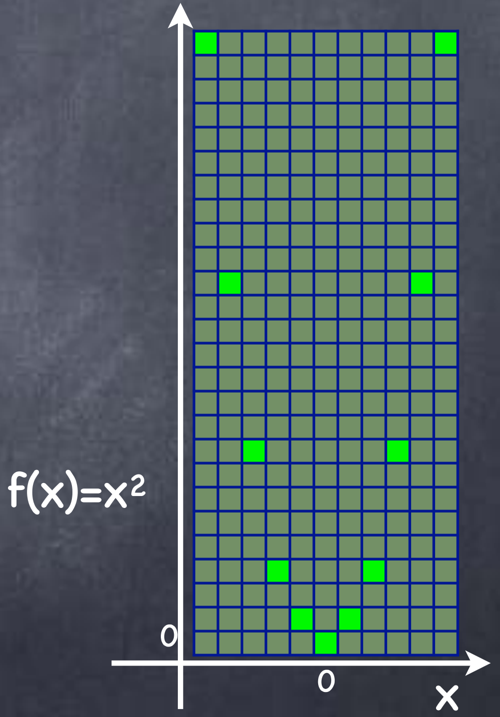
Functions



Functions

- For each element in a universe (domain), a predicate assigns one of two values, True and False.
- "Co-domain" is {True,False}
- Functions: more general co-domains
  - $f : A \to B$
- A function maps each element in the domain to an element in the co-domain
- To specify a function, should specify domain, co-domain and the "table" itself

pair$\in$AIW^2	Likes(pair)
(Alice, Alice)	TRUE
(Alice, Jabberwock)	FALSE
(Alice, Flamingo)	TRUE
(Jabberwock, Alice)	FALSE
(Jabberwock, Jabberwock)	TRUE
(Jabberwock, Flamingo)	FALSE
(Flamingo, Alice)	FALSE
(Flamingo, Jabberwock)	FALSE
(Flamingo, Flamingo)	TRUE

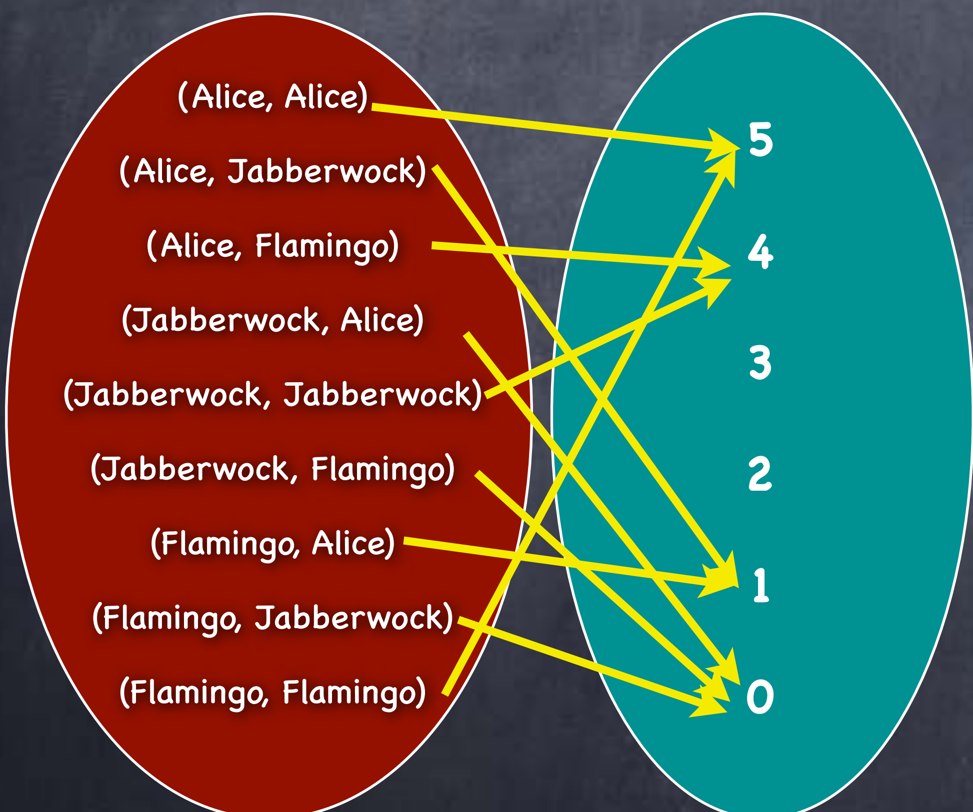
Functions

- eg: Extent of liking, $f: AIW^2 \rightarrow \{0,1,2,3,4,5\}$
  - Note: no empty slot, no slot with more than one entry
  - Not all values from the co-domain need be used
- Image: set of values in the co-domain that do get used
  - For $f:A \rightarrow B$, $Im(f) \subseteq B$ s.t. $Im(f) = \{ y \in B \mid \exists x \in A \;\; f(x) = y \}$

x∈Domain	f(x)∈Co-Domain
(Alice, Alice)	5
(Alice, Jabberwock)	1
(Alice, Flamingo)	4
(Jabberwock, Alice)	0
(Jabberwock, Jabberwock)	4
(Jabberwock, Flamingo)	0
(Flamingo, Alice)	1
(Flamingo, Jabberwock)	0
(Flamingo, Flamingo)	5

Functions

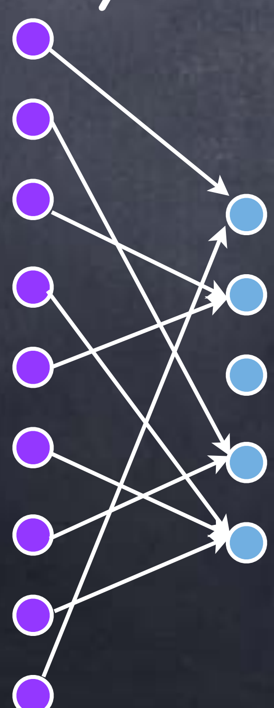
- eg: Extent of liking, $f: AIW^2 \rightarrow \{0,1,2,3,4,5\}$

