UNIT-I (CO1)

LOGICS

A proposition or a statement is a declarative sentence (that is, a sentence that declares a fact) that is either True or False, but not both.

Example:

9 is less than 15 → True.

9 is greater than 15 → False.

Do you speak English? → Not a proposition.

$3 - x = 5$ → It is a declarative statement but not a proposition as it can be either true or false depending on value of x.

Truth Value → truth or falsity of a proposition.

Examples: France borders Belgium → True → truth value of 1st Statement.

Barcelona is in Italy → False → truth value of 2nd Statement

→ Logical Connectives.

(i) Logical AND ($p \wedge q$ read as: p and q)

- Logical conjunction / multiplication.

Truth Table: (AND)

p	q	r
0	0	0
0	1	0
1	0	0
1	1	1

Truth Table: (OR)

p	q	r
T	T	T
T	F	T
F	T	T
F	F	F

(ii) Logical OR ($p \vee q$ read as: p OR q)

- Logical Disjunction / Addition

(iii) Logical NOT ($\sim p$ / $\neg p$ · read as: not p)

- Logical complementation.

Truth Table.

p	$\sim p$
T	F
F	T

Exercise: p: "It is cold" q: "It is raining"

$\sim p$: It is not cold.

$p \wedge q$: It is cold and It is raining

$p \vee q$: It is cold or It is raining.

$q \wedge p$: It is raining and It is cold.

$q \vee p$: It is raining or It is cold.

$q \vee \sim p$: It is raining or It is not cold.

→ Conditional Proposition.

If $p \rightarrow q$ (read as: if p then q) is the proposition,

p is called the hypothesis

q is called the conclusion

This proposition is false only when p is true and q is false

truth table:-

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

→ Biconditional Proposition.

If $p \leftrightarrow q$ is the proposition.

This true only when both p and q are true or both are false.

Otherwise it is false.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

→ Exclusive OR (XOR)

If p XOR q is the proposition.

It is true only when one is true and other one is false.

Rest is false (if both are T or both are false)

p	q	p XOR q
T	T	F
T	F	T
F	T	T
F	F	F

→ Simple proposition : statements that do not contain any logical connectives.

Example : p: 5 is a prime no.

q: 5 is a composite no.

→ Compound proposition: the new proposition obtained by using logical connectives.

Example: 5 is a prime no. or 5 is a composite no. ($p \vee q$)

→ Components or primitives: original propositions from which a compound proposition.

Exercise: p: "The election is decided"

q: "The votes have been counted."

(a) $\neg p$

"The election is not decided"

(b) $p \vee q$

The election is decided or the votes have been counted.

(c) $\neg p \wedge q$

The election is not decided and the votes have ~~not~~ been counted.

(d) $q \rightarrow p$

If the votes have been counted then the election is decided.

(e) $\neg q \rightarrow \neg p$

If the votes have not been counted then the election is not decided.

(f) $\neg p \rightarrow \neg q$

If the election is not decided then the votes have not been counted.

(g) $p \leftrightarrow q$

The votes have been counted if and only if the election is decided.

(h) $\neg q \vee (\neg p \wedge q)$

~~The election is not~~ The votes have not been counted or the election is not decided and the votes have been counted.

If p then q = q, if p
∴ p comes first.

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Exercise: p: "It is below freezing"

but → ∧

q: "It is snowing"

(a) $p \wedge q$
(b) $p \wedge \neg q$
(c) $\neg p \wedge \neg q$
(d) $p \vee q$
(e) $p \rightarrow q$
(f) ~~$(p \vee q) \wedge (\neg q \rightarrow p)$~~ $(p \vee q) \wedge (p \rightarrow \neg q)$
(g) $p \leftrightarrow q$

Exercise: Construct truth table: $(p \vee \neg q) \rightarrow (p \wedge q)$ [4 marks]

p	q	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	?
T	F	?
F	T	?
F	F	?

p	q	$\neg q$	(A) $p \vee \neg q$	(B) $p \wedge q$	$A \rightarrow B$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Homework: (i) $(p \vee \neg q) \rightarrow q$
(ii) $(p \vee q) \rightarrow (p \wedge q)$
(iii) $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$

2/3 marks or MCQ

(i) $(p \vee \neg q) \rightarrow q$

p	q	$\neg q$	(A) $p \vee \neg q$	$A \rightarrow q$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	F
F	F	T	T	T

(ii)

(iii)

(ii) $(p \vee q) \longrightarrow (p \wedge q)$

p	q	$p \vee q$ (A)	$p \wedge q$ (B)	$A \rightarrow B$
T	T	T	T	T
T	F	T	F	T
F	T	T	F	T
F	F	F	F	T

(iii) $(p \rightarrow q) \longleftrightarrow (\neg q \longrightarrow \neg p)$

p	q	$\neg p$	$\neg q$	$p \rightarrow q$ (A)	$\neg q \rightarrow \neg p$ (B)	$A \longleftrightarrow B$
T	T	F	F	T	T	T
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	T	T	T

Questions :-

$(p \vee q) \longrightarrow (p \oplus q)$

$(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$

$(p \vee q) \wedge \neg r$

2/3 marks or MCQs

→ Converse.

