Sets & Relations

Basics of Sets

Sets & Relations in action

Relational Database

x	y	Likes(x,y)
Alice	Alice	TRUE
	Jabberwock	FALSE
	Flamingo	TRUE
Jabberwock	Alice	FALSE
	Jabberwock	TRUE
	Flamingo	FALSE
Flamingo	Alice	FALSE
	Jabberwock	FALSE
	Flamingo	TRUE

Relational DB Table

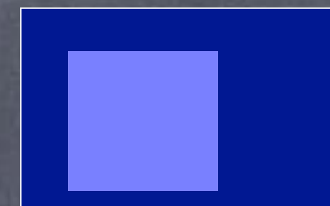
Likes	
x	y
Alice	Alice
Alice	Flamingo
Jabberwock	Jabberwock
Flamingo	Flamingo

- Queries to the DB are set/logical operations
  - SELECT x
    FROM Likes
    WHERE y='Alice' OR y='Flamingo'
  - $\{ x \mid (x,\text{Alice}) \in \text{Likes} \} \cup \{ x \mid (x,\text{Flamingo}) \in \text{Likes} \}$

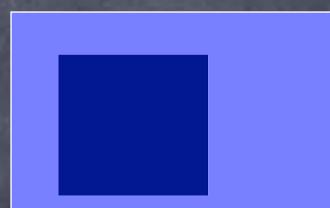
Sets: Basics

- Unordered collection of "elements"
  - e.g.: $\mathbb{Z}$, $\mathbb{R}$ (infinite sets), $\emptyset$ (empty set), $\{1, 2, 5\}$, ...
- Will always be given an implicit or explicit universe (universal set) from which the elements come
  - (Aside: In developing the foundations of mathematics, often one starts from "scratch", using only set theory to create the elements themselves)
- Set membership: e.g. $0.5 \in \mathbb{R}$, $0.5 \notin \mathbb{Z}$, $\emptyset \notin \mathbb{Z}$
- Set inclusion: e.g. $\mathbb{Z}, \subseteq \mathbb{R}$, $\emptyset \subseteq \mathbb{Z}$
- Set operations: complement, union, intersection, difference

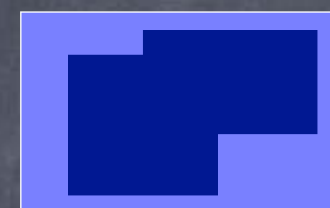
Set Operations



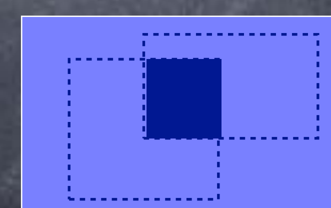
$\overline{S}$



S

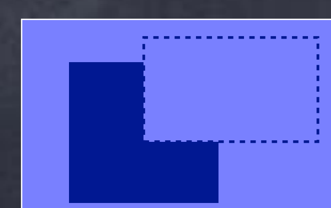


$S \cup T$



$S \cap T$

T



$S - T$

Sets as Predicates

x	Winged(x)	Flies(x)	Pink(x)	inClub(x)
Alice	FALSE	FALSE	FALSE	TRUE
Jabberwock	TRUE	TRUE	FALSE	FALSE
Flamingo	TRUE	TRUE	TRUE	TRUE

- Given a predicate can define the set of elements for which it holds
  - WingedSet = { x | Winged(x) } = {J'wock, Flamingo}
  - FliesSet = { x | Flies(x) } = {J'wock, Flamingo}
  - PinkSet = { x | Pink(x) } = {Flamingo}
- Conversely, given a set, can define a membership predicate for it e.g. given set Club = {Alice, Flamingo}. Then, define predicate inClub(x) s.t. inClub(x) = True iff $x \in$ Club

Set Operations

Unary operator

Binary operators

Associative

S complement
Symbol: $\overline{S}$

$in\overline{S}(x) \equiv \neg inS(x)$

S union T
Symbol: $S \cup T$

$inS{\cup}T(x)$
$\equiv inS(x) \vee inT(x)$

S intersection T
Symbol: $S \cap T$

$inS{\cap}T(x)$
$\equiv inS(x) \wedge inT(x)$

S difference T
Symbol: $S - T$
(Alternately: $S \setminus T$)

$inS{-}T(x)$
$\equiv inS(x) \wedge \neg inT(x)$
$\equiv inS(x) \nrightarrow inT(x))$

$S-T = S \cap \overline{T}$

S symmetric diff. T
Symbol: $S \,\Delta\, T$

$inS\Delta T(x)$
$\equiv inS(x) \oplus inT(x)$

Note: Notation inS(x) used only to explicate the connection with predicate logic. Will always write $x \in S$ later.

De Morgan's Laws

- $\overline{S \cup T} = \overline{S} \cap \overline{T}$
  - $x \in \overline{S \cup T} \equiv \neg(x \in S \cup T)$

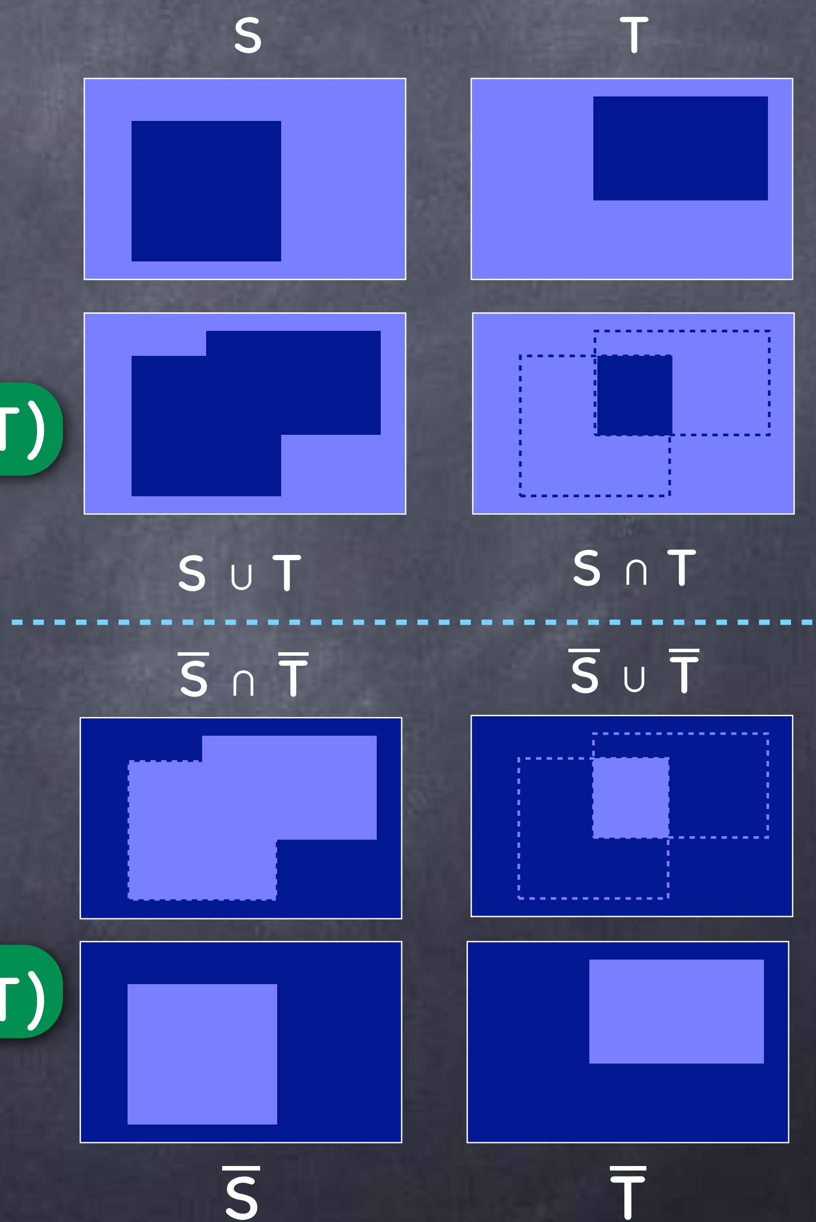
    $\equiv \neg(x \in S \lor x \in T) \equiv \neg(x \in S) \land \neg(x \in T)$

