Problem Set 7c

Released: November 1, 2021

1. Show that a tree has at most one perfect matching.

   Hint: Use induction.

   Solution: We will prove the claim by inducting on the number of vertices n in tree T. For $n \leq 2$, the claim is trivially true because there is at max one edge possible in T and hence one matching. Let us assume that the claim is true for $n \leq k$ where $k \geq 2$. Now consider a tree T with $n = k + 1$ many vertices. Since T has at least 2 vertices, it must have at least 2 leaves. Consider a leaf l and it's neighbour p in T. In a perfect matching of T, l must be mapped to p because that is the only edge that l is a part of. Let us call the subgraph induced by removing l and p from T as F. F is still acylic but may not be connected because other vertices in the subtree of p might have gotten disconnected from the rest of the graph. Therefore, F must be a forest with m many trees $T_1, T_2, ...T_m$ with at most $k-1$ vertices.

   It is easy to see that a perfect matching for T can be obtained by adding edge (l, p) to a perfect matching of F and a perfect matching for F can be obtained by removing the edge (l, p) from a perfect matching for T. Also, a perfect matching for F can only be obtained by combining the perfect matchings of trees $T_1, T_2, ...T_k$. By the induction hypothesis, we can say that each of these trees have at most one perfect matching which implies that the forest F has at most one perfect matching. Thus, T has at most one perfect matching.

2. Show that a tree is a bipartite graph.

   Hint: Consider the distance of each node from a fixed node. The two parts correspond to even and odd distances. Where do you use the fact that the graph is a tree?

   Solution: Consider a node u in G. Let distance(v_1, v_2) denote distance of vertex v_1 from v_2 in G. Now, let's define the following 2 sets -

   $$\text{Odd} = \{\ v \mid v \in V \text{ and distance}(u,\ v) \text{ is odd }\}$$
   $$\text{Even} = \{\ v \mid v \in V \text{ and distance}(u,\ v) \text{ is even }\}$$

   Color the vertices in Odd red and the vertices in Even blue. Color u blue. We need to prove that this is a valid coloring of G. For the sake of contradiction, assume that there is an edge between 2 vertices v_1 and v_2 of the same color. Since these 2 vertices have the same color, they must be at distances d_1 and d_2 from u s.t. $d_2 - d_1$ is even (including 0) or one of v_1 or v_2 must be u. In the second case, it can be shown that distance of the other vertex is 1 from u (because of the edge from v_1 to v_2) which implies that it should have different color from u. In the first case, we can consider the paths P_1 from u to v_1 and P_2 from u to v_2 and then we can get a cycle in the graph by considering $P = u...v_1v_2...u$ by using P_1 and P_2. If no vertex repeats in P, then it is a cycle. Else, the vertex that repeats will give a cycle. Hence, this must represent a valid 2-coloring of the graph which in turn implies that it is a bipartite graph.

3. A *spanning tree* of a graph G is a subgraph G' which has all the nodes in G and is a tree. Show that every connected graph G has a spanning tree.

   Hint: Induct on the number of cycles in G. Use the previous problem.

   Solution: We strong induct on the number of cycles in G.
   Base Case: G has 0 cycles. In this case, G is a tree, so it is its own spanning tree.

   Induction Hypothesis: For a natural number n, it is true that for all $m < n$, if a connected graph has m cycles then it has a spanning tree.

   Induction Step: Suppose G is a spanning tree with n cycles, where n is as above. Since $n \geq 1$, it has at least one cycle C. Let e be an edge of the cycle. Remove this edge e to obtain a graph G'. From the previous problem, G' is connected. Also, G' has strictly less than n cycles, because the cycle C no longer exists. By induction hypothesis, G' has a spanning tree T. Since G' is a subgraph of G with all nodes of G, so T is a spanning tree of G as well. This completes the proof.

4. Prove that for a graph with n vertices, any two of the following imply the third:

(a) G is connected.

(b) G is acyclic.

(c) G has $n-1$ edges.

Solution: **(a) & (b) together imply (c):** Let G be a connected, acyclic graph with n vertices. We induct on n. If G has only one vertex, then it clearly has $0 = 1-1$ edges, and (c) holds. Otherwise, suppose $n \geq 2$, and the assertion holds for all graphs with less than n vertices. We show first that G has a leaf. Take a maximal path $P = u_1u_2\cdots u_k$ in G. u_1 cannot have a neighbour outside u_is, else P would not be maximal. Also, u_1 cannot have $u_3, u_4, \cdots u_k$ as neighbours, or else a cycle is formed. So u_1 has degree 1, and we found a leaf. So delete u_1 from G. This gives a connected acyclic graph G' on $n-1$ vertices, so this graph G' has $n-2$ edges. Therefore G has $(n-2)+1 = n-1$ edges, as required.

(b) & (c) together imply (a): Let G be an acyclic graph with n vertices and $n-1$ edges. Suppose G has k connected components $G_1, G_2, \cdots, G_k$, with $n_1, n_2, \cdots, n_k$ vertices respectively. Then, each connected component G_i is also acyclic, so from the previous part, G_i has $n_i - 1$ edges. This means G has $(n_1-1)+(n_2-1)+\cdots+(n_k-1) = n-k$ edges, but we know that G has $n-1$ edges. Therefore $k=1$, so G is connected.

(c) & (a) together imply (b): Let G be a connected graph with n vertices and $n-1$ edges. Suppose G is cyclic. Remove an edge from a cycle; from the previous to previous problem, this new graph G_1 is connected. If G_1 also has a cycle, remove an edge from G_1 to obtain a connected graph G_2, and so on. This process cannot keep going on forever, because the number of cycles strictly decreases every step. Let G' be the final graph obtained. This graph is connected and acyclic, but has strictly less than $n-1$ edges, which contradicts the first part. We infer that our assumption is wrong. Hence, G is acyclic.

5. What is the maximum size of $|S|$ such that there is a poset $(S, \preceq)$ of height h and width w? Construct such a poset.

   Solution: Since the width of the poset is w, by Dilworth's theorem, there is a decomposition into w chains. Since, max. length of chain is h, maximum number of elements in S $= h * w$.

   As an example of a poset which attains the equality, consider, divisibility poset with $S = \cup_{i=0}^{w}\{p_i, p_i^2, .., p_i^h\}$, where $p_i's$ are w prime numbers.

6. Use Dilworth's theorem to show that any set of 5 natural numbers either contains numbers of the form x, xy and xyz, or contains 3 numbers which are mutually indivisible by each other?

   Solution: By Dilworth's theorem, size of largest anti-chain is equal to the smallest chain decomposition in any poset. Given a set of 5 naturals, consider the divisibility poset S on them. If any three of them are mutually indivisible then the required statement is true. Else, the maximal anti-chain in S has size less than or equal to 2. Which means, S can be decomponsed into atmost 2 chains. By PHP, since there are 5 numbers in total, in at least of one the chain there must be 3 elements. Since, they form a divisibility chain they must be of the form x, xy, xyz. Hence, the statement is true.