Combinatorics

Content :

- ⟶ Addition & Multiplication rule
- ⟶ Permutation & combination basics.
- ⟶ Properties of nC_r
- ⟶ Find $nCr \% M$ { Pascal's Triangle }
- ⟶ Find $nCr \% P$ { using Fermat's Theorem }

- ⟶ Excel column Tile { Easy will do if time permits }

Q> Given 3 T/F . How many ways we can answer them.

__ __ __

any = 8

F F F
F F T
F T F
F T T
T F F
T F T
T T F
T T T

1 & 2 & 3

__ __ __

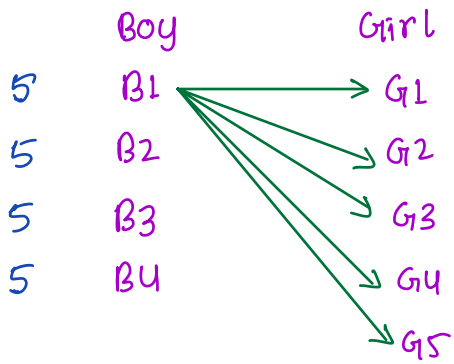
↓ ↓ ↓

T/F T/F T/F

2 x 2 x 2

⇒ Multiplication Rule = AND *

Q Given 5G and 4B . How many pairs B and G __ __

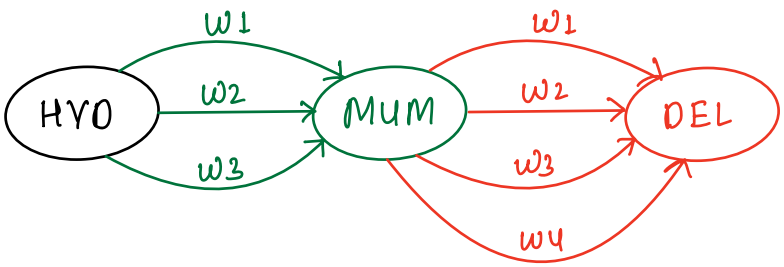


4 x 5

↓ ↓

$B_1 - B_4$ $G_1 - G_5$

Q>



ways HYD → DELHI via MUM

→ # HYD to MUM = 3

→ # MUM to DEL = 4

→ HYD to MUM and MUM to DEL

→ 3 × 4

= 12

Q> In a class 9G & 8B.
# of ways we can select a girl or a boy.

9G or 8B

→ a boy or a girl

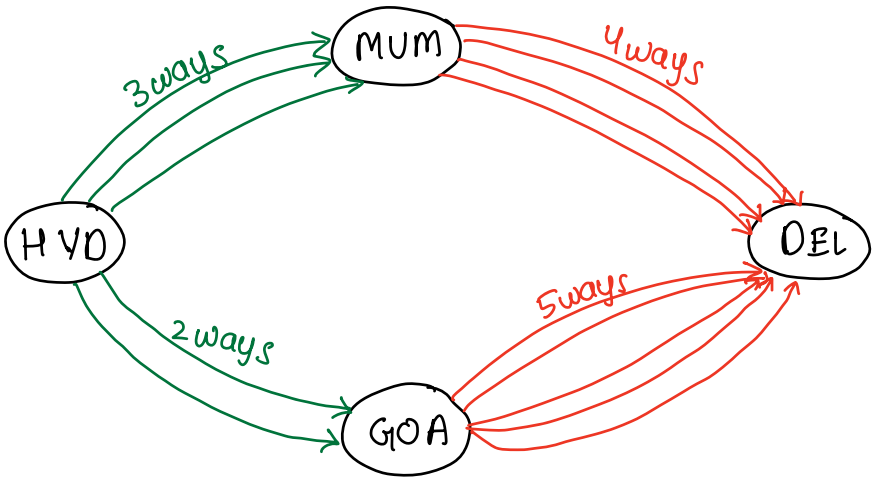
↓

8 + 9

Addition Rule

→ OR

\+



ways HYD – DELHI

HYD to MUM = 3
MUM to DEL = 4
HYD – DEL via MUM = 3 x 4

HYD to GOA = 2
GOA to DEL = 5
HYD – DEL via GOA = 2 x 5

HYD – DEL via MUM = 3 x 4

OR

HYD – DEL via GOA = 2 x 5

$\Rightarrow$ 12 + 10 = 22.

