# Problem Set 4

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1. Given a set $S$, its powerset is defined as the set of all subsets of $S$. That is, $\mathcal{P}(S) = \{T \mid T \subseteq S\}$. Describe $\mathcal{P}(\{1,2\}))$ explicitly.

   **Solution:** The different subsets of $S = \{1,2\}$ are $\emptyset, \{1\}, \{2\}$ and $\{1,2\}$. Thus, $\mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$.

2. **Subsets as bit strings.** Given a set $S$ with $|S| = n$, we may use bit-strings to conveniently represent the subsets of $S$. For this, we fix an arbitrary ordering of the $n$ elements in $S$. Then, $T \subseteq S$ is represented by the $n$-bit string $x_T$ such that the $i^{\text{th}}$ bit of $x_T$ is 1 iff the $i^{\text{th}}$ element of $S$ (in the order we have fixed) is in $T$. In answering the following, you can use boolean operators to $n$-bit strings, where the operation is applied bit-wise (e.g., $001 \oplus 010 = 011$, $\neg 001 = 110$).

   (a) Express $x_{A\cap B}, x_{A\cup B}$ and $x_{A-B}$ in terms of $x_A$ and $x_B$.

   **Solution:** $x_{A\cap B} = x_A \wedge x_B$ because the $i^{\text{th}}$ element of $S \in A \cap B$ iff it belongs to both $A$ and $B$. Similarly, $x_{A\cup B} = x_A \vee x_B$ i.e. the bit-wise OR of the 2 bit strings. Also, the $i^{\text{th}}$ element $\in x_{A-B}$ iff it belongs to $A$ but not in $B$. Thus, $x_{A-B} = x_A \wedge (\neg x_B)$.

   (b) Describe the set $T$ in terms of $A, B, C$, if $x_T = x_A \oplus x_B \oplus x_C$.

   **Solution:** For any 3 bits $p, q$ and $r$, $p \oplus q \oplus r =$ True iff odd number of variables are True i.e. either only $p, q$ or $r$ are True or all 3 of them are True. This translates to T as all the elements which are present in any one of $A$, $B$ or $C$ along with the elements which are present in all the 3 sets.

3. **A Set representing Prime Factorization.** For every positive integer $n$, define a set $PF_n \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$ to denote the prime factors of $n$, as follows.

$$PF_n = \{(p,i) : p \text{ is prime}, i \in \mathbb{Z}^+ \text{ and } (p^i \mid n)\}.$$

   (a) What is $PF_1$?

   **Solution:** $PF_1 = \emptyset$ because the only integer that divides 1 is 1 itself and 1 is not a prime.

   (b) Explicitly write down $PF_{12}$ and $PF_{30}$.

   **Solution:** $PF_{12} = \{(2,1),(2,2),(3,1)\}$ as $12 = 2^2 3^1$. Similarly, $PF_{30} = \{(2,1),(3,1),(5,1)\}$ as $30 = 2^1 3^1 5^1$.

   (c) Write down $PF_{\text{gcd}(12,30)}$.

   **Solution:** One can calculate $\text{gcd}(12,30) = 6$. Hence, $PF_{\text{gcd}(12,30)} = PF_6 = \{(2,1),(3,1)\}$ because $6 = 2^1 3^1$.

   (d) Write down $PF_{\text{lcm}(12,30)}$.

   **Solution:** One can calculate $\text{lcm}(12,30) = 60$. Hence, $PF_{\text{lcm}(12,30)} = PF_{60} = \{(2,1),(2,2),(3,1),(5,1)\}$ because $60 = 2^2 3^1 5^1$.

   (e) Suppose $a|b$ for positive integers $a, b$. What is the relation between $PF_a$ and $PF_b$?
   **Solution:** $PF_a \subseteq PF_b$. To see this, consider $(p,i) \in PF_a$. By definition of $PF_a$, $p^i \mid a$. But since $a|b$, we have $p^i \mid b$. Thus $(p,i) \in PF_b$.