    $\equiv x \in \overline{S} \land x \in \overline{T} \equiv x \in \overline{S} \cap \overline{T}$

- $\overline{S \cap T} = \overline{S} \cup \overline{T}$
  - $x \in \overline{S \cap T} \equiv \neg(x \in S \cap T)$

    $\equiv \neg(x \in S \land x \in T) \equiv \neg(x \in S) \lor \neg(x \in T)$

    $\equiv x \in \overline{S} \lor x \in \overline{T} \equiv x \in \overline{S} \cup \overline{T}$



Distributivity

- $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$
  - $x \in R \cap (S \cup T) \equiv$

    $\equiv x \in R \wedge (x \in S \vee x \in T) \equiv (x \in R \wedge x \in S) \vee (x \in R \wedge x \in T)$

    $\equiv x \in (R \cap S) \cup (R \cap T)$
- $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$
  - $x \in R \cup (S \cap T) \equiv$

    $\equiv x \in R \vee (x \in S \wedge x \in T) \equiv (x \in R \vee x \in S) \wedge (x \in R \vee x \in T)$

    $\equiv x \in (R \cup S) \cap (R \cup T)$

Set Inclusion

x	Winged(x)	Flies(x)	Pink(x)
Alice	FALSE	FALSE	FALSE
Jabberwock	TRUE	TRUE	FALSE
Flamingo	TRUE	TRUE	TRUE

- PinkSet $\subseteq$ FliesSet = WingedSet
- S $\subseteq$ T same as the proposition $\underline{\forall x \;\; x \in S \rightarrow x \in T}$
- S $\supseteq$ T same as the proposition $\underline{\forall x \;\; x \in S \leftarrow x \in T}$
- S = T same as the proposition $\underline{\forall x \;\; x \in S \leftrightarrow x \in T}$

Set Inclusion

- $S \subseteq T$ same as the proposition $\underline{\forall x \;\; x \in S \rightarrow x \in T}$
- If $S = \emptyset$, and T any arbitrary set, $S \subseteq T$
  - $\forall x$, vacuously we have $x \in S \rightarrow x \in T$
- If $S \subseteq T$ and $T \subseteq R$, then $S \subseteq R$
  - Consider arbitrary $x \in S$. Since $S \subseteq T$, $x \in T$. Then since $T \subseteq R$, $x \in R$.

If no such x, already done

- $S \subseteq T \longleftrightarrow \overline{T} \subseteq \overline{S}$
  - $\forall x \;\; \underline{x \in S \rightarrow x \in T} \;\equiv\; \forall x \;\; \underline{x \notin T \rightarrow x \notin S}$ (contrapositive)

$$\equiv \; \forall x \;\; \underline{x \in \overline{T} \rightarrow x \in \overline{S}}$$

Proving Set Equality

- To prove S = T, show $S \subseteq T$ and $T \subseteq S$
- e.g., $L(a,b) = \{ x : \exists u,v \in \mathbb{Z} \;\; x=au+bv \}$

  $M(a,b) = \{ x : (\gcd(a,b) \mid x) \}$
- [Recall] **Theorem**: $L(a,b) = M(a,b)$
- Proof in two parts:
  - $L(a,b) \subseteq M(a,b)$: i.e., $\forall x \in \mathbb{Z} \;\; x \in L(a,b) \rightarrow x \in M(a,b)$
  - $M(a,b) \subseteq L(a,b)$: i.e., $\forall x \in \mathbb{Z} \;\; x \in M(a,b) \rightarrow x \in L(a,b)$

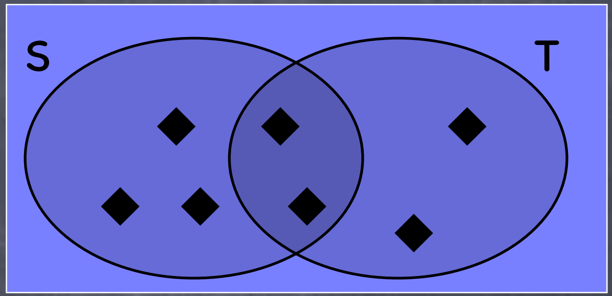
First show that $g \in L(a,b)$ (as the smallest +ve element in $L(a,b)$)

Let $x=ng$. But $g=au+bv \Rightarrow x=au'+bv'$

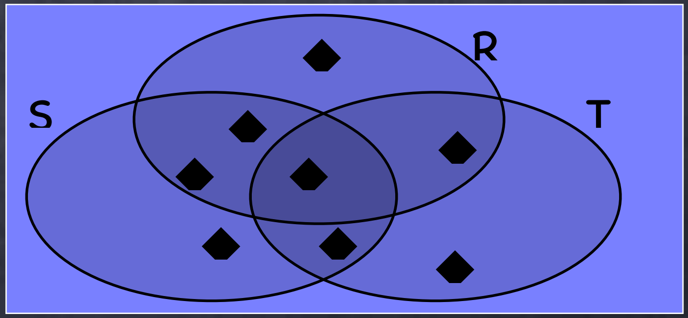
Let $x=au+bv$. $g|a, g|b \Rightarrow g|x$

Inclusion-Exclusion

- $|S| + |T|$ counts every element that is in S or in T
  - But it double counts the number of elements that are in both: i.e., elements in $S \cap T$
- So, $|S|+|T| = |S \cup T| + |S \cap T|$
- Or, $|S \cup T| = |S| + |T| - |S \cap T|$

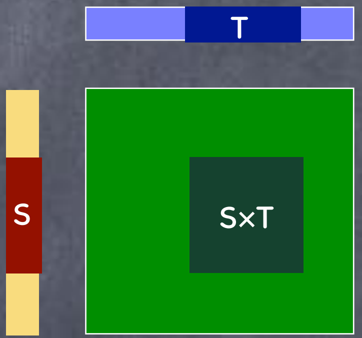


- $|R \cup S \cup T| = |R|+|S|+|T| - |R \cap S| - |S \cap T| - |T \cap R| + |R \cap S \cap T|$
  - $|R \cup S \cup T| = |R| + |S \cup T| - |R \cap (S \cup T)|$
    $= |R| + |S \cup T| - |(R \cap S) \cup (R \cap T)|$
    $= |R| + |S| + |T| - |S \cap T|$
    $-(|R \cap S| + |R \cap T| - |R \cap S \cap T|)$



Cartesian Product

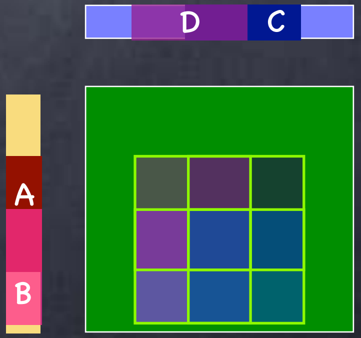
- $S \times T = \{ (s,t) \mid s\in S \text{ and } t\in T \}$
  - $(S= \emptyset \vee T= \emptyset) \leftrightarrow S \times T = \emptyset$
  - $|S \times T| = |S|\cdot|T|$



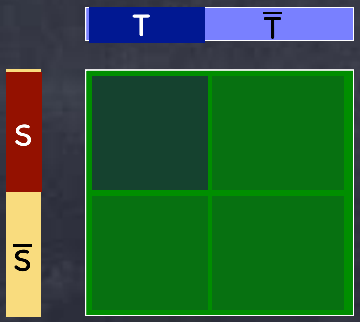
- $R \times S \times T = \{ (r,s,t) \mid r\in R,\ s\in S,\ t\in T \}$
  - Not the same as $(R \times S) \times T$ (but "essentially" the same)

$\{ ((r,s),t) \mid r\in R,\ s\in S,\ t\in T \}$

- $(A\cup B) \times C = (A\times C) \cup (B\times C)$. Also, $(A\cap B) \times C = (A\times C) \cap (B\times C)$
  - $(A\cup B) \times (C\cup D) = (A\times(C\cup D)) \cup (B\times(C\cup D))$
    $= (A\times C) \cup (A\times D) \cup (B\times C) \cup (B\times D)$



- Complement: $\overline{S \times T} = ?$
  - $(\overline{S}\times\overline{T}) \cup (\overline{S}\times T) \cup (S\times\overline{T})$



Sets & Relations

Relations

Relations: Basics

More commonly written as:
x Likes y, $x \sqsubset y$, $x \geq y$, $x \sim y$, xLy, ...

