Problem Set 7b

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1. **Matching Number.** For a graph G, its *matching number* is the size of a maximum matching in G. For each of the following graphs, compute its matching number: C_5, K_5, W_5, $K_{4,5}$.

   Solution: Suppose a graph G has n vertices. Clearly, the number of independent edges cannot exceed $\frac{n}{2}$; if there are $> \frac{n}{2}$ edges, then by pigeonhole principle, some two of them share a vertex. Therefore the matching number of a graph cannot exceed $\frac{n}{2}$.

   Consider C_5. The matching number cannot exceed 2. Also, if the graph can be written as $v_1v_2v_3v_4v_5$, then the set $\{v_1v_2, v_4v_5\}$ forms an independent set. Therefore the matching number of C_5 is 2. Also, it can be shown that the matching number of C_n is $\lfloor \frac{n}{2} \rfloor$.

   The matching number for K_5 is 2 again; the same subgraph mentioned for C_5 works here too. Also, it can be shown that the matching number of K_n is $\lfloor \frac{n}{2} \rfloor$.

   W_5 is the wheel graph on 6 vertices. The matching number for W_5 is at most 3. Now, if the cycle in W_5 is $v_1v_2 \cdots v_5$, and the central vertex is x, Then $\{xv_1, v_2v_3, v_4v_5\}$ is an independent edge set. So the matching number of W_5 is 3. More generally, W_n, the wheel graph with $n+1$ vertices, has matching number $\lfloor \frac{n+1}{2} \rfloor$.

   Let the 2 vertex sets of $K_{4,5}$ be V_1 and V_2 such that $|V_1| = 4$ and $|V_2| = 5$. We start off by noting that the matching number cannot exceed 4. Now, if the vertices in V_1 are $\{a_1, a_2, a_3, a_4\}$ and the vertices in V_2 are $\{b_1, b_2, b_3, b_4, b_5\}$, then $\{a_1b_1, a_2b_2, a_3b_3, a_4b_4\}$ forms a valid independent edge set. Therefore 4 is the matching number of $K_{4,5}$. In fact, it is easy to show that the matching number of $K_{m,n}$ is $\min(m, n)$.

2. How many different perfect matchings exist in each of the following graphs.

   (a) C_{2n}

   (b) $K_{n,n}$

   (c) K_{2n} *Hint: Any ordering of the $2n$ vertices as $v_1, v_2, \ldots, v_{2n-1}, v_{2n}$ can be interpreted as describing a matching, consisting of all edges of the form $\{v_{2i-1}, v_{2i}\}$. Account for the number of different orderings which result in the same matching.*

   (d) W_{2n-1}

   Solution:

   (a) Pick a vertex v in the cycle. It can be matched to either of its two neighbours. Once this matching is determined, the remaining $2n-2$ vertices form a path, and there is only one possible perfect matching. So the answer to the problem is 2.

   (b) If the vertices in the two partite sets U and V are $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$, then any matching corresponds to a permutation σ of the first n elements, where the matching has edges of the form $\{u_i, v_{\sigma(i)}\}$. Conversely, any such permutation σ yields a perfect matching. Therefore, the number of perfect matchings in $K_{n,n}$ is $n!$.

   (c) Suppose we order the $2n$ vertices in some manner; say $v_1, v_2, \cdots, v_{2n-1}, v_{2n}$. Any such ordering acts as a description of a matching $\{v_1, v_2\}, \{v_3, v_4\}, \cdots, \{v_{2n-1}, v_{2n}\}$. This should give $(2n)!$ matchings. However, any matching might be counted multiple times in this scheme.