If $p \Rightarrow q$ (read as: p implies q)

$q \Rightarrow p$ is called it's converse.

→ Contrapositive

If $p \Rightarrow q$.

$\neg q \Rightarrow \neg p$ is called it's contrapositive

→ Inverse

If $p \Rightarrow q$

$\neg p \Rightarrow \neg q$ is called it's inverse.

"The home town wins whenever it is raining"

Exercise: p : the home town wins } ×
q : It is raining

If it is raining then the home team wins.
p : It is raining } ✓
q : the home team wins

converse : The home town wins if it is raining
contrapositive : If the home town does not win, then it is not raining
inverse : If it is not raining then the home team does not win

"If I study hard then I will pass the exam"
p : I study hard.
q : I will pass the exam

converse : I will pass the exam if I study hard.
contrapositive : I will not pass the exam if I will study hard.
inverse : If I will not study hard the I will not pass the exam

(i) "I'll do your work if you give me an ice cream"
(ii) "I'll do your work only if you give me an ice cream"

(i) q : I'll do your work.
p : I'll give you an ice cream

(The word "only" will change the meaning of the sentence.)

(ii) p : I'll give you an ice cream
q : I'll do your work.

Note :-

If p and q are 2 premises, then,
- $p \to q$:- q if p (also written as if p then q)
- $q \to p$:- q only if p.

Write the truth table for $p \to q$, $q \to p$, $\neg p \to \sim q$, $\sim q \to \sim p$.

p	q	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Note :- which of the statements are equivalent ? (Logical equivalence)

$q \to p = \neg p \to \neg q$.

$p \to q = \neg q \to \neg p$

→ Tautology : If a given statement is always true for all possible values of it's propositional variables.

→ Contradiction : If a given statements is always false.

→ Contigency : If a given premise is either true or false depending on the truth values of it's propositional variables.

Exercise: $(p \to q) \leftrightarrow (\neg q \to \neg p)$ is a tautology.

p	q	$p \to q$	$\sim p$	$\sim q$	$\neg q \to \neg p$	$(p \to q) \leftrightarrow (\neg q \to \neg p)$
0	0	T	F	F	T	T
0	1	F	F	T	F	T
1	0	T	T	F	T	T
1	1	T	T	T	T	T

$(p \to q) \wedge (p \vee q)$ is a contingency.

HW

Q. $(p \wedge q) \wedge \neg(p \wedge q)$ is a contradiction.

Q. $p \vee \sim(p \wedge q)$ is a tautology.

Q. $[(p \rightarrow q) \wedge (q \rightarrow r)] \longrightarrow (p \rightarrow r)$ is a tautology.

→

→ Logical Equivalence:

Two statements p and q are logically equivalent if they always have the same truth values.

($p \equiv q$ or $p \Leftrightarrow q$) read as: p is logically equivalent to q.)

i.e., Two logical formulas p and q are logically equivalent if and only if $p \leftrightarrow q$ is a tautology

Example: $\neg(\neg p) \Leftrightarrow p$.

$p \vee p \Leftrightarrow p$

$p \wedge p \Leftrightarrow p$

Q: — $p \oplus q \Leftrightarrow (p \vee q) \wedge (\neg(p \wedge q))$

p	q	$p \vee q$	$p \wedge q$	$\neg(p \wedge q)$	$(p \vee q) \wedge \neg(p \wedge q)$	$p \oplus q$
T	T	T	T	F	F	F
T	F	T	F	T	T	T
F	T	T	F	T	T	T
F	F	F	F	T	F	F

Q. Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent: (De Morgan's Theorem:)

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Conclusion:

Q. Show that $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent: (De Morgan's Theorem:)

In general, if $p_1, p_2, \ldots p_n$ are propositions, the De Morgan's Law can be extended as:

$\neg(p_1 \vee p_2 \vee \ldots \vee p_n) \Leftrightarrow (\neg p_1 \wedge \neg p_2 \wedge \ldots \wedge \neg p_n)$

$\neg(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \Leftrightarrow (\neg p_1 \vee \neg p_2 \vee \ldots \vee \neg p_n)$

H.W. Q. $p \rightarrow q \Leftrightarrow \sim p \vee q$

$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

→ Some Basic Logical Equivalences:

$p \wedge T \equiv p$
$p \vee F \equiv p$ } Identity Laws.

