Problem Set 7a

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1. **Degree Sequence.** In each of the following problems, either show that the given sequence cannot be the degree sequence of a graph, or give an example of a graph with that degree sequence.

   (a) $(1, 1, 1, 1, 0)$

   Solution: Consider $V = \{1, 2, 3, 4, 5\}$ and $E = \{(1, 2), (3, 4)\}$. All the 4 vertices $\{1, 2, 3, 4\}$ have degree 1 while vertex $\{5\}$ has degree 0.

   (b) $(2, 2, 2, 2, 2)$

   Solution: The cycle C_5 has 5 vertices and each of these vertices have degree 2.

   (c) $(3, 3, 2, 2, 1)$

   Solution: This is not possible because $\sum_{v \in V} deg(v) = 3+3+2+2+1 = 11$ which is odd but $\sum_{v \in V} deg(v) = 2|E|$ which implies that the sum must be even.

   (d) $(4, 4, 3, 2, 1)$

   Solution: This is not possible because there are 2 vertices which are connected to every other vertex implying that the degree of each vertex must be at least 2.

   (e) $(4, 3, 3, 3, 3)$

   Solution: The wheel W_4 has this degree sequence.

2. Define a pseudograph to be an undirected graph with one or more self-loops allowed in each node. The degree of a node in a pseudograph is defined by counting each self-loop as two edges incident on the node.

   Show that every (sorted) sequence of non-negative integers with an even sum of its terms is the degree sequence of a pseudograph.

   Hint: Construct such a graph by first adding as many self-loops as possible at each vertex. What does the residual degree sequence (i.e., degrees that remain to be satisfied) look like?

   Solution: Suppose there are k odd numbers and $n-k$ even numbers in the degree sequence. Since the sum of the terms is given to be even, so k is even.

   Now, for each even degree $2d$ in the degree sequence, where $d \geq 0$, put d self-loops on a vertex, and join it to no other vertex. For each odd degree $2d+1$ in the degree sequence, where $d \geq 0$, put d self-loops on a vertex. This leaves us with k vertices (corresponding to the odd degrees) that still have one more edge to be incident on them. Since k is even, we can simply make $\frac{k}{2}$ disjoint edges within themselves (a matching, to be precise), and we are done.

3. **Regular Graphs.**

   (a) For any integer $n \geq 3$ and any even integer d with $2 \leq d \leq n-1$, show that there exists a d-regular graph with n nodes, by giving an explicit graph (V, E), where $V = \mathbb{Z}_n$ and E is formally defined using modular arithmetic. (You may find it convenient to use S_a to denote $\{1, \ldots, a\} \subseteq \mathbb{Z}_n$.)

   Hint: What would you do for $d = 2$? Then consider adding additional edges for larger values of d.

   Solution: Consider the cycle C_n such that the neighbours of $i \in \mathbb{Z}_n$ are $i-1 \pmod n$ and $i+1 \pmod n$. This is a 2-regular graph on n vertices.

   One can similarly get a d-regular graph where d is even. For each node $i \in Z_n$, its neighbours are $i - \frac{d}{2} \pmod n, i - \frac{d}{2} + 1 \pmod n, ..., i-1, i+1, ..., i + \frac{d}{2} - 1 \pmod n, i + \frac{d}{2} \pmod n$.
   Formally, let $E = \{\{i, j\} \mid i-j \in S_{d/2} \text{ or } j-i \in S_{d/2}\}$. Then $\Gamma[i] = (i+S_{d/2}) \cup (i-S_{d/2})$, where $i \pm S = \{i \pm x | x \in S\}$. Note that these two sets are each of size $d/2$ and they are disjoint: Otherwise $i + x = i - y$ where $x, y \in S_{d/2}$, implying $x + y \equiv 0 \pmod n$, which is not possible as $x + y \in \{2, \cdots, n-1\}$.

(b) For any even integer $n \geq 2$ and any integer d with $1 \leq d \leq n-1$ show that there exists a d-regular graph with n nodes.

Hint: Use the previous part for even d. For odd d, use the previous part to first construct a $(d-1)$-regular graph, and find a way to add new edges so that all nodes have their degree incremented by 1.

