CS 228 : Logic in Computer Science

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Recap

- $DFA = NFA = \epsilon$-NFA=regular languages
- Closure properties of regular languages

Agenda

- FO-definable $\Rightarrow$ regular
- Given an FO formula φ, construct a DFA A_φ such that $L(\varphi) = L(A_\varphi)$
- If $L(A_\varphi) = \emptyset$, then φ is unsatisfiable
- If $L(A_\varphi) \neq \emptyset$, then φ is satisfiable

FO to Regular Languages

- ▶ Every FO sentence φ over words can be converted into a DFA A_φ such that $L(\varphi) = L(A_\varphi)$.
- ▶ Start with atomic formulae, construct DFA for each of them.
- ▶ Conjunctions, disjunctions, negation of formulae easily handled via union, intersection and complementation of of respective DFA
- ▶ Handling quantifiers?

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- Think of this as $\begin{array}{c} baab \\ 0010 \end{array}$ or $\begin{array}{c} baab \\ 0100 \end{array}$
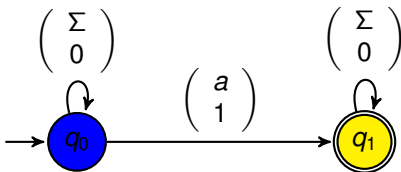
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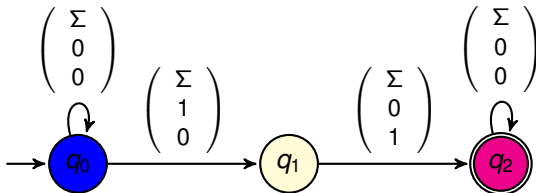|	*bab*		*bab*
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	010		001

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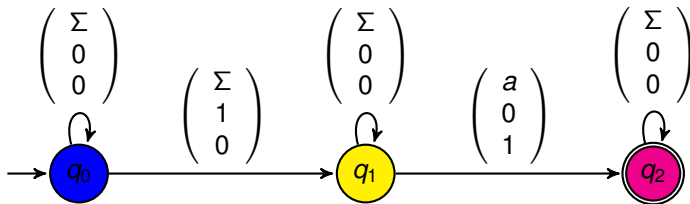
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Atomic Formulae to DFA : $x < y$

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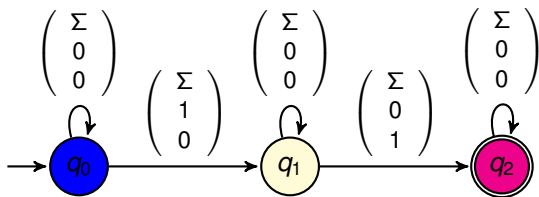
Simple Formulae to DFA

- $x < y \wedge Q_a(y)$
- $\Sigma' = \Sigma \times \{0,1\} \times \{0,1\}$
- Obtain intersection of DFA for $x < y$ and $Q_a(y)$

Formulae to DFA

- Given $\varphi(x_1, \ldots, x_n)$, a FO formula over Σ, consider the extended alphabet

$$\Sigma' = \Sigma \times \{0, 1\}^n$$

- Assign values to x_i at every position as seen in the cases of atomic formulae
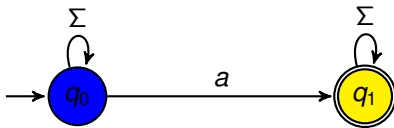- Keep in mind that every x_i can be assigned 1 at a unique position

Quantifiers
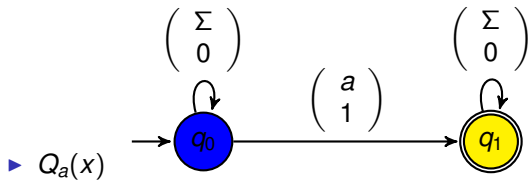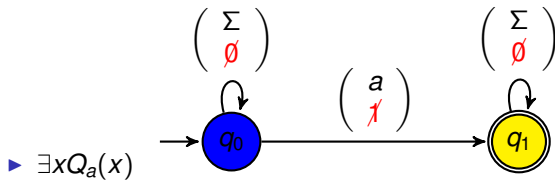
- $Q_a(x)$

