Quiz 5

CS 207 :: Autumn 2021
November 25, 2021

1. Honour code.

2. Suppose A, B are infinite sets, and $f : A \to B$ and $g : B \to A$ are functions such that $g \circ f$ is the identity function. Then:

- [ ] A. $|A| \leq |B|$
- [ ] B. $|B| \leq |A|$
- [ ] C. $|A| = |B|$
- [ ] D. None of the above necessarily holds

3. Suppose A, B are infinite sets, and $f : A \to B$ and $g : B \to A$ are functions such that $f \circ g$ is the identity function. Then:

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4. $16/\binom{2/5}{3}$
$=$ ____?

5. $250\binom{4/5}{3}$
$=$ ____?

6. $256\binom{1/4}{3}$
$=$ ____?

7. Suppose $f(k) = g(k+1)$ for all $k \geq 0$. Then the generating functions of f and g are related as:

- [ ] A. $G_f(X) = G_g(X+1) - g(0)$
- [ ] B. $G_f(X) = X(G_g(X) - g(0))$
- [ ] C. $G_f(X) = G_g(X) - g(1)X + f(0)$
- [ ] D. $G_f(X) = (G_g(X) - g(0))/X$
- [ ] E. None of the above

8. Suppose $g(k) = f(k+1)$ for all $k \geq 0$. Then the generating functions of f and g are related as:

- [ ] A. $G_g(X) = G_f(X) - f(1)X + g(0)$
- [ ] B. $G_g(X) = G_f(X+1) - f(0)$
- [ ] C. $G_g(X) = (G_f(X) - f(0))/X$
- [ ] D. $G_g(X) = X(G_f(X) - f(0))$
- [ ] E. None of the above

9. In Cantor's diagonal slash argument, given any function $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$, it is shown that f is not onto by showing that a certain set $X \subseteq \mathbb{N}$ is not in the image of f. Define the set X.

   Solution: The set X is defined as all the natural numbers which are not contained in the set given by their mappings. In other words,
   $$X = \{x \in \mathbb{N} \mid x \notin f(x)\}$$

10. Show that there exists a bijection between the set of all real numbers and the closed interval $[0, 1]$.
    Hint: You need not construct an explicit bijection.

    Solution: First, note that the function $f(x) = e^x$ is a one-to-one function from the set of all real numbers to the set of positive reals. Next, note that the function $g(x) = e^{-x}$ is a one-to-one function from the set of positive real numbers to the closed interval $[0, 1]$. Therefore, the function $g \circ f$ is a one-to-one function from the set of reals to $[0, 1]$. Next, note that the identity function $h(x) = x$ is a one-to-one function from $[0, 1]$ to reals. Hence, using the CSB theorem we can say that there is a bijection between the set of all real numbers and the closed interval $[0, 1]$.

11. Find the generating function $G_f(X)$ for the function f defined below:
    $$af(n-1) - bf(n) = c^n \text{ for } n \geq 1, \text{ and } f(0) = 1$$
    Then, use it to derive a closed form expression for $f(n)$, when $a = 6, b = 1, c = 3$.

    Solution: After multiplying by x^n on both sides and applying the summation from $n = 1$ to ∞, we get
    $$\begin{aligned} \sum_{n=1}^{\infty}(af(n-1) - bf(n))x^n &= \sum_{n=1}^{\infty} c^n x^n \\ ax\sum_{n=1}^{\infty} f(n-1)x^{n-1} - b\sum_{n=1}^{\infty} f(n)x^n &= (\sum_{n=0}^{\infty}(cx)^n) - 1 \\ axG_f(x) - b(G_f(x) - f(0)) &= cx/(1-cx) \\ G_f(x) &= (cx(1+b) - b)/(1-cx)(ax-b) \end{aligned}$$
    When $a = 6, b = 1, c = 3$, we get
    $$G_f(x) = (6x-1)/(1-3x)(6x-1) = 1/(1-3x) = \sum_{n=0}^{\infty} 3^n x^n$$
    Therefore, $f(n) = 3^n$ for this case.

12. Suppose the running time of a recursive algorithm satisfies the following:

$$T(n) = \sqrt{n}\, T(\sqrt{n}) + 1$$

and $T(n) = 1$ for $n \leq 2$. Give a $\Theta(n)$ bound on $T(n)$. Justify your solution in detail.

You may consider n of appropriate form, so that only integers are encountered when you unroll the recursion.

