Problem Set 2

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1. Prove that $((p \to r) \wedge (r \to q)) \to (p \to q)$, by three methods:

   (a) First, prove it by expanding this expression using distributive properties and conclude that it is equivalent to True.

   (b) Secondly, prove it by analysing two cases based on the truth value of r.

   (c) Finally, prove it by analysing 8 cases based on the truth values of p, q, r.

Solution: (a) Firstly, note that

$$\alpha \to (\beta \to \gamma) \equiv \neg\alpha \vee \neg\beta \vee \gamma \equiv (\alpha \wedge \beta) \to \gamma$$

Hence,

$$\begin{aligned}
((p \to r) \wedge (r \to q)) \to (p \to q) &\equiv ((p \to r) \wedge (r \to q) \wedge p) \to q \\
&\equiv ((p \wedge r) \wedge (r \to q)) \to q && \text{since } (p \to r) \wedge p \equiv p \wedge r \\
&\equiv (p \wedge r \wedge q) \to q && \text{since } r \wedge (r \to q) \equiv r \wedge q \\
&\equiv T && \text{since } (\alpha \wedge q) \to q \equiv \neg\alpha \vee \neg q \vee q \equiv T
\end{aligned}$$

Hence proved.

(b) We now prove that the formula evaluates to tautology based on case study of truth value of r. We use the following, for any proposition p, we have,

$$\begin{aligned}
p \to T &\equiv T \\
T \to p &\equiv p \\
p \to F &\equiv \neg p \\
F \to p &\equiv T
\end{aligned}$$

Each of the above equality follows from properties of implication.

<u>Case 1:</u> $r = T$.

In this case, the formula evaluates as,

$$\begin{aligned}
((p \to r) \wedge (r \to q)) \to (p \to q) &\equiv ((p \to T) \wedge (T \to q)) \to (p \to q) \\
&\equiv (T \wedge q) \to (p \to q) \\
&\equiv q \to (p \to q) \\
&\equiv \neg q \vee (p \to q) \\
&\equiv \neg q \vee (\neg p \vee q) \\
&\equiv (\neg q \vee q) \vee p \\
&\equiv T
\end{aligned}$$

Therefore, the formula evaluates to tautology in this case.

<u>Case 2:</u> $r = F$.

In this case, the formula evaluates as,

$$\begin{aligned}
((p \to r) \wedge (r \to q)) \to (p \to q) &\equiv ((p \to F) \wedge (F \to q)) \to (p \to q) \\
&\equiv (\neg p \wedge T) \to (p \to q) \\
&\equiv \neg p \to (p \to q) \\
&\equiv \neg(\neg p) \vee (p \to q) \\
&\equiv p \vee (\neg p \vee q) \\
&\equiv (p \vee \neg p) \vee q \\
&\equiv T
\end{aligned}$$

Therefore, the formula also evaluates to tautology in this case. Since the above two cases cover all possibilities for truth values assignments of the propositions p, q, r, we have that the formula evaluates to tautology. Hence, proved.

(c) The formula needs to be evaluated for each of the 8 possible assignments to the propositions p, q, r. We see that it turns out to be true in each case.

p	q	r	$\alpha := p \to r$	$\beta := r \to q$	$\gamma := p \to q$	$\delta := \alpha \wedge \beta$	$\delta \to \gamma$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	T	F	F	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	F	T	F	T
F	F	F	T	T	T	T	T

2. **Contrapositive.** Prove each of the following by stating and proving its contrapositive.

(a) If x and y are real numbers such that the product xy is an irrational number, then either x or y must be an irrational number.

Solution: Given that the domain we are working in is the set of real number, let p be the statement "xy is an irrational number" and let q be the statement "either x or y is an irrational number". We want to show that $p \to q$. The contrapositive of this statement is $\neg q \to \neg p$, that is, "If neither x nor y is an irrational number, then xy is not an irrational number". This can be further simplified to "If x and y are rational numbers, then xy is a rational number".

Proving this is an easy job; suppose x and y are rationals, say $x = \frac{p}{q}$ and $y = \frac{r}{s}$ where p, q, r, s are integers, and q, s are non-zero. Then $xy = \frac{pr}{qs} = \frac{m}{n}$, where $m = pr$ and $n = qs$ are integers and n is non-zero. Therefore xy is rational too.

