \frac{1}{2}\int \cos(4x+10)\,dx$$

$$= \frac{1}{2}x - \frac{1}{2}\left(\frac{\sin(4x+10)}{4}\right) + C$$

$$= \frac{1}{2}x - \frac{1}{8}\sin(4x+10) + C$$

**Question 2:**

$$\sin 3x \cos 4x$$

Answer

It is known that,
$$\sin A \cos B = \frac{1}{2}\{\sin(A+B)+\sin(A-B)\}$$

$$\therefore \int \sin 3x \cos 4x\,dx = \frac{1}{2}\int\{\sin(3x+4x)+\sin(3x-4x)\}\,dx$$

$$= \frac{1}{2}\int\{\sin 7x + \sin(-x)\}\,dx$$

$$= \frac{1}{2}\int\{\sin 7x - \sin x\}\,dx$$

$$= \frac{1}{2}\int \sin 7x\,dx - \frac{1}{2}\int \sin x\,dx$$

$$= \frac{1}{2}\left(\frac{-\cos 7x}{7}\right) - \frac{1}{2}(-\cos x) + C$$

$$= \frac{-\cos 7x}{14} + \frac{\cos x}{2} + C$$

**Question 3:**

cos 2$x$ cos 4$x$ cos 6$x$

Answer

It is known that, $\cos A \cos B = \dfrac{1}{2}\{\cos(A+B)+\cos(A-B)\}$

$$\therefore \int \cos 2x(\cos 4x \cos 6x)dx = \int \cos 2x \left[\frac{1}{2}\{\cos(4x+6x)+\cos(4x-6x)\}\right]dx$$

$$= \frac{1}{2}\int \{\cos 2x \cos 10x + \cos 2x \cos(-2x)\}\,dx$$

$$= \frac{1}{2}\int \{\cos 2x \cos 10x + \cos^2 2x\}\,dx$$

$$= \frac{1}{2}\int \left[\left\{\frac{1}{2}\cos(2x+10x)+\cos(2x-10x)\right\}+\left(\frac{1+\cos 4x}{2}\right)\right]dx$$

$$= \frac{1}{4}\int (\cos 12x + \cos 8x + 1 + \cos 4x)\,dx$$

$$= \frac{1}{4}\left[\frac{\sin 12x}{12}+\frac{\sin 8x}{8}+x+\frac{\sin 4x}{4}\right]+C$$

**Question 4:**

sin$^3$ (2$x$ + 1)

Answer

Let $I = \int \sin^3 (2x+1)$

$$\Rightarrow \int \sin^3 (2x+1)dx = \int \sin^2 (2x+1)\cdot \sin(2x+1)dx$$

$$= \int (1-\cos^2 (2x+1))\sin(2x+1)dx$$

Let $\cos(2x+1)=t$

$$\Rightarrow -2\sin(2x+1)dx = dt$$

$$\Rightarrow \sin(2x+1)dx = \frac{-dt}{2}$$

$$\Rightarrow I = \frac{-1}{2}\int \left(1 - t^2\right)dt$$

$$= \frac{-1}{2}\left\{t - \frac{t^3}{3}\right\}$$

$$= \frac{-1}{2}\left\{\cos\left(2x+1\right) - \frac{\cos^3\left(2x+1\right)}{3}\right\}$$

$$= \frac{-\cos\left(2x+1\right)}{2} + \frac{\cos^3\left(2x+1\right)}{6} + C$$

**Question 5:**

$\sin^3 x \cos^3 x$

Answer

$$\text{Let } I = \int \sin^3 x \cos^3 x \cdot dx$$

$$= \int \cos^3 x \cdot \sin^2 x \cdot \sin x \cdot dx$$

$$= \int \cos^3 x \left(1 - \cos^2 x\right)\sin x \cdot dx$$

Let $\cos x = t$

$$\Rightarrow -\sin x \cdot dx = dt$$

$$\Rightarrow I = -\int t^3 \left(1 - t^2\right)dt$$

$$= -\int \left(t^3 - t^5\right)dt$$

$$= -\left\{\frac{t^4}{4} - \frac{t^6}{6}\right\} + C$$

$$= -\left\{\frac{\cos^4 x}{4} - \frac{\cos^6 x}{6}\right\} + C$$

$$= \frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C$$

**Question 6:**

$\sin x \sin 2x \sin 3x$

Answer

It is known that,
$$\sin A \sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}$$

$$\therefore \int \sin x \sin 2x \sin 3x \, dx = \int \left[\sin x \cdot \frac{1}{2}\{\cos(2x-3x) - \cos(2x+3x)\}\right] dx$$

$$= \frac{1}{2}\int \left(\sin x \cos(-x) - \sin x \cos 5x\right) dx$$

$$= \frac{1}{2}\int \left(\sin x \cos x - \sin x \cos 5x\right) dx$$

$$= \frac{1}{2}\int \frac{\sin 2x}{2} \, dx - \frac{1}{2}\int \sin x \cos 5x \, dx$$

$$= \frac{1}{4}\left[\frac{-\cos 2x}{2}\right] - \frac{1}{2}\int \left\{\frac{1}{2}\sin(x+5x) + \sin(x-5x)\right\} dx$$

$$= \frac{-\cos 2x}{8} - \frac{1}{4}\int \left(\sin 6x + \sin(-4x)\right) dx$$

$$= \frac{-\cos 2x}{8} - \frac{1}{4}\left[\frac{-\cos 6x}{3} + \frac{\cos 4x}{4}\right] + C$$

$$= \frac{-\cos 2x}{8} - \frac{1}{8}\left[\frac{-\cos 6x}{3} + \frac{\cos 4x}{2}\right] + C$$

$$= \frac{1}{8}\left[\frac{\cos 6x}{3} - \frac{\cos 4x}{2} - \cos 2x\right] + C$$

**Question 7:**

sin 4$x$ sin 8$x$

Answer

It is known that,
$$\sin A \sin B = \frac{1}{2}\cos(A-B) - \cos(A+B)$$

$$\therefore \int \sin 4x \sin 8x \, dx = \int \left\{ \frac{1}{2} \cos \left( 4x - 8x \right) - \cos \left( 4x + 8x \right) \right\} dx$$

$$= \frac{1}{2} \int \left( \cos \left( -4x \right) - \cos 12x \right) dx$$

$$= \frac{1}{2} \int \left( \cos 4x - \cos 12x \right) dx$$

$$= \frac{1}{2} \left[ \frac{\sin 4x}{4} - \frac{\sin 12x}{12} \right]$$

**Question 8:**

$$\frac{1 - \cos x}{1 + \cos x}$$

Answer

$$\frac{1 - \cos x}{1 + \cos x} = \frac{2 \sin^2 \dfrac{x}{2}}{2 \cos^2 \dfrac{x}{2}} \qquad \left[ 2 \sin^2 \frac{x}{2} = 1 - \cos x \text{ and } 2 \cos^2 \frac{x}{2} = 1 + \cos x \right]$$

$$= \tan^2 \frac{x}{2}$$

$$= \left( \sec^2 \frac{x}{2} - 1 \right)$$

$$\therefore \int \frac{1 - \cos x}{1 + \cos x} dx = \int \left( \sec^2 \frac{x}{2} - 1 \right) dx$$

$$= \left[ \frac{\tan \dfrac{x}{2}}{\dfrac{1}{2}} - x \right] + C$$

$$= 2 \tan \frac{x}{2} - x + C$$

**Question 9:**

$$\frac{\cos x}{1 + \cos x}$$

Answer

$$\frac{\cos x}{1+\cos x} = \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}} \qquad \left[ \cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \text{ and } \cos x = 2\cos^2 \frac{x}{2} - 1 \right]$$

$$= \frac{1}{2}\left[ 1 - \tan^2 \frac{x}{2} \right]$$

$$\therefore \int \frac{\cos x}{1+\cos x} dx = \frac{1}{2}\int \left( 1 - \tan^2 \frac{x}{2} \right) dx$$

$$= \frac{1}{2}\int \left( 1 - \sec^2 \frac{x}{2} + 1 \right) dx$$

$$= \frac{1}{2}\int \left( 2 - \sec^2 \frac{x}{2} \right) dx$$

$$= \frac{1}{2}\left[ 2x - \frac{\tan \frac{x}{2}}{\frac{1}{2}} \right] + C$$

$$= x - \tan \frac{x}{2} + C$$

**Question 10:**

$\sin^4 x$

Answer

$$\sin^4 x = \sin^2 x \sin^2 x$$

$$= \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{4}(1-\cos 2x)^2$$

$$= \frac{1}{4}\left[1+\cos^2 2x - 2\cos 2x\right]$$

$$= \frac{1}{4}\left[1+\left(\frac{1+\cos 4x}{2}\right) - 2\cos 2x\right]$$

$$= \frac{1}{4}\left[1+\frac{1}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right]$$

$$= \frac{1}{4}\left[\frac{3}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right]$$

$$\therefore \int \sin^4 x \, dx = \frac{1}{4}\int\left[\frac{3}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right] dx$$

$$= \frac{1}{4}\left[\frac{3}{2}x+\frac{1}{2}\left(\frac{\sin 4x}{4}\right) - \frac{2\sin 2x}{2}\right] + C$$

$$= \frac{1}{8}\left[3x+\frac{\sin 4x}{4} - 2\sin 2x\right] + C$$

$$= \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

**Question 11:**

$\cos^4 2x$

Answer

$$\cos^4 2x = \left(\cos^2 2x\right)^2$$

$$= \left(\frac{1+\cos 4x}{2}\right)^2$$

$$= \frac{1}{4}\left[1 + \cos^2 4x + 2\cos 4x\right]$$

$$= \frac{1}{4}\left[1 + \left(\frac{1+\cos 8x}{2}\right) + 2\cos 4x\right]$$

$$= \frac{1}{4}\left[1 + \frac{1}{2} + \frac{\cos 8x}{2} + 2\cos 4x\right]$$

$$= \frac{1}{4}\left[\frac{3}{2} + \frac{\cos 8x}{2} + 2\cos 4x\right]$$

$$\therefore \int \cos^4 2x\, dx = \int \left(\frac{3}{8} + \frac{\cos 8x}{8} + \frac{\cos 4x}{2}\right) dx$$

$$= \frac{3}{8}x + \frac{\sin 8x}{64} + \frac{\sin 4x}{8} + C$$

**Question 12:**

$$\frac{\sin^2 x}{1+\cos x}$$

Answer

$$\frac{\sin^2 x}{1+\cos x} = \frac{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}} \qquad \left[\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2};\ \cos x = 2\cos^2\frac{x}{2} - 1\right]$$

$$= \frac{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

$$= 2\sin^2\frac{x}{2}$$

$$= 1 - \cos x$$

$$\therefore \int \frac{\sin^2 x}{1+\cos x}\, dx = \int (1 - \cos x)\, dx$$

$$= x - \sin x + C$$

**Question 13:**

$$\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$$

Answer

$$\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} = \frac{-2\sin\dfrac{2x+2\alpha}{2}\sin\dfrac{2x-2\alpha}{2}}{-2\sin\dfrac{x+\alpha}{2}\sin\dfrac{x-\alpha}{2}} \qquad \left[\cos C - \cos D = -2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}\right]$$

$$= \frac{\sin(x+\alpha)\sin(x-\alpha)}{\sin\left(\dfrac{x+\alpha}{2}\right)\sin\left(\dfrac{x-\alpha}{2}\right)}$$

$$= \frac{\left[2\sin\left(\dfrac{x+\alpha}{2}\right)\cos\left(\dfrac{x+\alpha}{2}\right)\right]\left[2\sin\left(\dfrac{x-\alpha}{2}\right)\cos\left(\dfrac{x-\alpha}{2}\right)\right]}{\sin\left(\dfrac{x+\alpha}{2}\right)\sin\left(\dfrac{x-\alpha}{2}\right)}$$

$$= 4\cos\left(\dfrac{x+\alpha}{2}\right)\cos\left(\dfrac{x-\alpha}{2}\right)$$

$$= 2\left[\cos\left(\dfrac{x+\alpha}{2}+\dfrac{x-\alpha}{2}\right)+\cos\dfrac{x+\alpha}{2}-\dfrac{x-\alpha}{2}\right]$$

$$= 2\left[\cos(x)+\cos\alpha\right]$$

$$= 2\cos x + 2\cos\alpha$$

$$\therefore \int\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}dx = \int 2\cos x + 2\cos\alpha$$

$$= 2\left[\sin x + x\cos\alpha\right] + C$$

**Question 14:**

$$\frac{\cos x - \sin x}{1 + \sin 2x}$$

Answer

$$\frac{\cos x - \sin x}{1 + \sin 2x} = \frac{\cos x - \sin x}{\left(\sin^2 x + \cos^2 x\right) + 2\sin x \cos x}$$

$$\left[\sin^2 x + \cos^2 x = 1; \ \sin 2x = 2\sin x \cos x\right]$$

$$= \frac{\cos x - \sin x}{\left(\sin x + \cos x\right)^2}$$

Let $\sin x + \cos x = t$

$\therefore \left(\cos x - \sin x\right)dx = dt$

$$\Rightarrow \int \frac{\cos x - \sin x}{1 + \sin 2x}dx = \int \frac{\cos x - \sin x}{\left(\sin x + \cos x\right)^2}dx$$

$$= \int \frac{dt}{t^2}$$

$$= \int t^{-2}dt$$

$$= -t^{-1} + C$$

$$= -\frac{1}{t} + C$$

$$= \frac{-1}{\sin x + \cos x} + C$$

**Question 15:**

$\tan^3 2x \sec 2x$

Answer

$$\tan^3 2x \sec 2x = \tan^2 2x \tan 2x \sec 2x$$
$$= \left(\sec^2 2x - 1\right) \tan 2x \sec 2x$$
$$= \sec^2 2x \cdot \tan 2x \sec 2x - \tan 2x \sec 2x$$
$$\therefore \int \tan^3 2x \sec 2x \, dx = \int \sec^2 2x \tan 2x \sec 2x \, dx - \int \tan 2x \sec 2x \, dx$$
$$= \int \sec^2 2x \tan 2x \sec 2x \, dx - \frac{\sec 2x}{2} + C$$

Let $\sec 2x = t$

$\therefore 2 \sec 2x \tan 2x \, dx = dt$

$$\therefore \int \tan^3 2x \sec 2x \, dx = \frac{1}{2} \int t^2 dt - \frac{\sec 2x}{2} + C$$
$$= \frac{t^3}{6} - \frac{\sec 2x}{2} + C$$
$$= \frac{\left(\sec 2x\right)^3}{6} - \frac{\sec 2x}{2} + C$$

**Question 16:**

$\tan^4 x$

Answer

$\tan^4 x$
$$= \tan^2 x \cdot \tan^2 x$$
$$= \left(\sec^2 x - 1\right) \tan^2 x$$
$$= \sec^2 x \tan^2 x - \tan^2 x$$
$$= \sec^2 x \tan^2 x - \left(\sec^2 x - 1\right)$$
$$= \sec^2 x \tan^2 x - \sec^2 x + 1$$

$$\therefore \int \tan^4 x \, dx = \int \sec^2 x \tan^2 x \, dx - \int \sec^2 x \, dx + \int 1 \cdot dx$$
$$= \int \sec^2 x \tan^2 x \, dx - \tan x + x + C \qquad \ldots(1)$$

Consider $\int \sec^2 x \tan^2 x \, dx$

Let $\tan x = t \Rightarrow \sec^2 x \, dx = dt$

$$\Rightarrow \int \sec^2 x \tan^2 x \, dx = \int t^2 dt = \frac{t^3}{3} = \frac{\tan^3 x}{3}$$

From equation (1), we obtain

$$\int \tan^4 x \, dx = \frac{1}{3} \tan^3 x - \tan x + x + C$$

**Question 17:**

$$\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$$

Answer

$$\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} = \frac{\sin^3 x}{\sin^2 x \cos^2 x} + \frac{\cos^3 x}{\sin^2 x \cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x}$$

$$= \tan x \sec x + \cot x \operatorname{cosec} x$$

$$\therefore \int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} \, dx = \int (\tan x \sec x + \cot x \operatorname{cosec} x) \, dx$$

$$= \sec x - \operatorname{cosec} x + C$$

**Question 18:**

$$\frac{\cos 2x + 2 \sin^2 x}{\cos^2 x}$$

Answer

$$\frac{\cos 2x + 2 \sin^2 x}{\cos^2 x}$$

$$= \frac{\cos 2x + (1 - \cos 2x)}{\cos^2 x} \qquad \left[\cos 2x = 1 - 2\sin^2 x\right]$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

$$\therefore \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} \, dx = \int \sec^2 x \, dx = \tan x + C$$

**Question 19:**

$$\frac{1}{\sin x \cos^3 x}$$

Answer

$$\frac{1}{\sin x \cos^3 x} = \frac{\sin^2 x + \cos^2 x}{\sin x \cos^3 x}$$

$$= \frac{\sin x}{\cos^3 x} + \frac{1}{\sin x \cos x}$$

$$= \tan x \sec^2 x + \frac{1 \cos^2 x}{\frac{\sin x \cos x}{\cos^2 x}}$$

$$= \tan x \sec^2 x + \frac{\sec^2 x}{\tan x}$$

$$\therefore \int \frac{1}{\sin x \cos^3 x} dx = \int \tan x \sec^2 x \; dx + \int \frac{\sec^2 x}{\tan x} \; dx$$

Let $\tan x = t \Rightarrow \sec^2 x \; dx = dt$

$$\Rightarrow \int \frac{1}{\sin x \cos^3 x} dx = \int t \, dt + \int \frac{1}{t} dt$$

$$= \frac{t^2}{2} + \log|t| + C$$

$$= \frac{1}{2} \tan^2 x + \log|\tan x| + C$$

**Question 20:**

$$\frac{\cos 2x}{(\cos x + \sin x)^2}$$

Answer

$$\frac{\cos 2x}{\left(\cos x + \sin x\right)^2} = \frac{\cos 2x}{\cos^2 x + \sin^2 x + 2\sin x \cos x} = \frac{\cos 2x}{1 + \sin 2x}$$

$$\therefore \int \frac{\cos 2x}{\left(\cos x + \sin x\right)^2}\, dx = \int \frac{\cos 2x}{\left(1 + \sin 2x\right)}\, dx$$

Let $1 + \sin 2x = t$

$\Rightarrow 2\cos 2x\, dx = dt$

$$\therefore \int \frac{\cos 2x}{\left(\cos x + \sin x\right)^2}\, dx = \frac{1}{2} \int \frac{1}{t}\, dt$$

$$= \frac{1}{2}\log|t| + C$$

$$= \frac{1}{2}\log|1 + \sin 2x| + C$$

$$= \frac{1}{2}\log\left|\left(\sin x + \cos x\right)^2\right| + C$$

$$= \log|\sin x + \cos x| + C$$

**Question 21:**

$\sin^{-1}(\cos x)$

Answer

$\sin^{-1}(\cos x)$

Let $\cos x = t$

Then, $\sin x = \sqrt{1 - t^2}$

$$\Rightarrow (-\sin x)\,dx = dt$$

$$dx = \frac{-dt}{\sin x}$$

$$dx = \frac{-dt}{\sqrt{1-t^2}}$$

$$\therefore \int \sin^{-1}(\cos x)\,dx = \int \sin^{-1} t \left( \frac{-dt}{\sqrt{1-t^2}} \right)$$

$$= -\int \frac{\sin^{-1} t}{\sqrt{1-t^2}}\,dt$$

Let $\sin^{-1} t = u$

$$\Rightarrow \frac{1}{\sqrt{1-t^2}}\,dt = du$$

$$\therefore \int \sin^{-1}(\cos x)\,dx = \int 4\,du$$

$$= -\frac{u^2}{2} + C$$

$$= \frac{-\left(\sin^{1} t\right)^2}{2} + C$$

$$= \frac{-\left[\sin^{-1}(\cos x)\right]^2}{2} + C \qquad \ldots(1)$$

It is known that,

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$\therefore \sin^{-1}(\cos x) = \frac{\pi}{2} - \cos^{-1}(\cos x) = \left( \frac{\pi}{2} - x \right)$$

Substituting in equation (1), we obtain

$$\int \sin^{-1}(\cos x)\, dx = \frac{-\left[\frac{\pi}{2} - x\right]^2}{2} + C$$

$$= -\frac{1}{2}\left(\frac{\pi^2}{2} + x^2 - \pi x\right) + C$$

$$= -\frac{\pi^2}{8} - \frac{x^2}{2} + \frac{1}{2}\pi x + C$$

$$= \frac{\pi x}{2} - \frac{x^2}{2} + \left(C - \frac{\pi^2}{8}\right)$$

$$= \frac{\pi x}{2} - \frac{x^2}{2} + C_1$$

**Question 22:**

$$\frac{1}{\cos(x-a)\cos(x-b)}$$

Answer

$$\frac{1}{\cos(x-a)\cos(x-b)} = \frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\sin\left[(x-b)-(x-a)\right]}{\cos(x-a)\cos(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}\left[\tan(x-b) - \tan(x-a)\right]$$

$$\Rightarrow \int \frac{1}{\cos(x-a)\cos(x-b)}\, dx = \frac{1}{\sin(a-b)}\int\left[\tan(x-b) - \tan(x-a)\right]dx$$

$$= \frac{1}{\sin(a-b)}\left[-\log|\cos(x-b)| + \log|\cos(x-a)|\right]$$

$$= \frac{1}{\sin(a-b)}\left[\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right] + C$$

**Question 23:**

$$\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}\, dx$$ is equal to

**A.** tan $x$ + cot $x$ + C

**B.** tan $x$ + cosec $x$ + C

**C.** − tan $x$ + cot $x$ + C

**D.** tan $x$ + sec $x$ + C

Answer

$$\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}\, dx = \int \left( \frac{\sin^2 x}{\sin^2 x \cos^2 x} - \frac{\cos^2 x}{\sin^2 x \cos^2 x} \right) dx$$

$$= \int \left( \sec^2 x - \operatorname{cosec}^2 x \right) dx$$

$$= \tan x + \cot x + C$$

Hence, the correct Answer is A.

**Question 24:**

$$\int \frac{e^x (1+x)}{\cos^2 (e^x x)}\, dx$$ equals

**A.** − cot ($ex^x$) + C

**B.** tan ($xe^x$) + C

**C.** tan ($e^x$) + C

**D.** cot ($e^x$) + C

Answer

$$\int \frac{e^x (1+x)}{\cos^2 (e^x x)}\, dx$$

Let $ex^x = t$

$$\Rightarrow \left(e^x \cdot x + e^x \cdot 1\right) dx = dt$$

$$e^x \left(x + 1\right) dx = dt$$

$$\therefore \int \frac{e^x \left(1 + x\right)}{\cos^2 \left(e^x x\right)} \, dx = \int \frac{dt}{\cos^2 t}$$

$$= \int \sec^2 t \, dt$$

$$= \tan t + C$$

$$= \tan \left(e^x \cdot x\right) + C$$

Hence, the correct Answer is B.

**Exercise 7.4**

**Question 1:**

$$\dfrac{3x^2}{x^6+1}$$

Answer

Let $x^3 = t$

$\therefore 3x^2\ dx = dt$

$$\Rightarrow \int \dfrac{3x^2}{x^6+1}\,dx = \int \dfrac{dt}{t^2+1}$$
$$= \tan^1 t + C$$
$$= \tan^{-1}\left(x^3\right) + C$$

**Question 2:**

$$\dfrac{1}{\sqrt{1+4x^2}}$$

Answer

Let $2x = t$

$\therefore 2dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{1+4x^2}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}}$$

$$= \frac{1}{2}\left[\log\left|t+\sqrt{t^2+1}\right|\right]+C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|\right]$$

$$= \frac{1}{2}\log\left|2x+\sqrt{4x^2+1}\right|+C$$

**Question 3:**

$$\frac{1}{\sqrt{(2-x)^2+1}}$$

Answer

Let $2 - x = t$

$$\Rightarrow -dx = dt$$

$$\Rightarrow \int \frac{1}{\sqrt{(2-x)^2+1}}dx = -\int \frac{1}{\sqrt{t^2+1}}dt$$

$$= -\log\left|t+\sqrt{t^2+1}\right|+C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|\right]$$

$$= -\log\left|2-x+\sqrt{(2-x)^2+1}\right|+C$$

$$= \log\left|\frac{1}{(2-x)+\sqrt{x^2-4x+5}}\right|+C$$

**Question 4:**

$$\frac{1}{\sqrt{9-25x^2}}$$

Answer

Let $5x = t$

$\therefore 5dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{9-25x^2}}\, dx = \frac{1}{5} \int \frac{1}{9-t^2}\, dt$$
$$= \frac{1}{5} \int \frac{1}{\sqrt{3^2 - t^2}}\, dt$$
$$= \frac{1}{5}\sin^{-1}\left(\frac{t}{3}\right) + C$$
$$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$$

**Question 5:**

$$\frac{3x}{1+2x^4}$$

Answer

$$\text{Let } \sqrt{2}x^2 = t$$
$$\therefore\ 2\sqrt{2}x\, dx = dt$$

$$\Rightarrow \int \frac{3x}{1+2x^4}dx = \frac{3}{2\sqrt{2}} \int \frac{dt}{1+t^2}$$
$$= \frac{3}{2\sqrt{2}}\left[\tan^{-1} t\right] + C$$
$$= \frac{3}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x^2\right) + C$$

**Question 6:**

$$\frac{x^2}{1-x^6}$$

Answer

Let $x^3 = t$

$\therefore 3x^2\ dx = dt$

$$\Rightarrow \int \frac{x^2}{1-x^6}dx = \frac{1}{3}\int \frac{dt}{1-t^2}$$

$$= \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|\right]+C$$

$$= \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right|+C$$

**Question 7:**

$$\frac{x-1}{\sqrt{x^2-1}}$$

Answer

$$\int \frac{x-1}{\sqrt{x^2-1}}\ dx = \int \frac{x}{\sqrt{x^2-1}}\ dx - \int \frac{1}{\sqrt{x^2-1}}\ dx \qquad \qquad ...(1)$$

For $\int \frac{x}{\sqrt{x^2-1}}dx$, let $x^2-1=t \Rightarrow 2x\ dx = dt$

$$\therefore \int \frac{x}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{2}\int t^{-\frac{1}{2}}dt$$

$$= \frac{1}{2}\left[2t^{\frac{1}{2}}\right]$$

$$= \sqrt{t}$$

$$= \sqrt{x^2-1}$$

From (1), we obtain

$$\int \frac{x-1}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx - \int \frac{1}{\sqrt{x^2-1}}dx$$

$$= \sqrt{x^2-1} - \log\left|x+\sqrt{x^2-1}\right| + C$$

$$\left[\int \frac{1}{\sqrt{x^2-a^2}}dt = \log\left|x+\sqrt{x^2-a^2}\right|\right]$$

**Question 8:**

$$\frac{x^2}{\sqrt{x^6+a^6}}$$

Answer

Let $x^3 = t$

$\Rightarrow 3x^2\ dx = dt$

$$\therefore \int \frac{x^2}{\sqrt{x^6+a^6}}dx = \frac{1}{3}\int \frac{dt}{\sqrt{t^2+\left(a^3\right)^2}}$$

$$= \frac{1}{3}\log\left|t+\sqrt{t^2+a^6}\right| + C$$

$$= \frac{1}{3}\log\left|x^3+\sqrt{x^6+a^6}\right| + C$$

**Question 9:**

$$\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$$

Answer

Let $\tan x = t$

$\therefore \sec^2 x\ dx = dt$

$$\Rightarrow \int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} dx = \int \frac{dt}{\sqrt{t^2 + 2^2}}$$

$$= \log \left| t + \sqrt{t^2 + 4} \right| + C$$

$$= \log \left| \tan x + \sqrt{\tan^2 x + 4} \right| + C$$

**Question 10:**

$$\frac{1}{\sqrt{x^2 + 2x + 2}}$$

Answer

$$\int \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \int \frac{1}{\sqrt{(x+1)^2 + (1)^2}} dx$$

Let $x + 1 = t$

$\therefore \quad dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \int \frac{1}{\sqrt{t^2 + 1}} dt$$

$$= \log \left| t + \sqrt{t^2 + 1} \right| + C$$

$$= \log \left| (x+1) + \sqrt{(x+1)^2 + 1} \right| + C$$

$$= \log \left| (x+1) + \sqrt{x^2 + 2x + 2} \right| + C$$

**Question 11:**

$$\frac{1}{\sqrt{9x^2 + 6x + 5}}$$

Answer

$$\int \frac{1}{9x^2 + 6x + 5} dx = \int \frac{1}{(3x+1)^2 + (2)^2} dx$$

Let $(3x+1) = t$

$\therefore \ 3dx = dt$

$$\Rightarrow \int \frac{1}{(3x+1)^2 + (2)^2} dx = \frac{1}{3} \int \frac{1}{t^2 + 2^2} dt$$

$$= \frac{1}{3} \left[ \frac{1}{2} \tan^{-1} \left( \frac{t}{2} \right) \right] + C$$

$$= \frac{1}{6} \tan^{-1} \left( \frac{3x+1}{2} \right) + C$$

**Question 12:**

$$\frac{1}{\sqrt{7 - 6x - x^2}}$$

Answer

$7 - 6x - x^2$ can be written as $7 - (x^2 + 6x + 9 - 9)$.