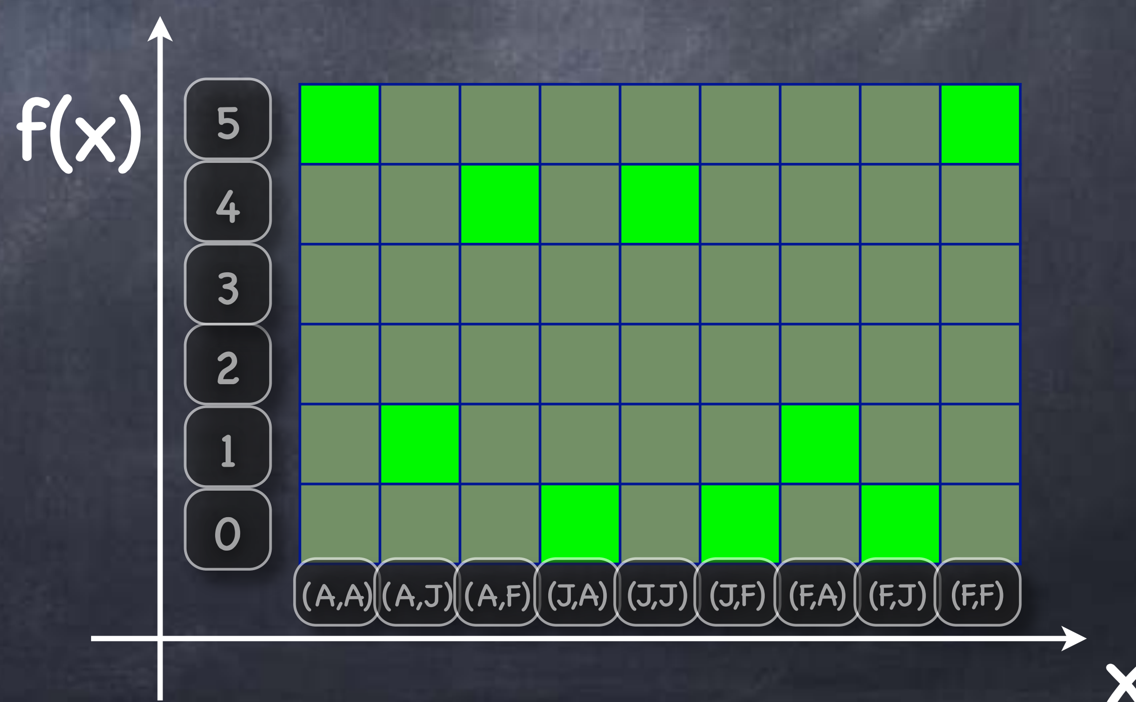


$x \in$ Domain	$f(x) \in$ Co-Domain
(Alice, Alice)	5
(Alice, Jabberwock)	1
(Alice, Flamingo)	4
(Jabberwock, Alice)	0
(Jabberwock, Jabberwock)	4
(Jabberwock, Flamingo)	0
(Flamingo, Alice)	1
(Flamingo, Jabberwock)	0
(Flamingo, Flamingo)	5

Function as a Relation

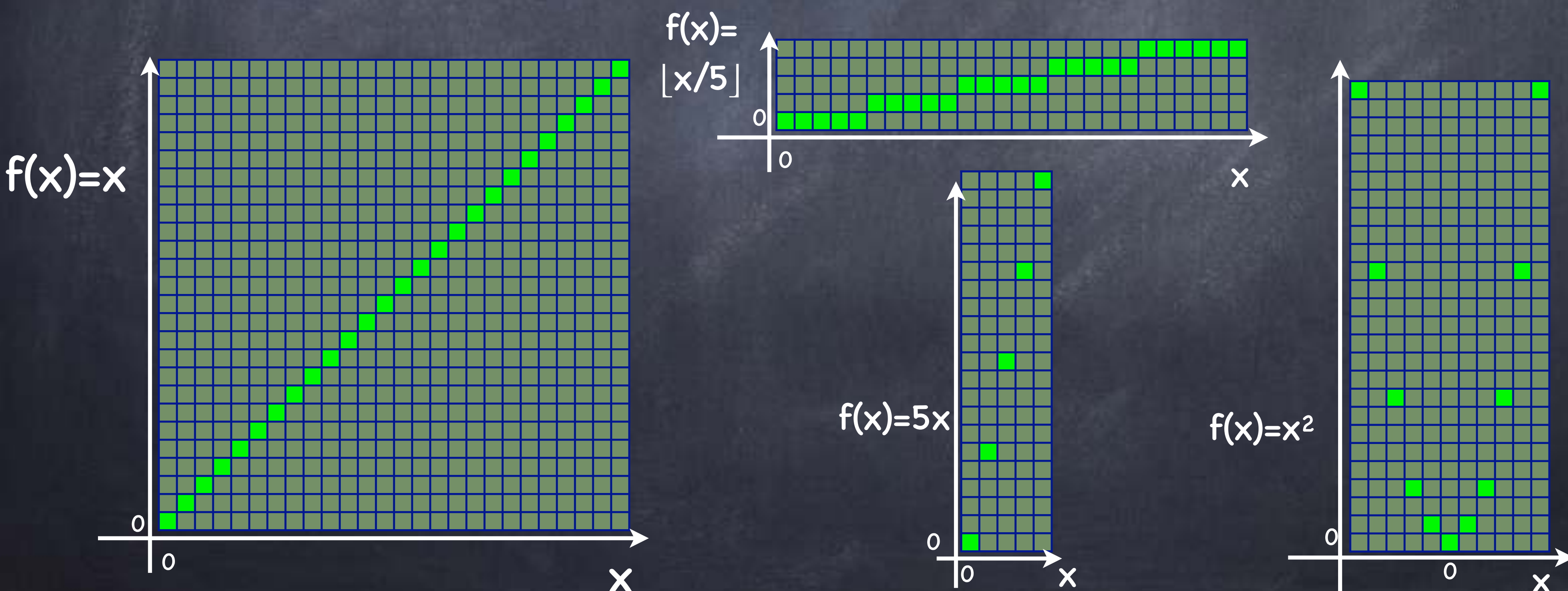
- As a relation between domain & co-domain, $R_f \subseteq$ domain $\times$ co-domain
  $R_f = \{ (x,f(x)) \mid x \in \text{domain} \}$
  - Special property of R_f: every x has a unique y s.t. $(x,y) \in R_f$
- Can be represented using a matrix
  - Convention: domain on the "x-axis", co-domain on the "y-axis"
  - Every column has exactly one cell "switched on"





