Problem Set 5

Released: October 11, 2021

1. **Equivalence Closure**

   (a) Show that the transitive closure of the symmetric closure of the reflexive closure of a relation R is the smallest equivalence relation that contains R.

   Solution: Let R_r be the reflexive closure of R, R_{rs} be the symmetric closure of R_r and R_{rst} be the transitive closure of R_{rs}. First we need to show that R_{rst} is an equivalence relation.

   i. Reflexive - Since $R_r \subseteq R_{rst}$, hence R_{rst} is reflexive.
   ii. Symmetric - We need to show that the transitive closure of a symmetric set is symmetric. For $(a,b) \in R_{rst}$, either $(a,b) \in R_{rs}$ or $\exists c$ s.t. $(a,c) \in R_{rs}$ and $(c,b) \in R_{rs}$. If $(a,b) \in R_{rs}$, then $(b,a) \in R_{rs}$ and hence $(b,a) \in R_{rst}$. If $(a,c) \in R_{rs}$ and $(c,b) \in R_{rs}$ then $(c,a) \in R_{rs}$ and $(b,c) \in R_{rs}$ which implies $(b,a) \in R_{rst}$. Hence, R_{rst} is symmetric.
   iii. Transitive - By definition, the transitive closure of a relation is transitive.

   Now we need to show that R_{rst} is the smallest equivalence relation which contains R. Let us consider an equivalence relation R_e which contains R. Then R_e must contain R_r because R_e is reflexive and contains R and by definition of reflexive closure, R_r is the smallest such relation. Thus, $R_r \subseteq R_e$. Also, R_e is symmetric and contains R_r. By definition of symmetric closure, R_{rs} is the smallest such relation. Hence, $R_{rs} \subseteq R_e$. Similarly, we can use the definition of transitive closure to claim $R_{rst} \subseteq R_e$.

   (b) Give an example such that the symmetric closure of the transitive closure of the reflexive closure of a relation R is not an equivalence relation.

   Solution: Consider a relation R on $\{1,2,3\}$ such that $R = \{(1,2),(3,2)\}$. The reflexive closure of R is $R_1 = \{(1,2),(3,2),(1,1),(2,2),(3,3)\}$. The transitive closure of R_1 is R_1 itself. The symmetric closure of R_1 is $R_2 = \{(1,2),(2,1),(3,2),(2,3),(1,1),(2,2),(3,3)\}$. Clearly, R_2 is not transitive as $(1,2) \in R_2$ and $(2,3) \in R_2$ but $(1,3) \notin R_2$. Hence, R_2 is not an equivalence relation.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $f((x,y)) = (y, y-x)$. Then define f^{-1}, or show that there is no unique inverse for f.

   Solution: We can define $f^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ as $f^{-1}(x,y) = (x-y, x)$. To show that this is the inverse function of f, we need to consider the compositions

   $$f^{-1} \circ f(x,y) = f^{-1}(f(x,y)) = f^{-1}(y, y-x) = (y-(y-x), y) = (x,y)$$

   By definition, f^{-1} is a valid inverse function for f. Also, since the image of f is the entire co-domain $\mathbb{R}^2$ (not very hard to verify), the inverse is unique.

3. Define a relation $\sim$ on the set of all functions from $\mathbb{R}$ to $\mathbb{R}$ by the rule $f \sim g$ if and only if there is a $z \in \mathbb{R}$ such that $f(x) = g(x)$ for every $x \geq z$. Prove that $\sim$ is an equivalence relation.

   Solution:

   (a) Reflexive - Pick any $z \in \mathbb{R}$. Trivially, $f(x) = f(x)$ for every $x \geq z$ which implies $f \sim f$.
   (b) Symmetric - Suppose $f \sim g$ where f and g are functions from $\mathbb{R}$ to $\mathbb{R}$. This means that $\exists z \in \mathbb{R}$ such that $f(x) = g(x)$ for every $x \geq z$. We can also write this as $g(x) = f(x)$ for every $x \geq z$. Hence, $g \sim f$.
   (c) Transitive - Consider $f \sim g$ and $g \sim h$ for f, g and h being functions from $\mathbb{R}$ to $\mathbb{R}$. Thus, $\exists z_1, z_2 \in \mathbb{R}$ s.t. $f(x) = g(x)\ \forall\ x \geq z_1$ and $g(x) = h(x)\ \forall\ x \geq z_2$. Consider $z = max\{z_1, z_2\}$. Then, $f(x) = h(x)\ \forall\ x \geq z$. Hence, $f \sim h$.

