Introduction

In this chapter, rotational motion of a rigid body about a fixed axis of rotation is discussed. A rigid body is a system of particles in which interparticle distances do not change and the body cannot be deformed no matter how large a force is applied to it. Although a solid body is not a rigid body, it can be so considered for most of the practical applications
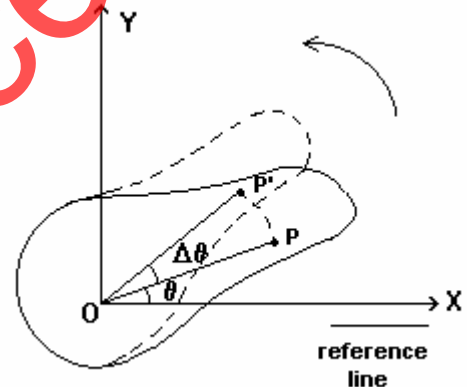
7.1 Rotational Kinematics and Dynamics

In rotational motion of a body, all its particles move on circular paths having centres on a definite straight line, called the axis of rotation. Kinematics deals with motion without considering its cause, whereas dynamics deals with motion alongwith its cause and properties of the body.

7.2 Relations between variables of rotational and linear motion

(a) Angular displacement:

The figure shows a rigid body rotating about a fixed axis OZ normal to the plane of the figure. P and P' are the positions of a particle of the body at time t and t + Δt. Angle θ made by the line joining the particle to the centre of its rotation with a reference line OX shows its angular position at time t. Similarly, angle θ + $\Delta\theta$ is its angular position at time t + Δt.

The change in angular position of a particle is called its angular displacement. The angular displacement of the particle P is $\Delta\theta$ in time Δ.

As the interparticle distances do not change in a rigid body, all its particles will have the same angular displacement in a given time. Hence, the angular displacement, $\Delta\theta$, of the particle P can be considered as the angular displacement of the rigid body.

(b) Angular speed and angular velocity:

The average angular speed of a particle or of the rigid body is defined as

$$< \omega > = \frac{\text{angular displacement}}{\text{time interval}} = \frac{\Delta \theta}{\Delta t}$$

The instantaneous angular speed of a particle or of the rigid body is given by

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t}$$ The unit of ω is radian / s or rotation / s.

The direction of angular velocity is given by the right handed screw rule. When a right handed screw is kept parallel to the axis of rotation and rotated in the direction of rotation of the body, the direction of advancement of screw gives the direction of angular velocity.

(c) Scalar relation between angular velocity and linear velocity:

As shown in the figure, the particle P covers a linear distance equal to the arc length PP'

in time Δt. Hence, average linear speed,

$$< v > = \frac{\text{arc length PP'}}{\Delta t} = \frac{r \Delta\theta}{\Delta t} = r < \omega >,$$ where r is the radius of the circular path.

The instantaneous linear speed is given by

$$v = \lim_{\Delta t \to 0} \frac{r \Delta\theta}{\Delta t} = \frac{r d\theta}{d t} = r \omega$$

(Note that the angular velocity of all particles of the rigid body rotating about a fixed axis of rotation is the same for all the particles, whereas the linear speed of a particle depends upon its distance (r) from the axis of rotation.)

Linear velocity is a vector quantity and its direction at any point on the path of motion is tangential to the path at that point. In the above equation, v, r and ω are the magnitudes of the vector quantities $\vec{v}$, $\vec{r}$ and $\vec{\omega}$.

Vector product of two vectors:

The vector product, also known as cross product, of two vectors $\vec{A}$ and $\vec{B}$ having an angle θ between them is defined as

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta \, \hat{n},$$

where $\hat{n}$ is the unit vector normal to the plane containing $\vec{A}$ and $\vec{B}$ having direction given by the right handed screw rule. When a right handed screw kept perpendicular to the plane containing $\vec{A}$ and $\vec{B}$ is rotated from $\vec{A}$ towards $\vec{B}$, the direction of advancement of screw gives the direction of $\hat{n}$.

Properties of vector product of two vectors:

(1) $\vec{A} \times \vec{B}$ $\vec{B} \times \vec{A}$

(2) $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

(3) The cross product of two parallel vectors, in the same or opposite directions, is a zero vector.

(4) If $\vec{A} \perp \vec{B}$, then $\vec{A} \times \vec{B} = AB \, \hat{n}$ ($\because \sin 90° = 1$)

(5) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$ and $\hat{i} \times \hat{j} = \hat{k},$ $\hat{j} \times \hat{k} = \hat{i},$ $\hat{k} \times \hat{i} = \hat{j}$

(6) If $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$ and $\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$, then
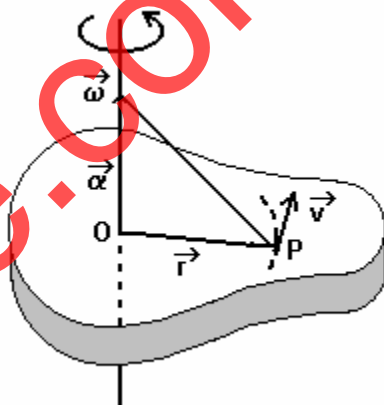
$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$$

$$\therefore \quad \overrightarrow{A} \times \overrightarrow{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

(d) Vector relation between angular velocity and linear velocity:

The position vector $\overrightarrow{r}$ w. r. t. the centre of the circular path of a particle, angular velocity $\overrightarrow{\omega}$ and linear velocity $\overrightarrow{v}$ are shown in the figure. As $\overrightarrow{v}$ is perpendicular to the plane formed by $\overrightarrow{\omega}$ and $\overrightarrow{r}$, the scalar relation $v = r\omega$ can be written in the vector form as

$$\overrightarrow{v} = \overrightarrow{\omega} \times \overrightarrow{r}$$

(e) Angular acceleration:

The average angular acceleration in the time interval Δt is

$$< \overrightarrow{\alpha} > = \frac{\text{change in angular velocity}}{\text{time interval}} = \frac{\Delta \overrightarrow{\omega}}{\Delta t}$$

and the instantaneous angular acceleration at time t is given by

$$\overrightarrow{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta \overrightarrow{\omega}}{\Delta t} = \frac{d \overrightarrow{\omega}}{d t} \quad \text{rate of change of angular velocity.}$$

$\overrightarrow{\alpha}$ is in the direction of $\Delta \overrightarrow{\omega}$ and in the case of fixed axis of rotation, both are parallel to the axis. The unit of $\overrightarrow{\alpha}$ is rad / s^2 or $rotation / s^2$.

(f) Relation between linear acceleration and angular acceleration :

Differentiating the equation $\overrightarrow{v} = \overrightarrow{\omega} \times \overrightarrow{r}$ with respect to time gives the linear acceleration

$$\overrightarrow{a} = \frac{d \overrightarrow{v}}{dt} = \frac{d}{dt} (\overrightarrow{\omega} \times \overrightarrow{r}) = \overrightarrow{\omega} \times \frac{d \overrightarrow{r}}{dt} + \frac{d \overrightarrow{\omega}}{dt} \times \overrightarrow{r}$$

$$= (\overrightarrow{\omega} \times \overrightarrow{v}) + (\overrightarrow{\alpha} \times \overrightarrow{r})$$

From the figure on the previous page and using right handed screw rule it can be found that the direction of $\overrightarrow{\omega} \times \overrightarrow{v}$ is radial towards the centre. Hence, it is called the radial component, $\overrightarrow{a_r}$, of the linear acceleration $\overrightarrow{a}$.

Similarly, the direction of $\vec{\alpha} \times \vec{r}$ is tangential to the circular path at the position of the particle. Hence it is called the tangential component, $\vec{a_T}$, of the linear acceleration $\vec{a}$.

$$\therefore \ \vec{a} = \vec{a_r} + \vec{a_T} \qquad \therefore \ a = \sqrt{a_r{}^2 + a_T{}^2} \quad (\because \ \vec{a_r} \perp \vec{a_T})$$
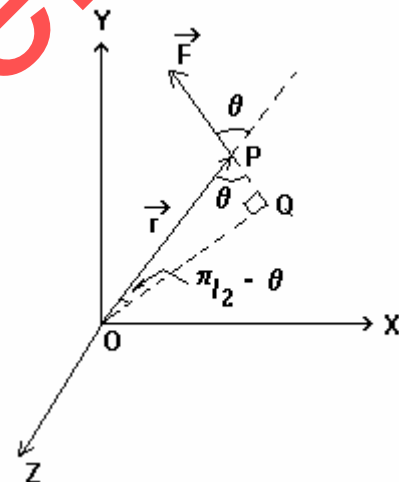
Even if $\alpha = 0$, that is the angular velocity is constant, $\vec{a_r}$ is not zero. As the angular displacement, θ, angular velocity, $\vec{\omega}$ and angular acceleration, $\vec{\alpha}$ are the same for all the particles of a rigid body, they are known as variables of the rotational kinematics.

7.4 Torque

Torque is a physical quantity of rotational dynamics whose role in rotational motion is similar to the role of force in translational motion.

(a) Torque acting on a particle:

Consider a particle P having position vector $\vec{r}$ w.r.t. the origin O of a co-ordinate system on which a force $\vec{F}$ acts as shown in the figure. The angle between $\vec{r}$ and $\vec{F}$ is θ.

The torque $\vec{\tau}$ acting on the particle w. r. t. point O is defined as

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$\therefore \ \tau = r\,F\sin\theta = (r\sin\theta) = F(OQ)$
    $= (F)(\text{Perpendicular distance of the line of action of force from O})$
    $= \text{Moment of force w. r. t. point O}$

Thus, torque is the moment of force w. r. t. an arbitrarily selected reference point.

$\vec{\tau}$ is a vector quantity. Its dimensional formula is $M^1 \, L^2 \, T^{-2}$ and its unit is Nm.

The direction of $\vec{\tau}$ is perpendicular to the plane formed by $\vec{r}$ and $\vec{F}$ and is obtained using the right handed screw rule. The magnitude and direction of torque depends on the selection of the reference point and hence it is necessary to mention the reference point while defining the torque of a particle.

