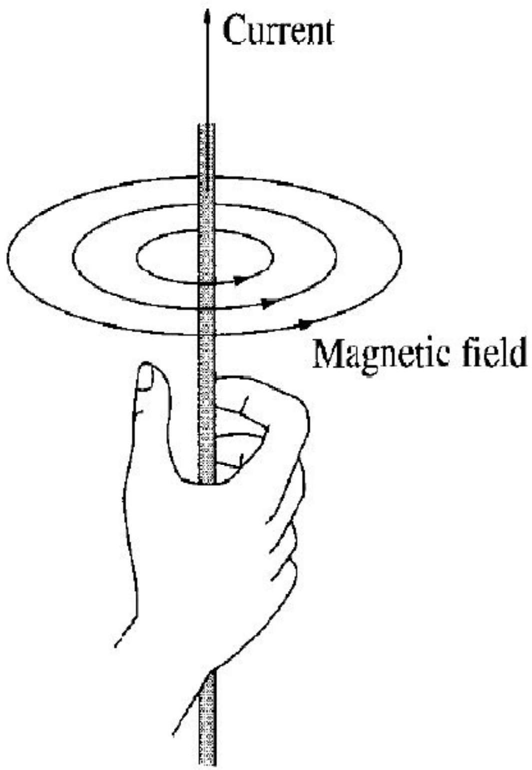
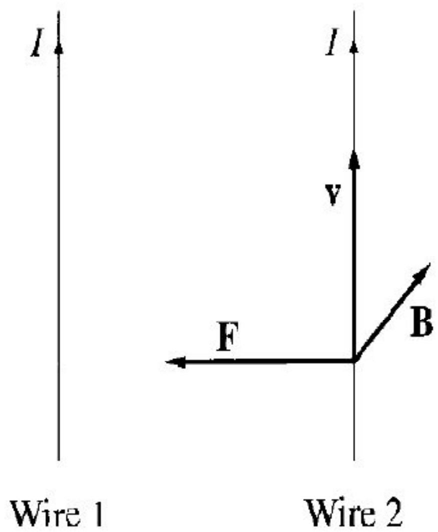
Magneto statics

A stationary charges produces only an electric field E in the space around it, a moving charge generates, in addition a magnetic field **B**.





The magnetic force in a charge Q, moving with velocity **v** in a magnetic field **B** is

$$\boxed{\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B}).}$$

This is known as the **Lorentz force Law**. In the presence of both electric and magnetic fields, the net force on Q would be

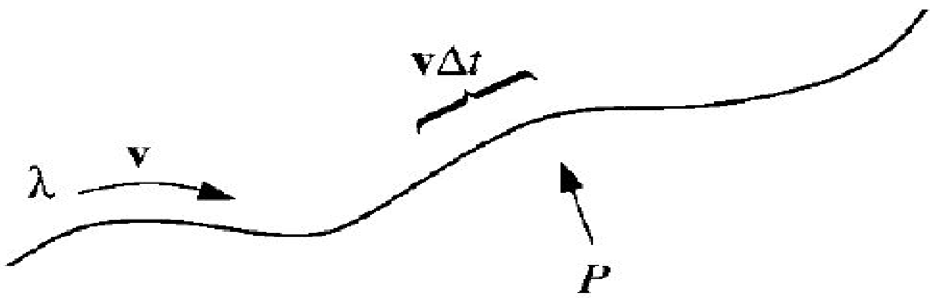
$$\mathbf{F} = Q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})].$$

The current I is defined as charge per unit time passing a given point.

A line charge λ traveling down a wire at speed v constitutes a current

$$I = \lambda v,$$

The magnetic force on a segment of current- carrying wire is evidently



$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\, dq = \int (\mathbf{v} \times \mathbf{B})\lambda\, dl = \int (\mathbf{I} \times \mathbf{B})\, dl.$$