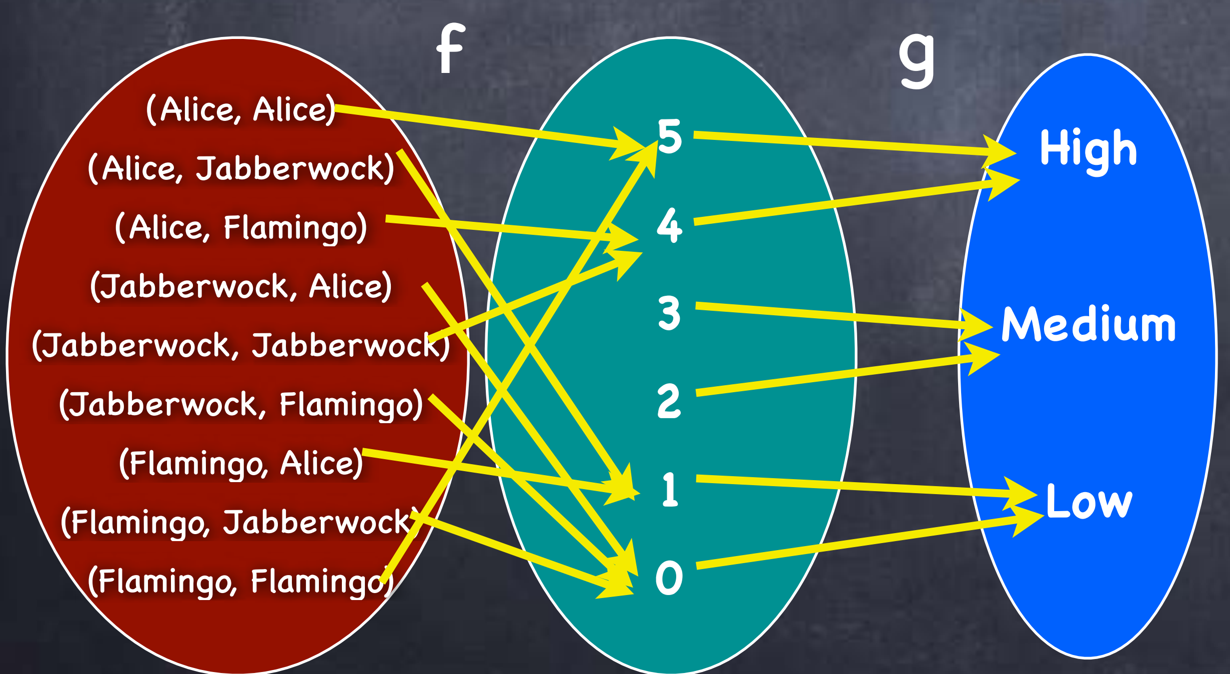
Plotting a Function

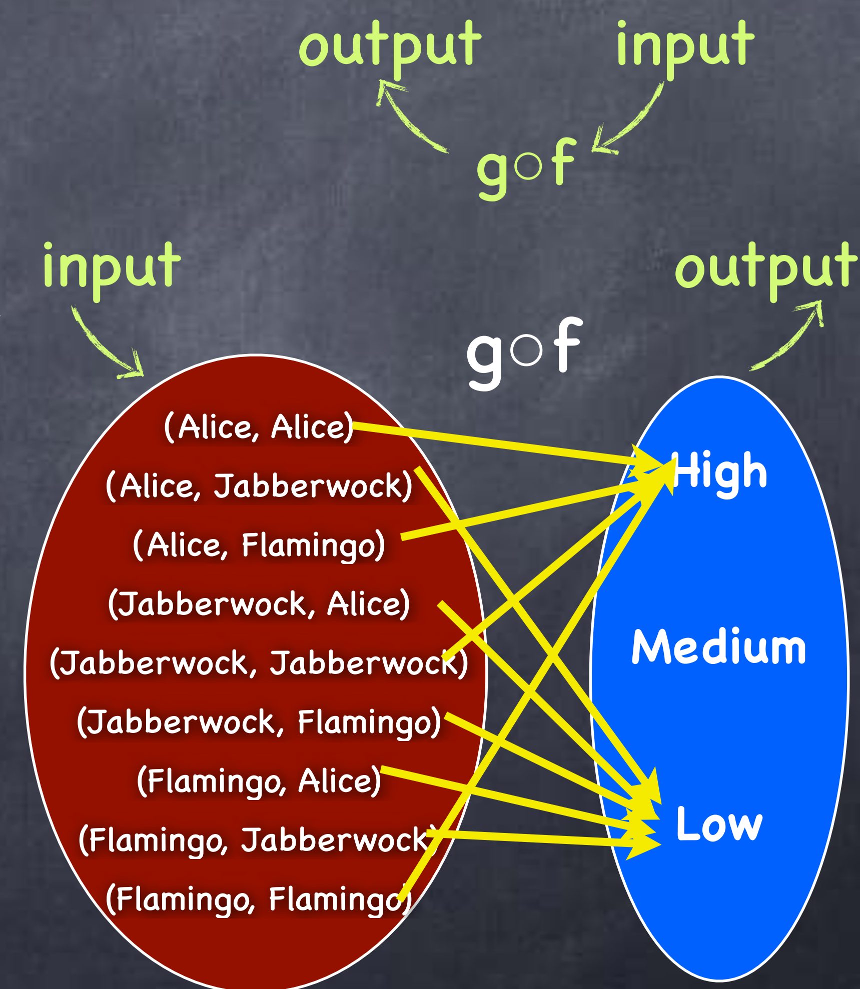
- When both domain and co-domain are numerical (or otherwise totally ordered), we often "plot" the function
  - Shows only part of domain/codomain when they are infinite (here $f:\mathbb{Z}\to\mathbb{Z}$)



Composition

- Composition of functions f and g: $g \circ f$: Domain(f) $\rightarrow$ Co-domain(g)
  - $g \circ f(x) \triangleq g(f(x))$