Therefore,

$7 - (x^2 + 6x + 9 - 9)$

$= 16 - (x^2 + 6x + 9)$

$= 16 - (x+3)^2$

$= (4)^2 - (x+3)^2$

$$\therefore \ \int \frac{1}{\sqrt{7 - 6x - x^2}} dx = \int \frac{1}{\sqrt{(4)^2 - (x+3)^2}} dx$$

Let $x + 3 = t$

$\Rightarrow dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{(4)^2 - (x+3)^2}} dx = \int \frac{1}{\sqrt{(4)^2 - (t)^2}} dt$$

$$= \sin^{-1} \left( \frac{t}{4} \right) + C$$

$$= \sin^{-1} \left( \frac{x+3}{4} \right) + C$$

**Question 13:**

$$\frac{1}{\sqrt{(x-1)(x-2)}}$$

Answer

$(x-1)(x-2)$ can be written as $x^2 - 3x + 2$.

Therefore,

$x^2 - 3x + 2$

$= x^2 - 3x + \dfrac{9}{4} - \dfrac{9}{4} + 2$

$= \left(x - \dfrac{3}{2}\right)^2 - \dfrac{1}{4}$

$= \left(x - \dfrac{3}{2}\right)^2 - \left(\dfrac{1}{2}\right)^2$

$\therefore \displaystyle\int \frac{1}{\sqrt{(x-1)(x-2)}}\, dx = \int \frac{1}{\sqrt{\left(x - \dfrac{3}{2}\right)^2 - \left(\dfrac{1}{2}\right)^2}}\, dx$

Let $x - \dfrac{3}{2} = t$

$\therefore dx = dt$

$\Rightarrow \displaystyle\int \frac{1}{\sqrt{\left(x - \dfrac{3}{2}\right)^2 - \left(\dfrac{1}{2}\right)^2}}\, dx = \int \frac{1}{\sqrt{t^2 - \left(\dfrac{1}{2}\right)^2}}\, dt$

$\qquad\qquad = \log\left| t + \sqrt{t^2 - \left(\dfrac{1}{2}\right)^2}\right| + C$

$\qquad\qquad = \log\left| \left(x - \dfrac{3}{2}\right) + \sqrt{x^2 - 3x + 2}\right| + C$

**Question 14:**

$$\frac{1}{\sqrt{8 + 3x - x^2}}$$

Answer

$8 + 3x - x^2$ can be written as $8 - \left( x^2 - 3x + \dfrac{9}{4} - \dfrac{9}{4} \right)$.

Therefore,

$8 - \left( x^2 - 3x + \dfrac{9}{4} - \dfrac{9}{4} \right)$

$= \dfrac{41}{4} - \left( x - \dfrac{3}{2} \right)^2$

$\Rightarrow \displaystyle\int \dfrac{1}{\sqrt{8 + 3x - x^2}}\, dx = \int \dfrac{1}{\sqrt{\dfrac{41}{4} - \left( x - \dfrac{3}{2} \right)^2}}\, dx$

Let $x - \dfrac{3}{2} = t$

$\therefore\ dx = dt$

$\Rightarrow \displaystyle\int \dfrac{1}{\sqrt{\dfrac{41}{4} - \left( x - \dfrac{3}{2} \right)^2}}\, dx = \int \dfrac{1}{\sqrt{\left( \dfrac{\sqrt{41}}{2} \right)^2 - t^2}}\, dt$

$= \sin^{-1}\left( \dfrac{t}{\dfrac{\sqrt{41}}{2}} \right) + C$

$= \sin^{-1}\left( \dfrac{x - \dfrac{3}{2}}{\dfrac{\sqrt{41}}{2}} \right) + C$

$= \sin^{-1}\left( \dfrac{2x - 3}{\sqrt{41}} \right) + C$

**Question 15:**

$\dfrac{1}{\sqrt{(x - a)(x - b)}}$

Answer

$(x-a)(x-b)$ can be written as $x^2 - (a+b)x + ab$.

Therefore,

$x^2 - (a+b)x + ab$

$= x^2 - (a+b)x + \dfrac{(a+b)^2}{4} - \dfrac{(a+b)^2}{4} + ab$

$= \left[ x - \left( \dfrac{a+b}{2} \right) \right]^2 - \dfrac{(a-b)^2}{4}$

$\Rightarrow \displaystyle\int \dfrac{1}{\sqrt{(x-a)(x-b)}} dx = \int \dfrac{1}{\sqrt{\left\{ x - \left( \dfrac{a+b}{2} \right) \right\}^2 - \left( \dfrac{a-b}{2} \right)^2}} dx$

Let $x - \left( \dfrac{a+b}{2} \right) = t$

$\therefore\ dx = dt$

$\Rightarrow \displaystyle\int \dfrac{1}{\sqrt{\left\{ x - \left( \dfrac{a+b}{2} \right) \right\}^2 - \left( \dfrac{a-b}{2} \right)^2}} dx = \int \dfrac{1}{\sqrt{t^2 - \left( \dfrac{a-b}{2} \right)^2}} dt$

$= \log \left| t + \sqrt{t^2 - \left( \dfrac{a-b}{2} \right)^2} \right| + C$

$= \log \left| \left\{ x - \left( \dfrac{a+b}{2} \right) \right\} + \sqrt{(x-a)(x-b)} \right| + C$

**Question 16:**

$\dfrac{4x+1}{\sqrt{2x^2 + x - 3}}$

Answer

Let $4x + 1 = A \dfrac{d}{dx} (2x^2 + x - 3) + B$

$\Rightarrow 4x + 1 = A(4x+1) + B$

$\Rightarrow 4x + 1 = 4Ax + A + B$

Equating the coefficients of *x* and constant term on both sides, we obtain

$4A = 4 \Rightarrow A = 1$

$A + B = 1 \Rightarrow B = 0$

Let $2x^2 + x - 3 = t$

$\therefore (4x + 1)\, dx = dt$

$$\Rightarrow \int \frac{4x+1}{\sqrt{2x^2 + x - 3}}\, dx = \int \frac{1}{\sqrt{t}}\, dt$$
$$= 2\sqrt{t} + C$$
$$= 2\sqrt{2x^2 + x - 3} + C$$

**Question 17:**

$$\frac{x+2}{\sqrt{x^2 - 1}}$$

Answer

$$\text{Let } x + 2 = A\frac{d}{dx}\left(x^2 - 1\right) + B \qquad \dots(1)$$
$$\Rightarrow x + 2 = A(2x) + B$$

Equating the coefficients of $x$ and constant term on both sides, we obtain

$$2A = 1 \Rightarrow A = \frac{1}{2}$$
$$B = 2$$

From (1), we obtain

$$(x+2) = \frac{1}{2}(2x) + 2$$

Then, $\int \frac{x+2}{\sqrt{x^2-1}} dx = \int \frac{\frac{1}{2}(2x)+2}{\sqrt{x^2-1}} dx$

$$= \frac{1}{2} \int \frac{2x}{\sqrt{x^2-1}} dx + \int \frac{2}{\sqrt{x^2-1}} dx \qquad ...(2)$$

In $\frac{1}{2} \int \frac{2x}{\sqrt{x^2-1}} dx$, let $x^2 - 1 = t \implies 2x\,dx = dt$

$$\frac{1}{2} \int \frac{2x}{\sqrt{x^2-1}} dx = \frac{1}{2} \int \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{2} \left[ 2\sqrt{t} \right]$$

$$= \sqrt{t}$$

$$= \sqrt{x^2-1}$$

Then, $\int \frac{2}{\sqrt{x^2-1}} dx = 2 \int \frac{1}{\sqrt{x^2-1}} dx = 2\log\left|x+\sqrt{x^2-1}\right|$

From equation (2), we obtain

$$\int \frac{x+2}{\sqrt{x^2-1}} dx = \sqrt{x^2-1} + 2\log\left|x+\sqrt{x^2-1}\right| + C$$

**Question 18:**

$$\frac{5x-2}{1+2x+3x^2}$$

Answer

Let $5x-2 = A\dfrac{d}{dx}\left(1+2x+3x^2\right) + B$

$\implies 5x-2 = A(2+6x) + B$

Equating the coefficient of *x* and constant term on both sides, we obtain

$$5 = 6A \Rightarrow A = \frac{5}{6}$$

$$2A + B = -2 \Rightarrow B = -\frac{11}{3}$$

$$\therefore 5x - 2 = \frac{5}{6}(2 + 6x) + \left(-\frac{11}{3}\right)$$

$$\Rightarrow \int \frac{5x - 2}{1 + 2x + 3x^2} dx = \int \frac{\frac{5}{6}(2 + 6x) - \frac{11}{3}}{1 + 2x + 3x^2} dx$$

$$= \frac{5}{6} \int \frac{2 + 6x}{1 + 2x + 3x^2} dx - \frac{11}{3} \int \frac{1}{1 + 2x + 3x^2} dx$$

Let $I_1 = \int \frac{2 + 6x}{1 + 2x + 3x^2} dx$ and $I_2 = \int \frac{1}{1 + 2x + 3x^2} dx$

$$\therefore \int \frac{5x - 2}{1 + 2x + 3x^2} dx = \frac{5}{6} I_1 - \frac{11}{3} I_2 \qquad \ldots(1)$$

$$I_1 = \int \frac{2 + 6x}{1 + 2x + 3x^2} dx$$

Let $1 + 2x + 3x^2 = t$

$$\Rightarrow (2 + 6x) dx = dt$$

$$\therefore I_1 = \int \frac{dt}{t}$$

$$I_1 = \log|t|$$

$$I_1 = \log|1 + 2x + 3x^2| \qquad \ldots(2)$$

$$I_2 = \int \frac{1}{1 + 2x + 3x^2} dx$$

$1+2x+3x^2$ can be written as $1+3\left(x^2+\dfrac{2}{3}x\right)$.

Therefore,

$$1+3\left(x^2+\frac{2}{3}x\right)$$

$$=1+3\left(x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}\right)$$

$$=1+3\left(x+\frac{1}{3}\right)^2-\frac{1}{3}$$

$$=\frac{2}{3}+3\left(x+\frac{1}{3}\right)^2$$

$$=3\left[\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right]$$

$$=3\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]$$

$$I_2=\frac{1}{3}\int\frac{1}{\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]}dx$$

$$=\frac{1}{3}\left[\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right)\right]$$

$$=\frac{1}{3}\left[\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) \qquad\qquad \ldots(3)$$

Substituting equations (2) and (3) in equation (1), we obtain

$$\int \frac{5x-2}{1+2x+3x^2}dx = \frac{5}{6}\Big[\log\big|1+2x+3x^2\big|\Big] - \frac{11}{3}\Big[\frac{1}{\sqrt{2}}\tan^{-1}\Big(\frac{3x+1}{\sqrt{2}}\Big)\Big] + C$$

$$= \frac{5}{6}\log\big|1+2x+3x^2\big| - \frac{11}{3\sqrt{2}}\tan^{-1}\Big(\frac{3x+1}{\sqrt{2}}\Big) + C$$

**Question 19:**

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}}$$

Answer

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}} = \frac{6x+7}{\sqrt{x^2-9x+20}}$$

$$\text{Let } 6x+7 = A\frac{d}{dx}\big(x^2-9x+20\big)+B$$

$$\Rightarrow 6x+7 = A(2x-9)+B$$

Equating the coefficients of *x* and constant term, we obtain

$2A = 6 \Rightarrow A = 3$

$-9A + B = 7 \Rightarrow B = 34$

$\therefore 6x + 7 = 3 (2x - 9) + 34$

$$\int \frac{6x+7}{\sqrt{x^2-9x+20}} = \int \frac{3(2x-9)+34}{\sqrt{x^2-9x+20}}dx$$

$$= 3\int \frac{2x-9}{\sqrt{x^2-9x+20}}dx + 34\int \frac{1}{\sqrt{x^2-9x+20}}dx$$

Let $I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2-9x+20}}dx$

$\therefore \int \frac{6x+7}{\sqrt{x^2-9x+20}} = 3I_1 + 34I_2$              ...(1)

Then,

$$I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}dx$$

Let $x^2-9x+20 = t$

$\Rightarrow (2x-9)dx = dt$

$\Rightarrow I_1 = \frac{dt}{\sqrt{t}}$

$I_1 = 2\sqrt{t}$

$I_1 = 2\sqrt{x^2-9x+20}$              ...(2)

and $I_2 = \int \frac{1}{\sqrt{x^2-9x+20}}dx$

$x^2 - 9x + 20$ can be written as $x^2 - 9x + 20 + \dfrac{81}{4} - \dfrac{81}{4}$.

Therefore,

$$x^2 - 9x + 20 + \frac{81}{4} - \frac{81}{4}$$

$$= \left(x - \frac{9}{2}\right)^2 - \frac{1}{4}$$

$$= \left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

$$\Rightarrow I_2 = \int \frac{1}{\sqrt{\left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx$$

$$I_2 = \log \left| \left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20} \right| \qquad \ldots(3)$$

Substituting equations (2) and (3) in (1), we obtain

$$\int \frac{6x + 7}{\sqrt{x^2 - 9x + 20}} dx = 3 \left[ 2\sqrt{x^2 - 9x + 20} \right] + 34 \log \left[ \left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20} \right] + C$$

$$= 6\sqrt{x^2 - 9x + 20} + 34 \log \left[ \left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20} \right] + C$$

**Question 20:**

$$\frac{x + 2}{\sqrt{4x - x^2}}$$

Answer

Let $x + 2 = A \dfrac{d}{dx}\left(4x - x^2\right) + B$

$\Rightarrow x + 2 = A(4 - 2x) + B$

Equating the coefficients of *x* and constant term on both sides, we obtain

$$-2A = 1 \Rightarrow A = -\frac{1}{2}$$

$$4A + B = 2 \Rightarrow B = 4$$

$$\Rightarrow (x+2) = -\frac{1}{2}(4-2x) + 4$$

$$\therefore \int \frac{x+2}{\sqrt{4x-x^2}}dx = \int \frac{-\frac{1}{2}(4-2x)+4}{\sqrt{4x-x^2}}dx$$

$$= -\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}dx + 4\int \frac{1}{\sqrt{4x-x^2}}dx$$

Let $I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx$ and $I_2 \int \frac{1}{\sqrt{4x-x^2}}dx$

$$\therefore \int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}I_1 + 4I_2 \qquad\qquad ...(1)$$

Then, $I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx$

Let $4x - x^2 = t$

$$\Rightarrow (4-2x)dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{4x-x^2} \qquad\qquad ...(2)$$

$$I_2 = \int \frac{1}{\sqrt{4x-x^2}}dx$$

$$\Rightarrow 4x - x^2 = -(-4x + x^2)$$

$$= (-4x + x^2 + 4 - 4)$$

$$= 4 - (x-2)^2$$

$$= (2)^2 - (x-2)^2$$

$$\therefore I_2 = \int \frac{1}{\sqrt{(2)^2 - (x-2)^2}}dx = \sin^{-1}\left(\frac{x-2}{2}\right) \qquad\qquad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{4x-x^2}}\,dx = -\frac{1}{2}\left(2\sqrt{4x-x^2}\right) + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

$$= -\sqrt{4x-x^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

**Question 21:**

$$\frac{x+2}{\sqrt{x^2+2x+3}}$$

Answer

$$\int \frac{(x+2)}{\sqrt{x^2+2x+3}}\,dx = \frac{1}{2}\int \frac{2(x+2)}{\sqrt{x^2+2x+3}}\,dx$$

$$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+2x+3}}\,dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}\,dx + \frac{1}{2}\int \frac{2}{\sqrt{x^2+2x+3}}\,dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}\,dx + \int \frac{1}{\sqrt{x^2+2x+3}}\,dx$$

Let $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}\,dx$ and $I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}\,dx$

$$\therefore \int \frac{x+2}{\sqrt{x^2+2x+3}}\,dx = \frac{1}{2}I_1 + I_2 \qquad \qquad ...(1)$$

Then, $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}\,dx$

Let $x^2 + 2x + 3 = t$

$\Rightarrow (2x + 2)\,dx = dt$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2 + 2x + 3} \qquad \qquad ...(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2 + 2x + 3}} dx$$

$$\Rightarrow x^2 + 2x + 3 = x^2 + 2x + 1 + 2 = (x+1)^2 + \left(\sqrt{2}\right)^2$$

$$\therefore I_2 = \int \frac{1}{\sqrt{(x+1)^2 + \left(\sqrt{2}\right)^2}} dx = \log\left|(x+1) + \sqrt{x^2 + 2x + 3}\right| \qquad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{x^2 + 2x + 3}} dx = \frac{1}{2}\left[2\sqrt{x^2 + 2x + 3}\right] + \log\left|(x+1) + \sqrt{x^2 + 2x + 3}\right| + C$$

$$= \sqrt{x^2 + 2x + 3} + \log\left|(x+1) + \sqrt{x^2 + 2x + 3}\right| + C$$

**Question 22:**

$$\frac{x+3}{x^2 - 2x - 5}$$

Answer

$$\text{Let } (x+3) = A\frac{d}{dx}\left(x^2 - 2x - 5\right) + B$$

$$(x+3) = A(2x - 2) + B$$

Equating the coefficients of *x* and constant term on both sides, we obtain

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

$$-2A + B = 3 \Rightarrow B = 4$$

$$\therefore (x+3) = \frac{1}{2}(2x - 2) + 4$$

$$\Rightarrow \int \frac{x+3}{x^2 - 2x - 5} dx = \int \frac{\frac{1}{2}(2x-2) + 4}{x^2 - 2x - 5} dx$$

$$= \frac{1}{2}\int \frac{2x - 2}{x^2 - 2x - 5} dx + 4\int \frac{1}{x^2 - 2x - 5} dx$$

Let $I_1 = \int \frac{2x-2}{x^2-2x-5} dx$ and $I_2 = \int \frac{1}{x^2-2x-5} dx$

$\therefore \int \frac{x+3}{\left(x^2-2x-5\right)} dx = \frac{1}{2} I_1 + 4 I_2$                    ...(1)

Then, $I_1 = \int \frac{2x-2}{x^2-2x-5} dx$

Let $x^2 - 2x - 5 = t$

$\Rightarrow (2x-2) dx = dt$

$\Rightarrow I_1 = \int \frac{dt}{t} = \log |t| = \log |x^2 - 2x - 5|$                    ...(2)

$I_2 = \int \frac{1}{x^2-2x-5} dx$

$= \int \frac{1}{\left(x^2-2x+1\right)-6} dx$

$= \int \frac{1}{(x-1)^2 + \left(\sqrt{6}\right)^2} dx$

$= \frac{1}{2\sqrt{6}} \log \left( \frac{x-1-\sqrt{6}}{x-1+\sqrt{6}} \right)$                    ...(3)

Substituting (2) and (3) in (1), we obtain

$\int \frac{x+3}{x^2-2x-5} dx = \frac{1}{2} \log |x^2-2x-5| + \frac{4}{2\sqrt{6}} \log \left| \frac{x-1-\sqrt{6}}{x-1+\sqrt{6}} \right| + C$

$= \frac{1}{2} \log |x^2-2x-5| + \frac{2}{\sqrt{6}} \log \left| \frac{x-1-\sqrt{6}}{x-1+\sqrt{6}} \right| + C$

**Question 23:**

$\frac{5x+3}{\sqrt{x^2+4x+10}}$

Answer

Let $5x + 3 = A\dfrac{d}{dx}\left(x^2 + 4x + 10\right) + B$

$\Rightarrow 5x + 3 = A(2x + 4) + B$

Equating the coefficients of $x$ and constant term, we obtain

$2A = 5 \Rightarrow A = \dfrac{5}{2}$

$4A + B = 3 \Rightarrow B = -7$

$\therefore 5x + 3 = \dfrac{5}{2}(2x + 4) - 7$

$\Rightarrow \displaystyle\int \dfrac{5x + 3}{\sqrt{x^2 + 4x + 10}}\,dx = \int \dfrac{\dfrac{5}{2}(2x + 4) - 7}{\sqrt{x^2 + 4x + 10}}\,dx$

$\qquad\qquad = \dfrac{5}{2}\displaystyle\int \dfrac{2x + 4}{\sqrt{x^2 + 4x + 10}}\,dx - 7\int \dfrac{1}{\sqrt{x^2 + 4x + 10}}\,dx$

Let $I_1 = \displaystyle\int \dfrac{2x + 4}{\sqrt{x^2 + 4x + 10}}\,dx$ and $I_2 = \int \dfrac{1}{\sqrt{x^2 + 4x + 10}}\,dx$

$\therefore \displaystyle\int \dfrac{5x + 3}{\sqrt{x^2 + 4x + 10}}\,dx = \dfrac{5}{2}I_1 - 7I_2$ $\qquad\qquad$ ...(1)

Then, $I_1 = \displaystyle\int \dfrac{2x + 4}{\sqrt{x^2 + 4x + 10}}\,dx$

Let $x^2 + 4x + 10 = t$

$\therefore (2x + 4)\,dx = dt$

$\Rightarrow I_1 = \displaystyle\int \dfrac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2 + 4x + 10}$ $\qquad\qquad$ ...(2)

$I_2 = \displaystyle\int \dfrac{1}{\sqrt{x^2 + 4x + 10}}\,dx$

$\quad = \displaystyle\int \dfrac{1}{\sqrt{\left(x^2 + 4x + 4\right) + 6}}\,dx$

$\quad = \displaystyle\int \dfrac{1}{\sqrt{(x + 2)^2 + \left(\sqrt{6}\right)^2}}\,dx$

$\quad = \log\left|(x + 2)\sqrt{x^2 + 4x + 10}\right|$ $\qquad\qquad$ ...(3)

Using equations (2) and (3) in (1), we obtain

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}\left[2\sqrt{x^2+4x+10}\right] - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$

$$= 5\sqrt{x^2+4x+10} - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$

**Question 24:**

$$\int \frac{dx}{x^2+2x+2}$$ equals

**A.** $x \tan^{-1}(x+1) + C$

**B.** $\tan^{-1}(x+1) + C$

**C.** $(x+1)\tan^{-1}x + C$

**D.** $\tan^{-1}x + C$

Answer

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x^2+2x+1)+1}$$

$$= \int \frac{1}{(x+1)^2+(1)^2}dx$$

$$= \left[\tan^{-1}(x+1)\right] + C$$

Hence, the correct Answer is B.

**Question 25:**

$$\int \frac{dx}{\sqrt{9x-4x^2}}$$ equals

**A.** $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

**B.** $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$

**C.** $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

**D.** $\dfrac{1}{2}\sin^{-1}\left(\dfrac{9x-8}{9}\right)+C$

Answer

$$\int \frac{dx}{\sqrt{9x-4x^2}}$$

$$= \int \frac{1}{\sqrt{-4\left(x^2-\dfrac{9}{4}x\right)}}dx$$

$$= \int \frac{1}{-4\left(x^2-\dfrac{9}{4}x+\dfrac{81}{64}-\dfrac{81}{64}\right)}dx$$

$$= \int \frac{1}{\sqrt{-4\left[\left(x-\dfrac{9}{8}\right)^2-\left(\dfrac{9}{8}\right)^2\right]}}dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\left(\dfrac{9}{8}\right)^2-\left(x-\dfrac{9}{8}\right)^2}}dx$$

$$= \frac{1}{2}\left[\sin^{-1}\left(\frac{x-\dfrac{9}{8}}{\dfrac{9}{8}}\right)\right]+C \qquad\qquad \left(\int \frac{dy}{\sqrt{a^2-y^2}}=\sin^{-1}\frac{y}{a}+C\right)$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$$

Hence, the correct Answer is B.

**Exercise 7.5**

**Question 1:**

$$\frac{x}{(x+1)(x+2)}$$

Answer

Let $\frac{x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$

$\Rightarrow x = A(x+2) + B(x+1)$

Equating the coefficients of *x* and constant term, we obtain

$A + B = 1$

$2A + B = 0$

On solving, we obtain

$A = -1$ and $B = 2$

$\therefore \frac{x}{(x+1)(x+2)} = \frac{-1}{(x+1)} + \frac{2}{(x+2)}$

$\Rightarrow \int \frac{x}{(x+1)(x+2)} dx = \int \frac{-1}{(x+1)} + \frac{2}{(x+2)} dx$

$= -\log|x+1| + 2\log|x+2| + C$

$= \log(x+2)^2 - \log|x+1| + C$

$= \log \frac{(x+2)^2}{(x+1)} + C$

**Question 2:**

$$\frac{1}{x^2-9}$$

Answer

Let $\frac{1}{(x+3)(x-3)} = \frac{A}{(x+3)} + \frac{B}{(x-3)}$

$1 = A(x-3) + B(x+3)$

Equating the coefficients of $x$ and constant term, we obtain

$A + B = 0$

$-3A + 3B = 1$

On solving, we obtain

$$A = -\frac{1}{6} \text{ and } B = \frac{1}{6}$$

$$\therefore \frac{1}{(x+3)(x-3)} = \frac{-1}{6(x+3)} + \frac{1}{6(x-3)}$$

$$\Rightarrow \int \frac{1}{(x^2-9)}dx = \int \left( \frac{-1}{6(x+3)} + \frac{1}{6(x-3)} \right) dx$$

$$= -\frac{1}{6}\log|x+3| + \frac{1}{6}\log|x-3| + C$$

$$= \frac{1}{6}\log \left| \frac{(x-3)}{(x+3)} \right| + C$$

**Question 3:**

$$\frac{3x-1}{(x-1)(x-2)(x-3)}$$

Answer

Let $\dfrac{3x-1}{(x-1)(x-2)(x-3)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-2)} + \dfrac{C}{(x-3)}$

$$3x-1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \qquad \dots(1)$$

Substituting $x = 1$, 2, and 3 respectively in equation (1), we obtain

$A = 1$, $B = -5$, and $C = 4$

$$\therefore \frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{4}{(x-3)}$$

$$\Rightarrow \int \frac{3x-1}{(x-1)(x-2)(x-3)}dx = \int \left\{ \frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{4}{(x-3)} \right\} dx$$

$$= \log|x-1| - 5\log|x-2| + 4\log|x-3| + C$$

**Question 4:**

$$\frac{x}{(x-1)(x-2)(x-3)}$$

Answer

Let $$\frac{x}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

$$x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \qquad \ldots(1)$$

Substituting $x = 1$, 2, and 3 respectively in equation (1), we obtain

$$A = \frac{1}{2}, \ B = -2, \ \text{and} \ C = \frac{3}{2}$$

$$\therefore \frac{x}{(x-1)(x-2)(x-3)} = \frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)}$$

$$\Rightarrow \int \frac{x}{(x-1)(x-2)(x-3)}\,dx = \int \left\{ \frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)} \right\} dx$$

$$= \frac{1}{2}\log|x-1| - 2\log|x-2| + \frac{3}{2}\log|x-3| + C$$

**Question 5:**

$$\frac{2x}{x^2 + 3x + 2}$$

Answer

Let $$\frac{2x}{x^2 + 3x + 2} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$$

$$2x = A(x+2) + B(x+1) \qquad \ldots(1)$$

Substituting $x = -1$ and $-2$ in equation (1), we obtain

$A = -2$ and $B = 4$

$$\therefore \frac{2x}{(x+1)(x+2)} = \frac{-2}{(x+1)} + \frac{4}{(x+2)}$$

$$\Rightarrow \int \frac{2x}{(x+1)(x+2)} dx = \int \left\{ \frac{4}{(x+2)} - \frac{2}{(x+1)} \right\} dx$$

$$= 4\log|x+2| - 2\log|x+1| + C$$

**Question 6:**

$$\frac{1-x^2}{x(1-2x)}$$

Answer

It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing $(1 - x^2)$ by $x(1 - 2x)$, we obtain

$$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{2}\left( \frac{2-x}{x(1-2x)} \right)$$

Let $$\frac{2-x}{x(1-2x)} = \frac{A}{x} + \frac{B}{(1-2x)}$$

$$\Rightarrow (2-x) = A(1-2x) + Bx \qquad\qquad ...(1)$$

Substituting $x = 0$ and $\dfrac{1}{2}$ in equation (1), we obtain

$A = 2$ and $B = 3$

$$\therefore \frac{2-x}{x(1-2x)} = \frac{2}{x} + \frac{3}{1-2x}$$

Substituting in equation (1), we obtain

$$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{2}\left\{\frac{2}{x} + \frac{3}{(1-2x)}\right\}$$

$$\Rightarrow \int \frac{1-x^2}{x(1-2x)} dx = \int \left\{\frac{1}{2} + \frac{1}{2}\left(\frac{2}{x} + \frac{3}{1-2x}\right)\right\} dx$$

$$= \frac{x}{2} + \log|x| + \frac{3}{2(-2)} \log|1-2x| + C$$

$$= \frac{x}{2} + \log|x| - \frac{3}{4} \log|1-2x| + C$$

**Question 7:**

$$\frac{x}{(x^2+1)(x-1)}$$

Answer

Let $\dfrac{x}{(x^2+1)(x-1)} = \dfrac{Ax+B}{(x^2+1)} + \dfrac{C}{(x-1)}$

$$x = (Ax+B)(x-1) + C(x^2+1)$$

$$x = Ax^2 - Ax + Bx - B + Cx^2 + C$$

Equating the coefficients of $x^2$, $x$, and constant term, we obtain

$A + C = 0$

$-A + B = 1$

$-B + C = 0$

On solving these equations, we obtain

$$A = -\frac{1}{2}, B = \frac{1}{2}, \text{ and } C = \frac{1}{2}$$

From equation (1), we obtain

$$\therefore \frac{x}{\left(x^2+1\right)\left(x-1\right)} = \frac{\left(-\frac{1}{2}x+\frac{1}{2}\right)}{x^2+1} + \frac{\frac{1}{2}}{\left(x-1\right)}$$

$$\Rightarrow \int \frac{x}{\left(x^2+1\right)\left(x-1\right)} = -\frac{1}{2}\int \frac{x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x-1}dx$$

$$= -\frac{1}{4}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + C$$

Consider $\int \frac{2x}{x^2+1}dx$, let $\left(x^2+1\right) = t \Rightarrow 2x\,dx = dt$

$$\Rightarrow \int \frac{2x}{x^2+1}dx = \int \frac{dt}{t} = \log|t| = \log|x^2+1|$$

$$\therefore \int \frac{x}{\left(x^2+1\right)\left(x-1\right)} = -\frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + C$$