- Informally, a <u>relation</u> is specified as what is related to what
- Formally, a predicate over the domain S×S
  - e.g. Likes(x,y)
- Alternately, a subset of S×S, namely the pairs for which the relation holds
  - Likes = { (Alice,Alice), (Alice, Flamingo), (J'wock,J'wock), (Flamingo,Flamingo) }

Homogeneous and binary (the default notion for us)

x,y	Likes(x,y)
Alice, Alice	TRUE
Alice, Jabberwock	FALSE
Alice, Flamingo	TRUE
Jabberwock, Alice	FALSE
Jabberwock, Jabberwock	TRUE
Jabberwock, Flamingo	FALSE
Flamingo, Alice	FALSE
Flamingo, Jabberwock	FALSE
Flamingo, Flamingo	TRUE

Many ways to look at it!

$R \subseteq S \times S$

a set of ordered-pairs

$\{ (a,b) \mid a \sqsubset b \}$

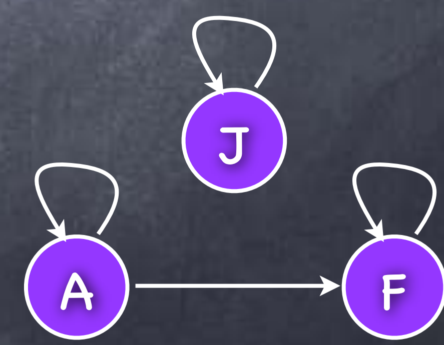
$\{ (A,A), (A,F), (J,J), (F,F) \}$

Boolean matrix,

$M_{a,b} = T$ iff $a \sqsubset b$

	A	J	F
A	T	F	T
J	F	T	F
F	F	F	T

(directed) graph

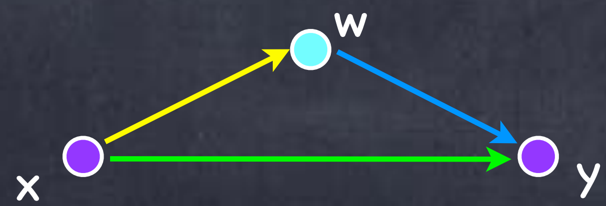


Operations on Relations

- Since a relation is a set, namely $R \subseteq S \times S$, all set operations extend to relations
  - Complement (with the universe being $S \times S$), Union, Intersection, Symmetric Difference
- Converse (a.k.a. Transpose)

$M^T_{xy} = M_{yx}$

  - $R^T = \{ (x,y) \mid (y,x) \in R \}$
- Composition



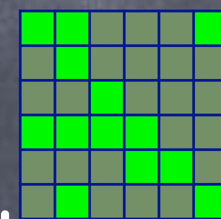
"Boolean matrix multiplication"
$(M \circ M')_{xy} = \bigvee_w (M_{xw} \wedge M'_{wy})$

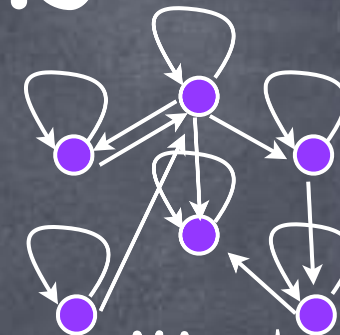
  - $R \circ R' = \{ (x,y) \mid \exists w \in S \;\; (x,w) \in R \text{ and } (w,y) \in R' \}$

Types of Relations

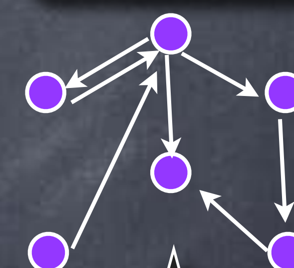
(Ir)Reflexive Relations

- Reflexive (e.g. Knows, ≤)
  - The kind of relationship that everyone has with themselves





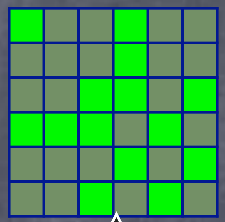
- Irreflexive (e.g. Gave birth to, ≠)
  - The kind that nobody has with themselves



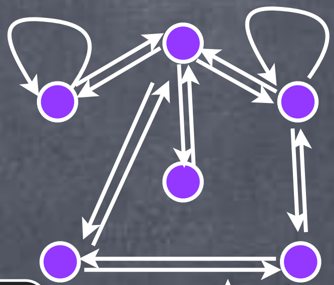
- Neither (e.g. is a prime factor of)
  - Some, but not all, have this relationship with themselves

(Anti)Symmetric Relations

- Symmetric (e.g. sits next to)
  - The relationship is reciprocated



symmetric matrix



self-loops & bidirectional edges only

- Anti-symmetric (e.g. parent of, prime factor of, $\subseteq$)
  - No reciprocation (except possibly with self)

no bidirectional edges

- Neither (e.g. likes)
  - Reciprocated in some pairs (with distinct members) and only one-way in other pairs

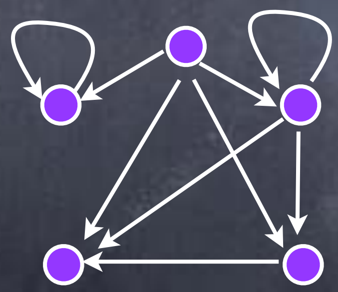
some bidirectional, some unidirectional

- Both (e.g., =)
  - Each one related only to self (if at all)

no edges except self-loops

Transitive Relations

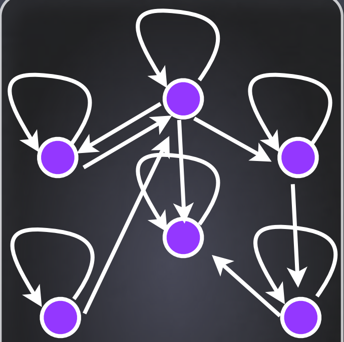
- Transitive (e.g., Ancestor of, subset of, divides, $\leq$)
  - if a is related to b and b is related to c, then a is related to c
- R is transitive $\leftrightarrow$ $R \circ R \subseteq R$ $\leftrightarrow$ $\forall k>1\ R^k \subseteq R$



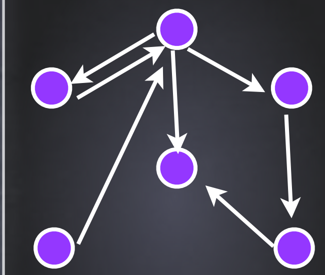
if there is a "path" from a to z, then there is edge (a,z)

- Intransitive: Not transitive

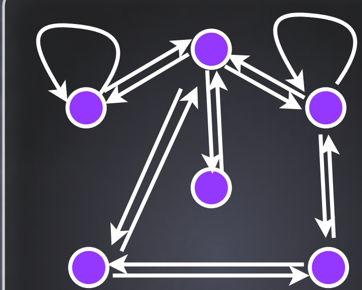
Types of Relations



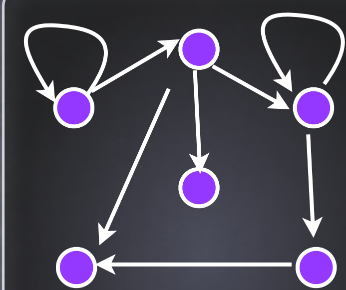
Reflexive:
All self-loops



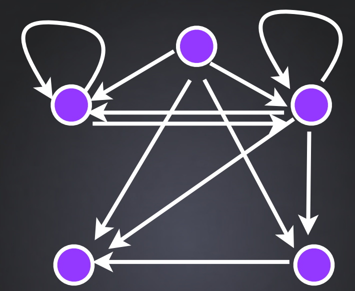
Irreflexive:
No self-loops



Symmetric:
Only self-loops & bidirectional edges



Anti-symmetric:
No bidirectional edges



Transitive:
Path from a to b implies edge (a,b)

