Logic Optimization

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CS-230: Digital Logic Design & Computer Architecture

Lecture 10 (31 January 2022)

CADSL

Logic Minimization.

Objective. → Cost ✓
Delay.

1, 2, 3, 4. variable.

ab \ cd	00	01	11	10
00	m_0	m_1	m_3	m_2
01	m_4	m_5	m_7	m_6
11	m_{12}	m_{13}	m_{15}	m_{14}
10	m_8	m_9	m_{11}	m_{10}

PI

SPI

EPI ✓

RPI ✗

Five-Variable Function

- F(A,B,C,D,E)

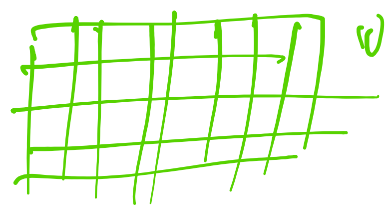
  $= \sum m(0,1,4,5,6,13,14,15,22,24,25,28,29,30,31)$

a bcde +
ā bcde
⇓
bcde

A = 0 (B across the last two columns, C across the middle two columns, D down the last two rows, E down the middle two rows)

0 1	4 1	12	8
1 1	5 1	13 1	9
3	7	15 1	11
2	6 1	14 1	10

ā b c e .



A = 1 (B across the last two columns, C across the middle two columns, D down the last two rows, E down the middle two rows)

	00	01	11	10
00	16	20	28 1	24 1
01	17	21	29 1	25 1
11	19	23	31 1	27
10	18	22 1	30 1	26

abce

$F = \bar{A}\,\bar{B}\,\bar{D} + A\,B\,\bar{D} + B\,C\,E + C\,D\,\bar{E}$

·bce

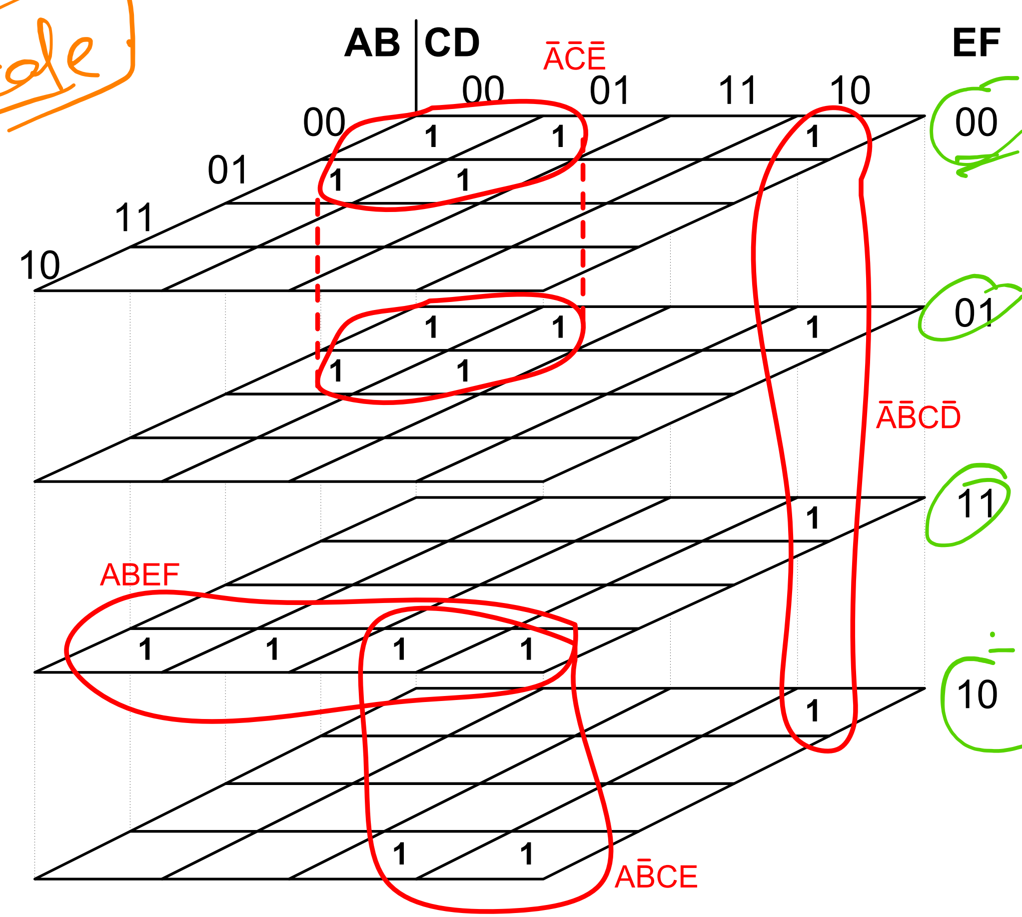
Why Not More Than Five Variables?

- Too hard to visualize in 3+ dimensions
- Systematic approach: "Tabular Methods"
  - Used in computer programs
- Why K-Maps at all?
  - Faster for quick optimizations
  - Understand Boolean logic and HW design
  - Design with simplification in mind

Scale

f(e,f,a,b,c,d)