   More precisely, if we change the ordering among v_{2i-1} and v_{2i}, the matching remains the same. For example, if $n = 3$, then $v_1, v_2, v_3, v_4, v_5, v_6$ and $v_1, v_2, v_4, v_3, v_6, v_5$ both yield the same matching. Furthermore, if we permute the vertices such that the n pairs $\{v_{2i-1}, v_{2i}\}$ remain the same, even then the matching remains the same. For example, $v_1, v_2, v_3, v_4, v_5, v_6$ and $v_3, v_4, v_5, v_6, v_1, v_2$ yield the same matching. In all, the matching remains same by either a permutation of the n sets $\{v_{2i-1}, v_{2i}\}$, or by a permutation within each 2-element set $\{v_{2i-1}, v_{2i}\}$. Therefore, the answer is $\frac{(2n)!}{2^n \cdot n!}$.

   (d) The vertex in the centre can be matched to any of the other $2n-1$ vertices. Once this is decided, there is a path left, which has a unique perfect matching. Therefore, the number of matchings of W_{2n-1} is $2n-1$.

3. A pack of $m \times n$ cards with m *values* and n *colours* consists of one card of each value and colour. The cards are arranged in an array with n rows and m columns, such that no two cards in a column have the same colour. Then, show that there exists a set of m cards, one in each column, such that they all have distinct values.

   Solution: Consider any arrangement of mn cards into n rows, m columns with no two colours in the same column. Since, the total number of distinct colours in the cards is n, every column of the arrangement must contain all the distinct n colours. Otherwise, by PHP, there will be at least two cards with same colour, which is a contradiction. Now, consider cards in the arrangement with a particular colour, say c. Since, there are m cards with this colour and exactly one card of this colour in a column, it follows that they are all in different columns. Also, since all these cards have the same colour c, they must have different values. Thus the desired set of cards exists.

4. A *doubly stochastic matrix* is a square matrix with non-negative real numbers such that the entries in each row adds up to 1, as does the entries in each column. A *permutation matrix* is a square matrix in which each row and each column has a single entry that is *equal to* 1, and all the other entries are 0. Show that any doubly stochastic matrix M can be written as $M = p_1Q_1 + \ldots p_tQ_t$, where $Q_1, \ldots, Q_t$ are permutation matrices and $p_1, \ldots, p_t$ are positive real numbers that add up to 1.

   Hint: It is enough to show that $M = \sum_i p_iQ_i$ for some positive real numbers p_i and permutation matrices Q_i. (Argue separately that $\sum_{i=1}^t p_t = 1$ must hold.) A special case of this problem was solved in the lecture, when M was the adjacency matrix of a d-regular bipartite graph (scaled by $1/d$, to make it doubly stochastic). Here, use induction on the number of positive entries in M. As base case, consider an $n \times n$ doubly stochastic matrix with exactly n non-zero entries.

   Solution: We induct on the number of non-zero entries in the doubly stochastic matrix. Note that if the number of non-zero entries is less than n, then some row has no non-zero entry. This would mean that the common sum of the matrix is 0, so all entries of the matrix are 0, which is a contradiction. So we assume that the number of non-zero entries in the matrix is at least n.

   For any doubly-stochastic $n \times n$ matrix M, define the bipartite graph $G_M = (R, C, E)$ where $R = \{r_1, \cdots, r_n\}$, $C = \{c_1, \cdots, c_n\}$ (formally, we may let $R = \{0\} \times [n]$, $C = \{1\} \times [n]$, $r_i = (0, i)$ and $c_j = (1, j)$) and $E = \{\{r_i, c_j\} \mid M_{ij} > 0\}$.

   Claim: For any doubly stochastic matrix M, the graph G_M has a perfect matching.

   Proof: We shall use Hall's theorem to prove that there is a perfect matching in $G_M = (R, C, E)$. Consider any $S \subset R$, and let $T = \Gamma(S)$. We claim that $|T| \geq |S|$.

   Now, let us count the sum of entries $W = \sum_{r_i \in S, c_j \in T} M_{ij}$ in two different ways. Firstly,

   $$W = \sum_{r_i \in S, c_j \in C} M_{ij} = |S|$$

   where the first equality follows from the fact that if $c_j \notin T = \Gamma(S)$ then in particular, $\{r_i, c_j\} \notin E$ and hence $M_{ij} = 0$; the second equality follows from stochasticity (specifically, for all r_i, $\sum_{c_j \in C} M_{ij} = 1$).

