UNIT-2 SET THEORY AND GRAPH THEORY

A set is well defined an ordered collection of objectives

Eg: $A = \{1, 2, 3, 4, 5\}$ → Finite set
$B = \{2, 4, 5 \ldots\}$ → Infinite set

Cordinality :
The number of elements of a set is known as cordinality of set. Eg: $A = \{a, e, i, o, u\}$ $|A| = 5$

Power Set
Collection of all subsets of a set is known as power set. Eg: $A = \{1, 2, 3\}$
$P(A) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}$
$|P(A)| = 8$

Cartesian Product:
Let A and B be non empty sets then the Cartesian product of A and B is given by A X B is given by set of ordered pair such that a and b are elements of set.

$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$
Eg: $A = \{1, 2\}$ $B = \{4, 5\}$
$A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5)\}$

Relations:
Let A and B be non empty set, a relation from A to B is a subset of A X B

Eg 1) Let $A = \{2, 4, 6\}$
$B = \{1, 2, 3, 4, 5, 6\}$
$R = \{(a\ b) \mid a < b\}$

$R = \{(2,3), (2,4), (2,5), (2,6), (4,5), (4,6)\}$

$|R| = 6$

Eg 2) :- $A = \{1, 2, 3\}$

$B = \{1, 2\}$

$R = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)\}$

$|R| = 6$

Note: If $R \subseteq A \times A$, then R is known as Relation on A

Domain and Range of R :-

- Let R be a relation from A-B, the domain of R is denoted by Dom(R) and is given by the set of elements in A. That are related to some element in B.

or

Dom(R) is a subset of A, which contains all the first elements in the pair (a, b) that constitute R

- The Range of 'R' is denoted by Ran(R) is given by the set of elements in B which are related to some element of A

or

Ran(R) is a subset of B which contains all the second element in the pair (a, b) that forms R

→ **Solve**

1 Let $A = \{1, 2, 3, 4\}$ Here aRb is given if and $B = \{3, 4, 5\}$ only if ~~if~~ a < b. Find the domain and Range.

= $R = \{(a\ b) / a < b\}$

$R = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5)\}$ → $|R| = 9$

$Dom(R) = \{1, 2, 3, 4\}$
$Ran(R) = \{3, 4, 5\}$

→ Matrix of a Relation:-

- We can represent the relation between two finite sets by means of a matrix.
- Let $A = \{a_1, a_2, a_3, a_4 \dots a_m\} \Rightarrow |A| = m$
  $B = \{b_1, b_2, b_3, b_4 \dots b_m\} \Rightarrow |B| = n$

~~Types~~ Two A and B two finite sets,
If R is a ~~relation~~ relation from A - B then R can be represented by $m \times n$ matrix and which is denoted by $M_R = [M_{ij}]_{m \times n}$
where i varies from $i = 1, 2, \dots m$
$j = 1, 2, \dots n$

$$m_{ij} = \begin{cases} 1 & \text{if } a_i R b_j \\ 0 & \text{if } a_i \text{ is not related to } b_j \end{cases}$$

→ Solve

1. Let $A = \{1, 2, 3\}$, $B = \{4, 5\}$ given relation R as aRb iff $a + b = 6$. Find M_R

Solution: Given $R = \{(a, b) \mid a + b = 6\}$
$R = \{(1, 5), (2, 4)\}$
$aRb =$ iff $a + b = 6$

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

2. Let $A \{1, 2, 3, 4\}$ define R on A such that $R = \{(a, b) \mid a + b < 5\}$. Find Range and domain

= Given = $A = \{1, 2, 3, 4\} = B$
~~$R = \{1,4\} \{1,2\}$~~ $R = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, \dots$ $(3, 1)$

$|R| = 6$

$Ran(R) = \{1, 2, 3\}$

$Dom(R) = \{1, 2, 3, 4\}$

$$M_R = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

3. Let $A = \{1, 2, 3, 4\}$ aRb iff a divides b. Find domain, Range and Matrix.

Given : $A = \{1, 2, 3, 4\}$ aRb a divides b $(a|b)$

$A = \{1, 2, 3, 4\}$

$B = \{1, 2, 3, 4\}$

$R = \{(a, b) / (a|b)\}$

$= \{(1,1) (1,2) (1,3), (1,4), (2,2), (2,4) (3,3), (4,4)\}$

$|R| = 8$.

$Dom(R) = \{1, 2, 3, 4\}$

$Ran(R) = \{1, 2, 3, 4\}$

$$M_R = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

4. Let $A = \{0, 1, 2, 3, 4\}$, $B = \{1, 2, 3\}$ $R = \{(a, b) / \gcd(a, b) = 1\}$

$A = \{0, 1, 2, 3, 4\}$

$B = \{0, 1, 2, 3\}$

$R = \{(a, b) / \gcd(a, b) = 1\}$

$R = \{(1,1) (1,2) (1,3) (2,1) (2,3) (3,1) (3,2) (4,1) (4,3)\}$

$|R| = 9$

$Dom(R) = \{1, 2, 3, 4\}$
$Ran(R) = \{1, 2, 3\}$

$$M_R = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

5. Find the domain, range, matrix when $A = B$, the digraph of the relation 'R' of the following:

i) $A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$, $R = \{(a,1), (a,2), (b,1), (c,$ ... $(d,1)\}$

$Dom(R) = \{a, b, c, d\}$
$Ran(R) = \{1, 2\}$

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Digraph: Not possible (as A, B - 2 are there)

Continued

→ **Representing Relation as Di-graph.**

If A is a finite set and R is the relation on A, then we can represent R as a di-graph as follows:-

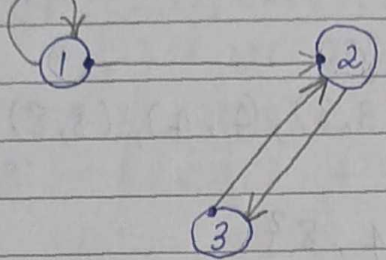
Step 1: Represent each element of set as verticies or points and draw a small circle of a for each element of A

Step 2: Draw an edge (lines) from vertex a_i to a_j if and only if a_i is in relation with a_j. The resulting representation of 'R' is called 'Di-graph' or directed

*Self loop have both in degree and out degree

graph.

For example: Let $A = \{1, 2, 3\}$ and $R = \{(1,1), (1,2)(2,3), (3,2)\}$



→ **In Degree and Out degree**

- If R is an relation on a set A and $a \in A$ then the in degree of A is a no of $b \in A$ (some) such that $(b, a) \in R$.

- The outdegree of A is a number of $b \in A$ such that $(a,b) \in R$

- In case of graph theory; in degree means no of edges (lines) coming in to the point / verticies. and out degree means no of edges going out of the point.

→ ii) $A = \{1, 2, 3, 4\}$ $B = \{1, 4, 6, 8, 9\}$ a R B iff $b = a^2$

= Let R = pair of (a, b) such that $b = a^2$

$R = \{(1,1), (2,4), (3,9)\}$

$Dom(R) = \{1, 2, 3\}$

$Range = Ran(R) = \{1, 4, 9\}$

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 4 & 6 & 8 & 9 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Digraph: Not possible.

iii) $A = \{1, 2, 3, 4, 8\}$ and aRb iff $a = b$

Given $A = \{1, 2, 3, 4, 8\}$

$B = \{1, 2, 3, 4, 8\}$

$R = \{(a, b) \mid a = b\}$

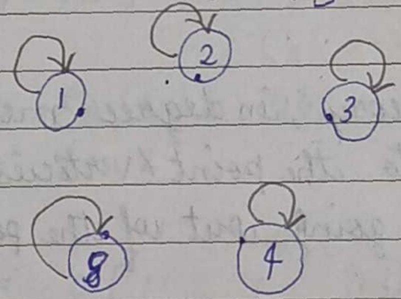
$\therefore R = \{(1,1), (2,2), (3,3), (4,4), (8,8)\}$

$Dom(R) = \{1, 2, 3, 4, 8\}$

$Ran(R) = \{1, 2, 3, 4, 8\}$

$$M_R = \begin{array}{c|c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 8 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 8 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Digraph



(a divides b)

iv) $A = \{1, 2, 3, 4, 8\}$ $B = \{1, 4, 6, 9\}$ aRb iff a/b

= Let $R = \{(a, b) \mid a \text{ divides } b\}$

$\therefore R = \{(1,1), (1,4), (1,6), (1,9), (2,4), (2,6), (3,6)(3,9), (4,4$

~~(8~~.

