

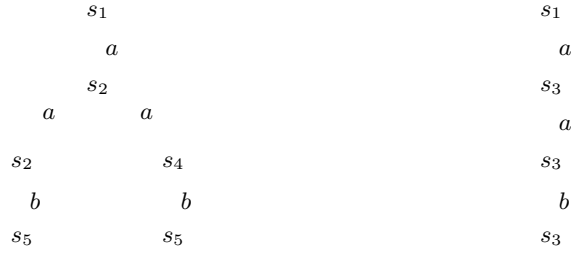
Problem Set 3

1. For each of the following, draw an NFA and/or FO formula whenever possible:
 - (a) The set of strings over $x \in \{0, 1\}^*$ such that $\#_0(x) \cdot \#_1(x)$ is even. The automaton can have no more than 4 states.
 - (b) the set of strings over $\{a, b\}$ containing atleast 3 occurrences of two consecutive a 's, overlapping allowed (for example, $aaaa$ is accepted). While drawing the automaton, draw one which has 7 states.
 - (c) $(0 + 1(01^*0)^*1)^*$
2. Let L_1, L_2 be FO-definable. Is $L_1.L_2$ FO-definable?
3. Let $h : \Sigma^* \rightarrow \Gamma^*$ be a mapping defined as follows. $h(\epsilon) = \epsilon$, h is defined on all symbols of Σ and $h(a)$ is a unique word for each $a \in \Sigma$. Moreover, $h(w_1w_2) = h(w_1)h(w_2)$ for all words w_1w_2 . If $L \subseteq \Sigma^*$, define $h(L) = \{h(w) \mid w \in L\}$.
 - (a) If L is regular, is $h(L)$ regular?
 - (b) If L is FO-definable, is $h(L)$ FO-definable?
4. Which of the following formulae are valid? Which are satisfiable? In each case, answer your question by constructing equivalent automata.
 - $\exists y[\neg first(y) \wedge Q_b(y) \wedge \forall x[x < y \rightarrow Q_a(x)]]$
 - $\forall x[Q_a(x) \rightarrow \exists y[(x < y) \wedge Q_b(y) \wedge \exists z_1 \exists z_2(z_1 < z_2) \wedge x < z_1 \wedge \neg(Q_b(z_1) \wedge Q_b(z_2))]]]$
5. Let us define a new kind of automaton inspired from logic, called $\forall\exists$ automaton as follows: A $\forall\exists$ automaton is a finite state automaton $(Q, \Sigma, \Delta, S, F)$ where the set of states Q is partitioned into two sets Q_\forall and Q_\exists . Let there be n states in Q . The transitions coming out of a Q_\exists state are called "or" transitions, and the transitions coming out of a Q_\forall state are called "and" transitions.

An "or" transition has the form $\Delta(q, a) = q_{i_1} \vee \dots \vee q_{i_j}$, $q \in Q_\exists$, while, an "and" transition has the form $\Delta(q, a) = q_{j_1} \wedge \dots \wedge q_{j_l}$, $q \in Q_\forall$.

For example, let $Q = \{s_1, s_2, s_3, s_4, s_5\}$, $S = \{s_1\}$, $F = \{s_5\}$, with transitions $\Delta(s_1, a) = s_2 \vee s_3$, $\Delta(s_2, a) = s_2 \wedge s_4$, $\Delta(s_2, b) = s_5$, $\Delta(s_4, a) = s_4$, $\Delta(s_4, b) = s_5$, $\Delta(s_3, a) = s_3$, $\Delta(s_3, b) = s_3$. The following are two run trees of the word aab :

Note that each time an "and" transition $\delta(q, a) = q_{i_1} \wedge \dots \wedge q_{i_j}$ is used, we spawn j threads, and maintain all the states q_{i_1}, \dots, q_{i_j} . All these threads then evolve in parallel. A word w is accepted by a run tree if a final state



is encountered in all the threads when you finish reading w . An “or” transition is the usual non-deterministic choice you have, you can pick any one of the choices. In the above example, the first run tree is accepting, while the second is not. A word is accepted if it has atleast one accepting run tree.

- (a) Compare the expressiveness of $\forall\exists$ automata and NFA.
 - (b) How do you complement a $\forall\exists$ automata?
6. Find the smallest r such that the following pairs of words are r -distinguishable ($w_1 \approx_r w_2$) with respect to $FO[<]$ by playing a game:
- (a) $((ab)^n b(ab)^n, (ab)^m (ab)^m)$
 - (b) $((aa)^n b(aaa)^n, (aaa)^n b(aa)^n)$
 - (c) $((aabbacb)^n, (aabbacb)^m aab(aabbacb)^k)$
- In each case, write the $FO[<]$ that distinguishes the two words.