$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + C$$

**Question 8:**

$$\frac{x}{\left(x-1\right)^2\left(x+2\right)}$$

Answer

Let $\dfrac{x}{\left(x-1\right)^2\left(x+2\right)} = \dfrac{A}{\left(x-1\right)} + \dfrac{B}{\left(x-1\right)^2} + \dfrac{C}{\left(x+2\right)}$

$$x = A\left(x-1\right)\left(x+2\right) + B\left(x+2\right) + C\left(x-1\right)^2$$

Substituting $x = 1$, we obtain

$$B = \frac{1}{3}$$

Equating the coefficients of $x^2$ and constant term, we obtain

$A + C = 0$

$-2A + 2B + C = 0$

On solving, we obtain

$$A = \frac{2}{9} \text{ and } C = \frac{-2}{9}$$

$$\therefore \frac{x}{(x-1)^2 (x+2)} = \frac{2}{9(x-1)} + \frac{1}{3(x-1)^2} - \frac{2}{9(x+2)}$$

$$\Rightarrow \int \frac{x}{(x-1)^2 (x+2)} dx = \frac{2}{9} \int \frac{1}{(x-1)} dx + \frac{1}{3} \int \frac{1}{(x-1)^2} dx - \frac{2}{9} \int \frac{1}{(x+2)} dx$$

$$= \frac{2}{9} \log|x-1| + \frac{1}{3} \left( \frac{-1}{x-1} \right) - \frac{2}{9} \log|x+2| + C$$

$$= \frac{2}{9} \log\left| \frac{x-1}{x+2} \right| - \frac{1}{3(x-1)} + C$$

**Question 9:**

$$\frac{3x+5}{x^3 - x^2 - x + 1}$$

Answer

$$\frac{3x+5}{x^3 - x^2 - x + 1} = \frac{3x+5}{(x-1)^2 (x+1)}$$

Let $\dfrac{3x+5}{(x-1)^2 (x+1)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-1)^2} + \dfrac{C}{(x+1)}$

$$3x+5 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

$$3x+5 = A(x^2 - 1) + B(x+1) + C(x^2 + 1 - 2x) \qquad ...(1)$$

Substituting $x = 1$ in equation (1), we obtain

$B = 4$

Equating the coefficients of $x^2$ and $x$, we obtain

$A + C = 0$

$B - 2C = 3$

On solving, we obtain

$$A = -\frac{1}{2} \text{ and } C = \frac{1}{2}$$

$$\therefore \frac{3x+5}{(x-1)^2(x+1)} = \frac{-1}{2(x-1)} + \frac{4}{(x-1)^2} + \frac{1}{2(x+1)}$$

$$\Rightarrow \int \frac{3x+5}{(x-1)^2(x+1)} dx = -\frac{1}{2} \int \frac{1}{x-1} dx + 4 \int \frac{1}{(x-1)^2} dx + \frac{1}{2} \int \frac{1}{(x+1)} dx$$

$$= -\frac{1}{2} \log|x-1| + 4\left(\frac{-1}{x-1}\right) + \frac{1}{2} \log|x+1| + C$$

$$= \frac{1}{2} \log\left|\frac{x+1}{x-1}\right| - \frac{4}{(x-1)} + C$$

**Question 10:**

$$\frac{2x-3}{(x^2-1)(2x+3)}$$

Answer

$$\frac{2x-3}{(x^2-1)(2x+3)} = \frac{2x-3}{(x+1)(x-1)(2x+3)}$$

Let $\dfrac{2x-3}{(x+1)(x-1)(2x+3)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x-1)} + \dfrac{C}{(2x+3)}$

$$\Rightarrow (2x-3) = A(x-1)(2x+3) + B(x+1)(2x+3) + C(x+1)(x-1)$$

$$\Rightarrow (2x-3) = A(2x^2+x-3) + B(2x^2+5x+3) + C(x^2-1)$$

$$\Rightarrow (2x-3) = (2A+2B+C)x^2 + (A+5B)x + (-3A+3B-C)$$

Equating the coefficients of $x^2$ and $x$, we obtain

$$B = -\frac{1}{10}, \ A = \frac{5}{2}, \text{ and } C = -\frac{24}{5}$$

$$\therefore \frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{5}{2(x+1)} - \frac{1}{10(x-1)} - \frac{24}{5(2x+3)}$$

$$\Rightarrow \int \frac{2x-3}{(x^2-1)(2x+3)} dx = \frac{5}{2}\int \frac{1}{(x+1)}dx - \frac{1}{10}\int \frac{1}{x-1}dx - \frac{24}{5}\int \frac{1}{(2x+3)}dx$$

$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{24}{5\times 2}\log|2x+3|$$

$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{12}{5}\log|2x+3| + C$$

**Question 11:**

$$\frac{5x}{(x+1)(x^2-4)}$$

Answer

$$\frac{5x}{(x+1)(x^2-4)} = \frac{5x}{(x+1)(x+2)(x-2)}$$

Let $\dfrac{5x}{(x+1)(x+2)(x-2)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+2)} + \dfrac{C}{(x-2)}$

$$5x = A(x+2)(x-2) + B(x+1)(x-2) + C(x+1)(x+2) \qquad \ldots(1)$$

Substituting $x = -1, -2$, and 2 respectively in equation (1), we obtain

$$A = \frac{5}{3}, \ B = -\frac{5}{2}, \text{ and } C = \frac{5}{6}$$

$$\therefore \frac{5x}{(x+1)(x+2)(x-2)} = \frac{5}{3(x+1)} - \frac{5}{2(x+2)} + \frac{5}{6(x-2)}$$

$$\Rightarrow \int \frac{5x}{(x+1)(x^2-4)}dx = \frac{5}{3}\int \frac{1}{(x+1)}dx - \frac{5}{2}\int \frac{1}{(x+2)}dx + \frac{5}{6}\int \frac{1}{(x-2)}dx$$

$$= \frac{5}{3}\log|x+1| - \frac{5}{2}\log|x+2| + \frac{5}{6}\log|x-2| + C$$

**Question 12:**

$$\frac{x^3+x+1}{x^2-1}$$

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Answer

It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing $(x^3 + x + 1)$ by $x^2 - 1$, we obtain

$$\frac{x^3 + x + 1}{x^2 - 1} = x + \frac{2x + 1}{x^2 - 1}$$

Let $\dfrac{2x + 1}{x^2 - 1} = \dfrac{A}{(x+1)} + \dfrac{B}{(x-1)}$

$$2x + 1 = A(x-1) + B(x+1) \qquad \dots(1)$$

Substituting $x = 1$ and $-1$ in equation (1), we obtain

$$A = \frac{1}{2} \text{ and } B = \frac{3}{2}$$

$$\therefore \frac{x^3 + x + 1}{x^2 - 1} = x + \frac{1}{2(x+1)} + \frac{3}{2(x-1)}$$

$$\Rightarrow \int \frac{x^3 + x + 1}{x^2 - 1} dx = \int x\, dx + \frac{1}{2} \int \frac{1}{(x+1)} dx + \frac{3}{2} \int \frac{1}{(x-1)} dx$$

$$= \frac{x^2}{2} + \frac{1}{2} \log|x+1| + \frac{3}{2} \log|x-1| + C$$

**Question 13:**

$$\frac{2}{(1-x)(1+x^2)}$$

Answer

Let $\dfrac{2}{(1-x)(1+x^2)} = \dfrac{A}{(1-x)} + \dfrac{Bx+C}{(1+x^2)}$

$$2 = A(1+x^2) + (Bx+C)(1-x)$$

$$2 = A + Ax^2 + Bx - Bx^2 + C - Cx$$

Equating the coefficient of $x^2$, $x$, and constant term, we obtain

$A - B = 0$

$B - C = 0$

$A + C = 2$

On solving these equations, we obtain

$A = 1$, $B = 1$, and $C = 1$

$$\therefore \frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$$

$$\Rightarrow \int \frac{2}{(1-x)(1+x^2)} dx = \int \frac{1}{1-x} dx + \int \frac{x}{1+x^2} dx + \int \frac{1}{1+x^2} dx$$

$$= -\int \frac{1}{x-1} dx + \frac{1}{2} \int \frac{2x}{1+x^2} dx + \int \frac{1}{1+x^2} dx$$

$$= -\log|x-1| + \frac{1}{2} \log|1+x^2| + \tan^{-1} x + C$$

**Question 14:**

$$\frac{3x-1}{(x+2)^2}$$

Answer

$$\text{Let } \frac{3x-1}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$$

$$\Rightarrow 3x-1 = A(x+2) + B$$

Equating the coefficient of $x$ and constant term, we obtain

$A = 3$

$2A + B = -1 \Rightarrow B = -7$

$$\therefore \frac{3x-1}{(x+2)^2} = \frac{3}{(x+2)} - \frac{7}{(x+2)^2}$$

$$\Rightarrow \int \frac{3x-1}{(x+2)^2} dx = 3\int \frac{1}{(x+2)} dx - 7\int \frac{x}{(x+2)^2} dx$$

$$= 3\log|x+2| - 7\left(\frac{-1}{(x+2)}\right) + C$$

$$= 3\log|x+2| + \frac{7}{(x+2)} + C$$

**Question 15:**

$$\frac{1}{x^4-1}$$

Answer

$$\frac{1}{(x^4-1)} = \frac{1}{(x^2-1)(x^2+1)} = \frac{1}{(x+1)(x-1)(1+x^2)}$$

$$\text{Let } \frac{1}{(x+1)(x-1)(1+x^2)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{Cx+D}{(x^2+1)}$$

$$1 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x^2-1)$$

$$1 = A(x^3+x-x^2-1) + B(x^3+x+x^2+1) + Cx^3 + Dx^2 - Cx - D$$

$$1 = (A+B+C)x^3 + (-A+B+D)x^2 + (A+B-C)x + (-A+B-D)$$

Equating the coefficient of $x^3$, $x^2$, $x$, and constant term, we obtain

$$A+B+C=0$$
$$-A+B+D=0$$
$$A+B-C=0$$
$$-A+B-D=1$$

On solving these equations, we obtain

$$A = -\frac{1}{4}, B = \frac{1}{4}, C = 0, \text{ and } D = -\frac{1}{2}$$

$$\therefore \frac{1}{x^4 - 1} = \frac{-1}{4(x+1)} + \frac{1}{4(x-1)} - \frac{1}{2(x^2+1)}$$

$$\Rightarrow \int \frac{1}{x^4 - 1} dx = -\frac{1}{4}\log|x-1| + \frac{1}{4}\log|x-1| - \frac{1}{2}\tan^{-1}x + C$$

$$= \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1}x + C$$

**Question 16:**

$\dfrac{1}{x(x^n + 1)}$ [Hint: multiply numerator and denominator by $x^{n-1}$ and put $x^n = t$]

Answer

$$\frac{1}{x(x^n + 1)}$$

Multiplying numerator and denominator by $x^{n-1}$, we obtain

$$\frac{1}{x(x^n + 1)} = \frac{x^{n-1}}{x^{n-1}x(x^n + 1)} = \frac{x^{n-1}}{x^n(x^n + 1)}$$

Let $x^n = t \Rightarrow x^{n-1}dx = dt$

$$\therefore \int \frac{1}{x(x^n + 1)} dx = \int \frac{x^{n-1}}{x^n(x^n + 1)} dx = \frac{1}{n}\int \frac{1}{t(t+1)} dt$$

Let $\dfrac{1}{t(t+1)} = \dfrac{A}{t} + \dfrac{B}{(t+1)}$

$$1 = A(1+t) + Bt \qquad\qquad \ldots(1)$$

Substituting $t = 0, -1$ in equation (1), we obtain

$A = 1$ and $B = -1$

$$\therefore \frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{(1+t)}$$

$$\Rightarrow \int \frac{1}{x(x^n+1)}dx = \frac{1}{n}\int\left\{\frac{1}{t}-\frac{1}{(t+1)}\right\}dx$$

$$= \frac{1}{n}\Big[\log|t|-\log|t+1|\Big]+C$$

$$= -\frac{1}{n}\Big[\log|x^n|-\log|x^n+1|\Big]+C$$

$$= \frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+C$$

**Question 17:**

$$\frac{\cos x}{(1-\sin x)(2-\sin x)}$$ [Hint: Put sin $x$ = $t$]

Answer

$$\frac{\cos x}{(1-\sin x)(2-\sin x)}$$

Let $\sin x = t \Rightarrow \cos x \, dx = dt$

$$\therefore \int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx = \int \frac{dt}{(1-t)(2-t)}$$

Let $$\frac{1}{(1-t)(2-t)} = \frac{A}{(1-t)}+\frac{B}{(2-t)}$$

$$1 = A(2-t)+B(1-t) \qquad\qquad ...(1)$$

Substituting $t$ = 2 and then $t$ = 1 in equation (1), we obtain

$A$ = 1 and $B$ = −1

$$\therefore \frac{1}{(1-t)(2-t)} = \frac{1}{(1-t)}-\frac{1}{(2-t)}$$

$$\Rightarrow \int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx = \int \left\{ \frac{1}{1-t} - \frac{1}{(2-t)} \right\}dt$$

$$= -\log|1-t| + \log|2-t| + C$$

$$= \log\left|\frac{2-t}{1-t}\right| + C$$

$$= \log\left|\frac{2-\sin x}{1-\sin x}\right| + C$$

**Question 18:**

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$$

Answer

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 - \frac{(4x^2+10)}{(x^2+3)(x^2+4)}$$

Let $\dfrac{4x^2+10}{(x^2+3)(x^2+4)} = \dfrac{Ax+B}{(x^2+3)} + \dfrac{Cx+D}{(x^2+4)}$

$$4x^2+10 = (Ax+B)(x^2+4) + (Cx+D)(x^2+3)$$

$$4x^2+10 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + 3Cx + Dx^2 + 3D$$

$$4x^2+10 = (A+C)x^3 + (B+D)x^2 + (4A+3C)x + (4B+3D)$$

Equating the coefficients of $x^3$, $x^2$, $x$, and constant term, we obtain

$A + C = 0$

$B + D = 4$

$4A + 3C = 0$

$4B + 3D = 10$

On solving these equations, we obtain

$A = 0$, $B = -2$, $C = 0$, and $D = 6$

$$\therefore \frac{4x^2+10}{(x^2+3)(x^2+4)} = \frac{-2}{(x^2+3)} + \frac{6}{(x^2+4)}$$

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 - \left( \frac{-2}{(x^2+3)} + \frac{6}{(x^2+4)} \right)$$

$$\Rightarrow \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}\, dx = \int \left\{ 1 + \frac{2}{(x^2+3)} - \frac{6}{(x^2+4)} \right\} dx$$

$$= \int \left\{ 1 + \frac{2}{x^2+\left(\sqrt{3}\right)^2} - \frac{6}{x^2+2^2} \right\}$$

$$= x + 2\left( \frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} \right) - 6\left( \frac{1}{2} \tan^{-1} \frac{x}{2} \right) + C$$

$$= x + \frac{2}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} - 3 \tan^{-1} \frac{x}{2} + C$$

**Question 19:**

$$\frac{2x}{(x^2+1)(x^2+3)}$$

Answer

$$\frac{2x}{(x^2+1)(x^2+3)}$$

Let $x^2 = t \Rightarrow 2x\, dx = dt$

$$\therefore \int \frac{2x}{(x^2+1)(x^2+3)}\, dx = \int \frac{dt}{(t+1)(t+3)} \qquad \dots (1)$$

$$\text{Let } \frac{1}{(t+1)(t+3)} = \frac{A}{(t+1)} + \frac{B}{(t+3)}$$

$$1 = A(t+3) + B(t+1) \qquad \dots (1)$$

Substituting $t = -3$ and $t = -1$ in equation (1), we obtain

$A = \dfrac{1}{2}$ and $B = -\dfrac{1}{2}$

$\therefore \dfrac{1}{(t+1)(t+3)} = \dfrac{1}{2(t+1)} - \dfrac{1}{2(t+3)}$

$\Rightarrow \displaystyle\int \dfrac{2x}{(x^2+1)(x^2+3)}\,dx = \int \left\{ \dfrac{1}{2(t+1)} - \dfrac{1}{2(t+3)} \right\} dt$

$\qquad\qquad\qquad\qquad = \dfrac{1}{2}\log\left|(t+1)\right| - \dfrac{1}{2}\log\left|t+3\right| + C$

$\qquad\qquad\qquad\qquad = \dfrac{1}{2}\log\left|\dfrac{t+1}{t+3}\right| + C$

$\qquad\qquad\qquad\qquad = \dfrac{1}{2}\log\left|\dfrac{x^2+1}{x^2+3}\right| + C$

**Question 20:**

$\dfrac{1}{x\left(x^4-1\right)}$

Answer

$\dfrac{1}{x\left(x^4-1\right)}$

Multiplying numerator and denominator by $x^3$, we obtain

$\dfrac{1}{x\left(x^4-1\right)} = \dfrac{x^3}{x^4\left(x^4-1\right)}$

$\therefore \displaystyle\int \dfrac{1}{x\left(x^4-1\right)}\,dx = \int \dfrac{x^3}{x^4\left(x^4-1\right)}\,dx$

Let $x^4 = t \Rightarrow 4x^3 dx = dt$

$\therefore \displaystyle\int \dfrac{1}{x\left(x^4-1\right)}\,dx = \dfrac{1}{4}\int \dfrac{dt}{t(t-1)}$

Let $\dfrac{1}{t(t-1)} = \dfrac{A}{t} + \dfrac{B}{(t-1)}$

$1 = A(t-1) + Bt$            ...(1)

Substituting $t = 0$ and 1 in (1), we obtain

$A = -1$ and $B = 1$

$\Rightarrow \dfrac{1}{t(t+1)} = \dfrac{-1}{t} + \dfrac{1}{t-1}$

$\Rightarrow \displaystyle\int \dfrac{1}{x(x^4-1)}dx = \dfrac{1}{4}\int \left\{\dfrac{-1}{t} + \dfrac{1}{t-1}\right\}dt$

$\qquad\qquad\qquad\quad = \dfrac{1}{4}\left[-\log|t| + \log|t-1|\right] + C$

$\qquad\qquad\qquad\quad = \dfrac{1}{4}\log\left|\dfrac{t-1}{t}\right| + C$

$\qquad\qquad\qquad\quad = \dfrac{1}{4}\log\left|\dfrac{x^4-1}{x^4}\right| + C$

**Question 21:**

$\dfrac{1}{(e^x-1)}$ [Hint: Put $e^x = t$]

Answer

$\dfrac{1}{(e^x-1)}$

Let $e^x = t \Rightarrow e^x\,dx = dt$

$\Rightarrow \displaystyle\int \dfrac{1}{e^x-1}dx = \int \dfrac{1}{t-1} \times \dfrac{dt}{t} = \int \dfrac{1}{t(t-1)}dt$

Let $\dfrac{1}{t(t-1)} = \dfrac{A}{t} + \dfrac{B}{t-1}$

$$1 = A(t-1) + Bt \qquad\qquad\qquad ...(1)$$

Substituting $t = 1$ and $t = 0$ in equation (1), we obtain

$A = -1$ and $B = 1$

$$\therefore \dfrac{1}{t(t-1)} = \dfrac{-1}{t} + \dfrac{1}{t-1}$$

$$\Rightarrow \int \dfrac{1}{t(t-1)}\, dt = \log\left|\dfrac{t-1}{t}\right| + C$$

$$= \log\left|\dfrac{e^x - 1}{e^x}\right| + C$$

**Question 22:**

$\displaystyle\int \dfrac{x\, dx}{(x-1)(x-2)}$ equals

A.  $\log\left|\dfrac{(x-1)^2}{x-2}\right| + C$

B.  $\log\left|\dfrac{(x-2)^2}{x-1}\right| + C$

C.  $\log\left|\left(\dfrac{x-1}{x-2}\right)^2\right| + C$

D.  $\log\left|(x-1)(x-2)\right| + C$

Answer

Let $\dfrac{x}{(x-1)(x-2)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-2)}$

$$x = A(x-2) + B(x-1) \qquad\qquad ...(1)$$

Substituting $x = 1$ and $2$ in (1), we obtain

$A = -1$ and $B = 2$

$$\therefore \frac{x}{(x-1)(x-2)} = -\frac{1}{(x-1)} + \frac{2}{(x-2)}$$

$$\Rightarrow \int \frac{x}{(x-1)(x-2)} dx = \int \left\{ \frac{-1}{(x-1)} + \frac{2}{(x-2)} \right\} dx$$

$$= -\log|x-1| + 2\log|x-2| + C$$

$$= \log \left| \frac{(x-2)^2}{x-1} \right| + C$$

Hence, the correct Answer is B.

**Question 23:**

$$\int \frac{dx}{x(x^2+1)} \text{ equals}$$

**A.** $\log|x| - \frac{1}{2}\log(x^2+1) + C$

**B.** $\log|x| + \frac{1}{2}\log(x^2+1) + C$

**C.** $-\log|x| + \frac{1}{2}\log(x^2+1) + C$

**D.** $\frac{1}{2}\log|x| + \log(x^2+1) + C$

Answer

$$\text{Let } \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$1 = A(x^2+1) + (Bx+C)x$$

Equating the coefficients of $x^2$, $x$, and constant term, we obtain

$A + B = 0$

$C = 0$

$A = 1$

On solving these equations, we obtain

$A = 1$, $B = -1$, and $C = 0$

$$\therefore \frac{1}{x\left(x^2+1\right)} = \frac{1}{x} + \frac{-x}{x^2+1}$$

$$\Rightarrow \int \frac{1}{x\left(x^2+1\right)} dx = \int \left\{ \frac{1}{x} - \frac{x}{x^2+1} \right\} dx$$

$$= \log|x| - \frac{1}{2}\log\left|x^2+1\right| + C$$

Hence, the correct Answer is A.

**Exercise 7.6**

**Question 1:**

$x \sin x$

Answer

Let $I = \int x \sin x \, dx$

Taking $x$ as first function and $\sin x$ as second function and integrating by parts, we obtain

$$I = x \int \sin x \, dx - \int \left\{ \left( \frac{d}{dx} x \right) \int \sin x \, dx \right\} dx$$

$$= x(-\cos x) - \int 1 \cdot (-\cos x) \, dx$$

$$= -x \cos x + \sin x + C$$

**Question 2:**

$x \sin 3x$

Answer

Let $I = \int x \sin 3x \, dx$

Taking $x$ as first function and $\sin 3x$ as second function and integrating by parts, we obtain

$$I = x \int \sin 3x \, dx - \int \left\{ \left( \frac{d}{dx} x \right) \int \sin 3x \, dx \right\}$$

$$= x \left( \frac{-\cos 3x}{3} \right) - \int 1 \cdot \left( \frac{-\cos 3x}{3} \right) dx$$

$$= \frac{-x \cos 3x}{3} + \frac{1}{3} \int \cos 3x \, dx$$

$$= \frac{-x \cos 3x}{3} + \frac{1}{9} \sin 3x + C$$

**Question 3:**

$x^2 e^x$

Answer

Let $I = \int x^2 e^x dx$

Taking $x^2$ as first function and $e^x$ as second function and integrating by parts, we obtain

$$I = x^2 \int e^x \, dx - \int \left\{ \left( \frac{d}{dx} x^2 \right) \int e^x dx \right\} dx$$

$$= x^2 e^x - \int 2x \cdot e^x dx$$

$$= x^2 e^x - 2 \int x \cdot e^x dx$$

Again integrating by parts, we obtain

$$= x^2 e^x - 2 \left[ x \cdot \int e^x dx - \int \left\{ \left( \frac{d}{dx} x \right) \cdot \int e^x dx \right\} dx \right]$$

$$= x^2 e^x - 2 \left[ xe^x - \int e^x dx \right]$$

$$= x^2 e^x - 2 \left[ xe^x - e^x \right]$$

$$= x^2 e^x - 2xe^x + 2e^x + C$$

$$= e^x \left( x^2 - 2x + 2 \right) + C$$

**Question 4:**

$x \log x$

Answer

Let $I = \int x \log x dx$

Taking log $x$ as first function and $x$ as second function and integrating by parts, we obtain

$$I = \log x \int x \, dx - \int \left\{ \left( \frac{d}{dx} \log x \right) \int x \, dx \right\} dx$$

$$= \log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx$$

$$= \frac{x^2 \log x}{2} - \int \frac{x}{2} dx$$

$$= \frac{x^2 \log x}{2} - \frac{x^2}{4} + C$$

**Question 5:**

*x* log 2*x*

Answer

Let $I = \int x \log 2x\, dx$

Taking log 2*x* as first function and *x* as second function and integrating by parts, we obtain

$$I = \log 2x \int x\, dx - \int\left\{\left(\frac{d}{dx} 2\log x\right) \int x\, dx\right\} dx$$

$$= \log 2x \cdot \frac{x^2}{2} - \int \frac{2}{2x} \cdot \frac{x^2}{2}\, dx$$

$$= \frac{x^2 \log 2x}{2} - \int \frac{x}{2}\, dx$$

$$= \frac{x^2 \log 2x}{2} - \frac{x^2}{4} + C$$

**Question 6:**

*x*$^2$ log *x*

Answer

Let $I = \int x^2 \log x\, dx$

Taking log *x* as first function and *x*$^2$ as second function and integrating by parts, we obtain

$$I = \log x \int x^2\, dx - \int\left\{\left(\frac{d}{dx} \log x\right) \int x^2 dx\right\} dx$$

$$= \log x \left(\frac{x^3}{3}\right) - \int \frac{1}{x} \cdot \frac{x^3}{3}\, dx$$

$$= \frac{x^3 \log x}{3} - \int \frac{x^2}{3}\, dx$$

$$= \frac{x^3 \log x}{3} - \frac{x^3}{9} + C$$

**Question 7:**

$x \sin^{-1} x$

Answer

Let $I = \int x \sin^{-1} x \, dx$

Taking $\sin^{-1} x$ as first function and $x$ as second function and integrating by parts, we obtain

$$I = \sin^{-1} x \int x \, dx - \int \left\{ \left( \frac{d}{dx} \sin^{-1} x \right) \int x \, dx \right\} dx$$

$$= \sin^{-1} x \left( \frac{x^2}{2} \right) - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} \, dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \frac{-x^2}{\sqrt{1-x^2}} \, dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \left\{ \frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \right\} dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \left\{ \sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}} \right\} dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \left\{ \int \sqrt{1-x^2} \, dx - \int \frac{1}{\sqrt{1-x^2}} \, dx \right\}$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \left\{ \frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x - \sin^{-1} x \right\} + C$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{x}{4} \sqrt{1-x^2} + \frac{1}{4} \sin^{-1} x - \frac{1}{2} \sin^{-1} x + C$$

$$= \frac{1}{4} \left( 2x^2 - 1 \right) \sin^{-1} x + \frac{x}{4} \sqrt{1-x^2} + C$$

**Question 8:**

$x \tan^{-1} x$

Answer

Let $I = \int x \tan^{-1} x \, dx$

Taking $\tan^{-1} x$ as first function and $x$ as second function and integrating by parts, we obtain

$$I = \tan^{-1} x \int x\, dx - \int \left\{ \left( \frac{d}{dx} \tan^{-1} x \right) \int x\, dx \right\} dx$$

$$= \tan^{-1} x \left( \frac{x^2}{2} \right) - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2} dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \left( \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2} \right) dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \left( 1 - \frac{1}{1+x^2} \right) dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \left( x - \tan^{-1} x \right) + C$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{x}{2} + \frac{1}{2} \tan^{-1} x + C$$

**Question 9:**

$$x \cos^{-1} x$$

Answer

Let $I = \int x \cos^{-1} x\, dx$

Taking $\cos^{-1} x$ as first function and $x$ as second function and integrating by parts, we obtain

$$I = \cos^{-1} x \int x \, dx - \int \left\{ \left( \frac{d}{dx} \cos^{-1} x \right) \int x \, dx \right\} dx$$

$$= \cos^{-1} x \frac{x^2}{2} - \int \frac{-1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} \, dx$$

$$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \frac{1-x^2-1}{\sqrt{1-x^2}} \, dx$$

$$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \left\{ \sqrt{1-x^2} + \left( \frac{-1}{\sqrt{1-x^2}} \right) \right\} dx$$

$$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \sqrt{1-x^2} \, dx - \frac{1}{2} \int \left( \frac{-1}{\sqrt{1-x^2}} \right) dx$$

$$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} I_1 - \frac{1}{2} \cos^{-1} x \qquad \ldots (1)$$

where, $I_1 = \int \sqrt{1-x^2} \, dx$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \int \frac{d}{dx} \sqrt{1-x^2} \int x \, dx$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \int \frac{-2x}{2\sqrt{1-x^2}} \cdot x \, dx$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}} \, dx$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}} \, dx$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \left\{ \int \sqrt{1-x^2} \, dx + \int \frac{-dx}{\sqrt{1-x^2}} \right\}$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \left\{ I_1 + \cos^{-1} x \right\}$$

$$\Rightarrow 2I_1 = x\sqrt{1-x^2} - \cos^{-1} x$$

$$\therefore I_1 = \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\cos^{-1} x$$