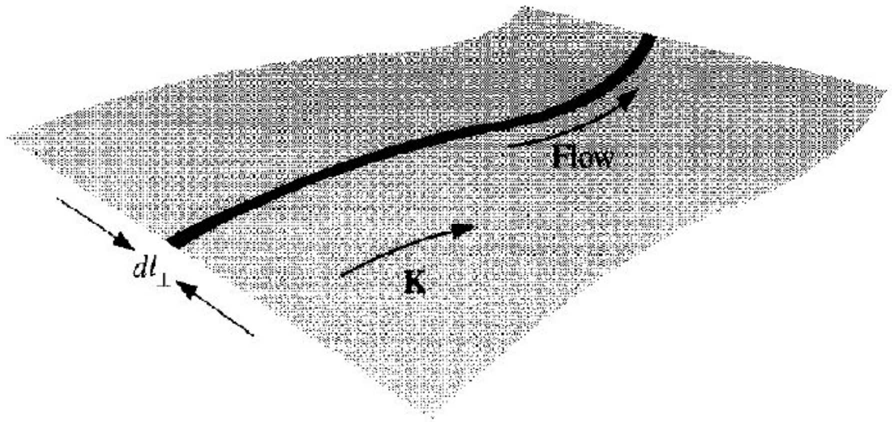
as I and dl both point in the same direction, so, it can be written as

$$\boxed{\mathbf{F}_{\text{mag}} = \int I\, (d\mathbf{l} \times \mathbf{B}).}$$

and since current is constant (in magnitude) along the wire therefore,

$$\mathbf{F}_{\text{mag}} = I \int (d\mathbf{l} \times \mathbf{B}).$$

When charges flows over a surface



we define surface current density K

$$\mathbf{K} \equiv \frac{d\mathbf{I}}{dl_\perp}$$

In particular, if the surface charge density is σ and its velocity is v, then

$$\mathbf{K} = \sigma \mathbf{v}.$$

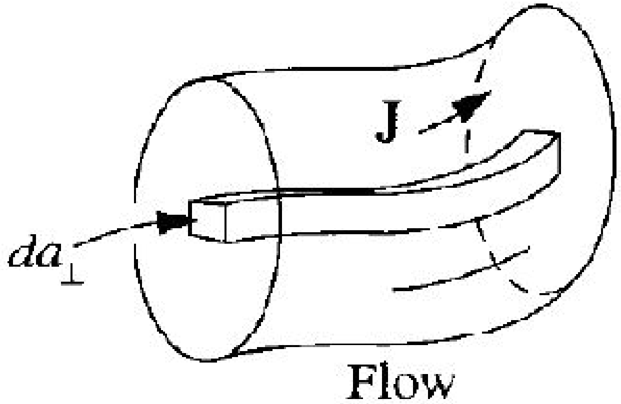
and the magnetic force on the surface current is

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\sigma\, da = \int (\mathbf{K} \times \mathbf{B})\, da.$$

Similarly, we can define magnetic force in the form of volume current density

$$\mathbf{J} \equiv \frac{d\mathbf{I}}{da_{\perp}} = \rho\mathbf{v}.$$

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\rho\, d\tau = \int (\mathbf{J} \times \mathbf{B})\, d\tau.$$



The current crossing a surface can be written as

$$I = \int_{\mathcal{S}} J \, da_{\perp} = \int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}.$$

and the total charge per unit time leaving a volume V is

$$\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{J}) \, d\tau.$$

Since charge is conserved, therefore

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{J})\, d\tau = -\frac{d}{dt}\int_{\mathcal{V}} \rho\, d\tau = -\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t}\right) d\tau.$$

The minus sign indicated outward flow decreases the charge left in V and since this can be applies to any volume, therefore

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.}$$

It is called **Equation of continuity** and reflects **local charge conservation.**

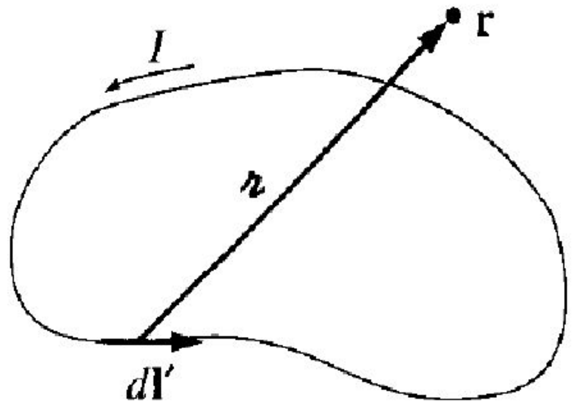
Bio-Savart Law :-

Stationary charges	$\Rightarrow$	constant electric fields: electrostatics.
Steady currents	$\Rightarrow$	constant magnetic fields: magnetostatics.

