

Probability Assignment-4

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Problem Assigned -: 12.13.5.7

Question—In an examination, 20 questions of true-false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answer true; if it falls tails, he answer false. Find the probability that he answers at least 12 questions correctly.

Now by identities,

$$\ln\left(\frac{np}{k}\right) = -\ln\left(1 + \frac{k - np}{np}\right) \quad (11)$$

$$\ln\left(\frac{n(1-p)}{n-k}\right) = -\ln\left(1 - \frac{k - np}{n(1-p)}\right) \quad (12)$$

SOLUTION -:

Let a binomial random variable be:

$$X \sim \text{Bin}(n, p) \quad (1)$$

$$\Rightarrow p = \frac{1}{2} = 0.5 \quad (2)$$

$$\Rightarrow n = 20 \quad (3)$$

where, p being the probability of answering question correctly.

n is the number of questions.

Proof of Gaussian Approximation on Binomial: By definition,

$$\Pr(X = k) = {}^nC_k p^k (1-p)^{n-k} \quad (4)$$

By Stirling's formula,

For $m \rightarrow \infty$,

$$m! \sim \sqrt{2\pi m} e^{-m} m^m \quad (5)$$

Hence,

$$\Pr(X = k) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \quad (6)$$

$$(7)$$

Hence,

$$\sim \frac{\sqrt{2\pi n} e^{-n} n^n p^k (1-p)^{n-k}}{\sqrt{2\pi k} e^{-k} k^k \sqrt{2\pi(n-k)} e^{-(n-k)} (n-k)^{n-k}} \quad (8)$$

$$= \left(\frac{p}{k}\right) \left(\frac{1-p}{n-k}\right)^{n-k} n^n \sqrt{\frac{n}{2\pi k(n-k)}} \quad (9)$$

$$= \left(\frac{np}{k}\right)^k \left(\frac{n(1-p)}{n-k}\right)^{n-k} \sqrt{\frac{n}{2\pi k(n-k)}} \quad (10)$$

By approximation,

For $y \rightarrow 0$

$$\ln(1+y) \sim y - \frac{y^2}{2} + \frac{y^3}{3} \dots \quad (13)$$

Hence,

$$\ln\left(\left(\frac{np}{k}\right)^k \left(\frac{n(1-p)}{n-k}\right)^{n-k}\right) = \quad (14)$$

$$-k \ln\left(1 + \frac{k - np}{np}\right) + (n-k) \ln\left(1 - \frac{k - np}{n(1-p)}\right) \quad (15)$$

\therefore ,

$$\ln\left(\left(\frac{np}{k}\right)^k \left(\frac{n(1-p)}{n-k}\right)^{n-k}\right) \sim -\frac{(k - np)^2}{2np(1-p)} \quad (16)$$

\therefore ,

$$\left(\frac{np}{k}\right)^k \left(\frac{n(1-p)}{n-k}\right)^{n-k} \sim e^{-\frac{(k - np)^2}{2np(1-p)}} \quad (17)$$

By approximating that, $k - np$ is of order \sqrt{n}

$$k - np \approx \sqrt{n} \quad (18)$$

$$n - k \approx n(1-p) - \sqrt{n} \quad (19)$$

$$k(n-k) \approx n^2 p(1-p) \quad (20)$$

$$\Rightarrow \sqrt{\frac{n}{2\pi k(n-k)}} \sim \frac{1}{\sqrt{2\pi np(1-p)}} \quad (21)$$

$$\Pr(X = x) = \frac{e^{-\frac{(x-np)^2}{2np(1-p)}}}{\sqrt{2\pi np(1-p)}} \quad (22)$$

$$\Pr(X = x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma}} \quad (23)$$

By Q-function,

$$\Pr(X \geq 12) = 1 - \Pr(X < 12) \quad (24)$$

$$= 1 - F_X(11) \quad (25)$$

Hence,

$$\mu = np \quad (26)$$

$$\Rightarrow \mu = 20 \times 0.5 = 10 \quad (27)$$

$$\sigma^2 = np(1-p) \quad (28)$$

$$\Rightarrow \sigma^2 = 20 \times 0.5 \times 0.5 = 5 \quad (29)$$

$$\sigma = \sqrt{np(1-p)} \quad (30)$$

$$\Rightarrow \sigma = \sqrt{5} \quad (31)$$

$$\Rightarrow Z = \frac{X - \mu}{\sigma} \quad (32)$$

$$Z = \frac{11 - 10}{\sqrt{5}} \quad (33)$$

$$Z = 0.4472135955 \quad (34)$$

By Z-score table, $\Pr(X \leq 11) = F_X(11) = 0.673$

Hence,

$$\Pr(X \geq 12) = 1 - 0.673 \quad (35)$$

$$= 0.327 \quad (36)$$

So, The answer differs by,

$$0.327 - 0.2517 = 0.0753$$