CS 228 : Logic in Computer Science

Krishna. S

Implications of the Game on FO definability

FO Definability

L is FO definable $\Rightarrow$ there exists an r such that for every (w_1, w_2) pair, such that $w_1 \in L, w_2 \notin L$, spoiler wins in r rounds

Implications of the Game on FO definability

FO Definability

L is FO definable $\Rightarrow$ there exists an r such that for every (w_1, w_2) pair, such that $w_1 \in L$, $w_2 \notin L$, spoiler wins in r rounds

Non FO Definability

For all $r \geqslant 0$, there exists a (w_1, w_2) pair with $w_1 \notin L$, $w_2 \in L$, duplicator wins in r rounds $\Rightarrow$ L is not FO definable

Non FO$[<]$ definability

- $FO[<, S] \subseteq FO[<]$
- Non definability in $FO[<]$ implies non definability in $FO[S, <]$

$(aa)^*$ is not $FO[<]$ Definable

- Assume that there is a sentence φ that defines words of even length, with $c(\varphi) = r$.
- Then, $a^i \models \varphi$ iff i is even
- Show that for all $r > 0$, $a^{2^r} \sim_r a^{2^r - 1}$

$(aa)^*$ is not $FO[<]$ Definable

- Base case : $(a, \emptyset)(a, \emptyset)$ and $(a, \emptyset)$ for $r = 1$
- In one round, duplicator wins on $(a, \emptyset)(a, \emptyset)$ and $(a, \emptyset)$

$(aa)^*$ is not $FO[<]$ Definable

- Base case : $(a, \emptyset)(a, \emptyset)$ and $(a, \emptyset)$ for $r = 1$
- In one round, duplicator wins on $(a, \emptyset)(a, \emptyset)$ and $(a, \emptyset)$
- Consider $(aaaa, aaa)$ for $r = 3$. Who wins?
- Consider $(aaaa, aaa)$ for $r = 2$. Who wins?

$(aa)^*$ is not $FO[<]$ Definable

- Show that for all $k \geqslant 2^r - 1$, duplicator has a winning strategy for the r-round game in (a^k, a^{k+1}), for all $r \geqslant 0$
- Induct on r
- If $r = 1$, then on (a, aa) duplicator wins in one round
- Assume now that the claim is true for $\leqslant r - 1$

$(aa)^*$ is not $FO[<]$ Definable

- Let $k \geqslant 2^r - 1$, and consider the structures

$$(a^k, a^{k+1})$$

- Spoiler puts pebble z_1 in one of the words obtaining

$$(a, \emptyset)^s (a, \{z_1\}) (a, \emptyset)^t$$

$(aa)^*$ is not $FO[<]$ Definable

- Let $k \geqslant 2^r - 1$, and consider the structures

$$(a^k, a^{k+1})$$

- Spoiler puts pebble z_1 in one of the words obtaining

$$(a, \emptyset)^s (a, \{z_1\})(a, \emptyset)^t$$

- $s \leqslant \frac{k-1}{2}$ or $t \leqslant \frac{k-1}{2}$

$(aa)^*$ is not $FO[<]$ Definable

- Assume $s \leqslant \frac{k-1}{2}$. Duplicator puts her pebble z_1 on the $(s+1)$th letter of the other word obtaining

$$(a, \emptyset)^s (a, \{z_1\})(a, \emptyset)^{t'}$$

where $t' = t + 1$ or $t' = t - 1$.

$(aa)^*$ is not $FO[<]$ Definable

- Assume $s \leqslant \frac{k-1}{2}$. Duplicator puts her pebble z_1 on the $(s+1)$th letter of the other word obtaining

$$(a,\emptyset)^s(a,\{z_1\})(a,\emptyset)^{t'}$$

  where $t' = t+1$ or $t' = t-1$.
- The structures after round 1 are thus

$$(a,\emptyset)^s(a,\{z_1\})(a,\emptyset)^{t}, (a,\emptyset)^s(a,\{z_1\})(a,\emptyset)^{t'}$$

$(aa)^*$ is not $FO[<]$ Definable

- Assume $s \leqslant \frac{k-1}{2}$. Duplicator puts her pebble z_1 on the $(s+1)$th letter of the other word obtaining

$$(a, \emptyset)^s (a, \{z_1\})(a, \emptyset)^{t'}$$

where $t' = t + 1$ or $t' = t - 1$.