The complete relation R = S × S is reflexive, symmetric and transitive

Reflexive closure of R: Minimal relation R′ ⊇ R s.t. R′ is reflexive

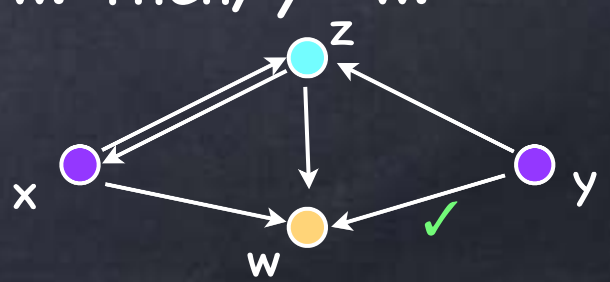
Symmetric closure of R: Minimal relation R′ ⊇ R s.t. R′ is symmetric

Transitive closure of R: Minimal relation R′ ⊇ R s.t. R′ is transitive

Each of these is unique [Why?]

Equivalence Relation

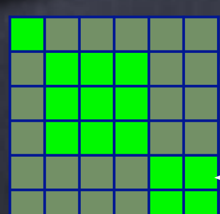
- A relation that is reflexive, symmetric and transitive
  - e.g. is a relative, has the same last digit, is congruent mod 7, ...
- Equivalence class of x: **$Eq(x) \triangleq \{y|x \sim y\}$**.
- Every element is in its own equivalence class (by reflexivity)
- Claim: If $Eq(x) \cap Eq(y) \neq \emptyset$, then $Eq(x) = Eq(y)$.
  - Let $z \in Eq(x) \cap Eq(y)$. To show $Eq(x) \subseteq Eq(y)$ [similarly, $Eq(y) \subseteq Eq(x)$]
  - Consider an arbitrary $w \in Eq(x)$: i.e., $x \sim w$.
  - By symmetry, $z \sim x$. Then, by transitivity, $z \sim w$. Then, $y \sim w$.
  - Thus, $w \in Eq(y)$. i.e., $Eq(x) \subseteq Eq(y)$.



Equivalence Relation

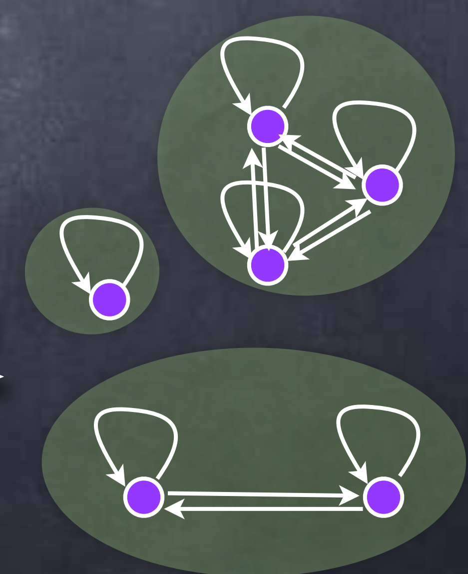
- A relation that is reflexive, symmetric and transitive
  - e.g. is a relative, has the same last digit, is congruent mod 7, ...
- **Equivalence class** of x: $\mathrm{Eq}(x) \triangleq \{y | x \sim y\}$.
- Every element is in its own equivalence class (by reflexivity)
- Claim: If $\mathrm{Eq}(x) \cap \mathrm{Eq}(y) \neq \emptyset$, then $\mathrm{Eq}(x) = \mathrm{Eq}(y)$.
- The equivalence classes partition the domain

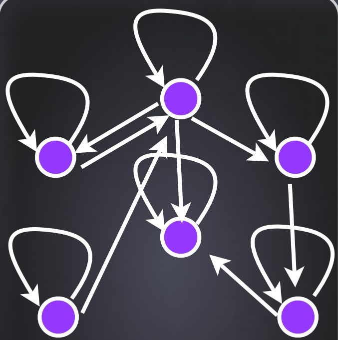
$P_1,..,P_t \subseteq S$
s.t.
$P_1 \cup .. \cup P_t = S$
$P_i \cap P_j = \emptyset$



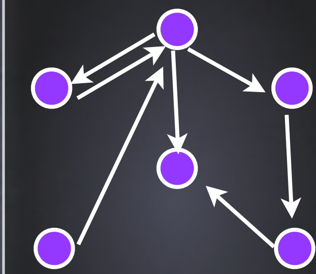
Square blocks along the diagonal, after sorting the elements by equivalence class

"Cliques" for each class

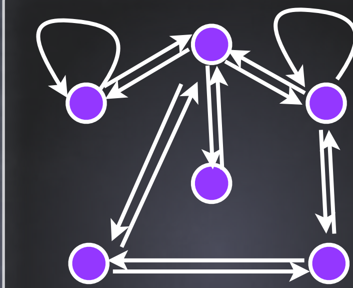




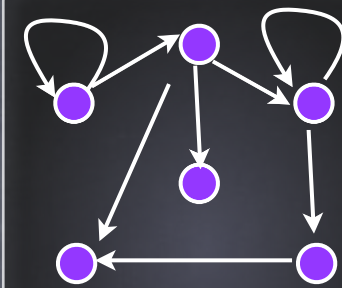
Reflexive: All self-loops



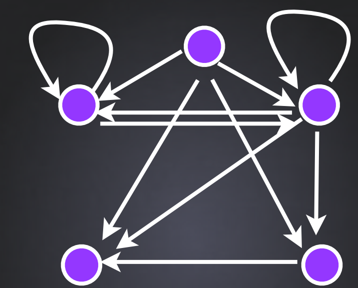
Irreflexive: No self-loops



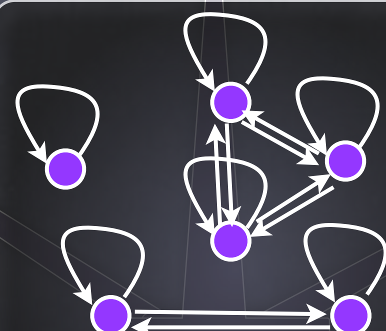
Symmetric: Only self-loops & bidirectional edges



Anti-symmetric: No bidirectional edges



Transitive: Path from a to b implies edge (a,b)



Equivalence: Cliques, disconnected from each other

Reflexive closure of R: Smallest relation R′ ⊇ R s.t. R′ is reflexive

Symmetric closure of R: Smallest relation R′ ⊇ R s.t. R′ is symmetric

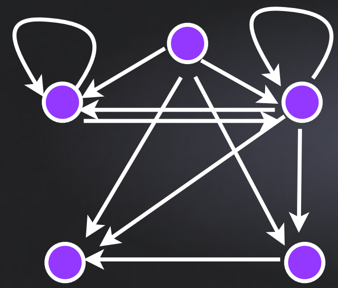
Transitive closure of R: Smallest relation R′ ⊇ R s.t. R′ is transitive