$p \vee T \equiv T$
$p \wedge F \equiv F$ } Domination Laws.

$p \vee p \equiv p$
$p \wedge p \equiv p$ } Idempotent Laws

$\neg(\neg p) \equiv p$ } Double negation Law

$p \vee q \equiv q \vee p$
$p \wedge q \equiv q \wedge p$ } Commutative Laws.

$(p \vee q) \vee r \equiv p \vee (q \vee r)$ } Associative Laws
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ } Distributive Laws.
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$p \vee (p \wedge q) \equiv p$ } Absorption laws
$p \wedge (p \vee q) \equiv p$

$p \vee \neg p \equiv T$ } Negation Laws.
$p \wedge \neg p \equiv F$

* $p \to q \equiv \neg p \vee q$ } Conditional Statements
* $p \to q \equiv \neg q \to \neg p$

* $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$ } Biconditional Statements.
* $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

→ Constructing New Logical Equivalence.
The logical equivalences discussed, can be used to construct additional logical equivalences.

Q. Show that $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.
→ (can't use truth table method for this)

$\neg(p \to q) \equiv \neg(\neg p \vee q)$ [(from above rules] [$p \to q \equiv \neg p \vee q$]
$\equiv \neg(\neg p) \wedge \neg q$ by deMorgan's theorem [$\neg(p \vee q) \equiv \neg p \wedge \neg q$]
$\equiv p \wedge \neg q$ since $\neg(\neg p) \equiv p$.

Thus, $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent.

Q. S.T $\neg p \to (q \to r)$ and $q \to (p \vee r)$ are logically equivalent using a series of logical equivalences.

$\neg p \to (q \to r) \equiv \neg p \to (\neg q \vee r)$ $[p \to q \equiv \neg p \vee q]$

$\equiv$ ~~p~~ q $\longrightarrow [p \to q \equiv \neg p$

$\equiv \neg(\neg p) \vee (\neg q \vee r)$

$\equiv p \vee (\neg q \vee r)$

$\equiv (p \vee \neg q) \vee r$ [Associative Law]

$\equiv (\neg q \vee p) \vee r$ [Commutative Law]

$\equiv \neg q \vee (p \vee r)$ [Associative Law]

$\equiv q \to (p \vee r)$ [by $\neg p \vee q \to p \to q$]

$\therefore$ They are logically equivalent

Q. S.T $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent using a series of logical equivalences.

$\neg(p \vee (\neg p \wedge q)) \equiv$ ~~$\neg(p \vee (\neg p \vee \neg q))$~~

$\equiv \neg p \wedge \neg(\neg p \wedge q)$

[DeMorgan's Law

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

$\equiv \neg p \wedge \neg(\neg p) \vee \neg q$

[DeMorgan's Law

$\neg(\neg p \wedge q) \equiv \neg(\neg p) \vee \neg q$]

$\equiv \neg p \wedge (p \vee \neg q)$ $[\neg(\neg p) \equiv p]$

$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ [Distributive

$\equiv F \vee (\neg p \wedge \neg q)$ $[(\neg p \wedge p) \equiv F]$

$\equiv (\neg p \wedge \neg q)$ $[F \vee p \equiv p]$

$\therefore$ They are logically equivalent.

Q. S.T $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$

$$(p \to q) \vee (p \to r) \equiv (\neg p \vee q) \vee (\neg p \vee r) \quad [\because p \to q \equiv \neg p \vee q]$$
$$\equiv (\neg p \vee \neg p) \vee (q \vee r) \quad [\because \text{Associative Property}]$$
$$\equiv (\neg p) \vee (q \vee r) \quad [\text{by } p \vee p \equiv p]$$
$$\equiv \neg(\neg p) \longrightarrow (q \vee r) \quad [\because p \to q \equiv \neg p \vee q]$$
$$\equiv p \longrightarrow (q \vee r)$$

H.W. Q. S.T $(p \wedge q) \to (p \vee q)$ is a tautology using a series of logical equivalences.