Substituting in $(1)$, we obtain

$$I = \frac{x\cos^{-1} x}{2} - \frac{1}{2}\left( \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\cos^{-1} x \right) - \frac{1}{2}\cos^{-1} x$$

$$= \frac{(2x^2-1)}{4}\cos^{-1} x - \frac{x}{4}\sqrt{1-x^2} + C$$

**Question 10:**

$$\left(\sin^{-1} x\right)^2$$

Answer

Let $I = \int \left(\sin^{-1} x\right)^2 \cdot 1 \, dx$

Taking $\left(\sin^{-1} x\right)^2$ as first function and 1 as second function and integrating by parts, we obtain

$$I = \left(\sin^{-1} x\right) \int 1 dx - \int \left\{ \frac{d}{dx}\left(\sin^{-1} x\right)^2 \cdot \int 1 \cdot dx \right\} dx$$

$$= \left(\sin^{-1} x\right)^2 \cdot x - \int \frac{2\sin^{-1} x}{\sqrt{1-x^2}} \cdot x \, dx$$

$$= x\left(\sin^{-1} x\right)^2 + \int \sin^{-1} x \cdot \left( \frac{-2x}{\sqrt{1-x^2}} \right) dx$$

$$= x\left(\sin^{-1} x\right)^2 + \left[ \sin^{-1} x \int \frac{-2x}{\sqrt{1-x^2}} \, dx - \int \left\{ \left( \frac{d}{dx}\sin^{-1} x \right) \int \frac{-2x}{\sqrt{1-x^2}} dx \right\} dx \right]$$

$$= x\left(\sin^{-1} x\right)^2 + \left[ \sin^{-1} x \cdot 2\sqrt{1-x^2} - \int \frac{1}{\sqrt{1-x^2}} \cdot 2\sqrt{1-x^2} \, dx \right]$$

$$= x\left(\sin^{-1} x\right)^2 + 2\sqrt{1-x^2} \, \sin^{-1} x - \int 2 \, dx$$

$$= x\left(\sin^{-1} x\right)^2 + 2\sqrt{1-x^2} \, \sin^{-1} x - 2x + C$$

**Question 11:**

$$\frac{x\cos^{-1} x}{\sqrt{1-x^2}}$$

Answer

Let $I = \int \frac{x\cos^{-1} x}{\sqrt{1-x^2}} dx$

$$I = \frac{-1}{2} \int \frac{-2x}{\sqrt{1-x^2}} \cdot \cos^{-1} x \, dx$$

Taking $\cos^{-1} x$ as first function and $\left( \frac{-2x}{\sqrt{1-x^2}} \right)$ as second function and integrating by parts, we obtain

$$I = \frac{-1}{2} \left[ \cos^{-1} x \int \frac{-2x}{\sqrt{1-x^2}} \, dx - \int \left\{ \left( \frac{d}{dx} \cos^{-1} x \right) \int \frac{-2x}{\sqrt{1-x^2}} \, dx \right\} dx \right]$$

$$= \frac{-1}{2} \left[ \cos^{-1} x \cdot 2\sqrt{1-x^2} - \int \frac{-1}{\sqrt{1-x^2}} \cdot 2\sqrt{1-x^2} \, dx \right]$$

$$= \frac{-1}{2} \left[ 2\sqrt{1-x^2} \cos^{-1} x + \int 2 \, dx \right]$$

$$= \frac{-1}{2} \left[ 2\sqrt{1-x^2} \cos^{-1} x + 2x \right] + C$$

$$= -\left[ \sqrt{1-x^2} \cos^{-1} x + x \right] + C$$

**Question 12:**

$$x \sec^2 x$$

Answer

Let $I = \int x \sec^2 x \, dx$

Taking $x$ as first function and $\sec^2 x$ as second function and integrating by parts, we obtain

$$I = x \int \sec^2 x \, dx - \int \left\{ \left\{ \frac{d}{dx} x \right\} \int \sec^2 x \, dx \right\} dx$$

$$= x \tan x - \int 1 \cdot \tan x \, dx$$

$$= x \tan x + \log |\cos x| + C$$

**Question 13:**

$$\tan^{-1} x$$

Answer

Let $I = \int 1 \cdot \tan^{-1} x \, dx$

Taking $\tan^{-1} x$ as first function and 1 as second function and integrating by parts, we obtain

$$I = \tan^{-1} x \int 1 \, dx - \int \left\{ \left( \frac{d}{dx} \tan^{-1} x \right) \int 1 \cdot dx \right\} dx$$

$$= \tan^{-1} x \cdot x - \int \frac{1}{1 + x^2} \cdot x \, dx$$

$$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1 + x^2} \, dx$$

$$= x \tan^{-1} x - \frac{1}{2} \log\left| 1 + x^2 \right| + C$$

$$= x \tan^{-1} x - \frac{1}{2} \log\left( 1 + x^2 \right) + C$$

**Question 14:**

$x \left( \log x \right)^2$

Answer

$$I = \int x \left( \log x \right)^2 dx$$

Taking $\left( \log x \right)^2$ as first function and 1 as second function and integrating by parts, we obtain

$$I = \left( \log x \right)^2 \int x \, dx - \int \left[ \left\{ \left( \frac{d}{dx} \log x \right)^2 \right\} \int x \, dx \right] dx$$

$$= \frac{x^2}{2} \left( \log x \right)^2 - \left[ \int 2 \log x \cdot \frac{1}{x} \cdot \frac{x^2}{2} \, dx \right]$$

$$= \frac{x^2}{2} \left( \log x \right)^2 - \int x \log x \, dx$$

Again integrating by parts, we obtain

$$I = \frac{x^2}{2}(\log x)^2 - \left[\log x \int x\,dx - \int\left\{\left(\frac{d}{dx}\log x\right)\int x\,dx\right\}dx\right]$$

$$= \frac{x^2}{2}(\log x)^2 - \left[\frac{x^2}{2}-\log x - \int\frac{1}{x}\cdot\frac{x^2}{2}dx\right]$$

$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{x^2}{4} + C$$

**Question 15:**

$$(x^2+1)\log x$$

Answer

Let $I = \int(x^2+1)\log x\,dx = \int x^2\log x\,dx + \int\log x\,dx$

Let $I = I_1 + I_2$ … (1)

Where, $I_1 = \int x^2\log x\,dx$ and $I_2 = \int\log x\,dx$

$$I_1 = \int x^2\log x\,dx$$

Taking log $x$ as first function and $x^2$ as second function and integrating by parts, we obtain

$$I_1 = \log x - \int x^2 dx - \int\left\{\left(\frac{d}{dx}\log x\right)\int x^2 dx\right\}dx$$

$$= \log x\cdot\frac{x^3}{3} - \int\frac{1}{x}\cdot\frac{x^3}{3}dx$$

$$= \frac{x^3}{3}\log x - \frac{1}{3}\left(\int x^2 dx\right)$$

$$= \frac{x^3}{3}\log x - \frac{x^3}{9} + C_1 \qquad\qquad …(2)$$

$$I_2 = \int\log x\,dx$$

Taking log $x$ as first function and 1 as second function and integrating by parts, we obtain

$$I_2 = \log x \int 1 \cdot dx - \int \left\{ \left( \frac{d}{dx} \log x \right) \int 1 \cdot dx \right\}$$

$$= \log x \cdot x - \int \frac{1}{x} \cdot x \, dx$$

$$= x \log x - \int 1 \, dx$$

$$= x \log x - x + C_2 \qquad \qquad \dots (3)$$

Using equations (2) and (3) in (1), we obtain

$$I = \frac{x^3}{3} \log x - \frac{x^3}{9} + C_1 + x \log x - x + C_2$$

$$= \frac{x^3}{3} \log x - \frac{x^3}{9} + x \log x - x + (C_1 + C_2)$$

$$= \left( \frac{x^3}{3} + x \right) \log x - \frac{x^3}{9} - x + C$$

**Question 16:**

$$e^x (\sin x + \cos x)$$

Answer

Let $I = \int e^x (\sin x + \cos x) \, dx$

Let $f(x) = \sin x$

☐ $f'(x) = \cos x$

☐ $I = \int e^x \{ f(x) + f'(x) \} \, dx$

It is known that, $\int e^x \{ f(x) + f'(x) \} \, dx = e^x f(x) + C$

$\therefore I = e^x \sin x + C$

**Question 17:**

$$\frac{xe^x}{(1+x)^2}$$

Answer

Let $$I = \int \frac{xe^x}{(1+x)^2} dx = \int e^x \left\{ \frac{x}{(1+x)^2} \right\} dx$$

$$= \int e^x \left\{ \frac{1+x-1}{(1+x)^2} \right\} dx$$

$$= \int e^x \left\{ \frac{1}{1+x} - \frac{1}{(1+x)^2} \right\} dx$$

Let $$f(x) = \frac{1}{1+x} \square \quad f'(x) = \frac{-1}{(1+x)^2}$$

$$\Rightarrow \int \frac{xe^x}{(1+x)^2} dx = \int e^x \{ f(x) + f'(x) \} dx$$

It is known that, $\int e^x \{ f(x) + f'(x) \} dx = e^x f(x) + C$

$$\therefore \int \frac{xe^x}{(1+x)^2} dx = \frac{e^x}{1+x} + C$$

**Question 18:**

$$e^x \left( \frac{1+\sin x}{1+\cos x} \right)$$

Answer

$$e^x \left( \frac{1+\sin x}{1+\cos x} \right)$$

$$= e^x \left( \frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2\sin \frac{x}{2} \cos \frac{x}{2}}{2\cos^2 \frac{x}{2}} \right)$$

$$= \frac{e^x \left( \sin \frac{x}{2} + \cos \frac{x}{2} \right)^2}{2\cos^2 \frac{x}{2}}$$

$$= \frac{1}{2} e^x \cdot \left( \frac{\sin \frac{x}{2} + \cos \frac{x}{2}}{\cos \frac{x}{2}} \right)^2$$

$$= \frac{1}{2} e^x \left[ \tan \frac{x}{2} + 1 \right]^2$$

$$= \frac{1}{2} e^2 \left( 1 + \tan \frac{x}{2} \right)^2$$

$$= \frac{1}{2} e^x \left[ 1 + \tan^2 \frac{x}{2} + 2\tan \frac{x}{2} \right]$$

$$= \frac{1}{2} e^x \left[ \sec^2 \frac{x}{2} + 2\tan \frac{x}{2} \right]$$

$$\frac{e^x (1+\sin x) dx}{(1+\cos x)} = e^x \left[ \frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \right] \qquad \qquad \dots(1)$$

Let $\tan \frac{x}{2} = f(x)$, $f'(x) = \frac{1}{2} \sec^2 \frac{x}{2}$

It is known that, $\int e^x \{ f(x) + f'(x) \} dx = e^x f(x) + C$

From equation (1), we obtain

$$\int \frac{e^x (1+\sin x)}{(1+\cos x)} dx = e^x \tan \frac{x}{2} + C$$

**Question 19:**

---

$$e^x \left( \frac{1}{x} - \frac{1}{x^2} \right)$$

Answer

Let $I = \int e^x \left[ \frac{1}{x} - \frac{1}{x^2} \right] dx$

Also, let $\frac{1}{x} = f(x)$ □ $f'(x) = \frac{-1}{x^2}$

It is known that, $\int e^x \left\{ f(x) + f'(x) \right\} dx = e^x f(x) + C$

$\therefore I = \frac{e^x}{x} + C$

**Question 20:**

$$\frac{(x-3)e^x}{(x-1)^3}$$

Answer

$\int e^x \left\{ \frac{x-3}{(x-1)^3} \right\} dx = \int e^x \left\{ \frac{x-1-2}{(x-1)^3} \right\} dx$

$$= \int e^x \left\{ \frac{1}{(x-1)^2} - \frac{2}{(x-1)^3} \right\} dx$$

Let $f(x) = \frac{1}{(x-1)^2}$ □ $f'(x) = \frac{-2}{(x-1)^3}$

It is known that, $\int e^x \left\{ f(x) + f'(x) \right\} dx = e^x f(x) + C$

$\therefore \int e^x \left\{ \frac{(x-3)}{(x-1)^2} \right\} dx = \frac{e^x}{(x-1)^2} + C$

**Question 21:**

$e^{2x} \sin x$

Answer

Let $I = \int e^{2x} \sin x \, dx$                                    ...(1)

Integrating by parts, we obtain

$$I = \sin x \int e^{2x} dx - \int \left\{ \left( \frac{d}{dx} \sin x \right) \int e^{2x} dx \right\} dx$$

$$\Rightarrow I = \sin x \cdot \frac{e^{2x}}{2} - \int \cos x \cdot \frac{e^{2x}}{2} dx$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} \int e^{2x} \cos x \, dx$$

Again integrating by parts, we obtain

$$I = \frac{e^{2x} \cdot \sin x}{2} - \frac{1}{2} \left[ \cos x \int e^{2x} dx - \int \left\{ \left( \frac{d}{dx} \cos x \right) \int e^{2x} dx \right\} dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} \left[ \cos x \cdot \frac{e^{2x}}{2} - \int (-\sin x) \frac{e^{2x}}{2} dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \cdot \sin x}{2} - \frac{1}{2} \left[ \frac{e^{2x} \cos x}{2} + \frac{1}{2} \int e^{2x} \sin x \, dx \right]$$

$$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} - \frac{1}{4} I \qquad \text{[From (1)]}$$

$$\Rightarrow I + \frac{1}{4} I = \frac{e^{2x} \cdot \sin x}{2} - \frac{e^{2x} \cos x}{4}$$

$$\Rightarrow \frac{5}{4} I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4}$$

$$\Rightarrow I = \frac{4}{5} \left[ \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} \right] + C$$

$$\Rightarrow I = \frac{e^{2x}}{5} [2 \sin x - \cos x] + C$$

**Question 22:**

$$\sin^{-1} \left( \frac{2x}{1 + x^2} \right)$$

Answer

Let $x = \tan \theta$  □  $dx = \sec^2 \theta \, d\theta$

$$\therefore \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta)$$
$$= 2\theta$$

$$\square \quad \int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int 2\theta \cdot \sec^2\theta\, d\theta = 2\int \theta \cdot \sec^2\theta\, d\theta$$

Integrating by parts, we obtain

$$2\left[\theta \cdot \int \sec^2\theta\, d\theta - \int\left\{\left(\frac{d}{d\theta}\theta\right)\int \sec^2\theta\, d\theta\right\}d\theta\right]$$

$$= 2\left[\theta \cdot \tan\theta - \int \tan\theta\, d\theta\right]$$

$$= 2\left[\theta \tan\theta + \log|\cos\theta|\right] + C$$

$$= 2\left[x\tan^{-1}x + \log\left|\frac{1}{\sqrt{1+x^2}}\right|\right] + C$$

$$= 2x\tan^{-1}x + 2\log\left(1+x^2\right)^{-\frac{1}{2}} + C$$

$$= 2x\tan^{-1}x + 2\left[-\frac{1}{2}\log\left(1+x^2\right)\right] + C$$

$$= 2x\tan^{-1}x - \log\left(1+x^2\right) + C$$

**Question 23:**

$\int x^2 e^{x^3}dx$ equals

(A) $\quad \frac{1}{3}e^{x^3} + C$ 

(B) $\quad \frac{1}{3}e^{x^2} + C$

(C) $\quad \frac{1}{2}e^{x^3} + C$ 

(D) $\quad \frac{1}{3}e^{x^2} + C$

Answer

Let $I = \int x^2 e^{x^3}dx$

Also, let $x^3 = t \,\square\, 3x^2 dx = dt$

$$\Rightarrow I = \frac{1}{3}\int e^t \, dt$$

$$= \frac{1}{3}\left(e^t\right) + C$$

$$= \frac{1}{3}e^{x^3} + C$$

Hence, the correct Answer is A.

**Question 24:**

$\int e^x \sec x \left(1 + \tan x\right) dx$ equals

$(A) \quad e^x \cos x + C$ $\qquad$ $(B) \quad e^x \sec x + C$

$(C) \quad e^x \sin x + C$ $\qquad$ $(D) \quad e^x \tan x + C$

Answer

$\int e^x \sec x \left(1 + \tan x\right) dx$

Let $I = \int e^x \sec x \left(1 + \tan x\right) dx = \int e^x \left(\sec x + \sec x \tan x\right) dx$

Also, let $\sec x = f(x) \;\square\; \sec x \tan x = f'(x)$

It is known that, $\int e^x \left\{ f(x) + f'(x) \right\} dx = e^x f(x) + C$

$\therefore I = e^x \sec x + C$

Hence, the correct Answer is B.

**Exercise 7.7**

**Question 1:**

$\sqrt{4-x^2}$

Answer

$\text{Let } I = \int \sqrt{4-x^2}\, dx = \int \sqrt{(2)^2 - (x)^2}\, dx$

$\text{It is known that, } \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2}\,\frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

$\therefore I = \frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2} + C$

$= \frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2} + C$

**Question 2:**

$\sqrt{1-4x^2}$

Answer

$\text{Let } I = \int \sqrt{1-4x^2}\, dx = \int \sqrt{(1)^2 - (2x)^2}\, dx$

$\text{Let } 2x = t \implies 2\, dx = dt$

$\therefore I = \frac{1}{2}\int \sqrt{(1)^2 - (t)^2}\, dt$

$\text{It is known that, } \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

$\Rightarrow I = \frac{1}{2}\left[\frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1}t\right] + C$

$= \frac{t}{4}\sqrt{1-t^2} + \frac{1}{4}\sin^{-1}t + C$

$= \frac{2x}{4}\sqrt{1-4x^2} + \frac{1}{4}\sin^{-1}2x + C$

$= \frac{x}{2}\sqrt{1-4x^2} + \frac{1}{4}\sin^{-1}2x + C$

**Question 3:**

$$\sqrt{x^2 + 4x + 6}$$

Answer

$$Let \ I = \int \sqrt{x^2 + 4x + 6} \ dx$$

$$= \int \sqrt{x^2 + 4x + 4 + 2} \ dx$$

$$= \int \sqrt{(x^2 + 4x + 4) + 2} \ dx$$

$$= \int \sqrt{(x+2)^2 + (\sqrt{2})^2} \ dx$$

It is known that, $\int \sqrt{x^2 + a^2} \, dx = \dfrac{x}{2}\sqrt{x^2 + a^2} + \dfrac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$

$$\therefore I = \frac{(x+2)}{2}\sqrt{x^2 + 4x + 6} + \frac{2}{2}\log\left|(x+2) + \sqrt{x^2 + 4x + 6}\right| + C$$

$$= \frac{(x+2)}{2}\sqrt{x^2 + 4x + 6} + \log\left|(x+2) + \sqrt{x^2 + 4x + 6}\right| + C$$

**Question 4:**

$$\sqrt{x^2 + 4x + 1}$$

Answer

$$Let \ I = \int \sqrt{x^2 + 4x + 1} \ dx$$

$$= \int \sqrt{(x^2 + 4x + 4) - 3} \ dx$$

$$= \int \sqrt{(x+2)^2 - (\sqrt{3})^2} \ dx$$

It is known that, $\int \sqrt{x^2 - a^2} \, dx = \dfrac{x}{2}\sqrt{x^2 - a^2} - \dfrac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$

$$\therefore I = \frac{(x+2)}{2}\sqrt{x^2 + 4x + 1} - \frac{3}{2}\log\left|(x+2) + \sqrt{x^2 + 4x + 1}\right| + C$$

**Question 5:**

$$\sqrt{1-4x-x^2}$$

Answer

$$\text{Let } I = \int \sqrt{1-4x-x^2}\, dx$$

$$= \int \sqrt{1-\left(x^2+4x+4-4\right)}\, dx$$

$$= \int \sqrt{1+4-\left(x+2\right)^2}\, dx$$

$$= \int \sqrt{\left(\sqrt{5}\right)^2-\left(x+2\right)^2}\, dx$$

It is known that, $\int \sqrt{a^2-x^2}\, dx = \dfrac{x}{2}\sqrt{a^2-x^2}+\dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}+C$

$$\therefore I = \frac{\left(x+2\right)}{2}\sqrt{1-4x-x^2}+\frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)+C$$

**Question 6:**

$$\sqrt{x^2+4x-5}$$

Answer

$$\text{Let } I = \int \sqrt{x^2+4x-5}\, dx$$

$$= \int \sqrt{\left(x^2+4x+4\right)-9}\, dx$$

$$= \int \sqrt{\left(x+2\right)^2-\left(3\right)^2}\, dx$$

It is known that, $\int \sqrt{x^2-a^2}\, dx = \dfrac{x}{2}\sqrt{x^2-a^2}-\dfrac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$$\therefore I = \frac{\left(x+2\right)}{2}\sqrt{x^2+4x-5}-\frac{9}{2}\log\left|\left(x+2\right)+\sqrt{x^2+4x-5}\right|+C$$

**Question 7:**

$$\sqrt{1+3x-x^2}$$

Answer

Let $I = \int \sqrt{1 + 3x - x^2} \, dx$

$= \int \sqrt{1 - \left( x^2 - 3x + \dfrac{9}{4} - \dfrac{9}{4} \right)} \, dx$

$= \int \sqrt{\left( 1 + \dfrac{9}{4} \right) - \left( x - \dfrac{3}{2} \right)^2} \, dx$

$= \int \sqrt{\left( \dfrac{\sqrt{13}}{2} \right)^2 - \left( x - \dfrac{3}{2} \right)^2} \, dx$

It is known that, $\int \sqrt{a^2 - x^2} \, dx = \dfrac{x}{2} \sqrt{a^2 - x^2} + \dfrac{a^2}{2} \sin^{-1} \dfrac{x}{a} + C$

$\therefore I = \dfrac{x - \dfrac{3}{2}}{2} \sqrt{1 + 3x - x^2} + \dfrac{13}{4 \times 2} \sin^{-1} \left( \dfrac{x - \dfrac{3}{2}}{\dfrac{\sqrt{13}}{2}} \right) + C$

$= \dfrac{2x - 3}{4} \sqrt{1 + 3x - x^2} + \dfrac{13}{8} \sin^{-1} \left( \dfrac{2x - 3}{\sqrt{13}} \right) + C$

**Question 8:**

$\sqrt{x^2 + 3x}$

Answer

Let $I = \int \sqrt{x^2 + 3x} \, dx$

$= \int \sqrt{x^2 + 3x + \dfrac{9}{4} - \dfrac{9}{4}} \, dx$

$= \int \sqrt{\left( x + \dfrac{3}{2} \right)^2 - \left( \dfrac{3}{2} \right)^2} \, dx$

It is known that, $\int \sqrt{x^2 - a^2} \, dx = \dfrac{x}{2}\sqrt{x^2 - a^2} - \dfrac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$

$$\therefore I = \dfrac{\left(x + \dfrac{3}{2}\right)}{2}\sqrt{x^2 + 3x} - \dfrac{\dfrac{9}{4}}{2}\log\left|\left(x + \dfrac{3}{2}\right) + \sqrt{x^2 + 3x}\right| + C$$

$$= \dfrac{(2x + 3)}{4}\sqrt{x^2 + 3x} - \dfrac{9}{8}\log\left|\left(x + \dfrac{3}{2}\right) + \sqrt{x^2 + 3x}\right| + C$$

**Question 9:**

$$\sqrt{1 + \dfrac{x^2}{9}}$$

Answer

$$\text{Let } I = \int\sqrt{1 + \dfrac{x^2}{9}}\,dx = \dfrac{1}{3}\int\sqrt{9 + x^2}\,dx = \dfrac{1}{3}\int\sqrt{(3)^2 + x^2}\,dx$$

It is known that, $\int \sqrt{x^2 + a^2}\,dx = \dfrac{x}{2}\sqrt{x^2 + a^2} + \dfrac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$

$$\therefore I = \dfrac{1}{3}\left[\dfrac{x}{2}\sqrt{x^2 + 9} + \dfrac{9}{2}\log\left|x + \sqrt{x^2 + 9}\right|\right] + C$$

$$= \dfrac{x}{6}\sqrt{x^2 + 9} + \dfrac{3}{2}\log\left|x + \sqrt{x^2 + 9}\right| + C$$

**Question 10:**

$$\int\sqrt{1 + x^2}\,dx \quad \text{is equal to}$$

A. $\dfrac{x}{2}\sqrt{1 + x^2} + \dfrac{1}{2}\log\left|x + \sqrt{1 + x^2}\right| + C$

B. $\dfrac{2}{3}\left(1 + x^2\right)^{\frac{2}{3}} + C$

C. $\dfrac{2}{3}x\left(1 + x^2\right)^{\frac{3}{2}} + C$

**D.** $\dfrac{x^2}{2}\sqrt{1+x^2}+\dfrac{1}{2}x^2\log\left|x+\sqrt{1+x^2}\right|+C$

Answer

It is known that, $\displaystyle\int\sqrt{a^2+x^2}\,dx=\dfrac{x}{2}\sqrt{a^2+x^2}+\dfrac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

$\therefore \displaystyle\int\sqrt{1+x^2}\,dx=\dfrac{x}{2}\sqrt{1+x^2}+\dfrac{1}{2}\log\left|x+\sqrt{1+x^2}\right|+C$

Hence, the correct Answer is A.

**Question 11:**

$\displaystyle\int\sqrt{x^2-8x+7}\,dx$ is equal to

**A.** $\dfrac{1}{2}(x-4)\sqrt{x^2-8x+7}+9\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

**B.** $\dfrac{1}{2}(x+4)\sqrt{x^2-8x+7}+9\log\left|x+4+\sqrt{x^2-8x+7}\right|+C$

**C.** $\dfrac{1}{2}(x-4)\sqrt{x^2-8x+7}-3\sqrt{2}\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

**D.** $\dfrac{1}{2}(x-4)\sqrt{x^2-8x+7}-\dfrac{9}{2}\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

Answer

Let $I=\displaystyle\int\sqrt{x^2-8x+7}\,dx$

$=\displaystyle\int\sqrt{(x^2-8x+16)-9}\,dx$

$=\displaystyle\int\sqrt{(x-4)^2-(3)^2}\,dx$

It is known that, $\displaystyle\int\sqrt{x^2-a^2}\,dx=\dfrac{x}{2}\sqrt{x^2-a^2}-\dfrac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$\therefore I=\dfrac{(x-4)}{2}\sqrt{x^2-8x+7}-\dfrac{9}{2}\log\left|(x-4)+\sqrt{x^2-8x+7}\right|+C$

Hence, the correct Answer is D.

