Problem Set 8

Released: November 16, 2021

1. Define a function $g : \mathbb{N} \to \mathbb{R}$ recursively as follows:

   - $g(1) = 1$
   - $g(n+1) = 1 + \frac{1}{g(n)}$ for all integers $n \geq 1$

   Use induction to prove that $g(n) = f(n+1)/f(n)$ for all $n \in \mathbb{Z}^+$, where $f(n)$ is the n^{th} Fibonacci number.

   Note: As n tends to infinity, $g(n)$ tends to the positive solution of the quadratic equation given by $x = 1 + \frac{1}{x}$. This number, $\frac{1+\sqrt{5}}{2} \approx 1.618$ is sometimes called the "golden ratio."

   Solution: We simply induct on n.
   Base Case: For $n = 1$, we have $g(1) = 1 = \frac{1}{1} = \frac{f(2)}{f(1)}$.

   Induction Hypothesis: For all $k \leq n$, where n is some natural number, we have that $g(k) = \frac{f(k+1)}{f(k)}$.

   Induction Step: We have

   $$g(n+1) = 1 + \frac{1}{g(n)} = 1 + \frac{f(n)}{f(n+1)} = \frac{f(n+1)+f(n)}{f(n+1)} = \frac{f(n+2)}{f(n+1)},$$

   as required. This finishes the proof.

2. Let $f(n)$ denote the n^{th} Fibonacci number. Then show (without using the closed form expression for $f(n)$) that

   (i) $f(0) - f(1) + f(2) - ... - f(2n-1) + f(2n) = f(2n-1) - 1$ where n is a positive integer.
   Solution: We induct on n.
   Base Case: For $n = 1$, we have $f(0) - f(1) + f(2) = 0 - 1 + 1 = 0 = f(1) - 1 = f(2 \cdot 1 - 1) - 1$, as required.

   Induction Hypothesis: For all $k \leq n$, where n is some natural number, the assertion holds for k.

   Induction Step: We have

   $$\begin{aligned}\sum_{i=0}^{2n+2} (-1)^i f(i) &= \sum_{i=0}^{2n} (-1)^i f(i) - f(2n+1) + f(2n+2)\\ &= f(2n-1) - 1 + f(2n+2) - f(2n+1)\\ &= f(2n-1) - 1 + f(2n)\\ &= f(2n-1) + f(2n) - 1\\ &= f(2n+1) - 1\end{aligned}$$

   as required. This proves the formula.

   (ii) $f(0)f(1) + f(1)f(2) + ... + f(2n-1)f(2n) = f(2n)^2$ where n is a positive integer.
   Solution: Again, this is an induction exercise.
   Base Case: For $n = 1$, we have $f(0)f(1) + f(1)f(2) = 0 \cdot 1 + 1 \cdot 1 = 1 = 1^2 = f(2 \cdot 1)^2$, as required.

   Induction Hypothesis: For all $k \leq n$, where n is some natural number, the assertion holds for k.

   Induction Step: We have

   $$\begin{aligned}\sum_{i=0}^{2n+1} f(i)f(i+1) &= \sum_{i=0}^{2n-1} f(i)f(i+1) + f(2n)f(2n+1) + f(2n+1)f(2n+2)\\ &= f(2n)^2 + f(2n)f(2n+1) + f(2n+1)f(2n+2)\\ &= f(2n)(f(2n) + f(2n+1)) + f(2n+1)f(2n+2)\\ &= f(2n)f(2n+2) + f(2n+1)f(2n+2)\\ &= (f(2n) + f(2n+1))f(2n+2)\\ &= f(2n+2)^2\end{aligned}$$

as desired.

3. A partition of a positive integer n is a way to write n as a sum of positive integers where the order of terms in the sum does not matter. For instance, $7 = 3 + 2 + 1 + 1$ is a partition of 7. Let $P(m)$ equal the number of different partitions of m, and let $Q(m, n)$ be the number of different ways to express m as the sum of positive integers not exceeding n.

   (i) What are the values of j such that $P(m) = Q(m, j)$ holds?
   Solution: If $j \geq m$, then $P(m) = Q(m, j)$ is certainly true; no term in a valid partition of m exceeds m anyway. On the other hand, if $j < m$, then the partition $m = m$ has a term that exceeds j, so $P(m) > Q(m, j)$.