Q. Determine whether $(\neg q \wedge (p \to q)) \longrightarrow \neg p$ is a tautology.

$$(\neg q \wedge (p \to q)) \longrightarrow \neg p \equiv \neg(\neg q \wedge (p \to q)) \vee \neg p$$
$$\equiv (\neg\neg q \vee \neg(p \to q)) \vee \neg p$$
$$\equiv (q \vee \neg(\neg p \vee q)) \vee \neg p$$
$$\equiv (q \vee (p \wedge \neg q)) \vee \neg p$$
$$\equiv (q \vee p) \wedge (q \vee \neg q) \vee \neg p$$
$$\equiv ((q \vee p) \wedge T) \vee \neg p$$
$$\equiv (q \vee p) \vee \neg p \equiv q \vee (p \vee \neg p)$$
$$\equiv q \vee T \equiv T.$$

$\therefore$ the expression is a tautology.

Imp Q. S.T $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$ by using a series of logical equivalences.

$$((\neg p \vee q) \wedge p)$$
$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$
$$\equiv (\neg p \vee q) \wedge (\neg q \vee p) \quad [\because p \to q \equiv \neg p \vee q]$$
$$\equiv ((\neg p \vee q) \wedge \neg q) \vee ((\neg p \wedge p) \vee (q \wedge p)) \quad [\text{By distributive law}]$$
$$\equiv ((\neg p \wedge \neg q) \vee (q \wedge \neg q)) \vee ((\neg p \wedge p) \vee (q \wedge p))$$
$$\equiv ((\neg p \wedge \neg q) \vee F) \vee ((F) \vee (q \wedge p)) \quad [\because p \wedge \neg p \equiv F]$$
$$\equiv (\neg p \wedge \neg q) \vee (q \wedge p) \quad \because p \vee F \equiv p$$
$$\equiv (p \wedge q) \vee (\neg p \wedge \neg q) \quad [\text{By commutative law, twice}]$$

*Hint: $p \leftrightarrow \neg q \equiv (p \rightarrow \neg q) \wedge (\neg q \rightarrow p)$

Imp. H.W. Q. S.T. $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$ by developing a series of logical expressions.

→ Method of proof and rules of inference.

Mathematical logic is often used for logical proofs.

Proofs are valid arguments that determine the truth values of mathematical.

Arguments:

A sequence of statements.

Premise 1: Statement 1

Premise 2: statement 2.

Conclusion: ∴ statement 3.

For example

1: All mammals have a backbone.

2: The dolphin is a mammal.

Conclusion: ∴ The dolphin has a backbone.

An argument can be valid or invalid.

It is valid if the conclusion logically follows from the prem

* If $p_1, p_2 \ldots p_n$ are premises and q is the conclusion, then $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q$ is a tautology

→ Rules of Inference:

provides templates or guidelines for constructing valid arguments from statements we already have.

① Modus Ponens (PM) : If $p \rightarrow q \quad \dfrac{p}{\therefore q}$

Eg: S1: If it rains, the ground will be wet.

S2: It is raining.

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② Modus Tollens (MT): If $p \to q$, $\frac{\neg q}{\therefore \neg p}$

Eg: S1: If it rains, "
S2: The ground is not wet
C: ∴ It is not raining.

③ Hypothetical Syllogism (HS): if $p \to q$ and $q \to r$, then $\therefore p \to r$.
Eg: S1: If I study, I will pass.
S2: If I pass, I will graduate
C: ∴ If I study, I will graduate.

④ Disjunctive Syllogism (DS): if $p \vee q$ and $\neg p$ is true, then $\therefore q$.
S1: It is either raining or snowing.
S2: But it is not snowing.
C: ∴ It is raining.

⑤ Conjunction: if p ~~and q~~, is true and q is true, then ∴ $p \wedge q$ is true
S1: I have a pen.
S2: I have a notebook.
C: ∴ I have a pen and a notebook.

⑥ Simplification: If $p \wedge q$ is true, then ∴ p is true.
S: I have a pen and a notebook.
C: ∴ I have a pen.

⑦ Addition: If p is true, then ∴ $p \vee q$ ~~may~~ will be true.
S: I have a pen.
C: ∴ I have a pen or notebook.

⑧ Resolution: If $p \vee q$ and $\neg p \vee r$ is true, then ∴ $q \vee r$ is true.
S1: Either it is raining or it is cloudy.
S2: Either it is not raining or I will carry an umbrella.
C: ∴ Either it is ~~not~~ cloudy or I will carry an umbrella.

Q. Establish the validity of arguments usings Rules of Inference.

(a) p

$p \rightarrow q$

$\neg q \vee r$

$\therefore r$

Ans:

Steps	Reason
1. p	given.
2. $p \rightarrow q$	given.
3. $\therefore q$	Modus Ponens (MP) on steps 1 and 2
4. $\sim q \vee r$	given
5. r	Disjunctive syllogisms [Here $p \vee q \equiv (\sim q \vee r)$ and this rule implies $p \vee q$ and $\neg p$ gives q $\therefore \sim\sim q$ (i.e. q) gives r]

Hence it is a valid argument.