(b) Torque acting on a system of particles:

As the internal forces between the particles of a system are equal in magnitude and opposite in direction, the resultant torque due to them is zero. The resultant torque acting on the system is the vector sum of torques acting on its particles having position vectors, $\vec{r_1}, \vec{r_2}, \dots, \vec{r_n}$ due to external forces, $\vec{F_1}, \vec{F_2}, \dots, \vec{F_n}$ respectively.

$$\therefore \ \vec{\tau} = \vec{r_1} \times \vec{F_1} + \vec{r_2} \times \vec{F_2} + \dots + \vec{r_n} \times \vec{F_n} = \vec{\tau_1} + \vec{\tau_2} + \dots + \vec{\tau_n}$$

(c) Torque acting on a rigid body:

The figure shows a rigid body rotating about a fixed axis OZ. $\vec{F_1}, \vec{F_2}, ... , \vec{F_n}$ are the forces acting on its particles with position vectors $\vec{r_1}, \vec{r_2}, ... , \vec{r_n}$ respectively.
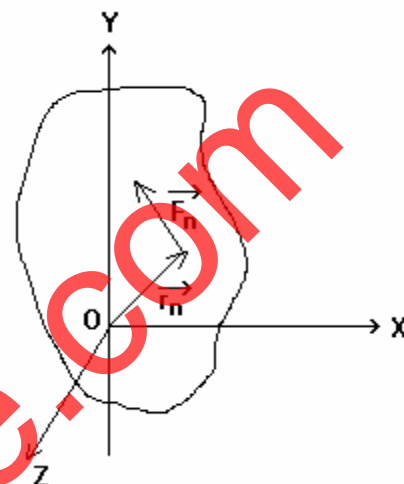
The torque, $\vec{\tau}$, acting on the rigid body is the vector sum of torques acting on its particles.

$$\therefore \vec{\tau} = \sum_n \vec{\tau_n} = \sum_n \vec{r_n} \times \vec{F_n}$$

$$= \sum_n \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_n & y_n & z_n \\ F_{nx} & F_{ny} & F_{nz} \end{vmatrix}$$

$$= \sum_n [y_n F_{nz} - z_n F_{ny}] \hat{i} + [z_n F_{nx} - x_n F_{nz}] \hat{j} + [x_n F_{ny} - y_n F_{nx}] \hat{k}$$
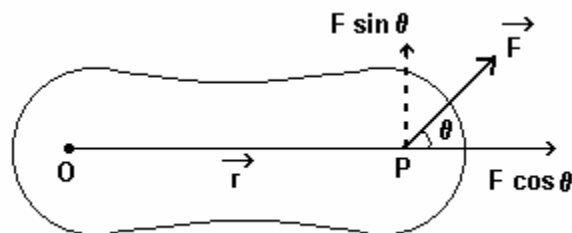
Here, as the rotation is considered about z-axis, only the z-component of torque is responsible for the rotation. In general, if rotation is about a fixed axis with $\hat{n}$ as the unit vector on it, the component of torque responsible for rotational motion is $\vec{\tau} \cdot \hat{n}$.

(d) Physical explanation of the definition of torque:

Suppose a force $\vec{F}$ acts on a particle P of a rigid body, having position vector $\vec{r}$ w. r. t. O, and the body rotates about a fixed axis passing through O and perpendicular to the plane of the figure. The angle between $\vec{F}$ and $\vec{r}$ is θ. The component of force, $\vec{F} \cos \theta$, parallel to $\vec{r}$ is ineffective in producing rotational motion. The component, $\vec{F} \sin \theta$, produces the rotational motion about the axis through O.
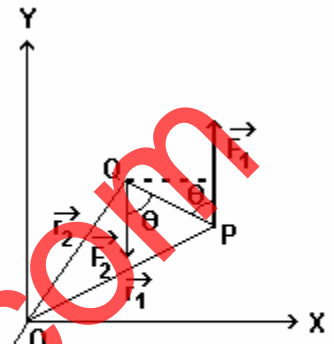
More effective will be the rotational motion, larger the value of $| \vec{F} |$, $| \vec{r} |$ and / or $\sin \theta$. Thus, the effectiveness of rotational motion depends on $rF\sin \theta$ which is known as torque.

Vectorially, Torque, $\vec{\tau} = \vec{r} \times \vec{F}$

Thus, torque is the measure of effectiveness of force in producing rotational motion.

(e) Couple:

"Two non-collinear forces, equal in magnitude and opposite in direction, constitute a couple. The figure shows forces, $\vec{F_1}$ and $\vec{F_2}$, acting at points P and Q of a rigid body having position vectors, $\vec{r_1}$ and $\vec{r_2}$, respectively w. r. t. to O. These forces are equal in magnitude and hence constitute a couple. Here,

$$\vec{\tau} = \vec{r_1} \times \vec{F_1} + \vec{r_2} \times \vec{F_2}$$

$$= \vec{r_1} \times \vec{F_1} - \vec{r_2} \times \vec{F_1} \qquad [\because \vec{F_2} = - \vec{F_1}]$$

$$= (\vec{r_1} - \vec{r_2}) \times \vec{F_1} = | \vec{r_1} - \vec{r_2} | (F_1) \sin (\pi - \theta)$$

$$= | \vec{r_1} - \vec{r_2} | F_1 \sin \theta, \text{ where } \pi - \theta \text{ is the angle between } \vec{r_1} - \vec{r_2} \text{ and } \vec{F_1} .$$

$\therefore$ **Moment of a couple = Magnitude of force constituting couple × perpendicular distance between the two forces.**