Solution: For any even integer n, consider a 1-regular graph on n nodes in $\mathbb{Z}_n$ such that $E = \{(0,1),(2,3),...,(n-2,n-1)\}$. This is a complete bi-partite graph on n nodes. Since this is the only possibility for $n=2$, we can reduce our proof for the cases of even $n \geq 3$ with $2 \leq d \leq n-1$. For even d, this problem reduces to part (a). Consider odd d such that $3 \leq d \leq n-1$ with even n. Our aim is to construct a $d-$regular graph on n vertices.

Consider the $(d-1)-$regular graph on n vertices from the construction in (a). Therefore, each $i \in \mathbb{Z}_n$ has neighbours $i - \frac{d-1}{2} \pmod n), ..., i-1, i+1, ..., i+\frac{d-1}{2} \pmod n$. Add a neighbour $i + \frac{n}{2} \pmod n$ to the vertex i i.e. the diagonally opposite node to i. This is possible because $d \leq n-1$ implying $\frac{d-1}{2} \leq \frac{n}{2} - 1$ which shows that $i + \frac{n}{2} \pmod n$ is not already a neighbour of i. The degrees of i and $i + \frac{n}{2} \pmod n$ have been increased from $d-1$ to d. Proceed similarly for each i with $0 \leq i < \frac{n}{2}$. Since n is even, a diagonally opposite element exists for each node. Hence, we have a $d-$regular graph on n vertices.

4. A graph with vertices $(v_1, \ldots, v_n)$ is said to be a *graph realization* of a sequence $d_1 \geq \ldots \geq d_n$ of non-negative integers, if for each i, $\deg(v_i) = d_i$ in the graph. There are efficient algorithms to check if a given sequence has a graph realization. In this problem you shall see one such algorithm.

a) Show that if $d_1 \geq \ldots \geq d_n$ has a graph realization, then it has a graph realization such that v_1 is adjacent to the d_1 nodes $v_2, \ldots, v_{d_1+1}$.

Hint: Among all the realizations, consider one which maximizes the sum of degrees of the nodes adjacent to v_1. If its vertices cannot be relabelled to be of the required form, then there are nodes v_i, v_j with $d_i > d_j$ and v_1 adjacent to v_j but not adjacent to v_i. **Solution:** Suppose the degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$ has a graph realisation. Among all graphs that realise this degree sequence, chose one which maximises the sum of degrees of those nodes which are adjacent to v_1. Call this graph G. We now assume the contrary, that is, suppose there exists no graph realising the given degree sequence such that v_1 is adjacent to $v_2, v_3, \cdots, v_{d_1+1}$. In particular, this is true for G as well. Consider the smallest index i for which v_1 is not adjacent to v_i. There must be some $j > i$ for which v_1 is adjacent to i (otherwise v_1 is adjacent to, and only to, $v_2, v_3, \cdots, v_{i-1}$, in which case $i = d_1 + 2$ and we are done).

If $d_i = d_j$, then we can swap the labels i and j, so that v_1 becomes adjacent to v_i (which was formerly v_j); this swap preserves the sum of degrees of the nodes adjacent to v_1. If we can keep doing such swaps and make v_1 adjacent to $v_2, v_3, \cdots, v_{d_1+1}$, then we arrive at a contradiction to our assumption. So the problem occurs when a swap is not possible, that is, there exist indices $i < j$ (these i and j may not be the i and j mentioned before) such that v_1 is adjacent to v_j but not to v_i, and $d_i > d_j$.

Since $d_i > d_j$, there exists some vertex v_k such that v_i is adjacent to v_k but v_j is not adjacent to v_k. Now, consider a new graph G' formed from G as follows: Remove the edges $v_1 v_j$ and $v_i v_k$, and add the edges $v_1 v_i$ and $v_j v_k$. This preserves the degree of each vertex as is. Furthermore, we now have v_i adjacent to v_1 and v_j not adjacent to v_1. The sum of the degrees of the nodes adjacent to v_1 changes by $d_i - d_j > 0$, which contradicts our choice of G.

Therefore, our assumption is wrong. This means if $d_1 \geq d_2 \geq \cdots \geq d_n$ has a graph realisation, then it has one such that v_1 is adjacent to the d_1 nodes $v_2, \cdots, v_{d_1+1}$.

b) Show that the sequence $d_1 \geq \ldots \geq d_n$ has a graph realization if and only if the sequence obtained by sorting $(d_2 - 1), ..., (d_{d_1+1} - 1), d_{d_1+2}, ..., d_n$ has a graph realization.