(b) $(\sim p \vee q) \longrightarrow r$

$r \longrightarrow (s \vee t)$

$\sim s \wedge \sim u$

$\sim u \rightarrow \sim t$

$\therefore p$

Ans:

Steps	Reason
1. $(\sim p \vee q) \rightarrow r$	given.
2. $r \rightarrow (s \vee t)$	given.
3. $\therefore (\sim p \vee q) \rightarrow (s \vee t)$	Hypothetical Syllogism (HS) on steps 1 and 2
4. $\sim s \wedge \sim u$	given
5. $\therefore \sim s$	Simplification Rule on step 4.
6. $\therefore \sim u$	Simplification Rule on step 4.
7. $\sim u \rightarrow \sim t$	given.
8. $\therefore \sim t$	Modus Ponens (MP) on steps 6 and 7
9. $\sim s \wedge \sim t \equiv \sim (s \vee t)$	De Morgan's Theorem on step 5 and 8
10. $\therefore \sim (\sim p \vee q)$	Using Modus Tollens on steps 3 and 9
11. $\sim(\sim p) \wedge \sim q \equiv p \wedge \sim q$	De Morgan's Law and negation law
12. p	Simplification on step 11.

$\therefore$ It is a valid argument.

(c) $p \wedge q$
$p \rightarrow (r \wedge q)$
$r \rightarrow (s \vee t)$
$\sim s$

$\therefore t$

Ans:

Steps	Reason
1. $p \wedge q$	given.
2. $p \rightarrow (r \wedge q)$	given.
3. $\therefore p$	Simplification on step 1.
4. $\therefore (r \wedge q)$	Modus Ponens on steps 2 and 3.
5. $\therefore r$	Simplification on step 4.
6. $r \rightarrow (s \vee t)$	given.
7. $\therefore (s \vee t)$	Modus Ponens on steps 5 and 6.
8. $\sim s$	given.
9. $\therefore t$	Disjunctive Syllogism on steps 7 and 8.

Hence it is a valid argument.

Q. give an argument to show that the hypothises/premises $p \wedge q, p \rightarrow (\sim q \vee r), r \rightarrow s$ leads to conclusion s.

$p \wedge q$.
$p \rightarrow (\sim q \vee r)$
$r \rightarrow s$

$\therefore s$

Ans.

Steps	Reason
1. $p \wedge q$	given.
2. $p \rightarrow (\sim q \vee r)$	given.
3. $\therefore p$	Simplification on step 1.
4. $\therefore (\sim q \vee r)$	Modus Ponens on steps 2 and 3.
5. $\therefore r$	Disjunctive syllogism ($\because \sim(\sim q)$ is T)
6. $r \rightarrow s$	given.
7. $\therefore s$	Modus Ponens on steps 5 and 6.

Hence it is a valid argument.

Q. Consider :

(a) "It's not sunny and It's colder than yesterday"

(b) "We will go swimming only if it's sunny."

(c) "If we don't go swimming then we will take canoe trip"

(d) "If we take canoe trip, then we will be home by sunset"

P.T conclusion : "We will be home by sunset" is valid

$$\begin{array}{l} \sim p \wedge q \\ r \rightarrow p \\ \sim r \rightarrow s \\ s \rightarrow t \\ \hline \therefore t \end{array}$$

Ans:

	Steps	Reason
1.	$\sim p \wedge q$	Given
2.	$r \rightarrow p$	Given
3.	$\therefore \sim p$	Simplification on Step 1.
4.	$\therefore \sim r$	Modus ~~Ponens~~ Tollens on Steps 2 and 3
5.	$\sim r \rightarrow s$	Given.
6.	$\therefore s$	Modus Ponens on steps 4 and 5
7.	$s \rightarrow t$	Given.
8.	$\therefore t$	Modus Ponens on steps 6 and 7.

$\therefore$ It is a valid argument.

Q. Consider:

(a) "If Dominic goes to the racetrack, then Helen will be mad"

(b) "If Ralph plays cards all night, then Carmela gets mad."