(f) Equilibrium of a rigid body:

If $\vec{F_1}, \vec{F_2}, ... , \vec{F_n}$ are the forces acting on a rigid body and $\vec{\tau_1} + \vec{\tau_2} + ... + \vec{\tau_n}$ are the torques on it due to these forces, then

(i) for translational equilibrium, $\displaystyle\sum_n F_{x_i} = \sum_n F_{y_i} = \sum_n F_{z_i} = 0$ and

(ii) for rotational equilibrium, $\vec{\tau} = \displaystyle\sum_n \vec{\tau_n} = 0$
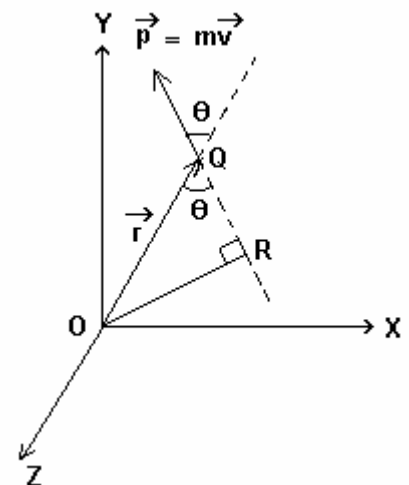
7.5 Angular momentum

(a) Angular momentum of a particle:

Consider a particle Q of mass m having position vector $\vec{OQ} = \vec{r}$ moving with velocity $\vec{v}$ and having linear momentum $\vec{p} = m \vec{v}$. The angle between $\vec{r}$ and $\vec{p}$ is θ. The coordinate axes are so chosen that the particle and its motion are in (x, y) plane.

The angular momentum of the particle w. r. t. point O is defined as

$$\vec{l} = \vec{r} \times \vec{p} = r p \sin \theta = p (OR)$$

= p × (perpendicular distance of linear momentum
        vector from the reference point)

= Moment of linear momentum

The unit of angular momentum is $kg\,m^2/s$ or Js.

Its dimensional formula is $M^1 L^2 T^{-1}$.

The direction of $\vec{l}$ is perpendicular to the plane formed by $\vec{r}$ and $\vec{p}$ and is obtained using

the right handed screw rule. In the present case, $\vec{l}$ is in the OZ direction.

The magnitude and direction of $\vec{l}$ depend on the selection of reference point and hence while defining the angular momentum of a particle it is necessary to mention the reference point.

(b) The relation between angular momentum of a particle and torque acting on it:

Differentiating equation $\vec{l} = \vec{r} \times \vec{p}$ w. r. t. to time

$$\frac{d\vec{l}}{dt} = \left(\vec{r} \times \frac{d\vec{p}}{dt} \right) + \left(\frac{d\vec{r}}{dt} \times \vec{p} \right)$$

But $\dfrac{d\vec{p}}{dt}$ = rate of change of linear momentum = force $\vec{F}$ and $\dfrac{d\vec{r}}{dt}$ = velocity $\vec{v}$

$\therefore \dfrac{d\vec{l}}{dt} = (\vec{r} \times \vec{F}) + (\vec{v} \times \vec{p})$

$\therefore \dfrac{d\vec{l}}{dt} = (\vec{r} \times \vec{F}) = \vec{\tau}$ (As $\vec{v}$ and $\vec{p}$ are in the same direction, $\vec{v} \times \vec{p} = 0$)

Thus, "the time rate of change of angular momentum is equal to the torque." This statement is similar to the Newton's Second Law of motion that "the time rate of change of linear momentum is equal to the force."

(c) Angular momentum of a system of particles:

The angular momentum of a system of n particles having angular momenta $\vec{l}_1$, $\vec{l}_2$, ..., $\vec{l}_n$ is

given by $\vec{L} = \vec{l}_1 + \vec{l}_2 + ... + \vec{l}_n$

$\therefore \dfrac{d\vec{L}}{dt} = \dfrac{d\vec{l}_1}{dt} + \dfrac{d\vec{l}_2}{dt} + ... + \dfrac{d\vec{l}_n}{dt}$

$= \vec{\tau}_1 + \vec{\tau}_2 + ... + \vec{\tau}_n$

$= \vec{\tau}$

Thus, the time rate of change of total angular momentum of a system of particles is equal to the resultant external torque acting on the system.

(d) Angular momentum of a rigid body:

All particles of a rigid body move on circular paths in a plane perpendicular to the axis of rotation and hence their linear momenta are in this plane only. Taking position vector of each particle w. r. t. the centre of its circular path, its angular momentum will be parallel to the axis of rotation. Also, for each particle, $\vec{r}$ and $\vec{p}$ are perpendicular to each other.

$$\therefore \quad \vec{L} = \vec{l_1} + \vec{l_2} + ... + \vec{l_n} = (\vec{r_1} \times \vec{p_1}) + (\vec{r_2} \times \vec{p_2}) + ... + (\vec{r_n} \times \vec{p_n})$$

$$\therefore |\vec{L}| = r_1 p_1 + r_2 p_2 + ... + r_n p_n \quad (\because \vec{r} \perp \vec{p} \text{ and all vectors on R.H.S. are parallel})$$

$$= r_1 m_1 v_1 + r_2 m_2 v_2 + ... + r_n m_n v_n \quad (\because p = mv)$$

$$= m_1 r_1^2 \omega + m_2 r_2^2 \omega + ... + m_n r_n^2 \omega \quad (\because v = r\omega)$$

$$= (m_1 r_1^2 + m_2 r_2^2 + ... + m_n r_n^2) \omega$$

$$\therefore |\vec{L}| = I |\vec{\omega}|, \quad \text{where } I = m_1 r_1^2 + m_2 r_2^2 + ... + m_n r_n^2$$

I is called the moment of inertia of the rigid body about the given axis of rotation.