Six-Variable K-Map



64

16, 16, 16, 16

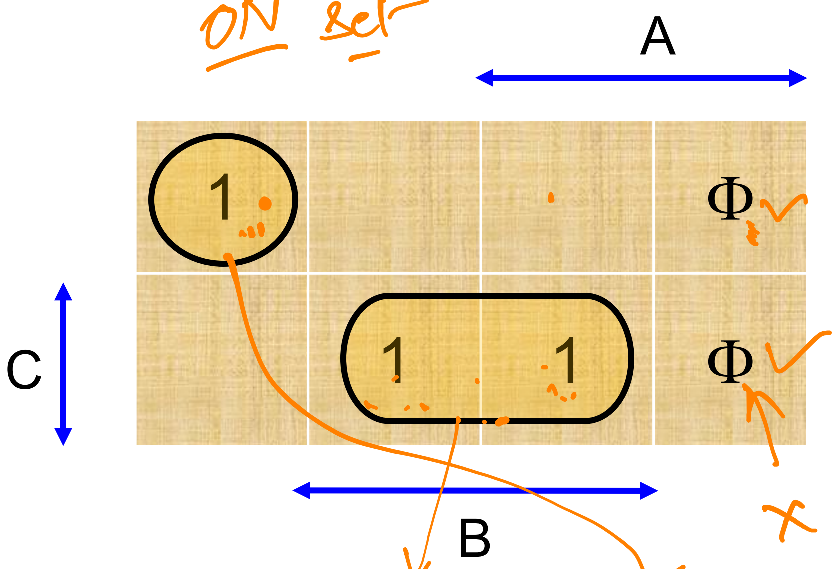
CADSL

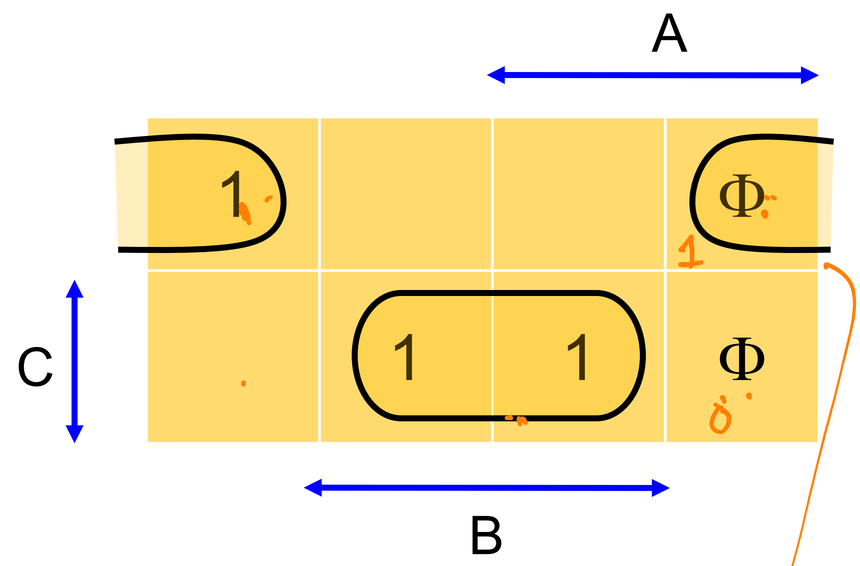
Functions with Don't Care Minterms

- $F(A,B,C) = \sum m(0,3,7) + d(4,5)$ DC Set
- Include don't care minterms when beneficial.

ON ·
DC
OFF

ON Set



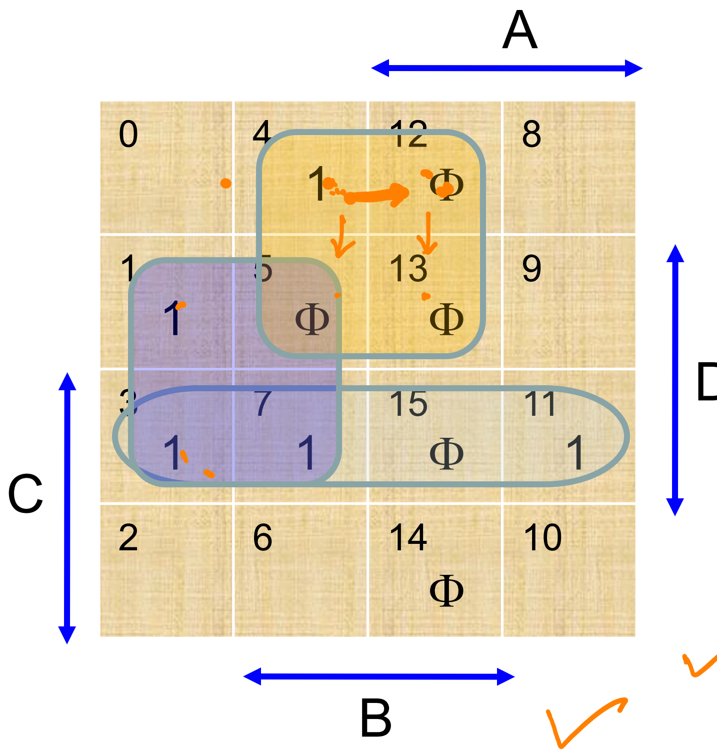


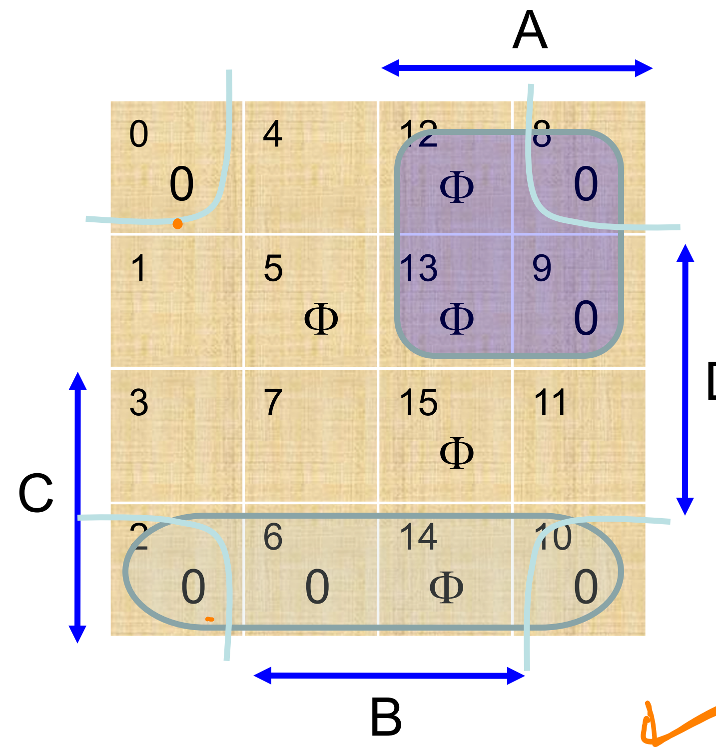
$F = B\,C + \bar{A}\;\bar{B}\;\bar{C}$

$F = B\,C + \bar{B}\;\bar{C}$

Minimized SOP and POS

- $F(A,B,C,D) = \sum m(1,3,4,7,11) + d(5,12,13,14,15)$
  $= \prod M(0,2,6,8,9,10)\ D(5,12,13,14,15)$





$F = \underbrace{B\,\overline{C} + \overline{A}D + C\,D}$

SOP'

