CS 228 : Logic in Computer Science

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Summary

- Started looking at FO nondefinability
- Defined maximal quantifier depth or quantifier rank of a formula
- Showed that there are finitely many FO formulae of rank r
- Introduced some new notations for words, mimicking assignments of values to free variables

Logical Equivalence

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- $\sim_r$ is an equivalence relation
- Finitely many equivalence classes : each class consists of words that behave the same way on formulae of rank $\leqslant r$

$\sim_r$ Equivalence Classes

- Let there be k formulae $\varphi_1, \ldots, \varphi_k$ of rank 0, ℓ formulae $\psi_1, \ldots, \psi_\ell$ of rank 1 and m formulae $\chi_1, \ldots, \chi_m$ of rank 2. As an example, consider w_1, w_2 s.t.
- $w_1, w_2 \models \varphi_{2i+1}, w_1, w_2 \not\models \varphi_{2i+2}$ for $0 \leqslant i \leqslant \frac{k}{2} - 1$,
- $w_1, w_2 \models \psi_j$ for all j
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Non-Expressibility in FO : The Game Begins

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- Duplicator wants to show that they are same ($w_1 \sim_r w_2$)
- Each player has r pebbles $z_1, \ldots, z_r$

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- The game ends after r rounds, when both players have used all their pebbles

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  - Spoiler : $(a, \{z_1\})(b, \emptyset)(a, \{z_2\})$
  - Duplicator : $(a, \{z_1, z_2\})(b, \emptyset)$ or $(a, \{z_1\})(b, \{z_2\})$

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- That is, $w_1' \sim_0 w_2'$
- Spoiler wins otherwise.

Winner

Given two word structures (w_1, w_2), duplicator wins on (w_1, w_2) if for every atomic formula α, $w_1 \models \alpha$ iff $w_2 \models \alpha$

Play continues

- Who won in the earlier play?
- We had
  - $(a, \{z_1\})(b, \emptyset)(a, \{z_2\})$ and $(a, \{z_1, z_2\})(b, \emptyset)$
  - $(a, \{z_1\})(b, \emptyset)(a, \{z_2\}) \models (z_1 < z_2)$
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- Spoiler wins in two rounds
- If the game was played only for one round, who will win?

Unique Winner

Given structures w_1, w_2, and a number of rounds r, exactly one of the players win.

Logical Equivalence and Winning

Let w_1, w_2 be $\mathcal{V}$-structures and let $r \geqslant 0$. Then $w_1 \sim_r w_2$ iff Duplicator has a winning strategy in the r-round game on (w_1, w_2).

Logical Equivalence and Winning

Assume $w_1 \sim_r w_2$, and induct on r

- Base : $r = 0$ and $w_1 \sim_0 w_2$. Duplicator wins, since by assumption, w_1, w_2 agree on all atomic formulae.

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- Assume for $r - 1$: $w_1 \sim_{r-1} w_2 \Rightarrow$ Duplicator has a winning strategy in a $r - 1$ round game

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- Now, let $w_1 \sim_r w_2$, and assume spoiler wins the r-round game on (w_1, w_2).
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  - In response, duplicator places her pebble somewhere on w_2

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  - By inductive hypothesis, $w_1' \nsim_{r-1} w_2'$

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  - Then $w_1' \models \psi$, $w_2' \not\models \psi$
  - We thus have

$$w_1 \models \exists z_1 \psi, w_2 \not\models \exists z_1 \psi$$

contradicting $w_1 \sim_r w_2$

Logical Equivalence and Winning : Converse

Assume Duplicator wins r-round game on (w_1, w_2) and induct on r

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- Now, let duplicator win in the r round game, but $w_1 \not\sim_r w_2$.
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  - In reply, duplicator keeps pebble z_1 on w_2 obtaining w_2'

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  - $w_1 \not\sim_r w_2 \Rightarrow$ there is some formula ψ, $c(\psi) = r$ such that $w_1 \models \psi$, $w_2 \not\models \psi$
  - Assume $\psi = \exists z_1 \varphi$. Then $c(\varphi) = r - 1$
  - Since $w_1 \models \exists z_1 \varphi$, spoiler can keep pebble z_1 somewhere in w_1 obtaining w_1' satisfying φ
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  - That is, either both w_1', w_2' satisfy φ, or both dont, a contradiction.

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- Consider the formula

$$[\psi_{v_1,w_1} \wedge \psi_{v_1,w_2} \wedge \cdots \wedge \psi_{v_1,w_n} \wedge \dots]$$

$$\vee$$

$$[\psi_{v_2,w_1} \wedge \psi_{v_2,w_2} \wedge \cdots \wedge \psi_{v_2,w_n} \wedge \dots]$$

$$\vee$$

$$\vdots$$

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- Hence the disjunction and conjunction are finite
- ψ_L is a proper formula (of finite size)
- ψ_L captures L since each $v \in L$ satisfies $\bigwedge_{w \notin L} \psi_{vw}$ while none of the $w \notin L$ satisfy $\bigwedge_{w \notin L} \psi_{vw}$

FO-definable languages

Given a property $\mathcal{K}$, if for any pair $v \in \mathcal{K}$ and $w \notin \mathcal{K}$, spoiler has a winning strategy in the k-round EF game on v and w, then there is a rank k FO formula $\varphi_{\mathcal{K}}$ that defines the property $\mathcal{K}$.

$$\varphi_{\mathcal{K}} = \bigvee_{v \in \mathcal{K}} \bigwedge_{w \notin \mathcal{K}} \psi_{vw}$$

where ψ_{vw} is as explained in the previous slide.

- Note that k is fixed in the above, and is independent of the choices of the words.

Implications of the Game on FO definability

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Non FO Definability

For all $r \geqslant 0$, there exists a (w_1, w_2) pair with $w_1 \notin L, w_2 \in L$, duplicator wins in r rounds $\Rightarrow$ L is not FO definable