An equivalence relation R is its own reflexive, symmetric and transitive closure

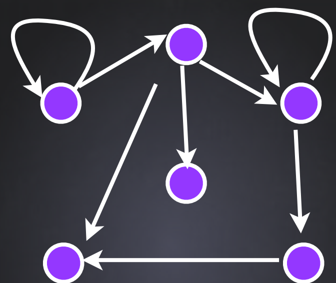
Sets & Relations

Posets

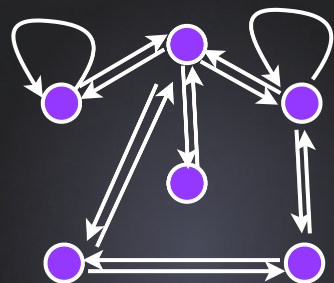
Landscape of Transitive Relations



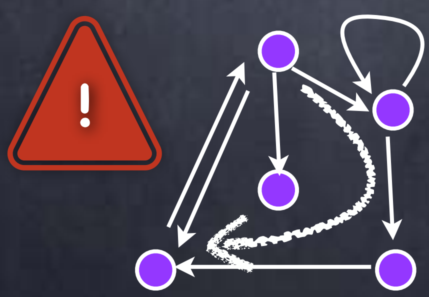
Transitive:
Path from a to b
implies edge (a,b)



Anti-symmetric:
No bidirectional
edges



Symmetric:
Only self-loops &
bidirectional edges



Acyclic
Cannot follow a sequence of non-self-loop edges and get back to where you started from

$\subseteq$ $\leq$

$<$

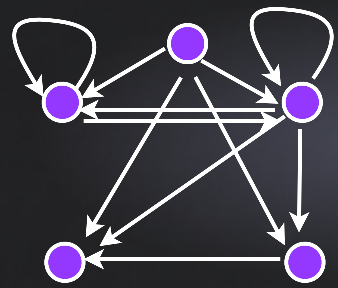
ancestor of

has same last name as

$\equiv$

Anti-Symmetric ⟷ Symmetric

Landscape of Transitive Relations



Transitive:
Path from a to b
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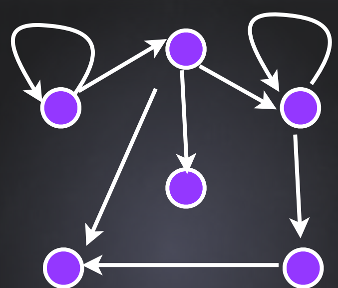
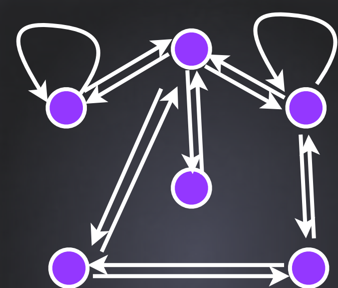
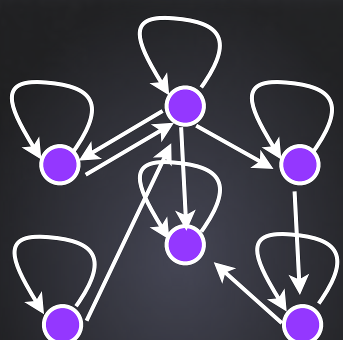


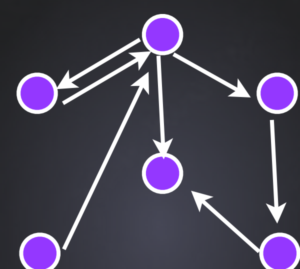
Anti-symmetric:
No bidirectional edges



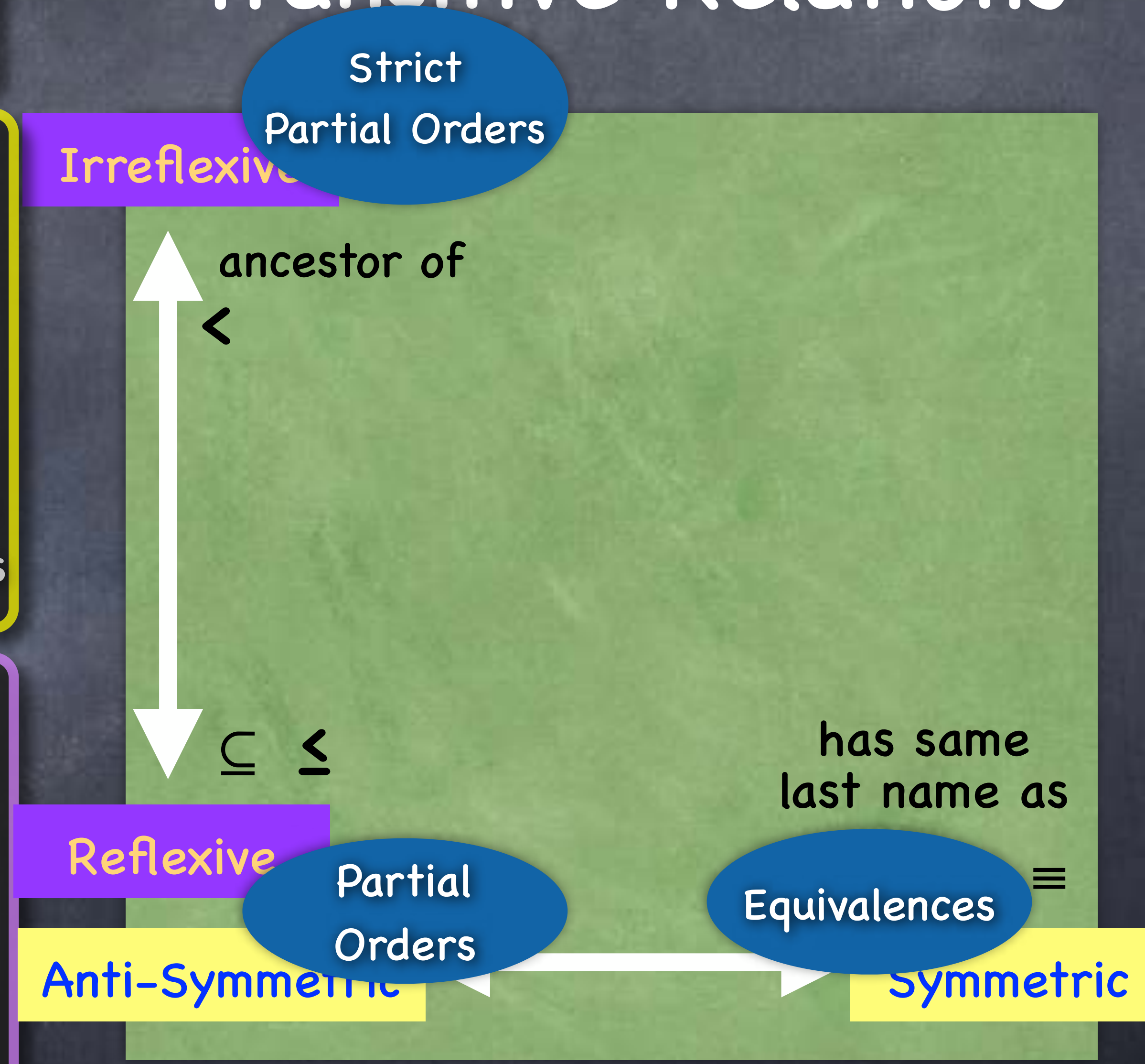
Symmetric:
Only self-loops & bidirectional edges



Reflexive:
All self-loops



Irreflexive:
No self-loops



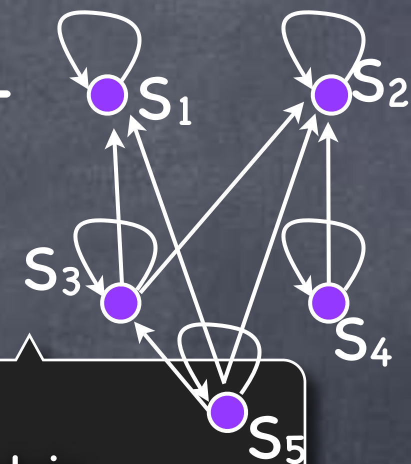
Partial Order

Strict partial order: irreflexive, rather than reflexive

- A transitive, anti-symmetric and reflexive relation
  - e.g. ≤ for integers, divides for integers, ⊆ for sets, "containment" for line-segments
- Equivalently, transitive and acyclic (and ir/reflexive) (a pair of bidirectional edges is a "cycle")
  - "Order" refers to these properties
- "Partial": not every two elements need be "comparable"
  - i.e., {a,b} s.t. neither $a \sqsubseteq b$ nor $b \sqsubseteq a$
    - e.g., neither $A \subseteq B$ nor $B \subseteq A$

