Counting

Permutations & Combinations

Strings

- Given an alphabet (a finite set) B, we can consider strings of length k, made up of characters from the alphabet
  - e.g., B = {a,b,c}, and a length-5 string σ = aacca
  - Formally, a length k string is a function $\sigma : \{1,...,k\} \rightarrow B$

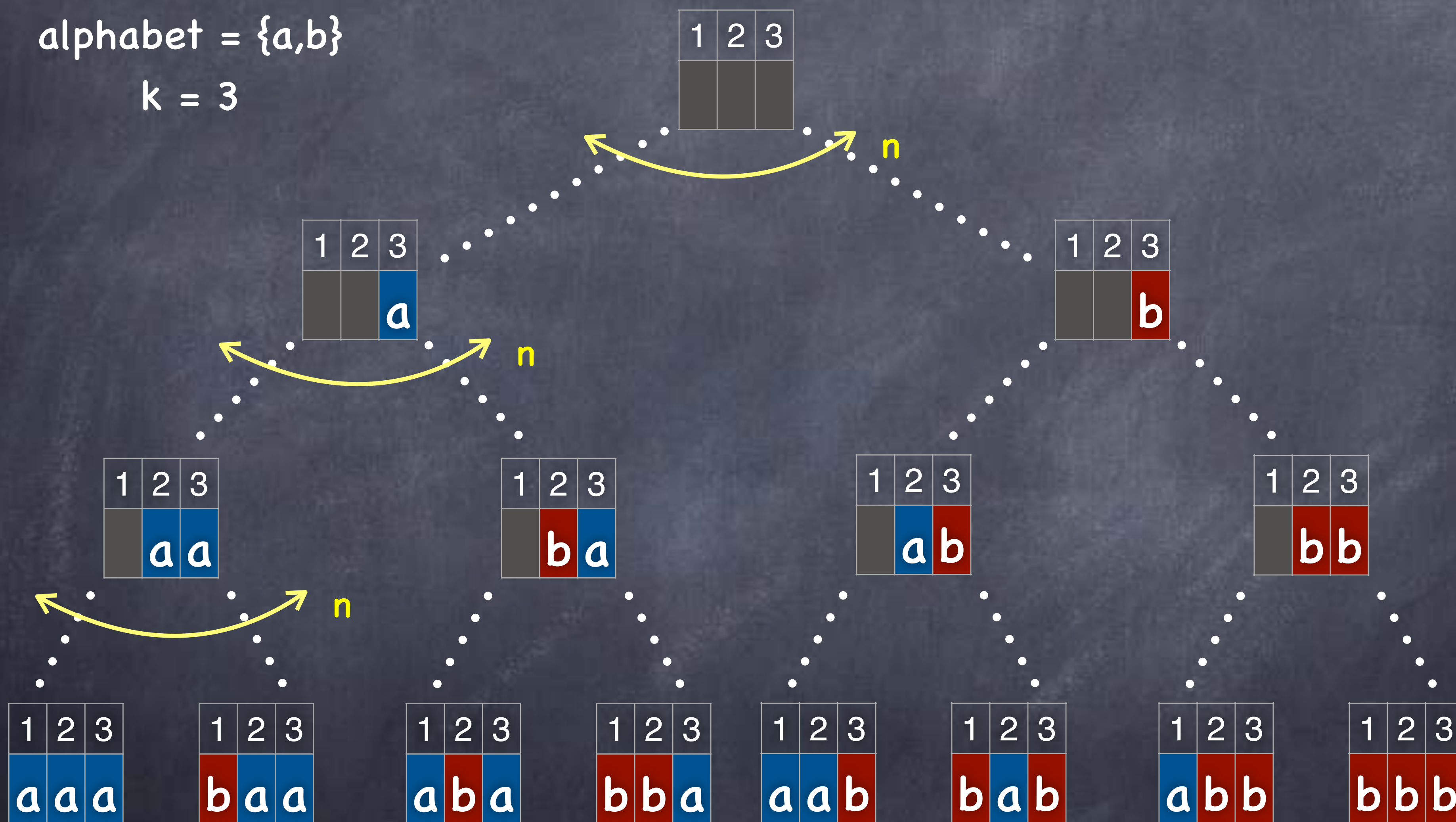
1	2	3	4	5
a	a	c	c	a

- How many length-k strings exist over an alphabet of size n?
  - n^k [Note: Grows exponentially with the length]
  - Proof by induction: Fix arbitrary alphabet size n. Let the number of k-long strings be $\alpha(k)$. Claim $\alpha(k) = n^k$.
    - $\alpha(1) = n$. For $k>1$, a k-long string consists of a $(k-1)$-long string followed by a single character. $\alpha(k) = \alpha(k-1) \cdot n$.

Strings

alphabet = {a,b}

k = 3



Binary Strings

- Binary string: A string with alphabet of size 2
  - Typically, alphabet {0,1}
- Number of length-k strings binary strings = 2^k
- A length-k binary string can be used to represent a subset of a set of size k
- Take the alphabet to be [k] $\triangleq$ {1,…,k}
- Subset associated with string σ:
  $S_\sigma = \{ i \mid \sigma_i = 1 \}$
