EFFICIENT CALIBRATION, HEDGING AND MULTI-INDEX PORTFOLIO SIMULATION USING GARCH(1,1) MODEL

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Introduction

The project focuses on model calibration and parameter estimation techniques on real data for GARCH(1,1) volatility model. The GARCH(1,1) model is one of the most popular ways to model time-varying volatility in financial returns. It captures a key property of financial data known as volatility clustering. In this project, we calibrate the parameters so that the model fits real data accurately and we explore how to make calibration faster and more accurate by comparing different optimization methods and computational techniques. We also do hedging using GARCH(1,1) and see the influence of α. Further, we also do an advanced python portfolio simulation by simulating a combined portfolio that holds stock in an index fund and also incorporates hedging strategies on options across multiple different stock indexes

GARCH(1,1) Model

The GARCH(1,1) model defines volatility σ_t and returns r_t varying along time as the following:

- $r_t = \mu + \sigma_t$
- $\epsilon_t = \sigma_t z_t$, $z_t \sim \mathcal{N}(0,1)$
- $\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2$

where,

- ω controls the long run average variance,
- α measures the short-term reaction to shocks,
- β measures the volatility persistence over time.

The main constaint in this model is $\alpha + \beta < 1$.

Efficient Calibration of GARCH(1,1) Model

We estimate the parameters $\theta =(\omega, \alpha, \beta)$ using Maximum Likelihood Estimation (MLE). The log-likelihood is given by

$$\log(L(\theta)) = -\frac{1}{2}\sum_{i=1}^{t}\left(\log(2\pi) + \log(\sigma_t^2) + \frac{(r_t-\mu)^2}{\sigma_t^2}\right)$$

We use the following three optimization methods:

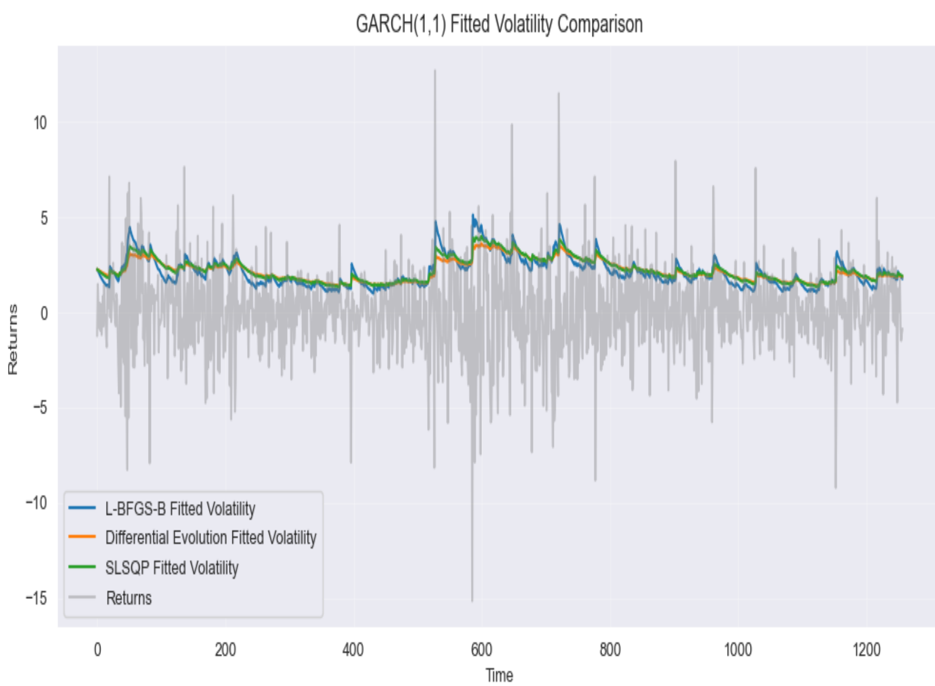
(1) L-BFGS-B – a fast, gradient-based local optimizer.

(2) Differential Evolution – a global optimizer that explores multiple solutions simultaneously.
(3) SLSQP - a local, gradient-based optimizer with full constraint support.

Results. We calibrate using the above three optimization methods measuring computation time and minimizing negative log-likelihood. These three methods successfully calibrate the parameters $\theta = (\omega, \alpha, \beta)$ and the results are the following:

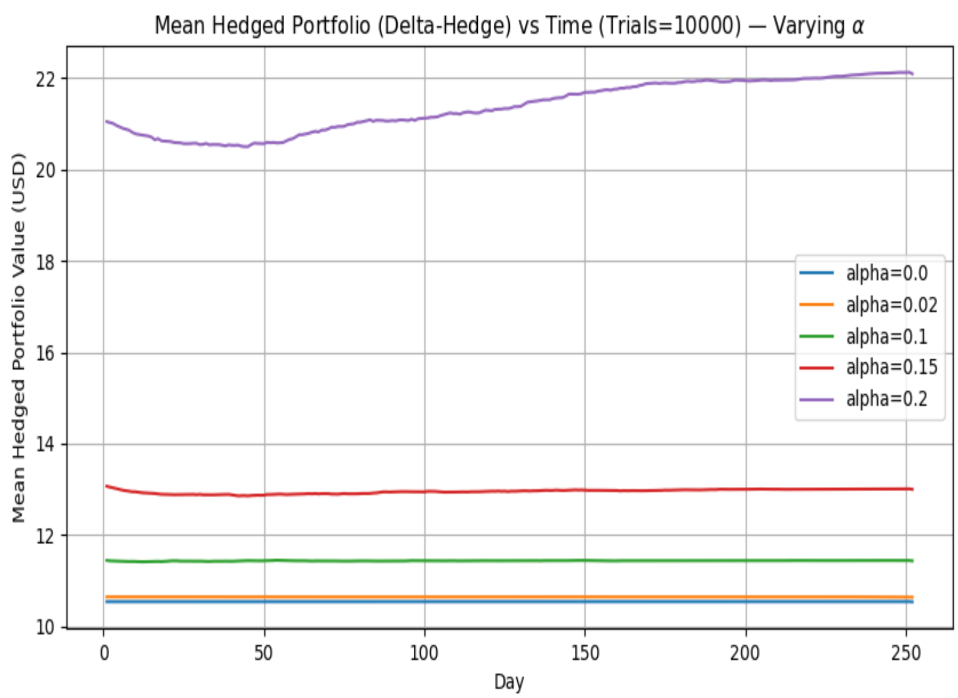
- L-BFGS-B is fast and good for quick, single-run calibration.
- Differential Evolution is more robust but slower.
- SLSQP is the fastest

To conclude, here is the graph comparing fitted volatility of the three optimization methods:



Delta Hedging using GARCH(1,1) Model: Influence of α

We simulate a GARCH(1,1) paths, then do delta-hedging of a European call using Black–Scholes deltas computed from the GARCH conditional volatility, and compare hedging performance for several alpha values. We get the following result.



It says the following:

(1) Higher α makes the stock price process look like fatter tails and higher peaks in returns, i.e. there is a greater chance of both very small and very large price movements.

(2) When α is high, volatility can spike dramatically after a large price move. The Black-Scholes delta used for hedging is a static sensitivity derived under constant volatility. It fails to account for the risk that volatility will change or that will be miscalculated due to sudden volatility shifts. This leads to P&L errors and the large positive drift observed for $\alpha = 0.2$.

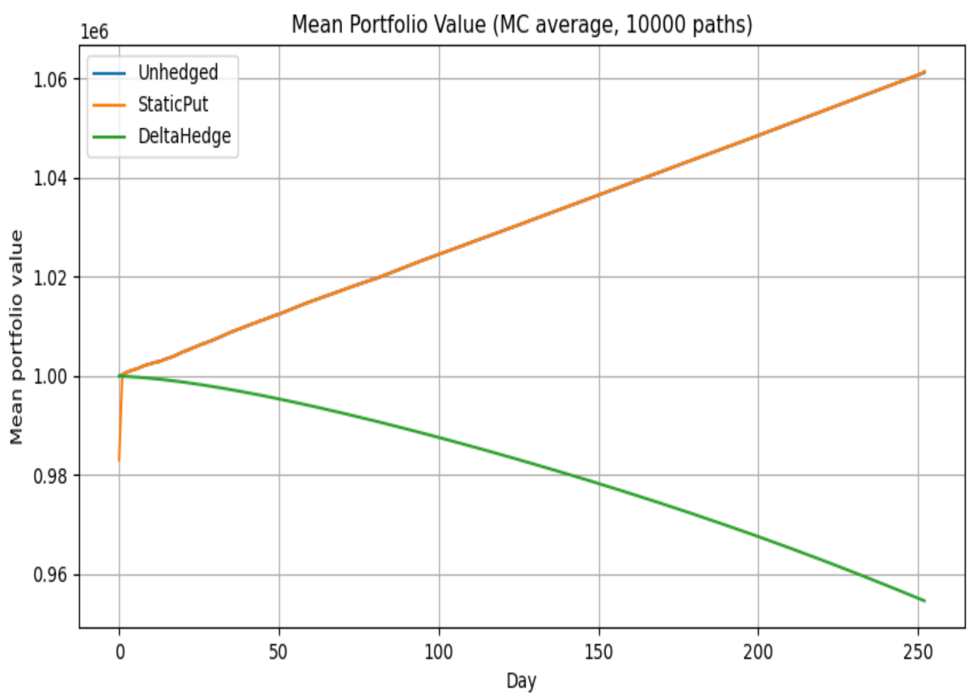
Portfolio Simulation using GARCH(1,1) Model

We do Portfolio Simulation using GARCH(1,1) Model where initial portfolio notional is allocated across indices via user weights. We then hold static quantities of index-fund shares proportional to the notional and initial spot. Further, we implement the following two hedging strategies.

(1) Static protective put — buy puts at t=0, held to expiry.

(2) Dynamic delta hedge — buy calls and hedge daily by trading the underlying to offset option delta. The implementation assumes frictionless trading.

We get the following comparison between these two strategies. It says the following.



(1) The line corresponding to static put represents the stock value plus the initial put cost. The upward slope is the underlying portfolio growing due to its positive expected return. The terminal value includes the potential upside from the stock and the downside protection from the put.

(2) The downward trend of the line corresponding to the delta hedge is the average cost of rebalancing that occurs when realized volatility (from GARCH) deviates from the implied volatility used for pricing and delta calculation.