Composition

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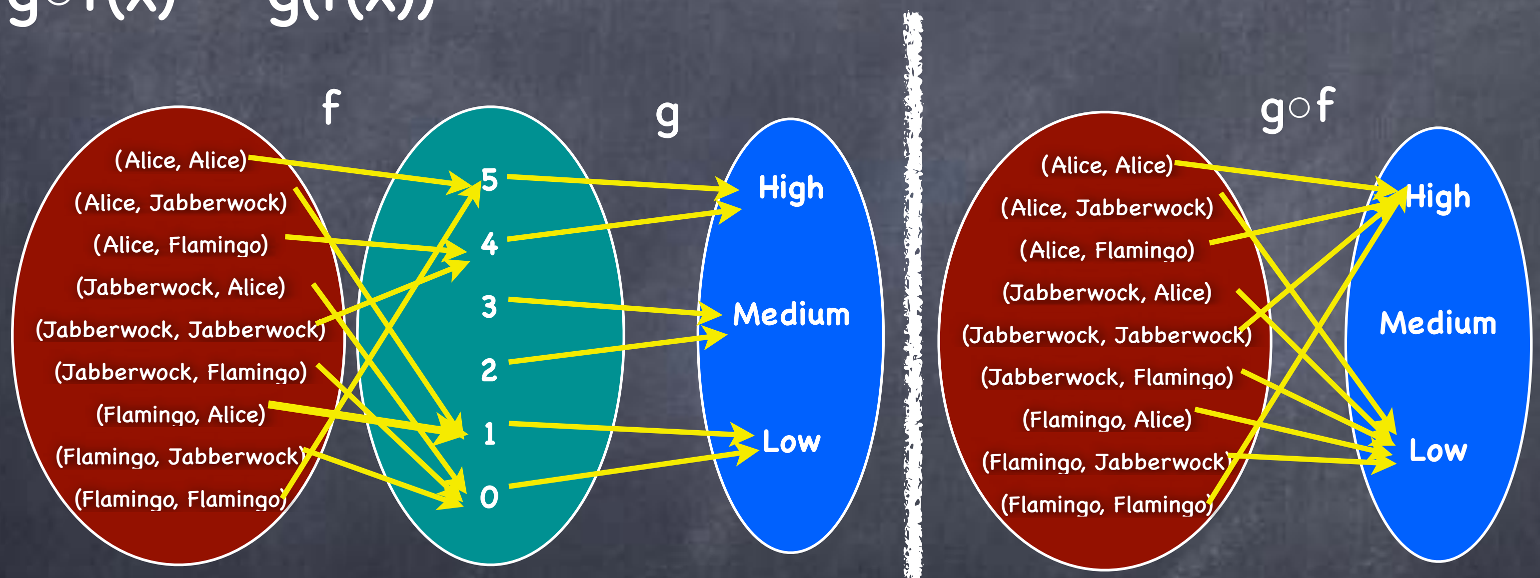
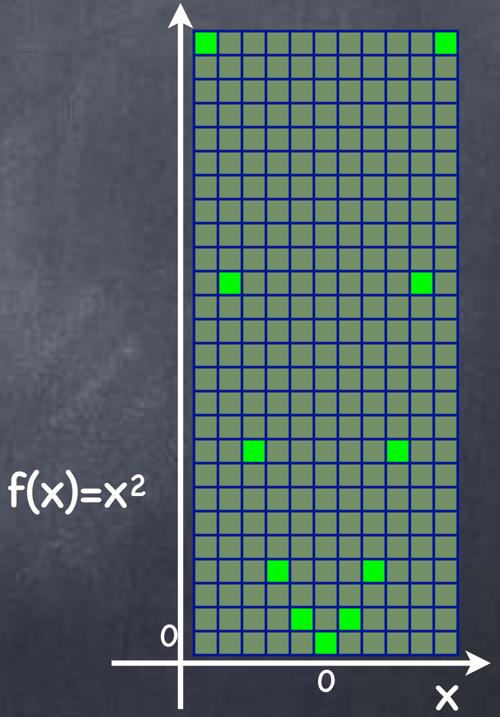


  - Defined only if Im(f) $\subseteq$ Domain(g)
    - Typically, Domain(g) = Co-domain(f)
  - $g \circ f$: Domain(f) → Co-domain(g)
  - Im($g \circ f$) $\subseteq$ Im(g)