- Number of subsets of [k] = 2^k

1	2	3	4	5
0	1	0	0	1

$\{2,5\} \subseteq [5]$

Permutations

- Permutations refer to arrangements of a set of symbols as a string, without repetition
  - e.g.,

1	2	3	4	5
c	a	d	e	b

(alphabet = {a,b,c,d,e})

  - A bijection from [n] = {1,...,n} to the alphabet of size n
  - Sometimes we want to consider shorter strings without repeating symbols

1	2	3
c	a	d

One-to-one

- How many length-k strings which do not have repeating symbols exist over an alphabet of size n?
  - $P(n,k) = \begin{cases} 0 & \text{if } k>n \\ n!/(n-k)! & \text{otherwise} \end{cases}$

$$n! = \begin{cases} 1 & \text{if } n=0 \\ n\cdot(n-1)! & \text{if } n>0 \end{cases}$$

Permutations

- How many length-k strings which do not have repeating symbols exist over an alphabet of size n?
  - $P(n,k) = \begin{cases} 0 & \text{if } k>n \\ n!/(n-k)! & \text{otherwise} \end{cases}$
- Proof by induction on n (for all k) [Exercise]
  - Base case, n=1
  - Induction step: Using $P(n,k) = n \cdot P(n-1,k-1)$
    - Alternately, $P(n,k) = P(n,k-1) \cdot (n-k+1)$
- $n!/(n-k)! = \underbrace{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}_{k \text{ times}}$