   (ii) Show that the following recursive definition for $Q(m, n)$ is correct :

$$Q(m,n) = \begin{cases} 1 & \text{if } m = 1 \\ 1 & \text{if } n = 1 \\ Q(m,m) & \text{if } m < n \\ 1 + Q(m, m-1) & \text{if } m = n > 1 \\ Q(m, n-1) + Q(m-n, n) & \text{if } m > n > 1 \end{cases}$$

   Solution:

   - If $m = 1$, then $n \geq 1 = m$, so by the previous exercise, $Q(m, n) = P(m) = P(1) = 1$.
   - If $n = 1$, then we want to count the number of partitions of m where each part is at most 1. There is exactly one such partition for any m, which is to express it as a sum of m ones.
   - If $m < n$, then by the previous exercise $Q(m, n) = P(m) = Q(m, m)$.
   - Suppose $m = n > 1$. All partitions of m have each of the parts not exceed $m - 1$, unless the partition of m was $m = m$. Therefore $Q(m, n) = 1 + Q(m, m - 1)$.
   - We want to evaluate $Q(m, n)$. We distinguish it into two cases:
     i. Suppose no part in a partition equals n. Then, no part exceeds $n - 1$ either, which means that the number of such partitions is $Q(m, n - 1)$.
     ii. Suppose some part in a partition equals n. Remove that part. The remaining parts add to $m - n$, and no part exceeds n. The number of such partitions is $Q(m - n, n)$.

     Therefore $Q(m, n) = Q(m, n - 1) + Q(m - n, n)$ if $m > n > 1$, as required.

   (iii) Find the number of partitions of 4 and of 5 using this recursive definition.
   Solution:

$$\begin{aligned} P(5) &= Q(5,5) \\ &= 1 + Q(5,4) \\ &= 1 + Q(5,3) + Q(1,4) \\ &= 1 + Q(5,3) + 1 = 2 + Q(5,3) \\ &= 2 + Q(5,2) + Q(2,3) = 2 + Q(5,2) + Q(2,2) \\ &= 2 + Q(5,2) + 1 + Q(2,1) = 3 + Q(5,2) + Q(2,1) \\ &= 3 + Q(5,2) + 1 = 4 + Q(5,2) \\ &= 4 + Q(5,1) + Q(3,2) \\ &= 4 + 1 + Q(3,2) = 5 + Q(3,2) \\ &= 5 + Q(3,1) + Q(1,2) \\ &= 5 + Q(3,1) + 1 = 6 + Q(3,1) \\ &= 6 + 1 \\ &= 7 \\ P(4) &= Q(4,4) \\ &= 1 + Q(4,3) \\ &= 1 + Q(4,2) + Q(1,3) \\ &= 1 + Q(4,2) + 1 = 2 + Q(4,2) \\ &= 2 + Q(4,1) + Q(2,2) \\ &= 2 + 1 + Q(2,2) = 3 + Q(2,2) \end{aligned}$$

$$
\begin{aligned}
&= 3 + 1 + Q(2,1) = 4 + Q(2,1) \\
&= 4 + 1 \\
&= 5
\end{aligned}
$$

4. Let us define a permutation π of the set $\{1, ..., n\}$ to be *fragmented* if there is a number k with $1 \le k < n$ such that π maps the subset $\{1, 2, ..., k\}$ into itself. Let $c(n)$ be the number of permutations over $\{1, \ldots, n\}$ that are *not* fragmented. Prove that

$$\sum_{i=1}^{n} c(i)(n-i)! = n!$$

Suppose $G_f(X) = \sum_{n \ge 1} n! X^n$ and $G_c(X) = \sum_{n \ge 1} c(n) X^n$ are the generating functions of the functions $f(n) = n!$ and $c(n)$ respectively (defined without a constant term). Then $G_c(X) = G_f(X)/(1 + G_f(X))$.

Hint: You can rewrite the recurrence relation as $n! - c(n) = \sum_{i=1}^{n-1} (n-i)! c(i)$ and the relation to prove as $G_f(X) - G_c(X) = G_f(X) G_c(X)$.

Solution: The summation relation to be proved can be re-written as

$$\sum_{i=1}^{n-1} c(i)(n-i)! = n! - c(n)$$

The term on the right hand side of the above equation determines the total number of permutations over $\{1, \ldots, n\}$ that *are* fragmented. According to the definition of a fragmented permutation, there should exist a number k with $1 \le k < n$ such that the permutation maps $\{1, \ldots, k\}$ into itself.