   On the other hand

   $$W \leq \sum_{r_i \in R, c_j \in T} M_{ij} = |T|$$

   where the inequality holds because $S \subseteq R$ and the equality follows from stochasticity. Putting these together, $|S| = W \leq |\Gamma(S)|$, as required. Hence G_M has a perfect matching, as claimed.

   Let us go back to the original question. We wanted to proceed via (strong) induction.

   Base Case: The number of non-zero entries in M is n.

   Clearly, no row and no column has all entries zero, as the common sum is 1. Therefore, each row and each column has exactly one non-zero entry, so M itself is a permutation matrix.

   Induction Step: Let M be a doubly-stochastic matrix with $k > n$ non-zero entries. From the claim, G_M has a perfect matching. This perfect matching corresponds to n entries in M, none of which share a row or a column. Suppose the smallest of these n entries is p. Consider the permutation matrix Q, which has a 1 as an entry wherever these n entries in M reside. Then, the matrix $M - pQ$ has the following properties:

   - All entries of $M - pQ$ are non-negative.
   - The sum of each row and each column of $M - pQ$ is $1 - p$.
   - $M - pQ$ has at least one less non-zero entry than M. This is because p was the smallest entry among the n entries we considered, so $M - pQ$ has a 0 in its place.

If $p = 1$, then we are already done, as then $M = Q$. Otherwise, Consider the doubly stochastic matrix $M' = \frac{M-pQ}{1-p}$, which has at least one non-zero entry less than M. By induction hypothesis,

$$M' = \sum_{i=1}^{t} p_i Q_i$$

for positive reals p_i and permutation matrices Q_i. But then

$$M = pQ + \sum_{i=1}^{t} p_i(1-p)Q_i$$

is also of the required form, because $p > 0$, $p_i(1-p) > 0$ for all i, and

$$p + \sum_{i=1}^{t} p_i(1-p) = p + (1-p)\sum_{i=1}^{t} p_i = p + (1-p) = 1$$

This ends the proof.

5. Let $G = (X, Y, E)$ be a bipartite graph such that $\deg(x) \geq 1\ \forall\ x \in X$ and $\deg(x) \geq \deg(y)\ \forall\ (x, y) \in E$ where $x \in X$ and $y \in Y$. Show that G has a complete matching from X into Y.

Solution: To every edge $e = \{x, y\}$ in the graph, we assign it a weight $\frac{1}{\deg(x)}$. Consider any subset S of X. We verify Hall's condition on the problem by counting the sum, W, of the weights of the edges across S and $\Gamma(S)$.

To start with, every $x \in S$ has all neighbours in $\Gamma(S)$, by definition of neighbourhood. Therefore

$$\begin{aligned}
W &= \sum_{x \in S, y \in \Gamma(S)} \frac{1}{\deg(x)} \\
&= \sum_{x \in S} \sum_{y \in \Gamma(S), xy \in E(G)} \frac{1}{\deg(x)} \\
&= \sum_{x \in S} \sum_{xy \in E(G)} \frac{1}{\deg(x)} \\
&= \sum_{x \in S} \deg(x) \cdot \frac{1}{\deg(x)} \\
&= \sum_{x \in S} 1 \\
&= |S|
\end{aligned}$$

On the other hand, it is also true that

$$\begin{aligned}
W &= \sum_{x \in S, y \in \Gamma(S)} \frac{1}{\deg(x)} \\
&= \sum_{y \in \Gamma(S)} \sum_{x \in S, xy \in E(G)} \frac{1}{\deg(x)} \\
&\leq \sum_{y \in \Gamma(S)} \sum_{xy \in E(G)} \frac{1}{\deg(x)} \\
&\leq \sum_{y \in \Gamma(S)} \sum_{xy \in E(G)} \frac{1}{\deg(y)} \\
&= \sum_{y \in \Gamma(S)} \deg(y) \cdot \frac{1}{\deg(y)} \\
&= \sum_{y \in \Gamma(S)} 1 \\
&= |\Gamma(S)|
\end{aligned}$$

Therefore $|S| \leq |\Gamma(S)|$ for all $S \subseteq X$. By Hall's Theorem, there is a complete matching in G from X to Y.