$\overline{F} = \overline{B}\,\overline{D} + C\,\overline{D} + A\,\overline{C}$

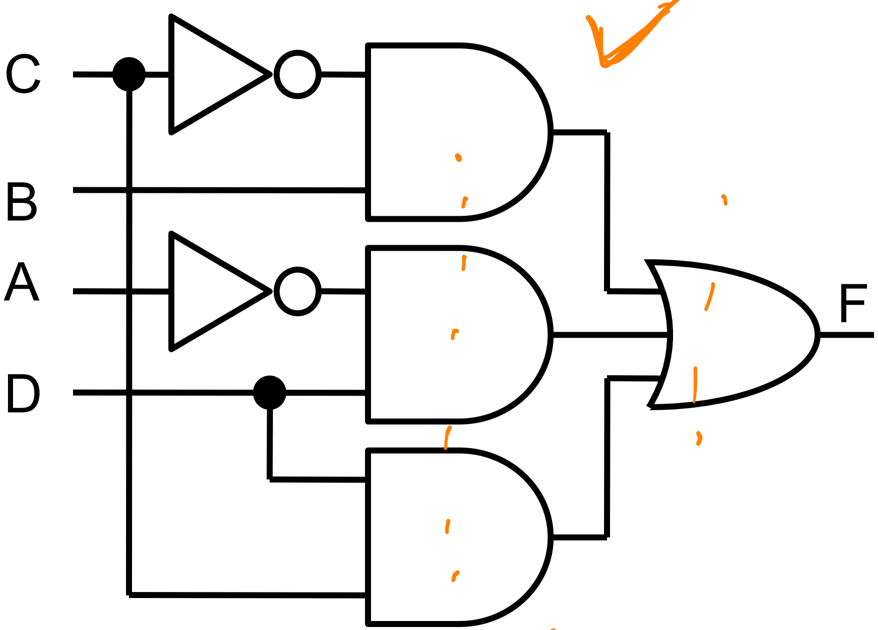
$F = (B + D)(\overline{C} + D)(\overline{A} + C)$

POS'

SOP and POS Circuits

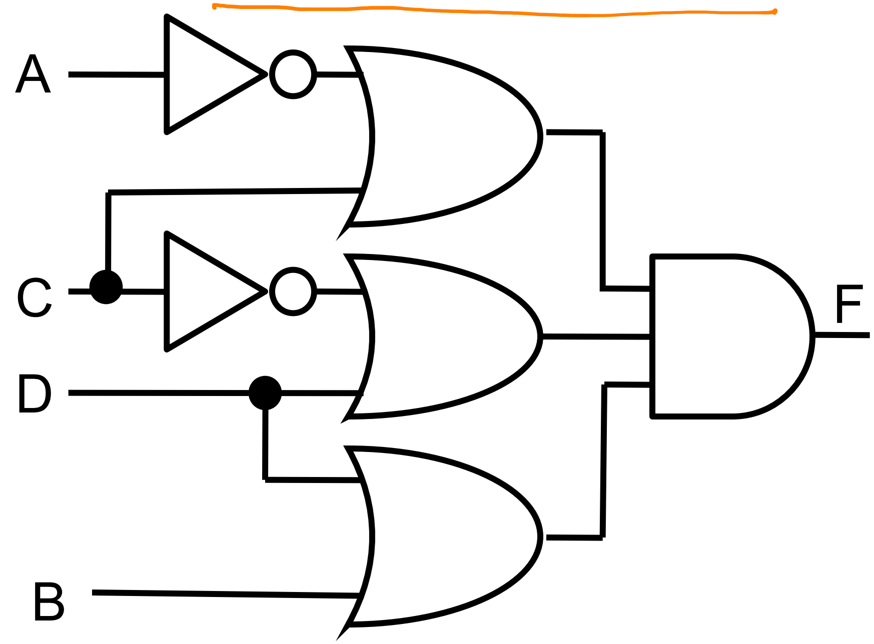
- $F(A,B,C,D) = \sum m(1,3,4,7,11)+d(5,12,13,14,15)$
  $= \prod M(0,2,6,8,9,10)\ D(5,12,13,14,15)$

$F = B\,\overline{C} + \overline{A}D + C\,D$



NAND-NAND

$F = (B + D)(\overline{C} + D)(\overline{A} + C)$



NOR-NOR

Multiple-Output Minimization

Multiple-Output Minimization

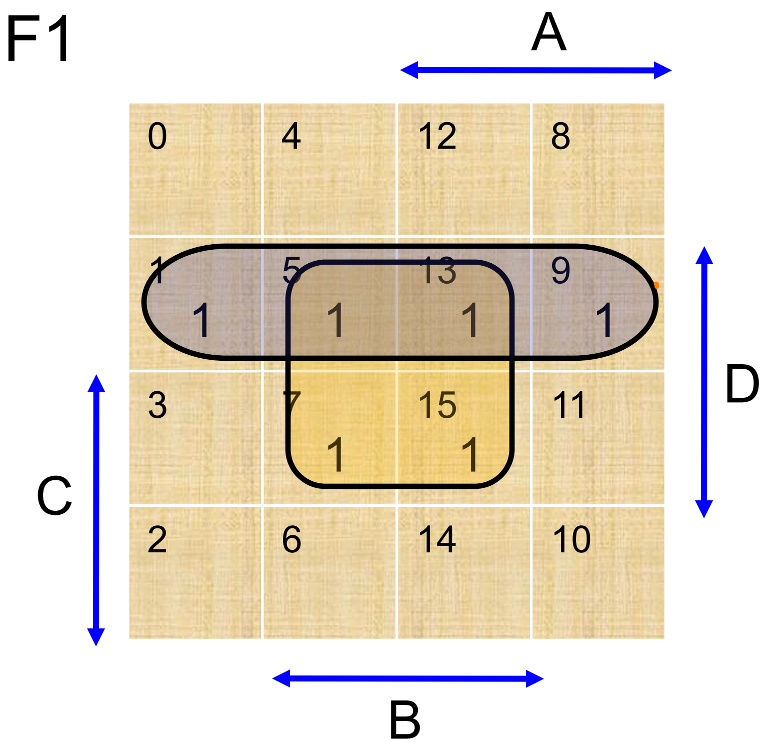
Inputs				Outputs	
A	B	C	D	F1	F2
0	0	0	0	0	0
0	0	0	1	1	0
0	0	1	0	0	0
0	0	1	1	0	0
0	1	0	0	0	0
0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	1	1	1
1	0	0	0	0	1
1	0	0	1	1	1
1	0	1	0	0	1
1	0	1	1	0	1
1	1	0	0	0	0
1	1	0	1	1	1
1	1	1	0	0	0
1	1	1	1	1	1