Solution: It can be seen that $n = 2^{2^k}$ for some $k \geq 0$, so that we always encounter integers upon unrolling the recursion. Then,

$$\begin{aligned}
T(2^{2^k}) &= 2^{2^{k-1}}\, T(2^{2^{k-1}}) + 1 \\
&= 2^{2^{k-1}+2^{k-2}}\, T(2^{2^{k-2}}) + 2^{2^{k-1}} + 1 \\
&\vdots \\
&= 2^{2^{k-1}+2^{k-2}+\ldots+2^0}\, T(2^{2^0}) + (k-1)2^{2^{k-1}} + (k-2)2^{2^{k-2}} + \ldots + 1 \\
&= 2^{2^k-1} + (k-1)2^{2^{k-1}} + (k-2)2^{2^{k-2}} + \ldots + 1 \\
&= 2^{2^k}\left(\frac{1}{2} + \frac{k-1}{2^{2^{k-1}}} + \frac{k-2}{2^{3.2^{k-2}}} + \ldots + \frac{k-i}{2^{2^k-2^{k-i}}} + \ldots \frac{1}{2^{2^k}}\right)
\end{aligned}$$

First of all note that by ignoring most of the terms in the summation, we get

$$T(2^{2^k}) \geq \frac{2^{2^k}}{2}$$

Secondly, since every term in the summation is upper-bounded by the first time, we get

$$\begin{aligned}
T(2^{2^k}) &\leq 2^{2^k}\left(\frac{1}{2} + \frac{(k-1)^2}{2^{2^{k-1}}}\right) \\
&\leq 2^{2^k}\left(\frac{1}{2} + \frac{1}{2}\right) \\
&\leq 2^{2^k}
\end{aligned}$$

where the second inequality follows for all large enough k, as the term in the summation is vanishing. In fact, one could have upper-bounded that term by any constant (and not just 1/2). Using the above two relations, we get

$$T(n) = \Theta(n)$$

13. Let $f(n)$ be the number of binary strings without 3 consecutive 0s. Derive a recursive definition of $f(n)$. Also, compute $f(6)$ using your recursion.

Solution: Let $a(n)$ denote the number of binary strings starting with a 0 but without 3 consecutive 0s. Similarly, let $b(n)$ define the number of such strings starting with a 1. Therefore, $f(n) = a(n) + b(n)$. For strings starting with 0, note that the second position can either be 1 or 0. But in case the second position is 0, the third position will have to be a 1. Therefore, $a(n) = b(n-1) + b(n-2)$. For strings starting with 1, the second position could be anything. Therefore, $b(n) = a(n-1) + b(n-1)$. The overall expressions become

$$a(n) = b(n-1) + b(n-2)$$
$$b(n) = b(n-1) + b(n-2) + b(n-3)$$
$$f(n) = b(n) + b(n-1) + b(n-2)$$

For $n = 6$, we get

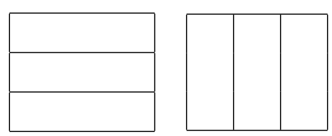
$$\begin{aligned} f(6) &= b(6) + b(5) + b(4) \\ &= b(5) + b(4) + b(3) + b(5) + b(4) \\ &= 2(b(4) + b(3) + b(2)) + 2b(4) + b(3) \\ &= 4(b(3) + b(2) + b(1)) + 3b(3) + 2b(2) \\ &= 7b(3) + 6b(2) + 4b(1) \end{aligned}$$

Note that $b(3) = 4, b(2) = 2, b(1) = 1$. Therefore, we get

$$f(6) = 7.4 + 6.2 + 4.1 = 44$$

14. Consider using 3×1 tiles to tile an $n \times 3$ rectangle. Let $f(n)$ be the number of such tilings. Give a recursive definition for $f(n)$.

Hint: Note that $f(3) = 2$, because of the two tilings shown below.



Solution: To tile an $n \times 3$ rectangle along the dimension of measure 3, there are only two options: place them breadth-wise or length-wise (as shown in the example above). In the first case, the problem reduces to tiling a $n - 1 \times 3$ rectangle after using a single tile. In the second case, it reduces to tiling an $n - 3 \times 3$ rectangle, after using 3 tiles. Therefore, the total number of tilings is given by

$$f(n) = f(n-1) + f(n-3)$$

To complete the definition, we also need to define the base cases. It can be easily seen that $f(1) = f(2) = 1, f(3) = 2$.