Solve all problems vectorially:

(1) Obtain the unit vectors perpendicular to each of $\bar{x} = (1, 2, -1)$ and $\bar{y} = (1, 0, 2)$.

$$\left[\text{Ans}: \pm\left(\frac{4}{\sqrt{29}}, \frac{-3}{\sqrt{29}}, \frac{-2}{\sqrt{29}} \right) \right]$$

(2) If α is the angle between two unit vectors $\bar{a}$ and $\bar{b}$, then prove that $|\bar{a} - \bar{b}\cos\alpha| = \sin\alpha$.

(3) If a vector $\bar{r}$ makes with X-axis and Y-axis angles of measures $45°$ and $60°$ respectively, then find the measure of the angle which $\bar{r}$ makes with Z-axis.

[Ans: $60°$ or $120°$]

(4) If $\bar{x}$ and $\bar{y}$ are non-collinear vectors of R^3, then prove that $\bar{x}$, $\bar{y}$ and $\bar{x} \times \bar{y}$ are non-coplanar.

(5) If the measure of angle between $\bar{x} = \bar{i} + \bar{j}$ and $\bar{y} = t\bar{i} - \bar{j}$ is $\frac{3\pi}{4}$, then find t.

[Ans: 0]

(6) Show that for any $a \in R$, the directions $(2, 3, 5)$ and $(a, a+1, a+2)$ cannot be the same or opposite.

(7) If θ is a measure of angle between unit vectors $\bar{a}$ and $\bar{b}$, prove that $\sin\frac{\theta}{2} = \frac{1}{2}|\bar{a} - \bar{b}|$.

(8) If $\bar{x}$, $\bar{y}$ and $\bar{z}$ are non-coplanar, prove that $\bar{x} + \bar{y}$, $\bar{y} + \bar{z}$ and $\bar{z} + \bar{x}$ are also non-coplanar.

(9) Show that the vectors (1, 2, 1), (1, 1, 4) and (1, 3, - 2) are coplanar. Also express each of these vectors as a linear combination of the other two.

$$\left[\text{Ans: } (1, 2, 1) = \frac{1}{2}(1, 1, 4) + \frac{1}{2}(1, 3, -2); \quad (1, 1, 4) = 2(1, 2, 1) - (1, 3, -2) \quad (1, 3, -2) = 2(1, 2, 1) - (1, 1, 4) \right]$$

[Note: These vectors are collinear besides being coplanar. Hence, any vector of R^3 which is not collinear with them cannot be expressed as a linear combination of these vectors even if it is coplanar with them.]

(10) Show that (1, 1, 0), (1, 0, 1) and (0, 1, 1) are non-coplanar vectors. Also express any vector (x, y, z) of R^3 as a linear combination of these vectors.

$$\left[\text{Ans: } (x, y, z) = \frac{x + y - z}{2}(1, 1, 0) + \frac{x - y + z}{2}(1, 0, 1) + \frac{y + z - x}{2}(0, 1, 1) \right]$$

(11) Prove that an angle in a semi-circle is a right angle.

(12) Prove that the three altitudes in a triangle are concurrent.

(13) If A - P - B and $\frac{AP}{PB} = \frac{m}{n}$, then prove that for any point O in space, $n(\overrightarrow{OA}) + m(\overrightarrow{OB}) = (m + n)\overrightarrow{OP}$.

(14) Prove that A (1, 5, 6), B (3, 1, 2) and C (4, - 1, 0) are collinear. Find also the ratio in which A divides $\overline{BC}$ from B.

[Ans: - 2 : 3]

(15) Find in which ratio and at which point does the XY-plane divide $\overline{AB}$ where A is (2, - 2, 1) and B is (1, 4, - 5).

$$\left[\text{Ans: } 1 : 5 \text{ from A at } \left(\frac{11}{6}, -1, 0 \right) \right]$$

(16) If $A(0, -1, -1)$, $B(16, -3, -3)$ and $C(-8, -1, -2)$ are given points, then find the point $D(x, y, z)$ in space so that $\vec{AB} = \vec{CD}$.

