# Discrete Structures :: CS 207 :: Autumn 2020

## Problem Set 1

Released: August 21, 2021

1. **Contrapositive.** Show that $p \to q \equiv \neg q \to \neg p$.

   This illustrates the equivalence of the statements "If today is a Sunday then today is a holiday" and "If today is not a holiday, then today is not a Sunday." (Note that these statements are **not equivalent** to "If today is not a Sunday, then today is not a holiday," or, $\neg p \to \neg q$.)

   **Solution:** We have $\neg q \to \neg p \equiv \neg(\neg q) \vee \neg p \equiv q \vee \neg p \equiv p \to q$.

2. **Distributive Property.** To show the equivalences below, you can derive the truth table of the formulas on the LHS and RHS, and compare them. Alternately, for a quicker argument, you can consider two cases, $p \equiv T$ and $p \equiv F$.

   (a) Show that $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

   (b) Show that $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

   (c) What is the condition on a binary operator $\star$ so that $\wedge$ distributes over $\star$ (i.e., $p \wedge (q \star r) \equiv (p \wedge q) \star (p \wedge r)$)? What is the condition for $\vee$ to distribute over $\star$?

   **Solution:** Restricted to $p \equiv T$, irrespective of what $\star$ is, we have $p \wedge (q \star r) \equiv q \star r \equiv (p \wedge q) \star (p \wedge r)$. However, for $p \equiv F$, we have $p \wedge (q \star r) \equiv F$, but $(p \wedge q) \star (p \wedge r) \equiv F \star F$. So, $\wedge$ distributes over $\star$ iff $F \star F \equiv F$. Similarly, $\vee$ distributes over $\star$ iff $T \star T \equiv T$.

   (d) Does $\wedge$ distribute over $\oplus$? Does $\vee$ distribute over $\oplus$?

   **Solution:** By the conditions above, $\wedge$ distributes over $\oplus$, but $\vee$ does not.

3. **Simplifying formulas.**

   Every formula in two variables is equivalent to a binary operator. Identify the operator in the following cases, and write down an equivalent expression.

   (Thus your answer should be one of the 16 possibilities: $T$, $F$, $p$, $q$, $\neg p$, $\neg q$, $p \oplus q$, $p \leftrightarrow q$, $p \wedge q$, $p \vee q$, $p \uparrow q$, $p \downarrow q$, $p \to q$, $q \to p$, $p \not\to q$ and $q \not\to p$.)

   You could prepare a truth table for each formula to help with the task. You could also employ the distributive property, De Morgan's law and other equivalences from the lecture.

   (a) $p \wedge \neg q$
   **Solution:** $p \wedge \neg q$ evaluates to $T$ exactly when $p \equiv T$ and $q \equiv F$. This corresponds to the truth table of $p \not\to q$. (Alternately, note that $p \not\to q \equiv \neg(p \to q) \equiv \neg(\neg p \vee q) \equiv p \wedge \neg q$.)

   (b) $(p \to q) \wedge \neg q$
   **Solution:** $\neg(p \vee q) \equiv p \downarrow q$.

   (c) $p \vee \neg(q \to p)$
   **Solution:** $p \vee q$.

   (d) $(p \wedge q) \to q$
   **Solution:** $T$.

   (e) $(p \wedge q) \leftrightarrow q$
   **Solution:** $q \to p$.

   (f) $(p \leftrightarrow q) \leftrightarrow ((p \wedge q) \vee (\neg p \wedge \neg q))$
   **Solution:** $T$.

4. **Functional Completeness.** A set of operators is *functionally complete* if all $n$-ary logical operations, for any $n > 0$, can be expressed as formulas that use only operators from this set. In other words, all possible truth tables *over any number of inputs* can be produced by formulas that use only these operators.

   Show that the set $\{\neg, \wedge, \vee\}$ is functionally complete.

   [ *Hint: First consider an $n$-ary operation which has a single row in its truth table evaluating to $T$. Can you design an equivalent formula with just $\neg$s and $\wedge$s? Next, if an operation's truth table has $k$ rows that evaluate to $T$, can you design a formula with $k$ terms of the above kind, combined using $\vee$s?* ]

**Solution:** Consider an arbitrary $n$-ary logical operation $f$, for an arbitrary integer $n > 0$. We shall construct a formula for $f(X_1, \ldots, X_n)$.