Posets

- Partially ordered set (a.k.a Poset)
  - A non-empty set and a partial order over it
  - Denoted like $(S, \preccurlyeq)$



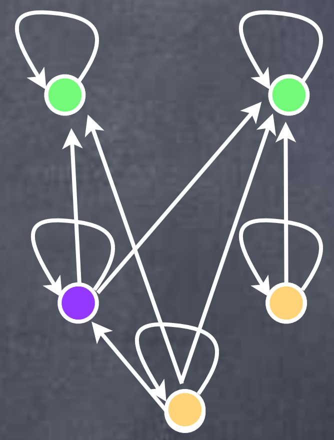
- e.g. $S = \{S_1, S_2, S_3, S_4, S_5\}$ where $S_1=\{0,1,2,3\}$, $S_2=\{1,2,3,4\}$, $S_3=\{1,2,3\}$, $S_4=\{3,4\}$, and $S_5 = \{2\}$. Poset $(S, \subseteq)$
  - More generally, $(S, \subseteq)$ where S is any set of sets
  - Verify: $P \subseteq P$; $P \subseteq Q \land Q \subseteq R \rightarrow P \subseteq R$; $P \subseteq Q \land Q \subseteq P \rightarrow P=Q$

Check:
- Anti-symmetric (no bidirectional edges),
- Transitive,
- Reflexive (all self-loops)

- e.g. Divisibility poset: $(\mathbb{Z}^+, \mid)$
  - Verify: $a \mid a$; $a \mid b \land b \mid c \rightarrow a \mid c$; $a \mid b \land b \mid a \rightarrow a=b$

Extremal & Extremum

- Maximal & minimal elements of a poset $(S, \preccurlyeq)$
  - $x \in S$ is maximal if $\nexists y \in S - \{x\}$ s.t. $x \preccurlyeq y$
  - $x \in S$ is minimal if $\nexists y \in S - \{x\}$ s.t. $y \preccurlyeq x$
    - Need not exist (e.g., in $(\mathbb{Z}, \leq)$).
    - Need not be unique when it exists (e.g., divisibility poset restricted to integers > 1)



- Claim: Every finite poset has at least one maximal and one minimal element

Useful in induction proofs about finite posets

  - Proof by induction on $|S|$ [Exercise]
- $x \in S$ is the greatest element if $\forall y \in S$, $y \preccurlyeq x$
- $x \in S$ is the least element if $\forall y \in S$, $x \preccurlyeq y$

Need not exist. Unique when one exists.

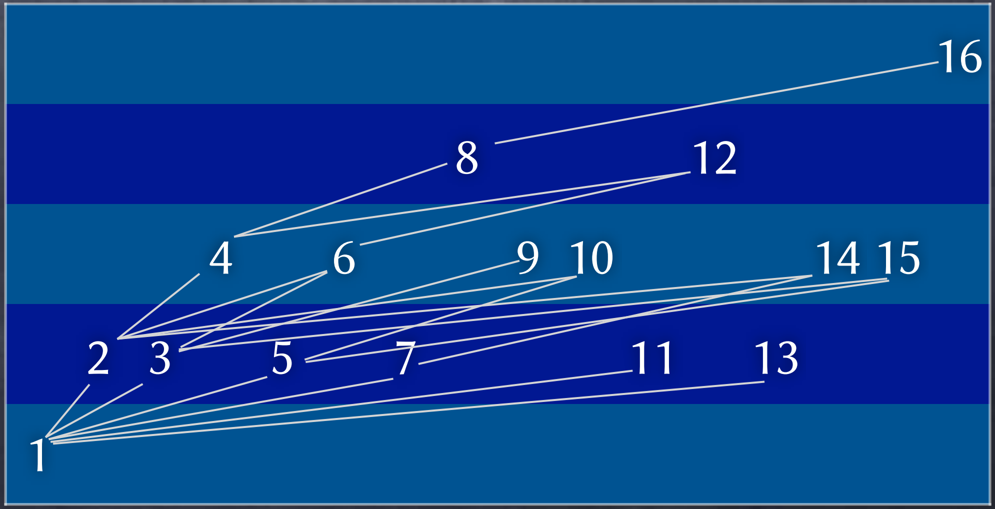
Other Relations from a Poset

- Consider partial order $\preccurlyeq$
- $\prec$ is the reflexive reduction of $\preccurlyeq$ iff $\preccurlyeq$ is the reflexive closure of $\prec$, and $\prec$ itself is irreflexive
  - $a \prec b$ iff $a \neq b$ and $a \preccurlyeq b$
- $\sqsubseteq$ is the transitive reduction of $\preccurlyeq$ iff $\preccurlyeq$ is the transitive closure of $\sqsubseteq$, and $\forall a,b$ ($a \sqsubseteq b \rightarrow \nexists m \notin \{a,b\}$, $a \preccurlyeq m \preccurlyeq b$)
  - Well-defined for finite posets: Define $a \sqsubseteq b$ iff $a \preccurlyeq b$ and $\nexists m \notin \{a,b\}$, $a \preccurlyeq m \preccurlyeq b$. [Prove by induction]
  - Need not exist for infinite sets (e.g., for $(\mathbb{R}, \leq)$, $\sqsubseteq$ defined as above is the equality relation)

Running Example

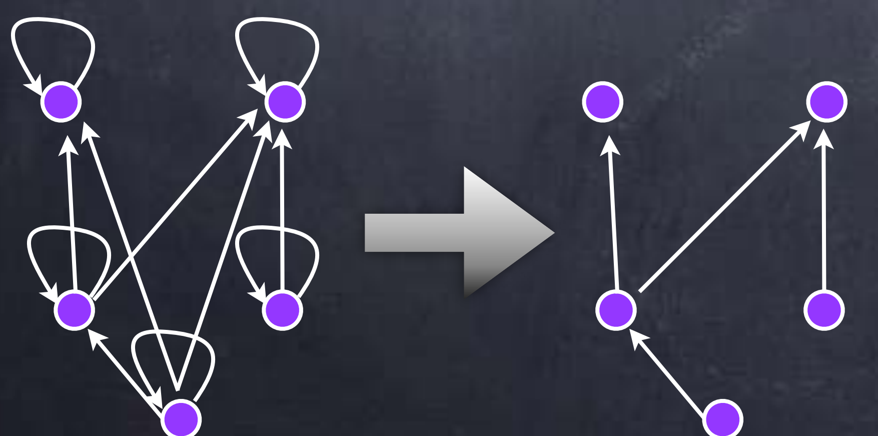
Divisibility poset: ($\mathbb{Z}^+$, |)

