Quiz 4

CS 207 :: Autumn 2021
November 2, 2021

1. Honour code.

2. Given positive integers n, k such that $k \leq n/2$, describe a forest with n nodes whose matching number (i.e., the largest possible number of edges in a matching) is exactly k. Justify your answer.

   Solution: Pick any $2k$ vertices among the n nodes, divide them into k pairs and the edges consist of these pairs only. Since these pairs form disjoint edges and the remaining vertices are disjoint, this satisfies the definition of a forest i.e., a disjoint union of trees.

3. An interval graph is specified by a set of closed intervals in the real number line as its vertices, and two such vertices have an edge between them iff they intersect. For example, if the set of vertices in an interval graph are $\{a, b, c\}$ where $a = [0, 5], b = [1, 2], c = [3, 5]$, then its set of edges are $\{\{a, b\}, \{a, c\}\}$.

   For each of the following graphs, indicate whether it can occur as an induced subgraph of some interval graph or not.

   (a) A clique with 5 nodes:

   - [x] A. Can occur
   - [ ] B. Cannot occur

   (b) A cycle with 4 nodes

   - [ ] A. Can occur
   - [x] B. Cannot occur

   (c) A path with 4 nodes

   - [x] A. Can occur
   - [ ] B. Cannot occur

   (d) A tree with 5 nodes of which 3 are leaves

   - [x] A. Can occur
   - [ ] B. Cannot occur

4. Recall that given two graphs $G = (U, E)$ and $H = (V, F)$, we define their box-product $G \square H$, as the graph over the vertex set $U \times V$, as follows:

   $G \square H$ has edges of the form $\{(u, v), (u', v')\}$ where either $\{u, u'\} \in E$ and $v = v'$, or $\{v, v'\} \in E$ and $u = u'$.

   Given $G = (U, E)$, $H = (V, F)$ and $K = (W, D)$ what is the degree of a node $(u, v, w) \in U \times V \times W$ in $G \square H \square K$? Express your answers in terms of the degrees in G, H and K. Briefly justify.

   Solution: Let $\deg_G(u)$ denote the degree of node u in graph $G = (U, E)$, for $u \in U$. Similarly we can define $\deg_H(v)$, for $v \in V$ and $\deg_K(w)$, for $w \in W$. Then, degree of node (u, v, w) in $G \square H \square K$ is given by $\deg_G(u) + \deg_H(v) + \deg_K(w)$. This is because an edge in the final graph is formed when two of the coordinates remain the same and edges are formed in the individual graph over the remaining coordinate.

5. A saturated hydrocarbon is a molecule with carbon and hydrogen atoms, such that each carbon atom has four bonds, and each hydrogen atom has a single bond. There are no cycles, nor double or triple bonds. Prove that such a molecule should have the chemical formula C_rH_{2r+2} for some r (i.e., if it has r carbon atoms, it has $2r+2$ hydrogen atoms).

   Solution: Let us assume that a saturated hydrocarbon molecule has r carbon atoms and s hydrogen atoms. We can visualize such a molecule as a graph $G = (V, E)$ with atoms representing vertices and bonds representing edges. Then, degree of a carbon atom is 4 and that of a hydrogen atom is 1. There are two more properties of this graph : it is acyclic and connected. Therefore, this must be a tree with $|V| - 1$ edges. Therefore,

$$\begin{aligned}|E| &= |V| - 1\\ 2|E| &= 2|V| - 2\\ 4r + s &= 2r + 2s - 2\\ 2r + 2 &= s\end{aligned}$$

   thus giving the required formula of C_rH_{2r+2}.

6. A bipartite graph (X, Y, E) is said to be *bi-regular* if every node in X has degree x and every node in Y has degree y (here x and y need not be equal). Show that such a graph has a complete matching from X into Y or a complete matching from Y into X.

   Solution: The number of edges going out of X is given by $x|X|$ and the number of edges coming into Y is given by $y|Y|$. These two quantities must be the same because the graph is bipartite. There are 3 possibilities: either $|X| = |Y|$, $|X| < |Y|$ or $|X| > |Y|$. When $|X| = |Y|$, it follows that $x = y$. For a subset $S \subseteq X$, let $N(S)$ denote the neighbourhood of S. Consequently, it follows that $|N(S)| = |S|$ (since $x = y$), for any subset $S \subseteq X$. Therefore, there is a perfect matching in this case.

   Consider the case when $|X| < |Y|$ and hence $x > y$. For any subset $S \subseteq X$, the number of edges going out of S is $x|S|$ and the number of edges coming into $N(S)$ is $y|N(S)|$. Since every edge going out of S goes into $N(S)$, it follows that $y|N(S)| \geq x|S|$. Since $y < x$, it must be the case that $|N(S)| > |S|$. Since the graph satisfies Hall's condition, there must be a matching from X into Y. The third case with $|X| > |Y|$ is exactly the same with order reversed. Hence, in that case there must be a matching from Y into X.

