CS 228 Second Half. Tutorial 1

a) $\Sigma = \{a, b\}$

equal "ab" and "ba"

$\Longleftrightarrow$ starts and ends with same letter

$$\cancel{\varphi_a}\,\phi(\text{first}) \equiv \exists x . [\underbrace{\forall y \; x \leq y}_{\text{first: existence}} \wedge \underbrace{\cancel{\varphi_a}\,\phi(x)}_{\text{property}}]$$

$$\varphi_a(\text{last}) \equiv \exists x [\forall y \; y \leq x \wedge \varphi_a(x)]$$

$$(\varphi_a(\text{first}) \wedge \varphi_a(\text{last})) \vee (\varphi_b(\text{first}) \wedge \varphi_b(\text{last}))$$

FO-definable $\subset$ Regular languages
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\parallel$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ MSO-definable

defined in terms of automata

will see later in course.

probably seen before.

- can also define in terms of regex
- can also define in terms of algebra, i.e. monoids

(most likely beyond CS 228 scope)

(Maybe not in CS 228)

Given a regular language, can run an algorithm to check if its FO definable.

↪ this algo can be constructive, but very costly, i.e. size blowup.

—x—

b) Single # *

everything before: a *

" after: b. *

$$\exists x \big(\varphi_{\#}(x) \wedge \forall y.\ y<x \Rightarrow \varphi_a(x)$$
$$\wedge\ \forall z.\ x<z \Rightarrow \varphi_b(x) \big)$$

Implicit in the formulae we write: We have clause

$\forall x.$ exactly one $(\varphi_a(x), \varphi_b(x))$

c) No occurrence of "ba" $\{a,b\}$

$$\forall x,y.\ S(x,y) \wedge \varphi_b(x) \Rightarrow \varphi_b(y)$$

Could try to do it like Q2

but be careful because

1) Empty word needs to satisfy

2) word with all a's needs to satisfy

Empty word is FO definable : ε

$\forall x.\ x \neq x$ Vacuously true for empty word structure

↑ the language of this sentence is $\{\varepsilon\}$

What is the language of

$\exists x.\ x \neq x$

There's no word structure for which this holds.

$\{\}$

d) $\phi(\text{second})$

$\exists x_2 \left(\underbrace{\exists x_1 . (\forall y.\ x_1 \leq y}_{x_1 \text{ is first}} \wedge \underbrace{S(x_1, x_2)}_{x_2 \text{ is second}} \right) \wedge \phi(x$

$\phi(\text{second\_last})$

$\exists x_2 \exists x_1 \forall y.\ y \leq x_1 \wedge S(x_2, x_1) \wedge \phi(x_2)$

$\varphi_0(\text{second}) \wedge \varphi_0(\text{second\_last})$

Count up to n^{th} position

$\exists x_n, x_{n-1}, \ldots x_1\ \forall y.\ x_1 \leq y \wedge S(x_1, x_2) \wedge S(x_2, x_3)$
$\ldots \wedge S(x_{n-1}, x_n)$

Purely with $\leq$: something like

$\forall y\ x_1 \leq y \wedge y \neq x_1 \Rightarrow x_2 \leq y$

e) $\Sigma = \left\{ \binom{0}{0}, \binom{0}{1}, \binom{1}{0}, \binom{1}{1} \right\}$

$w \in \Sigma^*$. top row is bigger.

1) exists a $\binom{1}{0}$ somewhere

2) before this, only $\binom{0}{0}$ and $\binom{1}{1}$

$$\exists x. \left(\varphi_{\binom{1}{0}}(x) \wedge \forall y.\ y < x \Rightarrow \left(\varphi_{\binom{0}{0}} y \vee \varphi_{\binom{1}{1}} y \right) \right)$$

—x—

2) FO ⊂ regular

regular languages are closed under

i) complement [given a DFA, flip accepting and non-accepting states]

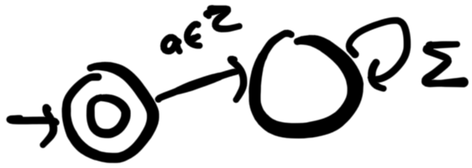
ii) union } Product of DFA

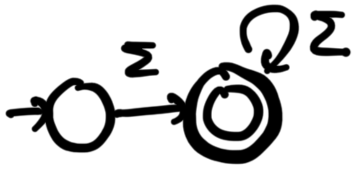
iii) intersection } wait for it in lectures.

(i) $\forall x\ (x \neq x)$

$\{\varepsilon\}$ $\mathcal{L}(\varphi)$

$\overline{\mathcal{L}(\varphi)}$: all non-empty words





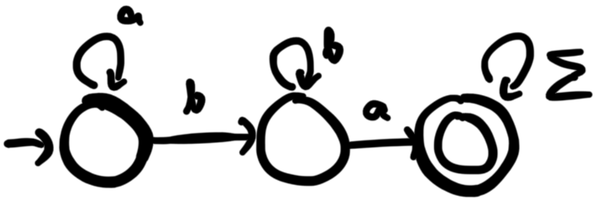
(2) $\exists x \exists y [x<y \wedge Q_b(x) \wedge Q_a(y)$

$\wedge \forall z (x<z<y \Rightarrow Q_a(z))$

~ baaa a ~

Case

$\Sigma^* b a^* a \Sigma^*$



There is an occurrence of ba

then for this consecutive pair, yellow clause is vacuously true

(3)

$\exists x [Q_a(x) \wedge \exists y [S(x,y) \wedge \forall z [z \leq y]]]$

x is the second last

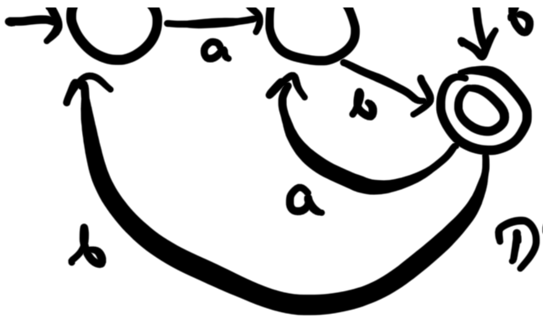
y is the last position.