$Dom(R) = \{1, 2, 3, 4\}$

$Ran(R) = \{1, 4, 6, 9\}$

$$M_R = \begin{array}{c|c} & \begin{array}{cccc} 1 & 4 & 6 & 9 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 8 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

v) $A = \{1, 2, 3, 4, 6\}$ aRb a is multiple of B

= Given $R = \{(a, b) \mid a \text{ is multiple of } b\}$

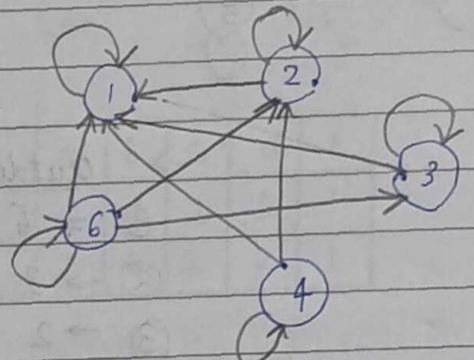
$R = \{(1,1)(2,1), (2,2), (3,1), (3,3), (4,1)(4,2), (4,4), (6,1)(6,2)(6,3), (6,6)\}$

$Dom(R) = \{1, 2, 3, 4, 6\}$

Range $= \{1, 2, 3, 4, 6\}$

$$M_R = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 6 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 & 0 \\ 4 & 1 & 1 & 0 & 1 & 0 \\ 6 & 1 & 1 & 1 & 0 & 1 \end{array}$$

Di-graph :



vi) Let $A = \{1, 2, 3, 4, 5, 6\}$ let R be the relation on A iff b is a multiple of a then find the followings.

(i). Domain and Range of R
ii) Matrix representation of R
iii) Draw digraph of R
iv) Find the indegree and outdegree of every vertex of R.

= Given $A = \{1, 2, 3, 4, 5, 6\}$

$B = \{1, 2, 3, 4, 5, 6\}$

$R = \{(a, b) \mid b \text{ is a multiple of } a\}$

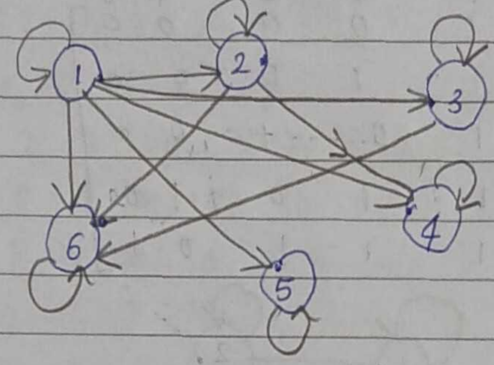
$R = \{(1,1), (1,2), (1,3)(1,4)(1,5), (1,6), (2,2), (2,4), (2,6), (3,3)(3,6), (4,4), (5,5)(6,6)\}$

~~M_R =~~ i) $Dom(R) = \{1, 2, 3, 4, 5, 6\}$

$Ran(R) = \{1, 2, 3, 4, 5, 6\}$

$$M_R = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 & 0 & 1 \\ 3 & 0 & 0 & 1 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 1 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Di-graph:



→ Indegree

1 → 1
2 → 2
3 → 2
4 → 3
5 → 2
6 → 4

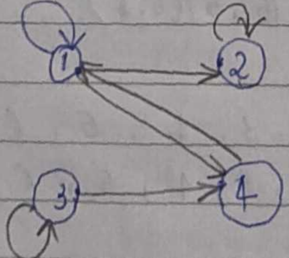
→ outdegree

1 → 6
2 → 3
3 → 2
4 → 1
5 → 1
6 → 1

vii) Let $A = \{1, 2, 3, 4\}$ and $M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 & 0 \\ 3 & 0 & 0 & 1 & 1 \\ 4 & 1 & 0 & 0 & 0 \end{array}$ give the relation 'R' on A and the also give the digraph, write Indegree and outdegree

= $A = \{1, 2, 3, 4\}$
$B = \{1, 2, 3, 4\}$
$R = \{(1,1), (1,2), (1,4), (2,2), (2,3), (3,3), (3,4), (4,1)\}$

Digraph



Vertices	1	2	3	4
Indegree	2	2	2	2
outdegree	3	2	2	1

viii) let $A = \{a, b, c, d, e\}$, write the relation, digraph indegree and outdegree

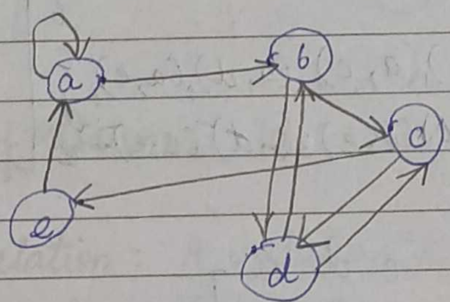
$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$A = \{a, b, c, d, e\}$

$B = \{a, b, c, d, e\}$

$R = \{(a,a), (a,b), (b,c), (b,d), (c,d), (c,e), (d,b), (d,c), (e,a)\}$

- Digraph



Vertices	a	b	c	d	e
Indegree	2	2	2	2	1
Out degree	2	2	2	2	1

→ **Paths in Relation**

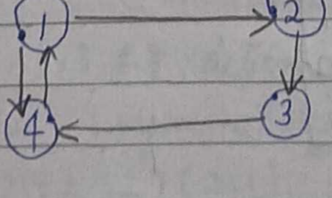
let 'R' be a relation on A, a path of length 'n' in R from a to b is a finite sequence $a, x_1, x_2, x_3 \ldots x_{n-1}$

In begining with a and ending at b ~~with~~ such that

$$aRx_1,\ x_1Rx_2,\ x_2Rx_3 \ldots x_{n-2}Rx_{n-1},\ x_{n-1}Rb$$

- A path of length 'n' from a to b is denoted by $aR^n b$

Eg: let $A = \{1, 2, 3, 4\}$ and digraph is

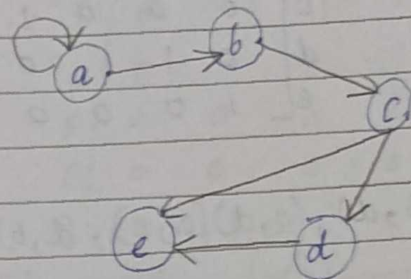


let $a = 1$ $b = 3$

→ Let $A = \{a, b, c, d, e\}$ and $R = \{(a,a), (b,b), (b,c), (c,e), (c,d), (d,e)\}$

Find R^2, R^∞, M_R^2, M_R^∞

=



length 2 - 2 step

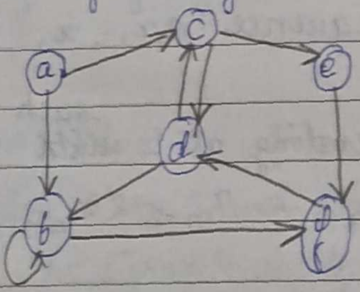
$R^{2} = \{(a,a), (a,b), (a,c), (b,d), ~~(b,e)~~, (b,e), (c,e)\}$

$$(M_R)^2 = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 1 & 1 & 1 & 0 & 0 \\ b & 0 & 0 & 0 & 1 & 1 \\ c & 0 & 0 & 0 & 0 & 1 \\ d & 0 & 0 & 0 & 0 & 0 \\ e & & & & & \end{array}$$

$R^\infty = \{(a,a)(a,b)(a,c), (a,d), (a,e), (b,c)(b,d)(b,e), (c,d)(c,e)(d,e)\}$

$$(M_R)^\infty = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 1 & 1 & 1 & 1 & 1 \\ b & 0 & 0 & 1 & 1 & 1 \\ c & 0 & 0 & 0 & 1 & 1 \\ d & 0 & 0 & 0 & 0 & 1 \\ e & 0 & 0 & 0 & 0 & 0 \end{array}$$

→ For the following digraph, compute $(MR)^2$ and $(MR)^\infty$



$R^2 = \{(a,b)(a,e), (a,d), (a,f), (b,b), (b,f)(b,d), (c,f), (c,c), (c,b), (d,d)(d,e), (d,f), (d,b), (e,d), (f,b), (f,c)\}$

$$(M_R)^2 = \begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & 0 & 1 & 0 & 1 & 1 & 1 \\ b & 0 & 1 & 0 & 1 & 0 & 1 \\ c & 0 & 1 & 1 & 0 & 0 & 1 \\ d & 0 & 1 & 0 & 1 & 1 & 1 \\ e & 0 & 0 & 0 & 1 & 0 & 0 \\ f & 0 & 1 & 1 & 0 & 0 & 0 \end{array}$$

$(R^\infty) = \{(a,c), (a,b), ~~(a,b)~~, (a,e), (a,d)(a,f), (b,b)(b,c)(b,d)(b,e)(b,f), (c,b)(c,c)(c,d), (c,e), (c,f)(d,b)(d,c)(d,e)(d,f), (e,b), (e,c), (e,d), (e,e), (e,f), (f,b)(f,c)\}$

$(f,d),(f,e),(f,f)\}$

$$(M_R)^{\infty} = \begin{array}{c|c} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

→ **Properties of Relation:**

- **Reflexive relation:** A relation 'R' on a set A is reflexive if $(a, a) \in R$ for all $a \in A$.

