Discrete Structures :: CS 207 :: Autumn 2021

Problem Set 4

Released: September 6, 2021

1. Given a set S, its powerset is defined as the set of all subsets of S. That is, $\mathcal{P}(S) = \{T \mid T \subseteq S\}$. Describe $\mathcal{P}(\{1,2\}))$ explicitly.

2. **Subsets as bit strings.** Given a set S with $|S| = n$, we may use bit-strings to conveniently represent the subsets of S. For this, we fix an arbitrary ordering of the n elements in S. Then, $T \subseteq S$ is represented by the n-bit string x_T such that the i^{th} bit of x_T is 1 iff the i^{th} element of S (in the order we have fixed) is in T. In answering the following, you can use boolean operators to n-bit strings, where the operation is applied bit-wise (e.g., $001 \oplus 010 = 011$, $\neg 001 = 110$).

   (a) Express $x_{A\cap B}, x_{A\cup B}$ and x_{A-B} in terms of x_A and x_B.

   (b) Describe the set T in terms of A, B, C, if $x_T = x_A \oplus x_B \oplus x_C$.

3. **A Set representing Prime Factorization.** For every positive integer n, define a set $PF_n \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$ to denote the prime factors of n, as follows.
$$PF_n = \{(p,i) : p \text{ is prime, } i \in \mathbb{Z}^+ \text{ and } (p^i \mid n)\}.$$

   (a) What is PF_1?

   (b) Explicitly write down PF_{12} and PF_{30}.

   (c) Write down $PF_{\gcd(12,30)}$.

   (d) Write down $PF_{\operatorname{lcm}(12,30)}$.

   (e) Suppose $a|b$ for positive integers a, b. What is the relation between PF_a and PF_b?

   (f) For any two positive integers m and n, give formulas for $PF_{\gcd(m,n)}$ and $PF_{\operatorname{lcm}(m,n)}$ in terms of PF_m and PF_n.

   (g) Conclude from the above that, if x is a common divisor and y is a common multiple of two positive integers m, n, then $x|\gcd(m,n)$ and $\operatorname{lcm}(m,n)|y$.

4. Give an example for each of the following, if such an example exists. Else prove why it cannot exist.

   (a) A relation that is irreflexive, antisymmetric and not transitive.

   (b) A relation that is neither symmetric nor antisymmetric.

   (c) An antisymmetric relation which has a symmetric relation as its subset.

   (d) Relations R_1 and R_2 on set S such that both are symmetric but $R_1 \cap R_2$ is not symmetric.

5. Given a relation R over a ground set S, and a subset $T \subseteq S$, define the relation $R|_T$ *induced* by R on T as follows:
$$R|_T = R \cap (T \times T).$$
   That is, for every pair $(a,b) \in T \times T$, $(a,b) \in R|_T$ iff $(a,b) \in R$. Which of the following statements are true for any relation R over S and any $T \subseteq S$? Justify your answer with a proof or a counterexample.

   (a) If R is symmetric, so is $R|_T$.

   (b) If R is irreflexive, so is $R|_T$.

   (c) If R is not reflexive, nor is $R|_T$.

   (d) If R is a partial order, so is $R|_T$.

6. Given a relation R, define R^2 as follows:

$$R^2 = \{(a, b) | \exists c \ (a, c) \in R \text{ and } (c, b) \in R\}.$$

Show the following.

(a) If R is symmetric, so is R^2.

(b) R^2 being symmetric does not imply that R is symmetric.

(c) If R is reflexive and transitive, $R = R^2$.

7. Let S be the set of all colourings of the 2×2 checkerboard where each of the four squares is coloured either red or blue. Note that S has 16 elements. Let R be a relation on S, so that $(C_1, C_2) \in R$ if and only if C_2 can be obtained from C_1 by rotating the checkerboard.

(a) Show that R is an equivalence relation.

(b) What are the equivalence classes of R? For each equivalence class, describe one member in the class and the size of the class.

8. Let $(S, \preceq)$ be a (non-empty) poset. We write $a \prec b$ if we have $a \preceq b$ and $a \neq b$. An element $a \in S$ is called *maximal* if $\nexists b \in S$ s.t. $a \prec b$. Similarly, an element $a \in S$ is called *minimal* if $\nexists b \in S$ s.t. $b \prec a$.

(a) Consider a restriction of the divisibility poset to a small set, $(\{2, 4, 5, 10, 12, 20, 25\}, |\})$. What are its maximal and minimal elements?

(b) Consider poset $(\mathcal{P}(S), \subseteq)$ for some set S. What are its maximal and minimal elements?

(c) Show that every maximal chain in a finite poset $(S, \preceq)$ contains a minimal element of S. (A maximal chain is a chain that is not a subset of a larger chain.)

9. In the context of relations, the term *lattice* is used to refer to a poset in which every finite set of elements has both a least upper bound and a greatest lower bound. Prove that the following posets are lattices. In each case, define the least upper bound and greatest lower bound of any finite set of elements.

(a) $(\mathcal{P}(X), \subseteq)$, the set of subsets of X with the inclusion relation.

(b) The divisibility poset, $(\mathbb{Z}^+, |)$.

Hint: You may use the fact from Problem 3(g) generalized to any finite number of integers.

10. Recall that the Mirsky's theorem stated in class states that in a poset P, the size of the largest chain in a poset P is of size k, is exactly equal to the smallest number of anti-chains that can partition P.

(a) Write out a formal proof for this, filling in all the details.

(b) Prove that any poset with n elements must have either (i) a chain *and* an anti-chain both of length equal to $\sqrt{n}$, or (ii) a chain *or* an anti-chain of length greater than $\sqrt{n}$.

(c) Consider the numbers from 1 to n arranged in an arbitrary order on a line. Prove that there must exist a $\sqrt{n}$-length subsequence of these numbers that is completely increasing or completely decreasing as you move from right to left. For example, the sequence 7, 8, 9, 4, 5, 6, 1, 2, 3 has an increasing subsequence of length 3, for example: 1, 2, 3, and a decreasing subsequence of length 3, for example: 9, 6, 3. *Hint: Define an appropriate poset that considers the value of each number as well as its position on the line.*