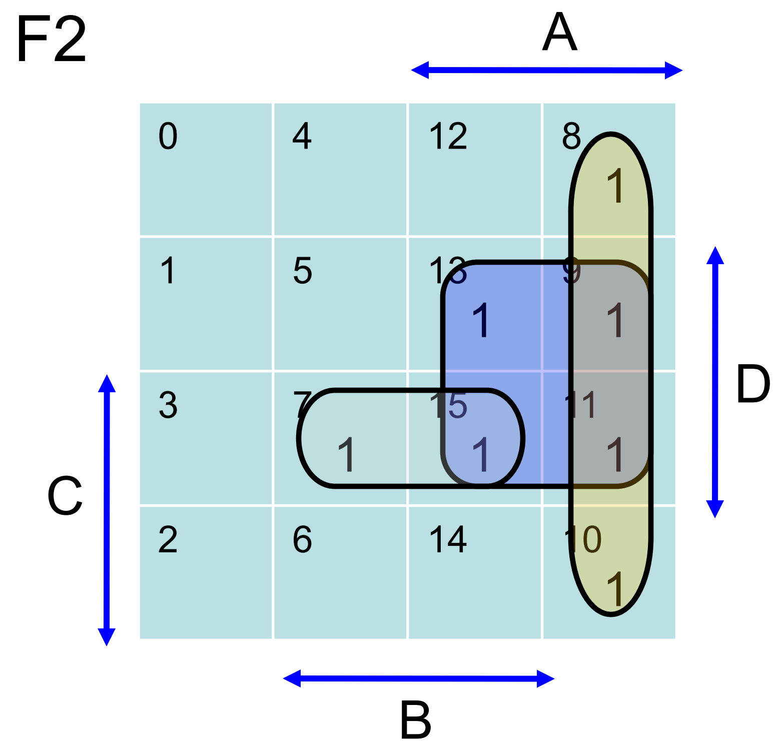
$f_1(A,B,C,D)$

$f_2(A,B,C,D)$

Individual Output Minimization

Need five products.