As $\vec{L}$ and $\vec{\omega}$ are parallel to the axis of rotation, $\vec{L} = I \vec{\omega}$.

$$\therefore \quad \frac{d\vec{L}}{dt} = I \frac{d\vec{\omega}}{dt} = I \vec{\alpha} = \vec{\tau}$$
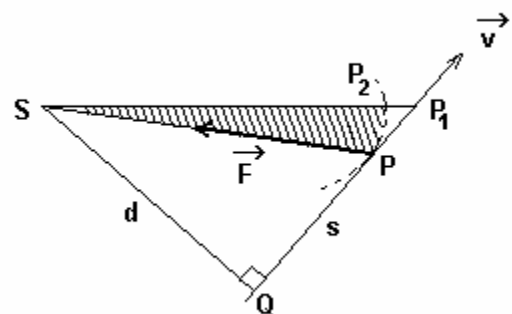
Law of conservation of angular momentum:

In the above equation, if $\vec{\tau} = 0$, $\vec{L}$ = constant. Hence, "In the absence of resultant external torque, the angular momentum of a rigid body remains constant." This statement is known as the law of conservation of angular momentum.

7.6 Geometrical representation of the law of conservation of angular momentum

Planet P having mass m moves in an elliptical orbit around the Sun with linear velocity $\vec{v}$ as shown in the figure. The perpendicular distance of $\vec{v}$ from the Sun is d.

$\therefore$ the angular momentum of the planet w. r. t. the Sun is

L = mvd … … … … … … … (1)

Area of the triangle SQP is $A = \dfrac{1}{2}(SQ)(PQ) = \dfrac{1}{2}(d)s$ $(\because PQ = s)$

$\therefore \quad \dfrac{dA}{dt} = \dfrac{1}{2}(d)\dfrac{ds}{dt} = \dfrac{1}{2}(d)v \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$

From equations (1) and (2), we get

$\dfrac{dA}{dt} = \dfrac{L}{2m}$ = constant   ($\because$ torque due to gravitational force on P by the Sun w. r. t. the Sun is zero and so L is constant.)

This equation represents Kepler's second law for planetary motion stated as "The area swept by the line joining a planet with the Sun per unit time, called areal velocity, is constant." This is the geometrical representation of the law of conservation of angular momentum.

7.7 Moment of inertia

If m_1, m_2, ..., m_n are the masses of the particles of a rigid body and r_1, r_2, ..., r_n are their perpendicular distances from a given axis respectively, then the moment of inertia of the body corresponding to the given axis is given by

$I = m_1 r_1^2 + m_2 r_2^2 + \ldots + m_n r_n^2 \displaystyle\sum_{i=1}^{n} m_i r_i^2$

The magnitude of moment of inertia depends on the selection of the axis and the distribution of mass about it. Its S I unit is $kg\,m$ and dimensional formula is $M^1 L^2 T^0$.

The equations $\overrightarrow{L} = I\overrightarrow{\omega}$ and $\overrightarrow{\tau} = I\overrightarrow{\alpha}$ are analogous to the equations of linear motion $\overrightarrow{P} = M\overrightarrow{v}$ and $\overrightarrow{F} = M\overrightarrow{a}$ respectively which shows that the moment of inertia plays the same role in rotational motion as the mass plays in linear motion. The moment of inertia is the inertia for rotational motion just as the mass is the inertia for linear motion.
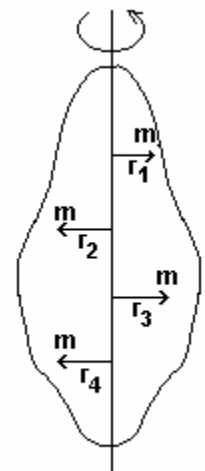
7.8 (a) Radius of gyration

Suppose the rigid body of mass M consists of n particles each of mass m $\therefore$ M = mn.

The moment of inertia of the body about a given axis,

$I = mr_1^2 + mr_2^2 + \ldots + mr_n^2 = \dfrac{M\left(r_1^2 + r_2^2 + \ldots + r_n^2\right)}{n} = MK^2$

where, K is called the radius of gyration corresponding to the given axis and is the mean of the squares of perpendicular distances of the particles of the body from the given axis.