(c) "If either Helen or Carmela get mad, then Veronica will be notified"

(d) "Veronica has not heard from either of them"

(e) Consequently, Dominic did not make it to the racetrack and Ralph didnt play cards all night.

$$\begin{array}{l} \cancel{p \to q} \\ \cancel{r \to s} \\ \cancel{(p \vee q) \to t} \\ \cancel{\sim t} \\ \hline \therefore \text{[illegible]} \end{array} \qquad \begin{array}{l} p \to q \\ r \to s \\ (q \vee s) \to t \\ \sim t \\ \hline \therefore \neg p \wedge \neg r \end{array}$$

Ans:

Steps	Reason
1. $(q \vee s) \to t$	Given.
2. $\sim t$	Given.
3. $\sim(q \vee s)$	Modus Tollens on steps 1 and 2.
4. $\sim q \vee \sim s$	De Morgan's Law on step 3.
5. $\therefore \sim q$	} Simplification on step 4.
6. $\therefore \sim s$	
7. $p \to q$	Given
8. $\therefore \sim p$	Modus Tollens on steps 5 and 7.
9. $r \to s$	Given.
10. $\therefore \sim r$	Modus ~~Ponens~~ Tollens on steps 6 and 9.
11. $\sim p \wedge \sim r$	Conjunction on steps 8 and 10.

IMP Q. "If you send me an e-mail message, then I will finish writing the program"
"If you do not send me an e-mail, then I will sleep early".
Conclusion: "If I do not finish writing the program then I will wake up feeling refreshed."
"If I go to sleep early I will wake up feeling refreshed"

$$\begin{array}{l} p \to q \\ \sim p \to r, \quad r \to s \\ \hline \therefore \sim q \to s \end{array}$$

Ans:

~~Steps~~	~~Reason~~
~~1. $p \to q$~~	~~Given~~
~~2. $\sim p \to r$~~	~~Given.~~

(P) →

Steps	Reason
1. $\sim p \rightarrow r$	given
2. $r \rightarrow s$	given
3. $\therefore \sim p \rightarrow s$	Hypothetical Syllogism on steps 1 and 2
4. $p \rightarrow q$	given.
5.	

HW Q. "If Rochelle gets the supervisor's position and works hard, then she will get a raise."
"If she gets the raise, then she'll buy a new car."
"She has not purchased a car."
∴ Either Rochelle did not get the supervisors position or she did not work hard.

→ Introduction to Proofs.

A proof is a valid argument that establishes the truth of mathematical statements and validate results.

Proofs types:

(i) Direct: $\underset{\text{Assume}}{p \to t}$ and prove q $(p \to q)$

(ii) Indirect:

(a) Contraposition: Assume $\neg q$ and prove $\neg p$ (i.e., $\neg q \to \neg p$)

(b) Contradiction: Assume $\neg p$ and prove F (i.e., $\neg p \to F$)

Q. Give direct proof of the theorem:

"If n is an odd integer, then n^2 is odd"

$p \to n$ is odd integer. $q \to n^2$ is odd.

To prove: $p \to q$.

Start with p: n is an odd integer.

then, $n = 2k+1$ $k \to$ is any integer. ⟶ (1)

Sq. both sides,

$n^2 = (2k+1)^2$

$= 4k^2+4k+1$

$= 2(2k^2+2k)+1$ which is in form of eqn (1)

$\therefore n^2$ is an odd integer

$\therefore q$.

$\therefore$ Hence $p \to q$.

Q. Give direct proof that: "If m and n are perfect sqs., then mn is also a perfect sq."

$p \to m$ and n are perfect sqs. $q \to mn$ is also a perfect sq.

To prove: $p \to q$

Start with p: m & n are perfect sqs.

Let $m = k^2$, $n = l^2$

$\therefore mn = k^2 l^2$

$= (k)(k).(l)(l)$

$= (k)(l).(k)(l)$

$= ((k)(l))^2$

$mn = (kl)^2$

$\therefore$ mn is also a perfect square.

$\therefore p \to q$.

Q. Give an Indirect proof:

"The product of 2 even integers is an even integers"

Rewritten as: "If m and n are even integers then mn is an even integer"

$p \rightarrow$ m and n are even integers $\quad q \rightarrow$ mn is an even integer

To prove: $\neg q \rightarrow \neg p$.

Start with $\neg q$: mn is not an even integer

$\therefore$ mn is an odd integer $\quad [2 (\nmid) mn]$ doesn't divide symbol

If mn is odd, it is ~~nod~~ not divisible by 2

$\frac{mn}{2}$ ~~and $m \nmid 2$ and $n \nmid 2$~~ $\quad \frac{m \times n}{2} = \frac{m}{2} \times \frac{n}{2}$

$\therefore 2 \nmid m$ and $2 \nmid n$.

$\therefore$ m and n are not even integers.

$\therefore \neg p$

$\therefore \neg q \rightarrow \neg p \equiv p \rightarrow q$.

