Problem Set 6

Released: October 11, 2021

1. How many relations are there on a set with n elements that are:

   (a) reflexive?
   (b) irreflexive?
   (c) symmetric?
   (d) antisymmetric?
   (e) asymmetric?
   (f) equivalence?

   Hint: Where appropriate, you may use $S(k,n)$, the Stirling number of the second kind*.*

   Solution: Let A be the domain for the relation R with n elements. Then are a total of n^2 elements in $A \times A$.

   (a) For a relation to be reflexive, it must contain pairs of the form (x,x) for all $x \in A$. There are n such pairs. For the remaining $n^2 - n$ pairs, both possibilities (of being present in R or not) do not affect the reflexive property. It follows that the total number of reflexive relations is 2^{n^2-n}.

   (b) For a relation to be irreflexive, it must not contain pairs of the form (x,x) for any $x \in A$. Any possibility is equally valid for the remaining pairs. Hence, the total number of irreflexive relations is 2^{n^2-n}.

   (c) In a symmetric relation, for $x \neq y$, either both (x,y) and (y,x) are in the relation or none of them is. Thus, there are two possibilities for $\frac{n^2-n}{2}$ tuples of unequal pairs i.e. tuples of the form $\{(x,y),(y,x)\}$ for $x \neq y$. The n pairs of the form (x,x) do not affect symmetricity. Therefore, the number of symmetric relations is $2^n 2^{\frac{n^2-n}{2}}$.

   (d) In an antisymmetric relation, for $x \neq y$, atmost one of (x,y) and (y,x) is present in the relation. Thus, there are three possibilities for $\frac{n^2-n}{2}$ tuples of unequal pairs i.e. exactly one of them is present (two possibilities) or none of them is. The n equal pairs do not affect the desired property. Therefore, the number of antisymmetric relations is $2^n 3^{\frac{n^2-n}{2}}$.

   (e) A asymmetric relation is like an antisymmetric relation along with the additional requirement that no pair of the form (x,x) can be present in the relation. Therefore, the total number of such relations is $3^{\frac{n^2-n}{2}}$.

   (f) An equivalence relation can be characterized by the equivalence partition of the domain. The number of partitions with exactly k parts is given by $S(n,k)$. Since the number of parts can range from 1 to n, the number of equivalence relations is $\sum_{k=1}^{n} S(n,k)$.

2. This problem considers proving that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$.

   (a) Give a combinatorial proof, by counting the number of ways to select a (non-empty) committee, with one member being the leader of the committee.

   (b) Prove this using the formula for $\binom{n}{k}$. First show that $k\binom{n}{k} = n\binom{n-1}{k-1}$.

   (c) Here is a trick we have not covered in the class, that uses your knowledge of calculus. Consider the polynomial $P(x) = (1+x)^n$. Let $P'(x)$ be the polynomial obtained as the derivative of $P(x)$. Write two expressions for $P'(x)$, and use them to evaluate $P'(1)$.

   Solution:

   (a) Given n persons, we need to select a non-empty committee and a leader for the committee. One way to do that would be to first select a leader by choosing a person from the n people. All the remaining $n-1$ persons may or may not be part of the committee without affecting its non-emptiness (since the leader is already a member). This gives us $n2^{n-1}$ ways. Another possibility would be to first form the committee and then the leader. Let the size of the committee be k. Such a committee can be selected in $\binom{n}{k}$ ways. The leader of this committee could be selected in k ways. Since k can range from 1 to n, this gives us $\sum_{k=1}^{n} k\binom{n}{k}$ ways.

(b) First, we can prove that

$$\begin{aligned} k\binom{n}{k} &= k\frac{n!}{k!(n-k)!} \\ &= \frac{n!}{(k-1)!(n-k)!} \\ &= n\frac{(n-1)!}{(k-1)!(n-k)!} \\ &= n\binom{n-1}{k-1} \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{k=1}^{n} k\binom{n}{k} &= \sum_{k=1}^{n} n\binom{n-1}{k-1} \\ &= n\sum_{k=1}^{n}\binom{n-1}{k-1} \\ &= n2^{n-1} \end{aligned}$$

(c) Using binomial theorem, we can write

$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$$

Taking the derivative on both sides,

$$n(1+x)^{n-1} = \sum_{i=1}^{n}\binom{n}{k}kx^{k-1}$$

Evaluating this equation at $x = 1$,

$$n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$$

3. How many ways are there to travel in $xyzw$ space from the origin (0,0,0,0) to the point (4,3,5,4) by taking steps one unit in the positive x, positive y, positive z, or positive w direction? **Solution:** Each step will look like (0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0) or (1, 0, 0, 0). Hence, a total of 16 steps will be taken to reach (4, 3, 5, 4) across each path from the origin. Each step can be labelled as $1_x, 1_y, 1_z$ or 1_z depending along which direction the step is taken. A path can be labelled as a permutation of these symbols with 4 1_x, 3 1_y, 5 1_z and 4 1_z symbols occurring in each permutation. Hence, the total number of such permutations are $\frac{16!}{4!3!5!4!}$.

4. A sequence of integers is said to be *smooth* if any two consecutive integers in the sequence differ by exactly 1. For instance, $5, 4, 5, 6, 5, 4$ is a smooth sequence of length 6.

   How many smooth sequences of length 16 are there that start with 5 and end with 10? **Solution:** The differences form a ±1 sequence of length 15. The number of $+1$s should be 5 more than the number of -1s. Solving $x+y = 15$, $x-y = 5$, we have $x = 10$ $+1$s and $y = 5$ -1s. So $\binom{15}{5}$ ways.