6. Suppose that $G = (X, Y, E)$ is a bipartite graph. For each $S \subseteq X$, define $\text{shrinkage}(S) = \max\{0, |S| - |\Gamma(S)|\}$. Show that the the size of the largest subset of X which has a complete matching into Y is $|X| - \max_{S\subseteq X} \text{shrinkage}(S)$.

   Hint: Hall's theorem is a special case when $\text{shrinkage}(S) = 0$ for all $S \subseteq X$. To prove the above general formulation, if the largest subset of X which has a complete matching is of size $|X| - t$, consider applying Hall's theorem to a larger graph formed by adding t new vertices to Y which are all connected to every vertex in X.

   Solution: Call the maximum among all shrinkages to be the *deficit* d, that is, define

   $$d = \max_{S\subseteq X} \text{shrinkage}(S)$$

   This means that for every $S \subseteq X$, $|\Gamma(S)| \geq |S| - d$, by definition.

   First, we show that there exists a subset of X which has a complete matching into Y with size at least $|X| - d$. For this, consider a new bipartite graph G', that adds d new vertices to Y. Each of these d new vertices is connected to every vertex in X. In this new graph G', it is easy to see that for any $S \subseteq X$, we have

   $$|\Gamma_{G'}(S)| = |\Gamma_G(S)| + d \geq (|S| - d) + d = |S|$$

   where we add indices G and G' to avoid confusion. By Hall's Theorem, G' has a matching that saturates all vertices of X. In this matching, remove the edges that are joined to any of the d new vertices added. This gives a matching in G of size at least $|X| - d$, as required.

   Now, we show that $|X| - d$ is the best we can do. Assume the contrary, that is, suppose the maximum matching has size $|X| - k$ for some $k < d$, and this subset of X belonging to this matching is X'. Now, let S be any subset of X. Write S as $S = (S \cap X') \cup (S - X')$; note that this is a disjoint union. Now, every vertex in $S \cup X'$ is in X', so it has a corresponding matching vertex in Y from the maximum matching. Also, $|S - X'|$ cannot exceed k, because X' has $|X| - k$ elements. Therefore,

   $$|\Gamma(S)| \geq |\Gamma(S \cap X')| \geq |S \cap X'| = |S| - |S - X'| \geq |S| - k$$

   which means that the deficit is at most $k < d$, a contradiction.

7. Let $G = (X, Y, E)$ be a bipartite graph. Suppose that $S \subseteq X$, $T \subseteq Y$ such that G has a complete matching from S to Y, and also has a complete matching from T to X. Prove that there exists a matching which contains a complete matching from S to Y as well as a complete matching from T to X.

   Hint: Start with an arbitrary complete matching from T to X. Can you extend this to a matching which contains a complete matching from S to Y?

   Solution: Let M_1, M_2 be arbitrary complete matchings from S to Y, T to X respectively. Consider the edges in the graph $G' = M_1 \cup M_2$. Since, each vertex has a degree atmost 1 in either of M_1, M_2, each vertex has a degree of 2 in G'. Now, we will find a matching M in G' covering each vertex in $S \cup T$. It can been seen that if such a matching exists then it contains the required matchings.

   Since, each vertex has atmost degree 2 in G', we conclude that G' can be a collection of isolated vertices, isolated edges, cycles or chains and has no other configuration of edges. Let v be any vertex in $S \cup T$. Clearly, it has at least degree 1 in G'. Consider the sub graph of G' containing v. If it is an isolated edge, including it in M covers v. If it is a cycle, it has be an even cycle (as G' is bipartite). Thus, taking alternating edges of the cycle in M cover all the vertices in the sub graph and also v. Similar subgraphs can be constructed for odd-length (edge-wise) chains by taking alternating edges.

