Quiz 3

CS 207 :: Autumn 2021
October 12, 2021

1. Honour code.

2. Given finite sets of integers A, B and an onto function $f : A \to B$, let $g : B \to A$ be defined as follows:

$$g(b) = \min\{\ a | f(a) = b\ \}$$

Indicate which of the following statements are true (select false if not guaranteed to be true):

(a) $f \circ g \circ f = f$

(b) $g \circ f \circ g = g$

(c) $g \circ f : A \to A$ is the identity function.

(d) $f \circ g : B \to B$ is the identity function.

3. Given finite sets of integers A, B, indicate which of the following statements are true (select false if not guaranteed to be true):

(a) $|A - B| + |B - A| = |A \cap B| \leftrightarrow |A| + |B| = 3|A \cap B|$.

(b) Let $A + B$ be defined as the set

$$A + B = \{a + b \mid a \in A, b \in B\}$$

Then, $|A + B| \geq |A \cup B|$.

(c) $|A - B| = |B| \to |A| = 2|B|$.

(d) $|A \cap B| = \min\{|A|, |B|\} \leftrightarrow |A \cup B| = \max\{|A|, |B|\}$.

4. Suppose $T_1, \ldots, T_n$ are finite sets. For each $J \subseteq [n]$, let $S_J = \bigcap_{k \in J} T_k$.

(a) Let $\alpha_J = |S_J|$. Then express $|\bigcup_{k \in [n]} T_k|$ in terms of α_J.

Solution: By the inclusion-exclusion principle, we have that

$$|\bigcup_{k \in [n]} T_k| = \sum_{J \subseteq [n], J \neq \emptyset} (-1)^{|J|+1} \alpha_J$$

(b) Let $R_J = \bigcup_{k \in J} T_k$. Let $\beta_J = |S_J - R_{[n]-J}|$. Then express $|\bigcup_{k \in [n]} T_k|$ in terms of β_J.

Solution: It can be shown that

$$|\bigcup_{k \in [n]} T_k| = \sum_{J \subseteq [n], J \neq \emptyset} \beta_J$$

5. Let $N(k,n)$ denote the number of onto functions from $[k]$ to $[n]$. Show that for all $k,n > 1$,

$$N(k,n) = n(N(k-1,n) + N(k-1,n-1)).$$

   Solution: For defining an onto function, we will first fix a mapping for the element $1 \in [k]$. There are n possibilities for this mapping. It is easy to see that the rest of the function (i.e., ignoring the mapping of 1) is either an onto function from $[k]\backslash\{1\}$ to $[n]$ or to $[n]\backslash\{a\}$, where a denotes the mapping of 1 chosen earlier. Note that the remaining function cannot be an onto function of both kinds. There can thus be $N(k-1,n) + N(k-1,n-1)$ possibilities for the remaining function. Hence, the total number of onto functions are given by the right hand side of the equation in the description.

6. Suppose 4 students A, B, C and D are to be assigned to 3 hostel rooms, Room-1, Room-2 and Room-3. Each room can accommodate upto 3 students. A rule requires that Room-3 can have a student only if the other two rooms are not empty. How many different assignments are possible?

   Solution: Let us first count the number of assignments with no student in Room-3. The goal here is to divide 4 students into 2 rooms so that each room can occupy upto 3 students. There are 3 cases:

   - Room-1 has 3 and Room-2 has 1 student. Such an assignment is completely determined by choosing a student for Room-2, which can be done in 4 ways.
   - Both rooms have 2 students each. This can be done in $\binom{4}{2} = 6$ ways.
   - Room-1 has 1 while Room-2 has 3 students. Since this is symmetric to case 1, there will be 4 such assignments.

   Now let us consider the cases when Room-3 is non-empty. For this to happen, both Room-1 and Room-2 should have at least 1 student. There are 3 possibilities such that two rooms will have a single student while one room will have 2 students. Such a division can be done in $\binom{4}{2}\binom{2}{1} = 12$ ways, irrespective of the case. Therefore, the total number of such assignments are $3 * 12 = 36$.

   Overall, there are $6 + 4 + 4 + 36 = 50$ such assignments.

7. Let n, d be positive integers.

   (a) How many solutions does the equation $x + y + z = n$ have in which x, y, z are all integers greater than or equal to $-d$? Show your reasoning. «««< HEAD

   Solution: Since x, y and z are integers greater than or equal to $-d$, we can write

$$x = x' - d, y = y' - d, z = z' - d$$

   such that x', y' and z' are greater than or equal to 0. Then the equation becomes

$$x' + y' + z' = 3d + n$$

   in which x', y' and z' are integers greater than or equal to 0.

   This is exactly like the "stars and bars" problem. There are $3d + n$ ones (similar to balls) which need to be placed into the 3 integers (similar to bins). If V ones are placed into a variable x, then the value of the variable becomes V.

This problem can be formulated as a "stars and bars" problem with $3d + n$ stars and 2 bars representing the boundaries between the integers. The total number of such combinations can be counted by choosing the positions for the 2 bars in $\binom{3d+n+2}{2}$ ways and the rest of the positions will be occupied by stars.

(b) How many solutions does the equation $x + y + z = n$ have in which x, y, z are all integers in the range $[0, d]$ (inclusive)? You may assume $2(d + 1) \leq n \leq 3d$. Show your reasoning.
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(c) How many solutions does the equation $x + y + z = n$ have in which x, y, z are all integers in the range $[0, d]$ (inclusive)? You may assume $2(d + 1) \leq n \leq 3d$. Show your reasoning.

