CS 228 : Logic in Computer Science

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Recap

- Given FO formula φ, build an automaton A_φ preserving the language
- Satisfiability of FO reduces to non-emptiness of underlying automaton
- Starting today : non FO-definability

FO Definability

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- Quantifier free formulae written in DNF : $C_1 \vee C_2 \vee \cdots \vee C_n$

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- $c(\exists\varphi) = c(\varphi) + 1$
- Quantifier free formulae written in DNF : $C_1 \vee C_2 \vee \cdots \vee C_n$
- Formulae of quantifier rank $k + 1$ written as a disjunction of the conjunction of formulae, each formula of the form $\exists x\varphi, \neg\exists x\varphi$ or φ, with $c(\varphi) \leqslant k$. Eliminate repeated disjuncts/conjunts
- $(\exists x\varphi_1 \wedge \exists y\varphi_2) \vee (\neg\exists z\varphi_3)$, $c(\varphi_1), c(\varphi_2), c(\varphi_3) \leqslant k$.

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- All possible disjuncts using each C_i : formulae in DNF of rank 0

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- $2m$ atomic/negated atomic formulae
- Number of conjunctions C_i possible $\leqslant 2^{2m}$
- Number of formulae in DNF $\leqslant 2^{2^{2m}}$ $(c = 0)$

Rank 1

Let there be p formulae φ of rank 0.

- $2p$ formulae of the form $\exists x \varphi, \neg \exists x \varphi$
- 2^{2p} conjunctions of rank 1
- Conjuncting any one of the p formulae of rank 0 gives all conjuncts of rank 1 : $p2^{2p}$ more
- Possible conjuncts of rank 1 is $q = (p+1)2^{2p}$
- Possible disjuncts of these : 2^q

Number of FO formulae of rank c

Let $\mathcal{V}$ be a finite set of first order variables, and let $c \geqslant 0$. There are finitely many FO formulae in DNF with rank c over $\mathcal{V}$.

Some Notation

Given a word $w = a_1 \ldots a_n$, and a finite set of variables $\mathcal{V}$, define a $\mathcal{V}$-enriched-word with respect to w as

- $(a_1, U_1)(a_2, U_2) \ldots (a_n, U_n)$ where
- $\bigcup_i U_i = \mathcal{V}$
- $U_i \cap U_j = \emptyset$

- A $\mathcal{V}$-enriched-word is over the alphabet $\Sigma \times 2^{\mathcal{V}}$
- $(a, \{x\})(b, \{y, z\})(c, \emptyset)(d, \{u, v\})$ is a $\{x, y, z, u, v\}$-enriched word with respect to the word *abcd*.
- We will refer to $\mathcal{V}$-enriched-word structures as $\mathcal{V}$-structures from here on

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- $w \models x < y$ iff there exists $i < j$ such that $x \in S_i, y \in S_j$
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  - $(a, \{x\})(b, \{y, z\})(c, \emptyset) \models x < y$
- $w \models \exists x Q_a(x)$ iff there exists i such that $(a_1, S_1) \ldots (a_i, S_i \cup \{x\}) \ldots (a_n, S_n) \models Q_a(x)$
  - $(b, \{y, z\})(a, \{u\})(c, \emptyset) \models \exists x Q_a(x)$ since $(b, \{y, z\})(a, \{x, u\})(c, \emptyset) \models Q_a(x)$

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Similarly, $(a,\emptyset)(a,\{x\})(b,\{y\}) \models (Q_a(x) \rightarrow [(x < y) \wedge Q_b(y)])$ and $(a,\emptyset)(a,\emptyset)(b,\{x,y\}) \models (Q_a(x) \rightarrow [(x < y) \wedge Q_b(y)])$