Q. Prove that: "If n is an integer and $3n+2$ is odd, then n is odd".

(This statement cannot be proven directly)

$p \rightarrow 3n+2$ is odd $\quad q \rightarrow$ n is odd.

Since this cannot be proven directly, we use indirect proof

To prove: $\neg q \rightarrow \neg p$.

Start with: $\neg q$: n is not odd.

$\therefore$ n is even.

Can be written in 2k form

$n = 2k \quad (k \rightarrow \text{integer})$

Then, $3n+2 = 3(2k)+2$

$= 6k+2$

$= 2(3k+1)$

$= 2(K)$ – is even. [K is an integer]

$\therefore 3n+2$ is even.

$\therefore \neg p$.

$\therefore \neg q \rightarrow \neg p \equiv p \rightarrow q$

Q. Prove that: "If n is an integer and n^2 is odd, then n is odd".

$p \rightarrow n^2$ is odd $\quad$ $q \rightarrow n$ is odd. (can prove ~~i~~directly)

To prove: $\neg q \rightarrow \neg p$.

Start with: $\neg q$

$\therefore$ n is not odd.

which means n is even.

we can write $n = 2k$ ($k \rightarrow$ integer).

$\therefore n^2 = (2k)^2$

$n^2 = 4k^2$

$n^2 = 2(2k)$ $\quad$ ($2k \rightarrow$ integer).

$\therefore$ n^2 is also even.

$\therefore$ n^2 is not odd

which implies $\neg p$

$\therefore \neg q \rightarrow \neg p \equiv p \rightarrow q$

Q. Prove that: If $n = ab$, where a and b are integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

$p \rightarrow n = ab$ $\quad$ $q \rightarrow a \leq \sqrt{n}$ $\quad$ $r \rightarrow b \leq \sqrt{n}$.

To prove: $p \rightarrow (q \vee r)$

Contrapositive: $\neg(q \vee r) \rightarrow \neg p$.

Start with: $\neg(q \vee r) =$

De Morgan's Law: ~~$\neg q \wedge \neg r$~~ $\neg q \wedge \neg r$.

~~$\neg q \wedge \neg r$~~

$\equiv a > \sqrt{n}$ and $b > \sqrt{n}$.

multiply.

$\equiv ab > \sqrt{n}\sqrt{n} = n$.

$\equiv ab > n$.

$\rightarrow ab \neq n \equiv \sim p$ (since $ab \neq n$)

Hence $\sim(q \vee r) \rightarrow \sim p \equiv p \rightarrow (q \vee r)$

if $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

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All rational nos. are in the form $\frac{p}{q}, q \neq 0$.
p and q have no common factors and are in minimum simplest form.

Imp Q. P.T $\sqrt{2}$ is irrational by giving a proof by contradiction.

$p \rightarrow \sqrt{2}$ is irrational.

Start with : $\neg p$: $\sqrt{2}$ is not irrational.

$\therefore \sqrt{2}$ is rational.

If $\sqrt{2}$ is rational, there will exist 2 nos. a and b such that $\boxed{\sqrt{2} = a/b}$ such that $b \neq 0$ and a and b have no common factors.

Squaring both sides,

$$2 = a^2/b^2$$

$$2b^2 = a^2 \longrightarrow (1)$$

If $a^2 = 2$ into smth it must mean that a^2 is even. so $\therefore$ a also must be even.

Then $a = 2k$ (It can be written in this form)

sub. in eqn (1)

$$2b^2 = 4k^2$$

$$\therefore b^2 = 2k^2$$

similarly b^2 is even so. b must be even.

So $\therefore$ if both a and b are even, then both are divisible by 2 $\therefore$ a and b have a common factor 2.

$\therefore$ this a contradiction to our assumption that a and b have no common factors.

$\therefore \neg p \rightarrow F$.

$\therefore$ The statement "$\sqrt{2}$ is rational" is false.

$\therefore$ "$\sqrt{2}$ is irrational" is true.

Hence proved.

Step 1:

Step 2:

→ Mathematical Induction

It is a mathematical technique which is used to prove a statement, a formula or a theorem is true for every natural nos.

Steps Involved:

Step 1: (Base step): It proves that a statement is true for the initial value.

Step 2: (Inductive step): It proves that if the statement is true for the nth iteration (for number n), then it is also true for the $(n+1)$th iteration.

Q. S.T by mathematical induction, that for all $n \geq 1$.

$$1+2+3+\dots\dots+n = \frac{n(n+1)}{2}.$$

Let $P(n): 1+2+3+\dots+n = \frac{n(n+1)}{2}$

Base Step:

Put $n=1$. We must show $p(1)$ is true

$p(1): 1 = \frac{1(1+1)}{2} = \underline{\underline{1}}$ ∴ which is clearly true.