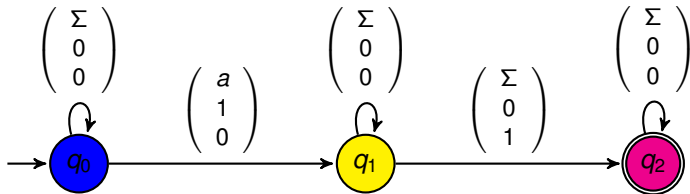# Handling Quantifiers

- $\exists x Q_a(x)$
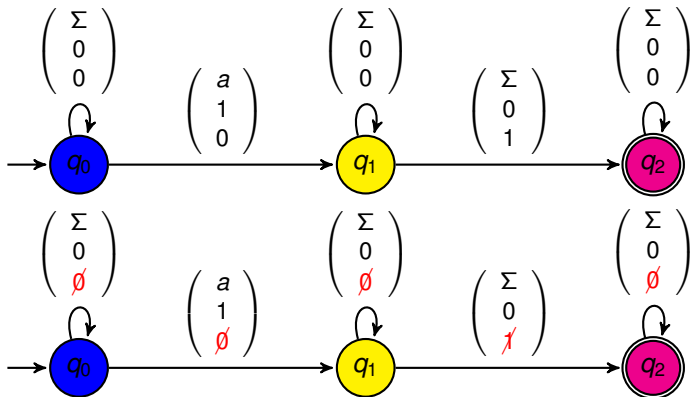
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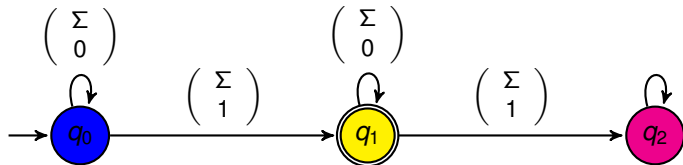
Handling Quantifiers: $\forall x (x \neq x)$

- $(x = x)$

$$\left(\begin{array}{c} \Sigma \\ 0 \end{array} \right) \qquad \left(\begin{array}{c} \Sigma \\ 0 \end{array} \right)$$

$$\xrightarrow{} q_0 \xrightarrow{\left(\begin{array}{c} \Sigma \\ 1 \end{array} \right)} q_1 \xrightarrow{\left(\begin{array}{c} \Sigma \\ 1 \end{array} \right)} q_2$$
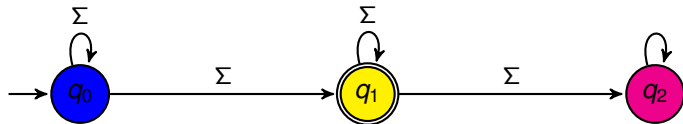
- $\exists x (x = x)$
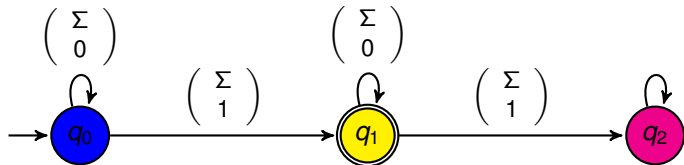
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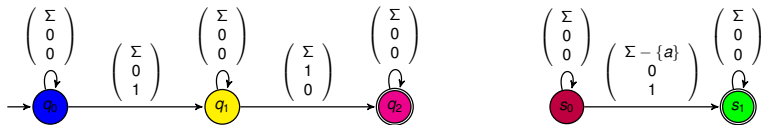

