CS 228 tut 2

4.3 Propositional Logic. Word Problem Proof

$$\left.\begin{array}{l}(G \land S) \Rightarrow D \\ (S \Rightarrow D) \Rightarrow P \\ G\end{array}\right\} \Sigma$$

P

Prove $\Sigma \vdash P$

1) $\Sigma \vdash (G \land S) \Rightarrow D$
2) $\Sigma \vdash (S \Rightarrow D) \Rightarrow P$
3) $\Sigma \vdash G$

Assumption (1–3)

4) $\Sigma \cup \{S\} \vdash S$ Assumption
5) $\Sigma \cup \{S\} \vdash G$ Monotonic on 3
6) $\Sigma \cup \{S\} \vdash G \land S$ $\land$-intro on 5,4
7) $\Sigma \cup \{S\} \vdash (G \land S) \Rightarrow D$ Monotonic on 1
8) $\Sigma \cup \{S\} \vdash D$ $\Rightarrow$-elim on 6,7
9) $\Sigma \vdash S \Rightarrow D$ $\Rightarrow$-intro on 8
10) $\Sigma \vdash P$ $\Rightarrow$ elim on 2,9

4.4.1 Redundant Proof Rules?

Show that $\lor$-symm is expendable.

$$\frac{\Sigma \vdash F \vee G}{\Sigma \vdash G \vee F} \quad \vee\text{-symm}$$

Hint 1: We have an $\wedge$-symm

$$\frac{\Sigma \vdash F \wedge G}{\Sigma \vdash G \wedge F} \quad \wedge\text{-symm} \qquad \frac{\text{Premise}}{\text{Conclusion}} \text{ cond } G$$

Hint 2: We have a rule to relate $\wedge$ and $\vee$

$$\frac{\Sigma \vdash F \vee G}{\Sigma \vdash \neg(\neg F \wedge \neg G)} \quad \vee\text{-def}^{n} \,*$$

GOES BOTH WAY

$$\frac{\Sigma \vdash \neg(\neg F \wedge \neg G)}{\Sigma \vdash F \vee G} \quad \overset{?}{\rightsquigarrow} \quad \Sigma \vdash \neg(\neg G \wedge \neg F)$$

can use symmetry on $\neg G \wedge \neg F$ & will to get $\neg F \wedge \neg G$

$$\Sigma \vdash \neg(\neg F \wedge \neg G)$$
$$\downarrow ?$$
$$\Sigma \vdash \neg(\neg G \wedge \neg F)$$

The derived rule of contrapositive

$$\text{contrapositive} \quad \frac{\Sigma \cup \{F\} \vdash G}{\Sigma \cup \{\neg G\} \vdash \neg F}$$

to unlock, prove without using $\vee$-symm

1) $\Sigma \cup \{F\} \vdash G$ Premise

2) $\Sigma \vdash F \Rightarrow G$ $\Rightarrow$-intro on 1

3) $\Sigma \vdash \neg F \vee G$ $\Rightarrow$-def'n on 2

4) $\Sigma \cup \{\neg G\} \vdash \neg F \vee G$ Monotonic on 3

5) $\Sigma \cup \{\neg G, \neg F\} \vdash \neg F$ Assumption

6) $\Sigma \cup \{\neg G, G\} \vdash \neg G$ Assumption

7) " $\vdash \neg G \vee \neg F$ $\vee$-intro 6

8) " $\vdash G \Rightarrow \neg F$ $\Rightarrow$-def 7

9) " $\vdash G$ Assumption

10) $\Sigma \cup \{\neg G, G\} \vdash \neg F$ $\Rightarrow$-elim on 8, 9

11) $\Sigma \cup \{\neg G\} \vdash \neg F$ $\vee$-elim on 5, 10, 4

The plan to use contrapositive

$\Sigma \cup \{(\neg G \wedge \neg F)\} \vdash \neg F \wedge \neg G$ $\wedge$-symm

$\Sigma \cup \{\neg(\neg F \wedge \neg G)\} \vdash \neg(\neg G \wedge \neg F)$ contrapositi

Relating using $\vee$-def'n.

$\Sigma \vdash G \vee \neg F$

$\neg G \Rightarrow \neg F$ needs work

$\neg\neg G \vee \neg F$ rule direct

5.4 Lots of formal proofs!

Fill in reasons yourself please

1) $\Sigma \vdash p \Rightarrow q$

$\Sigma \vdash p \vee q$

$\Sigma \cup \{p\} \vdash p$

$\Sigma \cup \{p\} \vdash q$

$\Sigma \cup \{q\} \vdash q$

$\Sigma \vdash q$

2) Prove $\neg F$

$\Sigma \cup \{F\} \vdash G$

$\vdash \neg G$

$\Sigma \cup \{\neg r \wedge p\} \vdash \neg r$

$\vdash p$

$\vdash q$

$\vdash r$

loop

3) $\Sigma \vdash q \vee (r \wedge s)$

$\Sigma \vdash q \Rightarrow t$

$\Sigma \vdash t \Rightarrow s$

$\vee$-elim

q — $\Rightarrow$-elim

$r \wedge s$ — $\wedge$-elim

Tip: Indent the lines of your proof while using $\vee$-elim.