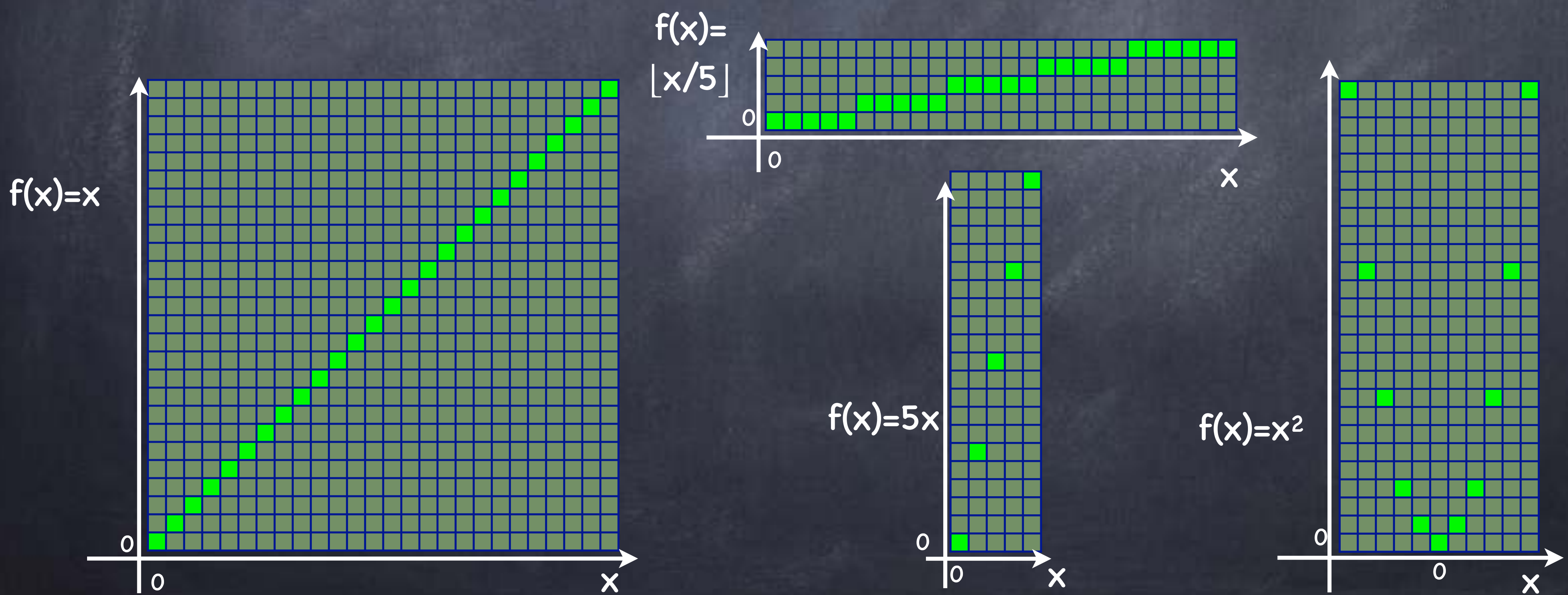
Functions

One-to-one, Onto, Bijections



Types of Functions

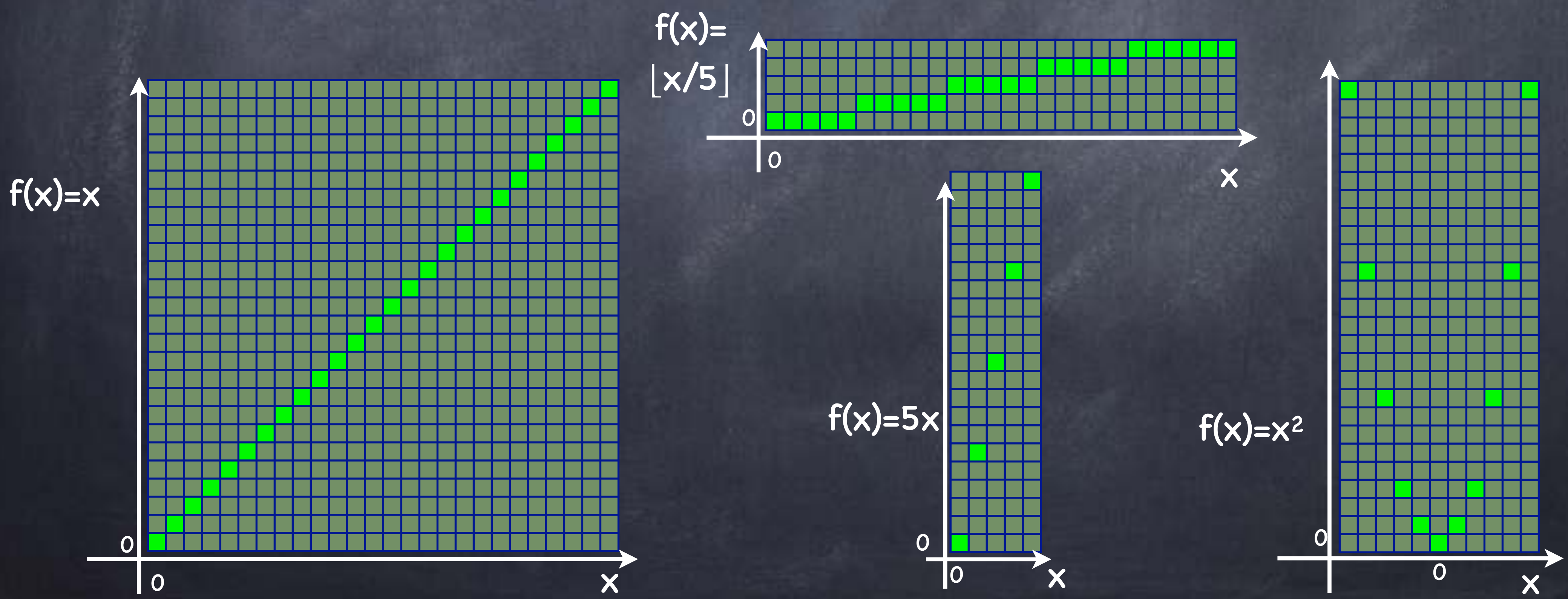
- Function viewed as a matrix: every column has exactly one cell "on"
- **Onto Function** (surjection): Every row has <u>at least</u> one cell "on"
- **One-to-One function** (injection): Every row has <u>at most</u> one cell "on"
- **Bijection**: Every row has <u>exactly</u> one cell "on"



Surjective Functions

$\forall y \in B\ \exists\ x \in A\ f(x)=y$

- **Onto Function** (surjection): Every row has <u>at least</u> one cell "on"
- Given $f:A \rightarrow B$, one can always define an "equivalent" onto function $f':A \rightarrow Im(f)$ such that $\forall x \in A\ f(x)=f'(x)$

