Discrete Structures :: CS 207 :: Autumn 2021

Problem Set 7a

Released: October 17, 2021

1. **Degree Sequence.** In each of the following problems, either show that the given sequence cannot be the degree sequence of a graph, or give an example of a graph with that degree sequence.

   (a) $(1, 1, 1, 1, 0)$

   (b) $(2, 2, 2, 2, 2)$

   (c) $(3, 3, 2, 2, 1)$

   (d) $(4, 4, 3, 2, 1)$

   (e) $(4, 3, 3, 3, 3)$

2. Define a pseudograph to be an undirected graph with one or more self-loops allowed in each node. The degree of a node in a pseudograph is defined by counting each self-loop as two edges incident on the node.

   Show that every (sorted) sequence of non-negative integers with an even sum of its terms is the degree sequence of a pseudograph.

   Hint: Construct such a graph by first adding as many self-loops as possible at each vertex. What does the residual degree sequence (i.e., degrees that remain to be satisfied) look like?

3. **Regular Graphs.**

   (a) For any integer $n \geq 3$ and any even integer d with $2 \leq d \leq n-1$, show that there exists a d-regular graph with n nodes, by giving an explicit graph (V, E), where $V = \mathbb{Z}_n$ and E is formally defined using modular arithmetic. (You may find it convenient to use S_a to denote $\{1, \dots, a\} \subseteq \mathbb{Z}_n$.)

   Hint: What would you do for $d = 2$? Then consider adding additional edges for larger values of d.

   (b) For any even integer $n \geq 2$ and any integer d with $1 \leq d \leq n-1$ show that there exists a d-regular graph with n nodes.

   Hint: Use the previous part for even d. For odd d, use the previous part to first construct a $(d-1)$-regular graph, and find a way to add new edges so that all nodes have their degree incremented by 1.

4. A graph with vertices $(v_1, \dots, v_n)$ is said to be a *graph realization* of a sequence $d_1 \geq \dots \geq d_n$ of non-negative integers, if for each i, $\deg(v_i) = d_i$ in the graph. There are efficient algorithms to check if a given sequence has a graph realization. In this problem you shall see one such algorithm.

   a) Show that if $d_1 \geq \dots \geq d_n$ has a graph realization, then it has a graph realization such that v_1 is adjacent to the d_1 nodes $v_2, \dots, v_{d_1+1}$.

   Hint: Among all the realizations, consider one which maximizes the sum of degrees of the nodes adjacent to v_1. If its vertices cannot be relabelled to be of the required form, then there are nodes v_i, v_j with $d_i > d_j$ and v_1 adjacent to v_j but not adjacent to v_i.

   b) Show that the sequence $d_1 \geq \dots \geq d_n$ has a graph realization if and only if the sequence obtained by sorting $(d_2 - 1), \dots, (d_{d_1+1} - 1), d_{d_1+2}, \dots, d_n$ has a graph realization.

   Note: This reduces the problem of checking realizability of n-long sequences to a problem of $(n-1)$-long sequences. This leads to a recursive algorithm.

5. **Complement of a Graph.** We define the *complement of a graph* as a graph which has the same vertex set, but with exactly those edges that are absent from the original graph. Formally, if $G = (V, E)$, its complement $\overline{G} = (V, \overline{E})$, such that $\overline{E} = K_V - E$ where $K_V = \{\{a, b\} | a \in V, b \in V, a \neq b\}$.

   Show that if a graph with n vertices is isomorphic to its complement, then $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$.

6. Match each graph on the left with a description of its complement:

   (a) K_4
   (b) C_4
   (c) $K_{1,3}$
   (d) P_4

   (a) A graph with no edges.
   (b) A graph with a single edge.
   (c) A path with two edges.
   (d) A matching with two edges.
   (e) A graph isomorphic to its complement.
   (f) A complete graph.
   (g) A cyclic graph.

7. **What is Wrong With this Proof?**

   Claim: If every vertex in a graph has degree at least 1, then the graph is connected.

   Proof. We use induction. Let $P(n)$ be the proposition that if every vertex in an n-vertex graph has degree at least 1, then the graph is connected.

   Base case: There is only one graph with a single vertex and it has degree 0. Therefore, $P(1)$ is vacuously true.

   Inductive step: We must show that $P(n)$ implies $P(n+1)$ for all $n \geq 1$.

   Consider an n-vertex graph G in which every vertex has degree at least 1. By the induction hypothesis, G is connected; that is, there is a path between every pair of vertices. Now we add one more vertex x to G to obtain an $(n+1)$-vertex graph H. Since x must have degree at least one, there is an edge from x to some other vertex; call it y. Since y is connected to every other node in the graph, x will be connected to every other node in the graph. QED

   - ☐ A. The proof needs to consider base case $n = 2$.
   - ☐ B. The proof needs to use strong induction.
   - ☐ C. The proof should instead induct on the degree of each node.
   - ☐ D. The proof only considers $(n+1)$ node graphs with minimum degree 1 from which deleting a vertex gives a graph with minimum degree 1.
   - ☐ E. The proof only considers n node graphs with minimum degree 1 to which adding a vertex with non-zero degree gives a graph with minimum degree 1.
   - ☐ F. This is a trick question. There is nothing wrong with the proof!

8. **Prove using Induction.** Prove that for any positive integer n, for any triangle-free graph $G = (V, E)$ with $|V| = 2n$, it must be the case that $|E| \leq n^2$.

9. **Walks and Paths.** In this problem, you shall prove that for any graph G and any two nodes a and b in G, if there is a walk from a to b, then there is a path from a to b.

   (a) Prove this using strong induction. Induct on the length of the walk.

   (b) Prove this using the well-ordering principle, and by proving a stronger statement: A shortest walk from a to b is a path from a to b.

10. **Connectivity and Cycles.** Show that if a graph has a cycle, then deleting any edge in that cycle results in a graph which has the same connectivity relation (i.e., if there is a walk from u to v before deleting the edge, then after deleting the edge too there is such a walk).

11. Show that any two *maximum* length paths in a connected graph should have a common vertex.

    Hint: Consider a shortest path that connects the two paths.

12. **Triangle-Free and Claw-Free Graphs.** Recall that an *induced subgraph* of G is obtained by removing zero or more vertices of G as well as all the edges incident on the removed vertices. (No further edges can be removed.) Formally, $G' = (V', E')$ is an induced subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' = \{\{a, b\} \mid a \in V', b \in V', \{a, b\} \in E\}$.

    A graph G is said to be H-*free* if no induced subgraph of G is isomorphic to H. For example, $G = (V, E)$ is K_3-free (or triangle free) if and only if there are no three distinct vertices a, b, c in V such that $\{\{a, b\}, \{b, c\}, \{c, a\}\} \subseteq E$.

    Prove that the complement of a K_3-free graph is a $K_{1,3}$-free graph.[1]

    Hint: Prove the contrapositive.

13. If a graph G has chromatic number $k > 1$, prove that its vertex set can be partitioned into two nonempty sets V_1 and V_2, such that

    $$\chi(G(V_1)) + \chi(G(V_2)) = k$$

    where $G(V_1)$ denotes the induced subgraph of G with vertex set V_1.

14. The union of 2 graphs on the same vertex set $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ is defined as $(V, E_1 \cup E_2)$. Prove that the chromatic number of the union of G_1 and G_2 is at most $\chi(G_1)\chi(G_2)$.

[1] The graph $K_{1,3}$ is often called the "claw" graph. So this problem can be restated as asking you to prove that the complement of a triangle-free graph is a claw-free graph.