(b) If x and y are two integers whose product is odd, then both must be odd.

Solution: The domain we are working in is the set of integers. Let p be the statement "xy is odd" and let q be the statement "x and y are both odd". To show $p \to q$, we state and prove its contrapositive as we did before. The contrapositive is $\neg q \to \neg p$, that is, "If either x or y is not odd, then xy is not odd". Since an integer that is not odd is necessarily even, this can be written equivalently as "If either x or y is even, then xy is even".

Now we prove this reformulation. Suppose one of x or y is even, say x is even (without loss of generality). Then $x = 2k$ for some integer k. Then, $xy = 2ky = 2m$, where $m = ky$ is an integer. This means xy is even, as required.

(c) If n is a positive integer such that n leaves a remainder of 2 when divided by 3, then n is not a perfect square.

Solution: The domain we are working in is the set of natural numbers. Let p be the statement "n leaves a remainder of 2 when divided by 3", and let q be the statement "n is not a perfect square". Again, we need to show $p \to q$, and the contrapositive $\neg q \to \neg p$ can be written as "If n is a perfect square, then n does **not** leave a remainder of 2 when divided by 3".

Let us try to prove this statement. Suppose $n = m^2$ for some positive integer m. We take cases on the remainder when m is divided by 3. Suppose m leaves a remainder of 0 when divided by 3, that is, suppose $m = 3k$ for some integer k. Then, $n = (3k)^2 = 3(3k^2)$, so n divided by 3 leaves a remainder 0. Instead, suppose m leaves a remainder of 1 when divided by 3, that is, $m = 3k + 1$ for some integer k. Then, $n = (3k + 1)^2 = 3(3k^2 + 2k) + 1$, so n leaves remainder 1 when divided by 3. Finally, suppose m leaves a remainder of 2 when divided by 3, that is, $m = 3k + 2$ for some integer k. Then, $n = (3k + 2)^2 = 3(3k^2 + 4k + 1) + 1$, so n leaves remainder 1 when divided by 3.

In either of the three cases, n does not leave remainder 2 when divided by 3, so we are done.

(d) If n is a positive integer such that n leaves a remainder of 2 or 3 on division by 4, then n is not a perfect square.

Solution: The domain we are working in is the set of positive integers. Let p be the statement "n leaves a remainder of 2 or 3 when divided by 4", and let q be the statement "n is not a perfect square". Again, we need to show $p \to q$, and the contrapositive $\neg q \to \neg p$ can be written as "If n is a perfect square, then n does not leave remainder 2 or 3 on division by 4".

This can be proved in a similar fashion as the previous problem; let $n = m^2$ for some positive integer m, and take cases modulo 4 for m. However, we can reduce case-work by taking cases modulo 2 instead.

Suppose m leaves remainder 0 when divided by 2. Then $m = 2k$ for some integer k. This means $n = 4k^2$, so n leaves remainder 0 when divided by 4. Instead, suppose m leaves remainder 1 when divided by 2, that is, $m = 2k + 1$ for some integer k. This means $n = (2k + 1)^2 = 4(k^2 + k) + 1$, so n leaves remainder 1 when divided by 4.

In either case, n does not leave remainder 2 or 3 upon division by 4, so we are done.

3. **Proof by Contradiction.**

(a) There are no positive integer solutions to the equation $x^2 - y^2 = 10$. (Such a problem, when an integral solution is sought for a polynomial equation, the equations is called a *Diophantine equation.*)

Solution: Suppose, for the sake of contradiction, that there is an integer solution (x, y) such that $x^2 - y^2 = 10$. Then, we have $(x+y)(x-y) = 10$. Note that $x-y$ and $x+y$ are both even or both odd (because, $x-y = (x+y)-2y$). If both are odd, their product is odd; if both are even then their product is a multiple of 4. Hence, in both cases, their product cannot be 10, which is a contradiction.

(b) There is no rational solution to the equation $x^5 + x^4 + x^3 + x^2 + 1 = 0$.

[Hint: A rational number can be written as $\frac{p}{q}$ where p, q are integers which have no common factors.]