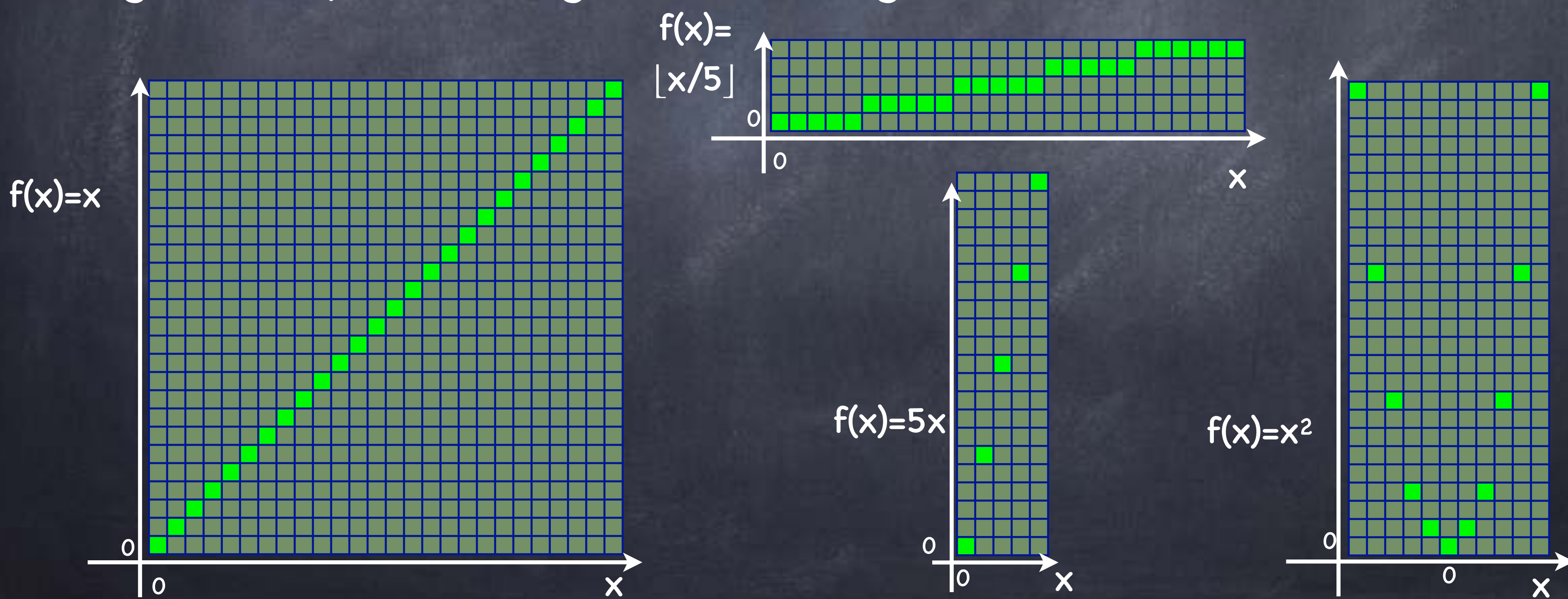


Injective Functions

$\forall x,x' \in A \;\; f(x)=f(x') \rightarrow x=x'$

$\forall y \in Im(f) \; \exists! \; x \in A \;\; f(x)=y$

- One-to-One function (injection): Every row has at most one cell "on"
- Domain matters : $\mathbb{Z} \rightarrow \mathbb{Z}$ defined as $f(x)=x^2$ is not one-to-one, but $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ defined as $f(x)=x^2$ is one-to-one
- E.g., strictly increasing or decreasing functions

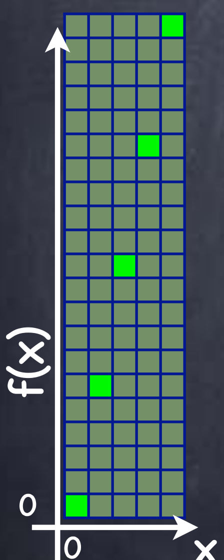


Injective $\longleftrightarrow$ Invertible

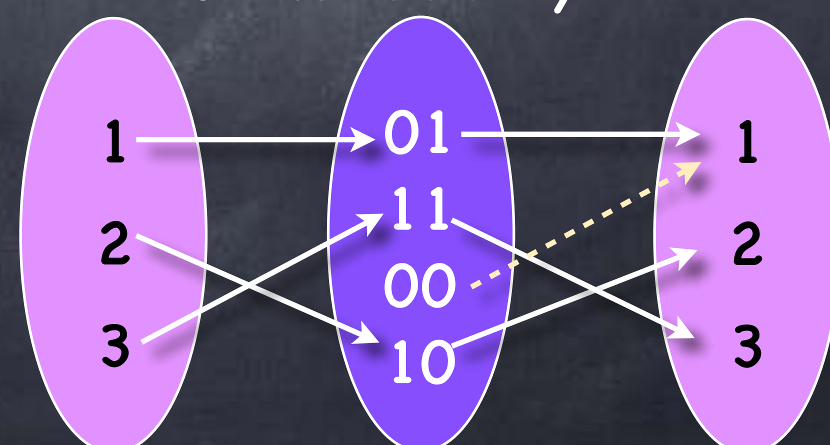
Can recover x from f(x):
f doesn't lose information

- f is said to be invertible if $\exists g$ s.t. $g \circ f \equiv Id$
- One-to-one functions are invertible
- Suppose $f : A \to B$ is one-to-one

$\forall y \in Im(f)\ \exists! x \in A\quad f(x) = y$



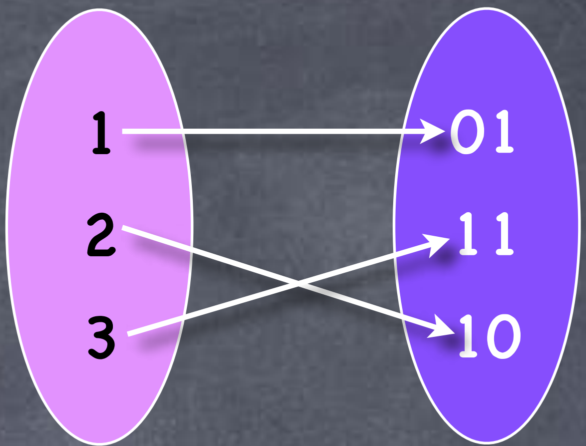
- Let $g : B \to A$ be defined as follows:
  for $y \in Im(f)$, $g(y)=x$ s.t. $f(x)=y$ (well-defined)
  for $y \notin Im(f)$, $g(y)$ = some arbitrary element in A
  - Then $g \circ f \equiv Id_A$, where $Id_A : A \to A$ is the identity function over A
  - g need not be invertible



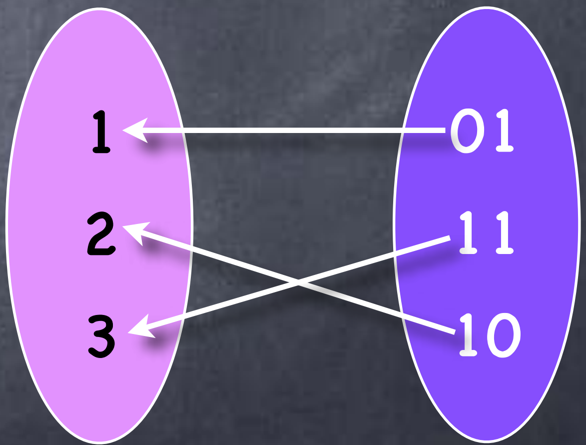
Injective $\longleftrightarrow$ Invertible

- f is said to be invertible if $\exists g$ s.t. $g \circ f \equiv Id$
- One-to-one functions are invertible
- And invertible functions are one-to-one
  - Suppose $f : A \rightarrow B$ is invertible
  - Let $g : B \rightarrow A$ be s.t. $g \circ f \equiv Id$
  - Now, for any $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $g(f(x_1)) = g(f(x_2))$
  - But $g(f(x)) = Id(x) = x$
  - Hence, $\forall x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$

Bijections



- Bijection: both onto and one-to-one
  - Every row and every column has exactly one cell "on"
  - Every element in the co-domain has exactly one pre-image
    - If $f : A \to B$, $f^{-1} : B \to A$ such that $f^{-1} \circ f : A \to A$ and $f \circ f^{-1} : B \to B$ are both identity functions
    - Both f and f^{-1} are invertible, and the inverses are unique
    - $(f^{-1})^{-1} = f$