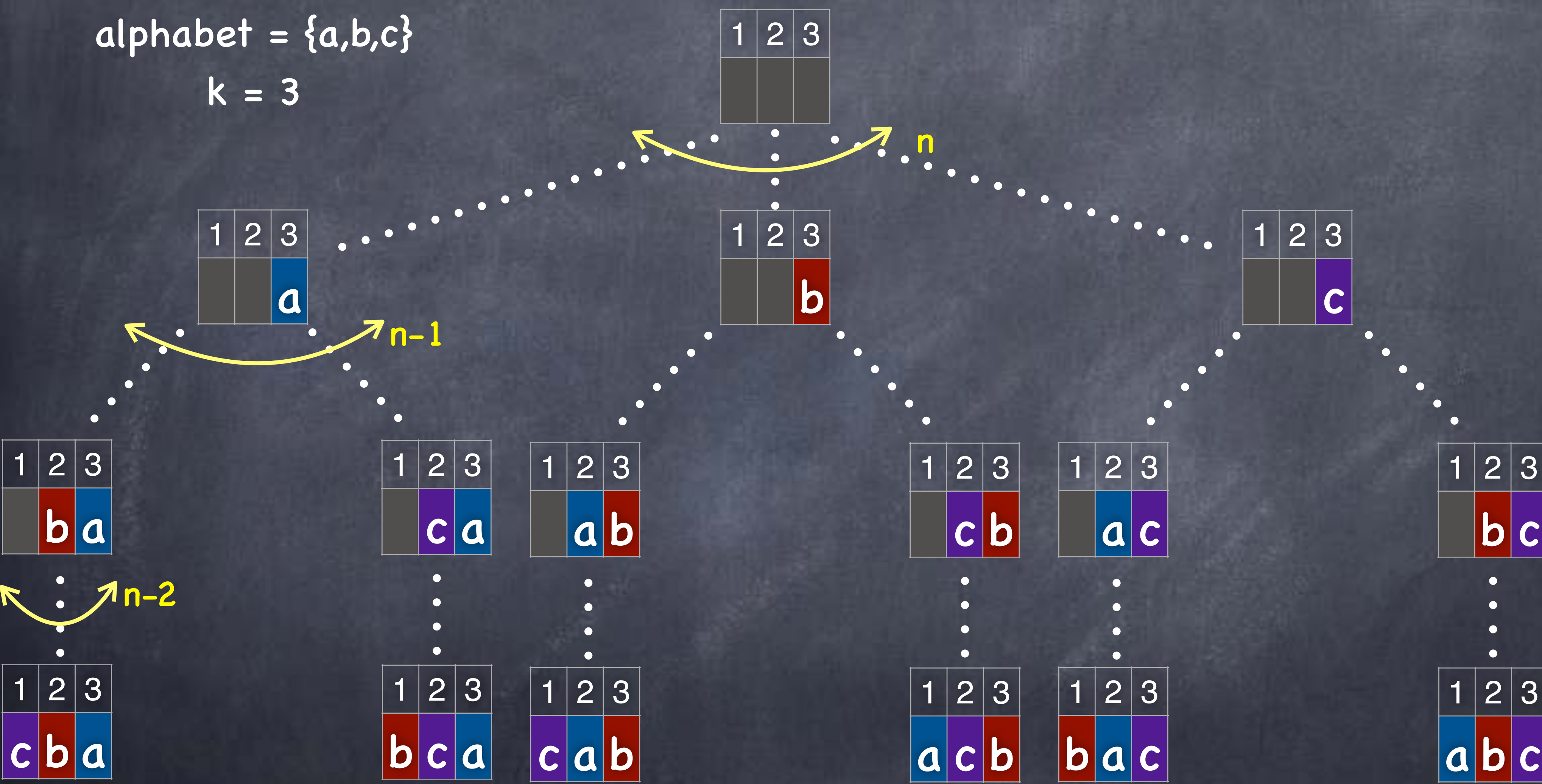
a.k.a. falling factorial, $(n)_k$

- $P(n,n) = n!$

Permutations

alphabet = {a,b,c}

k = 3



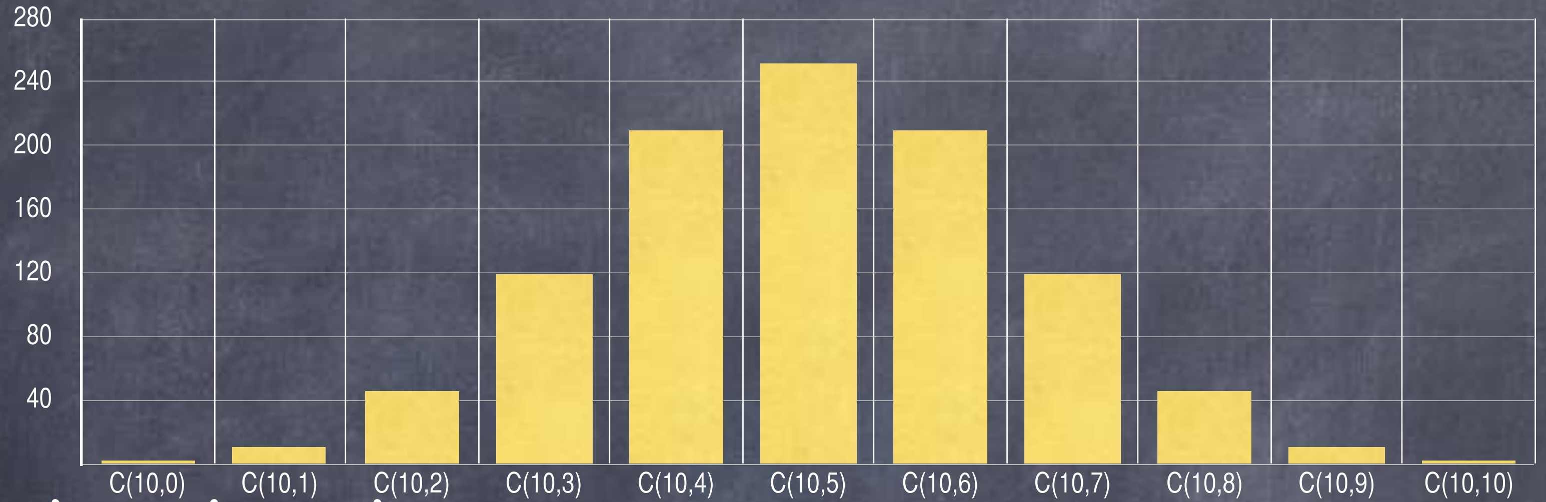
Combinations

- How many subsets of size k does a set of size n have?
- We can represent subsets as strings without repetitions
  - e.g., $\{a,c,d\} \subseteq \{a,b,c,d,e\}$ can be represented as acd
- But the same subset can be represented as multiple strings: adc, cad, ...
  - We know exactly how many ways
  - k! strings using the same k symbols
- # k-symbol subsets of n-symbol alphabet
  = # repetition-free strings of length k, divided by k!
  - $C(n,k) = P(n,k)/k! = n! / ((n-k)! \cdot k!)$

Also written $\binom{n}{k}$

$C(n,k)$

- For $n,k \in \mathbb{N}$, $C(n,k) = n!/(k!(n-k)!)$ if $k \leq n$, and 0 otherwise



- $C(n,k) = C(n,n-k)$
  - Selecting k out of n elements is the same as unselecting n-k out of n elements
- $C(n,0) = C(n,n) = 1$
  - In particular, $C(0,0) = 1$ (how many subsets of size 0 does $\emptyset$ have?)
- $C(n,0) + C(n,1) + \ldots + C(n,n-1) + C(n,n) = 2^n$

C(n,k)

- $(1+x)^n = \Sigma_{k=0 \text{ to } n} \; C(n,k)\; x^k$
- $(1+x)\cdot(1+x)\cdot(1+x) = (1+x)\cdot(1\cdot 1 + 1\cdot x + x\cdot 1 + x\cdot x)$