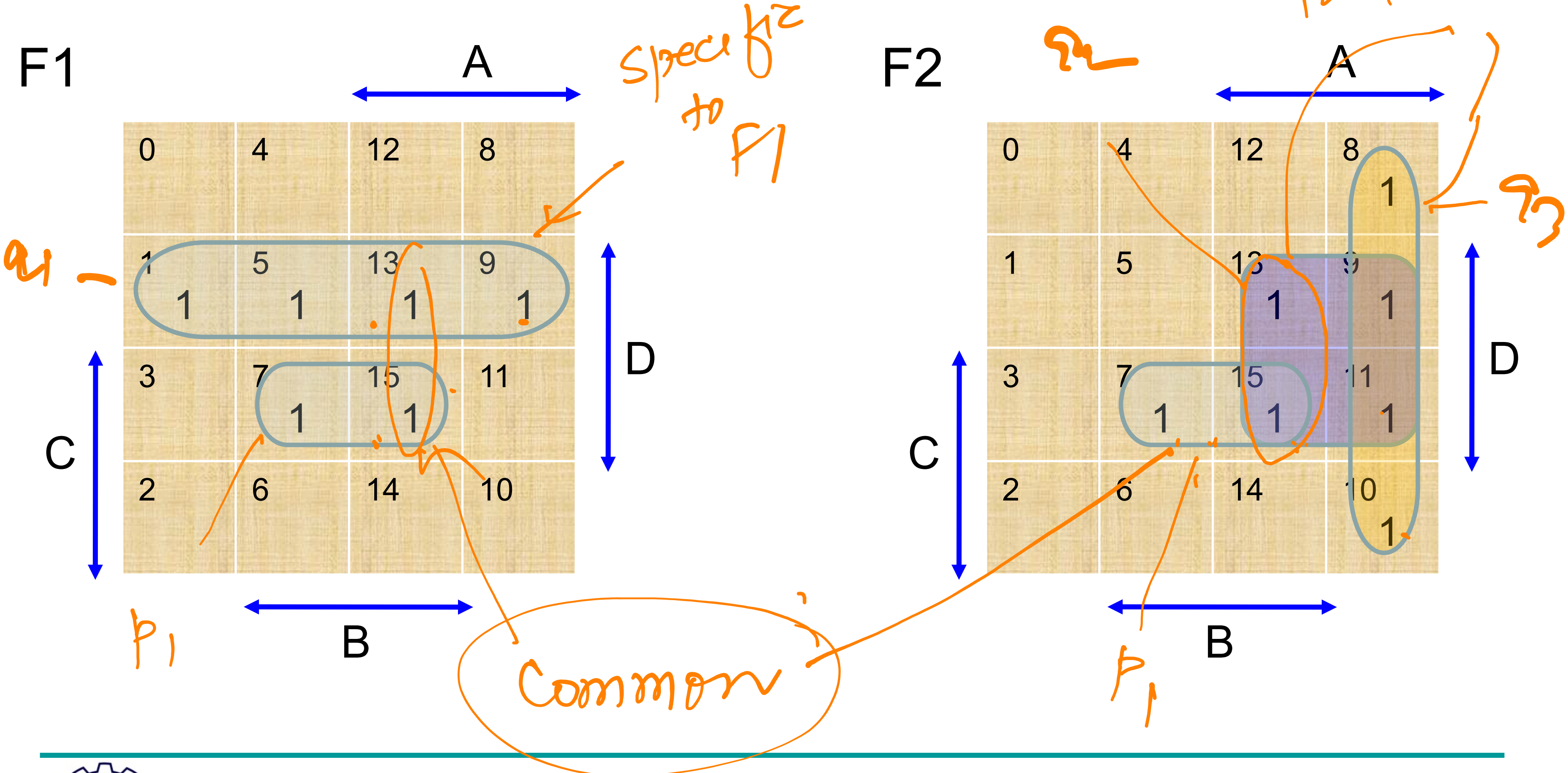


Global Minimization

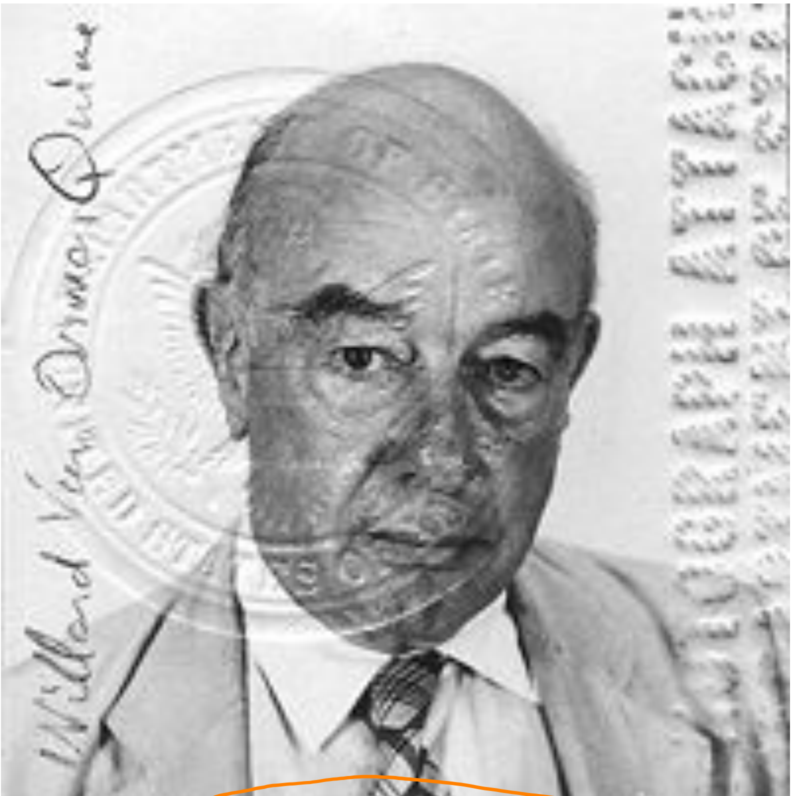
5 variable
6 Variable

F1 F2

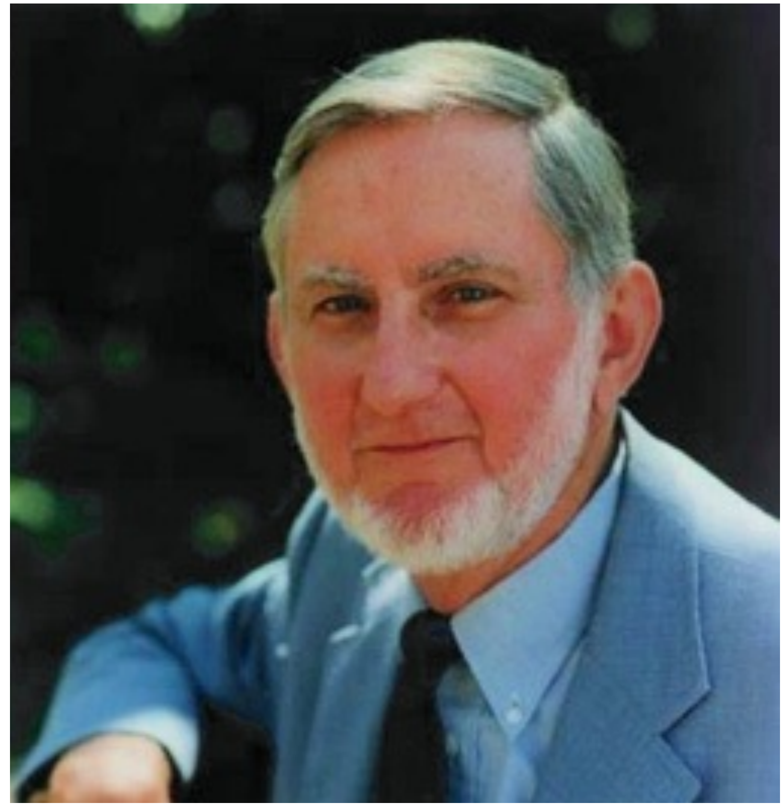
Need four products.



Quine-McCluskey



Willard V. O. Quine
1908 – 2000



Edward J. McCluskey
1929 -- 2016

Quine-McCluskey Tabular Minimization Method

- W. V. Quine, "The Problem of Simplifying Truth Functions," *American Mathematical Monthly*, vol. 59, no. 10, pp. 521-531, October 1952.

- E. J. McCluskey, "Minimization of Boolean Functions," *Bell System Technical Journal*, vol. 35, no. 11, pp. 1417-1444, November 1956.

Q-M Tabular Minimization

- Minimizes functions with many variables.
- Begin with minterms:
  - Step 1: Tabulate minterms in groups of increasing number of true variables.
  - Step 2: Conduct linear searches to identify all prime implicants (PI).
  - Step 3: Tabulate PI's vs. minterms to identify EPI's.
  - Step 4: Tabulate non-essential PI's vs. minterms not covered by EPI's. *Select* minimum number of PI's to cover all minterms.
- MSOP contains all EPI's and *selected* non-EPI's.

EPI

abc, $ab\bar{c}$ $\Rightarrow ab$

abc, $a\bar{b}c \Rightarrow ac$

abc, $\bar{a}bc$ bc

$F(A,B,C,D) = \sum m(2,4,6,8,9,10,12,13,15)$

- Q-M Step 1: Group minterms with 1 true variable, 2 true variables, etc.

0010 – 2
0100 – 4
1000 – 8

0100

Minterm	ABCD	Groups
2	0010	1: single 1
4	0100	
8	1000	
6	0110	2: two 1's
9	1001	
10	1010	
12	1100	
13	1101	3: three 1's
15	1111	4: four 1's

G1

G2

(2,6)

(2,10)

Q-M Step 2

- Find all implicants by combining minterms, and then combining products that differ in a single variable: For example,
  - 2 and 6, or $\bar{A}\ \bar{B}\ C\ \bar{D}$ and $\bar{A}\ B\ C\ \bar{D} \rightarrow \bar{A}\ C\ \bar{D}$, written as 0 – 1 0.
- Try combining a minterm (or product) with all minterms (or products) listed below in the table.
- Include resulting products in the next list.
- If minterm (or product) does not combine with any other, mark it as PI.
- Check the minterm (or product) and repeat for all other minterms (or products).