Note: This reduces the problem of checking realizability of n-long sequences to a problem of $(n-1)$-long sequences. This leads to a recursive algorithm. **Solution:** Suppose $d_1 \geq d_2 \geq \cdots \geq d_n$ has a graph realisation. From the previous problem, it has a graph realisation G such that if $V(G) = \{v_1, \cdots, v_n\}$ and $d_i = \deg(v_i)$ for every i, then v_1 is adjacent to $v_2, \cdots, v_{d_1+1}$.

Delete the vertex v_1 along with the edges incident on it. This leaves us with a graph G' with unsorted degree sequence $d_2 - 1, \cdots, d_{d_1+1} - 1, d_{d_1+2}, \cdots, d_n$. Therefore this degree list has a graph realisation too.

Conversely, suppose the unsorted degree sequence $d_2 - 1, \cdots, d_{d_1+1} - 1, d_{d_1+2}, \cdots, d_n$ has a graph realisation G'. Construct a graph G from G' by introducing a new vertex v, and joining v to the vertices with degrees

$d_2-1,\cdots,d_{d_1+1}-1$. This graph has degrees $d_1,d_2,\cdots,d_{d_1+1},d_{d_1+2},\cdots,d_n$. Therefore $d_1 \geq d_2 \geq \cdots \geq d_n$ has a graph realisation.

This polynomial time algorithm is called the **Havel-Hakimi** algorithm.

5. **Complement of a Graph.** We define the *complement of a graph* as a graph which has the same vertex set, but with exactly those edges that are absent from the original graph. Formally, if $G=(V,E)$, its complement $\overline{G}=(V,\overline{E})$, such that $\overline{E}=K_V-E$ where $K_V=\{\{a,b\}|a\in V, b\in V, a\neq b\}$.

   Show that if a graph with n vertices is isomorphic to its complement, then $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$.

   Solution: Suppose the graph $G=(V,E)$ with n vertices has e_1 edges and it's complement graph $\bar{G}$ has e_2 edges. Then,

   $$e_1+e_2=\binom{n}{2}=\frac{n(n-1)}{2}$$

   because $(v_1,v_2)\notin E$ iff $(v_1,v_2)\in\bar{E}$ implying all possible combinations of 2 vertices from n vertices is counted exactly once, either in e_1 (edges of G) or in e_2 (edges of $\bar{G}$). Also, 2 isomorphic graphs have the same number of edges implying

   $$e_1=e_2=\frac{n(n-1)}{4}$$

   Since the number of edges can only be an integer,

   $$n(n-1)\equiv 0 \pmod 4$$

   For any $n\geq 1$, n and $n-1$ are co-prime to each other because $\gcd(n,n-1)=1$ using Euclid's algorithm. Hence, either $n\equiv 0 \pmod 4$ or $n-1\equiv 0 \pmod 4$.

6. Match each graph on the left with a description of its complement:

   (a) K_4
   (b) C_4
   (c) $K_{1,3}$
   (d) P_4

   (a) A graph with no edges.
   (b) A graph with a single edge.
   (c) A path with two edges.
   (d) A matching with two edges.
   (e) A graph isomorphic to its complement.
   (f) A complete graph.
   (g) A cyclic graph.

   Solution:

   - The complement of K_4 is clearly the empty graph on 4 vertices. So (a) matches with (a).
   - The complement of C_4 is a graph with "opposite" vertices of C_4 now joined together. The graph has four vertices and two disjoint edges, that is, a matching with two edges. (b) matches with (d).
   - The complement of $K_{1,3}$ is an isolated vertex and a cycle on three vertices. (c) matches option (g), that is, the graph is cyclic (a graph is said to be cyclic if it has a cycle).
   - The complement of P_4 is P_4 again! Therefore (d) matches only (e).

7. **What is Wrong With this Proof?**

   Claim: If every vertex in a graph has degree at least 1, then the graph is connected.

   Proof. We use induction. Let $P(n)$ be the proposition that if every vertex in an n-vertex graph has degree at least 1, then the graph is connected.

   Base case: There is only one graph with a single vertex and it has degree 0. Therefore, $P(1)$ is vacuously true.