If the minimum number k for which this is happening is provided, then the total number of fragmented permutations become $c(k)(n-k)!$ because we don't want to count any fragmented permutation on the first k elements as that would imply there exists another $k^{'} < k$ for which $\{1, \ldots, k^{'}\}$ maps to itself. Such a minimum k can go from 1 to $n-1$ and hence the summation follows.

The result with the generating functions can be re-written as

$$G_f(X) - G_c(X) = G_f(X) G_c(X)$$

The left hand side summation evaluates to $\sum_{n \ge 1} (n! - c(n)) X^n$ which can be re-written as $\sum_{n \ge 1} (\sum_{i=1}^{n-1} c(i)(n-i)!) X^n$ using the above result. By carefully noting the coefficient of X^n in the above summation, one should be able to see that it is equal to the coefficient of X^n in the product of $G_f(X)$ and $G_c(X)$ because of which the result follows.

5. Let $s(n)$ be the number of sequences $(x_1, ..., x_k)$ of integers satisfying $1 \le x_i \le n$ for all i and $x_{i+1} \ge 2x_i$ for $i = 1, ..., k-1$. (The length of the sequence is not specified; in particular, the empty sequence is included.) Prove the recurrence

$$s(n) = s(n-1) + s(\lfloor n/2 \rfloor)$$

for $n \ge 1$, with $s(0) = 1$. Show that the generating function $G_s(X)$ satisfies $(1 - X) G_s(X) = (1 + X) G_s(X^2)$.

Solution: Consider all sequences which contribute to $s(n)$. These include those sequences with $1 \le x_i \le n - 1$ for all i and $x_{i+1} \ge 2x_i$ for $i = 1, ..., k-1$ and the number of such sequences is $s(n-1)$. The sequences which contribute to $s(n)$ also include the sequences in which there is the number n which can only occupy the last place as it is an increasing sequence. Also, since each consecutive term is at least twice the previous term, it means that the terms other than n will be less than or equal to $\lfloor n/2 \rfloor$ and the number of such sequences are $s(\lfloor n/2 \rfloor)$. These 2 cases together prove the recurrence.

To find the relation with the generating function $G_s(X)$, let us multiply both sides of the above equation by X^n and putting the appropriate limits with summation gives us

$$
\begin{aligned}
\sum_{n \ge 1} s(n) X^n &= \sum_{n \ge 1} s(n-1) X^n + \sum_{n \ge 1} s(\lfloor n/2 \rfloor) X^n \\
G_s(X) - s(0) &= X G_s(X) + \sum_{\substack{n \ge 2 \\ n = 2k}} s(n/2) X^n + \sum_{\substack{n \ge 1 \\ n = 2k+1}} s(\frac{n-1}{2}) X^n
\end{aligned}
$$

$$
\begin{aligned}
G_s(X) - 1 &= XG_s(X) + \sum_{k\geq 1} s(k)X^{2k} + \sum_{k\geq 0} s(k)X^{2k+1} \\
G_s(X) - 1 &= XG_s(X) + G_s(X^2) - s(0) + XG_s(X^2) \\
(1-X)G_s(X) &= (1+X)G_s(X^2)
\end{aligned}
$$

6. Find the generating function $G_f(X)$ for each f below.

(a) $\forall n \geq 0,\ f(n) = n$.

Solution: Consider the series

$$\sum_{n\geq 0} X^n = \frac{1}{1-X}$$

Taking derivative on both sides,

$$\sum_{n\geq 0}(n+1)X^n = \frac{1}{(1-X)^2}$$

Multiplying both sides by X, we get

$$\sum_{n\geq 1} nX^n = \sum_{n\geq 0} f(n)X^n = \frac{X}{(1-X)^2}$$

(b) $\forall n \geq 0,\ f(n) = n^2$.

Solution: Using the result from the previous part, we get

$$\sum_{n\geq 1} nX^n = \frac{X}{(1-X)^2}$$

Taking derivative on both sides, we get

$$\sum_{n\geq 0}(n+1)^2X^n = \frac{2X}{(1-X)^3} + \frac{1}{(1-X)^2} = \frac{1+X}{(1-X)^3}$$

Multiplying both sides by X, we get

$$\sum_{n\geq 1} n^2X^n = \sum_{n\geq 0} f(n)X^n = \frac{X(1+X)}{(1-X)^3}$$

(c) $f(0) = a$, and $\forall n > 0,\ f(n) = f(n-1) + b$.