The magnetic field of a steady line current is given by the **Biot-Savart law:**

$$\boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\hat{\boldsymbol{\imath}}}{\imath^2}\,dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}'\times\hat{\boldsymbol{\imath}}}{\imath^2}.}$$

The constant μ_0 is called the **permeability of free space:**



Prob. :-

Find the magnetic field a distance s from a long straight wire carrying a steady current I

Solution: In the diagram, $(d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}})$ points *out* of the page, and has the magnitude

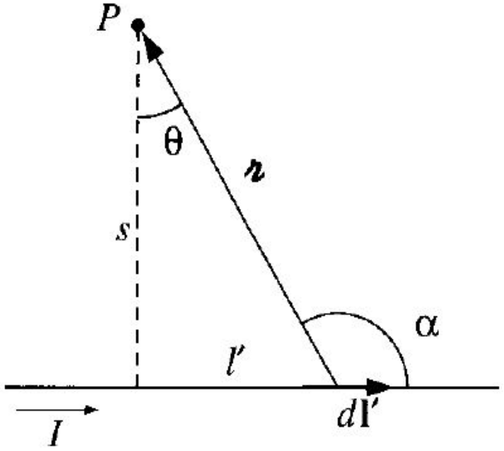
$$dl' \sin\alpha = dl' \cos\theta.$$

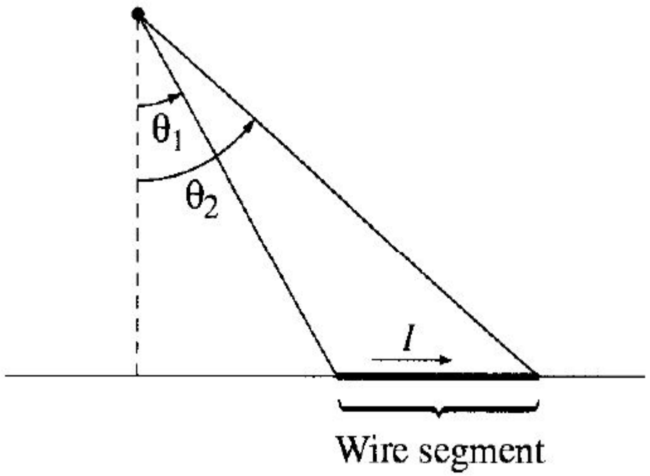
Also, $l' = s\tan\theta$, so

$$dl' = \frac{s}{\cos^2\theta}\, d\theta,$$

and $s = \mathscr{r}\cos\theta$, so

$$\frac{1}{\mathscr{r}^2} = \frac{\cos^2\theta}{s^2}.$$





Thus

$$B = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2}\left(\frac{\cos^2\theta}{s^2}\right)\left(\frac{s}{\cos^2\theta}\right)\cos\theta\, d\theta$$

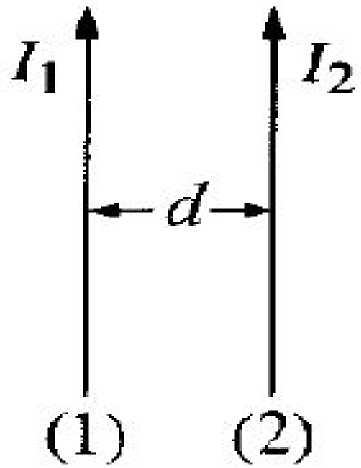
$$= \frac{\mu_0 I}{4\pi s}\int_{\theta_1}^{\theta_2}\cos\theta\, d\theta = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1).$$

In the case

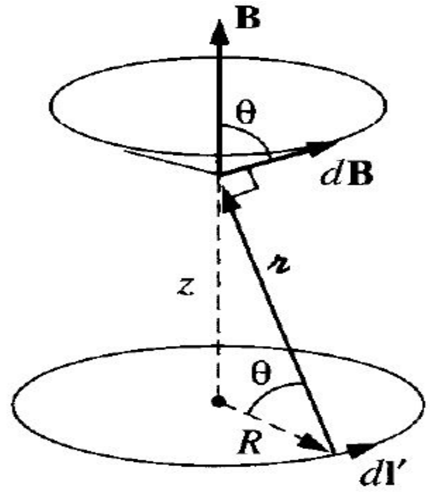
of an infinite wire, $\theta_1 = -\pi/2$ and $\theta_2 = \pi/2$, so we obtain

$$B = \frac{\mu_0 I}{2\pi s}.$$

Prob. Find the force of attraction between two long parallel wires at a distance d apart, carrying a current as shown in figure.