   (f) For any two positive integers $m$ and $n$, give formulas for $PF_{\text{gcd}(m,n)}$ and $PF_{\text{lcm}(m,n)}$ in terms of $PF_m$ and $PF_n$.
   **Solution:**
   CLAIM: $PF_{\text{gcd}(m,n)} = PF_m \cap PF_n$.
   PROOF: Firstly, we note that $a|b$ iff $PF_a \subseteq PF_b$. One direction was shown above. In the other direction note that $a = \prod_{(p,i)\in PF_a} p$. Hence if $PF_a \subseteq PF_b$, then $b = (\prod_{(p,i)\in PF_a} p)(\prod_{(p,i)\in PF_b - PF_a} p)$, and so $a|b$.
   Hence, $g$ is a common divisor of $m, n$ iff $PF_g \subseteq PF_m \cap PF_n$. Let $g^* = \prod_{(p,i)\in PF_m \cap PF_n} p$. That is, for each prime $p$, the highest power of $p$ that is a divisor for $g^*$ is the smaller of those for $m$ and $n$. Then it can be seen that $PF_{g^*} = PF_m \cap PF_n$. Hence $g^*$ is a common divisor of $m, n$. Further, any common divisor $g$ is such that $g|g^*$ and hence $g \leq g^*$. Thus $g^* = \text{gcd}(m,n)$.
   CLAIM: $PF_{\text{lcm}(m,n)} = PF_m \cup PF_n$.
   PROOF: Using the fact (argued above) that $a|b$ iff $PF_a \subseteq PF_b$, we have that $h$ is a common multiple of $m, n$ iff $PF_m \cup PF_n \subseteq PF_h$. Now let $h^* = \prod_{(p,i)\in PF_m \cup PF_n}$ so that $PF_{h^*} = PF_m \cup PF_n$. Hence $h^*$ is a common multiple of $m, n$ and further, for any common multiple $h$, we have $h^*|h$ and hence $h^* \leq h$. Thus $h^* = \text{lcm}(m,n)$.

(g) Conclude from the above that, if $x$ is a common divisor and $y$ is a common multiple of two positive integers $m, n$, then $x|\gcd(m,n)$ and $\text{lcm}(m,n)|y$.

4. Give an example for each of the following, if such an example exists. Else prove why it cannot exist.

(a) A relation that is irreflexive, antisymmetric and not transitive.

**Solution:** A simple finite example is a relation defined over a set $\{1,2,3\}$ by the set of pairs $\{(1,2),(2,3)\}$.
As an example over an infinite set, we define relation $a \sqsubset b$ to hold iff $b = a^2$ where $a, b \in \mathbb{Z}^+ \backslash \{1\}$. This relation is irreflexive as for any $a \geq 2$, $a^2 \neq a$. It is antisymmetric because if $b = a^2$ then $a \neq b^2$ as the only integer values for $b$ that solve this equation are $b = \{0,1\}$. It is not transitive because $b = a^2$ and $c = b^2$ implies $c = a^4$ and not $c = a^2$.

(b) A relation that is neither symmetric nor antisymmetric.

**Solution:** Suppose we define $a \sqsubseteq b$ as $\text{slope}(a) \geq \text{slope}(b)$ where the set $S$ is the set of all lines in 2-D. For 2 different lines $l_1, l_2$ such that $\text{slope}(l_1) = \text{slope}(l_2)$, both $l_1 \sqsubseteq l_2$ and $l_2 \sqsubseteq l_1$ but for 2 lines with different slopes, only one of these possibilities can happen.

(c) An antisymmetric relation which has a symmetric relation as its subset.

**Solution:** Given any antisymmetric relation $R_A$, consider all $(a,a) \in R_A$. All such elements taken together will give a symmetric relation (trivially) which will be a subset of $R_A$.

(d) Relations $R_1$ and $R_2$ on set $S$ such that both are symmetric but $R_1 \cap R_2$ is not symmetric.

**Solution:** We can prove that given 2 symmetric relations $R_1$ and $R_2$, $R_1 \cap R_2$ is also symmetric. For the sake of contradiction, assume that $R_1 \cap R_2$ is not symmetric. Thus, $\exists (a,b) \in R_1 \cap R_2$ such that $(b,a) \notin R_1 \cap R_2$. We can say that $(a,b) \in R_1$ and $(a,b) \in R_2$ (definition of intersection). Hence, $(b,a) \in R_1$ and $(b,a) \in R_2$ (both $R_1$ and $R_2$ are symmetric). Thus, $(b,a) \in R_1 \cap R_2$. Hence, we get a contradiction.