4) $\Sigma \vdash p \vee q$

$\Sigma \vdash r \vee s$

F: $(p \wedge r) \vee q \vee s$

$\Sigma \cup \{p\} \vdash p$

$\Sigma \cup \{p, r\} \vdash r$

$\vdash p \wedge r$

$\vdash F$

$\Sigma \cup \{p, s\} \vdash s$

$\vdash F$

$\Sigma \cup \{q\} \vdash q$

$\vdash F$

Key: getting one disjunct of F is crucial.

$$\Sigma \vdash \underbrace{((p \Rightarrow q) \Rightarrow q)}_{F} \Rightarrow q$$

Key: Prove $\{p\} \vdash (p \Rightarrow q) \Rightarrow q$

$p \Rightarrow q$

$\Sigma \cup \{p\} \vdash q$

$\Sigma \cup \{p\} \vdash F$

$\{p, F \Rightarrow q\} \vdash F$

Hint: When confronted with a formula

whose truth value is independent of a variable, BYCASES is useful

$\{p, q\} \vdash q$

$\vdash \neg(p \Rightarrow q) \vee q$

$\vdash (p \Rightarrow q) \Rightarrow q$

How to prove $\{p, \neg q\} \vdash (p \Rightarrow q) \Rightarrow q$

$\{p, \neg q, p \Rightarrow q\} \vdash p \Rightarrow q$

$\vdash p$

⇒-intro $\quad \vdash q \quad$ ⇒-elim

$\vdash \neg q$

$\{p, \neg q\} \vdash \neg(p \Rightarrow q)$ Pf. by contradicti

$\vdash \neg(p \Rightarrow q) \vee q$

$\vdash (p \Rightarrow q) \Rightarrow q$

By Cases $\quad \{p\} \vdash (p \Rightarrow q) \Rightarrow q$

$\Sigma \cup \{p\} \vdash (p \Rightarrow q) \Rightarrow q$

$\vdash ((p \Rightarrow q) \Rightarrow q) \Rightarrow q$

$\vdash q$

$\Sigma \vdash p \Rightarrow q$

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6. If you try to simplify with equivalence (TRY THIS ONLY IN ROUGH WORK)

you'll get $r \vee \neg r$ as a subformula

$\{r\} \vdash r \qquad \Sigma \vdash F \qquad \vdash G \Rightarrow F$

$\vdash q \vee r$ $\Sigma \cup \{g\} \vdash g$
$\vdash \neg p \vee (q \vee r)$ $\vdash F$
$\vdash p \Rightarrow (q \vee r)$ ↑ this is prob 7

$\{\neg r\} \vdash \neg r \vee \neg p$
$\vdash r \Rightarrow \neg p$

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8) $\Sigma \vdash p \Rightarrow (q \Rightarrow r)$ $(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$

$\Sigma \cup \{p \Rightarrow q\} \vdash p \Rightarrow q$

$\Sigma \cup \{p \Rightarrow q, p\} \vdash p \Rightarrow q$

$\vdash p$

$\vdash q$

$\vdash q \Rightarrow r$

$\vdash r$

$\Sigma \cup \{p \Rightarrow q\} \vdash p \Rightarrow r$

easy

10. By cases on r

r: $\neg t \vee u$ immediate

$\neg r$: use unitRes to get $s \wedge \neg t$

$t \Rightarrow u \leftarrow \neg t \vee u \leftarrow \neg t$

q $\neg\neg p \vee \neg q$ Target $q \Rightarrow p$

$\neg q$ $\neg q \vee p$

$\neg q \vee \neg\neg p$ $\vee$. elim ?

$\neg\neg p$

p [Rev. Double Negation]

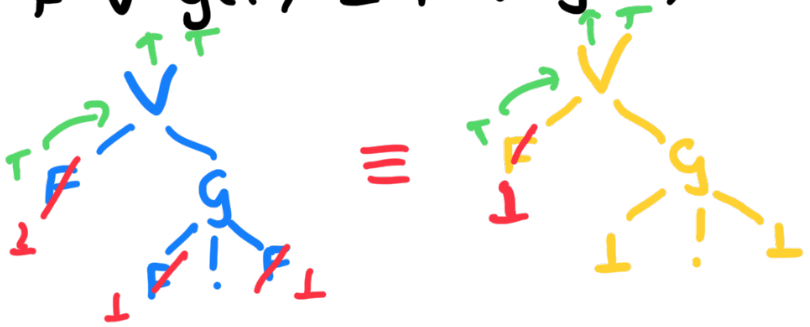
$\neg q \vee p$

Key: Go over the problems after an interval, internalise the patterns.

(and the available rules)

6.16 Substitutions, parse trees.

$$F \vee G(F) \equiv F \vee G(\bot)$$



$\rightarrow$ $m \models F$ then both formulae true

$\rightarrow$ $m \not\models F$ then

For this model ($m \models F$ iff $m \models \bot$)

Appeal to substitution thm

(——) : F and $\bot$ have the same truth value under m.

Other two: TWO CASES $\underset{T}{m \models F}$, $\underset{\bot}{m \not\models F}$

In one case, formula vacuously true, in the other, interesting

6.18 Convince yourself of 6.11

(Parity count, can do inductively)

odd number of p : p

even : $\bot$

$\underbrace{p \oplus \ldots \oplus p}_{n} \oplus p$ Induction

Carefully deal with parity

case analysis