- The structures after round 1 are thus

$$(a, \emptyset)^s (a, \{z_1\})(a, \emptyset)^{t}, (a, \emptyset)^s (a, \{z_1\})(a, \emptyset)^{t'}$$

- We have $2^r - 1 \leqslant k = min(t, t') + s + 1 \leqslant min(t, t') + \frac{k-1}{2} + 1$

$(aa)^*$ is not $FO[<]$ Definable

- Assume $s \leqslant \frac{k-1}{2}$. Duplicator puts her pebble z_1 on the $(s+1)$th letter of the other word obtaining

$$(a,\emptyset)^s(a,\{z_1\})(a,\emptyset)^{t'}$$

  where $t' = t+1$ or $t' = t-1$.
- The structures after round 1 are thus

$$(a,\emptyset)^s(a,\{z_1\})(a,\emptyset)^{t}, (a,\emptyset)^s(a,\{z_1\})(a,\emptyset)^{t'}$$

- We have $2^r - 1 \leqslant k = min(t,t') + s + 1 \leqslant min(t,t') + \frac{k-1}{2} + 1$
- Hence $min(t,t') \geqslant \frac{k-1}{2} \geqslant 2^{r-1} - 1$

$(aa)^*$ is not $FO[<]$ Definable

- Assume $s \leqslant \frac{k-1}{2}$. Duplicator puts her pebble z_1 on the $(s+1)$th letter of the other word obtaining

$$(a, \emptyset)^s (a, \{z_1\})(a, \emptyset)^{t'}$$

  where $t' = t + 1$ or $t' = t - 1$.
- The structures after round 1 are thus

$$(a, \emptyset)^s (a, \{z_1\})(a, \emptyset)^{t}, (a, \emptyset)^s (a, \{z_1\})(a, \emptyset)^{t'}$$

- We have $2^r - 1 \leqslant k = min(t, t') + s + 1 \leqslant min(t, t') + \frac{k-1}{2} + 1$
- Hence $min(t, t') \geqslant \frac{k-1}{2} \geqslant 2^{r-1} - 1$
- By inductive hypothesis, duplicator has a winning strategy for the $r - 1$ round game on $(a^t, a^{t'})$.

Duplicator's Win

- Use the duplicator's winning strategy for the $r-1$ round game on $(a^t, a^{t'})$, to obtain a winning strategy in $r-1$ rounds on

$$(a,\emptyset)^s(a,\{z_1\})(a,\emptyset)^t, (a,\emptyset)^s(a,\{z_1\})(a,\emptyset)^{t'}$$

Duplicator's Win

- Use the duplicator's winning strategy for the $r-1$ round game on $(a^t, a^{t'})$, to obtain a winning strategy in $r-1$ rounds on

$$(a,\emptyset)^s(a,\{z_1\})(a,\emptyset)^t, (a,\emptyset)^s(a,\{z_1\})(a,\emptyset)^{t'}$$

- Whenever spoiler plays on a structure on letter $i \leqslant s+1$, duplicator plays on the same position on the other structure

Duplicator's Win

- Use the duplicator's winning strategy for the $r-1$ round game on $(a^t, a^{t'})$, to obtain a winning strategy in $r-1$ rounds on

$$(a,\emptyset)^s(a,\{z_1\})(a,\emptyset)^t, (a,\emptyset)^s(a,\{z_1\})(a,\emptyset)^{t'}$$

- Whenever spoiler plays on a structure on letter $i \leqslant s+1$, duplicator plays on the same position on the other structure
- When spoiler plays at a position $i > s+1$ in either word, duplicator plays in the part of the other word $> s+1$ using her winning strategy in $(a^t, a^{t'})$

Duplicator's Win

- At the end of r rounds, we have structures w_1', w_2'.
- For $i \leqslant s+1$, pebble z_j appears at position i of w_1' iff pebble z_j appears at position i of w_2'
- Lets erase the first $s+1$ letters in w_1', w_2', obtaining v_1', v_2'
- v_1', v_2' are the words that result after $r' \leqslant r-1$ rounds of play on $(a^t, a^{t'})$. Recall that duplicator won this.
- Show that w_1', w_2' satisfy the same atomic formulae

Duplicator's Win

- Atomic Formulae : $Q_a(z_j)$: Both w_1', w_2' satisfy this.
- $w_1' \models z_i < z_j$. If z_i, z_j are in the first $s+1$ letters, then $w_2' \models z_i < z_j$.
- If z_i, z_j occur in the last $|w_1'| - s - 1$ positions, then $v_1' \models z_i < z_j$. By duplicator's win in $(a^t, a^{t'})$, $v_2' \models z_i < z_j$
- If z_i appears among the first $s+1$ letters and z_j after the first $s+1$ letters of w_1', same is true in w_2'.

Historically Speaking

The games that we saw are due to Ehrenfeucht and Fraïssé

Reference: Finite Automata, Formal Logic and Circuit Complexity, by Howard Straubing.