Solution: Suppose, for the sake of contradiction, that there is a rational solution, $x = \frac{p}{q}$, for integers p, q, which have no common factors. Then,

$$p^5 + p^4q + p^3q^2 + p^2q^3 + q^5 = 0.$$

At least one of p, q is odd (since they have no common factors). We consider 3 cases: both p, q are odd, only p is odd, and only q is odd. In the first case, all 5 terms on the LHS are odd, and hence their sum should be odd, and hence cannot be 0. In the second and third cases, exactly one term on the LHS (p^5 or q^5) is odd, and hence the sum is again odd, and cannot be 0.

(c) We say that a point $P = (x, y)$ in the Cartesian plane is rational if both x and y are rational. More precisely, P is rational if $P = (x, y) \in \mathbb{Q}^2$. An equation $F(x, y) = 0$ is said to have a rational point if there exists $x_0, y_0 \in \mathbb{Q}$ such that $F(x_0, y_0) = 0$. For example, the equation $x^2 + y^2 - 1 = 0$ has rational points $(0, \pm 1)$ and $(\pm 1, 0)$. Show that the equation $x^2 + y^2 - 3 = 0$ has no rational points.

[Hint: Prove by contradiction. It would be useful to consider whether the largest power of 3 that divides an integer is even or odd. Also, it will be useful to know what values can appear as the remainder of a perfect square when divided by 3.]

Solution: Suppose, for the sake of contradiction, that there is a rational solution $(x, y) = (\frac{p_1}{q_1}, \frac{p_2}{q_2})$ such that $x^2 + y^2 = 3$. Then, $p_1^2q_2^2 + p_2^2q_1^2 = 3(q_1^2q_2^2)$. The largest power of 3 that divides the RHS is odd. Suppose the largest power of 3 that divides p_1q_2 and p_2q_1 be a and b respectively. We analyze three cases based on a, b and derive a contradiction in each case.

Case 1, $a = b$: In this case the LHS is of the form $3^{2a}(c^2 + d^2)$, where c, d are not multiples of 3. Then c^2, d^2 leave a remainder of 1 when divided by 3 (because $(3n \pm 1)^2 = 3(3n^2 \pm 2n) + 1$), and hence $c^2 + d^2$ is not a multiple

Case 2, $a > b$: In this case the LHS is of the form $3^{2b}(c^2 + d^2)$ where c a multiple of 3 but d is not. Then $c^2 + d^2$ leaves a remainder of 1 when divided by 3. Hence, the largest power of 3 that divides the LHS is $2b$, which is even.

Case 3, $b > a$: This is analogous to the above case, and the largest power of 3 that divides the LHS is $2a$, which is even.

(d) Use (c) to show that $\sqrt{3}$ is irrational.

4. **Weak Induction**. Prove by induction that the following hold for every positive integer n:

(a) $1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}\dfrac{n(n+1)}{2}$.

Solution: Base Case :

$$1^2 = (-1)^0 \frac{1(2)}{2}$$
$$1 = 1$$

Induction Step : Assume that $1^2 - 2^2 + 3^2 - \ldots + (-1)^{n-1}n^2 = (-1)^{n-1}\dfrac{n(n+1)}{2}$ for some $n > 1$.

$$\begin{aligned}
1^2 - 2^2 + 3^2 - \ldots + (-1)^n(n+1)^2 &= \left((-1)^{n-1}\frac{n(n+1)}{2}\right) + \left((-1)^n(n+1)^2\right) \\
&= (-1)^{n-1}(n+1)\left[\frac{n}{2} - (n+1)\right] \\
&= (-1)^{n-1}(n+1)\left[\frac{n-2n-2}{2}\right] \qquad (1) \\
&= (-1)^{n-1}(n+1)\left[\frac{(-1)(n+2)}{2}\right] \\
&= (-1)^n\frac{(n+1)(n+2)}{2}
\end{aligned}$$

(b) if $h > -1$, then $1 + nh \leq (1+h)^n$.

Solution:

Base case : $1 + 0 \leq (1+h)^0$

Induction Step : Assume that

$$1 + nh \leq (1+h)^n$$

for some $n > 1$. Then,

$$1 + (n+1)h = 1 + nh + h \leq (1+h)^n + h$$

To complete the proof, it suffices to show that

$$h \leq h(1+h)^n$$

. Consider 2 cases :

i. $-1 < h < 0$: It follows that

$$0 < 1 + h < 1$$

Hence

$$0 < (1+h)^n < 1$$

for $n \geq 1$. Thus, $h(1+h)^n$ will be a negative number with a smaller magnitude than h.

ii. $h \geq 0$: This means that $1 + h \geq 1$ and hence

$$h \leq h(1+h)^n$$

for $n \geq 1$.