   The only case where we can't assign an edge is when the sub graph containing v is an even-length chain with both the end points belonging to $S \cup P$. We claim that this can't happen and prove this by contradiction. Suppose, there is a connected-component of G' which is an $2n$ even-length chain $a, c_0, c_1, ..., c_{2*n-2}, b$ with a, b belonging to $S \cup T$. W.l.g, let $a \in S$. Then, the path from a to b must be an alternating sequence between M_1, M_2 starting with M_1 from a and ending with M_2 at b. Since, it is of even length, a, b will be on same bipartite set, hence, $b \in S$. Thus, there will be an edge in M_1 covering b, but this can't be the edge ending in the path as then, two edges from M_1 would have a common vertex, which is a contradiction. Thus, in all cases, we can find a matching M from X to Y , which contains all vertices from $S \cup T$. Hence, proved.

8. Given a graph $G = (V, E)$, its *line graph* (or "edge graph") $L(G)$ is defined as the graph whose vertices are the edges of G, and two such vertices are connected if they share a vertex in G. That is, $L(G) = (E, E')$, where $E' = \{\{e_1, e_2\} \mid e_1, e_2 \in E,\ e_1 \cap e_2 \neq \emptyset\}$.

(a) Describe $L(C_n)$ and $L(K_{1,n})$. What is $L(C_3)$ and $L(K_{1,3})$?

Solution: We can see that $L(C_n)$ is C_n. $L(K_{1,n})$ is K_n. Thus, $L(C_3)$ is triangle C_3 and $L(K_{1,3})$ is also a triangle $K_3(= C_3)$.

(b) Show that if G is a d-regular bipartite graph, then $L(G)$ has a d-colouring.

Hint: Argue that a colouring of $L(G)$ corresponds to a partition of G into matchings.

Solution: Let G be a d-regular bipartite graph with sets (A, B, E). We know, by repeated applications of Hall's theorem, that E is comprised of d disjoint perfect matching from A to B.

Now, every disjoint set of edges in G corresponds to an independent vertex set in $L(G)$. This is because, if the corresponding vertices in $L(G)$ have an edge in between then the edges must have common vertex, which is a contradiction. Thus, each of d-disjoint perfect matching in G forms an independent vertex set. Since, the perfect matchings cover the entire edge-set, these d-independent vertex sets form a vertex cover. Now, colouring each independent vertex set with a distinct colour gives a valid colouring of $L(G)$. Hence, $L(G)$ has a d-colouring.

(c) Show that the size of the largest matching in G is the same as the size of the largest independent set in $L(G)$.

Solution: From above, we see that disjoint set of edges in G corresponds to an independent set in $L(G)$. By similar reasoning, we have that every independent set in $L(G)$ corresponds to a disjoint set of edges in $L(G)$. Since, every disjoint set of edges in G is a matching, we have that the largest matching in G corresponds to the largest independent set in $L(G)$. Hence, they have same size.

(d) Given the above problem, one may wonder if algorithms for finding the size of the largest matching in a graph can be used to find the size of the largest independent sets in graphs. Unfortunately, this does not always work. (Indeed, finding the size of the largest matching is an "easy" problem, while finding the size of the largest independent set is a "hard" problem.) In particular, there are graphs which are not line graphs for any graph. Show that $K_{1,3}$ (the "claw graph") is not the line graph of any graph.

Solution: Denote the vertices of $K_{1,3}$ as v_1, v_2, v_3, v_4, with degree of $v_1 = 3$. Suppose, G is a graph whose line graph is $K_{1,3}$. Let e_1, e_2, e_3, e_4 be their corresponding edges in G. Then, by edges in $K_{1,3}$, we must have that, e_1 is adjacent to each of e_2, e_3, e_4 and none of them are adjacent to each other. Since, e_1 has only two end-points, by PHP, two of $e_i, i \geq 2$ must have a common end point, which is a contradiction. Thus, $K_{1,3}$ isn't a line graph for any graph.

9. In a graph, if every vertex has degree at least 2, then it must have a cycle (as it has no leaves). Show that if every vertex has degree at least 3, then it must have a cycle of even length.