Step 2 Executed on Example

List 1			List 2			List 3		
Minterm	ABCD	PI?	Minterms	ABCD	PI?	Minterms	ABCD	PI?
2	0010	X	2, 6	0-10	PI_2	8,9,12,13	1-0-	PI_1
4	0100	X	2,10	-010	PI_3			
8	1000	X	4,6	01-0	PI_4			
6	0110	X	4,12	-100	PI_5			
9	1001	X	8,9	100-	X			
10	1010	X	8,10	10-0	PI_6			
12	1100	X	8,12	1-00	X			
13	1101	X	9,13	1-01	X			
15	1111	X	12,13	110-	X			
			13,15	11-1	PI_7			

EPL / SPL

Step 3: Identify EPI's

Covered by EPI →				x	x		x	x	x
Minterms →	2	4	6	8	9	10	12	13	15
PI_1 is EPI				x	x		x	x	
PI_2	x		x						
PI_3	x					x			
PI_4		x	x						
PI_5		x					x		
PI_6				x		x			
PI_7 is EPI								x	x

PI_1 & PI_7 EPI

Step 4: Cover Remaining Minterms

	Remaining minterms →	2	4	6	10
x_2	PI_2	x		x	
x_3	PI_3	x			x
x_4	PI_4		x	x	
x_5	PI_5		x		
x_6	PI_6				x

Integer linear program (ILP), available from MATLAB and other sources: Define integer {0,1} variables, x_k = 1, select PI_k; x_k = 0, do not select PI_k.

Minimize $\sum_k x_k$, subject to constraints:

$x2 + x3 \geq 1$
$x4 + x5 \geq 1$
$x2 + x4 \geq 1$
$x3 + x6 \geq 1$

$\min(x_2 + x_3 + x_4 + x_5 + x_6)$

A solution is x3 = x4 = 1, x2 = x5 = x6 = 0, or select PI_3, PI_4

Linear Programming (LP)

- A mathematical optimization method for problems where some “cost” depends on a large number of variables.
- An easy to understand introduction is:
  - S. I. Gass, *An Illustrated Guide to Linear Programming*, New York: Dover Publications, 1970.
- Very useful tool for a variety of engineering design problems.
- Available in software packages like MATLAB.

Step 4: Cover Remaining Minterms

	Remaining minterms →	2	4	6	10
x_2	PI_2	x		x	
x_3	PI_3	x			x
x_4	PI_4		x	x	
x_5	PI_5		x		
x_6	PI_6				x

Patrick's Method

Logical.