(c) 12 divides $n^4 - n^2$.

Solution: Let $D(n) = n^4 - n^2$ and $P(n)$ denote the predicate which evaluates to true when 12 divides $D(n)$.

Base Case : $D(1) = 0$ implies $P(1)$ holds trivially.

Induction Step : Assume that $P(k)$ holds for some integer $k \geq 1$. We shall prove that $P(k+1)$ also holds. For $P(k)$ to hold, it must be the case that $D(k)$ is divisible by both 3 and 4. Note that $D(k) = k^2(k^2-1)$, a product of two consecutive integers. Two consecutive integers are always co-prime i.e. they have no common factors. Hence, there are 4 possibilities :

i. k^2 is divisible by both 3 and 4. It follows that k is divisible by both 3 and 2. Hence, both k and $k+2$ are even. Note that $D(k+1) = k(k+1)^2(k+2)$. Therefore, $D(k+1)$ is divisible by 3 (via k) and also divisible by 4 (via k and $k+2$).

ii. $k^2 - 1$ is divisible by both 3 and 4. Since $k^2 - 1 = (k-1)(k+1)$, it follows that both $k-1$ and $k+1$ are even. Also, either $k-1$ (and hence $k+2$) or $k+1$ are divisible by 3. Combining the above results, it can be claimed that $P(k+1)$ holds.

iii. k^2 is divisible by 3 and $k^2 - 1$ is divisible by 4. It follows that both $k-1$ and $k+1$ are even. Note that $D(k+1)$ is divisible by 4 because $k+1$ is even. For k^2 to be divisible by 3, k must also be divisible by 3 (as 3 is a prime number). Hence, $P(k+1)$ holds.

iv. k^2 is divisible by 4 and $k^2 - 1$ is divisible by 3. Either $k-1$ or $k+1$ should be divisible by 3. It follows that either $k+2$ or $k+1$ is divisible by 3. When k^2 is divisible by 4, both k and $k+2$ are even. Hence, $P(k+1)$ holds.

Here is a proof which does not use induction. A quantity is divisible by 12 if it has a factor of 4 and a factor of 3. Note that $n^4 - n^2 = n^2(n^2-1) = n^2(n-1)(n+1)$. If n is even, then n^2 has a factor of 4. If n is odd, then both $n-1$ and $n+1$ are even, and hence $(n-1)(n+1)$ has a factor of 4. In either case $4|(n^4 - n^2)$. Also, among $n-1, n, n+1$, one of them is a multiple of 3. Hence $3|n(n-1)(n+1)$, and hence $3|(n^4 - n^2)$. Hence $12|(n^4 - n^2)$.

5. **Strong induction.** An $a \times b$ chocolate bar is a rectangular piece of chocolate consisting of ab square pieces of chocolate. Your job is to break this chocolate into the ab individual square pieces. At any point during this task, you will have one or more pieces of the chocolate bar; you can pick any piece and break it into two, along a vertical or horizontal line separating the square pieces. For instance, if you start with a 2×2 bar, you can first break it vertically to get two 2×1 bars; then each of them you can break once horizontally, to end up with all 4 individual squares. In this process you made 3 breaks in all (one vertical, two horizontal).

Show that to completely break an $a \times b$ bar into individual squares, you need exactly $ab - 1$ breaks, no matter which breaks you make.

[Hint: Induct on the number of squares. A single break splits a piece of chocolate into two smaller pieces with the same total number of squares.]

Solution:

Base Case: Consider $n = 1$. If we have 1 chocolate square, number of breaks = 0

Induction Step: For some $k \geq 1$, assume that claim is true for all chocolate bars with $n \leq k$ squares: that is, no matter what sequence of breaks are made, any such chocolate bar needs exactly $k - 1$ breaks. We need to show that claim is true for all chocolate bars with $n = k + 1$ squares.

Consider an arbitrary chocolate bar with $k + 1$ squares. Consider any sequence of breaks that break it into individual squares. The first break breaks the bar into two pieces of n_1 and n_2 squares for some integers $n_1, n_2 > 0$ such that $n_1 + n_2 = k + 1$. Thus $n_1 \leq k$ and $n_2 \leq k$.