Hint: Consider a maximal path.

Solution: Let $G = (V, E)$ be a graph with $\deg(v) \geq 3 \; \forall v \in V$. Consider a maximal path in G and call it $P = v_0 v_1 ... v_k$. Clearly, v_1 is a neighbour of v_0 in G because they are consecutive vertices in a path. Consider 2 more neighbours of v_0 - v_i and v_j for $i < j$ (WLOG). It can be seen that both v_i and v_j must be part of P because otherwise we could extend the path.

Consider the cycle $v_0 v_i ... v_j v_0$ where the path from v_i to v_j is obtained from P. If this cycle has even length, then we have proved the claim. If not, we have proved that the path between v_i and v_j in P has odd length. Now consider the cycle $v_1 ... v_i v_0 v_1$. If this cycle has odd length, then we have proved that the path from v_1 to v_i in P is of odd length. From the above 2 results, it is easy to see that the path from v_1 to v_j in P must be of even length. Then the cycle $v_1 ... v_j v_0 v_1$ must be of even length.

10. **An application of Kőnig's theorem.**

(a) Show that every bipartite graph with m edges and maximum degree d has a matching of size at least m/d.

Solution: Consider any subset of vertices S in $G = (V, E)$. Let $E(S) = \{(u, v) | (u, v) \in E \text{ and } u \in S \text{ or } v \in S\}$. Since maximum degree of any vertex is d, it implies that the maximum number of edges of G that can be covered by S are $d|S|$ implying that $|E(S)| \leq d|S|$. If S was a vertex cover, it should cover all edges in G thus implying that $|E(S)| = m$. Hence, $m \leq d|S|$ if S is a vertex cover of G implying that the size of the vertex cover must be at least m/d. From Kőnig's theorem, the size of the maximum matching is same as the size of the minimum vertex cover implying that the maximum matching will have size at least m/d.

(b) Show that a bipartite graph (X, Y, E) with $|X| = |Y| = n$ and $|E| > n(k-1)$ should have a matching of size at least k.

Hint: Use the previous part.

Solution: Apply the above result with $m = n(k-1)+1$ and $d = n$.

11. Give an alternate proof for Hall's Theorem, using Kőnig's theorem.

(In class, we proved Kőnig's theorem using Hall's theorem; so this may appear to be circular reasoning. But it is possible to prove Kőnig's theorem directly, by induction.)

Hint: If a bipartite graph $G = (X, Y, E)$ does not have a complete matching from X to Y, the largest matching has size $< |X|$, and hence, by Kőnig, G has a vertex cover C of that size. Can you show that $X - C$ must be shrinking?

Solution: Suppose for contradiction, a bi-partite graph $G = (A, B, E)$ satisfies Hall's condition but doesn't have a complete matching from X to Y. Thus, by Kőnig's theorem, there exists a vertex cover C with $|C| < |X|$. Let C_A, C_B are the vetices of C in A, B respectively. Consider the set $X - C_A$, which is non-empty. Since, $C = C_A \cup C_B$ is the vertex cover, we must have that, every edge incident on $X - C_A$ must be incident on some vertex in C_B. Thus $\tau(X - C_A) \subseteq C_B$ and hence, $|\tau(X - C_A)| < |C_B| \leq |X| - |C_A| \leq |X - C_A|$. Thus, $X - C_A$ is a shrinking set in G, which is a contradiction. Thus, all bi-partite graphs which satisfy the Hall's condition must have a complete matching. Hence, Hall's theorem is true.

12. A maximal matching can be smaller than a maximum matching. In this problem we explore how much smaller it can be.

(a) Show that for any graph G, a maximal matching is at least half as large as a maximum matching. Prove this directly (by contradicting the maximality of a matching which is less than half the size of the maximum matching), and then repeat the proof using the connections between matchings and vertex cover.