Prob.2:- Find the magnetic field at a distance z above the center of a circular loop o radius R, which carries a steady current I as in fig.



Solution: The field $d\mathbf{B}$ attributable to the segment $d\mathbf{l}'$ points as shown. As we integrate $d\mathbf{l}'$ around the loop, $d\mathbf{B}$ sweeps out a cone. The horizontal components cancel, and the vertical components combine to give

$$B(z) = \frac{\mu_0}{4\pi} I \int \frac{dl'}{\mathscr{r}^2} \cos\theta.$$

(Notice that $d\mathbf{l}'$ and $\hat{\mathscr{r}}$ are perpendicular, in this case; the factor of $\cos\theta$ projects out the vertical component.) Now, $\cos\theta$ and $\mathscr{r}^2$ are constants, and $\int dl'$ is simply the circumference, $2\pi R$, so

$$B(z) = \frac{\mu_0 I}{4\pi} \left(\frac{\cos\theta}{\mathscr{r}^2} \right) 2\pi R = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}}.$$

For surface and volume currents the Biot-Savart law becomes

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2}\, da' \quad \text{and} \quad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2}\, d\tau',$$

For a moving point charge,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{q\mathbf{v} \times \hat{\boldsymbol{\imath}}}{\imath^2},$$

Is it correct ?

The Divergence and Curl of B

The integral of **B** around a circular path of radius s, centered at the wire, is

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \frac{\mu_0 I}{2\pi s} \oint dl = \mu_0 I.$$

If we have bundle of straight wires, the line integral will then be

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}},$$

where I_{enc} stands for the total current enclosed by the integration path. If the flow of charge is represented by a volume current density **J**, the enclosed current is

$$I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{a},$$

with the integral taken over the surface bounded by the loop. Applying Stokes' theorem

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a},$$

and hence

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

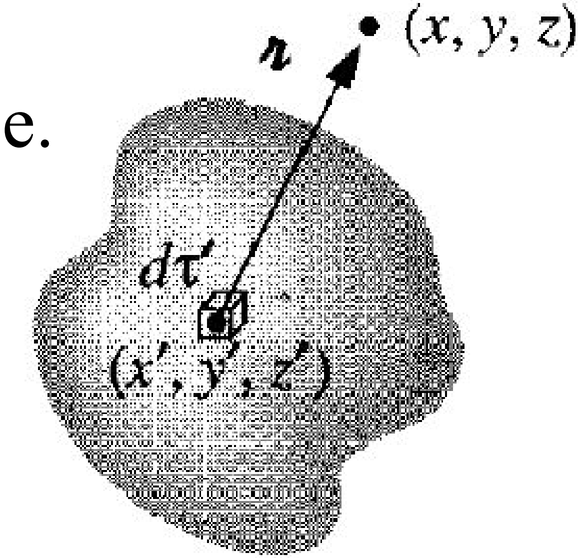
proof is for the simplest case of infinitely long straight wires.

In general case:-

The Biot-Savart law for the general case of a volume current reads

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2} d\tau'.$$

This formula gives the magnetic field at a point r = (x, y, z) in terms of an integral over the current distribution J (x', y', z'). i.e.



$$\mathbf{B} \text{ is a function of } (x, y, z),$$

$$\mathbf{J} \text{ is a function of } (x', y', z'),$$

$$\boldsymbol{\mathscr{r}} = (x - x')\,\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} + (z - z')\,\hat{\mathbf{z}},$$

$$d\tau' = dx'\,dy'\,dz'.$$

The integration is over the *primed* coordinates; the divergence and the curl are to be taken with respect to the *unprimed* coordinates.

$$\nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) d\tau'.$$

since

$$\nabla \cdot \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\imath}}}{\imath^2}\right) = \frac{\hat{\boldsymbol{\imath}}}{\imath^2} \cdot (\nabla \times \mathbf{J}) - \mathbf{J} \cdot \left(\nabla \times \frac{\hat{\boldsymbol{\imath}}}{\imath^2}\right).$$