Eg: Let $A = \{1, 2, 3\}$ → $R = \{(1,1), (2,2), (3,3), (1,2)\ (2,1), (1,3)\}$

- **Symmetric relation:** A relation 'R' on a set A is symmetric whenever $(a, b) \in R$, $(b, a) \in R$.

For eg: $A = \{1, 2, 3\}$ $R = \{(1,2), (2,1)(2,2), (3,3), (1,1)\}$

$A = \{1, 2, 3\}$ $R = \{(1,2)(2,1)(2,2)\ (3,3), (1,1)\}$

- **Transitive relation:** A relation 'R' on a set A is transitive if $(a, b) \in R$, $(b, c) \in R$ then $(a, c) \in R$

For eg: Let $A = \{1, 2, 3, 4\}$ → is transitive

$R = \{(1,1)\ (1,2)\ (2,1)\ (3,4)\ (4,3)\ (3,3)\}$

If a relation 'R' is Reflexive, symmmetric, and transitive then it is known as **Equivalence relation**

1. If $A = \{1, 2, 3, 4\}$, determine whether the following relations are reflexive, symmetric and transitive.

i) $R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}$

- here R is reflexive because $(a, a) \in R$ for all $a \in A$
- here R is symmetric because $(a, b) \in R$, $(b, a) \in R$
- here R is transitive because $(a, b) \in R$, $(b, c) \in R$, $(a, c) \in R$

ii) $R = \{(1,3)(1,1)(3,1)(1,2)(2,3)(4,4)\}$

The given R is not reflexive, not symmetric but it is transitive.

iii) $R = \{(1,1), (2,2), (3,3)\}$
This is not reflexive, not transitive, not symmetric.

iv) $R = \phi$ (null set)
It is not reflexive, not symmetric, not transitive.

v) $R = A \times A$
$R = \{(1,1)(1,2)(1,3)(1,4)(2,1)(2,2),(2,3),(2,4),(3,1),(3,2)$
$(3,3)(3,4)(4,1)(4,2)(4,3)(4,4)\}$
↳ It is symmetric, reflexive and transitive.

vi) $R = \{(1,2)(1,3)(3,1)(1,1)(3,3)(3,2)(1,4),(4,2),(3,4)\}$
It is not reflexive, not symmetric it is ~~not~~ transitive

- Reflexive Closure
Let 'R' be a relation on a set A, R may or may not be satisfy the property P,

If there is a relation 'S' with property P, it contains all the elements
is called clos
If R
then the new set is known as reflexive closure

- Symmetric Closure

- Transition closure.

Sure Question

Warshall's Algorithm

Warshall's Algorithm

6M

1. $A = \{1, 2, 3, 4\}$

$R = \{(1,2)(2,3)(3,4)(2,1)\}$ find the transitive closure using Warshall's Algorithm

⇒ Solution: Given $R = \{(1,2)(2,3)(3,4)(2,1)\}$

$$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array} = W_0$$

C_1: Location of 1's in first collumn : 2

R_1: Location of 1's in first row : 2

$(C_1, R_1) = (2, 2)$

$$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

C_2 : 1, 2

R_2 : 1, 2, 3

$(C_2, R_2) = \{(1,1)(1,2)(1,3)(2,1)(2,2),(2,3)$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = W_2$$

$C_3 : 1, 2$ $\qquad (C_3, R_3) = (1,4)(2,4)$

$R_3 : 4$

$$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 : 1, 2$

$R_4 :$ Null $= w_3 = w_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R^{\infty} = \{(1,1)(1,2)(1,3)(1,4)(2,1)(2,2)(2,3)(2,4)(3,4)\}$

2. Let $A = \{1,2,3,4\}$ find the matrix of transitive closure by Warshall's algorithm for the following matrices

i) $M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = W_0$

$C_1 : 1, 2$ $\qquad (C_1, R_1) = (1,1)(1,4)(2,1)(2,4)$

$R_1 : 1, 4$

$$W_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 : 2$

$R_2 : 1, 2, 4$

$\cancel{C_2 = 4, 2, 3}$ $\qquad \cancel{(C_2 R_2)}$ $\qquad (C_2, R_2) = (2,1)(2,2)(2,4)$

$\cancel{R_2 = 1, 2, 4}$

$$W_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 = 3$ $\qquad (C_3\ R_3) = (3, 3)$

$R_3 = 3$

$$W_3 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 : 1, 2, 4$
$R4 : 4$

$(C_4, R_4) - (1,4)(2,4)(4,4)$

$$W_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = R_R^{\infty}$$

$R^{\infty} = \{(1,1)(1,4)(2,1)(2,2)(2,4)(3,3)(4,4)\}$

ii) $$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = W_0$$

$C_1 = 1, 2$ $\quad (C_1 R_1) = (1,1)(1,2)(2,1)(2,2)$
$R_1 = 1, 2$

$$W_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$C_2 : 1, 2$ $\quad (C_2 R_2) = \{(1,1)(1,2)(2,1)(2,2)\}$
$R_2 : 1, 2$

$$W_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$C_3 = 4$
$R_3 =$ Null $\quad (C_3, R_3) = (4, 0)$

$$W_3 = W_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$C_4 : 4$
$R_4 : 3$
$(C_4, R_4) : (4, 3)$

$$w_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = M_R^{\infty}$$

$R = \{(1,1)(1,2)(2,1)(2,2)(4,3)\}$

iii)
$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = M_R = W_0$$

$C_1 = 1, 4$ $\quad (C_1 R_1) = (1,1)(1,4)(4,1)(4,4)$

$R_1 = 1, 4$

$$W_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 : 2, 3$

$R_2 : 2, 3$

$(C_2 R_2) : (2,2)(2,3)(3,2)(3,3)$

$$W_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 = 2, 3$

$R_3 = 2, 3$

$(C_3 R_3) = (2,2)(2,3)(3,2)(3,3)$

$$W_3 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 : 1, 4$

$R_4 : 1, 4$

$(C_4 R_4) : (1,1)(1,4)(4,1)(4,4)$

$$W_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = M_R^{\infty}$$

$R^{\infty} = \{(1,1)(1,4)(2,2)(2,3)(3,2)(3,3)(4,1)(4,4)\}$

iv)
$$M_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = W_0$$

$C_1 : 2$ $\qquad (C_1\ R_1) - (2, 4)$

$R_1 : 4$

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{matrix} C_2 = \cancel{1,2,4}\ 3 \\ R_2 = 1, 4 \end{matrix}$$

$(C_2\ R_2) = (3,1)(3,4)$

$$W_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$C_3 - 3, 4$

$R_3 - 1, 2, 3, 4$

$(C_3, R_3) = (3,1)(3,2)(3,3)(3,4)$
$(4,1)(4,2)(4,3)(4,4)$

$$W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$C_4 - 1, 2, 3, 4$

$R_4 \quad 1, 2, 3, 4$

$(C_4\ R_4) = (1,1)(1,2)(1,3)(1,4)(2,1)$
$(2,2)(2,3)(2,4)(3,1)(3,2)$
$(3,3)(3,4)(4,1)(4,2)(4,3)(4,4)$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$R^{\infty} = \{(1,1)(1,2)(1,3)(1,4)(2,1)(2,2)(2,3)(2,4)(3,1)(3,2)(3,3)(3,4)$
$(4,1)(4,2)(4,3)(4,4)\}$

→ **Functions**

- Let A and B be non empty sets, a function from A to B is an assignment of exactly one element of set B to each element of set A, that is $f(a) = b$

- If f is a function from A to B then we write it as $f: A \to B$, where A is known as domain and B is known as ~~range~~ Co domain of f.