4. If functions $f : A \to B$ and $g : B \to C$ are such that $g \circ f$ is onto, then prove that g is onto. Use precise mathematical notation to prove this, starting from the definitions of onto and composition.

   Solution: Given that the function, $g \circ f : A \to C$ is an onto function. This means, $\forall y \in C, \exists x \in A\ (g \circ f)(x) = y$. By definition of composition we have, $(g \circ f)(x) = g(f(x))$. Therefore, we conclude that, $\forall y \in C, \exists x \in A\ g(f(x)) = y$. Since, $\forall x \in A, f(x) \in B$, we have $\forall y \in C, \exists z \in B\ g(z) = y$. Hence, g is an onto function.

5. Suppose $f : A \to B$ and $g : B \to C$ are such that $g \circ f$ is one-to-one. Is f necessarily one-to-one? Is g necessarily one-to-one? Justify.

Solution: Yes, f is necessarily a one-one function. We prove this as follows. Suppose, for $x_1, x_2 \in A$, $f(x_1) = f(x_2)$. Then,

$$\begin{aligned} &\implies g(f(x_1)) = g(f(x_2)) && \text{(applying g on both sides.)} \\ &\implies (g \circ f)(x_1) = (g \circ f)(x_2) && \text{(By definition of composition)} \\ &\implies x_1 = x_2 && \text{(As } g \circ f \text{ is one-one)} \end{aligned}$$

Therefore, $\forall x_1, x_2 \in A, f(x_1) = f(x_2) \implies x_1 = x_2$. Hence, f is one-one.

No, g need not be one-one. We prove this by giving a counter example. let $f : \mathbb{N}^+ \to \mathbb{N}^+$ be $f(x) = x+1$ and $g : \mathbb{N}^+ \to \mathbb{N}^+$ be $g(1) = 2$ and $g(x) = x$, if $x \geq 2$. Now, g is not one-one as $g(1) = g(2)$ but $(g \circ f)(x) = x + 1$ is one-one. Hence, g need not be one-one.

6. Suppose $f : A \to A$ is a function and $f \circ f$ is a bijection. Is f necessarily a bijection?

Solution: Yes, f is necessarily a bijection. Given that, $f \circ f : A \to A$ is a bijection, which means, $f \circ f$ is both one-one and onto. By Q4, we have that f is one-one and by Q5, we have that f is onto. Therefore, f is necessarily a bijection.

7. Given a function $f : A \to B$, define another function $f' : \mathcal{P}(A) \to \mathcal{P}(B)$ (where $\mathcal{P}(A)$ stands for the power-set of A), as follows: for any $S \subseteq A$, $f'(S) = \{f(x) | x \in S\}$. Show that $f'(S \cap T) \subseteq f'(S) \cap f'(T)$. Give an example of f and S, T such that $f'(S \cap T) \neq f'(S) \cap f'(T)$.

Solution: Suppose $b \in f'(S \cap T)$. By definition of f', $b = f(a)$ for some $a \in S \cap T$. Since $a \in S \cap T$, $a \in S$ and $a \in T$. Again, by definition of f', this means $b = f(a)$ is an element of $f'(S)$ and of $f'(T)$. Therefore $b \in f'(S) \cap f'(T)$. Therefore $f'(S \cap T) \subseteq f'(S) \cap f'(T)$.

Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(n) = n^2$ for all integers n. Take $S = \{0, 1\}$ and $T = \{0, -1\}$. Then,

$$f'(S \cap T) = f'(\{0\}) = \{f(0)\} = \{0\}$$

On the other hand

$$f'(S) \cap f'(T) = f'(\{0, 1\}) \cap f'(\{0, -1\}) = \{0, 1\} \cap \{0, 1\} = \{0, 1\}$$

Clearly, $f'(S \cap T) \neq f'(S) \cap f'(T)$.