But $\nabla \times \mathbf{J} = 0$, because $\mathbf{J}$ doesn't depend on the unprimed variables (x, y, z),

and

$$\nabla \times (\hat{\boldsymbol{\imath}}/\imath^2) = 0$$

so,

$$\boxed{\nabla \cdot \mathbf{B} = 0.}$$

Applying the curl to Eq. $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2} d\tau'.$

we get,

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\imath}}}{\imath^2}\right) d\tau'.$$

Using the appropriate product rule, expand the integrand

$$\nabla \times \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\imath}}}{\imath^2} \right) = \mathbf{J} \left(\nabla \cdot \frac{\hat{\boldsymbol{\imath}}}{\imath^2} \right) - (\mathbf{J} \cdot \nabla) \frac{\hat{\boldsymbol{\imath}}}{\imath^2}.$$

Since **J** does not depends on x, y, and z, therefore, the second term integrates to zero and

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\imath}}}{\imath^2} \right) = 4\pi \delta^3 (\boldsymbol{\imath}).$$

Thus

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') 4\pi \delta^3 (\mathbf{r} - \mathbf{r}') \, d\tau' = \mu_0 \mathbf{J}(\mathbf{r}),$$

Ampère's Law

The equation for the curl of **B**

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J},}$$

is called **Ampère's law** (in differential form). It can be converted to integral form by the usual device of applying one of the fundamental theorems—in this case Stokes' theorem:

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}.$$

Now, $\int \mathbf{J} \cdot d\mathbf{a}$ is the total current passing through the surface (the **current enclosed** by the **amperian loop**). Thus

$$\boxed{\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}.}$$

This is the integral version of Ampère's law;

Prob.:- Find the magnetic field at a distance s from a long straight wire (Fig. a), carrying a steady current I .using Ampere's circuital law.

Solution: We know the direction of **B** is "circumferential," circling around the wire as indicated by the right hand rule. By symmetry, the magnitude of **B** is constant around an amperian loop of radius s, centered on the wire. So Ampère's law gives

$$\oint \mathbf{B} \cdot d\mathbf{l} = B \oint dl = B 2\pi s = \mu_0 I_{\text{enc}} = \mu_0 I,$$

or

$$B = \frac{\mu_0 I}{2\pi s}.$$

Magnetic Vector Potential

Just as $\nabla \times \mathbf{E} = 0$ permitted us to introduce a scalar potential (V) in electrostatics,

$$\mathbf{E} = -\nabla V,$$

so $\nabla \cdot \mathbf{B} = 0$ invites the introduction of a *vector* potential $\mathbf{A}$ in magnetostatics:

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}.}$$

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}.$$

The definition **B = curl A** specifies the curl of A but it does not say anything about divergence so we have liberty to pick that as we see fit, and zero is ordinarily the simplest choice.

With this condition on **A** , Ampere's law becomes

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}.}$$

This *again* is nothing but Poisson's equation—or rather, it is *three* Poisson's equations, one for each Cartesian[13] component. Assuming $\mathbf{J}$ goes to zero at infinity, we can read off the solution:

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau'.}$$

For line and surface currents,

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}}{\mathscr{r}}\,dl' = \frac{\mu_0 I}{4\pi}\int \frac{1}{\mathscr{r}}\,d\mathbf{l}'; \qquad \mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}}{\mathscr{r}}\,da'.$$

Diamagnets, Paramagnets, Ferromagnets

All the magnetic phenomena are due to electric charges in motion, i.e. electrons orbiting around nuclei and electrons spinning about their axes. These currents loops are so small that we may treat them as magnetic dipoles. Ordinarily, they cancel each other out because of the random orientation of the atoms. But when magnetic field is applied, a net alignment of these magnetic dipoles occurs, and the medium becomes magnetically polarized, or magnetized.

Torques and Forces on Magnetic Dipoles

$$\boxed{\mathbf{N} = \mathbf{m} \times \mathbf{B},}$$

where $m = Iab$ is the magnetic dipole moment of the loop.

This equation gives the exact torque on any localized current distribution, in the presence of a uniform field. In a non-uniform field it is the exact torque (about the center) for a perfect dipole of infinitesimal size.