  $= 1\cdot 1\cdot 1 + 1\cdot 1\cdot x + 1\cdot x\cdot 1 + 1\cdot x\cdot x$

  $+ x\cdot 1\cdot 1 + x\cdot 1\cdot x + x\cdot x\cdot 1 + x\cdot x\cdot x$

  - Each term is of the form ?·?·? where each ? is 1 or x
- Coefficient of x^k = number of strings with exactly k x's out of the n positions = C(n,k)
- Proof by induction on n:
  coefficient of x^k in $(1+x)\cdot(\ldots + ax^{k-1} + bx^k + \ldots)$ is a+b
  - a = coefficient of x^{k-1} in $(1+x)^{n-1}$ = C(n−1,k−1)
    b = coefficient of x^k in $(1+x)^{n-1}$ = C(n−1,k)
  - C(n,k) = C(n−1,k−1) + C(n−1,k) (where n,k ≥ 1)

C(n,k)

- $C(n,k) = C(n-1,k-1) + C(n-1,k)$ (where $n,k \geq 1$)
  - Easy derivation: Let $|S|=n$ and $a \in S$.
    $C(n,k) =$ # k-sized subsets of S containing a
    + # k-sized subsets of S not containing a
- In fact, gives a recursive definition of C(n,k)
  - Base case (to define for $k \leq n$):
    $C(n,0) = C(n,n) = 1$ for all $n \in \mathbb{N}$
  - Or, to define it for all $(n,k) \in \mathbb{N} \times \mathbb{N}$
    Base case: $C(n,0)=1$, for all $n \in \mathbb{N}$,
    and $C(0,k)=0$ for all $k \in \mathbb{Z}^+$

n \ k	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	1	1	0	0	0	0	0
2	1	2	1	0	0	0	0
3	1	3	3	1	0	0	0
4	1	4	6	4	1	0	0
5	1	5	10	10	5	1	0
6	1	6	15	20	15	6	1