Now, the sequence of breaks we are considering must break these two pieces into individual squares. Further, any break affects (pieces obtained from) only one of these two pieces. Thus the subsequent breaks can be partitioned into two sequences, one which breaks the n_1-square piece completely, and one which breaks the n_2-square piece completely. By the induction hypothesis, the first sequence has exactly $n_1 - 1$ breaks and the second sequence has exactly $n_2 - 1$ breaks.

So the total number of breaks in the original sequence (including the first break) is $1 + (n_1 - 1) + (n_2 - 1) = n_1 + n_2 - 1 = k$.

Thus we have shown that for any chocolate bar with $k + 1$ squares, any sequence of breaks that break it into individual bars must have exactly k breaks. This completes the induction step.

6. **Well Ordering Principle.** Prove the Well-Ordering Principle – that every non-empty subset of $\mathbb{Z}^+$ has a minimum element – using mathematical induction.

   [Hint: Use *strong induction* to prove the contrapositive of the above statement, i.e. if a subset of $\mathbb{Z}^+$ does not have a least element, then it must be empty.]

   Solution:

   We need to show that if a subset S of $\mathbb{Z}^+$ does not have a minimum element, then it must be empty. In other words, we need to prove that $P(k)$ holds for all $k \in \mathbb{Z}^+$, where the predicate P is defined by $P(k) \leftrightarrow k \notin S$.

   Base Case: $P(1)$ holds because if $1 \in S$ then 1 will be the minimum element of the set S, as $n \geq 1$ for all $n \in \mathbb{Z}^+$. Since $S \subseteq \mathbb{Z}^+$, this implies that $n \geq 1$ for all $n \in S$.

   Induction Step: Assume that $P(m)$ holds for all $1 \leq m \leq k$ for some integer $k \geq 1$. We shall now prove that $P(k+1)$ also holds.

   For the sake of contradiction, assume that $P(k+1)$ does not hold, implying that $k+1 \in S$. By the inductive hypothesis, all positive integers $1 \leq m \leq k$ satisfy that $m \notin S$. Since the set S only consists of positive integers, it follows that $n \geq k + 1$ for all $n \in S$. Combining this result with the induction hypothesis, we obtain that $k + 1$ is the least element of the set S, thus leading to contradiction.

7. Suppose that 9 bits – five ones and four zeros – are arranged around a circle in some order. Between any two equal bits you insert a 0 and between any two unequal bits you insert a 1 to produce nine new bits. Then you erase the nine original bits. Show that when you iterate this procedure, you can never get nine zeros. [Hint: Prove using the well ordering principle, or by mathematical induction.]

   Solution: We prove this using mathematical induction. Let Z_i denote the number of zeroes and O_i denote the number of ones respectively around the circle in the i^{th} configuration. We claim that $Z_i \geq 1$ and $O_i \geq 1 \ \forall i \geq 1$.

   (a) Base case : $Z_0 = 4$ and $O_0 = 5$.

   (b) Let's assume that $Z_i \geq 1$ and $O_i \geq 1$ for some $i > 1$. We need to prove that $Z_{i+1} \geq 1$ and $O_{i+1} \geq 1$.

      i. Let's assume that $O_{i+1} = 0$. This means that all the bits present in the $(i+1)^{th}$ configuration are zeroes. The only way in which this is possible is if all the bits in the i^{th} iteration are equal. That leads to a contradiction since the i^{th} iteration has at least one zero and a one. So, $O_{i+1} \geq 1$.

      ii. Let's assume that $Z_{i+1} = 0$. This means all bits are ones in the $(i+1)^{th}$ configuration. This is possible only when every possible pair of 2 consecutive bits in the i^{th} iteration are opposite (a zero and a one). Let's suppose we start traversing the circle in the clockwise direction with the bit one (this is possible because $O_i \geq 1$). Then it's neighbour must be a zero. The neighbour of this bit zero must be a one. Thus, all bits traversed in the odd turn are ones and those traversed in the even turns are zeroes. A total of 9 bits means that the 9^{th} bit must be a one which will also be the neighbour of the first one we traversed. This leads to a contradiction as we've found a pair of 2 consecutive equal bits. So, $Z_{i+1} \geq 1$.