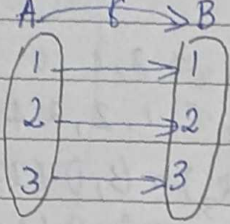
- If $f(a) = b$, then b is known as image of a, a is known as pre image of b

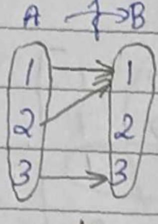
- Types of functions :

i) One - One (Injective)

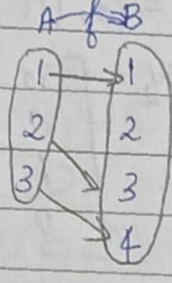
A function f is said to be one-one, if and only if $f(a) = f(b) \rightarrow a = b$ for all (a, b) in the domain of f

Eg:





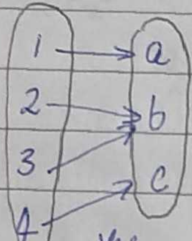
Not



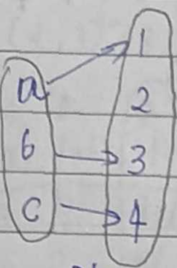
ii) Onto

A function 'f' from A to B is called onto iff for every element $b \in B$, there is an element $a \in A$ such that $f(a) = b$

Eg:



Yes

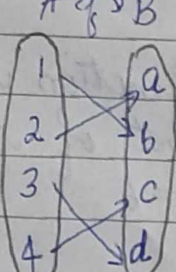


No

iii) Bijection

If a function 'f' is both one-one and onto, then it is known as bijection.

Eg: $A \xrightarrow{f} B$



- Composition of function:

let 'g' be a function defined from B to C and f be a function defined from $A \rightarrow B$, then the composition of the function f and g is given by '$f \circ g$' is defined as $f \circ g(x) = f(g(x))$

1) Eq: $f: Z \to Z$ Given $f(x) = 2x+3$ and $g(x) = 3x+2$
$g: Z \to Z$ then find fog and gof.

= $fog(x) = f(g(x))$
$= f(3x+2)$
$= 2(3x+2)+3$
$= 6x+4+3$
$= \underline{6x+7}$

$gof(x) = g(f(x))$
$= 3(2x+3)+2$
$= 6x+9+2$
$= \underline{6x+11}$

2. Let $f: R \to R$ Given $f(x) = x^3$ and $g(x) = 4x^2+1$
$g: R \to R$ Find fog and gof.

$(a+b)^3$
$= a^3+b^3+3a^2b+3ab^2$

$fog(x) = f(g(x))$
$= f(4x^2+1)$
$= (4x^2+1)^3$
$= (4x^2)^3+(1)^3+3(4x^2)^2(1)+3(4x^2)(1)^2$
$= \underline{64x^6+48x^4+12x^2+1}$

$gof(x) = g(f(x))$
$= g(x^3)$
$= 4(x^3)^2+1$
$= \underline{4x^6+1}$

3. Let $h: R \to R$ and $g: R \to R$ and $f: R \to R$
$f(x) = x^2$, $g(x) = (2x+1)$, $h(x) = 7x-2$
Find gof, fog, $(fog)oh$, $go(foh)$

= $fog(x) = f(g(x))$
$= f(2x+1)$
$= (2x+1)^2$
$= 4x^2+1^2+4x$
$= \underline{4x^2+1+4x}$

$gof = g(f(x))$
$= g(x^2)$
$= \underline{2x^2+1}$

$(fog)oh = fog(7x-2)$
$= (2(7x-2)+1)^2$
$= (14x-4+1)^2$
$= \underline{(14x-3)^2}$

$$
\begin{aligned}
g\circ(f\circ h) &= g((f\circ h)(x)) \\
&= g(f(h(x))) \\
&= g(f(7x-2)) \\
&= g((7x-2)^2) \qquad = \underline{\underline{2(7x-2)^2+1}} \\
&= \cancel{(2((7x-2)^2)+1)^2} \\
&\neq \cancel{((14x-4)^2+1)^2}
\end{aligned}
$$

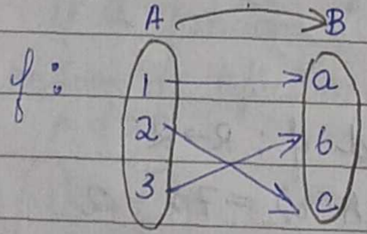
4. Let $g:\{a,b,c\}\to\{a,b,c\} \to g(a)=b,\ g(b)=c,\ g(c)=$
$f:\{a,b,c\}\to\{1,2,3\}$
find $f\circ g(a)$, $f\circ g(b)$ and $f\circ g(c)$

$f\circ g(a) = f(g(a)) = f(b) = \underline{2}$ $\qquad$ $f\circ g(b) = f(g(b)) = f(c) = \underline{\underline{3}}$ $\qquad$ $f\circ g(c) = f(g(c)) = f(a) = \underline{\underline{1}}$

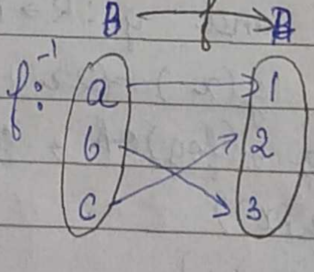
→ **Invertible Function:**

A function f from $A\to B$ is said to be invertible function if its inverse relation g from $B\to A$ is also a function and we can write g as f inverse (f^{-1})

1) Eg: $f:\{1,2,3\}\to\{a,b,c\}$
$f:\{(1,a),(2,c)(3,b)\}$

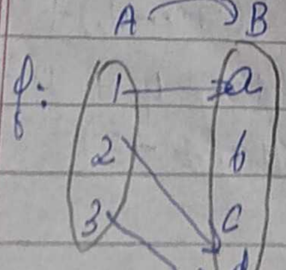


$f^{-1}=\{(a,1),(c,2),(b,3)\}$

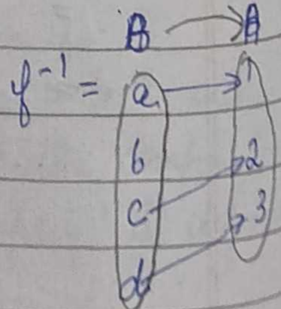


f inverse exists

2) $f:\{1,2,3\}\to\{a,b,c,d\}$
$f:\{(1,a)(2,c)(3,d)\}$



$f^{-1}=\{(a,1)\ (c,2)\ (d,3)\}$



This a not a function
∴ f^{-1} does not exist.

Note: f^{-1} exists iff 'f' is bijective function.

→ **Permutation function:**

A bijective function from set A to itself is called Permutation function. Suppose A has n elements $A = \{a_1, a_2, a_3, \ldots a_n\}$ and P is a bijection on A. Then the representation of Permutation function is given by $\begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ P(a_1) & P(a_2) & \ldots & P(a)_n \end{pmatrix}$

where $P(a_1), P(a_2) \cdots P(a_n)$ are function values of $a_1, a_2 \cdots a_n$

1] Let $A = \{1, 2, 3\}$ then find all the permutations of A.

$I_A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ $P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ $P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

$P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ $P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ $P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

Note: i) $P(n)^2 = P_n \circ P_n$ ii) $P_i \circ P_j \neq P_j \circ P_i \ (i \neq j)$

→ **Cyclic Permutation:**

Let $b_1, b_2, \ldots b_r$ are distinct elements of set A then the permutation function $P: A \to A$ is of the form $P(b_1) = b_2$, $P(b_2) = b_3$, $P(b_3) = b_4 \cdots P(b_{r-1}) = b_r$, $P b_r = b_r$ is called cyclic permutation of length r and it is denoted by $(b_1, b_2 \ldots b_r)$

Eg: Let $A = \{1, 2, 3, 4, 5\}$ and cycle is $(1, 3, 5)$
$b_1 \ b_2 \ b_3$

$P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix}$

1] Let $A = \{1, 2, 3, 4, 5, 6\}$ then compute.
i) $(4, 1, 3, 5) \circ (5, 6, 3)$
ii) $(5, 6, 3) \circ (4, 1, 3, 5)$

i) $(4, 1, 3, 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 1 & 4 & 6 \end{pmatrix}$

Page

i) $(4,1,3,5)\circ(5,6,3) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 1 & 4 & 6 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 4 & 6 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 4 & 1 & 6 & 5 \end{pmatrix}$

ii) $(5,6,3)\circ(4,1,3,5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 4 & 6 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 1 & 4 & 6 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 6 & 1 & 4 & 3 \end{pmatrix}$

- **Note** : The composition of two permutation usually reffered as the product of these two permutations.