»»»> dc81a0b0ffe8b6f38e7dd0d011e55a0855dc27a0 Further, confirm your answer by enumerating all the solutions for the case of $n = 10, d = 4$.

Solution: The total number of solutions when $x, y, z \geq 0$ are given by $\binom{n+2}{2}$. Let us denote by S_x the solutions where $x \geq d + 1$ and $y, z \geq 0$ i.e.,

$$S_x = \{(x, y, z) \mid x + y + z = n, x \geq d + 1, y, z \geq 0\}$$

It is easy to see that $|S_x| = \binom{n-(d+1)+2}{2}$. Similarly, we can define S_y and S_z. Note that $|S_x| = |S_y| = |S_z|$ due to symmetry. It can be seen that the undesired solutions are given by $S_x \cup S_y \cup S_z$. Therefore, the desired number of solutions are given by $\binom{n+2}{2} - |S_x \cup S_y \cup S_z|$. Using the inclusion-exclusion principle, we can write

$$\begin{aligned}
|S_x \cup S_y \cup S_z| &= |S_x| + |S_y| + |S_z| - |S_x \cap S_y| - |S_y \cap S_z| - |S_x \cap S_z| + |S_x \cap S_y \cap S_z| \\
&= 3\binom{n - (d+1) + 2}{2} - 3\binom{n - 2(d+1) + 2}{2} + 0 \\
&= 3\left[\binom{n - d + 1}{2} - \binom{n - 2d}{2}\right]
\end{aligned}$$

where $|S_x \cap S_y \cap S_z| = 0$ because $n \leq 3d$. Therefore, the desired number of solutions are given by

$$\binom{n+2}{2} - 3\left[\binom{n - d + 1}{2} - \binom{n - 2d}{2}\right]$$

For $n = 10$ and $d = 4$, the answer comes out to be

$$\binom{12}{2} - 3\left[\binom{7}{2} - \binom{2}{2}\right] = 66 - 3[21 - 1] = 6$$

Note that for the sum to be 10, at least one of the variables will have to be 4; otherwise the sum would have been at most 9. For the remaining values, there are two possiblities - both variables are 3 or one is 4 and another is 2. Therefore, the possible values of the variables are (4, 4, 2) and (4, 3, 3). For the ordering, there are 3 possibilities in both cases which implies that overall there are 6 solutions.

8. Let S and T be two finite sets of integers, with $|S| = m$ and $|T| = n$. Let f be some arbitrary function from S to T.

Define a relation $\preccurlyeq$ over S, so that $x \preccurlyeq y$ iff $f(x) < f(y)$ or $x = y$.

(a) Argue that $(S, \preccurlyeq)$ is a poset.

Solution: We need to argue 3 properties:

- Reflexive: for all $x \in S$, $x \preccurlyeq x$ because $x = x$.
- Anti-symmetric: for any $x, y \in S$ s.t. $x \neq y$, $x \preccurlyeq y \implies f(x) < f(y)$ and $y \preccurlyeq x \implies f(y) < f(x)$. Thus both these being true leads to a contradiction.
- Transitive: for any $x, y, z \in S$, $x \preccurlyeq y \implies f(x) < f(y)$ and $y \preccurlyeq z \implies f(y) < f(z)$. Combining the two, we get that $f(x) < f(z)$ and hence $x \preccurlyeq z$.

(b) What is the smallest possible value (for fixed S and T, over all choices of f) for the maximum size of an antichain in this poset? Justify your answer.

Solution: Using Dilworth's theorem, we can say that the size of the largest antichain is exactly the least number of chains needed to partition S. Note that in any chain C, the elements are ordered based on their images i.e., if $C = (x_1, \ldots, x_l)$ then $f(x_1) < \ldots < f(x_l)$. Since there can only be at most $|T| = n$ different possible values for these images, we can say that $|C| \leq n$. Therefore, the least number of chains needed to partition S are $|S|/n = m/n$.

9. Let $(S, \preccurlyeq)$ be a finite poset and let f be a function from S to itself such that $x \preccurlyeq f(x)$ for all $x \in S$. Let f^k denote f composed with itself k times (i.e., $f^1 = f$, and recursively, $f^k = f \circ f^{k-1}$).

(a) For each $x \in S$, define the set

$$K(x) = \{ \ y \mid \exists k, f^k(x) = y \ \}$$

Show that $K(x)$ is a chain in the poset.

Solution: Consider any $y, z \in K(x)$ s.t. $y \neq z$, which means that there exist m, n s.t. $m \neq n$, and $f^m(x) = y$ and $f^n(x) = z$. WLOG we can assume that $m < n$. Therefore,

$$y = f^m(x) \ \square f^{m+1}(x) \ \square \ \ldots \ \square \ f^n(x) = z$$

Since $\square$ is a transitive relation (by definition of a poset), we have that $y \ \square \ z$.

(b) Let h be the size of the largest chain in the poset. Characterize the image of f^h. Briefly justify your answer. (A full proof is not required.)

Solution: Note that any chain in the poset must involve applications of the function f to the same original element x. Therefore, chains of the type $K(x)$, for all $x \in S$, are the largest chains in the poset. Such chains end in fixed points of f i.e., $y \in S$ s.t. $f(y) = y$. Therefore, starting from an element x, such a chain consists of application of the function f a total of h times until a fixed point y is reached. Hence, the image of f^h is some fixed point of f.