7. A vertex t in a graph is said to be the *median* of a triple of vertices (u, v, w) if there exist shortest paths between pairs (u, v), (v, w) and (u, w) all of which pass through t.

   Show that for any three distinct vertices u, v, w in the n-dimensional hypercube graph $(n > 1)$, there is a *unique median*. Describe the median of u, v, w as a bit string in terms of the bit strings corresponding to u, v, w (using the natural representation of nodes in a hypercube as bit strings).

   Solution: If vertices u, v, w are represented by bit strings s_u, s_v, s_w respectively, then the bit string given by $s_t = \mathrm{Maj}(s_u, s_v, s_w)$ represents the unique median of these vertices. Here, $\mathrm{Maj}(a, b, c)$ denotes the bit-wise majority function applied on the strings a, b and c. Let us show why there exists a shortest path between u and v that passes through t, as defined above. First of all, any vertex lying on the shortest path between u and v should have the same bit as those positions where u and v have the same bit values. This is true for t as at that position, the bit value in t is the majority of the values of u, v and w. At the remaining positions, the bit value equals either that of u, or that of v. Hence, while designing a shortest path from u to v, one can pass through t by flipping only those bits where t differs from u. Since these bits had to be flipped anyways to reach u, this is still a shortest path. The argument for the remaining two pairs follows similarly.

Now we will show why the median is unique. For any 3 bits, there is a bit which occurs at least twice. For any bit position, there is a bit which occurs at least twice from amongst u, v and w. For t, the bit value at this position must equal that bit since it lies on the shortest path between those two vertices (which have that bit at that position). Therefore, given u, v and w, the median is uniquely defined.

8. Suppose M is a (non-empty) matching in a graph $G = (V, E)$ and L is a matching in its complement graph. Show that the graph $(V, M \cup L)$ is bipartite.

   Solution: To show that a graph is bipartite, it suffices to show that the graph does not contain any odd cycles. For the sake of contradiction, assume that the graph contains an odd cycle $(v_1, \ldots, v_l)$, with l being odd. WLOG lets assume that the edge $(v_1, v_2) \in M$. It follows that the edge $(v_2, v_3) \in L$, since the vertex v_2 cannot be part of more than one edge from the same matching. Reasoning similarly, we find that the edge $(v_l, v_1) \in M$, as l is odd. This leads to a contradiction as v_1 is part of two different edges from the same matching M.

9. Let G be a graph with vertex set V, and let $U \subseteq V$. Let $\mathrm{odd}(U)$ be defined as the number of connected components in the subgraph induced by $V - U$ that have an odd number of vertices.

   Show that G cannot have a matching with more than

   $$(|V| - \mathrm{odd}(U) + |U|)/2$$

   edges.

   Solution: Let us consider the subgraph induced by $V - U$. Let $N_1, N_2, \ldots, N_{\mathrm{odd}(U)}$ denote the total number of vertices in each connected component that have odd number of vertices in this subgraph. Then the i^{th} such component can have at most $(N_i - 1)/2$ matching edges present in it for any matching of G. Therefore, the total number of matching edges present in these components with odd vertices are

   $$\sum_{i=1}^{\mathrm{odd}(U)} \frac{N_i - 1}{2} = \frac{\left(\sum_{i=1}^{\mathrm{odd}(U)} N_i\right) - \mathrm{odd}(U)}{2}$$

   Similarly, let $\mathrm{even}(U)$ denote the number of connected components in the subgraph induced by $V - U$ that have even number of vertices. Let $N'_1, N'_2, \ldots, N'_{\mathrm{even}(U)}$ denote the total number of vertices in each such component, which can have at most $N'_i/2$ matching edges present in it for any matching of G. The total number of these edges can be at most

   $$\sum_{i=1}^{\mathrm{even}(U)} \frac{N'_i}{2}$$

   Therefore, we get an upper bound for the number of matching edges present in the subgraph induced by $V - U$:

   $$\frac{\left(\sum_{i=1}^{\mathrm{odd}(U)} N_i\right) - \mathrm{odd}(U)}{2} + \frac{\sum_{i=1}^{\mathrm{even}(U)} N'_i}{2} = \frac{|V| - |U| - \mathrm{odd}(U)}{2}$$

   Note that matching edges in G can be partitioned into two categories: edges which lie in the subgraph induced by $V - U$ and the remaining edges. Every remaining edge must be incident on at least one vertex from U. Also note that each vertex in U can only be part of at most one matching edge. This gives us an upper bound of $|U|$ on the number of remaining edges. Since we have an upper bound for both kinds of edges, we have an upper bound for the overall number of matching edges as:

$$\frac{|V| - |U| - \mathrm{odd}(U)}{2} + |U| = \frac{|V| + |U| - \mathrm{odd}(U)}{2}$$