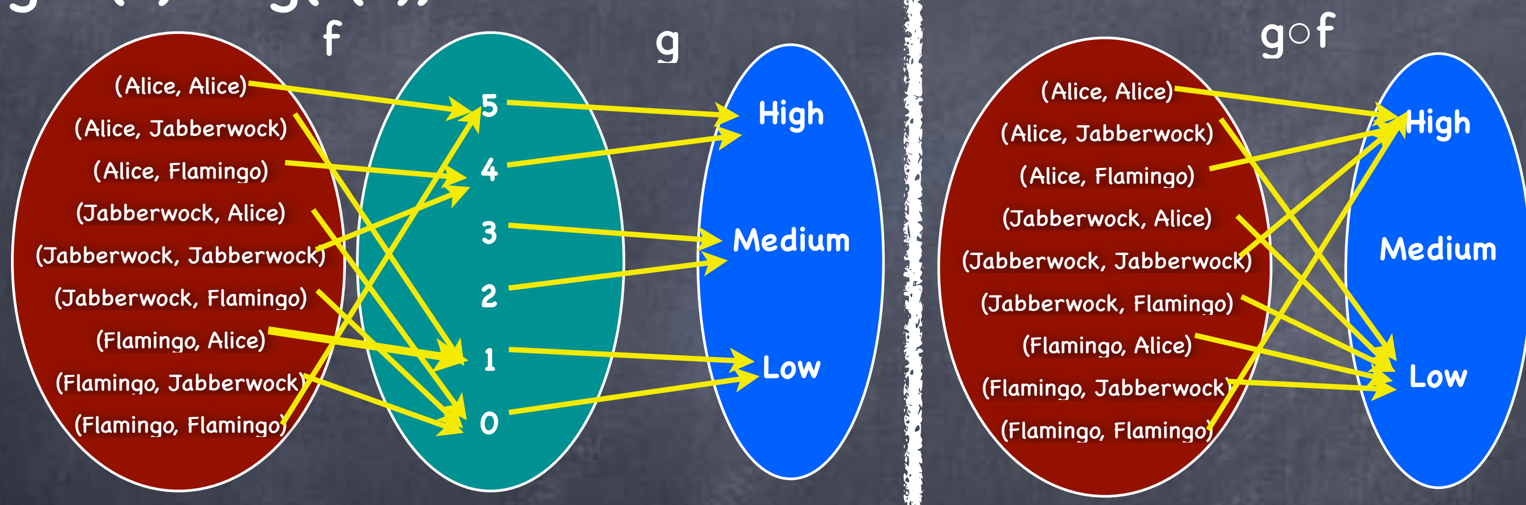


Domain & Co-Domain Sizes

- Suppose $f : A \rightarrow B$ where A, B are finite
- $|Im(f)| \leq |A|$, with equality holding iff f is one-to-one
- $|Im(f)| \leq |B|$, with equality holding iff f is onto
- If f is onto, then $|A| \geq |B|$
  - f onto $\Rightarrow Im(f) = B \Rightarrow |B| \leq |A|$
- If f is one-to-one, then $|A| \leq |B|$
  - f one-to-one $\Leftrightarrow |Im(f)| = |A|$. But $|Im(f)| \leq |B| \Rightarrow |A| \leq |B|$
  - Contrapositive: If $|A| > |B|$, then f not one-to-one
    - Pigeonhole principle
- If f is a bijection, then $|A| = |B|$
- If $|A| = |B|$, then f is onto $\equiv$ f in one-to-one $\equiv$ f is a bijection

Composition

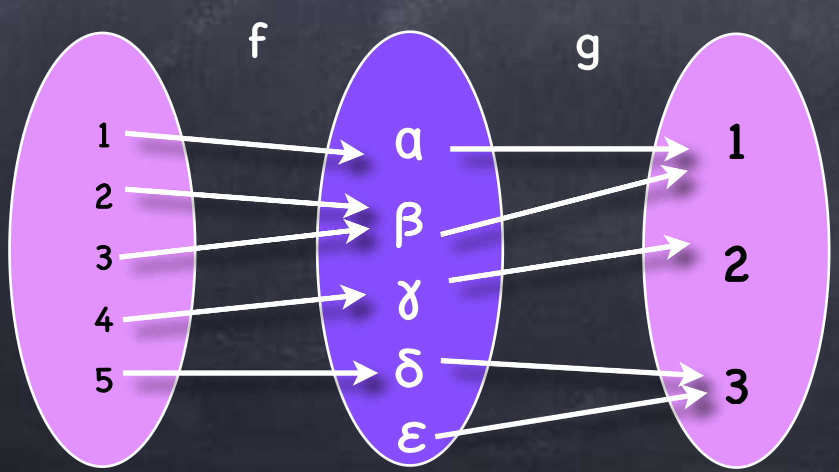
- Composition of functions f and g: $g \circ f$: Domain(f) $\rightarrow$ Co-domain(g)
  - $g \circ f(x) \triangleq g(f(x))$



- Defined only if Im(f) $\subseteq$ Domain(g)
  - Typically, Domain(g) = Co-domain(f)
- $g \circ f$: Domain(f) $\rightarrow$ Co-domain(g)
- Im($g \circ f$) $\subseteq$ Im(g)

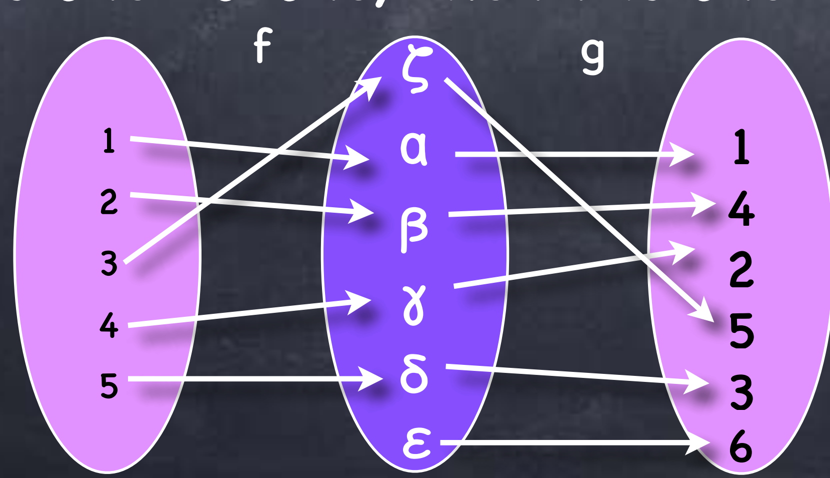
Composition & Onto/One-to-One

- Suppose Domain(g) = Co-Domain(f) (then g○f well-defined).
- Composition **"respects onto-ness"**
  - If f and g are onto, g○f is onto as well
  - If g○f is onto, then g is onto