5. Given a relation $R$ over a ground set $S$, and a subset $T \subseteq S$, define the relation $R|_T$ *induced* by $R$ on $T$ as follows:

$$R|_T = R \cap (T \times T).$$

That is, for every pair $(a,b) \in T \times T$, $(a,b) \in R|_T$ iff $(a,b) \in R$. Which of the following statements are true for any relation $R$ over $S$ and any $T \subseteq S$? Justify your answer with a proof or a counterexample.

(a) If $R$ is symmetric, so is $R|_T$.

**Solution:** True. Suppose $R$ is symmetric and let $(x,y) \in R|_T$. Then $(x,y) \in T \times T$ and $(x,y) \in R$. Since $R$ is symmetric, $(y,x) \in R$. Also, $(y,x) \in T \times T$. Therefore, $(y,x) \in R|_T$. Hence, $R|_T$ is symmetric.

(b) If $R$ is irreflexive, so is $R|_T$.

**Solution:** True. We prove this by contradiction. Suppose R is irreflexive and let $(x,x) \in R|_T$. Then, by definition, $(x,x) \in R$, which means $R$ is not irreflexive, which is a contradiction. Therefore, $R$ is irreflexive.

(c) If $R$ is not reflexive, nor is $R|_T$.

**Solution:**
False. Consider S = $\{1,2\}$, R = $\{(1,1),(1,2)\}$, T = $\{1\}$. Then, $R|_T = \{(1,1)\}$. Clearly, R is not reflexive over S, but $R|_T$ is over T. Therefore, the proposition is false.

(d) If $R$ is a partial order, so is $R|_T$.

**Solution:** True. In addition to (a), we prove the following.
(i) If R is reflexive, so is $R|_T$.
(ii) If R is transitive, so is $R|_T$.
Suppose, R is reflexive. Then $\forall x \in T \subseteq S$, $(x,x) \in R$. But as $(x,x) \in T \times T$, $(x,x) \in R|_T$. Hence, (i) is true.
Suppose R is transitive and $(x,y),(y,z) \in R|_T$. Then, $(x,y),(y,z) \in R$ and $x,y,z \in T$. Since, R is transitive, $(x,z) \in R$ and $(x,z) \in T \times T$. Therefore, $(x,z) \in R|_T$. Hence, (ii) is true.
Now, if R is a partial order, then R is reflexive, symmetric and transitive. From above, this implies that $R|_T$ is reflexive, symmetric and transitive. Therefore $R|_T$ is also a partial order.

6. Given a relation $R$, define $R^2$ as follows:

$$R^2 = \{(a,b) | \exists c\ (a,c) \in R \text{ and } (c,b) \in R\}.$$

Show the following.

(a) If $R$ is symmetric, so is $R^2$.

**Solution:** Suppose $R$ is symmetric. Let $(x,y) \in R^2$. This implies $\exists z$ s.t $(x,z) \in R, (z,y) \in R$. Since, R is symmetric, this means $(y,z) \in R, (z,y) \in R$. Therefore, $(y,x) \in R^2$. Hence, $R^2$ is symmetric.

(b) $R^2$ being symmetric does not imply that $R$ is symmetric.

**Solution:** We prove this by giving a counter example. Let R = $\{(1,4),(1,2),(2,3),(2,1),(3,2)\}$ be a relation over S = $\{1,2,3,4\}$. Then, $R^2 = \{(1,3),(3,1)\}$. It can be seen that $R^2$ is symmetric but not $R$. Hence, the statement is true.

(c) If $R$ is reflexive and transitive, $R = R^2$.

**Solution:** Suppose $R$ is reflexive and transitive. We shall prove that $R \subseteq R^2$ and $R^2 \subseteq R$.
Let $(x,y) \in R$. Since, R is reflexive, $(y,y) \in R$. Therefore, by definition of $R^2$, $(x,y) \in R^2$. Thus, for all $(x,y) \in R$, $(x,y) \in R^2$.
Let $(x,y) \in R^2$, then by definition, $\exists z, (x,z) \in R, (z,y) \in R$. Since, $R$ is transitive, and $(x,z),(z,y) \in R$, $(x,y) \in R$. Thus, for all $(x,y) \in R^2$, $(x,y) \in R$.
From the above two, we conclude that, $R = R^2$.