Exercise 7.8

**Question 1:**

$$\int_a^b x \, dx$$

Answer

It is known that,

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\Big[f(a) + f(a+h) + \ldots + f(a+(n-1)h)\Big], \text{ where } h = \frac{b-a}{n}$$

Here, $a = a$, $b = b$, and $f(x) = x$

$$\therefore \int_a^b x \, dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\Big[a + (a+h)\ldots(a+2h)\ldots a+(n-1)h\Big]$$

$$= (b-a)\lim_{n\to\infty}\frac{1}{n}\Big[\underbrace{(a+a+a+\ldots+a)}_{n\ times} + \big(h+2h+3h+\ldots+(n-1)h\big)\Big]$$

$$= (b-a)\lim_{n\to\infty}\frac{1}{n}\Big[na + h\big(1+2+3+\ldots+(n-1)\big)\Big]$$

$$= (b-a)\lim_{n\to\infty}\frac{1}{n}\Big[na + h\Big\{\frac{(n-1)(n)}{2}\Big\}\Big]$$

$$= (b-a)\lim_{n\to\infty}\frac{1}{n}\Big[na + \frac{n(n-1)h}{2}\Big]$$

$$= (b-a)\lim_{n\to\infty}\frac{n}{n}\Big[a + \frac{(n-1)h}{2}\Big]$$

$$= (b-a)\lim_{n\to\infty}\Big[a + \frac{(n-1)h}{2}\Big]$$

$$= (b-a)\lim_{n\to\infty}\Big[a + \frac{(n-1)(b-a)}{2n}\Big]$$

$$= (b-a)\lim_{n\to\infty}\Big[a + \frac{\big(1-\frac{1}{n}\big)(b-a)}{2}\Big]$$

$$= (b-a)\Big[a + \frac{(b-a)}{2}\Big]$$

$$= (b-a)\Big[\frac{2a+b-a}{2}\Big]$$

$$= \frac{(b-a)(b+a)}{2}$$

$$= \frac{1}{2}\big(b^2 - a^2\big)$$

**Question 2:**

$$\int_0^5 (x+1)\, dx$$

Answer

Let $I = \int_0^5 (x+1)\, dx$

It is known that,

$$\int_a^b f(x)\, dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\big[f(a)+f(a+h)\ldots f(a+(n-1)h)\big], \text{ where } h=\frac{b-a}{n}$$

Here, $a=0$, $b=5$, and $f(x)=(x+1)$

$$\Rightarrow h=\frac{5-0}{n}=\frac{5}{n}$$

$$\therefore \int_0^5 (x+1)\, dx = (5-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f\left(\frac{5}{n}\right)+\ldots+f\left((n-1)\frac{5}{n}\right)\right]$$

$$= 5\lim_{n\to\infty}\frac{1}{n}\left[1+\left(\frac{5}{n}+1\right)+\ldots\left\{1+\left(\frac{5(n-1)}{n}\right)\right\}\right]$$

$$= 5\lim_{n\to\infty}\frac{1}{n}\left[\underbrace{(1+1+1\ldots1)}_{n\text{ times}}+\left[\frac{5}{n}+2\cdot\frac{5}{n}+3\cdot\frac{5}{n}+\ldots(n-1)\frac{5}{n}\right]\right]$$

$$= 5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}\{1+2+3\ldots(n-1)\}\right]$$

$$= 5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}\cdot\frac{(n-1)n}{2}\right]$$

$$= 5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5(n-1)}{2}\right]$$

$$= 5\lim_{n\to\infty}\left[1+\frac{5}{2}\left(1-\frac{1}{n}\right)\right]$$

$$= 5\left[1+\frac{5}{2}\right]$$

$$= 5\left[\frac{7}{2}\right]$$

$$= \frac{35}{2}$$

**Question 3:**

$$\int_2^3 x^2 dx$$

Answer

It is known that,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\Big[f(a)+f(a+h)+f(a+2h)...f\{a+(n-1)h\}\Big], \text{ where } h=\frac{b-a}{n}$$

Here, $a=2,\ b=3,$ and $f(x)=x^2$

$$\Rightarrow h=\frac{3-2}{n}=\frac{1}{n}$$

$$\therefore \int_2^3 x^2 dx = (3-2)\lim_{n\to\infty}\frac{1}{n}\left[f(2)+f\left(2+\frac{1}{n}\right)+f\left(2+\frac{2}{n}\right)...f\left\{2+(n-1)\frac{1}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[(2)^2+\left(2+\frac{1}{n}\right)^2+\left(2+\frac{2}{n}\right)^2+...\left(2+\frac{(n-1)}{n}\right)^2\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[2^2+\left\{2^2+\left(\frac{1}{n}\right)^2+2\cdot2\cdot\frac{1}{n}\right\}+...+\left\{(2)^2+\frac{(n-1)^2}{n^2}+2\cdot2\cdot\frac{(n-1)}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\underbrace{\left(2^2+...+2^2\right)}_{n\text{ times}}+\left\{\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+...+\left(\frac{n-1}{n}\right)^2\right\}+2\cdot2\cdot\left\{\frac{1}{n}+\frac{2}{n}+\frac{3}{n}+...+\frac{(n-1)}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{1}{n^2}\left\{1^2+2^2+3^2...+(n-1)^2\right\}+\frac{4}{n}\left\{1+2+...+(n-1)\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{1}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\}+\frac{4}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{n\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6}+\frac{4n-4}{2}\right]$$

$$=\lim_{n\to\infty}\left[4+\frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+2-\frac{2}{n}\right]$$

$$=4+\frac{2}{6}+2$$

$$=\frac{19}{3}$$

**Question 4:**

$$\int_1^4 \left(x^2 - x\right) dx$$

Answer

Let $I = \int_1^4 \left(x^2 - x\right) dx$

$$= \int_1^4 x^2 \, dx - \int_1^4 x \, dx$$

Let $I = I_1 - I_2$, where $I_1 = \int_1^4 x^2 dx$ and $I_2 = \int_1^4 x \, dx$ ...(1)

It is known that,

$$\int_a^b f(x) dx = (b-a) \lim_{n \to \infty} \frac{1}{n} \left[ f(a) + f(a+h) + f(a+(n-1)h) \right], \text{ where } h = \frac{b-a}{n}$$

For $I_1 = \int_1^4 x^2 dx$,

$a = 1, b = 4$, and $f(x) = x^2$

$$\therefore h = \frac{4-1}{n} = \frac{3}{n}$$

$$I_1 = \int_1^4 x^2 dx = (4-1) \lim_{n \to \infty} \frac{1}{n} \left[ f(1) + f(1+h) + \ldots + f(1+(n-1)h) \right]$$

$$= 3 \lim_{n \to \infty} \frac{1}{n} \left[ 1^2 + \left(1 + \frac{3}{n}\right)^2 + \left(1 + 2 \cdot \frac{3}{n}\right)^2 + \ldots + \left(1 + \frac{(n-1)3}{n}\right)^2 \right]$$

$$= 3 \lim_{n \to \infty} \frac{1}{n} \left[ 1^2 + \left\{ 1^2 + \left(\frac{3}{n}\right)^2 + 2 \cdot \frac{3}{n} \right\} + \ldots + \left\{ 1^2 + \left(\frac{(n-1)3}{n}\right)^2 + \frac{2 \cdot (n-1) \cdot 3}{n} \right\} \right]$$

$$= 3 \lim_{n \to \infty} \frac{1}{n} \left[ \underbrace{\left(1^2 + \ldots + 1^2\right)}_{n \text{ times}} + \left(\frac{3}{n}\right)^2 \left\{ 1^2 + 2^2 + \ldots + (n-1)^2 \right\} + 2 \cdot \frac{3}{n} \left\{ 1 + 2 + \ldots + (n-1) \right\} \right]$$

$$= 3 \lim_{n \to \infty} \frac{1}{n} \left[ n + \frac{9}{n^2} \left\{ \frac{(n-1)(n)(2n-1)}{6} \right\} + \frac{6}{n} \left\{ \frac{(n-1)(n)}{2} \right\} \right]$$

$$= 3 \lim_{n \to \infty} \frac{1}{n} \left[ n + \frac{9n}{6} \left( 1 - \frac{1}{n} \right) \left( 2 - \frac{1}{n} \right) + \frac{6n-6}{2} \right]$$

$$= 3 \lim_{n \to \infty} \left[ 1 + \frac{9}{6} \left( 1 - \frac{1}{n} \right) \left( 2 - \frac{1}{n} \right) + 3 - \frac{3}{n} \right]$$

$$= 3 \left[ 1 + 3 + 3 \right]$$

$$= 3 \left[ 7 \right]$$

$$I_1 = 21 \qquad \qquad \ldots(2)$$

For $I_2 = \int_1^4 x \, dx$,

$a = 1, b = 4,$ and $f(x) = x$

$$\Rightarrow h = \frac{4-1}{n} = \frac{3}{n}$$

$$\therefore I_2 = (4-1) \lim_{n \to \infty} \frac{1}{n} \left[ f(1) + f(1+h) + \ldots f(a + (n-1)h) \right]$$

$$= 3 \lim_{n \to \infty} \frac{1}{n} \left[ 1 + (1+h) + \ldots + (1 + (n-1)h) \right]$$

$$= 3 \lim_{n \to \infty} \frac{1}{n} \left[ 1 + \left( 1 + \frac{3}{n} \right) + \ldots + \left\{ 1 + (n-1) \frac{3}{n} \right\} \right]$$

$$= 3 \lim_{n \to \infty} \frac{1}{n} \left[ \underbrace{(1 + 1 + \ldots + 1)}_{n \text{ times}} + \frac{3}{n} (1 + 2 + \ldots + (n-1)) \right]$$

$$= 3 \lim_{n \to \infty} \frac{1}{n} \left[ n + \frac{3}{n} \left\{ \frac{(n-1)n}{2} \right\} \right]$$

$$= 3 \lim_{n \to \infty} \frac{1}{n} \left[ 1 + \frac{3}{2} \left( 1 - \frac{1}{n} \right) \right]$$

$$= 3 \left[ 1 + \frac{3}{2} \right]$$

$$= 3 \left[ \frac{5}{2} \right]$$

$$I_2 = \frac{15}{2} \qquad \qquad \ldots(3)$$

From equations (2) and (3), we obtain

$$I = I_1 + I_2 = 21 - \frac{15}{2} = \frac{27}{2}$$

**Question 5:**

$$\int_{-1}^{1} e^x dx$$

Answer

$$\text{Let } I = \int_{-1}^{1} e^x dx \qquad \qquad ...(1)$$

It is known that,

$$\int_{a}^{b} f(x) dx = (b-a) \lim_{n \to \infty} \frac{1}{n} \left[ f(a) + f(a+h) ... f(a+(n-1)h) \right], \text{ where } h = \frac{b-a}{n}$$

Here, $a = -1$, $b = 1$, and $f(x) = e^x$

$$\therefore h = \frac{1+1}{n} = \frac{2}{n}$$

$$\therefore I = (1+1)\lim_{n\to\infty}\frac{1}{n}\left[f(-1)+f\left(-1+\frac{2}{n}\right)+f\left(-1+2\cdot\frac{2}{n}\right)+...+f\left(-1+\frac{(n-1)2}{n}\right)\right]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}+e^{\left(-1+\frac{2}{n}\right)}+e^{\left(-1+2\cdot\frac{2}{n}\right)}+...e^{\left(-1+(n-1)\frac{2}{n}\right)}\right]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}\left\{1+e^{\frac{2}{n}}+e^{\frac{4}{n}}+e^{\frac{6}{n}}+e^{(n-1)\frac{2}{n}}\right\}\right]$$

$$= 2\lim_{n\to\infty}\frac{e^{-1}}{n}\left[\frac{e^{\frac{2n}{n}}-1}{e^{\frac{2}{n}}-1}\right]$$

$$= e^{-1}\times 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^2-1}{e^{\frac{2}{n}}-1}\right]$$

$$= \frac{e^{-1}\times 2\left(e^2-1\right)}{\lim_{\frac{2}{n}\to 0}\left(\frac{e^{\frac{2}{n}}-1}{\frac{2}{n}}\right)\times 2}$$

$$= e^{-1}\left[\frac{2\left(e^2-1\right)}{2}\right] \qquad\qquad \left[\lim_{h\to 0}\left(\frac{e^h-1}{h}\right)=1\right]$$

$$= \frac{e^2-1}{e}$$

$$= \left(e-\frac{1}{e}\right)$$

**Question 6:**

$$\int_0^4\left(x+e^{2x}\right)dx$$

Answer

It is known that,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+...+f(a+(n-1)h)\right], \text{ where } h=\frac{b-a}{n}$$

Here, $a=0$, $b=4$, and $f(x)=x+e^{2x}$

$$\therefore h = \frac{4-0}{n} = \frac{4}{n}$$

$$\Rightarrow \int_0^4 \left(x+e^{2x}\right)dx = (4-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f(h)+f(2h)+...+f\left((n-1)h\right)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\left(0+e^0\right)+\left(h+e^{2h}\right)+\left(2h+e^{2\cdot2h}\right)+...+\left\{(n-1)h+e^{2(n-1)h}\right\}\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[1+\left(h+e^{2h}\right)+\left(2h+e^{4h}\right)+...+\left\{(n-1)h+e^{2(n-1)h}\right\}\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\left\{h+2h+3h+...+(n-1)h\right\}+\left(1+e^{2h}+e^{4h}+...+e^{2(n-1)h}\right)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[h\left\{1+2+...(n-1)\right\}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\frac{\left(h(n-1)n\right)}{2}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\frac{4}{n}\cdot\frac{(n-1)n}{2}+\left(\frac{e^8-1}{e^{\frac{8}{n}}-1}\right)\right]$$

$$= 4(2)+4\lim_{n\to\infty}\frac{\left(e^8-1\right)}{\left(\dfrac{e^{\frac{8}{n}}-1}{\frac{8}{n}}\right)8}$$

$$= 8+\frac{4\cdot\left(e^8-1\right)}{8} \qquad\qquad \left(\lim_{x\to0}\frac{e^x-1}{x}=1\right)$$

$$= 8+\frac{e^8-1}{2}$$

$$= \frac{15+e^8}{2}$$

Exercise 7.9

**Question 1:**

$$\int_{-1}^{1} (x+1) dx$$

Answer

Let $I = \int_{-1}^{1} (x+1) dx$

$$\int (x+1)\, dx = \frac{x^2}{2} + x = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(1) - F(-1)$$

$$= \left(\frac{1}{2}+1\right) - \left(\frac{1}{2}-1\right)$$

$$= \frac{1}{2}+1-\frac{1}{2}+1$$

$$= 2$$

**Question 2:**

$$\int_{2}^{3} \frac{1}{x} dx$$

Answer

Let $I = \int_{2}^{3} \frac{1}{x}\, dx$

$$\int \frac{1}{x}\, dx = \log|x| = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(3) - F(2)$$

$$= \log|3| - \log|2| = \log\frac{3}{2}$$

**Question 3:**

$$\int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right)dx$$

Answer

$$\text{Let } I = \int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right)dx$$

$$\int \left(4x^3 - 5x^2 + 6x + 9\right)dx = 4\left(\frac{x^4}{4}\right) - 5\left(\frac{x^3}{3}\right) + 6\left(\frac{x^2}{2}\right) + 9(x)$$

$$= x^4 - \frac{5x^3}{3} + 3x^2 + 9x = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(2) - F(1)$$

$$I = \left\{2^4 - \frac{5 \cdot (2)^3}{3} + 3(2)^2 + 9(2)\right\} - \left\{(1)^4 - \frac{5(1)^3}{3} + 3(1)^2 + 9(1)\right\}$$

$$= \left(16 - \frac{40}{3} + 12 + 18\right) - \left(1 - \frac{5}{3} + 3 + 9\right)$$

$$= 16 - \frac{40}{3} + 12 + 18 - 1 + \frac{5}{3} - 3 - 9$$

$$= 33 - \frac{35}{3}$$

$$= \frac{99 - 35}{3}$$

$$= \frac{64}{3}$$

**Question 4:**

$$\int_0^{\frac{\pi}{4}} \sin 2x \, dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \sin 2x \, dx$$

$$\int \sin 2x \, dx = \left(\frac{-\cos 2x}{2}\right) = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F\left(\frac{\pi}{4}\right) - F(0)$$

$$= -\frac{1}{2}\left[\cos 2\left(\frac{\pi}{4}\right) - \cos 0\right]$$

$$= -\frac{1}{2}\left[\cos\left(\frac{\pi}{2}\right) - \cos 0\right]$$

$$= -\frac{1}{2}[0 - 1]$$

$$= \frac{1}{2}$$

**Question 5:**

$$\int_0^{\frac{\pi}{2}} \cos 2x \, dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \cos 2x \, dx$$

$$\int \cos 2x \, dx = \left(\frac{\sin 2x}{2}\right) = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F\left(\frac{\pi}{2}\right) - F(0)$$

$$= \frac{1}{2}\left[\sin 2\left(\frac{\pi}{2}\right) - \sin 0\right]$$

$$= \frac{1}{2}\left[\sin \pi - \sin 0\right]$$

$$= \frac{1}{2}[0 - 0] = 0$$

**Question 6:**

$$\int_4^5 e^x \, dx$$

Answer

Let $I = \int_4^5 e^x dx$

$\int e^x dx = e^x = F(x)$

By second fundamental theorem of calculus, we obtain

$I = F(5) - F(4)$

$= e^5 - e^4$

$= e^4(e-1)$

**Question 7:**

$\int_0^{\frac{\pi}{4}} \tan x \, dx$

Answer

Let $I = \int_0^{\frac{\pi}{4}} \tan x \, dx$

$\int \tan x \, dx = -\log|\cos x| = F(x)$

By second fundamental theorem of calculus, we obtain

$I = F\left(\frac{\pi}{4}\right) - F(0)$

$= -\log\left|\cos\frac{\pi}{4}\right| + \log|\cos 0|$

$= -\log\left|\frac{1}{\sqrt{2}}\right| + \log|1|$

$= -\log(2)^{-\frac{1}{2}}$

$= \frac{1}{2}\log 2$

**Question 8:**

$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos ec \, x \, dx$

Answer

Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos ec\, x\, dx$

$\int \cos ec\, x\, dx = \log|\cos ec\, x - \cot x| = F(x)$

By second fundamental theorem of calculus, we obtain

$I = F\left(\dfrac{\pi}{4}\right) - F\left(\dfrac{\pi}{6}\right)$

$= \log\left|\cos ec\, \dfrac{\pi}{4} - \cot \dfrac{\pi}{4}\right| - \log\left|\cos ec\, \dfrac{\pi}{6} - \cot \dfrac{\pi}{6}\right|$

$= \log\left|\sqrt{2} - 1\right| - \log\left|2 - \sqrt{3}\right|$

$= \log\left(\dfrac{\sqrt{2} - 1}{2 - \sqrt{3}}\right)$

**Question 9:**

$\int_{0}^{1} \dfrac{dx}{\sqrt{1 - x^2}}$

Answer

Let $I = \int_{0}^{1} \dfrac{dx}{\sqrt{1 - x^2}}$

$\int \dfrac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x = F(x)$

By second fundamental theorem of calculus, we obtain

$I = F(1) - F(0)$

$= \sin^{-1}(1) - \sin^{-1}(0)$

$= \dfrac{\pi}{2} - 0$

$= \dfrac{\pi}{2}$

**Question 10:**

$\int_{0}^{1} \dfrac{dx}{1 + x^2}$

Answer

Let $I = \int_0^1 \frac{dx}{1+x^2}$

$\int \frac{dx}{1+x^2} = \tan^{-1} x = F(x)$

By second fundamental theorem of calculus, we obtain

$I = F(1) - F(0)$

$= \tan^{-1}(1) - \tan^{-1}(0)$

$= \frac{\pi}{4}$

**Question 11:**

$\int_2^3 \frac{dx}{x^2-1}$

Answer

Let $I = \int_2^3 \frac{dx}{x^2-1}$

$\int \frac{dx}{x^2-1} = \frac{1}{2} \log\left|\frac{x-1}{x+1}\right| = F(x)$

By second fundamental theorem of calculus, we obtain

$I = F(3) - F(2)$

$= \frac{1}{2}\left[\log\left|\frac{3-1}{3+1}\right| - \log\left|\frac{2-1}{2+1}\right|\right]$

$= \frac{1}{2}\left[\log\left|\frac{2}{4}\right| - \log\left|\frac{1}{3}\right|\right]$

$= \frac{1}{2}\left[\log\frac{1}{2} - \log\frac{1}{3}\right]$

$= \frac{1}{2}\left[\log\frac{3}{2}\right]$

**Question 12:**

$$\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$$

**Answer**

Let $I = \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

$$\int \cos^2 x \, dx = \int \left( \frac{1 + \cos 2x}{2} \right) dx = \frac{x}{2} + \frac{\sin 2x}{4} = \frac{1}{2} \left( x + \frac{\sin 2x}{2} \right) = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = \left[ F\left( \frac{\pi}{2} \right) - F(0) \right]$$

$$= \frac{1}{2} \left[ \left( \frac{\pi}{2} - \frac{\sin \pi}{2} \right) - \left( 0 + \frac{\sin 0}{2} \right) \right]$$

$$= \frac{1}{2} \left[ \frac{\pi}{2} + 0 - 0 - 0 \right]$$

$$= \frac{\pi}{4}$$

**Question 13:**

$$\int_2^3 \frac{x \, dx}{x^2 + 1}$$

**Answer**

Let $I = \int_2^3 \frac{x}{x^2 + 1} dx$

$$\int \frac{x}{x^2 + 1} dx = \frac{1}{2} \int \frac{2x}{x^2 + 1} dx = \frac{1}{2} \log \left( 1 + x^2 \right) = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(3) - F(2)$$

$$= \frac{1}{2} \left[ \log \left( 1 + (3)^2 \right) - \log \left( 1 + (2)^2 \right) \right]$$

$$= \frac{1}{2} \left[ \log(10) - \log(5) \right]$$

$$= \frac{1}{2} \log \left( \frac{10}{5} \right) = \frac{1}{2} \log 2$$

**Question 14:**

$$\int_0^1 \frac{2x+3}{5x^2+1} dx$$

Answer

$$\text{Let } I = \int_0^1 \frac{2x+3}{5x^2+1} dx$$

$$\int \frac{2x+3}{5x^2+1} dx = \frac{1}{5} \int \frac{5(2x+3)}{5x^2+1} dx$$

$$= \frac{1}{5} \int \frac{10x+15}{5x^2+1} dx$$

$$= \frac{1}{5} \int \frac{10x}{5x^2+1} dx + 3 \int \frac{1}{5x^2+1} dx$$

$$= \frac{1}{5} \int \frac{10x}{5x^2+1} dx + 3 \int \frac{1}{5\left(x^2+\frac{1}{5}\right)} dx$$

$$= \frac{1}{5} \log(5x^2+1) + \frac{3}{5} \cdot \frac{1}{\frac{1}{\sqrt{5}}} \tan^{-1} \frac{x}{\frac{1}{\sqrt{5}}}$$

$$= \frac{1}{5} \log(5x^2+1) + \frac{3}{\sqrt{5}} \tan^{-1}\left(\sqrt{5}x\right)$$

$$= F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(1) - F(0)$$

$$= \left\{ \frac{1}{5} \log(5+1) + \frac{3}{\sqrt{5}} \tan^{-1}\left(\sqrt{5}\right) \right\} - \left\{ \frac{1}{5} \log(1) + \frac{3}{\sqrt{5}} \tan^{-1}(0) \right\}$$

$$= \frac{1}{5} \log 6 + \frac{3}{\sqrt{5}} \tan^{-1} \sqrt{5}$$

**Question 15:**

$$\int_0^1 xe^{x^2} dx$$

Answer

Let $I = \int_0^1 xe^{x^2} dx$

Put $x^2 = t \Rightarrow 2x\, dx = dt$

As $x \to 0, t \to 0$ and as $x \to 1, t \to 1$,

$\therefore I = \frac{1}{2}\int_0^1 e^t dt$

$\frac{1}{2}\int e^t dt = \frac{1}{2}e^t = F(t)$

By second fundamental theorem of calculus, we obtain

$I = F(1) - F(0)$

$= \frac{1}{2}e - \frac{1}{2}e^0$

$= \frac{1}{2}(e-1)$


**Question 16:**

$\int_0^1 \frac{5x^2}{x^2 + 4x + 3}$

Answer

Let $I = \int_1^2 \frac{5x^2}{x^2 + 4x + 3}dx$

Dividing $5x^2$ by $x^2 + 4x + 3$, we obtain

$I = \int_1^2 \left\{ 5 - \frac{20x+15}{x^2+4x+3} \right\}dx$

$= \int_1^2 5dx - \int_1^2 \frac{20x+15}{x^2+4x+3}dx$

$= \left[5x\right]_1^2 - \int_1^2 \frac{20x+15}{x^2+4x+3}dx$

$I = 5 - I_1$, where $I = \int_1^2 \frac{20x+15}{x^2+4x+3}dx$   ...(1)

Consider $I_1 = \int_1^2 \dfrac{20x+15}{x^2+4x+8} dx$

Let $20x+15 = A\dfrac{d}{dx}(x^2+4x+3)+B$

$$= 2Ax+(4A+B)$$

Equating the coefficients of $x$ and constant term, we obtain

A = 10 and B = −25

$\Rightarrow I_1 = 10\int_1^2 \dfrac{2x+4}{x^2+4x+3} dx - 25\int_1^2 \dfrac{dx}{x^2+4x+3}$

Let $x^2+4x+3 = t$

$\Rightarrow (2x+4)dx = dt$

$\Rightarrow I_1 = 10\int \dfrac{dt}{t} - 25\int \dfrac{dx}{(x+2)^2 - 1^2}$

$$= 10\log t - 25\left[\dfrac{1}{2}\log\left(\dfrac{x+2-1}{x+2+1}\right)\right]$$

$$= \left[10\log(x^2+4x+3)\right]_1^2 - 25\left[\dfrac{1}{2}\log\left(\dfrac{x+1}{x+3}\right)\right]_1^2$$

$$= \left[10\log 15 - 10\log 8\right] - 25\left[\dfrac{1}{2}\log\dfrac{3}{5} - \dfrac{1}{2}\log\dfrac{2}{4}\right]$$

$$= \left[10\log(5\times 3) - 10\log(4\times 2)\right] - \dfrac{25}{2}\left[\log 3 - \log 5 - \log 2 + \log 4\right]$$

$$= \left[10\log 5 + 10\log 3 - 10\log 4 - 10\log 2\right] - \dfrac{25}{2}\left[\log 3 - \log 5 - \log 2 + \log 4\right]$$

$$= \left[10 + \dfrac{25}{2}\right]\log 5 + \left[-10 - \dfrac{25}{2}\right]\log 4 + \left[10 - \dfrac{25}{2}\right]\log 3 + \left[-10 + \dfrac{25}{2}\right]\log 2$$

$$= \dfrac{45}{2}\log 5 - \dfrac{45}{2}\log 4 - \dfrac{5}{2}\log 3 + \dfrac{5}{2}\log 2$$

$$= \dfrac{45}{2}\log\dfrac{5}{4} - \dfrac{5}{2}\log\dfrac{3}{2}$$

Substituting the value of $I_1$ in (1), we obtain

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$$I = 5 - \left[ \frac{45}{2} \log \frac{5}{4} - \frac{5}{2} \log \frac{3}{2} \right]$$

$$= 5 - \frac{5}{2} \left[ 9 \log \frac{5}{4} - \log \frac{3}{2} \right]$$

**Question 17:**

$$\int_0^{\frac{\pi}{4}} \left( 2 \sec^2 x + x^3 + 2 \right) dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \left( 2 \sec^2 x + x^3 + 2 \right) dx$$

$$\int \left( 2 \sec^2 x + x^3 + 2 \right) dx = 2 \tan x + \frac{x^4}{4} + 2x = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F \left( \frac{\pi}{4} \right) - F(0)$$

$$= \left\{ \left( 2 \tan \frac{\pi}{4} + \frac{1}{4} \left( \frac{\pi}{4} \right)^4 + 2 \left( \frac{\pi}{4} \right) \right) - \left( 2 \tan 0 + 0 + 0 \right) \right\}$$

$$= 2 \tan \frac{\pi}{4} + \frac{\pi^4}{4^5} + \frac{\pi}{2}$$

$$= 2 + \frac{\pi}{2} + \frac{\pi^4}{1024}$$

**Question 18:**

$$\int_0^{\pi} \left( \sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx$$

Answer

Let $I = \int_0^\pi \left( \sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx$

$= -\int_0^\pi \left( \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \right) dx$

$= -\int_0^\pi \cos x \, dx$

$\int \cos x \, dx = \sin x = F(x)$

By second fundamental theorem of calculus, we obtain

$I = F(\pi) - F(0)$

$= \sin \pi - \sin 0$

$= 0$


**Question 19:**

$\int_0^2 \frac{6x+3}{x^2+4} dx$

Answer

Let $I = \int_0^2 \frac{6x+3}{x^2+4} dx$

$\int \frac{6x+3}{x^2+4} dx = 3 \int \frac{2x+1}{x^2+4} dx$

$= 3 \int \frac{2x}{x^2+4} dx + 3 \int \frac{1}{x^2+4} dx$

$= 3 \log (x^2 + 4) + \frac{3}{2} \tan^{-1} \frac{x}{2} = F(x)$

By second fundamental theorem of calculus, we obtain

$$I = F(2) - F(0)$$

$$= \left\{ 3\log\left(2^2 + 4\right) + \frac{3}{2}\tan^{-1}\left(\frac{2}{2}\right) \right\} - \left\{ 3\log(0 + 4) + \frac{3}{2}\tan^{-1}\left(\frac{0}{2}\right) \right\}$$

$$= 3\log 8 + \frac{3}{2}\tan^{-1} 1 - 3\log 4 - \frac{3}{2}\tan^{-1} 0$$

$$= 3\log 8 + \frac{3}{2}\left(\frac{\pi}{4}\right) - 3\log 4 - 0$$

$$= 3\log\left(\frac{8}{4}\right) + \frac{3\pi}{8}$$

$$= 3\log 2 + \frac{3\pi}{8}$$

**Question 20:**

$$\int_0^1 \left( xe^x + \sin\frac{\pi x}{4} \right) dx$$

Answer

$$\text{Let } I = \int_0^1 \left( xe^x + \sin\frac{\pi x}{4} \right) dx$$

$$\int \left( xe^x + \sin\frac{\pi x}{4} \right) dx = x\int e^x dx - \int \left\{ \left(\frac{d}{dx} x\right) \int e^x dx \right\} dx + \left\{ \frac{-\cos\frac{\pi x}{4}}{\frac{\pi}{4}} \right\}$$

$$= xe^x - \int e^x dx - \frac{4\pi}{\pi}\cos\frac{x}{4}$$

$$= xe^x - e^x - \frac{4\pi}{\pi}\cos\frac{x}{4}$$

$$= F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(1) - F(0)$$

$$= \left(1.e^1 - e^1 - \frac{4}{\pi}\cos\frac{\pi}{4}\right) - \left(0.e^0 - e^0 - \frac{4}{\pi}\cos 0\right)$$

$$= e - e - \frac{4}{\pi}\left(\frac{1}{\sqrt{2}}\right) + 1 + \frac{4}{\pi}$$

$$= 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}$$

**Question 21:**

$$\int_1^{\sqrt{3}} \frac{dx}{1+x^2} \text{ equals}$$

A. $\dfrac{\pi}{3}$

B. $\dfrac{2\pi}{3}$

C. $\dfrac{\pi}{6}$

D. $\dfrac{\pi}{12}$

Answer

$$\int \frac{dx}{1+x^2} = \tan^{-1} x = F(x)$$

By second fundamental theorem of calculus, we obtain

$$\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = F(\sqrt{3}) - F(1)$$

$$= \tan^{-1}\sqrt{3} - \tan^{-1} 1$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

Hence, the correct Answer is D.