[Ans: $(8, -3, -4)$]

(17) $A(0, -1, -4)$, $B(1, 2, 3)$ and $C(5, 4, -1)$ are given points. If D is the foot of perpendicular from A on $\overline{BC}$, find its position vector.

[Ans: $(3, 3, 1)$]

(18) If the position vectors A, B, C of triangle ABC are $\bar{a}$, $\bar{b}$, $\bar{c}$ respectively, then show that the area of triangle ABC $= \dfrac{1}{2}\left|(\bar{a}\times\bar{b}) + (\bar{b}\times\bar{c}) + (\bar{c}\times\bar{a})\right|$.

(19) Find the volume of a prism having a vertex at origin O and having coterminous edges $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, where A is $(4, 3, 1)$, B is $(3, 1, 2)$ and C is $(5, 2, 1)$.

[Ans: 10 cubic units]

(20) Find the volume of tetrahedron having vertices $V(1, 1, 3)$, $A(4, 3, 2)$, $B(5, 2, 7)$ and $C(6, 4, 8)$.

$$\left[\text{Ans : } \frac{14}{3} \text{ cubic units}\right]$$

(21) If the forces of magnitudes $\sqrt{2}$, 2 and $\sqrt{3}$ units are applied to a particle in the directions of vectors $(-1, 0, 1)$, $(1, 0, 1)$ and $(1, 1, -1)$ respectively, then find the magnitude and direction of the resultant force.

$$\left[\text{Ans : } \sqrt{5}\,,\ \left(\cos^{-1}\sqrt{\frac{2}{5}},\ \cos^{-1}\sqrt{\frac{1}{5}},\ \cos^{-1}\sqrt{\frac{2}{5}}\right)\right]$$

(22) A boat is sailing to the east with a speed of $10\sqrt{2}$ km / hr. A man on boat feels that the wind is blowing from the south-east with a speed of 5 km / hr. Find the true velocity of the wind.

$$\left[\text{Ans : } 5\sqrt{5} \text{ km/hr at an angle } \cos^{-1}\frac{3}{\sqrt{10}} \text{ with east towards north}\right]$$

(23) A force of magnitude $2\sqrt{10}$ units is acting on a particle in the direction $3\bar{i} - \bar{j}$ and a force of magnitude $3\sqrt{13}$ units is acting on the same particle in the direction $2\bar{i} + 3\bar{j}$. Under the influence of these forces, the particle is displaced from A (1, 2) to B (6, 4). Find the work done. [Ans: 74 units]

(24) Prove that the diagonals of a rhombus bisect each other orthogonally.

(25) If a pair of medians of a triangle are equal, then show that the triangle is isosceles.

(26) Show that the perpendicular bisectors of sides of any triangle are concurrent.

(27) Prove that the diagonals of a rhombus are bisectors of its angles.

(28) If $\overrightarrow{AD}$ is a bisector of $\angle BAC$ in triangle ABC and if $D \in \overline{BC}$, then show that $\frac{BD}{DC} = \frac{AB}{AC}$.

(29) ABCDEF is a regular hexagon. Prove that $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = 3\overrightarrow{AD}$.

(30) Show that centroid and in-centre of an equilateral triangle are the same. Find the incentre of the triangle with vertices (6, 4, 6), (12, 4, 0) and (4, 2, - 2).

$\left[\text{Ans : } \left(\frac{22}{3}, \frac{10}{3}, \frac{4}{3} \right) \right]$

(31) If A is (1, 2, 1) and B is (4, - 1, 2), then find S (x, y, z) such that $2\overrightarrow{AB} = \overrightarrow{AS}$.

[Ans: (7, - 4, 3)]

(32) Let A (1, 2, - 1) and B (3, 2, 2) be given points. Find in which ratios from A and at which points do the XY-, YZ- and ZX-planes divide $\overline{AB}$.