Solution: Suppose $M = \{u_1v_1, \cdots, u_kv_k\}$ is a maximum matching in G. Also, suppose M' is a maximal matching in M of size $k' < k/2$. Consider the set of vertices V' involved in the matching M'. This set has size $|V'| = 2k' < k$, so $|V'| \leq k-1$. Therefore, there exists an index i such that $1 \leq i \leq k$ and neither u_i nor v_i is in V'. But then, adding the edge u_iv_i to M' extends the matching M', contradicting its maximality. Therefore $k' \geq k/2$, as required.

(b) Give examples of graphs which have a maximal matching which is exactly half the size of a maximum matching. Specifically, for any n, describe a connected graph G with $2n$ nodes which has a perfect matching and also a maximal matching of size $\lceil n/2 \rceil$.

Hint: For $n = 2$, consider the "path graph" P_4.

Solution: We distinguish cases for $n = 4k$ and $n = 4k+2$, where k is a natural number (if $n = 2$, just take a P_2). Suppose $n = 4k$ for $k \geq 1$. Here is a formal description of the graph (the graph will be clear once you draw it). Take $4k$ vertices indexed as $u_{i,j}$, where $1 \leq i \leq k$ and $1 \leq j \leq 4$. For each i, join $u_{i,1}$ to $u_{i,2}$, $u_{i,2}$ to $u_{i,3}$, and $u_{i,3}$ to $u_{i,4}$. Also, join $u_{i,2}$ to $u_{i+1,2}$ for all $1 \leq i \leq k-1$. This gives a connected graph G on $4k$ vertices. Note that taking the edges $u_{i,1}u_{i,2}$ for all i and $u_{i,3}u_{i,4}$ for all i gives a perfect matching in G of size $2k$. Also, taking the edges $u_{i,2}u_{i,3}$ for all i gives a maximal matching of size $k = \lceil 2k/2 \rceil$.
The construction for $n = 4k+2$ is similar. Take the graph mentioned above. Now add two new vertices v and w, and join v to w. Also, join v to $u_{n,2}$. Call this graph G. Once again, the edges vw, $u_{i,1}u_{i,2}$ for all i, and $u_{i,3}u_{i,4}$ for all i, give a perfect matching in G of size $2k+1$. Also, the edges vw, and the edges $u_{i,2}u_{i,3}$ for all i, give a maximal matching of size $k+1 = \lceil (2k+1)/2 \rceil$.

(c) What is the size of the smallest maximal matching in C_6? What about in C_n?

Solution: We claim that the smallest maximal matching in C_n has size $\lceil n/3 \rceil$. In particular, the smallest maximal matching in C_6 has size 2. We first show the lower bound, and then construct a maximal matching of that size.

Let the vertices of the graph $C = u_1u_2\cdots u_n$. Let M' be a maximal matching of C. For notational rigour, we assign a variable $x_i = 1$ if the edge between u_i and u_{i+1} is present in M', and $x_i = 0$ otherwise (here, it is standard to assume that u_{n+1} means u_1). Note that the number of edges in M' is just $x_1 + x_2 + \cdots + x_n$.
Observe that $x_1 + x_2 + x_3 \geq 1$. Indeed, if this is not the case, then neither of u_1, u_2, u_3, u_4 is matched. So adding the edge

u_2u_3 in M' increases the size of the matching, which is a contradiction. Similarly, $x_2+x_3+x_4 \geq 1, \cdots, x_{n-1}+x_n+x_1 \geq 1$, and $x_n + x_1 + x_2 \geq 1$, Adding all these inequalities together, we get

$$3(x_1 + x_2 + \cdots + x_n) \geq n$$

or $x_1 + x_2 + \cdots + x_n \geq n/3$. Therefore M' has at least $n/3$ (and therefore at least $\lceil n/3 \rceil$ edges).

To show this is achieved, we provide a construction. Take the following edges in C: If $n = 3k$, take the edges $u_1u_2, u_4u_5, \cdots, u_{3k-2}u_{3k-1}$. If $n = 3k+1$ or $n = 3k+2$, take the edges $u_1u_2, u_4u_5, \cdots, u_{3k-2}u_{3k-1}, u_{3k}u_{3k+1}$.