Conventions for n=0 or k=0

- # of length-k strings over an alphabet of size n = n^k
  - What if k=0?
  - We define the empty string as a valid string
  - $n^0 = 1$ such string
  - What if n=0? Empty string can be defined over an empty alphabet as well. So, 1 again.
- The empty string has no repeating symbols: $P(n,0) = 1$
  - $P(n,0) = n!/(n-0)!$ still holds
  - $P(0,0) = 1$ holds too since $0! = 1$
- Size-0 subsets of a size-n set? There is just one: $\emptyset$
  - $C(n,0) = 1$. $C(n,0) = n!/(0!\cdot n!)$ still holds
  - $C(0,0) = 1$ (since $\emptyset \subseteq \emptyset$)

Counting

Balls and Bins

Balls and Bins

- How many ways can I throw a set of balls into a set of bins?
- Variants based on whether they are considered distinguishable (labelled) or indistinguishable

	Labelled balls	Unlabelled balls
Labelled bins	Function	Multiset
Unlabelled bins	Set Partition	Integer Partition

- Further variants: "no bin empty", "at most one ball in a bin"

Balls and Bins

	Labelled balls	Unlabelled balls
Labelled bins	Function	Multiset
Unlabelled bins	Set Partition	Integer Partition

- Each ball must be thrown into a single bin
  - Throwing: mapping a ball to a bin
  - A function with the set of balls as the domain and the set of bins as the co-domain
- Number of ways of throwing:
  - Number of functions from A to B
    - "Function table": A string of length |A|, over the alphabet B
    - $|B|^{|A|}$ such strings

$x \in A$	$f(x) \in B$
1	1
2	0
3	1
4	0

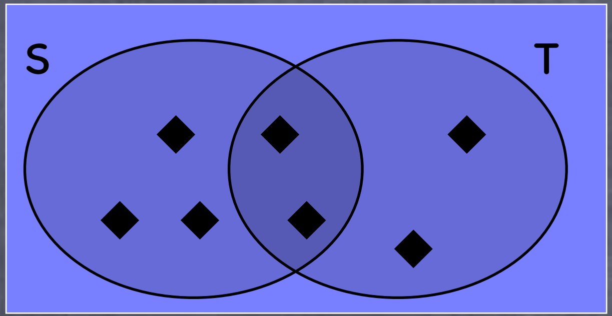
How many Functions?

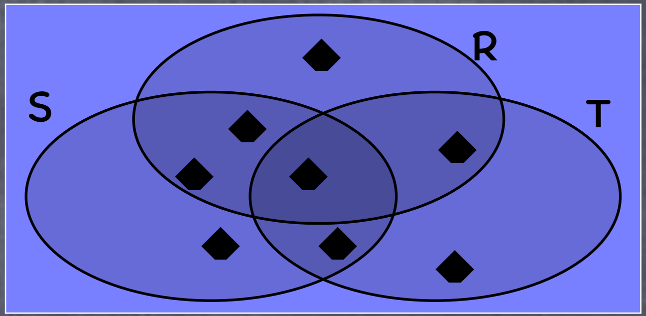
	Labelled balls	Unlabelled balls
Labelled bins	Function	Multiset
Unlabelled bins	Set Partition	Integer Partition

- Balls $\in$ A, bins $\in$ B. Let $|A|=k$, $|B|=n$.
- Unrestricted version:
  - # functions $f: A \to B$ = n^k
- Every bin can hold at most one ball: One-to-one functions
  - # one-to-one functions from A to B = $P(n,k)$
    - Recall Pigeonhole Principle: There is a one-to-one function from A to B only if $|B| \geq |A|$. $P(n,k) = 0$ for $k>n$
  - # bijections from A to B (only if $|A|=|B|$) is $P(n,n) = n!$
- No bin empty: Onto functions
  - # onto functions? A little more complicated.