$$(x_2+x_3)\cdot(x_4+x_5)\cdot(x_2+x_4)\cdot(x_3+x_6)=1$$

SOP expression — POS

$x_2=x_3=x_4=x_5=x_6=1$ ✗

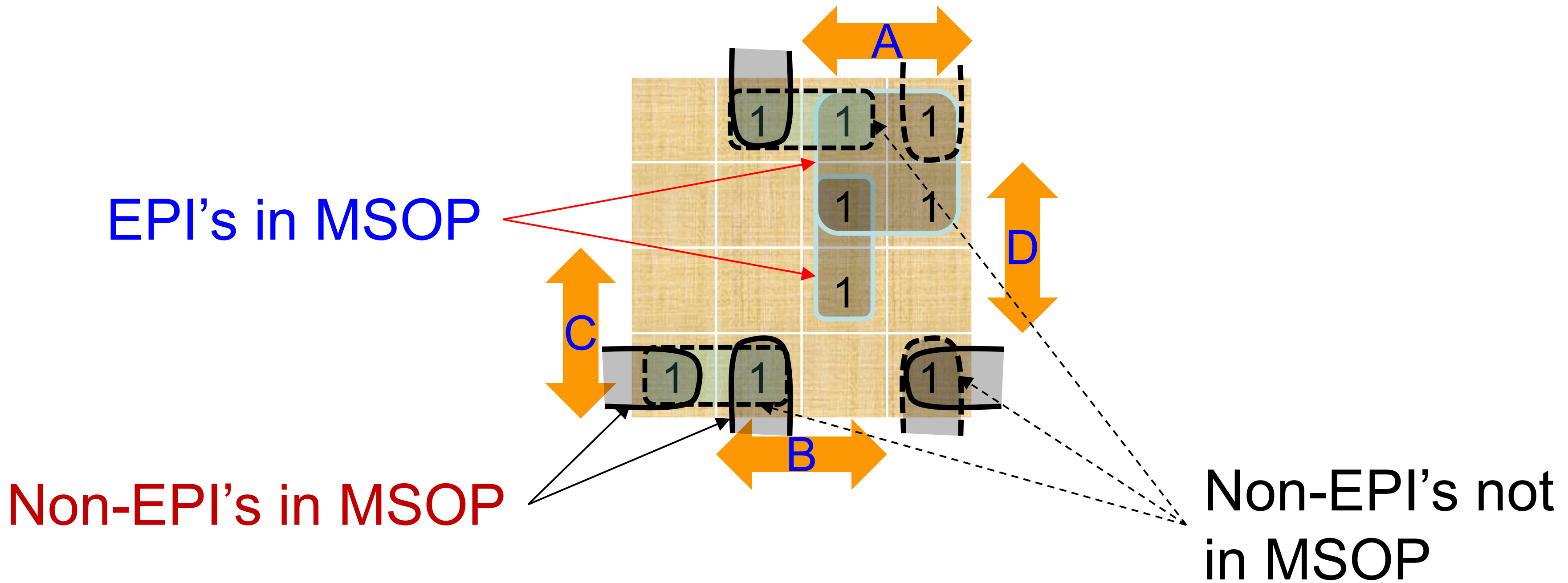
$$(x_2x_4+x_2x_5+x_3x_4+x_3x_5)$$

Q-M MSOP Solution and Verification

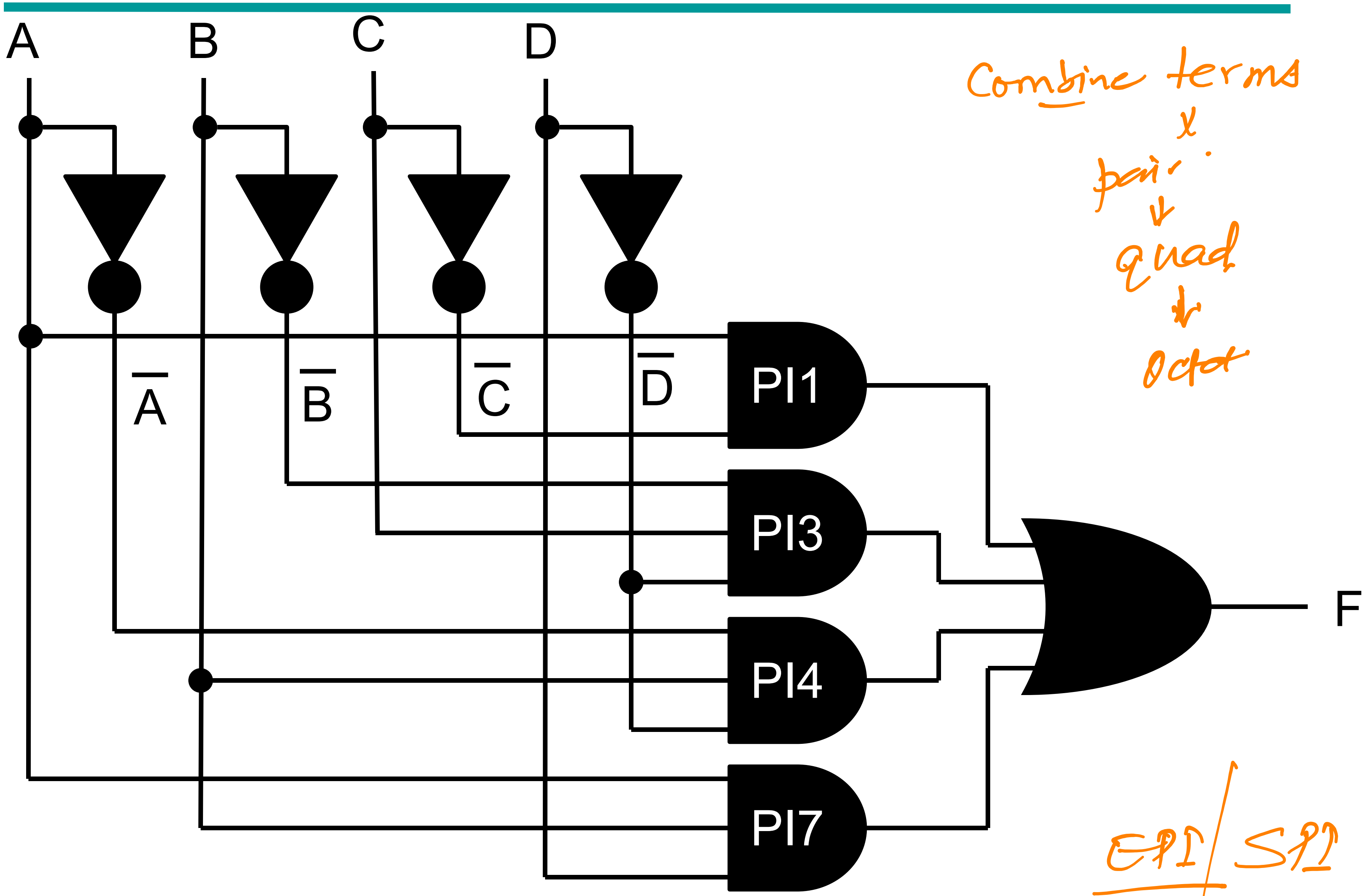
- F(A,B,C,D) = PI_1 + PI_3 + PI_4 + PI_7

  = 1-0- + -010 + 01-0 + 11-1

  = $A\ \overline{C} + \overline{B}\ C\ \overline{D} + \overline{A}\ B\ \overline{D} + A\ B\ D$
- See Karnaugh map.



Minimized Circuit



Combine terms
x
pair
quad
octet

EPI / SPI

Function with Don't Cares

$F(A,B,C,D) = \sum m(4,6,8,9,10,12,13) + \sum d(2, 15)$

- Q-M Step 1: Group "all" minterms with 1 true variable, 2 true variables, etc.

Minterm	ABCD	Groups
2	0010	1: single 1
4	0100	
8	1000	
6	0110	2: two 1's
9	1001	
10	1010	
12	1100	
13	1101	3: three 1's
15	1111	4: four 1's

Step 2: Same As Before on "All" Minterms

List 1			List 2			List 3		
Minterm	ABCD	PI?	Minterms	ABCD	PI?	Minterms	ABCD	PI?
2	0010	X	2, 6	0-10	PI2	8,9,12,13	1-0-	PI1
4	0100	X	2,10	-010	PI3			
8	1000	X	4,6	01-0	PI4			
6	0110	X	4,12	-100	PI5			
9	1001	X	8,9	100-	X			
10	1010	X	8,10	10-0	PI6			
12	1100	X	8,12	1-00	X			
13	1101	X	9,13	1-01	X			
15	1111	X	12,13	110-	X			
			13,15	11-1	PI7			

Step 3: Identify EPI's Ignoring Don't Cares

Covered by EPI →			x	x		x	x
Minterms →	4	6	8	9	10	12	13
PI1 is EPI			x	x		x	x
PI2		x					
PI3					x		
PI4	x	x					
PI5	x					x	
PI6			x		x		
PI7							x

Step 4: Cover Remaining Minterms

Remaining minterms →	4	6	10
PI_2		x	
PI_3			x
PI_4	x	x	
PI_5	x		
PI_6			x

Integer linear program (ILP), available from Matlab and other sources: Define integer {0,1} variables, xk = 1, select PI_k; xk = 0, do not select PI_k.

Minimize $\sum_k$ xk, subject to constraints:

$$x4 + x5 \geq 1$$
$$x2 + x4 \geq 1$$
$$x3 + x6 \geq 1$$

A solution is x3 = x4 = 1, x2 = x5 = x6 = 0, or select PI_3, PI_4

Step 4: Cover Remaining Minterms

Remaining minterms →	4	6	10
PI_2		x	
PI_3			x
PI_4	x	x	
PI_5	x		
PI_6			x

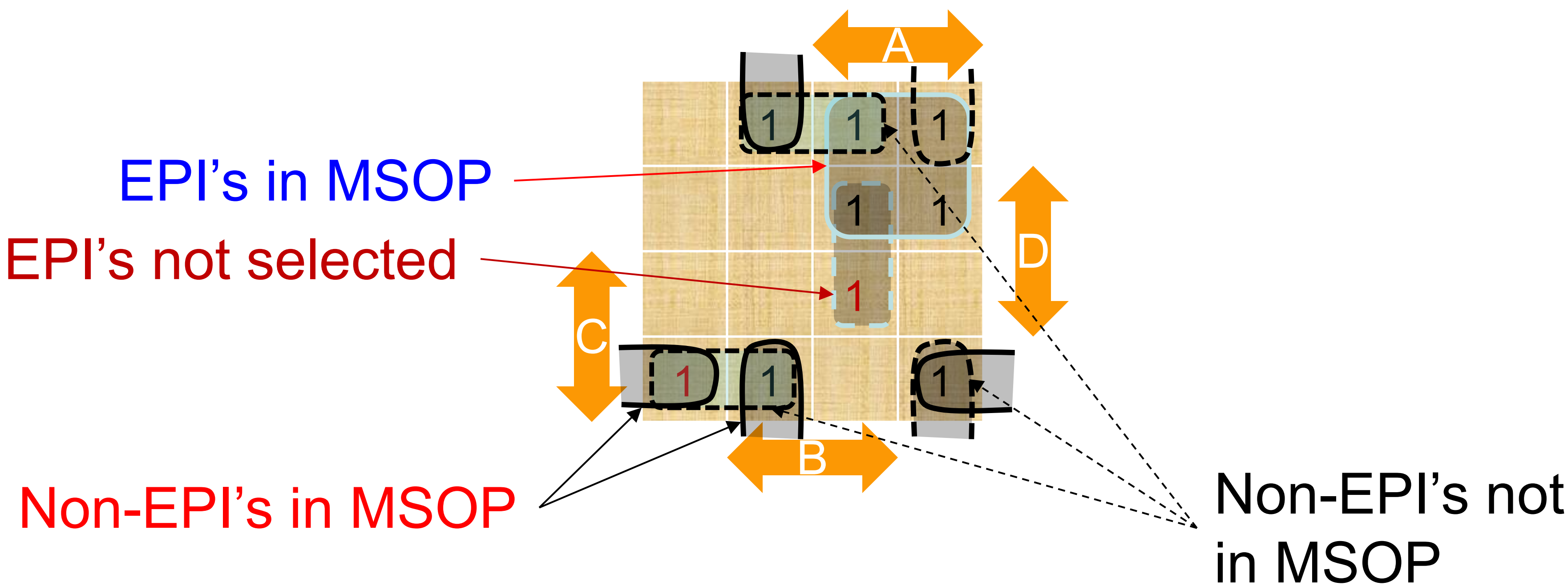
Patrick's Method

Q-M MSOP Solution and Verification

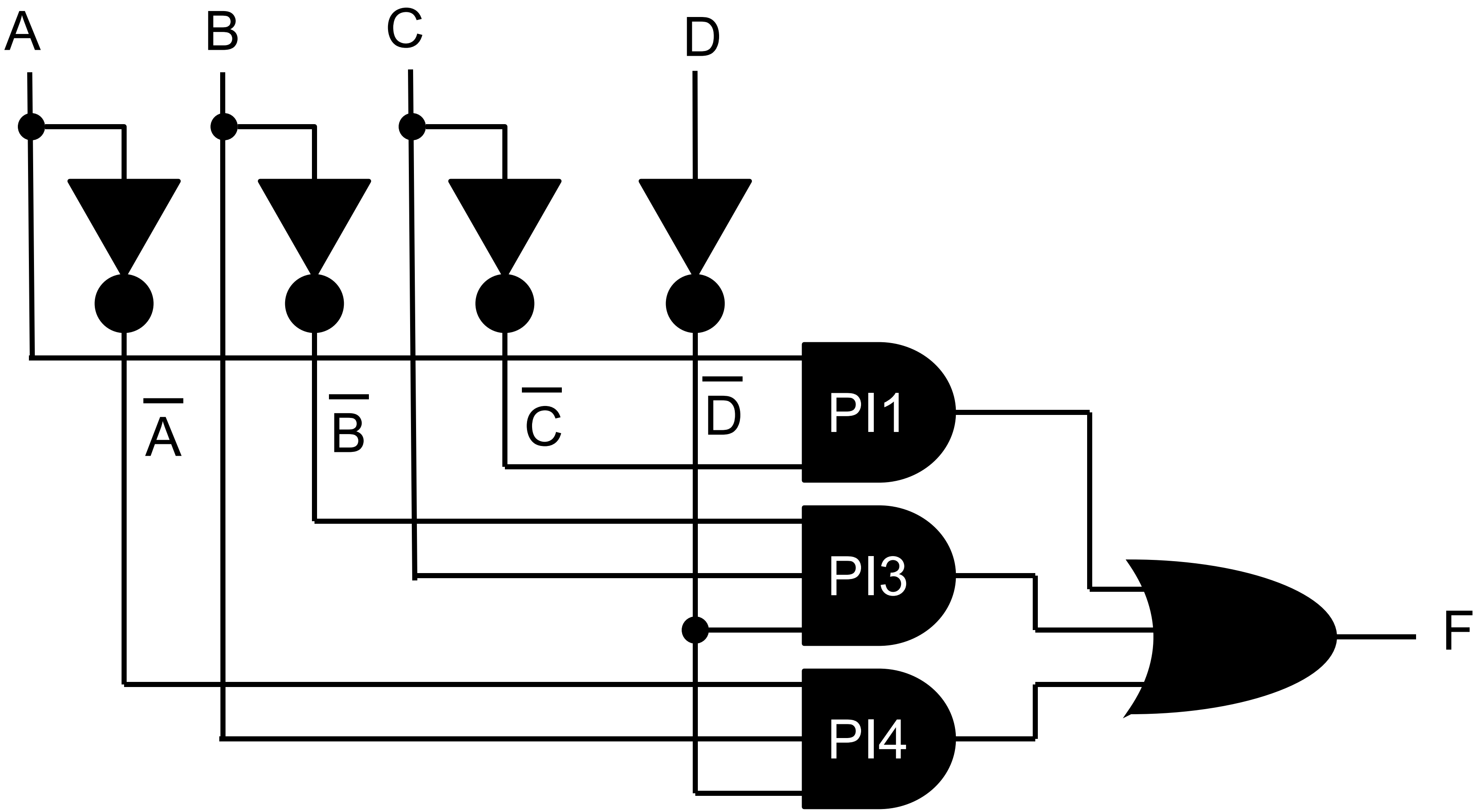
- F(A,B,C,D) = PI_1 + PI_3 + PI_4

  = 1-0- + -010 + 01-0

  = $A\,\overline{C} + \overline{B}\,C\,\overline{D} + \overline{A}\,B\,\overline{D}$
- See Karnaugh map.



Minimized Circuit



CADSL

QM Minimizer on the Web

- http://quinemccluskey.com/

Thank You