Let $N$ denote the number of rows in the truth table of $f$ which evaluate to $T$. Let the $i^{\text{th}}$ such row be indexed by a vector $(\alpha_{i,1}, \ldots, \alpha_{i,n}) \in \{T, F\}^n$, such that $f(\alpha_{i,1}, \ldots, \alpha_{i,n}) = T$. Then, for any vector $(x_1, \ldots, x_n) \in \{T, F\}^n$, we have that $f(x_1, \ldots, x_n) = T$ iff $(x_1, \ldots, x_n) \in \{(\alpha_{1,1}, \ldots, \alpha_{1,n}), \ldots, (\alpha_{N,1}, \ldots, \alpha_{N,n})\}$. Now, we construct a formula for $f$. For each $i \in \{1, \ldots, N\}$, define:

$$G_i(X_1, \ldots, X_n) \equiv (X_1 \leftrightarrow \alpha_{i,1}) \wedge \cdots \wedge (X_n \leftrightarrow \alpha_{i,N})).$$

Note that $G_i(x_1, \ldots, x_n) = T$ if and only if $(x_1, \ldots, x_n) = (\alpha_{i,1}, \ldots, \alpha_{i,n})$. Now let

$$F(X_1, \ldots, X_n) \equiv G_1(X_1, \ldots, X_n) \vee \cdots \vee G_N(X_1, \ldots, X_n).$$

We note that $F(x_1, \ldots, x_n) = T$ iff $(x_1, \ldots, x_n) \in \{(\alpha_{1,1}, \ldots, \alpha_{1,n}), \ldots, (\alpha_{N,1}, \ldots, \alpha_{N,n})\}$. Also, as noted above $f(x_1, \ldots, x_n) = T$ iff $(x_1, \ldots, x_n)$ belongs to the same set. Thus $f(X_1, \ldots, X_n) \equiv F(X_1, \ldots, X_n)$

As defined above, $F$ appears to use the operators $\wedge, \vee$ and $\leftrightarrow$. However, the last one is used only in the form $X_j \leftrightarrow \alpha_{i,j}$ where $\alpha_{i,j}$ is specified. If $\alpha_{i,j} \equiv T$, we write $X_j \leftrightarrow \alpha_{i,j}$ as $X_j$ and if $\alpha_{i,j} \equiv F$, we write $X_j \leftrightarrow \alpha_{i,j}$ as $\neg X_j$. Now, $F$ uses only the operators $\wedge$, $\vee$ and $\neg$. Since $f$ could be any $n$-ary operator for any $n > 0$, we conclude that the set $\{\wedge, \vee, \neg\}$ is functionally complete.

5. **A Tautology.** Prove that $\exists x \forall y\ P(x) \rightarrow P(y)$ is true no matter what the predicate $P$ is (assuming that the domain is non-empty).

   [Hint: consider two cases, depending on whether $\forall y\ P(y)$ is true or false.]

   **Solution:** There are two possible cases
   **Case 1:** $\forall y P(y)$ is true.

   - Since the domain is non-empty, there exists at least one element in the domain, let's say $w$.
   - Note that $P(w) \rightarrow P(y)$ for every $y$ since $P(y)$ is true for all $y$.
   - Hence, $(\forall y P(w) \rightarrow P(y))$ is true.
   - From this we can conclude that $\exists x \forall y P(x) \rightarrow P(y)$ is true.

   **Case 2:**

   - $\forall y P(y)$ is false which means $\exists y \neg P(y)$ is true. So, let $a$ be an element such that $\neg P(a)$ is true. Then $P(a)$ is false.
   - Since $P(a)$ is false, $P(a) \rightarrow P(y)$ is true for any $y$. That is, $\forall y,\ P(a) \rightarrow P(y)$ is true.
   - Since, $\forall y,\ P(a) \rightarrow P(y)$ is true, $\exists x \forall y,\ P(x) \rightarrow P(y)$ is true (by considering $x$ to be $a$).

6. **Pointless Games.** Suppose a game has the following structure: Alice specifies an integer $a$, then Bob specifies an integer $b$, and finally Alice specifies an integer $c$. Alice wins the game if $g(a, b, c) = 0$, where $g$ is a function associated with the game; if $g(a, b, c) \neq 0$ Bob wins.

   Alice is said to have a *winning strategy* if there is some way for her to play the game (i.e., pick $a$ and $c$) to ensure that she will win no matter how Bob plays (i.e., picks $b$). Note that Alice can pick $c$ *after seeing* Bob's number $b$.

   (a) Suppose $g(a, b, c) = a + b + c$. Specify a winning strategy for Alice.
   **Solution:** Alice chooses $a = 0$ and $b = -c$.

   (b) Suppose $g(a, b, c) = \max\{a + b, b + c\}$. Specify a winning strategy for Bob.
   **Solution:** Bob chooses $b = 1 - a$.

   (c) Express the proposition that Alice has a winning strategy in the language of first-order predicate calculus.
   **Solution:** $\exists a \in \mathbb{Z}\ \forall b \in \mathbb{Z}\ \exists c \in \mathbb{Z}\ g(a + b + c) = 0$.

   (d) Express the proposition that Bob has a winning strategy.
   **Solution:** $\forall a \in \mathbb{Z}\ \exists b \in \mathbb{Z}\ \forall c \in \mathbb{Z}\ g(a + b + c) \neq 0$.

   (e) Argue that, irrespective of what function $g$ is used, this is a "pointless game": either Alice or Bob has a winning strategy.
   **Solution:** The condition that Bob has a winning strategy is the negation of the condition that Alice has a winning strategy. So, if Alice does not have a winning strategy, the Bob has one.