   Inductive step: We must show that $P(n)$ implies $P(n+1)$ for all $n\geq 1$.

   Consider an n-vertex graph G in which every vertex has degree at least 1. By the induction hypothesis, G is connected; that is, there is a path between every pair of vertices. Now we add one more vertex x to G to obtain an $(n+1)$-vertex graph H. Since x must have degree at least one, there is an edge from x to some other vertex; call it y. Since y is connected to every other node in the graph, x will be connected to every other node in the graph. QED

- [ ] A. The proof needs to consider base case $n = 2$.
- [ ] B. The proof needs to use strong induction.
- [ ] C. The proof should instead induct on the degree of each node.
- [ ] D. The proof only considers $(n+1)$ node graphs with minimum degree 1 from which deleting a vertex gives a graph with minimum degree 1.
- [ ] E. The proof only considers n node graphs with minimum degree 1 to which adding a vertex with non-zero degree gives a graph with minimum degree 1.
- [ ] F. This is a trick question. There is nothing wrong with the proof!

Solution: Let us analyse each step carefully. $P(1)$ is true, so the base case of $n = 1$ is not wrong as such. Also, even if we consider the base case $n = 2$, the only graph with some vertex of degree at least 1 is the graph where the vertices are connected to each other, in which case the claim is true too. So the base case is probably not the problem.

The next two options also do not seem right; the fallacy in the proof has nothing to do with what should have been done in the proof itself.

Finally, option (F) is certainly wrong; there are graphs with smallest degree at least one, such that the graph is disconnected.

Here is the problem in the proof. When we want to show something to be true for $n + 1$, we must take an **arbitrary** graph G on $n+1$ vertices, because we want to show that the statement holds for all graphs on $n+1$ vertices. The proof given, however, starts with a graph on n vertices with minimum degree 1, and then adds a vertex to G to get a graph H on $n + 1$ vertices.

More rigorously, the only type of $n + 1$ node graphs considered are those that can be formed by adding a vertex to an n node graph with least degree 1. Consider the graph on 4 vertices which is isomorphic to the complement of C_4. This graph H is a matching on two edges, and has minimum degree 1. However, there exists no graph G on n vertices with minimum degree 1 to which a vertex can be added to form H.

So the true issue is option (D). The issue is not in considering n node graphs of other type. Even if you consider all graphs on n vertices, it does not quite solve the issue. Instead, the issue is not starting with an arbitrary $n + 1$ node graph.

8. **Prove using Induction.** Prove that for any positive integer n, for any triangle-free graph $G = (V, E)$ with $|V| = 2n$, it must be the case that $|E| \leq n^2$.

   Solution: Use weak induction on n to prove this. For $n = 1$, $V = \{1, 2\}$ and hence there can be at most one edge in G. Thus, $|E| \leq 1 = n^2$. Assume that for a positive integer $n = k > 1$, for any triangle-free graph $G = (V, E)$ with $|V| = 2k$, $|E| \leq k^2$. We need to prove that the same holds for $n = k + 1$.

   Given a triangle-free graph $G = (V, E)$ such that $|V| = 2k + 2$. Consider a pair of adjacent vertices $i, j \in V$ (if there are no such i, j, then the problem is trivially true). Consider the subgraph $G^{'}$ obtained by removing i, j and all edges incident on these 2 vertices from G. $G^{'}$ is also triangle-free because $G^{'}$ is itself an induced subgraph of G implying that if $G^{'}$ had a induced subgraph isomorphic to a triangle then G should also have this induced subgraph. Note that $|V^{'}| = 2k$.

   Using the inductive hypothesis, $|E^{'}| \leq k^2$. Now, since i and j are adjacent, and the graph G is triangle-free, therefore no vertex l in G' can be adjacent to both i and j because if that is the case then i, j and l will give rise to an induced subgraph isomorphic to a triangle. Therefore, the maximum number of edges added to $G^{'}$ in order to add i, j are $2k + 1$ (there is 1 edge between i and j and more than $2k$ vertices in $G^{'}$ linked to either of i or j by an edge would imply that there is a vertex linked to both via pigeonhole principle). Therefore, $|E| \leq k^2 + (2k + 1) = (k + 1)^2$.

9. **Walks and Paths.** In this problem, you shall prove that for any graph G and any two nodes a and b in G, if there is a walk from a to b, then there is a path from a to b.