**Question 22:**

$\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2}$ equals

**A.** $\frac{\pi}{6}$

**B.** $\frac{\pi}{12}$

**C.** $\frac{\pi}{24}$

**D.** $\frac{\pi}{4}$

Answer

$$\int \frac{dx}{4+9x^2} = \int \frac{dx}{(2)^2 + (3x)^2}$$

Put $3x = t \implies 3dx = dt$

$$\therefore \int \frac{dx}{(2)^2 + (3x)^2} = \frac{1}{3} \int \frac{dt}{(2)^2 + t^2}$$

$$= \frac{1}{3} \left[ \frac{1}{2} \tan^{-1} \frac{t}{2} \right]$$

$$= \frac{1}{6} \tan^{-1} \left( \frac{3x}{2} \right)$$

$$= F(x)$$

By second fundamental theorem of calculus, we obtain

$$\int_{0}^{\frac{2}{3}} \frac{dx}{4+9x^2} = F\left(\frac{2}{3}\right) - F(0)$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3}{2}\cdot\frac{2}{3}\right) - \frac{1}{6}\tan^{-1}0$$

$$= \frac{1}{6}\tan^{-1}1 - 0$$

$$= \frac{1}{6}\times\frac{\pi}{4}$$

$$= \frac{\pi}{24}$$

Hence, the correct Answer is C.

**Question 1:**

$$\int_0^1 \frac{x}{x^2+1}dx$$

Answer

$$\int_0^1 \frac{x}{x^2+1}dx$$

Let $x^2 + 1 = t \implies 2x\,dx = dt$

When $x = 0$, $t = 1$ and when $x = 1$, $t = 2$

$$\therefore \int_0^1 \frac{x}{x^2+1}dx = \frac{1}{2}\int_1^2 \frac{dt}{t}$$

$$= \frac{1}{2}\Big[\log|t|\Big]_1^2$$

$$= \frac{1}{2}\big[\log 2 - \log 1\big]$$

$$= \frac{1}{2}\log 2$$

**Question 2:**

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,\cos^5\phi\,d\phi$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\,d\phi = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,\cos^4\phi\,\cos\phi\,d\phi$$

Also, let $\sin\phi = t \implies \cos\phi\,d\phi = dt$

When $\phi = 0$, $t = 0$ and when $\phi = \dfrac{\pi}{2}$, $t = 1$

$$\therefore I = \int_0^1 \sqrt{t}\left(1-t^2\right)^2 dt$$

$$= \int_0^1 t^{\frac{1}{2}}\left(1+t^4-2t^2\right)dt$$

$$= \int_0^1\left[t^{\frac{1}{2}}+t^{\frac{9}{2}}-2t^{\frac{5}{2}}\right]dt$$

$$= \left[\dfrac{t^{\frac{3}{2}}}{\frac{3}{2}}+\dfrac{t^{\frac{11}{2}}}{\frac{11}{2}}-\dfrac{2t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1$$

$$= \dfrac{2}{3}+\dfrac{2}{11}-\dfrac{4}{7}$$

$$= \dfrac{154+42-132}{231}$$

$$= \dfrac{64}{231}$$

**Question 3:**

$$\int_0^1 \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)dx$$

Answer

Let $I = \int_0^1 \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)dx$

Also, let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

When $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \dfrac{\pi}{4}$

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$$I = \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sin^{-1}(\sin 2\theta)\sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\theta \cdot \sec^2\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \theta \cdot \sec^2\theta\,d\theta$$

Taking $\theta$ as first function and sec²$\theta$ as second function and integrating by parts, we obtain

$$I = 2\left[\theta\int\sec^2\theta\,d\theta - \int\left\{\left(\frac{d}{dx}\theta\right)\int\sec^2\theta\,d\theta\right\}d\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\theta\tan\theta - \int\tan\theta\,d\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\theta\tan\theta + \log|\cos\theta|\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\frac{\pi}{4}\tan\frac{\pi}{4} + \log\left|\cos\frac{\pi}{4}\right| - \log|\cos 0|\right]$$

$$= 2\left[\frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right) - \log 1\right]$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right]$$

$$= \frac{\pi}{2} - \log 2$$

**Question 4:**

$$\int_0^2 x\sqrt{x+2}\quad \left(\text{Put } x+2 = t^2\right)$$

Answer

$$\int_0^2 x\sqrt{x+2}\,dx$$

Let $x + 2 = t^2$ □ $dx = 2t\,dt$

When $x = 0$, $t = \sqrt{2}$ and when $x = 2$, $t = 2$

$$\therefore \int_0^2 x\sqrt{x+2}\,dx = \int_{\sqrt{2}}^2 \left(t^2-2\right)\sqrt{t^2}\,2t\,dt$$

$$= 2\int_{\sqrt{2}}^2 \left(t^2-2\right)t^2\,dt$$

$$= 2\int_{\sqrt{2}}^2 \left(t^4-2t^2\right)dt$$

$$= 2\left[\frac{t^5}{5}-\frac{2t^3}{3}\right]_{\sqrt{2}}^2$$

$$= 2\left[\frac{32}{5}-\frac{16}{3}-\frac{4\sqrt{2}}{5}+\frac{4\sqrt{2}}{3}\right]$$

$$= 2\left[\frac{96-80-12\sqrt{2}+20\sqrt{2}}{15}\right]$$

$$= 2\left[\frac{16+8\sqrt{2}}{15}\right]$$

$$= \frac{16\left(2+\sqrt{2}\right)}{15}$$

$$= \frac{16\sqrt{2}\left(\sqrt{2}+1\right)}{15}$$

**Question 5:**

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx$$

Answer

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx$$

Let cos $x = t$ □ $-\sin x\,dx = dt$

When $x = 0$, $t = 1$ and when $x = \dfrac{\pi}{2}, t = 0$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx = -\int_1^0 \frac{dt}{1+t^2}$$

$$= -\left[ \tan^{-1} t \right]_1^0$$

$$= -\left[ \tan^{-1} 0 - \tan^{-1} 1 \right]$$

$$= -\left[ -\frac{\pi}{4} \right]$$

$$= \frac{\pi}{4}$$

**Question 6:**

$$\int_0^2 \frac{dx}{x+4-x^2}$$

Answer

$$\int_0^2 \frac{dx}{x+4-x^2} = \int_0^2 \frac{dx}{-\left(x^2 - x - 4\right)}$$

$$= \int_0^2 \frac{dx}{-\left(x^2 - x + \frac{1}{4} - \frac{1}{4} - 4\right)}$$

$$= \int_0^2 \frac{dx}{-\left[\left(x - \frac{1}{2}\right)^2 - \frac{17}{4}\right]}$$

$$= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}$$

Let $x - \dfrac{1}{2} = t$ □ $dx = dt$

When $x = 0$, $t = -\dfrac{1}{2}$ and when $x = 2$, $t = \dfrac{3}{2}$

$$\therefore \int_0^2 \frac{dx}{\left(\dfrac{\sqrt{17}}{2}\right)^2 - \left(x - \dfrac{1}{2}\right)^2} = \int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{dt}{\left(\dfrac{\sqrt{17}}{2}\right)^2 - t^2}$$

$$= \left[\frac{1}{2\left(\dfrac{\sqrt{17}}{2}\right)} \log \frac{\dfrac{\sqrt{17}}{2} + t}{\dfrac{\sqrt{17}}{2} - t}\right]_{-\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{1}{\sqrt{17}}\left[\log \frac{\dfrac{\sqrt{17}}{2} + \dfrac{3}{2}}{\dfrac{\sqrt{17}}{2} - \dfrac{3}{2}} - \frac{\log \dfrac{\sqrt{17}}{2} - \dfrac{1}{2}}{\log \dfrac{\sqrt{17}}{2} + \dfrac{1}{2}}\right]$$

$$= \frac{1}{\sqrt{17}}\left[\log \frac{\sqrt{17} + 3}{\sqrt{17} - 3} - \log \frac{\sqrt{17} - 1}{\sqrt{17} + 1}\right]$$

$$= \frac{1}{\sqrt{17}} \log \frac{\sqrt{17} + 3}{\sqrt{17} - 3} \times \frac{\sqrt{17} + 1}{\sqrt{17} - 1}$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{17 + 3 + 4\sqrt{17}}{17 + 3 - 4\sqrt{17}}\right]$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{20 + 4\sqrt{17}}{20 - 4\sqrt{17}}\right]$$

$$= \frac{1}{\sqrt{17}} \log \left(\frac{5 + \sqrt{17}}{5 - \sqrt{17}}\right)$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{\left(5 + \sqrt{17}\right)\left(5 + \sqrt{17}\right)}{25 - 17}\right]$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{25 + 17 + 10\sqrt{17}}{8}\right]$$

$$= \frac{1}{\sqrt{17}} \log \left(\frac{42 + 10\sqrt{17}}{8}\right)$$

$$= \frac{1}{\sqrt{17}} \log \left(\frac{21 + 5\sqrt{17}}{4}\right)$$

**Question 7:**

$$\int_{-1}^{1} \frac{dx}{x^2 + 2x + 5}$$

Answer

$$\int_{-1}^{1} \frac{dx}{x^2 + 2x + 5} = \int_{-1}^{1} \frac{dx}{(x^2 + 2x + 1) + 4} = \int_{-1}^{1} \frac{dx}{(x+1)^2 + (2)^2}$$

Let $x + 1 = t \Rightarrow dx = dt$

When $x = -1$, $t = 0$ and when $x = 1$, $t = 2$

$$\therefore \int_{-1}^{1} \frac{dx}{(x+1)^2 + (2)^2} = \int_{0}^{2} \frac{dt}{t^2 + 2^2}$$

$$= \left[ \frac{1}{2} \tan^{-1} \frac{t}{2} \right]_{0}^{2}$$

$$= \frac{1}{2} \tan^{-1} 1 - \frac{1}{2} \tan^{-1} 0$$

$$= \frac{1}{2} \left( \frac{\pi}{4} \right) = \frac{\pi}{8}$$

**Question 8:**

$$\int_{1}^{2} \left( \frac{1}{x} - \frac{1}{2x^2} \right) e^{2x} dx$$

Answer

$$\int_{1}^{2} \left( \frac{1}{x} - \frac{1}{2x^2} \right) e^{2x} dx$$

Let $2x = t \Rightarrow 2dx = dt$

When $x = 1$, $t = 2$ and when $x = 2$, $t = 4$

$$\therefore \int_1^2 \left( \frac{1}{x} - \frac{1}{2x^2} \right) e^{2x} dx = \frac{1}{2} \int_2^4 \left( \frac{2}{t} - \frac{2}{t^2} \right) e^t dt$$

$$= \int_2^4 \left( \frac{1}{t} - \frac{1}{t^2} \right) e^t dt$$

Let $\frac{1}{t} = f(t)$

Then, $f'(t) = -\frac{1}{t^2}$

$$\Rightarrow \int_2^4 \left( \frac{1}{t} - \frac{1}{t^2} \right) e^t dt = \int_2^4 e^t \left[ f(t) + f'(t) \right] dt$$

$$= \left[ e^t f(t) \right]_2^4$$

$$= \left[ e^t \cdot \frac{2}{t} \right]_2^4$$

$$= \left[ \frac{e^t}{t} \right]_2^4$$

$$= \frac{e^4}{4} - \frac{e^2}{2}$$

$$= \frac{e^2 \left( e^2 - 2 \right)}{4}$$

**Question 9:**

The value of the integral $\int_{\frac{1}{3}}^1 \frac{\left( x - x^3 \right)^{\frac{1}{3}}}{x^4} dx$ is

**A.** 6

**B.** 0

**C.** 3

**D.** 4

Answer

Let $I = \int_{\frac{1}{3}}^{1} \frac{\left(x - x^3\right)^{\frac{1}{3}}}{x^4} dx$

Also, let $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

When $x = \frac{1}{3}$, $\theta = \sin^{-1}\left(\frac{1}{3}\right)$ and when $x = 1$, $\theta = \frac{\pi}{2}$

$$\Rightarrow I = \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{\left(\sin\theta - \sin^3\theta\right)^{\frac{1}{3}}}{\sin^4\theta} \cos\theta\, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(1 - \sin^2\theta\right)^{\frac{1}{3}}}{\sin^4\theta} \cos\theta\, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(\cos\theta\right)^{\frac{2}{3}}}{\sin^4\theta} \cos\theta\, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(\cos\theta\right)^{\frac{2}{3}}}{\sin^2\theta\sin^2\theta} \cos\theta\, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{\left(\cos\theta\right)^{\frac{5}{3}}}{\left(\sin\theta\right)^{\frac{5}{3}}} \operatorname{cosec}^2\theta\, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \left(\cot\theta\right)^{\frac{5}{3}} \operatorname{cosec}^2\theta\, d\theta$$

Let $\cot\theta = t$ ☐ $-\operatorname{cosec}2\theta\, d\theta = dt$

When $\theta = \sin^{-1}\left(\dfrac{1}{3}\right)$, $t = 2\sqrt{2}$ and when $\theta = \dfrac{\pi}{2}$, $t = 0$

$$\therefore I = -\int_{2\sqrt{2}}^{0} (t)^{\frac{5}{3}} dt$$

$$= -\left[\frac{3}{8}(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$

$$= -\frac{3}{8}\left[(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$

$$= -\frac{3}{8}\left[-\left(2\sqrt{2}\right)^{\frac{8}{3}}\right]$$

$$= \frac{3}{8}\left[\left(\sqrt{8}\right)^{\frac{8}{3}}\right]$$

$$= \frac{3}{8}\left[(8)^{\frac{4}{3}}\right]$$

$$= \frac{3}{8}[16]$$

$$= 3 \times 2$$

$$= 6$$

Hence, the correct Answer is A.

**Question 10:**

If $f(x) = \int_{0}^{x} t\sin t\, dt$, then $f'(x)$ is

**A.** cos *x* + *x* sin *x*

**B.** *x* sin *x*

**C.** *x* cos *x*

**D.** sin *x* + *x* cos *x*

Answer

$$f(x) = \int_{0}^{x} t \sin t\, dt$$

Integrating by parts, we obtain

$$f(x) = t\int_0^x \sin t\, dt - \int_0^x \left\{ \left( \frac{d}{dt} t \right) \int \sin t\, dt \right\} dt$$

$$= \left[ t(-\cos t) \right]_0^x - \int_0^x (-\cos t)\, dt$$

$$= \left[ -t\cos t + \sin t \right]_0^x$$

$$= -x\cos x + \sin x$$

$$\Rightarrow f'(x) = -\left[ \{x(-\sin x)\} + \cos x \right] + \cos x$$

$$= x\sin x - \cos x + \cos x$$

$$= x\sin x$$

Hence, the correct Answer is B.

Exercise 7.11

**Question 1:**

$$\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$$

Answer

$$I = \int_0^{\frac{\pi}{2}} \cos^2 x \, dx \qquad \qquad ...(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \cos^2 \left( \frac{\pi}{2} - x \right) dx \qquad \left( \int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin^2 x \, dx \qquad \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \left( \sin^2 x + \cos^2 x \right) dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 \cdot dx$$

$$\Rightarrow 2I = \left[ x \right]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

**Question 2:**

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

Answer

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

Let $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$          ...(1)

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} dx \qquad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos}}{\sqrt{\cos} + \sqrt{\sin x}} dx \qquad\qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 \cdot dx$$

$$\Rightarrow 2I = \left[x\right]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

**Question 3:**

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x\, dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}$$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \dfrac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$                    ...(1)

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \dfrac{\sin^{\frac{3}{2}}\left(\dfrac{\pi}{2} - x\right)}{\sin^{\frac{3}{2}}\left(\dfrac{\pi}{2} - x\right) + \cos^{\frac{3}{2}}\left(\dfrac{\pi}{2} - x\right)} dx$          $\left(\int_0^a f(x) dx = \int_0^a f(a-x) dx\right)$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \dfrac{\cos^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$                    ...(2)

Adding (1) and (2), we obtain

$2I = \int_0^{\frac{\pi}{2}} \dfrac{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 \, dx$

$\Rightarrow 2I = \left[x\right]_0^{\frac{\pi}{2}}$

$\Rightarrow 2I = \dfrac{\pi}{2}$

$\Rightarrow I = \dfrac{\pi}{4}$

**Question 4:**

$\int_0^{\frac{\pi}{2}} \dfrac{\cos^5 x \, dx}{\sin^5 x + \cos^5 x}$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \frac{\cos^5 x}{\sin^5 x + \cos^5 x} dx$       ...(1)

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^5\left(\frac{\pi}{2} - x\right)}{\sin^5\left(\frac{\pi}{2} - x\right) + \cos^5\left(\frac{\pi}{2} - x\right)} dx$      $\left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x}{\sin^5 x + \cos^5 x} dx$       ...(2)

Adding (1) and (2), we obtain

$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x + \cos^5 x}{\sin^5 x + \cos^5 x} dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 \cdot dx$

$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$

$\Rightarrow 2I = \frac{\pi}{2}$

$\Rightarrow I = \frac{\pi}{4}$

**Question 5:**

$\int_{-5}^{5} |x + 2| dx$

Answer

Let $I = \int_{-5}^{5} |x + 2| dx$

It can be seen that $(x + 2) \le 0$ on $[-5, -2]$ and $(x + 2) \ge 0$ on $[-2, 5]$.

$$\therefore I = \int_{-5}^{-2} -(x+2)dx + \int_{-2}^{5} (x+2)dx \qquad \left(\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x)\right)$$

$$I = -\left[\frac{x^2}{2} + 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$$

$$= -\left[\frac{(-2)^2}{2} + 2(-2) - \frac{(-5)^2}{2} - 2(-5)\right] + \left[\frac{(5)^2}{2} + 2(5) - \frac{(-2)^2}{2} - 2(-2)\right]$$

$$= -\left[2 - 4 - \frac{25}{2} + 10\right] + \left[\frac{25}{2} + 10 - 2 + 4\right]$$

$$= -2 + 4 + \frac{25}{2} - 10 + \frac{25}{2} + 10 - 2 + 4$$

$$= 29$$

**Question 6:**

$$\int_2^8 |x-5|\,dx$$

Answer

$$\text{Let } I = \int_2^8 |x-5|\,dx$$

It can be seen that $(x - 5) \le 0$ on $[2, 5]$ and $(x - 5) \ge 0$ on $[5, 8]$.

$$I = \int_2^5 -(x-5)dx + \int_2^8 (x-5)dx \qquad \left(\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x)\right)$$

$$= -\left[\frac{x^2}{2} - 5x\right]_2^5 + \left[\frac{x^2}{2} - 5x\right]_5^8$$

$$= -\left[\frac{25}{2} - 25 - 2 + 10\right] + \left[32 - 40 - \frac{25}{2} + 25\right]$$

$$= 9$$

**Question 7:**

$$\int_0^1 x(1-x)^n\,dx$$

Answer

Let $I = \int_0^1 x(1-x)^n \, dx$

$\therefore I = \int_0^1 (1-x)(1-(1-x))^n \, dx$

$= \int_0^1 (1-x)(x)^n \, dx$

$= \int_0^1 (x^n - x^{n+1}) \, dx$

$= \left[ \dfrac{x^{n+1}}{n+1} - \dfrac{x^{n+2}}{n+2} \right]_0^1$

$= \left[ \dfrac{1}{n+1} - \dfrac{1}{n+2} \right]$

$= \dfrac{(n+2)-(n+1)}{(n+1)(n+2)}$

$= \dfrac{1}{(n+1)(n+2)}$

$\left( \int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx \right)$

**Question 8:**

$\int_0^{\frac{\pi}{4}} \log(1+\tan x) \, dx$

Answer

Let $I = \int_0^{\frac{\pi}{4}} \log\,(1+\tan x)\,dx$  ...(1)

$\therefore I = \int_0^{\frac{\pi}{4}} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx$  $\left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\left\{1+\dfrac{\tan\dfrac{\pi}{4}-\tan x}{1+\tan\dfrac{\pi}{4}\tan x}\right\}dx$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\left\{1+\dfrac{1-\tan x}{1+\tan}\right\}dx$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\,\dfrac{2}{(1+\tan x)}dx$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log 2\,dx - \int_0^{\frac{\pi}{4}} \log\,(1+\tan x)\,dx$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log 2\,dx - I$  $\left[\text{From } (1)\right]$

$\Rightarrow 2I = \left[x\log 2\right]_0^{\frac{\pi}{4}}$

$\Rightarrow 2I = \dfrac{\pi}{4}\log 2$

$\Rightarrow I = \dfrac{\pi}{8}\log 2$

**Question 9:**

$\int_0^2 x\sqrt{2-x}\,dx$

Answer

Let $I = \int_0^2 x\sqrt{2-x}\,dx$

$I = \int_0^2 (2-x)\sqrt{x}\,dx$

$\left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$

$= \int_0^2 \left\{ 2x^{\frac{1}{2}} - x^{\frac{3}{2}} \right\} dx$

$= \left[ 2\left( \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} \right) - \dfrac{x^{\frac{5}{2}}}{\frac{5}{2}} \right]_0^2$

$= \left[ \dfrac{4}{3} x^{\frac{3}{2}} - \dfrac{2}{5} x^{\frac{5}{2}} \right]_0^2$

$= \dfrac{4}{3}(2)^{\frac{3}{2}} - \dfrac{2}{5}(2)^{\frac{5}{2}}$

$= \dfrac{4 \times 2\sqrt{2}}{3} - \dfrac{2}{5} \times 4\sqrt{2}$

$= \dfrac{8\sqrt{2}}{3} - \dfrac{8\sqrt{2}}{5}$

$= \dfrac{40\sqrt{2} - 24\sqrt{2}}{15}$

$= \dfrac{16\sqrt{2}}{15}$

**Question 10:**

$\int_0^{\frac{\pi}{2}} \left( 2\log \sin x - \log \sin 2x \right) dx$

Answer

Let $I = \int_0^{\frac{\pi}{2}} (2\log \sin x - \log \sin 2x)\, dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{2\log \sin x - \log(2\sin x \cos x)\}\, dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{2\log \sin x - \log \sin x - \log \cos x - \log 2\}\, dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{\log \sin x - \log \cos x - \log 2\}\, dx \qquad\qquad \dots(1)$

It is known that, $\left(\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right)$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{\log \cos x - \log \sin x - \log 2\}\, dx \qquad\qquad \dots(2)$

Adding (1) and (2), we obtain

$2I = \int_0^{\frac{\pi}{2}} (-\log 2 - \log 2)\, dx$

$\Rightarrow 2I = -2\log 2 \int_0^{\frac{\pi}{2}} 1\, dx$

$\Rightarrow I = -\log 2 \left[\frac{\pi}{2}\right]$

$\Rightarrow I = \frac{\pi}{2}(-\log 2)$

$\Rightarrow I = \frac{\pi}{2}\left[\log \frac{1}{2}\right]$

$\Rightarrow I = \frac{\pi}{2}\log \frac{1}{2}$

**Question 11:**

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\, dx$

Answer

Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\, dx$

As $\sin^2(-x) = (\sin(-x))^2 = (-\sin x)^2 = \sin^2 x$, therefore, $\sin^2 x$ is an even function.

It is known that if *f*(*x*) is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

$$I = 2\int_{0}^{\frac{\pi}{2}} \sin^2 x\,dx$$

$$= 2\int_{0}^{\frac{\pi}{2}} \frac{1-\cos 2x}{2}\,dx$$

$$= \int_{0}^{\frac{\pi}{2}} (1-\cos 2x)\,dx$$

$$= \left[ x - \frac{\sin 2x}{2} \right]_{0}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

**Question 12:**

$$\int_{0}^{\pi} \frac{x\,dx}{1+\sin x}$$

Answer

$$\text{Let } I = \int_{0}^{\pi} \frac{x\,dx}{1+\sin x} \qquad \ldots(1)$$

$$\Rightarrow I = \int_{0}^{\pi} \frac{(\pi-x)}{1+\sin(\pi-x)}\,dx \qquad \left( \int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx \right)$$

$$\Rightarrow I = \int_{0}^{\pi} \frac{(\pi-x)}{1+\sin x}\,dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^\pi \frac{\pi}{1+\sin x} dx$$

$$\Rightarrow 2I = \pi \int_0^\pi \frac{(1-\sin x)}{(1+\sin x)(1-\sin x)} dx$$

$$\Rightarrow 2I = \pi \int_0^\pi \frac{1-\sin x}{\cos^2 x} dx$$

$$\Rightarrow 2I = \pi \int_0^\pi \left\{ \sec^2 x - \tan x \sec x \right\} dx$$

$$\Rightarrow 2I = \pi \left[ \tan x - \sec x \right]_0^\pi$$

$$\Rightarrow 2I = \pi [2]$$

$$\Rightarrow I = \pi$$

**Question 13:**

$$\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \, dx$$

Answer

$$\text{Let } I = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \, dx \qquad \ldots(1)$$

As $\sin^7(-x) = (\sin(-x))^7 = (-\sin x)^7 = -\sin^7 x$, therefore, $\sin^2 x$ is an odd function.

It is known that, if $f(x)$ is an odd function, then $\int_{-a}^{a} f(x) dx = 0$

$$\therefore I = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \, dx = 0$$

**Question 14:**

$$\int_0^{2\pi} \cos^5 x \, dx$$

Answer

$$\text{Let } I = \int_0^{2\pi} \cos^5 x \, dx \qquad \ldots(1)$$

$$\cos^5 (2\pi - x) = \cos^5 x$$

It is known that,

$$\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx, \text{ if } f(2a-x) = f(x)$$

$$= 0 \text{ if } f(2a-x) = -f(x)$$

$$\therefore I = 2\int_0^{\pi} \cos^5 x\,dx$$

$$\Rightarrow I = 2(0) = 0 \qquad\qquad \left[\cos^5(\pi - x) = -\cos^5 x\right]$$

**Question 15:**

$$\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x}\,dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x}\,dx \qquad\qquad \ldots(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)}\,dx \qquad\qquad \left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \sin x \cos x}\,dx \qquad\qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \frac{0}{1 + \sin x \cos x}\,dx$$

$$\Rightarrow I = 0$$

**Question 16:**

$$\int_0^{\pi} \log(1 + \cos x)\,dx$$

Answer

$$\text{Let } I = \int_0^{\pi} \log(1 + \cos x)\,dx \qquad\qquad \ldots(1)$$

$$\Rightarrow I = \int_0^{\pi} \log(1 + \cos(\pi - x))\,dx \qquad\qquad \left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$$\Rightarrow I = \int_0^{\pi} \log(1 - \cos x)\,dx \qquad\qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^\pi \{\log(1 + \cos x) + \log(1 - \cos x)\} dx$$

$$\Rightarrow 2I = \int_0^\pi \log(1 - \cos^2 x) dx$$

$$\Rightarrow 2I = \int_0^\pi \log \sin^2 x \, dx$$

$$\Rightarrow 2I = 2\int_0^\pi \log \sin x \, dx$$

$$\Rightarrow I = \int_0^\pi \log \sin x \, dx \qquad\qquad\qquad ...(3)$$

$$\sin(\pi - x) = \sin x$$

$$\therefore I = 2\int_0^{\frac{\pi}{2}} \log \sin x \, dx \qquad\qquad\qquad ...(4)$$

$$\Rightarrow I = 2\int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx = 2\int_0^{\frac{\pi}{2}} \log \cos x \, dx \qquad\qquad ...(5)$$

Adding (4) and (5), we obtain

$$2I = 2\int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x + \log 2 - \log 2) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} (\log 2 \sin x \cos x - \log 2) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx - \int_0^{\frac{\pi}{2}} \log 2 \, dx$$

Let $2x = t \square 2dx = dt$

When $x = 0$, $t = 0$ and when $x = \frac{\pi}{2}$, $\pi =$

$$\therefore I = \frac{1}{2}\frac{\pi}{\int_0^\pi} \log \sin t \, dt - \frac{\pi}{2}\log 2$$

$$\Rightarrow I = \frac{1}{2}\frac{\pi}{I} - \frac{\pi}{2}\log 2$$

$$\Rightarrow \frac{I}{2} = -\frac{\pi}{2}\log 2$$

$$\Rightarrow I = -\pi \log 2$$

**Question 17:**

$$\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx$$

Answer

Let $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx$      ...(1)

It is known that, $\left( \int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$

$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} dx$      ...(2)

Adding (1) and (2), we obtain

$2I = \int_0^a \frac{\sqrt{x} + \sqrt{a-x}}{\sqrt{x} + \sqrt{a-x}} dx$

$\Rightarrow 2I = \int_0^a 1 \, dx$

$\Rightarrow 2I = [x]_0^a$

$\Rightarrow 2I = a$

$\Rightarrow I = \dfrac{a}{2}$

**Question 18:**

$$\int_0^4 |x-1| dx$$

Answer

$I = \int_0^4 |x-1| dx$

It can be seen that, $(x - 1) \le 0$ when $0 \le x \le 1$ and $(x - 1) \ge 0$ when $1 \le x \le 4$

$$I = \int_0^1 |x-1|\,dx + \int_1^4 |x-1|\,dx \qquad \left( \int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x) \right)$$

$$= \int_0^1 -(x-1)\,dx + \int_1^4 (x-1)\,dx$$

$$= \left[ x - \frac{x^2}{2} \right]_0^1 + \left[ \frac{x^2}{2} - x \right]_1^4$$

$$= 1 - \frac{1}{2} + \frac{(4)^2}{2} - 4 - \frac{1}{2} + 1$$

$$= 1 - \frac{1}{2} + 8 - 4 - \frac{1}{2} + 1$$

$$= 5$$

**Question 19:**

Show that $\int_0^a f(x)g(x)\,dx = 2\int_0^a f(x)\,dx$, if $f$ and $g$ are defined as $f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$

Answer

$$\text{Let } I = \int_0^a f(x)g(x)\,dx \qquad ...(1)$$

$$\Rightarrow I = \int_0^a f(a-x)g(a-x)\,dx \qquad \left( \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx \right)$$

$$\Rightarrow I = \int_0^a f(x)g(a-x)\,dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^a \{ f(x)g(x) + f(x)g(a-x) \}\,dx$$

$$\Rightarrow 2I = \int_0^a f(x)\{ g(x) + g(a-x) \}\,dx$$

$$\Rightarrow 2I = \int_0^a f(x) \times 4\,dx \qquad \left[ g(x) + g(a-x) = 4 \right]$$

$$\Rightarrow I = 2\int_0^a f(x)\,dx$$

**Question 20:**

The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3 + x\cos x + \tan^5 x + 1\right)dx$ is

**A.** 0

**B.** 2

**C.** π

**D.** 1

Answer

Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3 + x\cos x + \tan^5 x + 1\right)dx$

$\Rightarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^5 x dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \cdot dx$

It is known that if $f(x)$ is an even function, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ and

if $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)dx = 0$

$I = 0 + 0 + 0 + 2\int_0^{\frac{\pi}{2}} 1 \cdot dx$

$= 2\left[x\right]_0^{\frac{\pi}{2}}$

$= \frac{2\pi}{2}$

$\pi =$

Hence, the correct Answer is C.