8. Let $a_1, a_2, ...$ be a sequence of real numbers satisfying $a_{i+j} \leq a_i + a_j$ for all positive integers i and j. Use strong induction to prove :
$$a_n \leq a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \ldots + \frac{a_n}{n}.$$
[Hint: You can write $a_n \leq a_i + a_{n-i}$ for $i = 1, \ldots, n-1$. You will want to use all $n-1$ of these inequalities. Use the strong inductive hypothesis to reason about a_i or a_{n-i}. It may help to work out examples for small values of n.]

Solution: For $n = 1$, the claim is $a_1 \leq a_1$, which is true. Now, for any arbitrary $k \geq 1$, suppose for all $1 \leq n \leq k$, it holds that $a_n \leq \sum_{j=1}^{n} \frac{a_j}{j}$. We shall prove that $a_{k+1} \leq \sum_{j=1}^{k+1} \frac{a_j}{j}$, or equivalently, $ka_{k+1} \leq \sum_{j=1}^{k} \frac{a_j}{j}$.

We know that $a_{k+1} \leq a_i + a_{k+1-i}$ for $i = 1, \ldots, k$. Summing up these inequalities we have, $ka_{k+1} \leq \sum_{i=1}^{k}(a_i + a_{k+1-i}) = 2\sum_{i=1}^{k} a_i$. Applying the strong induction hypothesis to each a_i we have
$$\sum_{1=1}^{k} a_i \leq \sum_{i=1}^{k}\sum_{j=1}^{i} \frac{a_j}{j} = \sum_{j=1}^{k}\sum_{i=j}^{k} \frac{a_j}{j}$$
$$= \sum_{j=1}^{k}(k+1-j)\frac{a_j}{j}.$$
Adding to this $\sum_{1=1}^{k} a_i = \sum_{j=1}^{k} j\frac{a_j}{j}$, we get $2\sum_{i=1}^{k} a_i \leq \sum_{j=1}^{k}(k+1)\frac{a_j}{j}$. Hence, $ka_{k+1} \leq (k+1)\sum_{j=1}^{k} \frac{a_j}{j}$ as required.

9. There are n identical cars on a circular track, at arbitrary distances from each other. All of them together have just enough petrol required for one car to complete a lap. Show, using induction, that there is a car which can complete a lap by collecting petrol from the other cars on its way around.

[Hint: It will be helpful to prove a stronger statement, that there is a car which can complete a lap in the clockwise direction. Your proof in the induction step may have following steps:

- Consider an arbitrary configuration of $k + 1$ cars (satisfying the given condition).
- First argue that there is a car who can reach its clockwise neighbouring car with the petrol it has. (Use proof by contradiction.)
- Use these two cars to change the given instance of the problem into an instance with k cars.
- Use the induction hypothesis to get some solution of the smaller instance; translate it into a solution for the original instance of the problem.]

Solution: We shall inductively prove that there is always a solution in which a car moves clockwise. As the base case, note that if $n = 1$ – i.e., there is a single car – we are guaranteed that it has enough petrol to cover a full lap.

Now, fix any $k \geq 1$, and suppose the claim holds for every k car configuration with the total petrol being sufficient to make a full lap. We shall use this assumption to show that the same holds for any $k + 1$ car configuration.

Suppose we are given an arbitrary configuration with $k + 1$ cars. Firstly, following the hint, if all cars had strictly less petrol than needed to cover the distance to their clockwise neighbour, then the total petrol across all cars is strictly less than what is needed for a full lap. Hence, at least one car, say A has sufficient petrol to reach its clockwise neighbour B. Now we derive a k car configuration as follows. We remove the car B, and give all the petrol it had to A. Since the total amount of petrol has not changed, we can invoke the induction hypothesis to get a solution for this configuration in which a car moves clockwise. We shall use the same solution for our $k + 1$ car configuration.

Note that in this solution the car will have to reach A before reaching the position of B (because it cannot start at B, or strictly between A and B). Till it reaches A, the car has the same amount of petrol in the $k + 1$ car configuration as in the k car configuration. When it reaches A, in the $k + 1$ car configuration it collects less petrol than in the $k + 1$ car configuration. However, since A stores enough petrol to reach B, the car will not run out of petrol before it reaches B. On reaching B its petrol level rises to the same level as in the k car configuration. Thus we obtain a solution with a car moving in the clockwise direction for the given $k + 1$ car configuration, as desired.