(a) Prove this using strong induction. Induct on the length of the walk. **Solution:** We strong induct on the length l of the walk. If $l = 1$, then the walk is simply an edge from a to b, so we are done.
Suppose the statement is true for all naturals less than or equal to l, where $l \geq 1$. Consider, now, a walk of length $l+1$. If this walk has no repeated vertices, then it is also a path, so we are done. Otherwise, the walk $v_0 v_1 \cdots v_l v_{l+1}$, where $a = v_0$ and $b = v_{l+1}$, has some vertex occurring at least once. Suppose $i < j$ and $v_i = v_j$. Then, the following is also a walk: $v_0 v_1 \cdots v_i v_{j+1} \cdots v_{l+1}$. This walk from a to b has length $l + 1 - (j - i) < l + 1$. By strong induction hypothesis, there is a path from a to b, and we are done.

(b) Prove this using the well-ordering principle, and by proving a stronger statement: A shortest walk from a to b is a path from a to b. **Solution:** Let W defined as $a = v_0, v_1, \cdots, v_k = b$ be a shortest walk from a to b of length k. Suppose W is not a path. Then there exist indices $0 \leq i < j \leq k$ such that $v_i = v_j$. Then, the truncated sequence $v_0, v_1, \cdots, v_i, v_{j+1}, \cdots v_k$ is also a walk W' of length $k - (j - i) < k$, contradicting the choice of W. Therefore, W is a path.

Now, suppose there is a walk from a to b. By well-ordering on naturals, there is a walk of minimum possible length. This walk is a path from a to b, as required.

10. **Connectivity and Cycles.** Show that if a graph has a cycle, then deleting any edge in that cycle results in a graph which has the same connectivity relation (i.e., if there is a walk from u to v before deleting the edge, then after deleting the edge too there is such a walk).

Solution: Let $C = u_1 u_2 \cdots u_k$ be a cycle of length $k \geq 3$ in a graph G. Without any loss of generality, suppose the edge removed from the cycle is $u_1 u_k$; call this new graph G'. We want to show that any $v, w \in V(G) = V(G')$ are connected in G if and only if they are connected in G'.

Suppose v and w are connected in G', and let P be a path from v to w in G'. This path P is clearly a path in G as well, so v and w are connected in G as well.

Now suppose v and w are connected in G; let P be a path from v to w. If this path does not use the edge $u_1 u_n$, then P is present in G' too. This would mean v and w are connected in G as well. Otherwise, suppose P does use $u_1 u_n$ as an edge. Replace the edge $u_1 u_n$ in P by $u_1 u_2 \cdots u_n$. This new trail is a walk W from v to w not using the edge $u_1 u_n$. This means W is a walk in G' as well, so v and w are connected in G'.

11. Show that any two *maximum* length paths in a connected graph should have a common vertex.

Hint: Consider a shortest path that connects the two paths.

Solution: Consider 2 maximum length paths P_1 and P_2 of length k in G. WLOG we can label these paths as $u_1 u_2 ... u_{k+1}$ and $v_1 v_2 ... v_{k+1}$. Also, for the sake of contradiction assume that these paths have no common vertex. Since the graph is connected, there must be a shortest path $P = u_i ... v_j$ that connects the 2 paths P_1 and P_2 in G (since the graph is connected). It can be seen that P has no vertex in common with P_1 apart from u_i because in that case we can get a shorter path connecting P_1 and P_2. Similarly, P has no vertex in common with P_2 apart from v_j. Also, WLOG we can assume that $i \leq j$. Now consider the path $P^{'} = v_1 ... v_j ... u_i ... u_{k+1}$ taken from P_2, P and P_1 respectively. Since $i \leq j$, $k - i \geq k - j$ which implies that length of $P^{'}$ is at least $k + 1$ which is contradiction.

12. **Triangle-Free and Claw-Free Graphs.** Recall that an *induced subgraph* of G is obtained by removing zero or more vertices of G as well as all the edges incident on the removed vertices. (No further edges can be removed.) Formally, $G' = (V', E')$ is an induced subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' = \{\{a, b\} \mid a \in V', b \in V', \{a, b\} \in E\}$.