Inductive step:

We must show that for $k \geq 1$, if $p(k)$ is true then $p(k+1)$ is also true.

We assume that for some fixed value of $k \geq 1$.

$$\left(1+2+3+\dots+k = \frac{k(k+1)}{2}\right)$$

We need to show that that:

$$p(k+1) = 1+2+3+\dots+(k+1) = \frac{(k+1)((k+1)+1)}{2}$$

$$P(k+1);\ 1+2+3+\dots k+(k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+2)(k+1)}{2}$$

$$\therefore 1+2+3+\dots+(k+1) = \frac{(k+1)((k+1)+1)}{2}$$

Hence proved

$1+2^1+2^2$ ———

$1+2^0 = 2.$

Q. Use mathematical induction to show that:

$$1+2+2^2+ \ldots +2^n = 2^{n+1}-1.$$ for all non-negative integers.

√Imp Q.

$P(n) = 1+2+2^2+\ldots+2^n = 2^{n+1}-1$

Basis step:

Put $n=0$. We must first show that $P(0)$ is true.

$P(0) = 2^{0+1}-1 = 2-1 = 1.$

RHS.

$= 1+2+\ldots 2^n$

$= 1+2+\ldots 2^0$

$= 1 = LHS.$

$\therefore LHS = RHS$

Inductive step:

Assume: $P(K) = 1+2+2^2+\ldots+2^K = 2^{K+1}-1$. is true

We need need to prove that.

$P(K+1) = 1+2+2^2+ \ldots +2^K+2^{K+1} = 2^{(K+1)+1}-1$. is also true.

$\therefore P(K+1) = \underbrace{1+2+2^2+\ldots+2^K}_{2^{K+1}-1}+2^{K+1} = (2^{K+1}-1)+2^{K+1}$

$= 2\times 2^{K+1} -1$

$\therefore P(K+1) = 1+2+2^2+\ldots+2^K+2^{K+1}. = 2^{(K+1)+1}-1$

$\therefore$ LHS of $P(K+1)$ = RHS of $P(K+1)$.

Hence by principle of mathematical induction, $p(n)$ is true for all values of $n \geq 1$.

Imp Q. Find the least +ve integer n for which the statement is true and then prove that $(1+n^2) < 2^n$

To find least +ve integer, for $(1+n^2) < 2^n$

For $n=1$: $(1+1^2) = 2$ and $2^1 = 2$ But, $2 \nless 2$

For $n=2$: $(1+2^2) = 5$ and $2^2 = 4$ $5 \nless 4$

For $n=3$: $(1+3^2) = 10$ and $2^3 = 8$ $10 \nless 8$

For $n=4$: $(1+4^2) = 17$ and $2^4 = 16$ $17 \nless 16$

For $n=5$: $(1+5^2) = 26$ and $2^5 = 32$ Here, $26 < 32$ ✓

∴ The inequality holds true.

∴ The least +ve integer for which $(1+n^2) < 2^n$ is true is 5.

Basis Step:

We have $p(5) : (1+5^2) < 2^5$

i.e. $p(5) : 26 < 32$ which is clearly true.

Inductive Step:

For $(K>5)$

Assume $p(K) = (1+K^2) < 2^K$ is true.

We need to show that

$p(K+1) = (1+(K+1)^2) < 2^{K+1}$ is also true.

Consider,

$$p(K+1) : (1+(K+1)^2) = (1+K^2+2K+1)$$

$$= \underbrace{(1+K^2)} + (\underbrace{2K}+1)$$

$$< \underbrace{2^K + 2^K}_{2^{K+1}}$$

[$2K < 2^K$ for all values of K

Eg: $2 \times 3 < 2^3$

$2 \times 5 < 2^5$]

$\therefore p(K+1) : (1+(K+1)^2) < 2^{K+1}$

$\therefore p(K+1) < 2^{K+1}$.

Which is what we wanted to prove.

∴ LHS of $p(K+1)$ = RHS of $p(K+1)$

Hence by principal of mathematical induction, it follows that $P(n)$ is true for all values of $n \geq 5$.

Other questions:

1Q. $1^3+2^3+3^3+\ldots+n^3 = \frac{n^2(n+1)^2}{4}$, for all natural numbers n

2Q. $1^2+3^2+5^2+\ldots+(2n-1)^2 = \frac{n(2n+1)(2n-1)}{3}$

Imp 3Q. $1+a+a^2+\ldots+a^{n-1} = \frac{a^n-1}{a-1}$

vImp 4Q. Find the least n for which the statement is true and then prove that $10n < 3^n$.