- If $a_1, a_2 \ldots a_n$ is a set containing 'n' elements then there exists n-factorial permutations.

2. Let $A = \{1, 2, 3\}$ list the permutations of A and compute i) Inverse of P_2 and P_4
ii) $P_1 \circ P_5$, $P_5 \circ P_1$, P_2^2

$= I_A = P_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ $P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ $P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ $P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

i) $P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ $\cancel{P_2^{-1} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix}}$ $P_2^{-1} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

$\downarrow$

$P_2 = \{(1,2)(2,1)(3,3)\}$ $P_2^{-1} = \{(2,1)(1,2)(3,3)\}$

$P_4 = \{(1,2)(2,3)(3,1)\}$ $\quad P_4^{-1} = \{(2,1)(3,2)(1,3)\}$

$$P_4^{-1} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

ii) $P_1 \circ P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

$P_5 \circ P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

$P_2^2 = P_2 \circ P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

→ **Disjoint Cycles:**

Two cycles of a set A are said to be disjoint if no element of A, appears in both the cycles.

Eg: Let $A = \{1, 2, 3, 4, 5, 6\}$, then the cycles $(1, 2, 6)$, $(3, 4, 5)$ are disjoint, where as $(1, 2, 5)$ and $(5, 3, 4, 6)$ are not disjoint cycles.

1. Write the Permutation ~~P∈~~ 1, 2, 3, 4, 5, 6, 7, 8

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 6 & 5 & 2 & 1 & 8 & 7 \end{pmatrix}$$ of set A

$A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ as the product of disjoint cycle. ~~Given P=~~

= Given:

$= (1, 3, 6) \cdot (2, 4, 5)(7, 8)$

→ **Transpositions:**

A cycle of length 2 is known as transposition that is
Let $P = (a_i, a_j)$ then $P(a_i) = a_j$ and $P(a_j) = a_i$

Theorem:

- Every cycle can be written as product of transposition that is $(b_1, b_2, b_3, \ldots b_{r-1}, b_r) = (b_1, b_r)(b_1, b_{r-1})(b_1, b_{r-2}) \ldots (b_1, b_3)(b_1, b_2)$

1. Let $\{1, 2, 5, 6, 7\}$ be a cycle then the transpositions are $(1, 7) \circ (1, 6) \circ (1, 5) \circ (1, 2)$

2. Express the given permutation $P = (7, 8) \circ (2, 4, 5) \circ (1, 3, 6)$ as product of transpositions

$= (7, 8)(2, 5)(2, 4)(1, 6)(1, 3)$

→ **Odd and Even Permutation:**

A permutation of a finite set is called even if it can be product of an even number of transpositions ~~and it is called 'Even permutation'~~; odd number - odd permutation.

1. Verify if the following permutation is even or odd

i) $P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 5 & 7 & 6 & 3 & 1 \end{pmatrix}$

$(1, 2, 4, 7)(3, 5, 6)$

$(1, 7)(1, 4)(1, 2)(3, 6)(3, 5)$ ∴ Here we got five transpositions so it is odd.

ii) $P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 2 & 1 & 6 & 5 & 8 & 7 & 3 \end{pmatrix}$

$(1, 4, 6, 8, 3)(2)(5)(7)$

$(1, 3)(1, 8)(1, 6)(1, 4)(2)(5)(7)$ = 4 transpositions

= Even

iii) $P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 3 & 4 & 2 & 1 & 8 & 6 & 5 \end{pmatrix}$

$(1, 7, 6, 8, 5)(2, 3, 4)$ Even

$(1,5)(1,8)(1,6)(1,7)(2,4)(2,3) = 6$ transpositions

iv) $(6, 4, 2, 1, 5)$

$(6, 5)(6, 1)(6, 2)(6, 4) = \underline{4} = \underline{\text{Even}}$

v) $(4,8)(3,5,2,1)(2,4,7,1)$

$(4,8)(3,1)(3,2)(3,5)(2,1)(2,7)(2,4) = \underline{7} = \text{odd}$

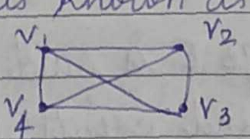
→ **Graph Theory:** $G = (V, E)$

A graph G is equal to (V, E) where V is the non empty set of the elements known as the vertices and E is set of edges.

Types of Graphs:

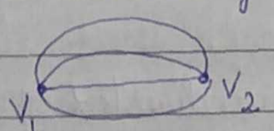
i) **Simple Graph:** A graph with no self loop or multiple edges between vertices is known as 'simple graph'.

Let G =

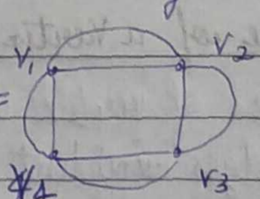


ii) **Multi Graph:** A graph that may have multiple edges or parallel edges connecting the same edges.

$G_1 =$

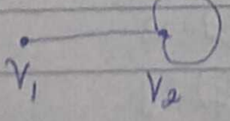


$G_2 =$

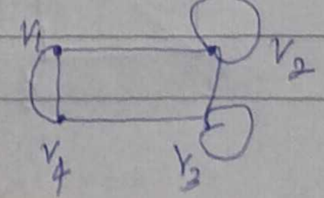


iii) **Pseudo Graph:** A graph which includes edges that connects a vertex to itself, such edges are called as loops and corresponding graphs are known as pseudo graphs

$G_1 =$



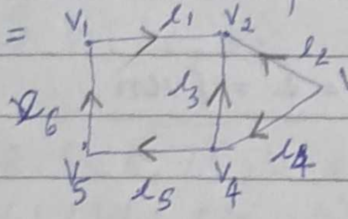
G_2



iv) **Directed Graph:** A graph $G = (V, E)$ consits of non empty set of elements V and a set of directed edges E, Each directed edge is associated with ordered pair of vertices.

If $E = (u, v)$, then u is said to be starting point and v is said to be end point.

Let G1 =

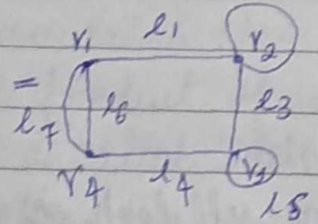


$V = \{v_1, v_2, v_3, v_4, v_5\}$
$E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

$e_1 = (v_1, v_2)$ $\quad e_3 = (v_4, v_2)$ $\quad e_5 = (v_4, v_5)$
$e_2 = (v_3, v_2)$ $\quad e_4 = (v_3, v_5)$ $\quad e_6 = (v_5, v_1)$

v) **Adjacent Vertices:** Two vertices u and v in an undirected graph G are said to be adjacent if (u and v) are end points of an edge in G

G =



v_1 and v_2 are adjacent
v_1 and v_3 are not adjacent
v_1 and v_4 are adjacent.

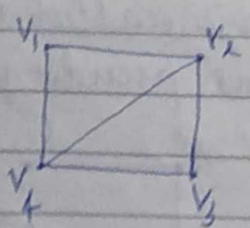
vi) **Incident Vertices:** A edge E is said to be incident to the vertices u and v are end points of v

In the above Eg: e_1 is incident to v_1 and v_2.

Degree of a Vertex:

Degree of vertex in an undirected graph is the number of edges incident to it and it is denoted by dig of V

Note: If self loop is present in the graph then its degree is 2



dig $(v_1) = 2$
dig $(v_2) = 3$
dig $(v_3) = 2$
dig $(v_4) = 3$

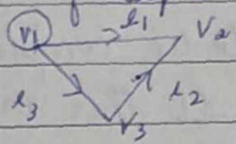
Date
Page

Note:

i) A vertex is said to be isolated if it has zero degree
ii) A vertex is said to be pendent vertex if its degree is 1

→ **Directed Graphs - In degree and outdegree of digraphs**

- Indegree :- No of edges which are coming into the vertex (v) or no of edges which has v as terminal point.

Eg:


Outdegree:
No of edges which are going out from the vertex and number of edges which has v as its intial point

→ **Handshaking Theorem:**

Imp) Let $G = (V, E)$ be an undirected graph with 'm' edges, then
$$2m = \sum_{v \in V} deg(v)$$

- **Theorem**

An undirected graph has an even number of vertices of odd degree.