Inclusion-Exclusion

- $|S \cup T| = |S| + |T| - |S \cap T|$





- $|R \cup S \cup T| = |R| + |S| + |T| - |R \cap S| - |S \cap T| - |T \cap R| + |R \cap S \cap T|$

- Given n finite sets $T_1, \ldots, T_n$

$$\left| \bigcup_{i \in [n]} T_i \right| = \sum_{J \subseteq [n],\ J \neq \emptyset} (-1)^{|J|+1} \left| \bigcap_{j \in J} T_j \right|$$

- Prove by induction on n [Exercise]
  - $|\ \cup_{i \in [n+1]} T_i\ | = |\ (\cup_{i \in [n]} T_i) \cup T_{n+1}\ |$

$= |\cup_{i \in [n]} T_i| + |T_{n+1}| - |\cup_{i \in [n]} Q_i|$ where $Q_i = T_i \cap T_{n+1}$ for $i \in [n]$

Onto Functions

	Labelled balls	Unlabelled balls
Labelled bins	Function	Multiset
Unlabelled bins	Set Partition	Integer Partition

- How many onto functions from A to B? Say $A=[k]$, $B=[n]$.
  - Let's call it $N(k,n)$

$n^k - C(n,1)\,(n-1)^k + C(n,2)\,(n-2)^k - \ldots$

- Claim: $N(k,n) = \Sigma_{i=0 \text{ to } n}\ (-1)^i\ C(n,i)\ (n-i)^k$
- Non-onto functions: $\cup_{i\in[n]} T_i$ where $T_i = \{\ f{:}A\rightarrow B \mid i \notin Im(f)\ \}$
- Inclusion-exclusion to count $|\cup_{i\in[n]}\ T_i|$

$|\ \cup_{i\in[n]} T_i\ | = \Sigma_{J\subseteq[n],J\neq\emptyset}\ (-1)^{|J|+1}\ |\cap_{j\in J} T_j|$

  - $|\cap_{j\in J} T_j\ | = (n-t)^k$ where $t=|J|$
    - $f \in T_{i_1} \cap\ldots\cap T_{i_t} \leftrightarrow Im(f) \subseteq [n] - \{i_1,\ldots,i_t\}$
    - $|T_{i_1} \cap \ \ldots \ \cap T_{i_t}| = (n-t)^k$
  - Number of $J\subseteq[n]$ s.t. $|J|=t$ is $C(n,t)$
  - $|\cup_{i\in[n]} T_i| = \Sigma_{t\in[n]}\ (-1)^{t+1}\ C(n,t)\ (n-t)^k$
- $N(k,n) = n^k - \Sigma_{t\in[n]}\ (-1)^{t+1}\ C(n,t)\ (n-t)^k = \Sigma_{t=0 \text{ to } n}\ (-1)^t\ C(n,t)\ (n-t)^k$

Balls and Bins

- How many ways to throw a set of k balls into a set of n bins?

<table>
<tr><th></th><th>Labelled balls</th><th>Unlabelled balls</th></tr>
<tr><td>Labelled bins</td><td>Function
<table>
<tr><td>all</td><td>n^k</td></tr>
<tr><td>1-to-1</td><td>$P(n,k)$</td></tr>
<tr><td>onto</td><td>$N(k,n)$</td></tr>
</table></td><td>Multiset</td></tr>
<tr><td>Unlabelled bins</td><td>Set Partition</td><td>Integer Partition</td></tr>
</table>

Balls and Bins

	Labelled balls	Unlabelled balls
Labelled bins	Function	Multiset
Unlabelled bins	Set Partition	Integer Partition

