Discrete Structures :: CS 207 :: Autumn 2021

Problem Set 8

Released: November 2, 2021

1. Define a function $g : \mathbb{N} \to \mathbb{R}$ recursively as follows:

   - $g(1) = 1$
   - $g(n+1) = 1 + \frac{1}{g(n)}$ for all integers $n \geq 1$

   Use induction to prove that $g(n) = f(n+1)/f(n)$ for all $n \in \mathbb{Z}^+$, where $f(n)$ is the n^{th} Fibonacci number.

   Note: As n tends to infinity, $g(n)$ tends to the positive solution of the quadratic equation given by $x = 1 + \frac{1}{x}$. This number, $\frac{1+\sqrt{5}}{2} \approx 1.618$ is sometimes called the "golden ratio."

2. Let $f(n)$ denote the n^{th} Fibonacci number. Then show (without using the closed form expression for $f(n)$) that

   (i) $f(0) - f(1) + f(2) - ... - f(2n-1) + f(2n) = f(2n-1) - 1$ where n is a positive integer.

   (ii) $f(0)f(1) + f(1)f(2) + ... + f(2n-1)f(2n) = f(2n)^2$ where n is a positive integer.

3. A partition of a positive integer n is a way to write n as a sum of positive integers where the order of terms in the sum does not matter. For instance, $7 = 3 + 2 + 1 + 1$ is a partition of 7. Let $P(m)$ equal the number of different partitions of m, and let $Q(m, n)$ be the number of different ways to express m as the sum of positive integers not exceeding n.

   (i) What are the values of j such that $P(m) = Q(m, j)$ holds?

   (ii) Show that the following recursive definition for $Q(m, n)$ is correct :

   $$Q(m,n) = \begin{cases} 1 & \text{if } m = 1 \\ 1 & \text{if } n = 1 \\ Q(m,m) & \text{if } m < n \\ 1 + Q(m, m-1) & \text{if } m = n > 1 \\ Q(m, n-1) + Q(m-n, n) & \text{if } m > n > 1 \end{cases}$$

   (iii) Find the number of partitions of 4 and of 5 using this recursive definition.

4. Let us define a permutation π of the set $\{1, ..., n\}$ to be *fragmented* if there is a number k with $1 \leq k < n$ such that π maps the subset $\{1, 2, ..., k\}$ into itself. Let $c(n)$ be the number of permutations over $\{1, \dots, n\}$ that are *not* fragmented. Prove that

   $$\sum_{i=1}^{n} c(i)(n-i)! = n!$$

   Suppose $G_f(X) = \sum_{n \geq 1} n! X^n$ and $G_c(X) = \sum_{n \geq 1} c(n) X^n$ are the generating functions of the functions $f(n) = n!$ and $c(n)$ respectively (defined without a constant term). Then $G_c(X) = G_f(X)/(1 + G_f(X))$.

   Hint: You can rewrite the recurrence relation as $n! - c(n) = \sum_{i=1}^{n-1} (n-i)! c(i)$ and the relation to prove as $G_f(X) - G_c(X) = G_f(X) G_c(X)$.

5. Let $s(n)$ be the number of sequences $(x_1, ..., x_k)$ of integers satisfying $1 \leq x_i \leq n$ for all i and $x_{i+1} \geq 2x_i$ for $i = 1, ..., k-1$. (The length of the sequence is not specified; in particular, the empty sequence is included.) Prove the recurrence

   $$s(n) = s(n-1) + s(\lfloor n/2 \rfloor)$$

   for $n \geq 1$, with $s(0) = 1$. Show that the generating function $G_s(X)$ satisfies $(1 - X)G_s(X) = (1 + X)G_s(X^2)$.

6. Find the generating function $G_f(X)$ for each f below.

   (a) $\forall n \geq 0,\ f(n) = n$.

   (b) $\forall n \geq 0,\ f(n) = n^2$.

(c) $f(0) = a$, and $\forall n > 0$, $f(n) = f(n-1) + b$.

(d) $f(0) = f(1) = 0$, $f(2) = 1$, and $\forall n > 2$, $f(n) = f(n-1) + f(n-2) + f(n-3)$.

(e) $f(0) = 0$, and $\forall n > 0$, $f(n) = 2f(n-1) + 3^n$.

7. If the generating functions of two functions f and g satisfy the identity $G_g(X) = G_f(X)(1-X)$, define g in terms of f.

8. Prove that for $n \in \mathbb{Z}^+$, $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$.

9. Use the extended binomial theorem to find the coefficient of X^{10} in the power series of each of the following expressions. (Express your answers without using $\binom{n}{k}$ for any $n \notin \mathbb{Z}^+$.)

   (a) $X^4/(1-3X)^3$

   (b) $X^4/(1-X^3)$

   (c) $1/(1-X^3)$

   (d) $1/\sqrt{1-4X}$

10. For the function f recursively defined in Problem 6(c), find a closed form for it using its generating function G_f.

11. Find the closed form expression for the n^{th} Fibonacci number.