Solution: Multiplying both sides by X^n and taking limits for summation as $n \geq 1$ (to make sure that the summation is indeed well-defined), we get

$$
\begin{aligned}
\sum_{n\geq 1} f(n)X^n &= \sum_{n\geq 1} f(n-1)X^n + \sum_{n\geq 1} bX^n \\
G_f(X) - f(0) &= XG_f(X) + \frac{bX}{1-X} \\
(1-X)G_f(X) &= a + \frac{bX}{1-X} \\
G_f(X) &= \frac{a}{1-X} + \frac{bX}{(1-X)^2}
\end{aligned}
$$

(d) $f(0) = f(1) = 0$, $f(2) = 1$, and $\forall n > 2$, $f(n) = f(n-1) + f(n-2) + f(n-3)$.

Solution: Applying a similar strategy as the previous part, we get

$$\begin{aligned}
\sum_{n\geq 3} f(n)X^n &= \sum_{n\geq 3} f(n-1)X^n + \sum_{n\geq 3} f(n-2)X^n + \sum_{n\geq 3} f(n-3)X^n \\
G_f(X) - f(0) - f(1)X - f(2)X^2 &= X(G_f(X) - f(0) - f(1)X) + X^2(G_f(X) - f(0)) + X^3 G_f(X) \\
G_f(X) - X^2 &= XG_f(X) + X^2 G_f(X) + X^3 G_f(X) \\
(1 - X - X^2 - X^3)G_f(X) &= X^2 \\
G_f(X) &= \frac{X^2}{1 - X - X^2 - X^3}
\end{aligned}$$

(e) $f(0) = 0$, and $\forall n > 0$, $f(n) = 2f(n-1) + 3^n$.

Solution: As done in the previous 2 parts,

$$\begin{aligned}
\sum_{n\geq 1} f(n)X^n &= 2\sum_{n\geq 1} f(n-1)X^n + \sum_{n\geq 1} 3^n X^n \\
G_f(X) - f(0) &= 2XG_f(X) + \frac{3X}{1-3X} \\
G_f(X) &= \frac{3X}{(1-2X)(1-3X)}
\end{aligned}$$

7. If the generating functions of two functions f and g satisfy the identity $G_g(X) = G_f(X)(1-X)$, define g in terms of f.

Solution: Given that for two functions f, g, we have, $G_g(X) = G_f(X)(1-X)$. Expanding the expressions we get,

$$\sum_{i=0}^{\infty} g(i)x^i = (\sum_{i=0}^{\infty} f(i)x^i)(1-x)$$

Simplifying RHS, we get,

$$\begin{aligned}
(\sum_{i=0}^{\infty} f(i)x^i)(1-x) &= \sum_{i=0}^{\infty} f(i)x^i - \sum_{i=0}^{\infty} f(i)x^{i+1} \\
&= f(0) + \sum_{i=1}^{\infty}(f(i) - f(i-1))x^i
\end{aligned}$$

Now, for this series to be equal to that LHS, the corresponding coefficients should be equal.
Thus, we have,

$$\begin{aligned}
g(0) &= f(0) \\
g(i) &= f(i) - f(i-1)
\end{aligned}$$

This computes g in terms of f.

8. Prove that for $n \in \mathbb{Z}^+$, $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$.

Solution:
By definition, we have, $\binom{-n}{k} = \frac{(-n)(-n-1)\ldots(-n-k+1)}{k!}$. Simplifying RHS, we get,

$$\begin{aligned}
\frac{(-n)(-n-1)\ldots(-n-k+1)}{k!} &= \frac{(-1)^k(n)(n+1)\ldots(n+k-1)}{k!} \\
&= (-1)^k \frac{(n-1)!(n)(n+1)\ldots(n+k-1)}{(n-1)!k!} \\
&= (-1)^k \frac{(n+k-1)!}{(n-1)!k!} \\
&= (-1)^k \binom{n+k-1}{k}
\end{aligned}$$

Thus, we have, $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$.

9. Use the extended binomial theorem to find the coefficient of X^{10} in the power series of each of the following expressions. (Express your answers without using $\binom{n}{k}$ for any $n \notin \mathbb{Z}^+$.)