7. Let $S$ be the set of all colourings of the $2 \times 2$ checkerboard where each of the four squares is coloured either red or blue. Note that $S$ has 16 elements. Let $R$ be a relation on $S$, so that $(C_1, C_2) \in R$ if and only if $C_2$ can be obtained from $C_1$ by rotating the checkerboard.

(a) Show that $R$ is an equivalence relation.

**Solution:** Let's define a turn as rotating the board by $90^o$ clockwise and a negative turn as rotating in anticlockwise.
As every board can be obtained by rotating 0 turns, $R$ is reflexive relation.
Let for board positions $C_1, C_2$, $(C_1, C_2) \in R$ and let $C_2$ can be obtained by rotating $C_1$ $x$ turns. Then rotating $C_2$ by $-x$ turns give $C_1$. Therefore, $(C_2, C_1) \in R$. Hence, $R$ is symmetric.
Let $(C_1, C_2)$ and $(C_2, C_3) \in R$ and $C_2$ is obtained by turning $C_1$ by $x$ turns and $C_3$ is obtained by turning $C_2$ by $y$ turns. Then, $C_3$ is obtained by turning $C_1$ $(x+y)$ turns. Hence, $(C_1, C_3)$ is in $R$.
Hence, $R$ is transitive.
From above, we conclude that $R$ is an equivalence relation.

(b) What are the equivalence classes of $R$? For each equivalence class, describe one member in the class and the size of the class.

**Solution:** In each of the equivalence classes of R each board position can be obtained from other by rotation. We denote an equivalence class, as 4-tuple, the colours of the tiles in clockwise direction. R stands for colour red and B stands for Blue. There 6 equivalence classes in R. They are,
1. All the tiles are coloured red. (R, R, R, R). It has 1 element.
2. All the tiles are coloured blue. (B, B, B, B). It has 1 element.
3. 3 tiles are coloured blue. (B, B, B, R). It has 4 element.
4. 3 tiles are coloured red. (R, R, R, B). It has 4 element.
5. 2 tiles are coloured red. (B, B, R, R). It has 4 elements.
6. 2 tiles are coloured red. (R, B, R, B). It has 2 elements.

8. Let $(S, \preceq)$ be a (non-empty) poset. We write $a \prec b$ if we have $a \preceq b$ and $a \neq b$. An element $a \in S$ is called *maximal* if $\nexists b \in S$ s.t. $a \prec b$. Similarly, an element $a \in S$ is called *minimal* if $\nexists b \in S$ s.t. $b \prec a$.

(a) Consider a restriction of the divisibility poset to a small set, $(\{2,4,5,10,12,20,25\}, |)$. What are its maximal and minimal elements?

**Solution:** The minimal elements are 2 and 5. The maximal elements are 12,20, and 25.

(b) Consider poset $(\mathcal{P}(S), \subseteq)$ for some set $S$. What are its maximal and minimal elements?

**Solution:** Clearly, the empty set $\phi$ is a minimal element of this poset. Also, for any other subset $X$ of $S$, $\phi \subseteq X$ and $X \neq \phi$, so $X$ cannot be a minimal element. Therefore, $\phi$ is the only minimal element of the poset.
Similarly, it is easy to see that $S$ is a maximal element of this poset. For any other subset $X$ of $S$, $X \subseteq S$ and $X \neq S$, so $X$ cannot be a maximal element. Therefore $S$ is the only maximal element of the poset.

(c) Show that every maximal chain in a finite poset $(S, \preceq)$ contains a minimal element of $S$. (A maximal chain is a chain that is not a subset of a larger chain.)

**Solution:** Let $C$ be a maximal chain in $S$. Consider the element $a$ in $C$ such that $a \preceq b$ for all $b$ in $C$. We claim $a$ is a minimal element in $S$.
Suppose not. Then there exists an element $s$ in $S$ such that $s \preceq a$ and $s \neq a$. Consider the subset $C' = C \cup \{s\}$. We claim $C'$ is a chain in $S$. Indeed, for any two elements $b, c$ in $C'$, if neither $b$ nor $c$ equals $s$, then $b, c$ are elements in $C$, so they are comparable. Furthermore, $s$ is comparable with any element in $C$. This is because for any element $b$ in $C$, $s \preceq a$ and $a \preceq b$, so $s \preceq b$ by transitivity.
However, $C'$ is now a chain strictly containing $C$, which is a maximal chain. This is a contradiction to the maximality of $C$. Therefore $a$ is a minimal element in $S$.