7.8 (b) Two theorems regarding moment of inertia

(i) Parallel axes theorem: The moment of inertia (I) of a body about a given axis is equal to the sum of its moment of inertia I_c about a parallel axis passing through its centre of mass and the product of its mass and square of perpendicular distance (d) between the two axes.
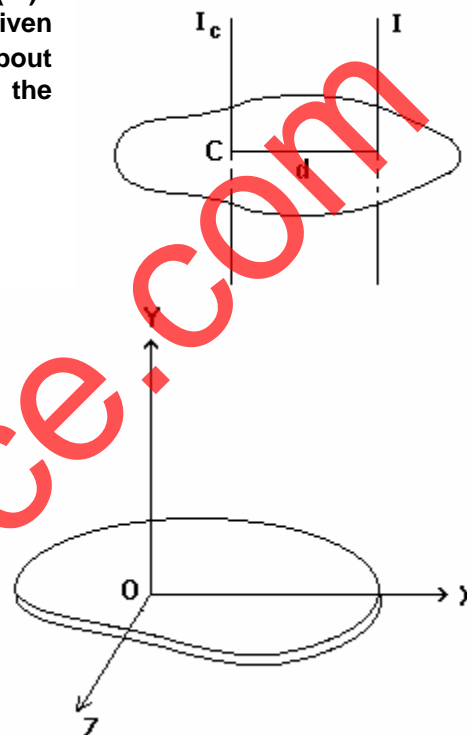
$$I = I_c + M d^2$$

(ii) Perpendicular axes theorem:

(a) <u>For laminar bodies:</u> For laminar bodies, the moment of inertia I_z about Z-axis normal to its plane is equal to the sum of its moments of inertia about X-axis, I_x and Y-axis, I_y.

$$I_z = I_x + I_y$$

(b) <u>For three-dimensional bodies:</u> The sum of moments of inertia of a three dimensional body about any three mutually perpendicular axes drawn through the same point is equal to twice the moment of inertia of the body about that point.

$$I_x + I_y + I_z = 2 I_0$$

7.9 Calculation of moment of inertia of certain symmetric objects
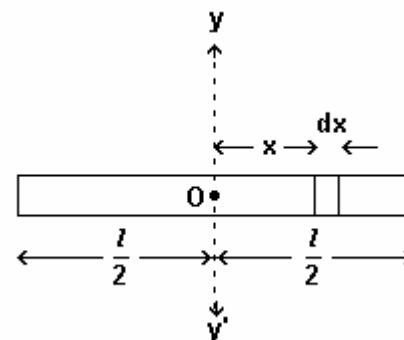
(a) Moment of inertia of a thin uniform rod about an axis, passing through its centre and perpendicular to its length:

To calculate moment of inertia of a thin rod of length l and mass M about an axis yy' passing through its centre O and perpendicular to its length, consider O as origin and X-axis along the length of the rod. A small element of length dx of the rod is at a distance x from O.

The moment of inertia of this element about yy' is

$$d I = \frac{M}{l} dx \cdot x^2 \quad \therefore \quad \text{moment of inertia of the rod,}$$

$$I = \int_{-\frac{l}{2}}^{+\frac{l}{2}} \frac{M}{l} dx \cdot x^2 = \frac{M}{l} \left[\frac{x^3}{3} \right]_{-\frac{l}{2}}^{+\frac{l}{2}}$$

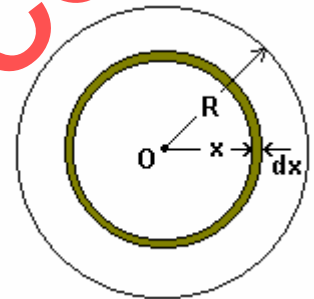$$= \frac{M}{3 l} \left[\frac{l^3}{8} + \frac{l^3}{8} \right] = \frac{M l^2}{12}$$

(b) Moment of inertia of a thin ring or a thin walled hollow cylinder or a thin walled hollow sphere:

As the entire mass, M, of a thin ring is at the same distance, equal to the radius R of the ring from its centre, the moment of inertia of a thin ring about an axis passing through its centre and perpendicular to its plane is MR^2. Similarly, the moments of inertia of a thin walled cylinder about its geometric axis or of a thin walled hollow sphere about its centre are also given by MR^2, where M represents their mass and R their radii.

(c) Moment of inertia of a disc or a solid cylinder:

To calculate moment of inertia of a disc of uniform thickness t, radius R and mass M about an axis passing through its centre and perpendicular to its plane, consider an element of the disc in the form of a thin ring of thickness dx at a distance x from its centre. Mass of this ring is $2 \pi x \, dx \cdot t \cdot \rho$, where ρ is the density of the material of the ring. Therefore, the moment of inertia of the ring about an axis passing through the centre, O, of the disc and perpendicular to its plane is

$$d I = (2 \pi x \, dx \cdot t \cdot \rho) x^2 = (2 \pi t \rho) x^3 \, dx$$

∴ moment of inertia of the disc about an axis passing through its centre and perpendicular to its plane is

$$I = \int_0^R (2 \pi t \rho) x^3 \, dx = (2 \pi t \rho) \left[\frac{x^4}{4} \right]_0^R = 2 \pi t \rho \frac{R^4}{4} = \frac{1}{2} \left(\pi R^2 t \rho \right) R^2$$

$$= \frac{1}{2} M R^2$$

Similarly, moment of inertia of a solid cylinder about its axis is also $\frac{1}{2} M R^2$.

(d) Moment of inertia of a thin walled hollow sphere about its diameter:

Moment of inertia of a thin walled hollow sphere about its centre is $I_0 = MR^2$.

By perpendicular axes theorem for three dimensional bodies, $2 I_0 = I_x + I_y + I_z$.

