CS 228 Quiz 2 Question 1 Solution

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1 Part (a):

Suppose L is regular, and let $A = (Q, \Sigma, \delta, F, q_0)$ be its DFA. Consider the ϵ-NFA $B = (Q', \Sigma, \delta', F', Q_0)$ such that $Q' = Q \times Q \times \{0, 1\}$, $Q_0 = \{(q, q, 0) \mid q \in Q\}$, $F' = \{(q, q, 1) \mid q \in Q\}$, and δ' is defined as follows:

$$\delta'((q_1, q_2, \eta), a) = (q_1, \delta(q_2, a), \eta) \text{ for any } q_1, q_2 \in Q,\ a \in \Sigma, \text{ and } \eta \in \{0, 1\}$$
$$\delta'((q_1, q_2, 0), \epsilon) = (q_1, q_0, 1) \text{ for any } q_1 \in Q \text{ and } q_2 \in F$$

and there are no other transitions.
Basically, we have created an ϵ-NFA $A'(q)$ for each state $q \in Q$ by attaching the final states of one copy of A to the initial state of another copy of A via an ϵ transition, and made the initial and final states q in the corresponding copies of A respectively (and removing all other initial and final states). Then we have taken a union of all these $A'(q)$ to get B.

We claim that $\mathcal{L}(B) = \text{Cycle}(L)$. Indeed, suppose $vu \in \text{Cycle}(L)$ is created from some $uv \in L$. By uniqueness of paths in a DFA, we can find $q_1 \in Q$ and $q_2 \in F$ such that u goes from q_0 to q_1 and v goes from q_1 to q_2 in A. Then, in $A'(q_1)$, v goes from q_1 to q_2, then we take an ϵ transition to q_0, and then u goes from q_0 to q_1. Thus vu is accepted by B.
Conversely, suppose some w is accepted by B, say by some $A'(q_1)$. Since the initial state and final state are in different copies of A in $A'(q_1)$, the path has to cross the ϵ transition at some point. After that, it cannot return to the initial copy of A since there are no back-transitions. Thus we can split $w = vu$ where v is the part in the first copy of A and u is the one in the second. Now it is not hard to see that uv is accepted by A using similar logic as above.

Thus, we see that $\mathcal{L}(B) = \text{Cycle}(L)$, and so $\text{Cycle}(L)$ is regular as well.
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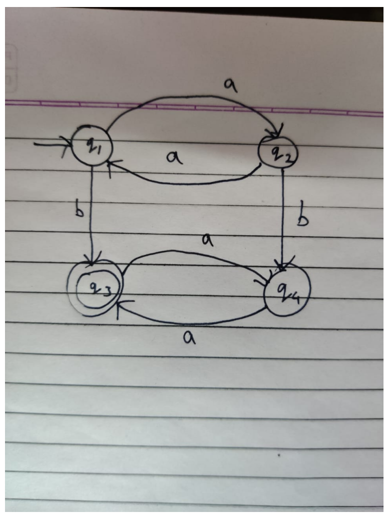
2 Part (b):

2.1 Method 1:

Suppose $\Sigma = \{a, b\}$, and consider $L = \{a^n ba^n \mid n \in \mathbb{N}\}$. By the hint, L is not regular. Another way to think about why this is true is the following: Suppose there is some DFA A of L. Let $q(n)$ be the state reached after $a^n b$ for all $n \in \mathbb{N}$ (this state is unique because we are looking at a DFA). Since there are only finitely many states in A, but infinitely many positive integers, by pigeonhole principle there are $m < n$ such that $q(m) = q(n) = q$ (say). Then, doing either a^m or a^n from q will lead to a final state, so something like $a^n ba^m$ is also accepted by A, contradiction! Thus L is not regular.

Now, note that Cycle(L) is just the set of all words containing a non-zero even number of as and exactly one b. This is regular, which can be seen by looking the following NFA:



which represents $\{a^n ba^n \mid n \geq 0\}$, and taking the intersection with the regular language $\{\bar{b}\}$ (complement of a finite language). Since regularity is preserved under intersection, Cycle(L) is regular.

So, Cycle(L) being regular dows not imply that L is regular.

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2.2 Method 2:

Let $\Sigma = \{a, b\}$. We will search for a counterexample in the set of all languages L with $\text{Cycle}(L) = \Sigma^*$. Clearly $\text{Cycle}(L)$ is regular in this case.

Now, break up the set of all words into equivalence classes such that u, v are in the same equivalence class iff they are a cyclic permutation of each other. Note that there are only finitely many words in each equivalence class, so there are infinitely many equivalence classes (in fact number of equivalence classes is $\aleph_0$, the cardinality of the natural numbers, because there are $\aleph_0$ many words in total). Now suppose we create L by choosing one word from each equivalence class. This L will satisfy $\text{Cycle}(L) = \Sigma^*$. Now, if a word contains both a and b, its equivalence class will have at least two members. Thus there are infinitely many classes with at least two members, and so the number of choices for such an L is at least $2^{\aleph_0}$ (in fact it would be exactly that many).

On the other hand, there are countably many DFAs (because each DFA has finitely many states and finitely many transitions, so we can list them in an increasing number of states order). Thus, there are countably many regular languages (since every DFA corrsponds to a unique language). Thus, the number of regular languages is $\aleph_0$, while the number of languages with $\text{Cycle}(L) = \Sigma^*$ is at least $2^{\aleph_0} > \aleph_0$. Thus there is a non-regular language L with $\text{Cycle}(L) = \Sigma^*$, which is regular.

So, $\text{Cycle}(L)$ being regular dows not imply that L is regular.

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