$\left[\text{Ans : } 1 : 2, \ \left(\frac{5}{3}, 2, 0 \right); \ -1 : 3, \ \left(0, 2, \frac{5}{2} \right); \ \overleftrightarrow{AB} \text{ is parallel to ZX-plane} \right]$

(33) Show that (6, 0, 1), (8, - 3, 7) and (2, - 5, 10) can be three vertices of some rhombus. Find the co-ordinates of the fourth vertex of this rhombus.

[Ans: (0, - 2, 4)]

(34) Show that (1, 2, 4), (- 1, 1, 1), (6, 3, 8) and (2, 1, 2) are the vertices of a trapezium. Find the area of this trapezium.

$\left[\text{Ans : } \frac{3}{2}\sqrt{59} \right]$

(35) Find the area of the parallelogram ABCD if $\overrightarrow{AC} = \bar{a}$ and $\overrightarrow{BD} = \bar{b}$.

$\left[\text{Ans : } \frac{1}{2} \left| \bar{a} \times \bar{b} \right| \right]$

(36) Find the volume of a prism having a vertex at origin and having edges $\overrightarrow{OA} = 2\bar{i} + \bar{j} + \bar{k}$, $\overrightarrow{OB} = 3\bar{i} - \bar{j} + \bar{k}$ and $\overrightarrow{OC} = -\bar{i} + \bar{j} - \bar{k}$.

[Ans : 4 cubic units]

(37) Show that (4, 5, 1), (0, - 1, - 1), (3, 9, 4) and (- 4, 4, 4) cannot be the vertices of any tetrahedron.

(38) Find the volume of the tetrahedron with vertices (4, 5, 1), (0, - 1, - 1), (3, 9, 4) and (1, 2, 3).

$\left[\text{Ans : } \frac{28}{3} \text{ cubic units} \right]$

(39) A mechanical boat is rowing towards the north with speed of 8 km / hr. If wind blows from the east with the speed of 10 km / hr, find the resulting speed of the boat and also the direction of resulting motion of the boat.

$\left[\text{Ans : } 2\sqrt{41} \text{ km / hr at an angle of } \pi - \cos^{-1}\left(\frac{5}{\sqrt{41}}\right) \text{ with east towards north} \right]$

(40) A river flows with a speed of 5 units. A person desires to cross the river in a direction perpendicular to its flow. Find in which direction should he swim if his speed is 8 units.

$\left[\text{Ans : At an angle of } \pi - \cos^{-1}\left(\frac{5}{8}\right) \text{ with the direction of flow of the river} \right]$

(41) If speed of a particle is 5 units towards the east and $\sqrt{8}$ units towards the south-west, then find the resultant speed of the particle and its direction.

$\left[\text{Ans : } \sqrt{13} \text{ units at an angle of } \cos^{-1}\frac{3}{\sqrt{13}} \text{ with east towards south} \right]$

(42) A boat speeds towards the north at $6\sqrt{2}$ units. A man on the boat feels that the wind is blowing from the south-east at 5 units. Find the true velocity of the wind.

$\left[\text{Ans : } \sqrt{157} \text{ units at an angle of } \pi - \cos^{-1}\left(\frac{5}{\sqrt{314}}\right) \text{ with east towards north} \right]$

(43) A steamer moves to the north-east with a speed of 40 units. A passenger on the steamer feels the wind to be blowing from the north with $25\sqrt{2}$ units. Find the true velocity of the wind.

$\left[\text{Ans : } 5\sqrt{34} \text{ units at an angle of } \cos^{-1}\frac{4}{\sqrt{17}} \text{ with east towards south} \right]$

(44) A particle is displaced from A (2, 1) to B (4, 2) when forces of magnitudes $4\sqrt{5}$ in the direction $2\bar{i} + \bar{j}$ and $6\sqrt{5}$ in the direction $\bar{i} - 2\bar{j}$ are applied. Find the work done.

[Ans: 20 units]