**Question 21:**

The value of $\int_0^{\frac{\pi}{2}} \log\left(\frac{4 + 3\sin x}{4 + 3\cos x}\right)dx$ is

**A.** 2

**B.** $\frac{3}{4}$

**C.** 0

**D.** $-2$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \log\left(\dfrac{4+3\sin x}{4+3\cos x}\right) dx$          ...(1)

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\left[\dfrac{4+3\sin\left(\dfrac{\pi}{2}-x\right)}{4+3\cos\left(\dfrac{\pi}{2}-x\right)}\right] dx$          $\left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\left(\dfrac{4+3\cos x}{4+3\sin x}\right) dx$          ...(2)

**Adding (1) and (2), we obtain**

$2I = \int_0^{\frac{\pi}{2}} \left\{\log\left(\dfrac{4+3\sin x}{4+3\cos x}\right) + \log\left(\dfrac{4+3\cos x}{4+3\sin x}\right)\right\} dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log\left(\dfrac{4+3\sin x}{4+3\cos x} \times \dfrac{4+3\cos x}{4+3\sin x}\right) dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log 1\, dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 0\, dx$

$\Rightarrow I = 0$

**Hence, the correct Answer is C.**

**Miscellaneous Solutions**

**Question 1:**

$$\frac{1}{x - x^3}$$

Answer

$$\frac{1}{x - x^3} = \frac{1}{x(1 - x^2)} = \frac{1}{x(1 - x)(1 + x)}$$

$$\text{Let } \frac{1}{x(1 - x)(1 + x)} = \frac{A}{x} + \frac{B}{(1 - x)} + \frac{C}{1 + x} \qquad ...(1)$$

$$\Rightarrow 1 = A(1 - x^2) + Bx(1 + x) + Cx(1 - x)$$

$$\Rightarrow 1 = A - Ax^2 + Bx + Bx^2 + Cx - Cx^2$$

Equating the coefficients of $x^2$, $x$, and constant term, we obtain

$-A + B - C = 0$

$B + C = 0$

$A = 1$

On solving these equations, we obtain

$$A = 1, B = \frac{1}{2}, \text{ and } C = -\frac{1}{2}$$

From equation (1), we obtain

$$\frac{1}{x(1-x)(1+x)} = \frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)}$$

$$\Rightarrow \int \frac{1}{x(1-x)(1+x)}dx = \int \frac{1}{x}dx + \frac{1}{2}\int \frac{1}{1-x}dx - \frac{1}{2}\int \frac{1}{1+x}dx$$

$$= \log|x| - \frac{1}{2}\log|(1-x)| - \frac{1}{2}\log|(1+x)|$$

$$= \log|x| - \log\left|(1-x)^{\frac{1}{2}}\right| - \log\left|(1+x)^{\frac{1}{2}}\right|$$

$$= \log\left|\frac{x}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}}\right| + C$$

$$= \log\left|\left(\frac{x^2}{1-x^2}\right)^{\frac{1}{2}}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right| + C$$

**Question 2:**

$$\frac{1}{\sqrt{x+a} + \sqrt{(x+b)}}$$

Answer

$$\frac{1}{\sqrt{x+a}+\sqrt{x+b}} = \frac{1}{\sqrt{x+a}+\sqrt{x+b}} \times \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}$$

$$= \frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)}$$

$$= \frac{\left(\sqrt{x+a}-\sqrt{x+b}\right)}{a-b}$$

$$\Rightarrow \int \frac{1}{\sqrt{x+a}-\sqrt{x+b}}dx = \frac{1}{a-b}\int\left(\sqrt{x+a}-\sqrt{x+b}\right)dx$$

$$= \frac{1}{(a-b)}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right]$$

$$= \frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}}\right] + C$$

**Question 3:**

$$\frac{1}{x\sqrt{ax-x^2}}$$ [Hint: Put $x = \frac{a}{t}$ ]

Answer

$$\frac{1}{x\sqrt{ax-x^2}}$$

$\text{Let } x = \frac{a}{t} \Rightarrow dx = -\frac{a}{t^2}dt$

$$\Rightarrow \int \frac{1}{x\sqrt{ax-x^2}}dx = \int \frac{1}{\frac{a}{t}\sqrt{a\cdot\frac{a}{t}-\left(\frac{a}{t}\right)^2}}\left(-\frac{a}{t^2}dt\right)$$

$$= -\int \frac{1}{at}\cdot\frac{1}{\sqrt{\frac{1}{t}-\frac{1}{t^2}}}dt$$

$$= -\frac{1}{a}\int \frac{1}{\sqrt{\frac{t^2}{t}-\frac{t^2}{t^2}}}dt$$

$$= -\frac{1}{a}\int \frac{1}{\sqrt{t-1}}dt$$

$$= -\frac{1}{a}\left[2\sqrt{t-1}\right]+C$$

$$= -\frac{1}{a}\left[2\sqrt{\frac{a}{x}-1}\right]+C$$

$$= -\frac{2}{a}\left(\frac{\sqrt{a-x}}{\sqrt{x}}\right)+C$$

$$= -\frac{2}{a}\left(\sqrt{\frac{a-x}{x}}\right)+C$$

**Question 4:**

$$\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$$

Answer

$$\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$$

Multiplying and dividing by $x^{-3}$, we obtain

$$\frac{x^{-3}}{x^2 \cdot x^{-3}\left(x^4+1\right)^{\frac{3}{4}}} = \frac{x^{-3}\left(x^4+1\right)^{\frac{-3}{4}}}{x^2 \cdot x^{-3}}$$

$$= \frac{\left(x^4+1\right)^{\frac{-3}{4}}}{x^5 \cdot \left(x^4\right)^{\frac{-3}{4}}}$$

$$= \frac{1}{x^5}\left(\frac{x^4+1}{x^4}\right)^{\frac{-3}{4}}$$

$$= \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{\frac{-3}{4}}$$

Let $\dfrac{1}{x^4} = t \;\Rightarrow\; -\dfrac{4}{x^5}dx = dt \Rightarrow \dfrac{1}{x^5}dx = -\dfrac{dt}{4}$

$$\therefore \int \frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}\,dx = \int \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{\frac{-3}{4}}dx$$

$$= -\frac{1}{4}\int (1+t)^{\frac{-3}{4}}\,dt$$

$$= -\frac{1}{4}\left[\frac{(1+t)^{\frac{1}{4}}}{\frac{1}{4}}\right] + C$$

$$= -\frac{1}{4}\frac{\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}}{\frac{1}{4}} + C$$

$$= -\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}} + C$$

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**Question 5:**

$$\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}\left[\text{Hint:}\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}=\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}\text{Put }x=t^6\right]$$

Answer

$$\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}=\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}$$

Let $x=t^6 \Rightarrow dx=6t^5 dt$

$$\therefore \int \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx = \int \frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}dx$$

$$=\int \frac{6t^5}{t^2(1+t)}dt$$

$$=6\int \frac{t^3}{(1+t)}dt$$

On dividing, we obtain

$$\int \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx = 6\int \left\{\left(t^2-t+1\right)-\frac{1}{1+t}\right\}dt$$

$$=6\left[\left(\frac{t^3}{3}\right)-\left(\frac{t^2}{2}\right)+t-\log|1+t|\right]$$

$$=2x^{\frac{1}{2}}-3x^{\frac{1}{3}}+6x^{\frac{1}{6}}-6\log\left(1+x^{\frac{1}{6}}\right)+C$$

$$=2\sqrt{x}-3x^{\frac{1}{3}}+6x^{\frac{1}{6}}-6\log\left(1+x^{\frac{1}{6}}\right)+C$$

**Question 6:**

$$\frac{5x}{(x+1)\left(x^2+9\right)}$$

Answer

Let $\dfrac{5x}{(x+1)(x^2+9)} = \dfrac{A}{(x+1)} + \dfrac{Bx+C}{(x^2+9)}$ ...(1)

$\Rightarrow 5x = A(x^2+9) + (Bx+C)(x+1)$

$\Rightarrow 5x = Ax^2 + 9A + Bx^2 + Bx + Cx + C$

Equating the coefficients of $x^2$, $x$, and constant term, we obtain

$A + B = 0$

$B + C = 5$

$9A + C = 0$

On solving these equations, we obtain

$A = -\dfrac{1}{2}, B = \dfrac{1}{2}$, and $C = \dfrac{9}{2}$

From equation (1), we obtain

$$\dfrac{5x}{(x+1)(x^2+9)} = \dfrac{-1}{2(x+1)} + \dfrac{\dfrac{x}{2}+\dfrac{9}{2}}{(x^2+9)}$$

$$\int \dfrac{5x}{(x+1)(x^2+9)}\,dx = \int \left\{ \dfrac{-1}{2(x+1)} + \dfrac{(x+9)}{2(x^2+9)} \right\} dx$$

$$= -\dfrac{1}{2}\log|x+1| + \dfrac{1}{2}\int \dfrac{x}{x^2+9}\,dx + \dfrac{9}{2}\int \dfrac{1}{x^2+9}\,dx$$

$$= -\dfrac{1}{2}\log|x+1| + \dfrac{1}{4}\int \dfrac{2x}{x^2+9}\,dx + \dfrac{9}{2}\int \dfrac{1}{x^2+9}\,dx$$

$$= -\dfrac{1}{2}\log|x+1| + \dfrac{1}{4}\log|x^2+9| + \dfrac{9}{2}\cdot\dfrac{1}{3}\tan^{-1}\dfrac{x}{3}$$

$$= -\dfrac{1}{2}\log|x+1| + \dfrac{1}{4}\log(x^2+9) + \dfrac{3}{2}\tan^{-1}\dfrac{x}{3} + C$$

**Question 7:**

$\dfrac{\sin x}{\sin(x-a)}$

Answer

$$\frac{\sin x}{\sin(x-a)}$$

Let $x - a = t \implies dx = dt$

$$\int \frac{\sin x}{\sin(x-a)} dx = \int \frac{\sin(t+a)}{\sin t} dt$$

$$= \int \frac{\sin t \cos a + \cos t \sin a}{\sin t} dt$$

$$= \int (\cos a + \cot t \sin a) dt$$

$$= t \cos a + \sin a \log|\sin t| + C_1$$

$$= (x-a)\cos a + \sin a \log|\sin(x-a)| + C_1$$

$$= x \cos a + \sin a \log|\sin(x-a)| - a \cos a + C_1$$

$$= \sin a \log|\sin(x-a)| + x \cos a + C$$

**Question 8:**

$$\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$$

Answer

$$\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} = \frac{e^{4\log x}\left(e^{\log x} - 1\right)}{e^{2\log x}\left(e^{\log x} - 1\right)}$$

$$= e^{2\log x}$$

$$= e^{\log x^2}$$

$$= x^2$$

$$\therefore \int \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} dx = \int x^2 dx = \frac{x^3}{3} + C$$

**Question 9:**

$$\frac{\cos x}{\sqrt{4 - \sin^2 x}}$$

Answer

$$\frac{\cos x}{\sqrt{4-\sin^2 x}}$$

Let $\sin x = t \implies \cos x \, dx = dt$

$$\Rightarrow \int \frac{\cos x}{\sqrt{4-\sin^2 x}} dx = \int \frac{dt}{\sqrt{(2)^2 - (t)^2}}$$

$$= \sin^{-1}\left(\frac{t}{2}\right) + C$$

$$= \sin^{-1}\left(\frac{\sin x}{2}\right) + C$$

**Question 10:**

$$\frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x}$$

Answer

$$\frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} = \frac{\left(\sin^4 x + \cos^4 x\right)\left(\sin^4 x - \cos^4 x\right)}{\sin^2 x + \cos^2 x - \sin^2 x \cos^2 x - \sin^2 x \cos^2 x}$$

$$= \frac{\left(\sin^4 x + \cos^4 x\right)\left(\sin^2 x + \cos^2 x\right)\left(\sin^2 x - \cos^2 x\right)}{\left(\sin^2 x - \sin^2 x \cos^2 x\right) + \left(\cos^2 x - \sin^2 x \cos^2 x\right)}$$

$$= \frac{\left(\sin^4 x + \cos^4 x\right)\left(\sin^2 x - \cos^2 x\right)}{\sin^2 x \left(1 - \cos^2 x\right) + \cos^2 x \left(1 - \sin^2 x\right)}$$

$$= \frac{-\left(\sin^4 x + \cos^4 x\right)\left(\cos^2 x - \sin^2 x\right)}{\left(\sin^4 x + \cos^4 x\right)}$$

$$= -\cos 2x$$

$$\therefore \int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx = \int -\cos 2x \, dx = -\frac{\sin 2x}{2} + C$$

**Question 11:**

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Answer

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Multiplying and dividing by $\sin(a-b)$, we obtain

$$\frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin\left[(x+a)-(x+b)\right]}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)\cdot\cos(x+b)-\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)}{\cos(x+a)}-\frac{\sin(x+b)}{\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\tan(x+a)-\tan(x+b)\right]$$

$$\int\frac{1}{\cos(x+a)\cos(x+b)}dx=\frac{1}{\sin(a-b)}\int\left[\tan(x+a)-\tan(x+b)\right]dx$$

$$=\frac{1}{\sin(a-b)}\left[-\log\left|\cos(x+a)\right|+\log\left|\cos(x+b)\right|\right]+C$$

$$=\frac{1}{\sin(a-b)}\log\left|\frac{\cos(x+b)}{\cos(x+a)}\right|+C$$

**Question 12:**

$$\frac{x^3}{\sqrt{1-x^8}}$$

Answer

$$\frac{x^3}{\sqrt{1-x^8}}$$

Let $x^4 = t \square 4x^3\ dx = dt$

$$\Rightarrow \int \frac{x^3}{\sqrt{1-x^8}} dx = \frac{1}{4} \int \frac{dt}{\sqrt{1-t^2}}$$

$$= \frac{1}{4} \sin^{-1} t + C$$

$$= \frac{1}{4} \sin^{-1}(x^4) + C$$

**Question 13:**

$$\frac{e^x}{(1+e^x)(2+e^x)}$$

Answer

$$\frac{e^x}{(1+e^x)(2+e^x)}$$

Let $e^x = t$ □ $e^x dx = dt$

$$\Rightarrow \int \frac{e^x}{(1+e^x)(2+e^x)} dx = \int \frac{dt}{(t+1)(t+2)}$$

$$= \int \left[ \frac{1}{(t+1)} - \frac{1}{(t+2)} \right] dt$$

$$= \log|t+1| - \log|t+2| + C$$

$$= \log \left| \frac{t+1}{t+2} \right| + C$$

$$= \log \left| \frac{1+e^x}{2+e^x} \right| + C$$

**Question 14:**

$$\frac{1}{(x^2+1)(x^2+4)}$$

Answer

$$\therefore \frac{1}{\left(x^2+1\right)\left(x^2+4\right)} = \frac{Ax+B}{\left(x^2+1\right)} + \frac{Cx+D}{\left(x^2+4\right)}$$

$$\Rightarrow 1 = \left(Ax+B\right)\left(x^2+4\right) + \left(Cx+D\right)\left(x^2+1\right)$$

$$\Rightarrow 1 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + Dx^2 + D$$

Equating the coefficients of $x^3$, $x^2$, $x$, and constant term, we obtain

$A + C = 0$

$B + D = 0$

$4A + C = 0$

$4B + D = 1$

On solving these equations, we obtain

$$A = 0, B = \frac{1}{3}, C = 0, \text{and } D = -\frac{1}{3}$$

From equation (1), we obtain

$$\frac{1}{\left(x^2+1\right)\left(x^2+4\right)} = \frac{1}{3\left(x^2+1\right)} - \frac{1}{3\left(x^2+4\right)}$$

$$\int \frac{1}{\left(x^2+1\right)\left(x^2+4\right)} dx = \frac{1}{3} \int \frac{1}{x^2+1} dx - \frac{1}{3} \int \frac{1}{x^2+4} dx$$

$$= \frac{1}{3} \tan^{-1} x - \frac{1}{3} \cdot \frac{1}{2} \tan^{-1} \frac{x}{2} + C$$

$$= \frac{1}{3} \tan^{-1} x - \frac{1}{6} \tan^{-1} \frac{x}{2} + C$$

**Question 15:**

$\cos^3 x e^{\log \sin x}$

Answer

$\cos^3 x e^{\log \sin x}$ = $\cos^3 x \times \sin x$

Let $\cos x = t \ \square\ -\sin x \, dx = dt$

$$\Rightarrow \int \cos^3 x\, e^{\log \sin x}\, dx = \int \cos^3 x \sin x\, dx$$

$$= -\int t \cdot dt$$

$$= -\frac{t^4}{4} + C$$

$$= -\frac{\cos^4 x}{4} + C$$

**Question 16:**

$$e^{3\log x}\left(x^4 + 1\right)^{-1}$$

Answer

$$e^{3\log x}\left(x^4 + 1\right)^{-1} = e^{\log x^3}\left(x^4 + 1\right)^{-1} = \frac{x^3}{\left(x^4 + 1\right)}$$

$$\text{Let } x^4 + 1 = t \;\Rightarrow\; 4x^3\, dx = dt$$

$$\Rightarrow \int e^{3\log x}\left(x^4 + 1\right)^{-1} dx = \int \frac{x^3}{\left(x^4 + 1\right)} dx$$

$$= \frac{1}{4}\int \frac{dt}{t}$$

$$= \frac{1}{4}\log|t| + C$$

$$= \frac{1}{4}\log\left|x^4 + 1\right| + C$$

$$= \frac{1}{4}\log\left(x^4 + 1\right) + C$$

**Question 17:**

$$f'(ax+b)\big[f(ax+b)\big]^n$$

Answer

$$f'(ax+b)\big[f(ax+b)\big]^n$$

Let $f(ax+b)=t \Rightarrow af'(ax+b)dx = dt$

$$\Rightarrow \int f'(ax+b)\big[f(ax+b)\big]^n dx = \frac{1}{a}\int t^n dt$$

$$= \frac{1}{a}\left[\frac{t^{n+1}}{n+1}\right]$$

$$= \frac{1}{a(n+1)}\big(f(ax+b)\big)^{n+1} + C$$

**Question 18:**

$$\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$$

Answer

$$\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} = \frac{1}{\sqrt{\sin^3 x (\sin x \cos\alpha + \cos x \sin\alpha)}}$$

$$= \frac{1}{\sqrt{\sin^4 x \cos\alpha + \sin^3 x \cos x \sin\alpha}}$$

$$= \frac{1}{\sin^2 x \sqrt{\cos\alpha + \cot x \sin\alpha}}$$

$$= \frac{\operatorname{cosec}^2 x}{\sqrt{\cos\alpha + \cot x \sin\alpha}}$$

Let $\cos\alpha + \cot x \sin\alpha = t \implies -\operatorname{cosec}^2 x \sin\alpha \, dx = dt$

$$\therefore \int \frac{1}{\sin^3 x \sin(x+\alpha)} dx = \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cos\alpha + \cot x \sin\alpha}} dx$$

$$= \frac{-1}{\sin\alpha} \int \frac{dt}{\sqrt{t}}$$

$$= \frac{-1}{\sin\alpha} \left[ 2\sqrt{t} \right] + C$$

$$= \frac{-1}{\sin\alpha} \left[ 2\sqrt{\cos\alpha + \cot x \sin\alpha} \right] + C$$

$$= \frac{-2}{\sin\alpha} \sqrt{\cos\alpha + \frac{\cos x \sin\alpha}{\sin x}} + C$$

$$= \frac{-2}{\sin\alpha} \sqrt{\frac{\sin x \cos\alpha + \cos x \sin\alpha}{\sin x}} + C$$

$$= -\frac{2}{\sin\alpha} \sqrt{\frac{\sin(x+\alpha)}{\sin x}} + C$$

**Question 19:**

$$\frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}, x \in [0,1]$$

Answer

Let $I = \int \dfrac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\, dx$

It is known that, $\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \dfrac{\pi}{2}$

$\Rightarrow I = \int \dfrac{\left(\dfrac{\pi}{2} - \cos^{-1}\sqrt{x}\right) - \cos^{-1}\sqrt{x}}{\dfrac{\pi}{2}}\, dx$

$= \dfrac{2}{\pi}\int\left(\dfrac{\pi}{2} - 2\cos^{-1}\sqrt{x}\right)dx$

$= \dfrac{2\pi}{\pi}\cdot\dfrac{1}{2}\int 1\cdot dx - \dfrac{4}{\pi}\int \cos^{-1}\sqrt{x}\, dx$

$= x - \dfrac{4}{\pi}\int \cos^{-1}\sqrt{x}\, dx \qquad\qquad \text{...(1)}$

Let $I_1 = \int \cos^{-1}\sqrt{x}\, dx$

Also, let $\sqrt{x} = t \;\Rightarrow\; dx = 2t\, dt$

$\Rightarrow I_1 = 2\int \cos^{-1} t \cdot t\, dt$

$= 2\left[\cos^{-1} t\cdot\dfrac{t^2}{2} - \int \dfrac{-1}{\sqrt{1-t^2}}\cdot\dfrac{t^2}{2}\, dt\right]$

$= t^2\cos^{-1} t + \int \dfrac{t^2}{\sqrt{1-t^2}}\, dt$

$= t^2\cos^{-1} t - \int \dfrac{1-t^2-1}{\sqrt{1-t^2}}\, dt$

$= t^2\cos^{-1} t - \int \sqrt{1-t^2}\, dt + \int \dfrac{1}{\sqrt{1-t^2}}\, dt$

$= t^2\cos^{-1} t - \dfrac{t}{2}\sqrt{1-t^2} - \dfrac{1}{2}\sin^{-1} t + \sin^{-1} t$

$= t^2\cos^{-1} t - \dfrac{t}{2}\sqrt{1-t^2} + \dfrac{1}{2}\sin^{-1} t$

From equation (1), we obtain

$$I = x - \frac{4}{\pi}\left[t^2 \cos t - \frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1} t\right]$$

$$= x - \frac{4}{\pi}\left[x\cos^{-1}\sqrt{x} - \frac{\sqrt{x}}{2}\sqrt{1-x} + \frac{1}{2}\sin^{-1}\sqrt{x}\right]$$

$$= x - \frac{4}{\pi}\left[x\left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right) - \frac{\sqrt{x-x^2}}{2} + \frac{\pi}{2}\sin^{-1}\sqrt{x}\right]$$

$$= x - 2x + \frac{4x}{\pi}\sin^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x-x^2} - \frac{2}{\pi}\sin^{-1}\sqrt{x}$$

$$= -x + \frac{2}{\pi}\left[(2x-1)\sin^{-1}\sqrt{x}\right] + \frac{2}{\pi}\sqrt{x-x^2} + C$$

$$= \frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x-x^2} - x + C$$

**Question 20:**

$$\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$$

Answer

$$I = \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$$

Let $x = \cos^2\theta \Rightarrow dx = -2\sin\theta\cos\theta\,d\theta$

$$I = \int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\,(-2\sin\theta\cos\theta)\,d\theta$$

$$= -\int \sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\,\sin 2\theta\,d\theta$$

$$= -\int \tan\frac{\theta}{2}\cdot 2\sin\theta\cos\theta\,d\theta$$

$$= -2\int \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\cos\theta\,d\theta$$

$$= -4 \int \sin^2 \frac{\theta}{2} \cos \theta \, d\theta$$

$$= -4 \int \sin^2 \frac{\theta}{2} \cdot \left( 2\cos^2 \frac{\theta}{2} - 1 \right) d\theta$$

$$= -4 \int \left( 2\sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} \right) d\theta$$

$$= -8 \int \sin^2 \frac{\theta}{2} \cdot \cos^2 \frac{\theta}{2} \, d\theta + 4 \int \sin^2 \frac{\theta}{2} \, d\theta$$

$$= -2 \int \sin^2 \theta \, d\theta + 4 \int \sin^2 \frac{\theta}{2} \, d\theta$$

$$= -2 \int \left( \frac{1 - \cos 2\theta}{2} \right) d\theta + 4 \int \frac{1 - \cos \theta}{2} \, d\theta$$

$$= -2 \left[ \frac{\theta}{2} - \frac{\sin 2\theta}{4} \right] + 4 \left[ \frac{\theta}{2} - \frac{\sin \theta}{2} \right] + C$$

$$= -\theta + \frac{\sin 2\theta}{2} + 2\theta - 2\sin \theta + C$$

$$= \theta + \frac{\sin 2\theta}{2} - 2\sin \theta + C$$

$$= \theta + \frac{2\sin \theta \cos \theta}{2} - 2\sin \theta + C$$

$$= \theta + \sqrt{1 - \cos^2 \theta} \cdot \cos \theta - 2\sqrt{1 - \cos^2 \theta} + C$$

$$= \cos^{-1} \sqrt{x} + \sqrt{1 - x} \cdot \sqrt{x} - 2\sqrt{1 - x} + C$$

$$= -2\sqrt{1 - x} + \cos^{-1} \sqrt{x} + \sqrt{x(1 - x)} + C$$

$$= -2\sqrt{1 - x} + \cos^{-1} \sqrt{x} + \sqrt{x - x^2} + C$$

**Question 21:**

$$\frac{2 + \sin 2x}{1 + \cos 2x} e^x$$

Answer

$$I = \int \left( \frac{2 + \sin 2x}{1 + \cos 2x} \right) e^x$$

$$= \int \left( \frac{2 + 2\sin x \cos x}{2\cos^2 x} \right) e^x$$

$$= \int \left( \frac{1 + \sin x \cos x}{\cos^2 x} \right) e^x$$

$$= \int \left( \sec^2 x + \tan x \right) e^x$$

Let $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$

$$\therefore I = \int \left( f(x) + f'(x) \right) e^x dx$$

$$= e^x f(x) + C$$

$$= e^x \tan x + C$$

**Question 22:**

$$\frac{x^2 + x + 1}{(x+1)^2 (x+2)}$$

Answer

Let $\dfrac{x^2 + x + 1}{(x+1)^2 (x+2)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+2)}$　　　...(1)

$\Rightarrow x^2 + x + 1 = A(x+1)(x+2) + B(x+2) + C(x^2 + 2x + 1)$

$\Rightarrow x^2 + x + 1 = A(x^2 + 3x + 2) + B(x+2) + C(x^2 + 2x + 1)$

$\Rightarrow x^2 + x + 1 = (A+C)x^2 + (3A + B + 2C)x + (2A + 2B + C)$

Equating the coefficients of $x^2$, $x$, and constant term, we obtain

$A + C = 1$

$3A + B + 2C = 1$

$2A + 2B + C = 1$

On solving these equations, we obtain

$A = -2$, $B = 1$, and $C = 3$

From equation (1), we obtain

$$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{-2}{(x+1)} + \frac{3}{(x+2)} + \frac{1}{(x+1)^2}$$

$$\int \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx = -2\int \frac{1}{x+1}\,dx + 3\int \frac{1}{(x+2)}\,dx + \int \frac{1}{(x+1)^2}\,dx$$

$$= -2\log|x+1| + 3\log|x+2| - \frac{1}{(x+1)} + C$$

**Question 23:**

$$\tan^{-1}\sqrt{\frac{1-x}{1+x}}$$

Answer

$$I = \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$$

Let $x = \cos\theta \implies dx = -\sin\theta\,d\theta$

$$I = \int \tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\,(-\sin\theta\,d\theta)$$

$$= -\int \tan^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\,\sin\theta\,d\theta$$

$$= -\int \tan^{-1}\tan\frac{\theta}{2}\cdot\sin\theta\,d\theta$$

$$= -\frac{1}{2}\int \theta\cdot\sin\theta\,d\theta$$

$$= -\frac{1}{2}\Big[\theta\cdot(-\cos\theta) - \int 1\cdot(-\cos\theta)\,d\theta\Big]$$

$$= -\frac{1}{2}\big[-\theta\cos\theta + \sin\theta\big]$$

$$= +\frac{1}{2}\theta\cos\theta - \frac{1}{2}\sin\theta$$

$$= \frac{1}{2}\cos^{-1}x\cdot x - \frac{1}{2}\sqrt{1-x^2} + C$$

$$= \frac{x}{2}\cos^{-1}x - \frac{1}{2}\sqrt{1-x^2} + C$$

$$= \frac{1}{2}\left(x\cos^{-1}x - \sqrt{1-x^2}\right) + C$$

**Question 24:**

$$\frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right)-2\log x\right]}{x^4}$$