Permutations — Arrangements (order matters)

$(i, j) \neq (j, i)$

ways to arrange p_1, p_2, p_3 =

p_1	p_2	p_3
p_1	p_3	p_2
p_2	p_1	p_3
p_2	p_3	p_1
p_3	p_1	p_2
p_3	p_2	p_1

3 2 1

$p_1 < p_2 - p_3$

$p_2 < p_1 - p_3$

$p_3 < p_1 - p_2$

$(n!)$

$\{1 \dots n\}$ product

ways to arrange p_1, p_2, p_3, p_4

4 3 2 1 $\longrightarrow$ $4 \times 3 \times 2 \times 1 = 4! = 24$

Q> # ways to arrange 2, from 4 people $\{p_1\ p_2\ p_3\ p_4\}$

p_1	p_2	p_2	p_1	p_3	p_1	p_4	p_1
	p_3		p_3		p_2		p_2
	p_4		p_4		p_4		p_3

$= 4 \times 3 = 12$

4 3

Q> # ways to arrange 3, from 5 people $\{P_1\ P_2\ P_3\ P_4\ P_5\}$

$\_\ \_\ \_$ $= 5 \times 4 \times 3 = 60$

5 4 3

Q> # ways to arrange r, from N people

r

1 2 3 .... r

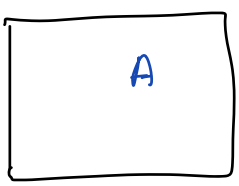
N $N-1$ $N-2$ .... $N-(r-1)$ $= N-r+1$

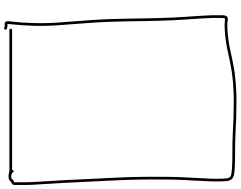
$$\Rightarrow \frac{N \times (N-1) \times (N-2) \times \ldots \times (N-r+1) \times (N-r) \times \ldots (2) \times 1}{(N-r) \times \ldots (2) \times 1}$$

$\Rightarrow$

# of ways to arrange r items from N distinct items $\quad \frac{N!}{(N-r)!}$

$0! = 1$ $\qquad N! =$ # arrange N items





Combinations — selection {order doesn't matter}

$(i, j) == (j, i)$

Say 4 people, how many ways to select 2 people?

P_1 P_2 P_3 P_4

P_1 P_2 P_2 P_3 P_3 P_4 6 ways.

P_1 P_3 P_2 P_4

P_1 P_4

Say 4 people, how many ways to select 3 people

P_1 P_2 P_3 P_4

ways to arrange 3 people

P_1 P_2 P_3
P_1 P_3 P_2
P_2 P_1 P_3
P_2 P_3 P_1
P_3 P_1 P_2
P_3 P_2 P_1

P_1 P_2 P_4
P_1 P_4 P_2
P_2 P_1 P_4
P_2 P_4 P_1
P_4 P_1 P_2
P_4 P_2 P_1

P_1 P_3 P_4
P_1 P_4 P_3
P_3 P_1 P_4
P_3 P_4 P_1
P_4 P_1 P_3
P_4 P_3 P_1

P_2 P_3 P_4
P_2 P_4 P_3
P_3 P_2 P_4
P_3 P_4 P_2
P_4 P_2 P_3
P_4 P_3 P_2

selection

arrange 3 items $\longrightarrow$ $3! \longrightarrow 1$

$24 \longrightarrow x$

$$3! \times x = 24 \times 1$$
$$x \times 3! = 24$$
$$x = \frac{24}{3!} = \frac{24}{6} = 4.$$

x = # select 3 items from 4 distinct items.

Given N distinct items, select r items from them.

selections

arrange r items $\longrightarrow$ $r! = 1$

$\longrightarrow$ $\frac{N!}{(N-r)!} = x$

$$x \times r! = \frac{N!}{(N-r)!} \times 1$$
$$x = \frac{N!}{(N-r)!\, r!} = \frac{{}^{N}P_{r}}{r!} = {}^{N}C_{r}$$

Given N distinct items, select r items from them.