Proof: Let V_1 and V_2 be the set of vertices of e1 degree and odd degree respectively. In an undirected graph $G = (V, E)$ with m edges, by handshaking theorem we have $2m = \sum deg(v)$
where $v \in V$

$$2m = \sum_{v \in V_1} deg(v) + \sum_{v \in V_2} deg(v)$$

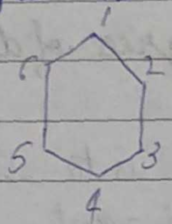
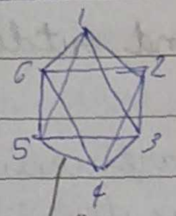
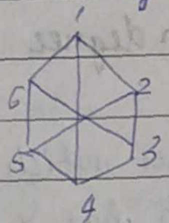
$$even = even + \sum_{v \in V_2} deg(v)$$

$$\sum_{v \in V_2} deg(v) = even$$

This is possible only if there are even number of vertices in V_2

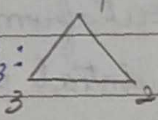
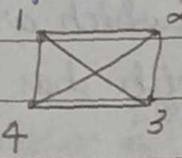
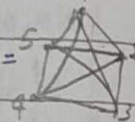
→ Some special cases of graphs

i) **Regular graph**: In any graph G, if all the vertices have the same degree then G is said to be regular graph.

Eg:   

K_6

ii) **Complete graph**: A graph G is said to be complete graph if all the vertices of G are adjacent to all other vertices in G and we denote complete graph by K_n

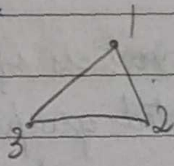
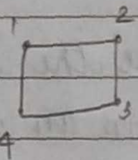
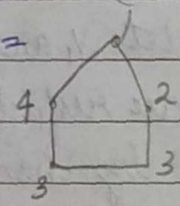
For example: K_3:  K_4 =  K_5 = 

Note:

· A complete graph, K_n with n vertices will be having $\frac{n(n-1)}{2}$ edges.

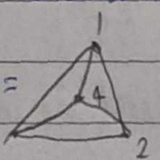
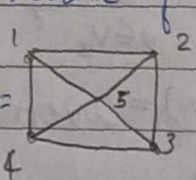
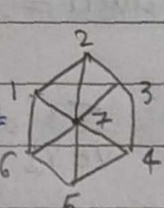
iii) **Cycle Graph**: A cycle C_n where $n >= 3$ consists of n vertices ~~the~~ $v_1, v_2, v_3 \ldots v_n$ and edges $(v_1, v_2)(v_2, v_3), (v_3, v_4) \ldots (v_{(n-1)}, v_n)(v_n, v_1)$

For eg:

C_3:  C_4:  C_5 = 

iv) **Wheel Graph**: A wheel graph is denoted by W_n and it is obtained by adding a new vertex to a cycle C_n and connecting this new vertex to each vertex of C_n

For eg:

W_3 =  W_4 =  W_6 = 

→ Representation of graphs in terms of Matrix

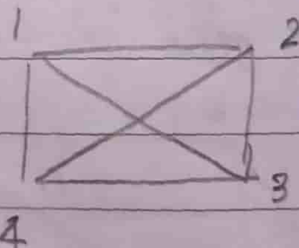
i) **Adjacency matrix**: Let $G = (V, E)$ be a graph with n vertices. ~~with~~ An adjacency matrix is a $n \times n$ matrix whose entries are as follows.

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

- The adjacency matrix is denoted by $A = [a_{ij}]_{n \times n}$

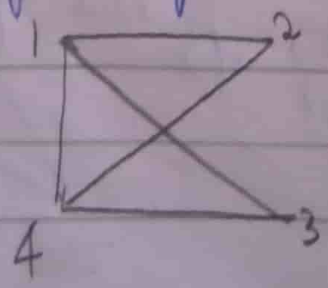
- An adjacency matrix is a symmetric matrix

$$[a_{ij}] = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

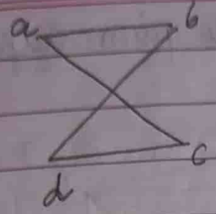


→ Use the adjacency matrix to represent the following graph.

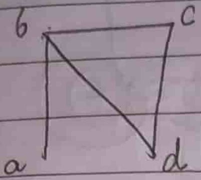
1)



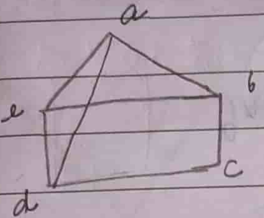
$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

2) 

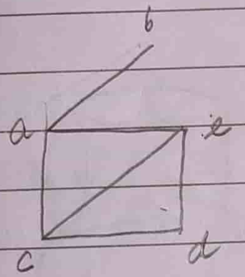
$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

3) 

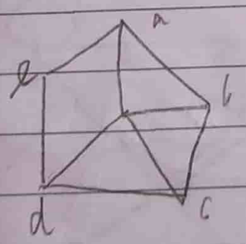
$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

4) 

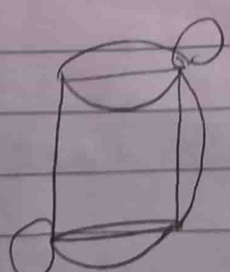
$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

5) 

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

6. 

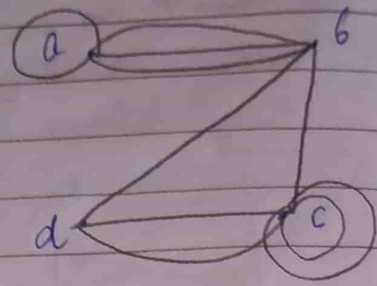
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

7) 

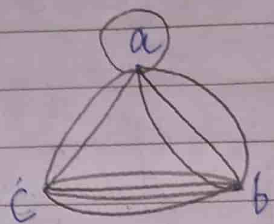
$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 1 \\ 3 & 1 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 1 & 0 & 3 & 1 \end{bmatrix} \end{array}$$

→ Draw an undirected graph for the given adjacency matrices

1) $$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix} \end{array}$$



2) $$\begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix} \end{array}$$



3) $$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

4) $$\begin{bmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$$

5) $$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

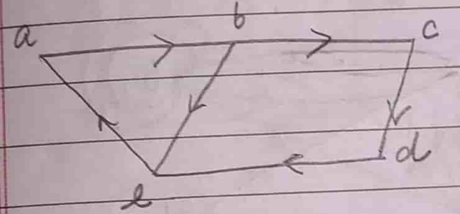
Adjacency Matrix of directed graph

let $G = (V, E)$ be a directed graph then adjacency matrix of G is a $n \times n$ matrix where n is no of vertices, the entries of the matrix G.

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is a directed edge} \\ 0 & \text{otherwise} \end{cases}$$

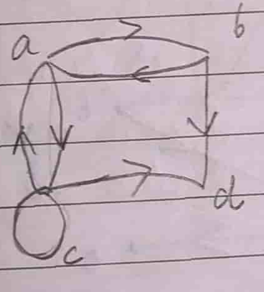
→ Represent the following graphs as adjacency matrices

i)



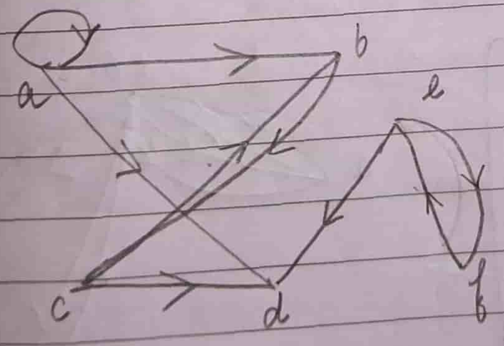
$$\begin{array}{c|c} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

2)



$$\begin{array}{c|c} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

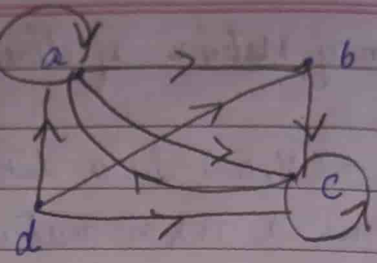
3)



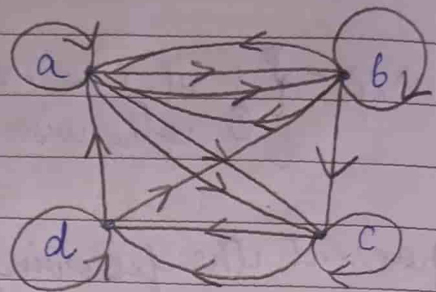
$$\begin{array}{c|c} & \begin{matrix} a & b & c & d & e & f \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \\ f \end{matrix} & \begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix} \end{array}$$