Answer

$$\frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right)-2\log x\right]}{x^4}=\frac{\sqrt{x^2+1}}{x^4}\left[\log\left(x^2+1\right)-\log x^2\right]$$

$$=\frac{\sqrt{x^2+1}}{x^4}\left[\log\left(\frac{x^2+1}{x^2}\right)\right]$$

$$=\frac{\sqrt{x^2+1}}{x^4}\log\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{x^3}\sqrt{\frac{x^2+1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

Let $1+\dfrac{1}{x^2}=t \Rightarrow \dfrac{-2}{x^3}dx=dt$

$$\therefore I=\int\frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)dx$$

$$=-\frac{1}{2}\int\sqrt{t}\log t\, dt$$

$$=-\frac{1}{2}\int t^{\frac{1}{2}}\cdot\log t\, dt$$

Integrating by parts, we obtain

$$I = -\frac{1}{2}\left[\log t \cdot \int t^{\frac{1}{2}}dt - \left\{\left(\frac{d}{dt}\log t\right)\int t^{\frac{1}{2}}dt\right\}dt\right]$$

$$= -\frac{1}{2}\left[\log t \cdot \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \int \frac{1}{t} \cdot \frac{t^{\frac{3}{2}}}{\frac{3}{2}}dt\right]$$

$$= -\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t - \frac{2}{3}\int t^{\frac{1}{2}}dt\right]$$

$$= -\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t - \frac{4}{9}t^{\frac{3}{2}}\right]$$

$$= -\frac{1}{3}t^{\frac{3}{2}}\log t + \frac{2}{9}t^{\frac{3}{2}}$$

$$= -\frac{1}{3}t^{\frac{3}{2}}\left[\log t - \frac{2}{3}\right]$$

$$= -\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}\left[\log\left(1+\frac{1}{x^2}\right) - \frac{2}{3}\right] + C$$

**Question 25:**

$$\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$$

Answer

$$I = \int_{\frac{\pi}{2}}^{\pi} e^x \left( \frac{1 - \sin x}{1 - \cos x} \right) dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} e^x \left( \frac{1 - 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}} \right) dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} e^x \left( \frac{\csc^2\frac{x}{2}}{2} - \cot\frac{x}{2} \right) dx$$

$$\text{Let } f(x) = -\cot\frac{x}{2}$$

$$\Rightarrow f'(x) = -\left( -\frac{1}{2}\csc^2\frac{x}{2} \right) = \frac{1}{2}\csc^2\frac{x}{2}$$

$$\therefore I = \int_{\frac{\pi}{2}}^{\pi} e^x \left[ f(x) + f'(x) \right] dx$$

$$= \left[ e^x \cdot f(x) dx \right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\left[ e^x \cdot \cot\frac{x}{2} \right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\left[ e^\pi \times \cot\frac{\pi}{2} - e^{\frac{\pi}{2}} \times \cot\frac{\pi}{4} \right]$$

$$= -\left[ e^\pi \times 0 - e^{\frac{\pi}{2}} \times 1 \right]$$

$$= e^{\frac{\pi}{2}}$$

**Question 26:**

$$\int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$$

Answer

Let $I = \int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\dfrac{(\sin x \cos x)}{\cos^4 x}}{\dfrac{(\cos^4 x + \sin^4 x)}{\cos^4 x}} dx$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\tan x \sec^2 x}{1 + \tan^4 x} dx$

Let $\tan^2 x = t \Rightarrow 2 \tan x \sec^2 x \, dx = dt$

When $x = 0$, $t = 0$ and $\quad$ when $x = \dfrac{\pi}{4}, t = 1$

$\therefore I = \frac{1}{2} \int_0^1 \frac{dt}{1+t^2}$

$= \frac{1}{2} \left[ \tan^{-1} t \right]_0^1$

$= \frac{1}{2} \left[ \tan^{-1} 1 - \tan^{-1} 0 \right]$

$= \frac{1}{2} \left[ \frac{\pi}{4} \right]$

$= \frac{\pi}{8}$

**Question 27:**

$\int_0^{\frac{\pi}{2}} \frac{\cos^2 x \, dx}{\cos^2 x + 4 \sin^2 x}$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x}\, dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4\left(1 - \cos^2 x\right)}\, dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4 - 4\cos^2 x}\, dx$

$\Rightarrow I = \frac{-1}{3} \int_0^{\frac{\pi}{2}} \frac{4 - 3\cos^2 x - 4}{4 - 3\cos^2 x}\, dx$

$\Rightarrow I = \frac{-1}{3} \int_0^{\frac{\pi}{2}} \frac{4 - 3\cos^2 x}{4 - 3\cos^2 x}\, dx + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4}{4 - 3\cos^2 x}\, dx$

$\Rightarrow I = \frac{-1}{3} \int_0^{\frac{\pi}{2}} 1\, dx + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4\sec^2 x - 3}\, dx$

$\Rightarrow I = \frac{-1}{3}\left[x\right]_0^{\frac{\pi}{2}} + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4\left(1 + \tan^2 x\right) - 3}\, dx$

$\Rightarrow I = -\frac{\pi}{6} + \frac{2}{3} \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1 + 4\tan^2 x}\, dx \qquad \qquad \qquad ...(1)$

Consider, $\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1 + 4\tan^2 x}\, dx$

Let $2\tan x = t \Rightarrow 2\sec^2 x\, dx = dt$

When $x = 0, t = 0$ and when $x = \dfrac{\pi}{2}, t = \infty$

$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1 + 4\tan^2 x}\, dx = \int_0^{\infty} \frac{dt}{1 + t^2}$

$\qquad\qquad\qquad\qquad = \left[\tan^{-1} t\right]_0^{\infty}$

$\qquad\qquad\qquad\qquad = \left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$

$\qquad\qquad\qquad\qquad = \frac{\pi}{2}$

Therefore, from $(1)$, we obtain

$I = -\frac{\pi}{6} + \frac{2}{3}\left[\frac{\pi}{2}\right] = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

**Question 28:**

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

Answer

Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$

$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)}{\sqrt{-(-\sin 2x)}} dx$

$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{-(-1+1-2\sin x \cos x)}} dx$

$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)}{\sqrt{1-(\sin^2 x + \cos^2 x - 2\sin x \cos x)}} dx$

$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x) dx}{\sqrt{1-(\sin x - \cos x)^2}}$

Let $(\sin x - \cos x) = t \Rightarrow (\sin x + \cos x) dx = dt$

When $x = \dfrac{\pi}{6}, t = \left(\dfrac{1-\sqrt{3}}{2}\right)$  and when $x = \dfrac{\pi}{3}, t = \left(\dfrac{\sqrt{3}-1}{2}\right)$

$I = \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$

$\Rightarrow I = \int_{-\left(\frac{\sqrt{3}-1}{2}\right)}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$

As $\dfrac{1}{\sqrt{1-(-t)^2}} = \dfrac{1}{\sqrt{1-t^2}}$ , therefore, $\dfrac{1}{\sqrt{1-t^2}}$ is an even function.

It is known that if $f(x)$ is an even function, then $\int_{-a}^{a} f(x) dx = 2\int_{0}^{a} f(x) dx$

$$\Rightarrow I = 2\int_0^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$= \left[2\sin^{-1}t\right]_0^{\frac{\sqrt{3}-1}{2}}$$

$$= 2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

**Question 29:**

$$\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$$

Answer

$$\text{Let } I = \int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$$

$$I = \int_0^1 \frac{1}{\left(\sqrt{1+x}-\sqrt{x}\right)} \times \frac{\left(\sqrt{1+x}+\sqrt{x}\right)}{\left(\sqrt{1+x}+\sqrt{x}\right)} dx$$

$$= \int_0^1 \frac{\sqrt{1+x}+\sqrt{x}}{1+x-x} dx$$

$$= \int_0^1 \sqrt{1+x}\,dx + \int_0^1 \sqrt{x}\,dx$$

$$= \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^1 + \left[\frac{2}{3}(x)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{2}{3}\left[(2)^{\frac{3}{2}}-1\right] + \frac{2}{3}[1]$$

$$= \frac{2}{3}(2)^{\frac{3}{2}}$$

$$= \frac{2\cdot 2\sqrt{2}}{3}$$

$$= \frac{4\sqrt{2}}{3}$$

**Question 30:**

$$\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$$

Answer

Let $I = \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$

Also, let $\sin x - \cos x = t \Rightarrow (\cos x + \sin x) dx = dt$

When $x = 0$, $t = -1$ and when $x = \frac{\pi}{4}$, $t = 0$

$\Rightarrow (\sin x - \cos x)^2 = t^2$

$\Rightarrow \sin^2 x + \cos^2 x - 2 \sin x \cos x = t^2$

$\Rightarrow 1 - \sin 2x = t^2$

$\Rightarrow \sin 2x = 1 - t^2$

$$\therefore I = \int_{-1}^{0} \frac{dt}{9 + 16(1 - t^2)}$$

$$= \int_{-1}^{0} \frac{dt}{9 + 16 - 16t^2}$$

$$= \int_{-1}^{0} \frac{dt}{25 - 16t^2} = \int_{-1}^{0} \frac{dt}{(5)^2 - (4t)^2}$$

$$= \frac{1}{4} \left[ \frac{1}{2(5)} \log \left| \frac{5 + 4t}{5 - 4t} \right| \right]_{-1}^{0}$$

$$= \frac{1}{40} \left[ \log(1) - \log \left| \frac{1}{9} \right| \right]$$

$$= \frac{1}{40} \log 9$$

**Question 31:**

$$\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x) dx$$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\,dx = \int_0^{\frac{\pi}{2}} 2\sin x \cos x \tan^{-1}(\sin x)\,dx$

Also, let $\sin x = t \Rightarrow \cos x\,dx = dt$

When $x = 0, t = 0$ and when $x = \dfrac{\pi}{2}, t = 1$

$$\Rightarrow I = 2\int_0^1 t \tan^{-1}(t)\,dt \qquad \qquad \ldots(1)$$

Consider $\int t \cdot \tan^{-1} t\,dt = \tan^{-1} t \cdot \int t\,dt - \int \left\{ \dfrac{d}{dt}\left(\tan^{-1} t\right) \int t\,dt \right\} dt$

$$= \tan^{-1} t \cdot \dfrac{t^2}{2} - \int \dfrac{1}{1+t^2} \cdot \dfrac{t^2}{2}\,dt$$

$$= \dfrac{t^2 \tan^{-1} t}{2} - \dfrac{1}{2} \int \dfrac{t^2 + 1 - 1}{1 + t^2}\,dt$$

$$= \dfrac{t^2 \tan^{-1} t}{2} - \dfrac{1}{2}\int 1\,dt + \dfrac{1}{2}\int \dfrac{1}{1+t^2}\,dt$$

$$= \dfrac{t^2 \tan^{-1} t}{2} - \dfrac{1}{2}\cdot t + \dfrac{1}{2}\tan^{-1} t$$

$$\Rightarrow \int_0^1 t \cdot \tan^{-1} t\,dt = \left[ \dfrac{t^2 \cdot \tan^{-1} t}{2} - \dfrac{t}{2} + \dfrac{1}{2}\tan^{-1} t \right]_0^1$$

$$= \dfrac{1}{2}\left[ \dfrac{\pi}{4} - 1 + \dfrac{\pi}{4} \right]$$

$$= \dfrac{1}{2}\left[ \dfrac{\pi}{2} - 1 \right] = \dfrac{\pi}{4} - \dfrac{1}{2}$$

From equation (1), we obtain

$$I = 2\left[ \dfrac{\pi}{4} - \dfrac{1}{2} \right] = \dfrac{\pi}{2} - 1$$

**Question 32:**

$$\int_0^\pi \dfrac{x \tan x}{\sec x + \tan x}\,dx$$

Answer

$Let\ I = \int_0^\pi \dfrac{x \tan x}{\sec x + \tan x} dx$     $\qquad ...(1)$

$I = \int_0^\pi \left\{ \dfrac{(\pi - x) \tan (\pi - x)}{\sec (\pi - x) + \tan (\pi - x)} \right\} dx$     $\qquad \left( \int_0^a f(x) dx = \int_0^a f(a - x) dx \right)$

$\Rightarrow I = \int_0^\pi \left\{ \dfrac{-(\pi - x) \tan x}{-(\sec x + \tan x)} \right\} dx$

$\Rightarrow I = \int_0^\pi \dfrac{(\pi - x) \tan x}{\sec x + \tan x} dx$     $\qquad ...(2)$

Adding (1) and (2), we obtain

$2I = \int_0^\pi \dfrac{\pi \tan x}{\sec x + \tan x} dx$

$\Rightarrow 2I = \pi \int_0^\pi \dfrac{\dfrac{\sin x}{\cos x}}{\dfrac{1}{\cos x} + \dfrac{\sin x}{\cos x}} dx$

$\Rightarrow 2I = \pi \int_0^\pi \dfrac{\sin x + 1 - 1}{1 + \sin x} dx$

$\Rightarrow 2I = \pi \int_0^\pi 1 . dx - \pi \int_0^\pi \dfrac{1}{1 + \sin x} dx$

$\Rightarrow 2I = \pi [x]_0^\pi - \pi \int_0^\pi \dfrac{1 - \sin x}{\cos^2 x} dx$

$\Rightarrow 2I = \pi^2 - \pi \int_0^\pi \left( \sec^2 x - \tan x \sec x \right) dx$

$\Rightarrow 2I = \pi^2 - \pi [\tan x - \sec x]_0^\pi$

$\Rightarrow 2I = \pi^2 - \pi [\tan \pi - \sec \pi - \tan 0 + \sec 0]$

$\Rightarrow 2I = \pi^2 - \pi [0 - (-1) - 0 + 1]$

$\Rightarrow 2I = \pi^2 - 2\pi$

$\Rightarrow 2I = \pi(\pi - 2)$

$\Rightarrow I = \dfrac{\pi}{2}(\pi - 2)$

**Question 33:**

$$\int_1^4 \left[ |x-1| + |x-2| + |x-3| \right] dx$$

Answer

Let $I = \int_1^4 \left[ |x-1| + |x-2| + |x-3| \right] dx$

$\Rightarrow I = \int_1^4 |x-1| dx + \int_1^4 |x-2| dx + \int_1^4 |x-3| dx$

$I = I_1 + I_2 + I_3$  ...(1)

where, $I_1 = \int_1^4 |x-1| dx$, $I_2 = \int_1^4 |x-2| dx$, and $I_3 = \int_1^4 |x-3| dx$

$I_1 = \int_1^4 |x-1| dx$

$(x-1) \geq 0$ for $1 \leq x \leq 4$

$\therefore I_1 = \int_1^4 (x-1) dx$

$\Rightarrow I_1 = \left[ \frac{x^2}{x} - x \right]_1^4$

$\Rightarrow I_1 = \left[ 8 - 4 - \frac{1}{2} + 1 \right] = \frac{9}{2}$  ...(2)

$I_2 = \int_1^4 |x-2| dx$

$x - 2 \geq 0$ for $2 \leq x \leq 4$ and $x - 2 \leq 0$ for $1 \leq x \leq 2$

$\therefore I_2 = \int_1^2 (2-x) dx + \int_2^4 (x-2) dx$

$\Rightarrow I_2 = \left[ 2x - \frac{x^2}{2} \right]_1^2 + \left[ \frac{x^2}{2} - 2x \right]_2^4$

$\Rightarrow I_2 = \left[ 4 - 2 - 2 + \frac{1}{2} \right] + [8 - 8 - 2 + 4]$

$\Rightarrow I_2 = \frac{1}{2} + 2 = \frac{5}{2}$  ...(3)

$I_3 = \int_1^4 |x-3| dx$

$x - 3 \geq 0$ for $3 \leq x \leq 4$ and $x - 3 \leq 0$ for $1 \leq x \leq 3$

$\therefore I_3 = \int_1^3 (3-x) dx + \int_3^4 (x-3) dx$

$\Rightarrow I_3 = \left[ 3x - \frac{x^2}{2} \right]_1^3 + \left[ \frac{x^2}{2} - 3x \right]_3^4$

$\Rightarrow I_3 = \left[ 9 - \frac{9}{2} - 3 + \frac{1}{2} \right] + \left[ 8 - 12 - \frac{9}{2} + 9 \right]$

$\Rightarrow I_3 = [6-4] + \left[ \frac{1}{2} \right] = \frac{5}{2}$                    ...(4)

From equations (1), (2), (3), and (4), we obtain

$I = \frac{9}{2} + \frac{5}{2} + \frac{5}{2} = \frac{19}{2}$

**Question 34:**

$\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log \frac{2}{3}$

Answer

Let $I = \int_1^3 \frac{dx}{x^2(x+1)}$

Also, let $\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

$\Rightarrow 1 = Ax(x+1) + B(x+1) + C(x^2)$

$\Rightarrow 1 = Ax^2 + Ax + Bx + B + Cx^2$

Equating the coefficients of $x^2$, $x$, and constant term, we obtain

$A + C = 0$

$A + B = 0$

$B = 1$

On solving these equations, we obtain

$A = -1$, $C = 1$, and $B = 1$

$$\therefore \frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)}$$

$$\Rightarrow I = \int_1^3 \left\{ -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)} \right\} dx$$

$$= \left[ -\log x - \frac{1}{x} + \log(x+1) \right]_1^3$$

$$= \left[ \log\left(\frac{x+1}{x}\right) - \frac{1}{x} \right]_1^3$$

$$= \log\left(\frac{4}{3}\right) - \frac{1}{3} - \log\left(\frac{2}{1}\right) + 1$$

$$= \log 4 - \log 3 - \log 2 + \frac{2}{3}$$

$$= \log 2 - \log 3 + \frac{2}{3}$$

$$= \log\left(\frac{2}{3}\right) + \frac{2}{3}$$

Hence, the given result is proved.

**Question 35:**

$$\int_0^1 xe^x dx = 1$$

Answer

Let $I = \int_0^1 xe^x \, dx$

Integrating by parts, we obtain

$$I = x \int_0^1 e^x dx - \int_0^1 \left\{ \left( \frac{d}{dx}(x) \right) \int e^x dx \right\} dx$$

$$= \left[ xe^x \right]_0^1 - \int_0^1 e^x dx$$

$$= \left[ xe^x \right]_0^1 - \left[ e^x \right]_0^1$$

$$= e - e + 1$$

$$= 1$$

Hence, the given result is proved.

**Question 36:**

$$\int_{-1}^{1} x^{17} \cos^4 x\, dx = 0$$

Answer

Let $I = \int_{-1}^{1} x^{17} \cos^4 x\, dx$

Also, let $f(x) = x^{17} \cos^4 x$

$\Rightarrow f(-x) = (-x)^{17} \cos^4(-x) = -x^{17} \cos^4 x = -f(x)$

Therefore, $f(x)$ is an odd function.

It is known that if $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\, dx = 0$

$\therefore I = \int_{-1}^{1} x^{17} \cos^4 x\, dx = 0$

Hence, the given result is proved.

**Question 37:**

$$\int_{0}^{\frac{\pi}{2}} \sin^3 x\, dx = \frac{2}{3}$$

Answer

Let $I = \int_{0}^{\frac{\pi}{2}} \sin^3 x\, dx$

$I = \int_{0}^{\frac{\pi}{2}} \sin^2 x \cdot \sin x\, dx$

$= \int_{0}^{\frac{\pi}{2}} (1 - \cos^2 x) \sin x\, dx$

$= \int_{0}^{\frac{\pi}{2}} \sin x\, dx - \int_{0}^{\frac{\pi}{2}} \cos^2 x \cdot \sin x\, dx$

$= \left[ -\cos x \right]_{0}^{\frac{\pi}{2}} + \left[ \frac{\cos^3 x}{3} \right]_{0}^{\frac{\pi}{2}}$

$= 1 + \frac{1}{3}[-1] = 1 - \frac{1}{3} = \frac{2}{3}$

Hence, the given result is proved.

**Question 38:**

$$\int_0^{\frac{\pi}{4}} 2\tan^3 x\,dx = 1 - \log 2$$

Answer

Let $I = \int_0^{\frac{\pi}{4}} 2\tan^3 x\,dx$

$I = 2\int_0^{\frac{\pi}{4}} \tan^2 x \tan x\,dx = 2\int_0^{\frac{\pi}{4}} \left(\sec^2 x - 1\right)\tan x\,dx$

$\quad = 2\int_0^{\frac{\pi}{4}} \sec^2 x \tan x\,dx - 2\int_0^{\frac{\pi}{4}} \tan x\,dx$

$\quad = 2\left[\dfrac{\tan^2 x}{2}\right]_0^{\frac{\pi}{4}} + 2\left[\log\cos x\right]_0^{\frac{\pi}{4}}$

$\quad = 1 + 2\left[\log\cos\dfrac{\pi}{4} - \log\cos 0\right]$

$\quad = 1 + 2\left[\log\dfrac{1}{\sqrt{2}} - \log 1\right]$

$\quad = 1 - \log 2 - \log 1 = 1 - \log 2$

Hence, the given result is proved.

**Question 39:**

$$\int_0^1 \sin^{-1} x\,dx = \frac{\pi}{2} - 1$$

Answer

Let $I = \int_0^1 \sin^{-1} x\,dx$

$\Rightarrow I = \int_0^1 \sin^{-1} x \cdot 1 \cdot dx$

Integrating by parts, we obtain

$$I = \left[\sin^{-1} x \cdot x\right]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}} \cdot x\, dx$$

$$= \left[x \sin^{-1} x\right]_0^1 + \frac{1}{2} \int_0^1 \frac{(-2x)}{\sqrt{1-x^2}}\, dx$$

Let $1 - x^2 = t \ \Box \ -2x\, dx = dt$

When $x = 0,\ t = 1$ and when $x = 1,\ t = 0$

$$I = \left[x \sin^{-1} x\right]_0^1 + \frac{1}{2} \int_1^0 \frac{dt}{\sqrt{t}}$$

$$= \left[x \sin^{-1} x\right]_0^1 + \frac{1}{2}\left[2\sqrt{t}\right]_1^0$$

$$= \sin^{-1}(1) + \left[-\sqrt{1}\right]$$

$$= \frac{\pi}{2} - 1$$

Hence, the given result is proved.


**Question 40:**

Evaluate $\int_0^1 e^{2-3x}\, dx$ as a limit of a sum.

Answer

Let $I = \int_0^1 e^{2-3x}\, dx$

It is known that,

$$\int_a^b f(x)\, dx = (b-a)\lim_{n \to \infty} \frac{1}{n}\left[f(a) + f(a+h) + \ldots + f\big(a + (n-1)h\big)\right]$$

Where, $h = \dfrac{b-a}{n}$

Here, $a = 0, b = 1,$ and $f(x) = e^{2-3x}$

$$\Rightarrow h = \frac{1-0}{n} = \frac{1}{n}$$

$$\therefore \int_0^1 e^{2-3x}\, dx = (1-0)\lim_{n \to \infty} \frac{1}{n}\left[f(0) + f(0+h) + \ldots + f\big(0 + (n-1)h\big)\right]$$

$$= \lim_{n \to \infty} \frac{1}{n}\left[e^2 + e^{2-3h} + \ldots e^{2-3(n-1)h}\right]$$

$$= \lim_{n \to \infty} \frac{1}{n} \left[ e^2 \left\{ 1 + e^{-3h} + e^{-6h} + e^{-9h} + \ldots e^{-3(n-1)h} \right\} \right]$$

$$= \lim_{h \to \infty} \frac{1}{n} \left[ e^2 \left\{ \frac{1 - \left( e^{-3h} \right)^n}{1 - \left( e^{-3h} \right)} \right\} \right]$$

$$= \lim_{n \to \infty} \frac{1}{n} \left[ e^2 \left\{ \frac{1 - e^{-\frac{3}{n} \times n}}{1 - e^{-\frac{3}{n}}} \right\} \right]$$

$$= \lim_{n \to \infty} \frac{1}{n} \left[ \frac{e^2 \left( 1 - e^{-3} \right)}{1 - e^{-\frac{3}{n}}} \right]$$

$$= e^2 \left( e^{-3} - 1 \right) \lim_{n \to \infty} \frac{1}{n} \left[ \frac{1}{e^{-\frac{3}{n}} - 1} \right]$$

$$= e^2 \left( e^{-3} - 1 \right) \lim_{n \to \infty} \left( -\frac{1}{3} \right) \left[ \frac{-\frac{3}{n}}{e^{-\frac{3}{n}} - 1} \right]$$

$$= \frac{-e^2 \left( e^{-3} - 1 \right)}{3} \lim_{n \to \infty} \left[ \frac{-\frac{3}{n}}{e^{-\frac{3}{n}} - 1} \right]$$

$$= \frac{-e^2 \left( e^{-3} - 1 \right)}{3} (1) \qquad \left[ \lim_{n \to \infty} \frac{x}{e^x - 1} \right]$$

$$= \frac{-e^{-1} + e^2}{3}$$

$$= \frac{1}{3} \left( e^2 - \frac{1}{e} \right)$$

**Question 41:**

$\displaystyle \int \frac{dx}{e^x + e^{-x}}$ is equal to

**A.** $\tan^{-1} \left( e^x \right) + C$

**B.** $\tan^{-1}\left(e^{-x}\right)+C$

**C.** $\log\left(e^{x}-e^{-x}\right)+C$

**D.** $\log\left(e^{x}+e^{-x}\right)+C$

Answer

Let $I = \int \dfrac{dx}{e^{x}+e^{-x}}\,dx = \int \dfrac{e^{x}}{e^{2x}+1}\,dx$

Also, let $e^{x}=t \Rightarrow e^{x}\,dx=dt$

$\therefore I = \int \dfrac{dt}{1+t^{2}}$

$\qquad = \tan^{-1}t + C$

$\qquad = \tan^{-1}\left(e^{x}\right)+C$

Hence, the correct Answer is A.


**Question 42:**

$\int \dfrac{\cos 2x}{\left(\sin x+\cos x\right)^{2}}\,dx$   is equal to

**A.** $\dfrac{-1}{\sin x+\cos x}+C$

**B.** $\log\left|\sin x+\cos x\right|+C$

**C.** $\log\left|\sin x-\cos x\right|+C$

**D.** $\dfrac{1}{\left(\sin x+\cos x\right)^{2}}$

Answer

Let $I = \dfrac{\cos 2x}{(\cos x + \sin x)^2}$

$I = \displaystyle\int \dfrac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2}\,dx$

$\quad = \displaystyle\int \dfrac{(\cos x + \sin x)(\cos x - \sin x)}{(\cos x + \sin x)^2}\,dx$

$\quad = \displaystyle\int \dfrac{\cos x - \sin x}{\cos + \sin x}\,dx$

Let $\cos x + \sin x = t \Rightarrow (\cos x - \sin x)\,dx = dt$

$\therefore I = \displaystyle\int \dfrac{dt}{t}$

$\quad = \log|t| + C$

$\quad = \log|\cos x + \sin x| + C$

Hence, the correct Answer is B.


**Question 43:**

If $f(a + b - x) = f(x)$, then $\displaystyle\int_a^b x\,f(x)\,dx$ is equal to

A. $\dfrac{a+b}{2}\displaystyle\int_a^b f(b-x)\,dx$

B. $\dfrac{a+b}{2}\displaystyle\int_a^b f(b+x)\,dx$

C. $\dfrac{b-a}{2}\displaystyle\int_a^b f(x)\,dx$

D. $\dfrac{a+b}{2}\displaystyle\int_a^b f(x)\,dx$

Answer

Let $I = \displaystyle\int_a^b x\,f(x)\,dx$             ...(1)

$$I = \int_a^b (a+b-x) f(a+b-x) dx \qquad \left( \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right)$$

$$\Rightarrow I = \int_a^b (a+b-x) f(x) dx$$

$$\Rightarrow I = (a+b) \int_a^b f(x) dx \qquad -I \qquad \left[ \text{Using} (1) \right]$$

$$\Rightarrow I + I = (a+b) \int_a^b f(x) dx$$

$$\Rightarrow 2I = (a+b) \int_a^b f(x) dx$$

$$\Rightarrow I = \left( \frac{a+b}{2} \right) \int_a^b f(x) dx$$

Hence, the correct Answer is D.


**Question 44:**

The value of $\int_0^1 \tan^{-1}\left( \frac{2x-1}{1+x-x^2} \right) dx$ is

**A.** 1

**B.** 0

**C.** − 1

**D.** $\dfrac{\pi}{4}$

Answer

$$\text{Let } I = \int_0^1 \tan^{-1}\left( \frac{2x-1}{1+x-x^2} \right) dx$$

$$\Rightarrow I = \int_0^1 \tan^{-1}\left( \frac{x-(1-x)}{1+x(1-x)} \right) dx$$

$$\Rightarrow I = \int_0^1 \left[ \tan^{-1} x - \tan^{-1}(1-x) \right] dx \qquad \text{...}(1)$$

$$\Rightarrow I = \int_0^1 \left[ \tan^{-1}(1-x) - \tan^{-1}(1-1+x) \right] dx$$

$$\Rightarrow I = \int_0^1 \left[ \tan^{-1}(1-x) - \tan^{-1}(x) \right] dx$$

$$\Rightarrow I = \int_0^1 \left[ \tan^{-1}(1-x) - \tan^{-1}(x) \right] dx \qquad \text{...}(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^1 \left( \tan^{-1} x + \tan^{-1}(1-x) - \tan^{-1}(1-x) - \tan^{-1} x \right) dx$$
$$\Rightarrow 2I = 0$$
$$\Rightarrow I = 0$$

Hence, the correct Answer is B.