There is an 'a' in second last position

$\Sigma^* a \Sigma$

...ab

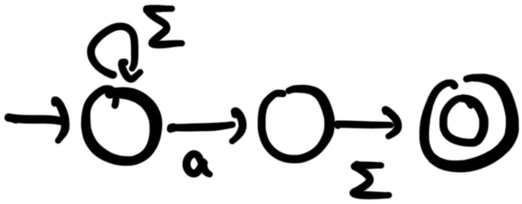
...aa



DFA

...aba

...abb



NFA

(4)

$$\exists x \forall y [x \leq y \wedge Q_a(x)]$$

first letter is an a

$$\wedge \ \exists x \forall y [y \leq x \wedge Q_b(x)]$$

last letter is a b

$$\wedge \ \forall x \forall y [Q_a(x) \wedge S(x,y) \Rightarrow Q_b(y)]$$

every "a" is immediately followed by a "b"

$$\wedge \ \forall x \forall y [Q_b(x) \wedge S(x,y) \Rightarrow Q_a(y)]$$

every "b" is immediatly followed by an "a"

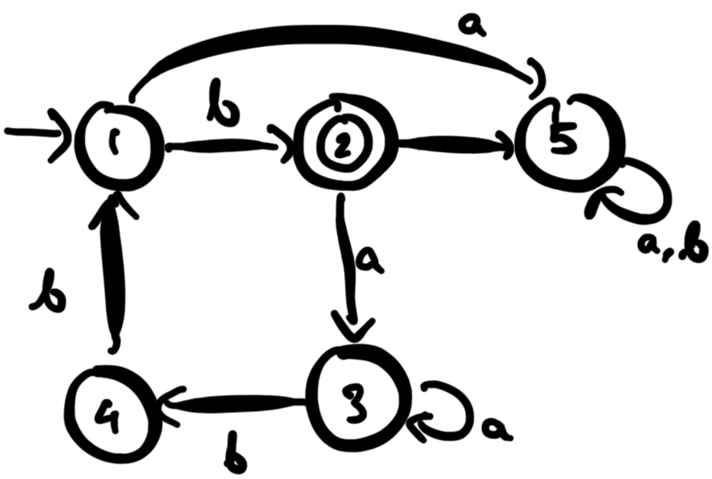
"ab" repeated a nonzero number of times

because of existential, empty word doesn't satisfy.

DFA

$ab(ab)^*$

Sink state. if you're here, your chances of matching the reqd pattern are irreparably ruined.



What exactly is going on

Key: Track what leads u to state "2"

Not DFA. No transition from "4" on reading "a"

But NFA.

As will see in course NFA are just as expressive as DFA, albeit more compact.

The only thing(s) going to 2 are

1) b

2) $b(aa^*bbb)^*$

b
repeat some finite $n \geq 0$ times
    a
    repeat some finite $n' \geq 0$ times
        a
    b
    b
    b

Eg. babbb, baabbbaaabbb

1) Starts with a "b"

2) If length > 1, then ends with "bbb"

3) For every "bbb", there is a nonzero block of "a" just before, and before that is the starting "b," or another "bbb"

1) $\exists x. \forall y. x \leq y \wedge Q_b(x)$

2) $\exists x, y\ S(x,y) \Rightarrow \exists x, y, z.$
$$\forall w.\ w \leq z \wedge S(x,y) \wedge S(y,z) \wedge Q_b(x) \wedge Q_b(y) \wedge Q_b(z)$$

3) $\forall x_1, x_2, x_3.$
$$S(x_1,x_2) \wedge S(x_2,x_3) \wedge Q_b(x_1) \wedge Q_b(x_2) \wedge Q_b(x_3)$$ for every "bbb"

$\Rightarrow \exists y_1$ there is a position y_1 such that

1) $\exists z.\ y_1 < z < x_1$ — there are positions b/w y_1 and x_1

$\wedge$

2) $\forall z.\ y_1 < z < x_1 \Rightarrow Q_a(x)$ — all these have "a"

$\wedge$

3) $Q_b(y_1)$ — y_1 is marked with "b"

$\wedge$

4) $(\ \forall z\ y_1 \leq z\ \vee$ — y_1 is either first position or

5) $\exists y_2, y_3.\ S(y_3, y_2) \wedge S(y_2, y_1) \wedge Q_b(y_1) \wedge Q_b(y_2) \wedge Q_b(y_3)$ — y_1 is the last of 3 consec. b's

Why can't there be 4 consecutive b's?

bbbb

$z \quad x_1 \quad x_2 \quad x_3$

Regardless of y_1

not marked with "a"

condition 2 violated

Point 2 asserts word ends in bbb.

Point 3 ensures that the pattern continues to be matched, by traversing word in RIGHT to LEFT manner.

Note: I'm sorry. The recording is

incomplete because I didn't realise I

incomplete, because I didn't realise I got disconnected. Besides, there are a bunch of bloopers (the academic content is fine, just some unnecessary stoppages, and possibly a rather loud amateur Bollywood concert in the background.)