13. An **edge cover** is a concept closely related to a matching. An edge cover of a graph $G = (V, E)$ is a set of edges such that every vertex in the graph is part of some edge in the set. That is, $L \subseteq E$ is an edge cover if $V = \bigcup_{e \in L} e$.

    (a) What is the condition on a graph $G = (V, E)$ so that E is an edge cover?

    Solution: It is easy to see that E is an edge cover if and only if there are no isolated vertices (with degree 0) in G.

    (b) Suppose $G = (V, E)$ is a graph which has an edge cover. Show that if L is the smallest edge cover for G, then the graph (V, L) is a forest, in which each connected component is a "star graph" – i.e., a tree in which every edge is incident on a leaf.

    Solution: Given that there exists an edge cover for G, let L be the smallest edge cover. We claim that L has no cycles. Because if it is a cycle then removing an edge from this cycle gives a smaller edge cover of G. Also, no edge in L connects two vertices each degree more than 1 in (V, L). Because, if there exists such an edge, dropping it from L still is an edge cover which is smaller. Hence, no edge in (V, L) connects vertices each of degree more than 1.
    From the above two observations on (V, L), we conclude it is a forest with each edge incident on a leaf.

    (c) Suppose G and L are as above. Then describe a matching M in G such that $|M| = |V| - |L|$.

    Hint: How many connected components are there in (V, L)?

    Solution: let k be the number of components in L. Since, it is a forest, it will contain $|V| - k$ edges $= |L|$. Thus, no. of components in $L = k = |V| - |L|$. Now, each component of L contains at least one edge (as G has no isolated vertices). Consider matching M in G as union of arbitrary edges from each component. Since, the components are disjoint, M is a matching. Now, $|M| =$ no. of components in $(V, L) = |V| - |L|$

    (d) Suppose $G = (V, E)$ is a graph which has an edge cover. Also, suppose M is a matching in G. Then show that G has an edge cover L such that $|L| \leq |V| - |M|$.

    Solution: Given a matching M in G with an edge cover. Consider the minimal edge cover set L. We will modify this edge-cover set to L' such that L' is also a forest and contains all edges of M and atmost one edge of M in a component.
    Now, we try to construct such an edge cover L' by modifying L. By (b), L is a forest with every edge on incident on a leaf. For each edge in M, either it is already present in L. In case it is present in L, the component in which it is present has non other edge in M (as each component is a "star"). In case it isn't present, it could be incident with a leaf in L with a non-leaf or between two non-leaves or two leaves. If it is incident between between two non-leaves, include it in L'. If it is incident between a leaf v_f and a non-leaf v_n, remove the connecting edge v_fv_n in L' and remove edge connecting v_f to it's initial component. If the edge connects between two leaves, remove the connecting edges and add this edge to L. If any isolated vertices are formed by these operation, add minimal edges not in M to connect it to some component. This is always possible as, the only edges removed are edges not in M. Because of their minimality, no cycles will formed after their addition hence, L' is still a forest.
    Now the constructed edge cover L' is a forest and every component has atmost one edge from matching M and all edges in M are covered. Hence, no. of components in L' is at least that of size of M. We have $|L'| = |V| -$ no. components in $L' \leq |V| - |M|$. Hence, proved.

    (e) Conclude from the above that if G is a graph which has an edge cover, then the size of a maximum matching and the size of a minimum edge cover add up to the number of nodes in the graph.

    Solution: Let L_{min}, M_{max} are the minimal edge-cover and maximal matching in G. By (c), we have there exists a matching M of size $|V| - |L_{min}|$. Thus, $|M_{max}| \geq |V| - |L_{min}|$.
    By (d), we have by taking M_{max} matching there exists an edge cover L of size $|V| - |M_{max}|$. Thus, $|L_{min}| \leq |V| - |M_{max}| \implies |M_{max}| \leq |V| - |L_{min}|$. Thus, from the inequalities, we conclude that $|M_{max}| = |L_{min}|$.