CS228 Midsem Marking Scheme and Solutions

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1 Question 8 - Pradipta

Solution:
We give a proof by contradiction. Consider domain $D = \{a, b\}$. Suppose there is a universal sentence $\forall x_1, x_2 \cdots x_n F$ such that:

$$\forall x_1, x_2 \cdots x_n F \equiv \exists x P(x)$$

Consider the model $m = (\{a, b\}, \{\}, \{\{a\}\})$ over the signature $S = (\{\}, \{P\})$
It is easy to see that $m \models \exists x P(x)$
Therefore

$$m \models \forall x_1, x_2 \cdots x_n F$$

Since F is quantifier free, any submodel of m obtained by restricting the domain should also satisfy $\forall x_1, x_2 \cdots x_n F$ as in that case the variables x_i will take values from the restricted domain.
Consider the restriction m' of m on the domain $D' = \{b\}$
m' will be equal to $(\{b\}, \{\}, \{\{\}\})$
We see that m' is a submodel of m formed by restricting m to the domain $\{b\}$.
Therefore

$$m' \models \forall x_1, x_2 \cdots x_n F$$

This means that

$$m' \models \exists x P(x)$$

But this is a contradiction as $P_{m'}$ is empty.
Thus our original assumption was wrong and there is no universal sentence $\forall x_1, x_2 \cdots x_n F$ which is equivalent to $\exists x P(x)$
Marking Scheme:
For the model solution above:
3 marks: Setup of the solution, by starting with contradiction and giving a model m which satisifies one side.
5 marks: Showing that submodel also satisifes the universal sentence (either formal proof by induction or reasoning in words will work)
4 marks: Considering a submodel and showing that it contradicts the equivalence.
Proofs without contradiction will also work, where one can directly prove that for a universal sentence, submodel of any model m satisfying it will also work (8 marks) and showing that this does not occur with $\exists x P(x)$ (4 marks).
Alternate solutions will get marks as per validity.
Common Mistakes

1. Using Skolemisation and stating that the skolemised formula is not equivalent is not correct. Skolemisation is a technique that gives one equisat formula, there can be other ways of getting an equivalent formula that is not the skolemised one (the fact that you cant do this is what you need to prove).

2. Simply writing $\exists x P(x)$ is the negation of a universal formula and stating it therefore cannot be equivalent to a universal formula will fetch you one mark without proof.
3. Simply writing the question again does not give marks.
4. Showing that we can go from for all to there exists is not correct (its the other direction)
5. F can have free variables! $\forall x_1, x_2 \cdots x_n F$ is a sentence, not F.
6. Converting to FOL CNF does not preserve equivalence and is therefore wrong.