9. In the context of relations, the term *lattice* is used to refer to a poset in which every finite set of elements has both a least upper bound and a greatest lower bound. Prove that the following posets are lattices. In each case, define the least upper bound and greatest lower bound of any finite set of elements.

(a) $(\mathcal{P}(X), \subseteq)$, the set of subsets of $X$ with the inclusion relation.

**Solution:** Consider a finite set $T \subseteq \mathcal{P}(X)$. Let $T = \{X_1, X_2, \cdots, X_k\}$ where each $X_i \in \mathcal{P}(X)$ (i.e., $X_i \subseteq X$). Let $W = X_1 \cup \cdots \cup X_k$ and $Z = X_1 \cap \cdots \cap X_k$. We claim that $W$ is the lowest upper bound and $Z$ the greatest lower bound of $T$. Firstly, $Z$ is a lower bound of $T$, because $Z \subseteq X_i$ for all $1 \leq i \leq k$. Furthermore, suppose $Y$ is some lower bound of $T$. Then $Y \subseteq X_i$ for all $1 \leq i \leq k$, and hence $Y \subseteq Z$. Thus, $Z$ is a lower bound of $T$ and for every lower bound $Y$, it holds that $Y \subseteq Z$. Thus $Z$ is the greatest lower bound of $T$. Similarly, $W$ can be shown to be the least upper bound of $T$.

(b) The divisibility poset, $(\mathbb{Z}^+, |)$.

*Hint: You may use the fact from Problem 3(g) generalized to any finite number of integers.*

**Solution:** Let $T = \{m_1, \cdots, m_k\}$, where each $m_i \in \mathbb{Z}^+$. We claim $g = \gcd(m_1, \cdots, m_k)$ is the greatest lower bound of $T$. Indeed, it is easy to see that $g$ is a lower bound, because $g \mid m_i$ for all $1 \leq i \leq k$. Furthermore, if $s$ is any other lower bound, then $s \mid m_i$ for all $1 \leq i \leq k$. That is $s$ is a common divisor of these $k$ numbers. Suppose the prime factorization of $s$ is $\prod_{i=1}^{t} p_i^{d_i}$. Then we know (using the hint) that $s \mid g$. That is $g$ is a lower bound for $T$ and for every lower bound $s$ it holds that $s \mid g$. Therefore $g$ is the greatest lower bound of $T$ in this poset. Similarly, it can be shown that $l = \text{lcm}(m_1, m_2, \cdots, m_k)$ is the least upper bound of $T$.
An alternate argument is to identify the divisibility poset as an inclusion poset over the power set of $P \times \mathbb{Z}^+$ where $P$ is the set of primes (using Problem 3), and then appeal to the previous part.

10. Recall that the Mirsky's theorem stated in class states that in a poset $P$, the size of the largest chain in a poset $P$ is of size $k$, is exactly equal to the smallest number of anti-chains that can partition $P$.

(a) Write out a formal proof for this, filling in all the details.

**Solution:** We prove the theorem by induction on $k$. For convenience of notation, let $P = (S, \preceq)$.
**Base Case:** If $k = 1$, then no two distinct elements in $S$ can be related to each other (otherwise those two elements form a chain of size $2 > k$). This means the entire poset $P$ is an anti-chain, so we need only one part to partition $P$ into anti-chains.
**Induction Hypothesis:** Now, suppose the result is true for $k = m$. We now need to show that if the size of the largest chain in $P$ is $m + 1$, then the smallest number of anti-chains needed to partition $P$ is equal to $m + 1$. Let this smallest number of anti-chains required be denoted by $a$. Also, let $C$ be a chain of size $m + 1$ in $P$; let the $m + 1$ distinct elements of $C$ be $c_1 \preceq c_2 \preceq \cdots \preceq c_{m+1}$.
**Induction Step:** The first observation is that $a \geq m + 1$. Indeed, in any anti-chain $A$ of $P$, no two elements of $C$ can both be in $A$ (as they are comparable). This means we need at least as many anti-chains to partition $P$ as the number of elements in $C$, which is $m + 1$.

Now, consider all minimal elements $M$ of $P$. Note that minimal elements form an anti-chain. Remove $M$ from the set $S$, and consider the poset $P'$ on the remaining elements. We claim that the maximum size of a chain in $P'$, denoted by $m'$, is equal to $m$.