- $\exists x(x = x)$

- $\neg\exists x(x = x)$

Handling Quantifiers : Summary

Quantifier Lemma

Let $L \subseteq (\Sigma \times \{0,1\}^n)^*$ be defined by $\varphi(x_1, \ldots, x_n)$. Let $f : (\Sigma \times \{0,1\}^n)^* \to (\Sigma \times \{0,1\}^{n-1})^*$ be the projection $f(w, c_1, \ldots, c_n) = (w, c_1, \ldots, c_{n-1})$. Then $\exists x_n \varphi(x_1, \ldots, x_{n-1})$ defines $f(L)$.
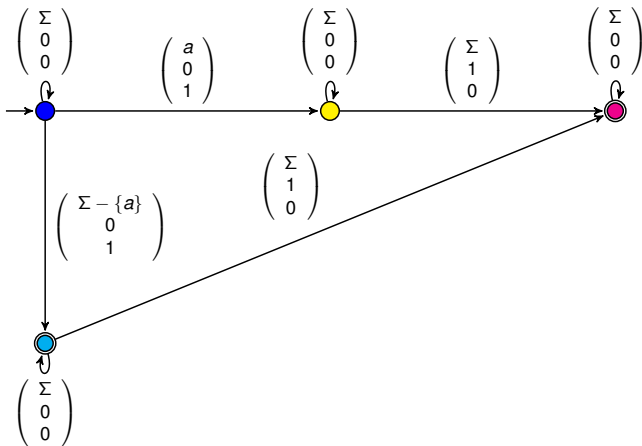
Handling Quantifiers

- $\exists y \forall x [x \leqslant y \wedge Q_a(y)] = \exists y [\neg \exists x [x > y \vee \neg Q_a(y)]]$
- $[x > y \vee \neg Q_a(y)]$
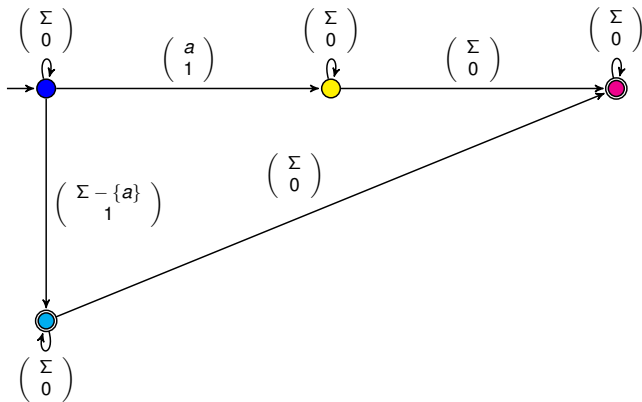- Check the automaton and correct/complete it!
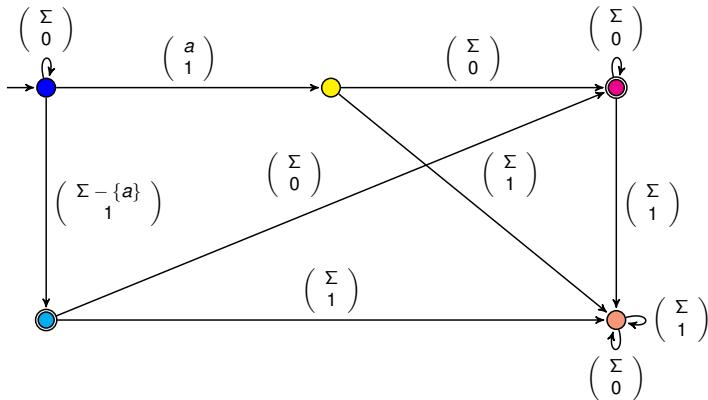
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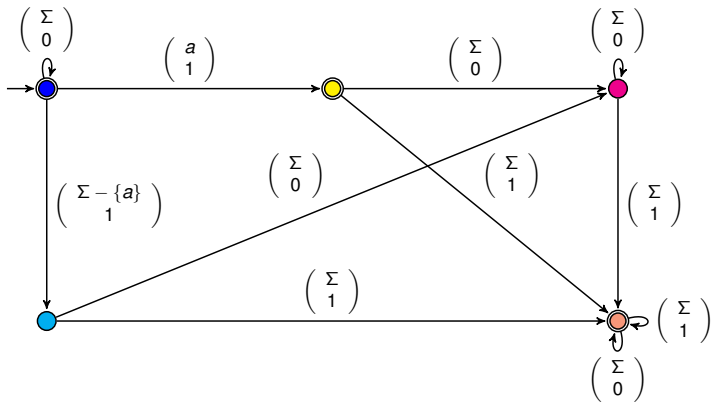
- $\exists x[x > y \vee \neg Q_a(y)]$