Properties of Combinations

$0! = 1$

1. # select 0 items from N items. $= {}^{N}C_{0}$

$$= \frac{N!}{(N-0)! \times 0!} = \frac{1}{0!} = 1$$

2. # select N items from N items. $= {}^{N}C_{N}$

$$= \frac{N!}{(N-N)!\ N!} = \frac{1}{0!} = 1$$

3. # select r items from N items. $= {}^{N}C_{r}$

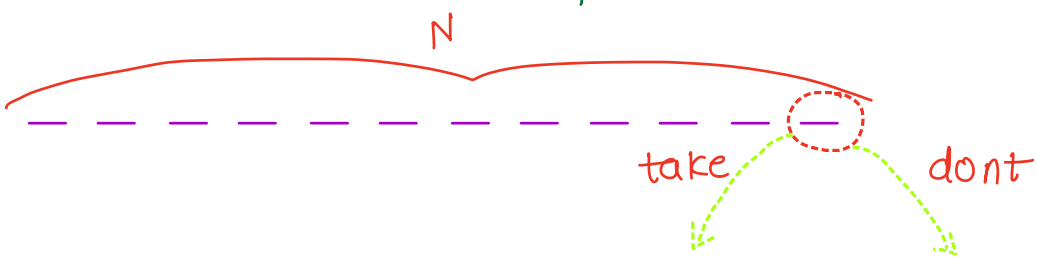
$$= \frac{N!}{(N-r)! \times r!}$$

4. # select n-r items from N items. ${}^{N}C_{N-r}$

$$= \frac{N!}{(N-(N-r))!\ (N-r)!} = \frac{N!}{r!\,(N-r)!}$$

$${}^{N}C_{r} == {}^{N}C_{N-r}$$

* Given N distinct items , select r items



take dont

$$^{N}C_{r} = {}^{N-1}C_{r-1} + {}^{N-1}C_{r}$$

Break — 8:30 am.

Q **Pascal Triangle**

$^0C_0$

$^1C_0$ $^1C_1$

$^2C_0$ $^2C_1$ $^2C_2$

$^3C_0$ $^3C_1$ $^3C_2$ $^3C_3$

$^4C_0$ $^4C_1$ $^4C_2$ $^4C_3$ $^4C_4$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

Give me the value of ^NC_r

Brute force $\quad {}^NC_r = \frac{N!}{(N-r)!\, r!}$

$\Rightarrow$ Calculate factorial of (N), $(N-r)$, (r)

$5! = 5\times4\times3\times2\times1 = 120$

$10! = 3628800$

$20! = 2.4 \times 10^{18}$

} factorials grow rapidly

$^0C_0$

$^1C_0$ $^1C_1$

$^2C_0$ $^2C_1$ $^2C_2$

$^3C_0$ $^3C_1$ $^3C_2$ $^3C_3$

$^4C_0$ $^4C_1$ $^4C_2$ $^4C_3$ $^4C_4$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

$^4C_2 = {}^3C_1 + {}^3C_2 =$

```
int nCr (int N, int R) {

    M[][]        // (N+1) (N+1)   init.

    for (int n = 0 ; n <= N ; n++) {
        for (r = 0 ; r <= n ; r++) {
            // edges
            if (r == 0 || r == n) {
                M[n][r] = 1
            }
            else {
                M[n][r] = M[n-1][r-1] +
                          M[n-1][r]
            }
        }
    }

    return M[N][R]
}
```

TC = $O(N^2)$

SC = $O(N^2)$

MSFT – Give the value of N_{C_r} mod P

```
int nCr ( int N , int R , int P ) {

    M[][]        // (N+1) (N+1)   init.

    for ( int n = 0 ; n <= N ; n++ ) {
        for ( r = 0 ; r <= n ; r++ ) {
            // edges
            if ( r == 0 || r == n ) {
                M[n][r] = 1
            }
            else {
                M[n][r] = M[n-1][r-1] +
                          M[n-1][r]

                M[N][R] %= P
            }
        }
    }

    return M[N][R]
}
```

– Give the value of N_{C_r} mod P

$$N_{C_r} = \frac{N!}{r!\,(N-r)!} \;\%\; P$$

numer = $(N!) \;\%\; P$

denom = $(r!\,(N-r)!) = X$

$= (r! \;\%\; P) \times ((N-r)! \;\%\; P)$

$$\Rightarrow \frac{A}{X} \;\%\; P = A \;\%\; P \times X^{-1} \;\%\; P$$

$=$

$$X^{-1} \;\%\; P = X^{P-2} \;\%\; P$$

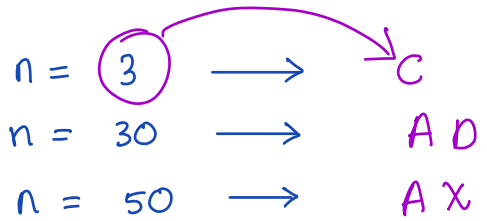
$N! = (((1 \times 2) \;\%\; P \times 3) \;\%\; P \ldots N) \;\%$

TC : $O(N + \log P)$

factorial — fermat

Q> Find n^{th} excel column title

1	2	3	.....	26	27	28	.....	52	53	54	.....
A	B	C	.....	Z	AA	AB	.....	AZ	BA	BB	.....