Now, $I_x = I_y = I_z = $ Moment of inertia, I, of the hollow sphere about its diameter.

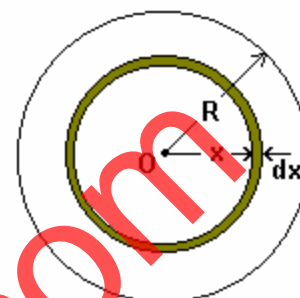∴ $I = \frac{2}{3} I_0 = \frac{2}{3} MR^2$.

(e) Moment of inertia of a solid sphere about its centre:

Let the solid sphere of mass M be of radius R. Consider a thin spherical shell of radius x and of thickness dx. The mass of this shell is $4 \pi x^2 \cdot dx \, \rho$, where ρ is the density of the material of the sphere. Hence the moment of inertia of the shell about the centre O is

$$d I_0 = 4 \pi x^2 \cdot dx \, \rho \cdot x^2$$

Therefore, the moment of inertia of the solid sphere about its centre is

$$I_0 = 4 \pi \rho \int_0^R x^4 \, dx = 4 \pi \rho \left[\frac{x^5}{5} \right]_0^R = \frac{4 \pi \rho}{5} R^5$$

$$= \frac{4}{3} \pi R^3 \rho \cdot \frac{3 R^2}{5} = \frac{3}{5} MR^2$$

(f) Moment of inertia of a solid sphere about its diameter:

The moments of inertia of the solid sphere about three mutually perpendicular axes passing through its centre given by I_x, I_y and I_z are all equal and represent the moment of inertia, I, of the solid sphere about the diameter.

$$\therefore \quad I = I_x = I_y = I_z$$

By theorem of perpendicular axes in three dimensions $2 I_0 = I_x + I_y + I_z = 3 I$

$$\therefore \quad I = \frac{2}{3} I_0 = \frac{2}{3} \times \frac{3}{5} MR^2 = \frac{2}{5} MR^2$$

(g) Moment of inertia of a solid cone about its geometric axis:

Consider a disc of radius r and thickness dy at a height y from the vertex of the cone.

Mass of disc = dm = volume × density = $\pi r^2 \, dy \cdot \rho$.

$$\therefore \quad \text{M. I. of disc} = dI = \frac{(dm)r^2}{2} = \frac{\pi r^4 \rho \, dy}{2}$$

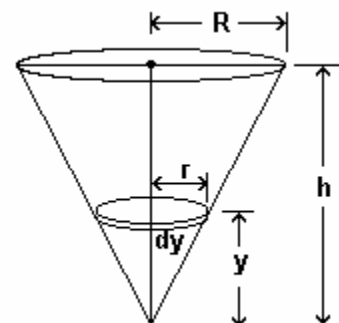**From the geometry of the figure,** $\dfrac{r}{y} = \dfrac{R}{h}$ $\therefore$ $r = \dfrac{R}{h} y$

$$\therefore dI = \frac{\pi R^4 \rho}{2 h^4} y^4 \, dy$$

$$\therefore \quad I = \frac{\pi R^4 \rho}{2 h^4} \int_0^h y^4 \, dy = \frac{\pi R^4 \rho}{2 h^4} \left[\frac{y^5}{5} \right]_0^h$$

$$= \frac{\pi R^4 \rho}{2 h^4} \frac{h^5}{5} = \frac{3}{10} \left[\frac{1}{3} \pi R^2 h \rho \right] R^2$$

$$= \frac{3}{10} MR^2$$

Moment of inertia and radius of gyration for some symmetric bodies

	Body	Axis	I	K
1	Thin rod of length L	Passing through its centre and perpendicular to its length	$\frac{1}{12}ML^2$	$\frac{L}{2\sqrt{3}}$
2	Ring of radius R $\rightarrow$ Thin-walled hollow cylinder of radius R $\rightarrow$	Passing through its centre and perpendicular to its plane Geometric axis	MR^2	R
3	Ring of radius R $\rightarrow$ Circular disc of radius R $\rightarrow$ Solid cylinder of radius R $\rightarrow$	Any diameter Passing through its centre and perpendicular to its plane Geometric axis	$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
4	Circular disc of radius R $\rightarrow$	Any diameter	$\frac{1}{4}MR^2$	$\frac{R}{2}$
5	Thin-walled hollow sphere of radius R $\rightarrow$	Any diameter	$\frac{2}{3}MR^2$	$\sqrt{\frac{2}{3}}R$
6	Solid sphere of radius R $\rightarrow$	Any diameter	$\frac{2}{5}MR^2$	$\sqrt{\frac{2}{5}}R$
7	Solid right circular cone of radius R $\rightarrow$	Geometric axis	$\frac{3}{10}R^2$	$\sqrt{\frac{3}{10}}R$