A graph G is said to be H*-free* if no induced subgraph of G is isomorphic to H. For example, $G = (V, E)$ is K_3-free (or triangle free) if and only if there are no three distinct vertices a, b, c in V such that $\{\{a, b\}, \{b, c\}, \{c, a\}\} \subseteq E$.

Prove that the complement of a K_3-free graph is a $K_{1,3}$-free graph.[1]

Hint: Prove the contrapositive.

Solution: The contrapositive of the above statement is that if a graph G has $K_{1,3}$ as an induced subgraph then it's complement graph $\bar{G}$ will have K_3 as an induced subgraph. To prove this, assume that the input graph G has $K_{1,3}$ as

[1] The graph $K_{1,3}$ is often called the "claw" graph. So this problem can be restated as asking you to prove that the complement of a triangle-free graph is a claw-free graph.

an induced subgraph. Consider this subgraph i.e. $V^{'} = \{v_1, v_2, v_3, v_4\}$ with $E^{'} = \{(v_1, v_2), (v_1, v_3), (v_1, v_4)\}$ without loss of generality. This implies that $(v_2, v_3), (v_2, v_4), (v_3, v_4) \notin E^{'}$ and hence not contained in E because this is an induced subgraph implying that all edges present in G between the vertices of the subgraph are also present in the subgraph. Therefore, $(v_2, v_3), (v_2, v_4), (v_3, v_4) \in \bar{E}$ and hence the induced subgraph of $\bar{G}$ with $\bar{V}^{'} = \{v_2, v_3, v_4\}$ is isomorphic to K_3.

13. If a graph G has chromatic number $k > 1$, prove that its vertex set can be partitioned into two nonempty sets V_1 and V_2, such that

$$\chi(G(V_1)) + \chi(G(V_2)) = k$$

where $G(V_1)$ denotes the induced subgraph of G with vertex set V_1.

Solution: Let $C : V \to [k]$ be a proper colouring of G. Note that C is onto, otherwise G would have a smaller chromatic number than k.
Let V_1 be the set of vertices v for which $C(v) = 1$. Let V_2 be all other vertices. Note that V_1 and V_2 are non-empty, because C is onto and $k > 1$.

Now, the chromatic number of $G(V_1)$ is at most (and hence exactly) 1, because the colouring C for G also acts as a proper colouring for $G(V_1)$, and the only colour used for the vertices of V_1 is the colour 1. Similarly, the chromatic number of $G(V_2)$ is at most $k - 1$. Suppose now that V_2 has chromatic number k', say. Let $C' : V_2 \to [k']$ be an onto proper colouring of $G(V_2)$. Consider the following colouring $C : V \to [k' + 1]$ for G; $C(v) = C'(v)$ for all $v \in V_2$, and $C(v) = k' + 1$ for all $v \in V_1$. It is easy to show that C is a proper colouring for G. Also note that every colour in $[k' + 1]$ is used for V. Therefore, we have $k' + 1 \geq k$, so $k' \geq k - 1$. This, along with the previous observation, proves that $k' = k - 1$, so

$$\chi(G(V_1)) + \chi(G(V_2)) = k$$

as required.

14. The union of 2 graphs on the same vertex set $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ is defined as $(V, E_1 \cup E_2)$. Prove that the chromatic number of the union of G_1 and G_2 is at most $\chi(G_1)\chi(G_2)$.

Solution: Let $C_1 : V \to \{1, 2, ..., \chi(G_1)\}$ be a proper coloring of G_1 and $C_2 : V \to \{1, 2, ..., \chi(G_2)\}$ be a proper coloring of G_2. Consider the coloring $C : V \to \{1, 2, ..., \chi(G_1)\} \times \{1, 2, ..., \chi(G_2)\}$ given by $C(v) = (C_1(v), C_2(v))$. Consider a pair of vertices $i, j \in V$ such that $C(i) = C(j)$ which can also be written as $(C_1(i), C_2(i)) = (C_1(j), C_2(j))$. This implies that $C_1(i) = C_1(j)$ and $C_2(i) = C_2(j)$. Since C_1 is a proper coloring of G_1, we can say that $(i, j) \notin E_1$ because i and j have the same color w.r.t C_1 and similarly we can say that $(i, j) \notin E_2$. Hence, $(i, j) \notin E_1 \cup E_2$. Thus, C is a proper coloring of $G_1 \cup G_2$ with at most $\chi(G_1)\chi(G_2)$ many colors.