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- $\neg \exists x[x > y \vee \neg Q_a(y)]$
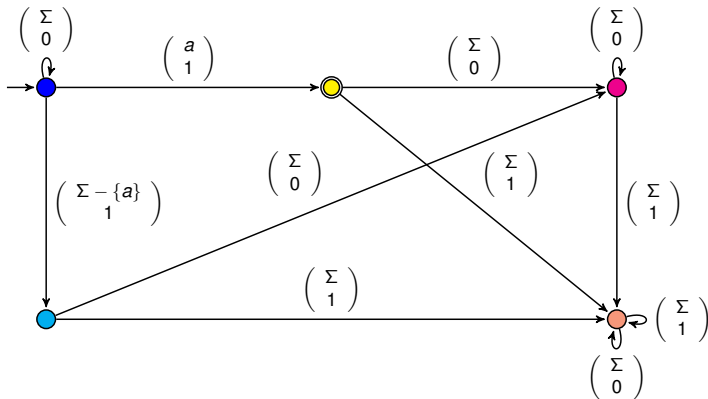- Deterministic Automaton corresponding to $\exists x[x > y \vee \neg Q_a(y)]$

Handling Quantifiers

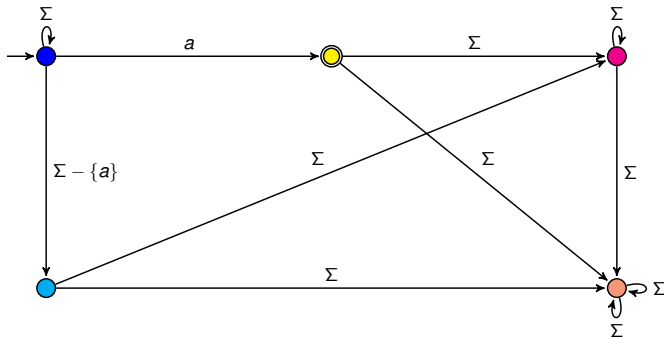- Incorrect deterministic Automaton corresponding to $\neg \exists x[x > y \vee \neg Q_a(y)]$

Handling Quantifiers

- $\neg\exists x[x > y \vee \neg Q_a(y)]$
- Deterministic Automaton corresponding to $\neg\exists x[x > y \vee \neg Q_a(y)]$
- Intersect with $\left(\begin{array}{c} \Sigma \\ 0 \end{array}\right)^* \left(\begin{array}{c} \Sigma \\ 1 \end{array}\right) \left(\begin{array}{c} \Sigma \\ 0 \end{array}\right)^*$

Handling Quantifiers

▶ $\exists y \neg \exists x[x > y \vee \neg Q_a(y)] = \exists y \forall x[x \leqslant y \wedge Q_a(y)]$

Points to Remember

- Given $\varphi(x_1, \ldots, x_n)$, construct automaton for atomic FO formulae over the extended alphabet $\Sigma \times \{0, 1\}^n$
- Intersect with the regular language where every x_i is assigned 1 exactly at one position
- Given a sentence $Q_{x_1} \ldots Q_{x_n} \varphi$, first construct the automaton for the formula $\varphi(x_1, \ldots, x_n)$
- Replace $\forall$ in terms of $\exists$

Points to Remember

- Given the automaton for $\varphi(x_1, \ldots, x_n)$, the automaton for $\exists x_i \varphi(x_1, \ldots, x_n)$ is obtained by projecting out the row of x_i
- This may result in an NFA
- Determinize it and complement it to get a DFA for $\neg \exists x_i \varphi(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$
- Intersect with the regular language where each of $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ are assigned 1 exactly at one position

The Computational Effort

Given NFAs A_1, A_2 each with almost n states,

- The union has almost $2n + 1$ states
- Intersection has almost n^2 states
- The complement has almost 2^n states
- The projection has almost n states

The Computational Effort

- $\psi = Q_1 \ldots Q_n \varphi$. If $Q_i = \exists$ for all i, then size of A_ψ is same the size of A_φ.
- When $Q_1 = \exists, Q_2 = \forall, \ldots$: each $\forall$ quantifier can create a 2^n blowup in automaton size
- Size of automaton is

$$2^{2^{2^{2^{2^{2^n}}}}}$$

  where the tower height k is the quantifier alternation size.
- This number is indeed a lower bound!

The Automaton-Logic Connection

Given any FO sentence φ, one can construct a DFA A_φ such that $L(\varphi) = L(A_\varphi)$.

Model checking