Problem Set 3

1. For each of the following, draw an NFA and/or FO formula whenever possible:
   (a) The set of strings over $x \in \{0,1\}^*$ such that $\#_0(x).\#_1(x)$ is even. The automaton can have no more than 4 states.
   (b) the set of strings over $\{a,b\}$ containing atleast 3 occurrences of two consecutive a's, overlapping allowed (for example, $aaaa$ is accepted). While drawing the automaton, draw one which has 7 states.
   (c) $(0 + 1(01^*0)^*1)^*$

2. Let L_1, L_2 be FO-definable. Is $L_1.L_2$ FO-definable?

3. Let $h : \Sigma^* \to \Gamma^*$ be a mapping defined as follows. $h(\epsilon) = \epsilon$, h is defined on all symbols of Σ and $h(a)$ is a unique word for each $a \in \Sigma$. Moreover, $h(w_1w_2) = h(w_1)h(w_2)$ for all words w_1w_2. If $L \subseteq \Sigma^*$, define $h(L) = \{h(w) \mid w \in L\}$.
   (a) If L is regular, is $h(L)$ regular?
   (b) If L is FO-definable, is $h(L)$ FO-definable?

4. Which of the following formulae are valid? Which are satisfiable? In each case, answer your question by constructing equivalent automata.
   – $\exists y[\neg first(y) \wedge Q_b(y) \wedge \forall x[x < y \to Q_a(x)]]$
   – $\forall x[Q_a(x) \to \exists y[(x < y) \wedge Q_b(y) \wedge \exists z_1 \exists z_2 (z_1 < z_2) \wedge x < z_1 \wedge \neg(Q_b(z_1) \wedge Q_b(z_2)))]]$

5. Let us define a new kind of automaton inspired from logic, called $\forall\exists$ automaton as follows: A $\forall\exists$ automaton is a finite state automaton $(Q, \Sigma, \Delta, S, F)$ where the set of states Q is partitioned into two sets $Q_\forall$ and $Q_\exists$. Let there be n states in Q. The transitions coming out of a $Q_\exists$ state are called "or" transitions, and the transitions coming out of a $Q_\forall$ state are called "and" transitions.

   An "or" transition has the form $\Delta(q,a) = q_{i_1} \vee \ldots \vee q_{i_j}$, $q \in Q_\exists$, while, an "and" transition has the form $\Delta(q,a) = q_{j_1} \wedge \ldots \vee q_{j_l}$, $q \in Q_\forall$.

   For example, let $Q = \{s_1, s_2, s_3, s_4, s_5\}$, $S = \{s_1\}$, $F = \{s_5\}$, with transitions $\Delta(s_1,a) = s_2 \vee s_3$, $\Delta(s_2,a) = s_2 \wedge s_4$, $\Delta(s_2,b) = s_5$, $\Delta(s_4,a) = s_4$, $\Delta(s_4,b) = s_5$, $\Delta(s_3,a) = s_3$, $\Delta(s_3,b) = s_3$. The following are two run trees of the word aab:
   Note that each time an "and" transition $\delta(q,a) = q_{i_1} \wedge \ldots \wedge q_{i_j}$ is used, we spawn j threads, and maintain all the states $q_{i_1}, \ldots, q_{i_j}$. All these threads then evolve in parallel. A word w is accepted by a run tree if a final state

$$\begin{array}{ccccc} & & s_1 & & \\ & & a & & \\ & & s_2 & & \\ & a & & a & \\ s_2 & & & & s_4 \\ b & & & & b \\ s_5 & & & & s_5 \end{array} \qquad\qquad \begin{array}{c} s_1 \\ a \\ s_3 \\ a \\ s_3 \\ b \\ s_3 \end{array}$$

is encountered in all the threads when you finish reading w. An "or" transition is the usual non-deterministic choice you have, you can pick any one of the choices. In the above example, the first run tree is accepting, while the second is not. A word is accepted if it has atleast one accepting run tree.

(a) Compare the expressiveness of $\forall\exists$ automata and NFA.

(b) How do you complement a $\forall\exists$ automata?

6. Find the smallest r such that the following pairs of words are r-distinguishable ($w_1 \nsim_r w_2$) with respect to $FO[<]$ by playing a game:
(a) $((ab)^n b(ab)^n, (ab)^m(ab)^m)$
(b) $((aa)^n b(aaa)^n, (aaa)^n b(aa)^n)$
(c) $((aabbacb)^n, (aabbacb)^m aab(aabbacb)^k)$
In each case, write the $FO[<]$ that distinguishes the two words.