In a *uniform* field, the net *force* on a current loop is zero:

$$\mathbf{F} = I \oint (d\mathbf{l} \times \mathbf{B}) = I \left(\oint d\mathbf{l} \right) \times \mathbf{B} = 0;$$

because the net displacement $\oint d\mathbf{l}$ around a closed loop vanishes.

For an *infinitesimal* loop, with dipole moment **m**, in a field **B**, the force is

$$\boxed{\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})}$$

Magnetization

In the presence of a magnetic field, matter becomes *magnetized*; that is, upon microscopic examination it will be found to contain many tiny dipoles, with a net alignment along some direction. We have discussed two mechanisms that account for this magnetic polarization: (1) paramagnetism (the dipoles associated with the spins of unpaired electrons experience a torque tending to line them up parallel to the field) and (2) diamagnetism (the orbital speed of the electrons is altered in such a way as to change the orbital dipole moment in a direction opposite to the field). Whatever the *cause*, we describe the state of magnetic polarization by the vector quantity

$$\mathbf{M} \equiv \textit{magnetic dipole moment per unit volume.}$$

M is called the **magnetization**; it plays a role analogous to the polarization **P**

Ampère's law in Magnetized Materials

any event, total current can be written as

$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f.$$

where $\mathbf{J_f}$ is free current because of actual transport of charge while $\mathbf{J_b}$ is bound current due to magnetization.

So Ampere's law can be written as

$$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J} = \mathbf{J}_f + \mathbf{J}_b = \mathbf{J}_f + (\nabla \times \mathbf{M}),$$

or, collecting together the two curls:

$$\nabla \times \left(\frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \right) = \mathbf{J}_f.$$

The quantity in parentheses is designated by the letter $\mathbf{H}$:

$$\boxed{\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}.}$$

In terms of $\mathbf{H}$, then, Ampère's law reads

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}_f,}$$

or, in integral form,

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\mathrm{enc}}},$$

where $I_{f_{\mathrm{enc}}}$ is the total *free* current passing through the Amperian loop.

Linear and Nonlinear Media

Magnetic Susceptibility and Permeability

In paramagnetic and diamagnetic materials, the magnetization is sustained by the field, when **B** is removed, **M** disappears. In fact, for most substances the magnetization is proportional to the field, provided the field is not too strong. Thus

$$\mathbf{M} = \frac{1}{\mu_0}\chi_m \mathbf{B} \quad \text{(incorrect!)}.$$

But custom dictates that it be written in terms of **H**, instead of **B**:

$$\boxed{\mathbf{M} = \chi_m \mathbf{H}.}$$

The constant of proportionality χ_m is called the **magnetic susceptibility**; it is a dimensionless quantity that varies from one substance to another—positive for paramagnets and negative for diamagnets. Typical values are around 10^{-5} (see Table 6.1).

Material	Susceptibility	Material	Susceptibility
Diamagnetic:		*Paramagnetic:*	
Bismuth	-1.6×10^{-4}	Oxygen	1.9×10^{-6}
Gold	-3.4×10^{-5}	Sodium	8.5×10^{-6}
Silver	-2.4×10^{-5}	Aluminum	2.1×10^{-5}
Copper	-9.7×10^{-6}	Tungsten	7.8×10^{-5}
Water	-9.0×10^{-6}	Platinum	2.8×10^{-4}
Carbon Dioxide	-1.2×10^{-8}	Liquid Oxygen ($-200°$ C)	3.9×10^{-3}
Hydrogen	-2.2×10^{-9}	Gadolinium	4.8×10^{-1}

Table 6.1 Magnetic Susceptibilities (unless otherwise specified, values are for 1 atm, 20° C). *Source: Handbook of Chemistry and Physics*, 67th ed. (Boca Raton: CRC Press, Inc., 1986).

Materials that obey the relation $\boxed{\mathbf{M} = \chi_m \mathbf{H}.}$ is called linear media.

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H},$$

for linear media. Thus **B** is *also* proportional to **H**:[5]

$$\mathbf{B} = \mu \mathbf{H},$$

where

$$\mu \equiv \mu_0(1 + \chi_m).$$

μ is called the **permeability** of the material.[6] In a vacuum, where there is no matter to magnetize, the susceptibility χ_m vanishes, and the permeability is μ_0. That's why μ_0 is called the **permeability of free space**.