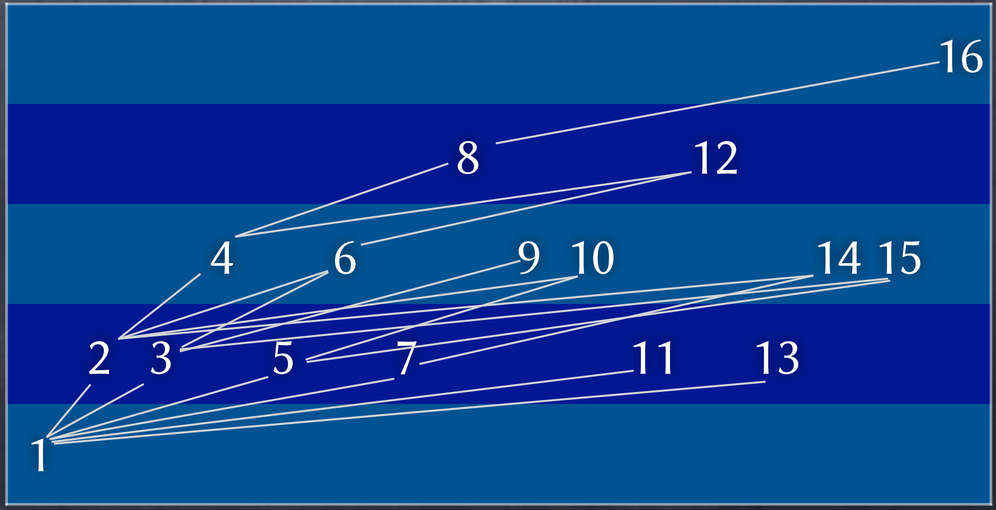
- Consider strict poset ($\mathbb{Z}^+$,$\sqsubset$), where a $\sqsubset$ b iff b/a is prime
- Claim: | is the transitive closure of the reflexive closure of $\sqsubset$ [Verify]
- Claim: $\sqsubset$ is the transitive reduction of the reflexive reduction of | [Verify]
  - Note: Divisibility poset has a transitive reduction even though it is infinite



Hasse Diagram

- For a poset (S, ≼), the transitive reduction of the reflexive reduction of ≼, if it exists, has all the information about the poset
  - Recall: For finite posets, guaranteed to exist
- Hasse Diagram: the graph of this relation (with arrowheads implicit)





Bounding Elements

- Given a poset $(S, \preccurlyeq)$ and $T \subseteq S$
- Maximal element in T : $x \in T$ s.t. $\forall y \in T$, $\underline{x \preccurlyeq y \rightarrow y = x}$

  Minimal element in T : $x \in T$ s.t. $\forall y \in T$, $\underline{y \preccurlyeq x \rightarrow y = x}$
- Greatest element in T : $x \in T$ s.t. $\forall y \in T$ $y \preccurlyeq x$

  Least element in T : $x \in T$ s.t. $\forall y \in T$, $x \preccurlyeq y$
- Upper Bound for T : $x \in S$ s.t. $\forall y \in T$, $y \preccurlyeq x$

  Lower Bound for T : $x \in S$ s.t. $\forall y \in T$, $x \preccurlyeq y$
- Least Upper Bound for T: Least in $\{x \mid x \text{ u.b. for } T\}$

  Greatest Lower Bound for T: Greatest in $\{x \mid x \text{ l.b. for } T\}$

Need not exist. Need not be unique when one exists.

Do exist in finite posets

Need not exist. Unique when one exists.

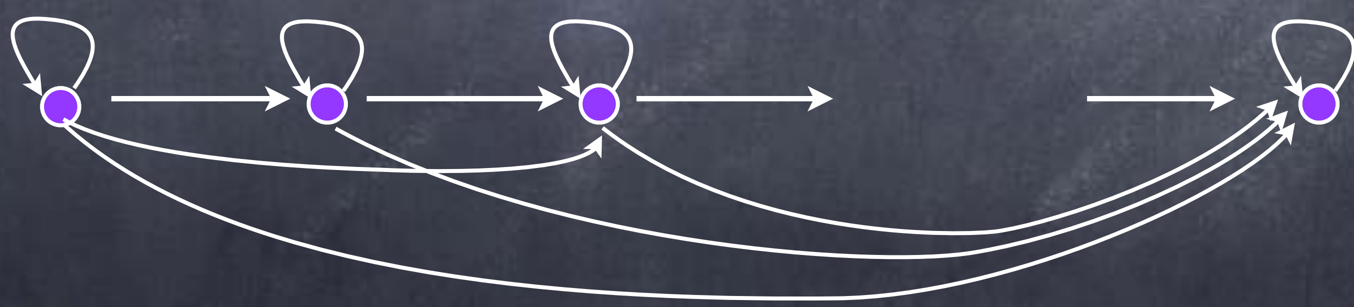
Running Example

Divisibility poset: $(\mathbb{Z}^+, \mid)$

- Lower bound
  - When is c a lower bound for T={a,b}?
    - $c \mid a$ and $c \mid b \Rightarrow$ c is a common divisor for {a,b}
  - Greatest lower bound for {a,b} = gcd(a,b)
- Upper bound
  - d is an upper bound for {a,b} $\Rightarrow a \mid d$, $b \mid d \Rightarrow$ d a common multiple for {a,b}
  - Least upper bound for {a,b} = lcm(a,b)

Total/Linear Order

- In some posets every two elements are "comparable": for {a,b}, either $a \sqsubseteq b$ or $b \sqsubseteq a$
  - Can arrange all the elements in a line, with all possible right-pointing edges (plus, all self-loops)



- If finite, has unique maximal and unique minimal elements (left and right ends)

Order Extension

- A poset $P'=(S,\leq)$ is an extension of a poset $P=(S,\preccurlyeq)$ if $\forall a,b \in S,\ a \preccurlyeq b \rightarrow a \leq b$
- Any finite poset can be extended to a total ordering (this is called <u>topological sorting</u>)
  - Prove by induction on $|S|$
    - Induction step: Remove a minimal element, extend to a total ordering, reintroduce the removed element as the minimum in the total ordering.
  - For infinite posets? The "Order Extension Principle" is typically taken as an axiom! (Unless an even stronger axiom called the "Axiom of Choice" is used)

Running Example

Divisibility poset: $(\mathbb{Z}^+, \mid)$

- The totally ordered set $(\mathbb{Z}^+, \leq)$, where $\leq$ is the standard "less-than-or-equals" relation, is an extension of the divisibility poset
  - Because $a|b \rightarrow a \leq b$
- Consider another totally ordered set $(\mathbb{Z}^+, \sqsubseteq)$:
  - For any $(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, $a \sqsubseteq b$ iff:
    - $a=1$, or
    - a,b both prime or both composite, and $a \leq b$, or
    - a prime and b composite
  - $(\mathbb{Z}^+, \sqsubseteq)$ extends the divisibility poset [Exercise]

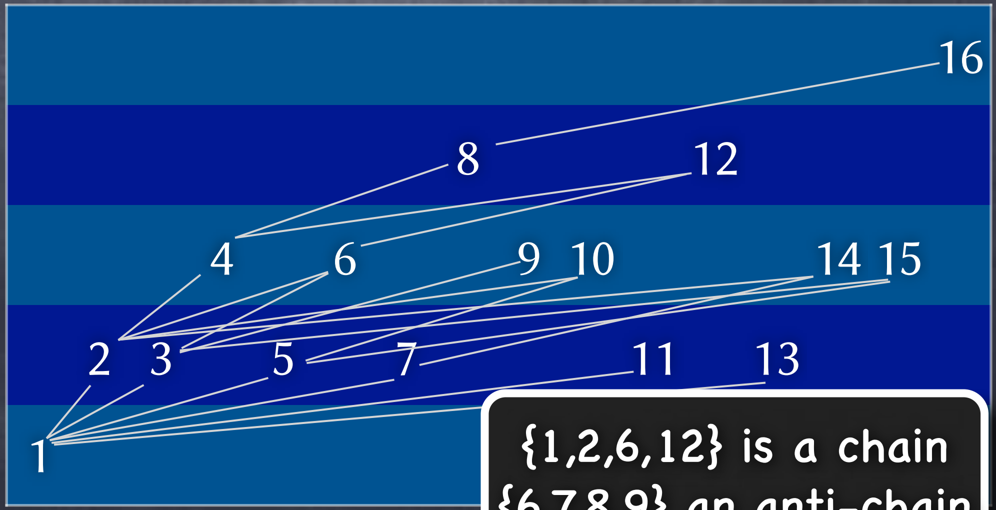
Sets & Relations

Chains and Anti-Chains

Chains & Anti-Chains

In a poset $(S,\preccurlyeq)$

- $C \subseteq S$ is said to be a chain if $\forall a,b \in C$, either $a \preccurlyeq b$ or $b \preccurlyeq a$
- i.e., $(C,\preccurlyeq)$ is a total order
- $A \subseteq S$ is an anti-chain if $\forall a,b \in A$, neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, unless $a=b$
- $(A,\preccurlyeq)$ is same as $(A,=)$
- Subset of a chain is a chain. Similarly for anti-chains.
- A singleton set is both a chain and an anti-chain
- For any chain C and anti-chain A, $|C \cap A| \leq 1$ (Why?)