→ Draw digraph for the following adjacency matri

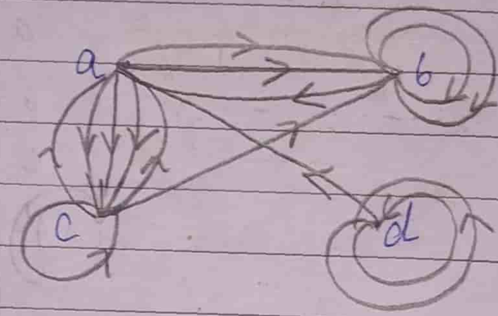
1) $$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$



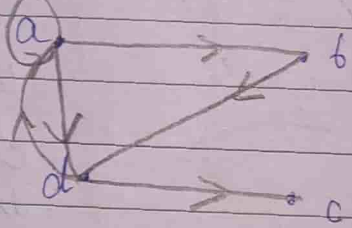
2) $$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$



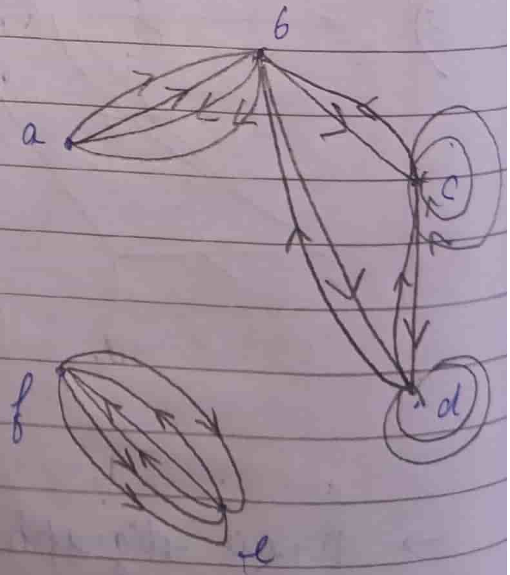
3) $$\begin{bmatrix} 0 & 2 & 3 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$



4) $$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$



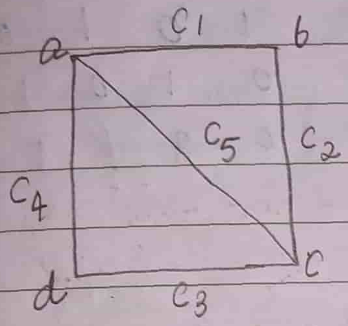
5) $$\begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 3 & 0 \end{bmatrix}$$



→ <u>Incident Matrix</u> :-

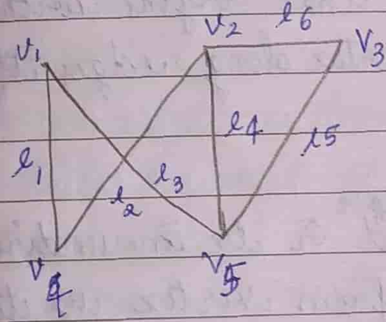
let $G = (V, E)$ be an undirected graph, let $V_1, V_2 \ldots V_n$ be the vertices and $E_1, E_2 \ldots E_m$ be the edges of graph G, then the incident matrix $M = [M_{ij}]_{n \times m}$ where $M_{ij} = \begin{cases} 1 & \text{if } ij \text{ is incident to } V_i \\ 0 & \text{otherwise.} \end{cases}$

→ write incident matrix for the following graphs



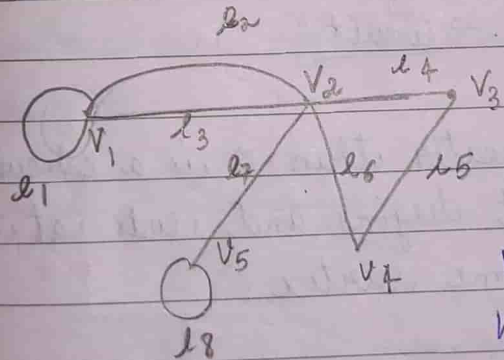
$$M = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

✓→ 2M



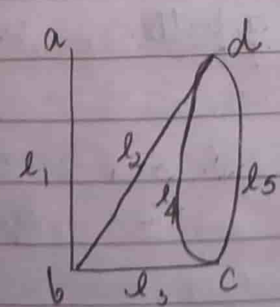
$$M = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

→

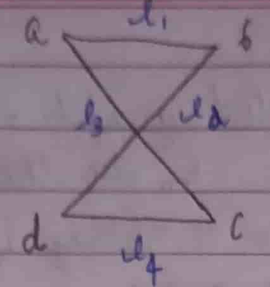


$$\begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

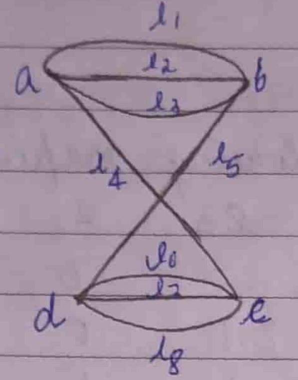
→



$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

→ 

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

→ 

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

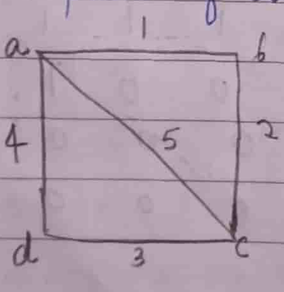
→ **Paths and Circuits:**

- Path is a sequence of edges that begins with a vert of a graph and travel from vertex along edges of the graph.

- Let 'n' be a non negative integer and G be an undirected graph a path of length n from vertex u to vertex is a sequence of edges. → Circuit.

→ Let A path of length greater than 0 is a circuit if it begins and ends at the same vertex

Eg:

G= 

→ abcd is a path, acd is also a path.

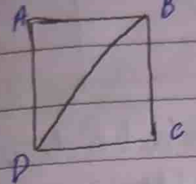
→ abda is a circuit

acda is also circuit.

→ Euler's path and circuit :-

- In a graph G, a simple path containing every edges of G → Euler's path.

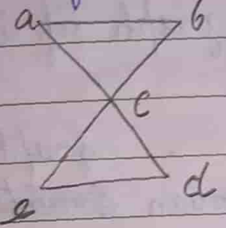
Eg:



- BCDBAD - euler's path

- Euler's circuit: In a graph G, a simple circuit containing every edge of G.

Eg:



abcdeca - Euler circuit.
But there does not exist a Euler path.

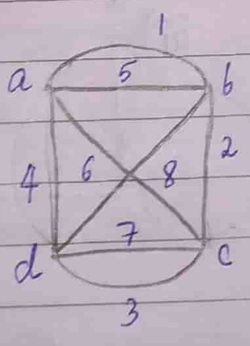
Theorem 1

- A connected multigraph with atleast two vertices has a Euler's circuit iff each its vertices has even degree.

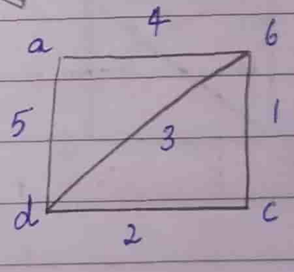
Theorem 2

- A connected multigraph has a ~~Euler~~ Euler path but not an Euler circuit iff it has exactly two vertices of odd degree.

Eg:



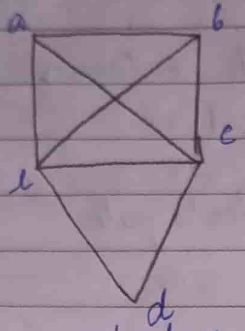
Eg:

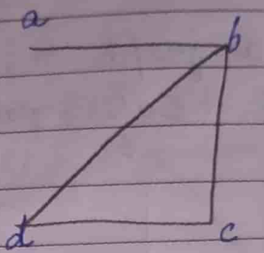


Hamilittonian Graph :

- A simple path in a graph G that passes through every vertex exactly ones is called Hamilttonian path.
- A simple circuit (starting and ending points are same) in a graph G that passes through every vertex exa

once is called 'Hamilttonian ~~vertices~~ circuit'.

i) 

ii) 

there, abcdea is a Hamilttonian circuit

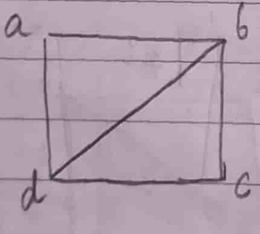
Here abcd is a Hamilttonian path.