- Throwing k unlabelled balls into n distinguishable bins is the same as assigning integers (number of balls) to each bin
  - But the total number of balls is fixed to k
- A multi-set (a.k.a "bag") is like a set, but allows an element in it to occur one or more times
  - Only multiplicity, not order, matters: e.g., [a,a,b] = [a,b,a]
  - Formally, specified as a multiplicity function: $\mu : B \to \mathbb{N}$ e.g., $\mu(a)=2$, $\mu(b)=1$, $\mu(x) = 0$ for other x.
  - Size of a multi-set: sum of multiplicities: $\Sigma_{x \in B}\ \mu(x)$
- Throwing: Making a multi-set of size k, with elements coming from a ground-set of n elements (the n bins)

Examples

	Labelled balls	Unlabelled balls
Labelled bins	Function	Multiset
Unlabelled bins	Set Partition	Integer Partition

- Making a multi-set of size k, with elements coming from a ground-set of n elements
  - Place orders for k books from a catalog of n books (may order multiple copies of the same book)
  - Fill a pencil box that can hold k pencils, using n types of pencils
  - Distribute k candies to n kids (kids are distinguishable, candies are not)
  - Solve the equation $x_1 + \ldots + x_n = k$ with $x_i \in \mathbb{N}$
    - Ground-set of size n, $\{a_1,\ldots,a_n\}$. $\mu(a_i)=x_i$.
    - Can think of $x_1,\ldots,x_n$ as the bins, and each ball as a 1

Stars and Bars

	Labelled balls	Unlabelled balls
Labelled bins	Function	Multiset
Unlabelled bins	Set Partition	Integer Partition

- How many ways can I throw k (indistinguishable) balls into n (distinguishable) bins?
- Each such combination can be represented using n-1 "bars" interspersed with k "stars"
  - e.g., 3 bins, 7 balls: ★ ★ ★ | ★ ★ ★ | ★
    Or, | | ★ ★ ★ ★ ★ ★ ★ (first two bins are empty)
  - Number of such combinations = ?
    - (n-1)+k places. Choose n-1 places for bars, rest get stars
    - C (n+k-1, k) ways

★	|	★	★	★	|	★	★	★

Example

	Labelled balls	Unlabelled balls
Labelled bins	Function	Multiset
Unlabelled bins	Set Partition	Integer Partition

- How many solutions are there for the equation x+y+z = 11, with $x,y,z \in \mathbb{Z}^+$?
- 3 bins, 11 balls: But no bin should be empty!
- First, throw one ball into each bin
- Now, how many ways to throw the remaining balls into 3 bins?
  - 3 bins, 8 balls
  - 2 bars and 8 stars: e.g., ★ | | ★ ★ ★ ★ ★ ★ ★
  - C(10,2) solutions
    - e.g., above distribution corresponds to x=2, y=1, z=8
- Same as k-n balls, n bins without the no-bin-empty restriction

Variants

	Labelled balls	Unlabelled balls
Labelled bins	Function	Multiset
Unlabelled bins	Set Partition	Integer Partition

- Unrestricted use of bins
  - Multi-set of size k, ground-set of size n
  - Stars and Bars: C(n+k-1,n-1)
- No bin empty
  - Multiset of size k, with every multiplicity ≥ 1
    - Multiset of size k-n (with multiplicities ≥ 0)
    - C(k-1,n-1)
- At most one ball in each bin
  - Set of size k
  - C(n,k)

Balls and Bins

- How many ways to throw a set of k balls into a set of n bins?

	Labelled balls			Unlabelled balls		
Labelled bins	Function	all	n^k	Multiset	all	C(n+k-1,k)
		1-to-1	P(n,k)		1-to-1	C(n,k)
		onto	N(k,n)		onto	C(k-1,n-1)
Unlabelled bins	Set Partition			Integer Partition		

Balls and Bins

	Labelled balls	Unlabelled balls
Labelled bins	Function	Multiset
Unlabelled bins	Set Partition	Integer Partition

- (Labelled) elements of the set A are partitioned into (unlabelled) bins
  - Recall: $\{P_1,\dots,P_d\}$ is a partition of A if $A = P_1 \cup \dots \cup P_d$, for all distinct i,j, $P_i \cap P_j = \emptyset$, and no part P_i is empty
- How many partitions does a set A of k elements have?
  - S(k,n): #ways A can be partitioned into exactly n parts
    - This corresponds to the "no bin empty" variant
  - #ways A can be partitioned into at most n parts: $\Sigma_{m\in[n]} S(k,m)$
  - Total number of partitions, $B_k = \Sigma_{m\in[k]} S(k,m)$

Stirling number of the 2nd kind

Bell number

How many Partitions?