Composition & Onto/One-to-One

- Suppose Domain(g) = Co-Domain(f) (then $g \circ f$ well-defined).
- Composition **"respects onto-ness"**
  - If f and g are onto, $g \circ f$ is onto as well
  - If $g \circ f$ is onto, then g is onto
- Composition **"respects one-to-one-ness"**
  - If f and g are one-to-one, $g \circ f$ is one-to-one as well
  - If $g \circ f$ is one-to-one, then f is one-to-one



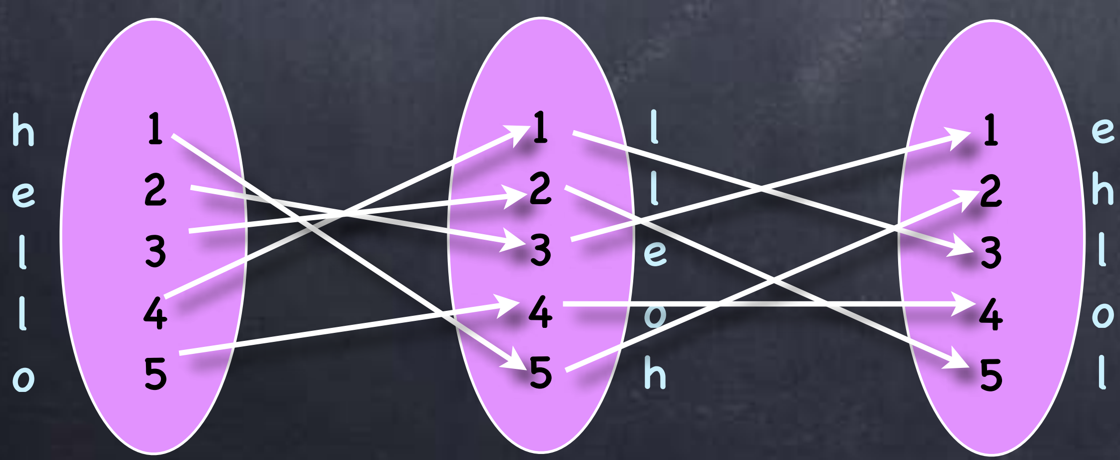
Composition & Onto/One-to-One

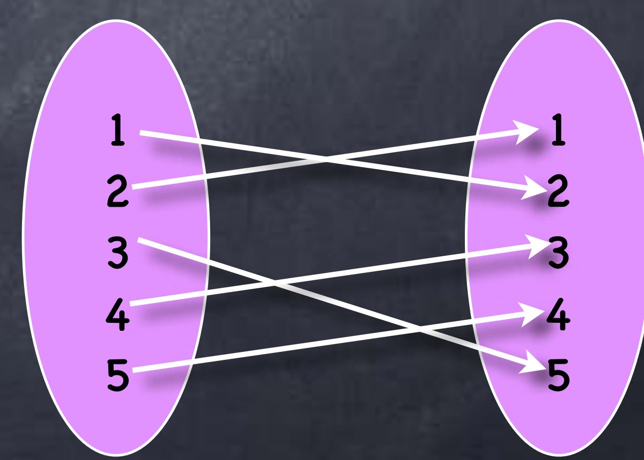
- Suppose Domain(g) = Co-Domain(f) (then $g \circ f$ well-defined).
- Composition "**respects onto-ness**"
  - If f and g are onto, $g \circ f$ is onto as well
  - If $g \circ f$ is onto, then g is onto
- Composition "**respects one-to-one-ness**"
  - If f and g are one-to-one, $g \circ f$ is one-to-one as well
  - If $g \circ f$ is one-to-one, then f is one-to-one
- Hence, composition "**respects bijections**"
  - If f and g are bijections then $g \circ f$ is a bijection as well
  - If $g \circ f$ is a bijection, then f is one-to-one and g is onto

Exercise: What if Domain(g) $\supsetneq$ Co-Domain(f)? What if Domain(g) = Im(f) and/or Co-Domain(f) = Im(f) ?

Permutation of a string

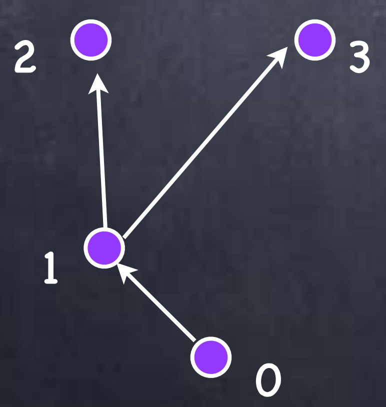
- To permute = to rearrange
  - e.g., $\pi_{53214}(\text{hello}) = \text{lleoh}$
  - e.g., $\pi_{35142}(\text{lleoh}) = \text{ehlol}$
- Permutations are essentially bijections from the set of positions (here $\{1,2,3,4,5\}$) to itself
  - A bijection from any finite set to itself is called a permutation
- Permutations compose to yield permutations (since bijections do so)
  - e.g., $\pi_{35142} \circ \pi_{53214} = \pi_{21534}$



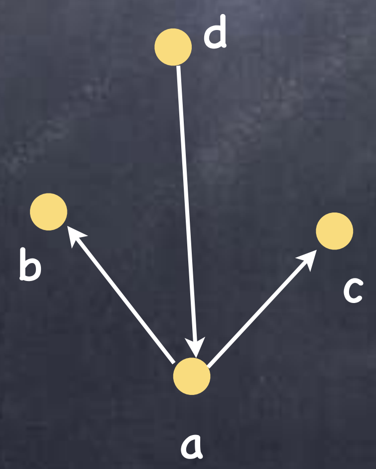


Isomorphism

- Bijection with additional "structure preserving properties"
  - "Structure": some relation(s)
- Consider sets S and S′ and relations $R \subseteq S \times S$ and $R' \subseteq S' \times S'$
- <u>An isomorphism between R and R′</u> is a bijection from S to S′ such that $\forall x,y \in S,\ R(x,y) \leftrightarrow R'(f(x),f(y))$



R



R′

S	S′
0	d
1	a
2	b
3	c

An isomorphism

S	S′
0	a
1	b
2	c
3	d

Not an isomorphism