Discrete Structures :: CS 207 :: Autumn 2021

Problem Set 6

Released: September 28, 2021

1. How many relations are there on a set with n elements that are:
   (a) reflexive?
   (b) irreflexive?
   (c) symmetric?
   (d) antisymmetric?
   (e) asymmetric?
   (f) equivalence?

   Hint: Where appropriate, you may use $S(k, n)$, the Stirling number of the second kind.

2. This problem considers proving that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$.
   (a) Give a combinatorial proof, by counting the number of ways to select a (non-empty) committee, with one member being the leader of the committee.
   (b) Prove this using the formula for $\binom{n}{k}$. First show that $k\binom{n}{k} = n\binom{n-1}{k-1}$.
   (c) Here is a trick we have not covered in the class, that uses your knowledge of calculus. Consider the polynomial $P(x) = (1+x)^n$. Let $P'(x)$ be the polynomial obtained as the derivative of $P(x)$. Write two expressions for $P'(x)$, and use them to evaluate $P'(1)$.

3. How many ways are there to travel in $xyzw$ space from the origin (0,0,0,0) to the point (4,3,5,4) by taking steps one unit in the positive x, positive y, positive z, or positive w direction?

4. A sequence of integers is said to be *smooth* if any two consecutive integers in the sequence differ by exactly 1. For instance, $5, 4, 5, 6, 5, 4$ is a smooth sequence of length 6.

   How many smooth sequences of length 16 are there that start with 5 and end with 10?

5. A sequence of positive integers $a_1, a_2, \ldots, a_m$ is said to be *decreasing* if for all i, we have $a_i \geq a_{i+1}$. A decreasing sequence is said to *strictly decreasing* if any integer appears at most once in the sequence. A decreasing sequence is said to be *almost strictly decreasing* if any integer appears at most twice in the sequence.
   (a) How many strictly decreasing sequences of positive integers are there with $a_1 = n$ (of all possible lengths)?
   (b) There are infinitely many decreasing sequences of positive integers that start with $a_1 = n$. How about almost strictly decreasing sequences?
   (c) How many strictly decreasing sequences of positive integers of length m exist with $a_1 = n$?
   (d) How many decreasing sequences of positive integers of length m exist with $a_1 = n$?

6. Consider the standard deck of 52 playing cards. A balanced hand is a subset of 13 cards containing four cards of one suit and three cards of each of the remaining three suits.
   (a) Find the number of balanced hands.
   (b) Find the number of ways of dealing the cards to four (distinguishable) players so that each player gets a balanced hand.

7. Suppose k universities are to be ranked by the Ministry of Education according to some arbitrary criteria. The ranking allows mutiple universities to be tied.

   For instance, universities $\{A, B, C, D\}$ may be ranked as $B > A = C > D$, to mean that B is top-ranked, A, C are tied below that, and D is ranked at the bottom; note that $B > C = A > D$ refers to the same ranking.

   What is the total number of such possible rankings? You may express your answer in the form of a summation, involving quantities used in the *balls-and-bins* problems.

8. A variant of the balls-and-bins problem, when the balls are distinguishable, is that within each bin, the balls are ordered. Let $L(k, n)$ denote the number of ways k labelled items can be distributed among n lists (i.e., bins with order), where the lists themselves are unlabelled, such that no list is empty. E.g., $L(3, 2) = 6$ since $\{a, b, c\}$ can be split into 2 lists as $\{(a), (b, c)\}$, $\{(a), (c, b)\}$, $\{(b), (a, c)\}$, $\{(b), (c, a)\}$, $\{(c), (a, b)\}$, or $\{(c), (b, a)\}$.

   Give a closed form expression for $L(k, n)$.

   Hint: First consider the case where the lists are labelled. You may use a variant of stars and bars, with labelled stars. To ensure that no list is empty, you may use stars themselves as bars.