Recall from problem 8 that $c_1$ is a minimal element in $S$. Also, no other element in $C$ can be a minimal element. Therefore, $C \backslash \{c_1\}$ is a chain of size $m$, which proves $m' \geq m$.
Now we show $m' \leq m$. Suppose not. Then $P'$ has a chain of size $m+1$, say $D$. This chain $D$ is also a chain in the original poset $P$ before deletion of the elements in $M$. But from problem 8, $D$ has a minimal element $d$. This means $d \in M$, which is a contradiction, as we had deleted all minimal elements. Therefore $m' \leq m$, and combined with $m' \geq m$, we get $m' = m$.

By induction hypothesis, $P'$ can be partitioned into $m$ anti-chains. Combined with $M$, we obtain a partition of $P$ into $m+1$ anti-chains, proving $a \leq m+1$. Therefore, $a = m+1$, as required.

(b) Prove that any poset with $n$ elements must have either (i) a chain *and* an anti-chain both of length equal to $\sqrt{n}$, or (ii) a chain *or* an anti-chain of length greater than $\sqrt{n}$.

**Solution:** Suppose $k$ is the size of the largest chain in a poset $P$ with $n$ elements. By Mirsky's Theorem, $P$ can be partitioned into $k$ anti-chains. By Pigeonhole Principle, one of these anti-chains, $A$, has size at least $\frac{n}{k}$.
If $k > \sqrt{n}$, then we have found a chain of length greater than $\sqrt{n}$. Else if $k < \sqrt{n}$, then the anti-chain $A$ has size strictly greater than $\frac{n}{\sqrt{n}} = \sqrt{n}$. Otherwise $k = \sqrt{n}$, so $\sqrt{n}$ is an integer. Now, $A$ has size at least $\sqrt{n}$ too; take a subset of $A$ of size $\sqrt{n}$; this subset is an anti-chain of size $\sqrt{n}$. Therefore we found a chain of size $\sqrt{n}$ and an anti-chain of size $\sqrt{n}$.

(c) Consider the numbers from 1 to $n$ arranged in an arbitrary order on a line. Prove that there must exist a $\sqrt{n}$-length subsequence of these numbers that is completely increasing or completely decreasing as you move from right to left. For example, the sequence 7, 8, 9, 4, 5, 6, 1, 2, 3 has an increasing subsequence of length 3, for example: 1, 2, 3, and a decreasing subsequence of length 3, for example: 9, 6, 3. *Hint: Define an appropriate poset that considers the value of each number as well as its position on the line.*

**Solution:** Each entry on the line can be represented by a pair of natural numbers $(v, p)$, where $v$ is the value of the number, and $p$ is the position of the number on the line. Here, $1 \leq v, p \leq n$. Note that there are $n$ such pairs of naturals; let the set containing these elements be $S$.
Define a relation $\preceq$ on the elements of $S$ as follows: $(v_1, p_1) \preceq (v_2, p_2)$ if $v_1 \leq v_2$ and $p_1 \leq p_2$. It is a straightforward check that $\preceq$ is in fact, a partial order on $S$. Let $P = (S, \preceq)$.
Using the previous problem, there exists either a chain of length at least $\sqrt{n}$ or an anti-chain of length at least $\sqrt{n}$, in $P$. Suppose the former is true, that is, $C$ is a chain of length $k \geq \sqrt{n}$ in $P$. Then the elements of $C$ can be written as $(v_1, p_1) \preceq (v_2, p_2) \preceq \cdots (v_k, p_k)$. This means the values $v_1, v_2, \cdots, v_k$ form an increasing sequence from left to right (that is, a decreasing sequence from right to left).
Otherwise, suppose the later is true, that is, $A$ is an antichain of length $k \geq \sqrt{n}$ in $P$. Let the elements of $A$ be $(v_1, p_1), (v_2, p_2), \cdots, (v_k, p_k)$, where $v_1 < v_2 < \cdots < v_k$ (note that $v_i \neq v_j$, because the line has all $n$ distinct naturals arranged on it). For any $i < j$, since $v_i < v_j$ and $(v_i, p_i)$ and $(v_j, p_j)$ are incomparable, we must have $p_i > p_j$. This means $p_1 > p_2 > \cdots > p_k$. This implies that the values $v_1, v_2, \cdots, v_k$ form an increasing sequence from right to left.