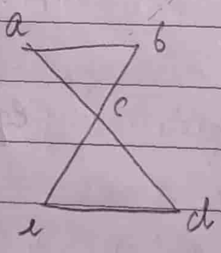
- A graph G is said to be Euler graph if it consists of Euler circuit

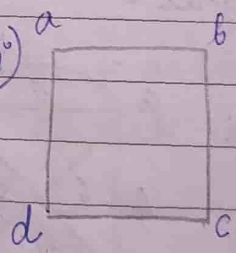
- A graph G is said to be Hamiltonian ~~circuit~~ graph if it consists of Hamilttonian circuit.

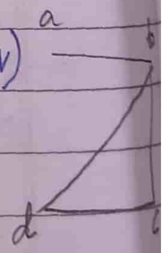
→ Write the graphs for the following :-

i) The graph which is Hamilttonian but not Euler.
ii) A graph which is Euler but not Hamilttonian.
iii) Both Euler and Hamilttonian
iv) Neither Euler nor Hamilttonian.

i) 

ii) 

iii) 

iv) 

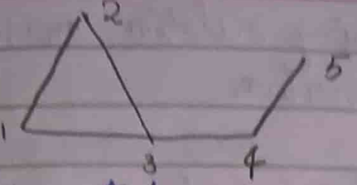
→ Connectidness of Graph:

If there exists a path for every pairs of vertices.

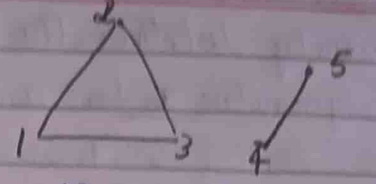
- Connectedness of an undirected graph means there exist a path between every pair of vertices. Allowing movement in either direction along edges.

For eg: Let G→

Let G =



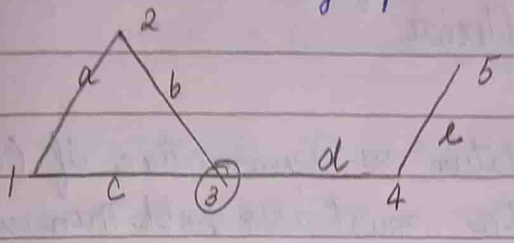
Connected graph



disconnected graph

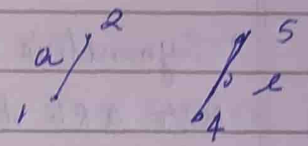
- **Cut point :**
  It is a vertex in a connected graph whose removal would disconnect the graph.

Let G =

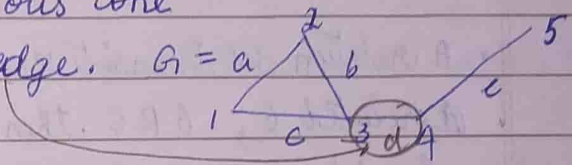


Let ③ be the cutpoint

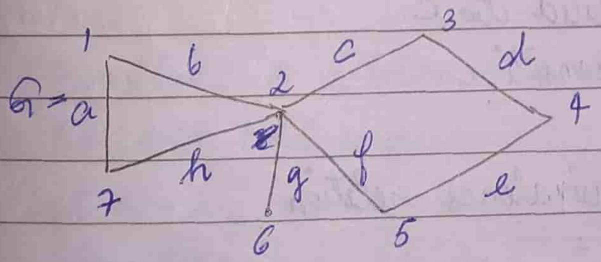
G' =



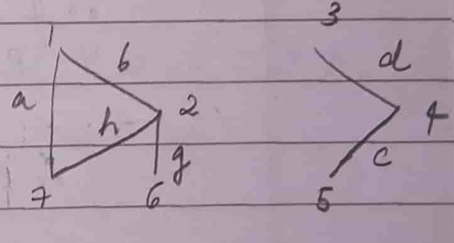
- **Cut edge**
  It is an edge that when removed disconnects the connected graph. Eg. Same as previous one
  → Let d be the cut edge. G =



- **Cut Set** : A set of cut edges is known as cut set



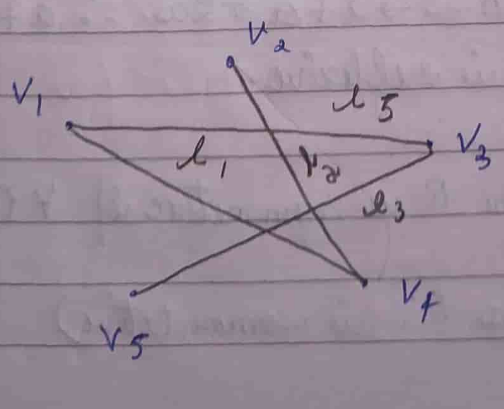
→ G' =



→ **Incidence matrix continuation**

Write the graph if its incidence matrix M =

$$\begin{array}{c|cccc} & e_1 & e_2 & e_3 & e_4 \\ \hline v_1 & 1 & 0 & 0 & 1 \\ v_2 & 0 & 1 & 0 & 0 \\ v_3 & 0 & 0 & 1 & 1 \\ v_4 & 1 & 1 & 0 & 0 \\ v_5 & 0 & 0 & 1 & 0 \end{array}$$



Equivalence

→ verify whether the following relation on set A is reflexive, symmetric and and transitive

1) $A = Z$: aRb ; aRb iff $a \le b = 1$

- Reflexive : $\forall a \in A$, aRa holds.

$a, R a$ iff $a \le a+1$ is true because the condition is true for any value of a

$\therefore$ R is reflexive

- Symmetric :- A relation is symmetric if $(a, b) \in A$ whenever aRb holds bRa must also hold. Then we need to write whether $b \le a+1$ is true or false. Let $a = 2$, $b = 0$

$a \le b+1 = 2 \le 0+1 = 2 \le 1$ false $\therefore (2,0)$ is not in R $(2,0) \notin R$

$b \le a+1 = 0 \le 2+1 = 10 \le 3 \Rightarrow$ True $(0, 2) \in R$ $\therefore$ R is not symmetric

- A relation is transitive if for all a, b, c in A, whenever a related b, bRC then aRC

Consider $a \le b+C$ and $b \le C+1$ ($\because aRb$ and bRC)

$a \le C+1+1$ or $a \le C+2$

$\therefore$ a is not related to c

R is not transitive

$\therefore$ R is not equivalence relation

2) $A = Z$: aRb iff $a+b$ is even

- Reflexive : A relation R is reflexive if $\forall a \in A$ then aRa holds $\Rightarrow a+b$ is even $\Rightarrow a+a = 2a$ $\therefore a+a$ is true

$\therefore$ R is reflexive.

- Symmetric : A relation R is symmetric if $\forall (a, b) \in A$ then aRb, bRa holds

wkt $a+b = b+a$ ($\because$ is commutative)

→ $b+a$ is even then bRa holds → R is symmetric

- Transitive : A relation R is transitive if $\forall a,b,c \in A$, aR[b] and bRc implies aRc holds.

Let $a+b$ and $b+c$ be even

$(a+b) \div (b+c) = a-c = a+c) =$ is even.

∴ $a+c$ is even aEc holds.

Here aRc holds → R is transitive

∴ R is equivalence

3) Let $a = R$; aRb iff $a^2+b^2 \cancel{\neq} = 4$

Reflexive : A relation R is reflexive if $\forall a \in A$ then aRa holds. $a^2+a^2=4$, $2a^2=4$ $= a^2=2$ $= a=\pm\sqrt{2}$

This is not true for all real numbers. Only for specific values it is true . ∴ It is not reflexive.

Symmetric : A relation R is symmetric if $\forall (a,b) \in A$ then aRb, bRa holds

Here $a^2+b^2=4$ is symmetric because $a^2+b^2=b^2+a^2$ (since addition is commutative) ∴ R is symmetric

Transitive: A relation R is transitive if $\forall a,b,c \in A$, aRb and bRC implies aRC holds.

Let $a=2$, $b=0$, $c=2$.

Consider $a^2+b^2 = 2^2+0=4$ True

$b^2+c^2 = 0^2+2^2=4$ True

$a^2+c^2 = 2^2+2^2=8 \neq 4$ = False

∴ a is not in Relation with R

∴ Not transitive