7.8 Comparison of translational motion and rotational motion

1	Linear displacement	$\vec{d}$	Angular displacement	θ
2	Linear velocity	$\vec{v}$	Angular velocity	$\vec{\omega}$
3	Linear acceleration	$\vec{a} = \dfrac{d\vec{v}}{dt}$	Angular acceleration	$\vec{\alpha} = \dfrac{d\vec{\omega}}{dt}$
4	Mass	m	Moment of inertia	I
5	Linear momentum	$\vec{p} = m\vec{v}$	Angular momentum	$\vec{L} = I\vec{\omega}$
6	Force	$\vec{F} = m\vec{a}$	Torque	$\vec{\tau} = I\vec{\alpha}$
7	Newton's Second Law of motion	$\vec{F} = \dfrac{d\vec{P}}{dt}$	Result similar to Newton's Second Law	$\vec{\tau} = \dfrac{d\vec{L}}{dt}$
8	Translational kinetic energy	$K = \dfrac{1}{2}mv^2$	Rotational kinetic energy	$K = \dfrac{1}{2}I\omega^2$
9	Work	$W = \vec{F} \cdot \vec{d}$	Work	$W = \tau\theta$

10	Power	$P = F v$	Power	$P = \tau \omega$
11	Equations of linear motion with constant linear acceleration	$v = v_0 + at$ $d = v_0 t + \dfrac{1}{2} at^2$ $2ad = v^2 - v_0^2$	Equations of rotational motion with constant angular acceleration	$\omega = \omega_0 + \alpha t$ $\theta = \omega_0 t + \dfrac{1}{2} \alpha t^2$ $2 \alpha \theta = \omega^2 - \omega_0^2$

7.11 Rigid (solid) bodies rolling without sliding

Let a body of mass m, radius R, moment of inertia I and radius of gyration K start rolling from rest without sliding from the top of a plane of height h inclined at an angle θ as shown in the figure.

The body performs combined translational and rotational motion achieving linear velocity v and angular velocity ω at the bottom of the inclined plane.

Now, the P. E. lost by the body   = mgh   where

g is acceleration due to gravity

and the K. E. gained by the body $= \dfrac{1}{2} mv^2 + \dfrac{1}{2} I \omega^2$

By the law of conservation of mechanical energy,

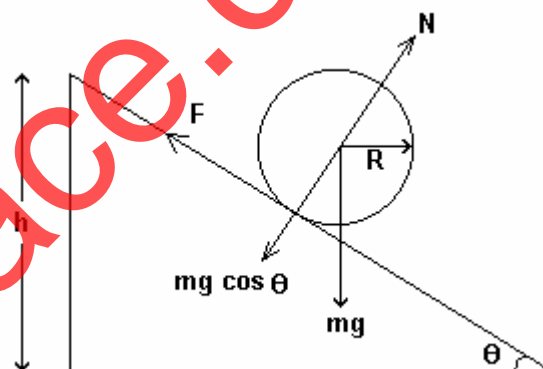$mgh = \dfrac{1}{2} mv^2 + \dfrac{1}{2} I \omega^2$ …   …   (1)

Putting $\omega = \dfrac{v}{R}$ and $I = m K^2$ in the equation (1) above, we get

$v^2 = \dfrac{2 g h}{\left(1 + \dfrac{K^2}{R^2} \right)}$ …   …   …   (2)

If d is the length of the inclined plane and a is the acceleration of the body along it, then

$v^2 = 2 a d = \dfrac{2 a h}{\sin \theta}$ …   …   (3)

Comparing equations (2) and (3), $a = \dfrac{g \sin \theta}{\left(1 + \dfrac{K^2}{R^2} \right)}$

$\therefore$ the decrease in the acceleration of the body due to the frictional force acting on it

$$= \ g \sin \theta \ - \ \frac{g \sin \theta}{\left(1 + \dfrac{K^2}{R^2} \right)} \quad = \quad g \sin \theta \left(\frac{K^2}{K^2 + R^2} \right)$$

$\therefore$ the frictional force,

$$F \ = \ m \, g \sin \theta \left(\frac{K^2}{K^2 + R^2} \right) \quad \ldots \quad (4)$$

The normal reaction N and $m \, g \cos \theta$ balance each other.

$\therefore$ $N \ = \ m \, g \cos \theta \qquad \ldots \quad \ldots \quad (5)$

Dividing equation (4) by equation (5),

$$\frac{F}{N} \ = \ \left(\frac{K^2}{K^2 + R^2} \right) \tan \theta$$

But $\dfrac{F}{N} \ \leq \ \mu_s$ (co-efficient of static friction)

(Note: Frictional force, F, increases as the angle of inclination, θ, of the inclined plane is increased till it reaches a maximum value = $\mu_s \, N$ at a certain maximum value of θ. For all values of θ less than this maximum value, $F \ < \ \mu_s \, N.$)

$$\therefore \ \left(\frac{K^2}{K^2 + R^2} \right) \tan \theta \ \leq \ \mu_s$$

or, $\mu_s \ \geq \ \left(\dfrac{K^2}{K^2 + R^2} \right) \tan \theta$ for body to roll without sliding.

Special   cases:

For thin ring or thin-walled hollow cylinder, radius of gyration, $K = R \ \Rightarrow \ \mu_s \ \geq \ \dfrac{1}{2} \tan \theta$

For circular disc or solid cylinder,   radius of gyration, $K = \dfrac{R}{\sqrt{2}} \ \Rightarrow \ \mu_s \ \geq \ \dfrac{1}{3} \tan \theta$

and for solid sphere,   radius of gyration, $K = \sqrt{\dfrac{2}{5}} \, R \ \Rightarrow \ \mu_s \ \geq \ \dfrac{2}{7} \tan \theta$