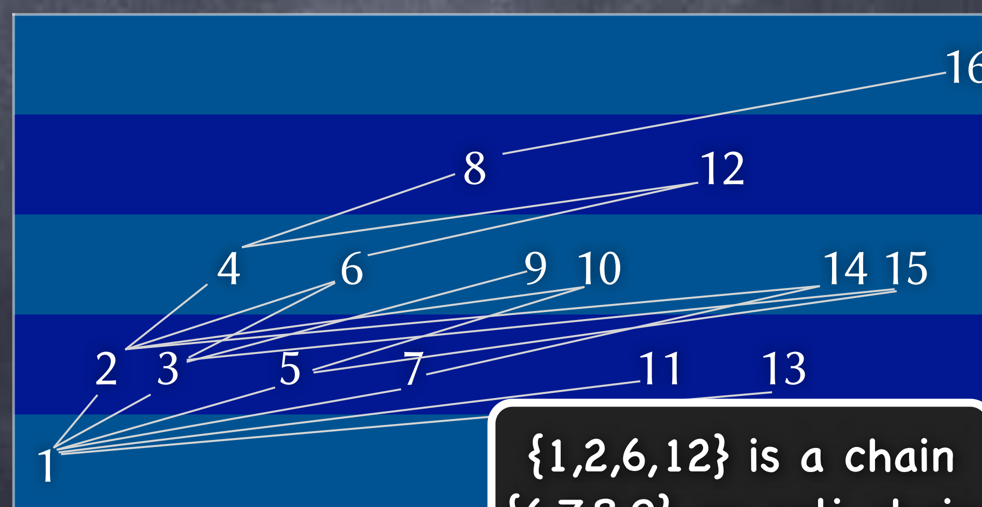


{1,2,6,12} is a chain
{6,7,8,9} an anti-chain
{2,6,7} neither
{6} both

Height in a Poset

Finite if S is finite

- In a poset $(S,\preccurlyeq)$, for any $a\in S$, we define
  height(a) = max size of a chain with a as the maximum
  - Note: every a has {a} as such a chain
- E.g., In $(\mathbb{Z}^+, \mid\ )$, height(1)=1, height(p)=2 for all primes p.
  For $m=p_1^{d1}\cdot\ldots\cdot p_t^{dt}$ (p_i primes), height(m) = $1+\Sigma_i\, d_i$
- Height <u>of</u> the poset $(S,\preccurlyeq)$
  = max { height(a) | $a\in S$}
  = max { |C| | chain C}
  - Size of the largest chain in the poset
  - Possibly ∞ (only if S infinite)



{1,2,6,12} is a chain
{6,7,8,9} an anti-chain
{2,6,7} neither
{6} both

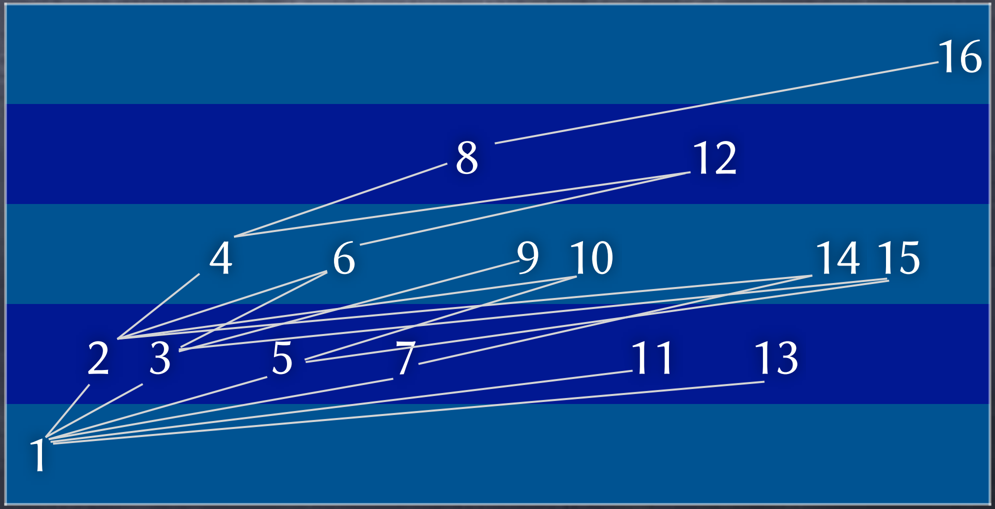
Anti-Chains from Height

- Let A_h = { a | height(a)=h }
- For every finite h, A_h is an anti-chain (possibly empty)
  - Otherwise, $\exists a \neq b$, $a \preccurlyeq b$ with height(a) = height(b) = h.
    height(a) = h $\Rightarrow$ $\exists$chain C s.t. a=max(C) and |C|=h

    $\Rightarrow$ $b \notin C$ and $C' = C \cup \{b\}$ is a chain with b=max(C')

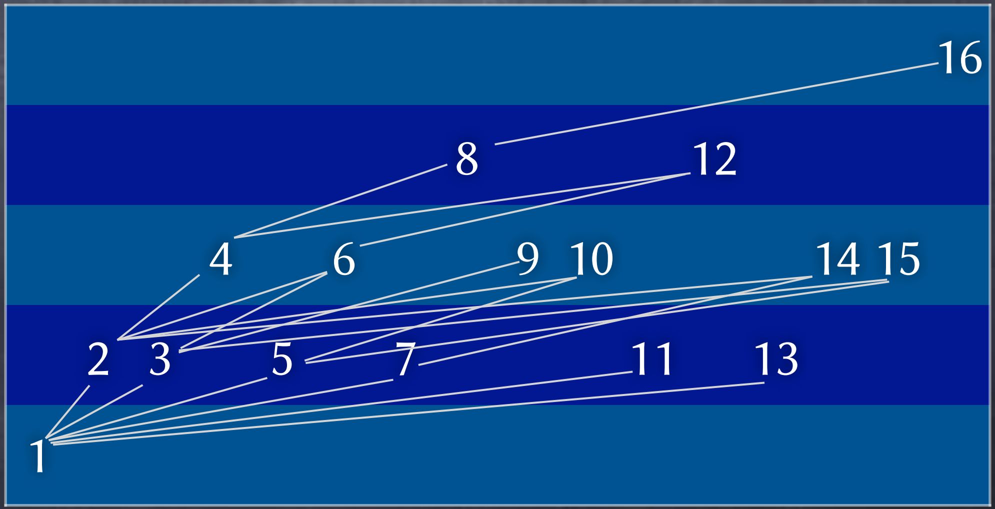
How?

$\Rightarrow$ height(b) $\geq$ h+1 !



Anti-Chains from Height

- Let $A_h = \{ a \mid \text{height}(a)=h \}$
- For every finite h, A_h is an anti-chain (possibly empty)
- $\max \{ h \mid A_h \neq \emptyset \}$ = height of the poset = $\max_{C \text{ chain}} |C|$
- In a finite poset, since every element has a finite height, every element appears in some A_h: i.e., A_h's partition S
- Mirsky's Theorem: The least number of anti-chains needed to partition S is exactly the size of a largest chain



  - For chain $C \subseteq S$, need $\geq |C|$ anti-chains to cover C, as $|C \cap A| \leq 1$ for anti-chain A

Partitioning with (Anti)Chains

- Mirsky's Theorem: The least number of anti-chains needed to partition S is exactly the size of a largest chain

Later

- Dilworth's Theorem: The least number of chains needed to partition S is exactly the size of a largest anti-chain