5. A sequence of positive integers $a_1, a_2, \ldots, a_m$ is said to be *decreasing* if for all i, we have $a_i \geq a_{i+1}$. A decreasing sequence is said to *strictly decreasing* if any integer appears at most once in the sequence. A decreasing sequence is said to be *almost strictly decreasing* if any integer appears at most twice in the sequence.

   (a) How many strictly decreasing sequences of positive integers are there with $a_1 = n$ (of all possible lengths)?
   Solution: Any subset of $\{1, \ldots, n-1\}$ when sorted gives exactly one such sequence. So, 2^{n-1} sequences.

   (b) There are infinitely many decreasing sequences of positive integers that start with $a_1 = n$. How about almost strictly decreasing sequences?
   Solution: An almost strictly decreasing sequence is obtained by including each element in $\{1, \ldots, n-1\}$ 0, 1 or 2 times, along with n, which can be included 1 or 2 times. So $2 \cdot 3^{n-1}$ times.

   (c) How many strictly decreasing sequences of positive integers of length m exist with $a_1 = n$?
   Solution: $\binom{n-1}{m-1}$.

(d) How many decreasing sequences of positive integers of length m exist with $a_1 = n$?

Solution: Choose $m-1$ elements with repetition from $\{1, \ldots, n\}$ (and sort them). Using "$m-1$ stars and $n-1$ bars," $\binom{n+m-2}{m-1}$ ways.

6. Consider the standard deck of 52 playing cards. A balanced hand is a subset of 13 cards containing four cards of one suit and three cards of each of the remaining three suits.

(a) Find the number of balanced hands.

Solution: The suit S_1 with 4 cards in the hand can be picked in 4 possible ways. The rest of the suits will have 3 cards each. Given S_1, 4 cards from S_1 can be selected in $\binom{13}{4}$ ways. For the remaining three suits, the cards can be selected in $\binom{13}{3}$ ways from each suit and hence the total number of balanced decks is given by $4\binom{13}{4}\binom{13}{3}^3$.

(b) Find the number of ways of dealing the cards to four (distinguishable) players so that each player gets a balanced hand.

Solution: Each player gets 4 cards from a unique suit. A unique suit could be assigned to a player in 4! ways. Once this assignment is done, the number of ways for P_1 to be dealt a balanced hand are given by $\binom{13}{4}\binom{13}{3}^3$. For P_2, the number of possibilities have reduced to $\binom{10}{4}\binom{9}{3}\binom{10}{3}^2$. For P_3, the number of ways are then given by $\binom{7}{4}\binom{7}{3}\binom{6}{3}^2$. Once these three players have been dealt balanced hands, the remaining cards will form a balanced hand which gives only 1 way for P_4. All this time, we have assumed an ordering among the players, which can be achieved in 4! ways. Hence, the total number of ways is given by $(4!)^2\binom{13}{4}\binom{13}{3}^3\binom{10}{4}\binom{10}{3}^2\binom{9}{3}\binom{7}{4}\binom{7}{3}\binom{6}{3}^2$.

7. Suppose k universities are to be ranked by the Ministry of Education according to some arbitrary criteria. The ranking allows mutiple universities to be tied.

For instance, universities $\{A, B, C, D\}$ may be ranked as $B > A = C > D$, to mean that B is top-ranked, A, C are tied below that, and D is ranked at the bottom; note that $B > C = A > D$ refers to the same ranking.

What is the total number of such possible rankings? You may express your answer in the form of a summation, involving quantities used in the *balls-and-bins* problems.

Solution: A ranking could be viewed as assigning a rank to each university, making sure that the set of ranks used is exactly $[n]$ for some n. For each $n \in [k]$, the number of rankings that use exactly n ranks is $N(k, n)$, the number of onto functions from $[k]$ to $[n]$. So the total number of rankings is $\sum_{n=1}^{k} N(k, n)$.

Alternately, a ranking involves first grouping together universities that are to be tied with each other, and then ordering these groups. There are $S(k, n)$ ways of partitioning the universities into exactly n groups, and then $n!$ ways of ordering the groups. So the total number of rankings is $\sum_{n=1}^{k} n!S(k, n)$.

8. A variant of the balls-and-bins problem, when the balls are distinguishable, is that within each bin, the balls are ordered. Let $L(k, n)$ denote the number of ways k labelled items can be distributed among n lists (i.e., bins with order), where the lists themselves are unlabelled, such that no list is empty. E.g., $L(3, 2) = 6$ since $\{a, b, c\}$ can be split into 2 lists as $\{(a), (b, c)\}$, $\{(a), (c, b)\}$, $\{(b), (a, c)\}$, $\{(b), (c, a)\}$, $\{(c), (a, b)\}$, or $\{(c), (b, a)\}$.

Give a closed form expression for $L(k, n)$.

Hint: First consider the case where the lists are labelled. You may use a variant of stars and bars, with labelled stars. To ensure that no list is empty, you may use stars themselves as bars.

Solution: When the lists are distinguishable, each distribution corresponds to an ordering of all the k elements obtained by concatenating the n lists, with the first item in each list marked. Thus we need to count the ways in which the k elements can be ordered, and n of them marked; but the first element is required to be marked, and we may only choose $n-1$ of the remaining $k-1$ to be marked. This can be done in $k!\binom{k-1}{n-1}$ ways. Compared to when the lists are indistinguishable, this counts each partition into n lists in $n!$. Hence $L(k, n) = \frac{k!}{n!}\binom{k-1}{n-1}$.