	Labelled balls	Unlabelled balls
Labelled bins	Function	Multiset
Unlabelled bins	Set Partition	Integer Partition

- S(k,n): #ways A can be partitioned into exactly n parts
  - Suppose we labeled the parts as 1,...,n
  - Such a partition is simply an onto function from A to [n]
    - N(k,n) ways
  - But in a partition, the parts are not labelled. With labelling, each partition was counted n! times.
  - S(k,n) = N(k,n) / n!

Balls and Bins

- How many ways to throw a set of k balls into a set of n bins?

	Labelled balls			Unlabelled balls		
Labelled bins	Function	all	n^k	Multiset	all	C(n+k-1,k)
		1-to-1	P(n,k)		1-to-1	C(n,k)
		onto	N(k,n)		onto	C(k-1,n-1)
Unlabelled bins	Set Partition	all	$\Sigma_{m\in[n]}$ S(k,m)	Integer Partition		
		1-to-1	0 or 1			
		onto	S(k,n)			

Balls and Bins

	Labelled balls	Unlabelled balls
Labelled bins	Function	Multiset
Unlabelled bins	Set Partition	Integer Partition

- Writing k as the sum of n non-negative integers
  - Integer solutions to $x_1 + \ldots + x_n = k$, s.t. $0 \leq x_1 \leq \ldots \leq x_n$
- "No bin empty" variant: x_i are positive integers
  - Number of such solutions called the partition number $p_n(k)$
- Number of solutions for the unrestricted variant: $p_n(k+n)$
  - $x_1 + \ldots + x_n = k$ s.t. $0 \leq x_1 \leq \ldots \leq x_n$
    $\leftrightarrow y_1 + \ldots + y_n = k+n$ s.t. $1 \leq y_1 \leq \ldots \leq y_n$ where $y_i = x_i+1$
- "At most one ball in a bin" variant: 1 if $n \geq k$, 0 otherwise

Partition Number

	Labelled balls	Unlabelled balls
Labelled bins	Function	Multiset
Unlabelled bins	Set Partition	Integer Partition

- $p_n(k) = |\{ (x_1,\ldots,x_n) \mid x_1+\ldots+x_n=k,\ 1 \le x_1 \le\ldots\le x_n\}|$
- $p_0(0) = 1$, if $k>0$ $p_0(k) = 0$, and if $k<n$ $p_n(k) = 0$
- $p_n(k) = p_n(k-n) + p_{n-1}(k-1)$
  - $|\{ (x_1,\ldots,x_n) \mid x_1+\ldots+x_n=k,$
    $1 < x_1 \le\ldots\le x_n \}|$
    $+ |\{ (x_1,\ldots,x_n) \mid x_1+\ldots+x_n=k,$
    $1 = x_1 \le\ldots\le x_n \}|$

n \ k	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	1	1
2	0	0	1	1	2	2	3	3	4
3	0	0	0	1	1	2	3	4	5
4	0	0	0	0	1	1	2	3	5
5	0	0	0	0	0	1	1	2	3
6	0	0	0	0	0	0	1	1	2
7	0	0	0	0	0	0	0	1	1
8	0	0	0	0	0	0	0	0	1

Balls and Bins

- How many ways to throw a set of k balls into a set of n bins?

	Labelled balls			Unlabelled balls		
Labelled bins	Function	all	n^k	Multiset	all	C(n+k-1,k)
		1-to-1	P(n,k)		1-to-1	C(n,k)
		onto	N(k,n)		onto	C(k-1,n-1)
Unlabelled bins	Set Partition	all	$\Sigma_{m\in[n]}$ S(k,m)	Integer Partition	all	$p_n(k+n)$
		1-to-1	0 or 1		1-to-1	0 or 1
		onto	S(k,n)		onto	$p_n(k)$