   (a) $X^4/(1-3X)^3$

   (b) $X^4/(1-X^3)$

   (c) $1/(1-X^3)$

   (d) $1/\sqrt{1-4X}$

   Solution:
   (a) By extended by binomial theorem, we have, $1/(1-3X)^3 = \sum_{k=0}^{\infty} \binom{-3}{k}(-3X)^k$. Thus, $X^4/(1-3X)^3 = \sum_{k=0}^{\infty} \binom{-3}{k}(-3)^k X^{k+4}$. For the term $X^{10}, k=6$. The corresponding coefficient is $\binom{-3}{6}3^6$. By Q8, $\binom{-3}{6} = \binom{3+6-1}{6} = \binom{8}{6}$. Thus, coefficient of X^{10} is $\binom{8}{6}3^6$.
   (b) By extended by binomial theorem, we have, $1/(1-X^3) = \sum_{k=0}^{\infty} \binom{-1}{k}(-X^3)^k$. Thus, $X^4/(1-X^3) = \sum_{k=0}^{\infty} \binom{-1}{k}(-X)^{3k+4}$. For the term $X^{10}, k=2$. The corresponding coefficient is $\binom{-1}{2}$. By Q8, $\binom{-1}{2} = \binom{1+2-1}{2} = \binom{2}{2} = 1$. Thus, coefficient of X^{10} is 1.
   (c) By extended by binomial theorem, we have, $1/(1-X^3) = \sum_{k=0}^{\infty} \binom{-1}{k}(-X^3)^k$. It can be seen that the series only contains terms with exponents of multiples of 3. Since, 10 is not a multiple of 3, coefficient of X^{10} is 0.
   (d) By extended by binomial theorem, we have, $1/\sqrt{1-4X} = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k}(-4X)^k$. For the term $X^{10}, k=10$. The corresponding coefficient is $\binom{-\frac{1}{2}}{10}(-4)^{10}$. We have, $\binom{-\frac{1}{2}}{10} = \frac{\frac{-1}{2} * \frac{-3}{2} * \cdots * \frac{-19}{2}}{10!} = \frac{19!}{2^{19}9!10!}$. Thus, coefficient of X^{10} is $\frac{19!}{2^{19}9!10!}4^{10} = 2\binom{19}{10}$.

10. For the function f recursively defined in Problem 6(c), find a closed form for it using its generating function G_f.

    Solution: The generating function G_f obtained for the function f as defined in the problem is

$$G_f(X) = \frac{a}{1-X} + \frac{bX}{(1-X)^2}$$

    Using the Extended Binomial Theorem, we can obtain the following relations:

$$\frac{1}{1-X} = \sum_{k \geq 0} X^k$$
$$\frac{1}{(1-X)^2} = \sum_{k \geq 0} (k+1)X^k$$

    Therefore, we can write

$$\begin{aligned} G_f(X) &= \sum_{k \geq 0} aX^k + \sum_{k \geq 0} b(k+1)X^{k+1} \\ &= a + \sum_{k \geq 1} aX^k + \sum_{k \geq 1} bkX^k \\ &= a + \sum_{k \geq 1} (a+bk)X^k \end{aligned}$$

    Using the expansion of the left hand side, we can say that $f(k) = a + bk$ for all $k \geq 0$.

11. Find the closed form expression for the n^{th} Fibonacci number.

    Solution: We know, that for n^{th} Fibonacci number, $f(n)$ satisfy, $f(n) = af(n-1) + bf(n-2)$, with $a = 1, b = 1$. Thus, as seen in class, $f(n) = px^n + qy^n$, for some p, q and x, y being solutions to $X^2 - X - 1 = 0$.

    Now, we try to find these parameters. Solutions to $X^2 - X - 1 = 0$ are $\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$. Thus, $f(n) = p(\frac{1+\sqrt{5}}{2})^n + q(\frac{1-\sqrt{5}}{2})^n$. Since, we have, $f(0) = 0, f(1) = 1$. We get, equations,

$$p + q = 0 \tag{1}$$
$$p(\frac{1+\sqrt{5}}{2}) + q(\frac{1-\sqrt{5}}{2}) = 1 \tag{2}$$

    Solving them, we get, $p = \frac{1}{\sqrt{5}}, q = -\frac{1}{\sqrt{5}}$

    Thus, closed form of n^{th}-Fibonacci number is given as, $\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$.