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idea – 26 base number system.

A	B		Z	AA
0	1	.....	25	26

% 26

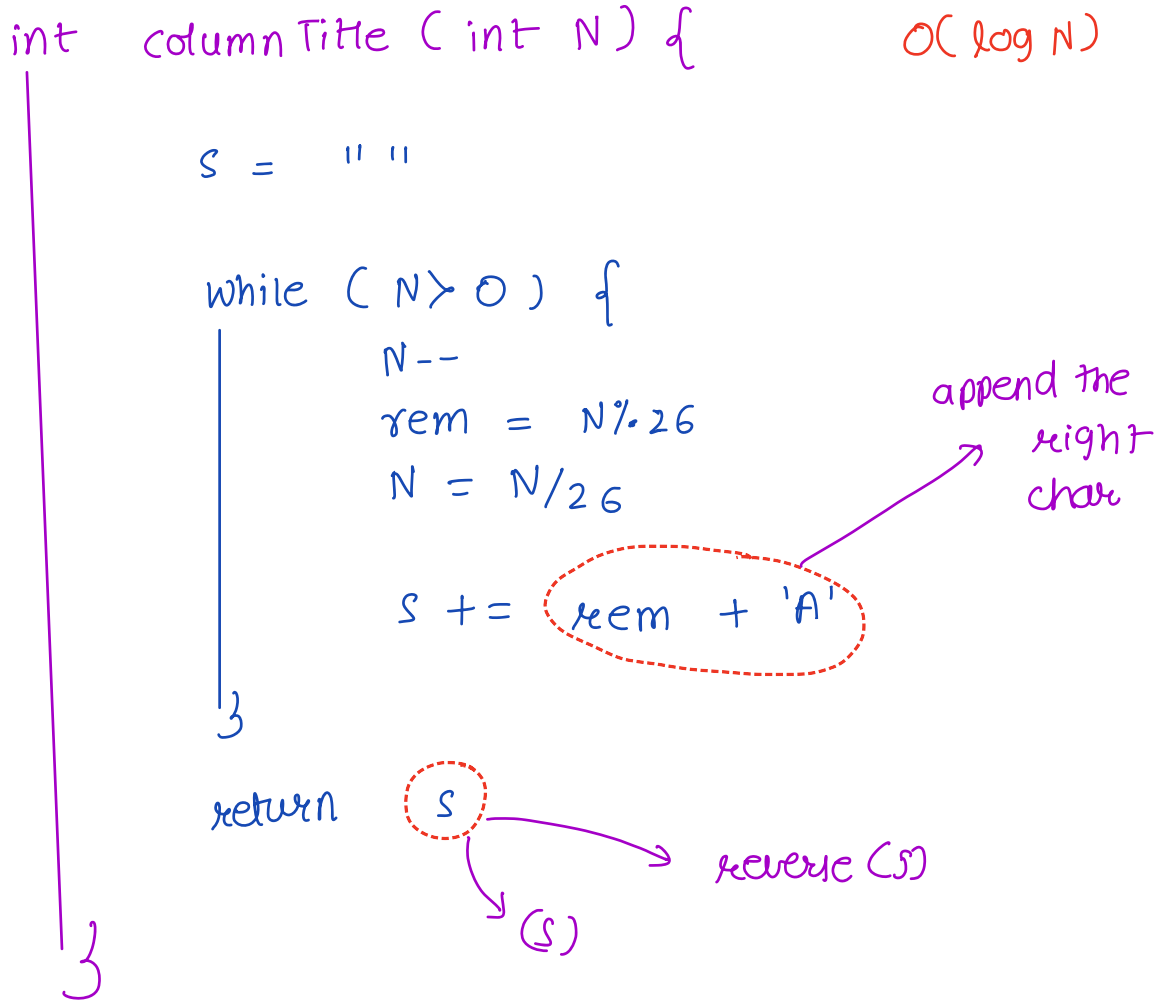
[0-25]

N = 1 N = 27

(N-1) % 26

⇒ 0 → A

(N-1) / 26 = 1



Read on stringbuilder.