Quiz 1

CS 207 :: Autumn 2021
August 24, 2021

1. Choose the expression equivalent to

   (a) $(p \to q) \to p$: [1 mark]

   - [ ] A. T
   - [ ] B. F
   - [ ] C. p
   - [ ] D. $\neg p$
   - [ ] E. None of the above

   (b) $p \to (q \to p)$: [1 mark]

   - [ ] A. T
   - [ ] B. F
   - [ ] C. p
   - [ ] D. $\neg p$
   - [ ] E. None of the above

2. $(\forall x\ P(x)) \to (\forall x\ Q(x))$ is equivalent to (assume non-empty domains):

   (a) $\forall x \exists y\ P(y) \to Q(x)$ [1 mark]

   - [ ] A. Equivalent
   - [ ] B. Not equivalent

   (b) $\exists x \forall y\ P(x) \to Q(y)$ [1 mark]

   - [ ] A. Equivalent
   - [ ] B. Not equivalent

   (c) $\forall x \exists y\ P(x) \to Q(y)$ [1 mark]

   - [ ] A. Equivalent
   - [ ] B. Not equivalent

3. Consider the following English statement. Statement: Every huge dinosaur is a sauropod and an adult.

   (a) Write the above statement in first-order logic using predicates $huge$, $adult$ and $sauropod$, defined over the domain D consisting of all dinosaurs (where $huge(x)$ stands for "x is huge," etc.). [1 mark]

   Solution: $huge(x) \to sauropod(x) \wedge adult(x)$.

(b) State the contrapositive of the above statement (as an English statement). [1 mark]

Solution: If a dinosaur is either not a sauropod or not an adult then it is not huge.

(c) State the negation of the above statement (as an English statement). [1 mark]

Solution: A huge dinosaur is either not a sauropod or not an adult.

4. A Conjunctive Normal Form (CNF) formula over a set of variables is a conjunction of disjunctions, or an AND of ORs. Conversely, a Disjunctive Normal Form (DNF) formula is a disjunction of conjunctions, or an OR of ANDs. This is an example of a CNF formula: [3 marks]

$$(p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee \neg q \vee r)$$

Similarly, the following is a DNF formula:

$$(\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$$

Construct CNF and DNF formulas, over the variables p, q, r, for the function f given by the following truth table:

p	q	r	$f(p,q,r)$
F	F	F	T
T	F	F	T
F	T	F	F
F	F	T	T
T	T	F	F
F	T	T	T
T	F	T	T
T	T	T	F

Solution: CNF formula:

$$(p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

DNF formula:

$$(\neg p \wedge \neg q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r)$$

5. Prove that there is no rational solution to the equation $x^5 + 2x^4 + 3x^3 + 4x^2 + 5 = 0$. [3 marks]

Solution: Let us prove this via contradiction. Assume that there is a rational solution, $x = p/q$ with $p, q \in \mathbb{Z}$, to the above equation. Without loss of generality, we can assume that p and q are co-prime to each other. This already rules out the possibility that both of them are even, as any two even numbers have 2 as a common factor. We can also assume that both p and q are non-zero as $x = 0$ is not a solution. Then, there are 3 cases:

- Both p and q are odd. Note that in this case, we have 3 non-zero odd terms and 2 non-zero even terms in the equation. Therefore, the sum can never be zero.

- p is even and q is odd. In this case, the first 4 terms in the equation are even and the final term is a non-zero odd number. Therefore, the sum can never be zero. The case with p being odd and q being even can be argued about similarly.

Since these 3 cases are exhaustive, we have shown that there cannot be a rational solution to the above equation.

6. Prove by strong induction that every positive integer n has a representation as the sum of *distinct* powers of 2. That is, for each n, there is a sequence of integers $(d_1, \ldots, d_t)$ such that $0 \leq d_1 < \ldots < d_t$, and $n = 2^{d_1} + \ldots + 2^{d_t}$. [3 marks]
Be as clear and precise as you can be. (One mark is reserved for good style.)

Solution: Base Case: Consider $n = 1$, since we want to prove the statement about positive integers. Since $1 = 2^0$, the representation for 1 is (0). This is valid as d_1 is allowed to be 0.
Induction Hypothesis: Assume that for every $n \leq k$, for some integer $k \geq 1$, there is a sequence of integers $(d_1, \ldots, d_t)$ such that $0 \leq d_1 < \ldots < d_t$, and $n = 2^{d_1} + \ldots + 2^{d_t}$.
Induction Step: We want to show that the above statement holds even for $n = k + 1$. There are 2 cases:

- n is even. Let $(d'_1, d'_2, \ldots, d'_t)$ be the sequence for $n/2$, as given by the induction hypothesis (as $(k + 1)/2 \leq k$). Therefore, we can write

$$\begin{aligned} n &= 2(2^{d'_1} + \ldots + 2^{d'_t}) \\ &= 2^{d'_1+1} + \ldots + 2^{d'_t+1} \end{aligned}$$

  Since the original sequence was composed of distinct integers, their increments should also be distinct. Hence, $(d'_1 + 1, \ldots, d'_t + 1)$ is a valid sequence for n.
- n is odd. Let $(d'_1, \ldots, d'_t)$ be the sequence for $n - 1$, which is even. It follows that $d'_1 \geq 1$, as only odd integers have d'_1 equal to zero. Consider the sequence $(0, d'_1, \ldots, d'_t)$. The sum of powers of 2 with this sequence evaluates to n. Since all distinct integers in the original sequence were greater than 0, it follows that the integers in the new sequence are also distinct. Therefore, the new sequence is a valid one for n.

7. Consider a set of n lines on a plane such that no two lines are parallel and no three lines intersect at the same point. They divide the plane into several regions (some of which may be unbounded).

   (a) Use strong induction to prove that there will be exactly $1 + n(n + 1)/2$ regions. [4 marks]
   Be as clear and precise as you can be. (One mark is reserved for good style.) You may state and use geometric facts without proof.

   Solution: Base Case: Consider the case with $n = 1$. A single line divides the plane into two regions. For $n = 1$, the expression also evaluates to 2 and hence, the statement holds.
   Induction Hypothesis: Assume that the statement holds for all $n \leq k$, for some $k \geq 1$.
   Induction Step: We want to show that the statement also holds for $n = k + 1$. Any line in this system has exactly one intersection point with every other line. Moreover, the intersection points with two different lines are different because no three lines intersect at a point. Therefore, every line has k intersection points with other lines. If we consider the system without this line, it is divided into $1 + k(k + 1)/2$ regions, as given by the induction hypothesis. Therefore, the addition

of this line adds exactly $k+1$ more regions, with these points being the corners of those regions. Thus, the new number of regions are

$$1 + k(k+1)/2 + (k+1) = 1 + (k+1)(k+2)/2$$

(b) Consider colouring these regions using two colours, red and blue. That is, each region is assigned one of these two colours. Use strong induction to prove that there is always a colouring such that no two adjacent regions (i.e., regions which share a common border) have the same colour. [4 marks]

Be as clear and precise as you can be. (One mark is reserved for good style.) You may state and use geometric facts without proof.