8. Given a function $f : A \to B$, we define another function $\mathrm{inv}_f : \mathcal{P}(B) \to \mathcal{P}(A)$ as follows: for any $S \subseteq B$, $\mathrm{inv}_f(S) = \{x | f(x) \in S\}$. Now, given functions $f : A \to B$ and $g : B \to C$, express $\mathrm{inv}_{g \circ f}$ in terms of inv_f and inv_g. Justify.

Solution: Since $g \circ f$ is a function from A to C, $\mathrm{inv}_{g \circ f}$ is a function from $\mathcal{P}(C)$ to $\mathcal{P}(A)$. For any subset S of C, we have

$$\mathrm{inv}_{g \circ f}(S) = \{x | g(f(x)) \in S\}$$

We claim

$$\mathrm{inv}_{g \circ f} = \mathrm{inv}_f \circ \mathrm{inv}_g$$

Let S be a subset of C. Suppose $x \in \mathrm{inv}_{g \circ f}(S)$. Then $g(f(x)) \in S$. This means, by definition, that $f(x) \in \mathrm{inv}_g(S)$. This in turn means $x \in \mathrm{inv}_f(\mathrm{inv}_g(S))$.
On the other hand, suppose $x \in \mathrm{inv}_f(\mathrm{inv}_g(S))$. This means $f(x) \in \mathrm{inv}_g(S)$. This in turn means $g(f(x)) \in S$, so $x \in \mathrm{inv}_{g \circ f}(S)$. Therefore $\mathrm{inv}_{g \circ f}(S) = \mathrm{inv}_f \circ \mathrm{inv}_g(S)$. Since S was arbitrary, we have proved the claim.

9. Construct a bijection $f : \mathbb{Z} \to \mathbb{Z}^+$.

Solution: Let us first construct the bijection informally; we want to arrange the integers on a sequence. A natural candidate would be

$$0, 1, -1, 2, -2, 3, -3, \cdots$$

Formally, the bijection $f : \mathbb{Z} \to \mathbb{Z}^+$ is given by

$$f(x) = \begin{cases} 2x & x > 0 \\ -2x + 1 & x \leq 0 \end{cases}$$

It is not hard to show this is indeed a bijection.

10. Construct a bijection $f : \mathbb{Z}^2 \to \mathbb{Z}$.

Solution: We want to provide a bijection from $\mathbb{Z}^2$ to $\mathbb{Z}$. It suffices to construct a bijection from ${\mathbb{Z}^+}^2$ to $\mathbb{Z}^+$. Infroamlly, we have lattice points in the first quadrant, and we want to arrange them in a sequence, such that every element in ${\mathbb{Z}^+}^2$ occurs exactly once. Consider the bijection

$$g(m, n) = \frac{(m + n - 1)(m + n - 2)}{2} + m$$

This is one possible bijection. To get a motivation as to how this bijection comes, observe that $f(1, 1) = 1, f(1, 2) = 2, f(2, 1) = 3, f(1, 3) = 4, f(2, 2) = 5$, etc. This forms a "diagonal" pattern over the lattice points.
There are many other bijections possible. The following is also a bijection:

$$f(m, n) = 2^{m-1} \cdot (2n - 1)$$

which relies on uniqueness of prime factorisation. Proving this is a bijection is easy. Suppose $f(m_1, n_1) = f(m_2, n_2)$. Then

$$2^{m_1 - 1} \cdot (2n_1 - 1) = 2^{m_2 - 1} \cdot (2n_2 - 1)$$

Comparing the powers of two both sides, we get $m_1 - 1 = m_2 - 1$, so $m_1 = m_2$. Cancelling, we get $2n_1 - 1 = 2n_2 - 1$, so $n_1 = n_2$. Therefore the function is injective.
Furthermore, for any natural number q, we can write it as $2^l \cdot k$, where $l \geq 0$, and k is odd. Then $l + 1$ and $\frac{k+1}{2}$ are both naturals, and

$$f\left(l + 1, \frac{k + 1}{2}\right) = 2^{(l+1)-1} \cdot \left(2 \cdot \frac{k + 1}{2} - 1\right) = q$$

Therefore f is surjective as well.