Probability Assignment-4

Name -: Shravan Yogesh Badgujar
Roll no -: CS22BTECH11010
Problem Assigned -: 12.13.5.7

***Question*—In an examination, 20 questions of true-false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answer true; if it falls tails,he answer false. Find the probability that he answers at least 12 questions correctly.**

SOLUTION -:

Let a binomial random variable be:

$$X \sim Bin(n, p) \tag{1}$$
$$\implies p = \frac{1}{2} = 0.5 \tag{2}$$
$$\implies n = 20 \tag{3}$$

where, p being the probability of answering question correctly.
n is the number of questions.

Proof of Gaussian Approximation on Binomial: By defination,

$$\Pr(X = k) = {}^{n}C_k p^k (1-p)^{n-k} \tag{4}$$

By Stirling's formula,
For $m \to \infty$,

$$m! \sim \sqrt{2\pi m} e^{-m} m^m \tag{5}$$

Hence,

$$\Pr(X = k) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \tag{6}$$
$$\tag{7}$$

Hence,

$$\sim \frac{\sqrt{2\pi n} e^{-n} n^n p^k (1-p)^{n-k}}{\sqrt{2\pi k} e^{-k} k^k \sqrt{2\pi(n-k)} e^{-(n-k)} (n-k)^{n-k}} \tag{8}$$
$$= \left(\frac{p}{k}\right)\left(\frac{1-p}{n-k}\right)^{n-k} n^n \sqrt{\frac{n}{2\pi k(n-k)}} \tag{9}$$
$$= \left(\frac{np}{k}\right)^k \left(\frac{n(1-p)}{n-k}\right)^{n-k} \sqrt{\frac{n}{2\pi k(n-k)}} \tag{10}$$

Now by identities,

$$ln\left(\frac{np}{k}\right) = -ln\left(1 + \frac{k-np}{np}\right) \tag{11}$$
$$ln\left(\frac{n(1-p)}{n-k}\right) = -ln\left(1 - \frac{k-np}{n(1-p)}\right) \tag{12}$$

By approximation,
For $y \to 0$

$$ln(1+y) \sim y - \frac{y^2}{2} + \frac{y^3}{3}... \tag{13}$$

Hence,

$$ln\left(\left(\frac{np}{k}\right)^k \left(\frac{n(1-p)}{n-k}\right)^{n-k}\right) = \tag{14}$$
$$-kln\left(1 + \frac{k-np}{np}\right) + (n-k)ln\left(1 - \frac{k-np}{n(1-p)}\right) \tag{15}$$

$\therefore$,

$$ln\left(\left(\frac{np}{k}\right)^k \left(\frac{n(1-p)}{n-k}\right)^{n-k}\right) \sim -\frac{(k-np)^2)}{2np(1-p)} \tag{16}$$

$\therefore$,

$$\left(\frac{np}{k}\right)^k \left(\frac{n(1-p)}{n-k}\right)^{n-k} \sim e^{-\frac{(k-np)^2)}{2np(1-p)}} \tag{17}$$

By approximating that, $k - np$ is of order $\sqrt{n}$

$$k - np \approx \sqrt{n} \tag{18}$$
$$n - k \approx n(1-p) - \sqrt{n} \tag{19}$$
$$k(n-k) \approx n^2 p(1-p) \tag{20}$$
$$\implies \sqrt{\frac{n}{2\pi k(n-k)}} \sim \frac{1}{\sqrt{2\pi np(n-p)}} \tag{21}$$

$$\Pr(X = x) = \frac{e^{-\frac{(x-np)^2}{2np(1-p)}}}{\sqrt{2\pi np(n-p)}} \tag{22}$$

$$\Pr(X = x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma}} \tag{23}$$

$$\Pr(X \geq 12) = 1 - \Pr(X < 12) \tag{24}$$

$$= 1 - F_X(11) \tag{25}$$

Hence,

$$\mu = np \tag{26}$$

$$\implies \mu = 20 \times 0.5 = 10 \tag{27}$$

$$\sigma^2 = np(1-p) \tag{28}$$

$$\implies \sigma^2 = 20 \times 0.5 \times 0.5 = 5 \tag{29}$$

$$\sigma = \sqrt{np(1-p)} \tag{30}$$

$$\implies \sigma = \sqrt{5} \tag{31}$$

$$\implies Z = \frac{X-\mu}{\sigma} \tag{32}$$

$$Z = \frac{11-10}{\sqrt{5}} \tag{33}$$

$$Z = 0.4472135955 \tag{34}$$

By Q-function,

$$Q(Z) = \int_Z^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \tag{35}$$

$$= 0.673 \tag{36}$$

$$\Pr(X < 11) = Q(Z) \tag{37}$$

$$\Pr(X \geq 12) = 1 - Q(Z) \tag{38}$$

$$\tag{39}$$

Hence,

$$\Pr(X \geq 12) = 1 - 0.673 \tag{40}$$

$$= 0.327 \tag{41}$$

So, The answer differs by,
$0.327 - 0.2517 = 0.0753$