Unit : 3

Z Transform :-

→ It is useful in solving differential equations which represent a discrete system.

→ It operates on a sequence u_n of discrete integer valued arguments.

$n = 0, \pm 1, \pm 2, \ldots$ unlike laplace and fourier transform which operates on continuous function.

Definition of Z transform :-

The z transform of a sequence u_n defined for discrete values $n = 0, 1, 2, \ldots$ and ($u_n = 0$ for $n < 0$). It denoted by $Z(u_n)$ and defined as,

$$\boxed{Z(u_n) = \sum_{n=0}^{\infty} u_n \cdot z^{-n} = U(z)}$$

whenever the series converges and u is a function of 'z' (complex) also the inverse z transform is written as :

$$\boxed{u_n = z^{-1}(U(z))}$$

Properties :-

1) $Z(a^n)$:

consider $Z(a^n) = \sum_{n=0}^{\infty} a^n z^{-n}$

$= \sum_{n=0}^{\infty} (a/z)^n$

$= 1 + \left(\frac{a}{z}\right)^1 + \left(\frac{a^2}{z}\right) + \dots$

$a + ar + ar^2 + \dots$

$S_\infty = \frac{a}{1-r}$.

$= \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}$

$$\boxed{Z(a^n) = \frac{z}{z-a}}$$

2) Recurrence formula :-

$$\boxed{Z(n^p) = -z\frac{d}{dz}(Z(n^{p-1}))}$$

we have $Z(n^p) = \sum_{n=0}^{\infty} n^p z^{-n}$ —— ①.

consider $Z(n^{p-1}) = \sum_{n=0}^{\infty} n^{p-1} z^{-n}$

Diff w.r.t z

$\frac{d}{dz} Z(n^{(p-1)}) = \sum_{n=0}^{\infty} n^{p-1} (-n) z^{-n-1}$

$= -z^{-1} \sum_{n=0}^{\infty} n^p z^{-n}$

from ①

$= -z^{-1} Z(n^p)$

$= Z(n^p) = -z \frac{d}{dz}(Z(n^{p-1}))$.

Special case on 1

1) $Z(1) = ?$

sub $a=1$ in prop 1

$$Z(1) = Z(1^n) = \frac{z}{z-1}$$

$$Z(1) = \frac{z}{z-1}$$

2) Put $p=1$ in property 2,

i.e $$Z(n^1) = -z\left(\frac{d}{dz}\right)\left(Z(n^{1-1})\right)$$

$$= -z\left(\frac{d}{dz}\right)(Z(1))$$

$$= -z\frac{d}{dz}\left(\frac{z}{z-1}\right)$$

using quotient rule

$$= -z\left(\frac{(z-1)(1) - z(1)}{(z-1)^2}\right)$$

$$= -z\frac{(z-1-z)}{(z-1)^2}$$

$$= \boxed{\frac{-z}{(z-1)^2} = Zn}$$

$u = z$

$v = z-1$

$\frac{du}{dz} = 1$

$\frac{dv}{dz} = 1$

$$\frac{d}{dz} = \left(\frac{z}{z-1}\right)\frac{u}{v}$$

$$\frac{d}{dz} = \frac{v \cdot \frac{d}{dz} - u\frac{d}{dz}}{v^2}$$

3) Put $P = 2$ in prop 2.

$$Z(n^2) = -z\frac{d}{dz}(Z(n))$$

$$= -z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right)$$

$$= -z\left(\frac{(z-1)^2 - z\,(2(z-1)(1))}{(z-1)^4}\right)$$

$$= \frac{-z\cancel{(z-1)}(z-1-2z)}{(z-1)^{\cancel{4}\,3}}$$

$$= \frac{-z^2+z+2z^2}{(z-1)^3}$$

$$Z(a^n) = \frac{z}{z-a} \qquad = \frac{z^2+z}{(z-1)^3} = \boxed{\frac{z(z+1)}{(z-1)^3} = Zn^2}$$

4) P(3)

Linearality Property :-

If a, b, c are any constants and u_n, v_n and w_n be any discrete functions then,

$$Z(a u_n + b v_n - c w_n) =$$

Consider $Z(a u_n + b v_n - c w_n) =$

$$= \sum_{n=0}^{\infty} (a u_n + b v_n - c w_n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a u_n z^{-n} + \sum_{n=0}^{\infty} b u_n z^{-n} - \sum_{n=0}^{\infty} c w_n z^{-n}$$

$$= a \sum_{n=0}^{\infty} u_n z^{-n} + b \sum_{n=0}^{\infty} u_n z^{-n} - c \sum_{n=0}^{\infty} w_n z^{-n}$$

$$\boxed{Z(a u_n + b V_n - c w_n) = a\, Z(u_n) + b\, Z(v_n) + c\, Z(w_n)}$$

Damping Rule :-

If $Z(u_n) = U(z)$ then $Z(a^{-n} u_n) = U(az)$

consider, $Z(a^{-n} u_n) = \sum_{n=0}^{\infty} a^{-n} u_n z^{-n}$

$= \sum_{n=0}^{\infty} u_n (az)^{-n} = U(az)$

$= \boxed{\Sigma u_n z^{-n} = U_z}$

Similarly, $Z(a^n u_n) = U(z/a)$

$Z(a^{-n} u_n) = U(az)$

10) Find the z transform of cos nθ and sin nθ

V.IMP
Theory Proof

$$Z(\cos(n\theta)) = Z\left[\frac{e^{in\theta} + e^{-in\theta}}{2}\right]$$

$Z(a^n) = \frac{z}{z-a}$

$\cos x = \frac{e^{ix} + e^{-ix}}{2}$

$a^{mn} = (a^m)^n$

$$= \frac{1}{2}\left[Z(e^{in\theta}) + Z(e^{-in\theta})\right]$$

$$= \frac{1}{2}\left[Z((e^{i\theta})^n) + Z((e^{-i\theta})^n)\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{i\theta}} + \frac{z}{z-e^{-i\theta}}\right]$$

$$= \frac{1}{2}\left[\frac{z(z-e^{-i\theta}) + z(z-e^{i\theta})}{(z-e^{i\theta})(z-e^{-i\theta})}\right]$$

$$= \frac{z}{2}\left[\frac{2z - (e^{+i\theta} + e^{-i\theta})}{z^2 - ze^{-i\theta} - ze^{i\theta} + e^{i\theta}e^{-i\theta}}\right]$$

$$= \frac{z}{2}\left[\frac{2z - (e^{i\theta} + e^{-i\theta})}{z^2 - z(e^{i\theta} + e^{-i\theta}) + 1}\right]$$

$$= \frac{z}{2}\left[\frac{2z - 2\cos\theta}{z^2 - z2\cos\theta + 1}\right]$$

$$= \frac{z\cancel{2}}{\cancel{2}}\left[\frac{z - \cos\theta}{z^2 - z2\cos\theta + 1}\right]$$

$$= \boxed{\frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}}$$

Now

$$Z(\sin(n\theta)) = Z\left(\frac{e^{in\theta} - e^{-in\theta}}{2i}\right)$$

$$\boxed{\sin x = \frac{e^{ix} - e^{-ix}}{2i}}$$

$$= \frac{1}{2i}\left[Z e^{in\theta} - Z e^{-in\theta}\right]$$

$$= \frac{1}{2i}\left[Z(e^{i\theta})^n - Z(e^{-i\theta})^n\right]$$

$$= \frac{1}{2i}\left[\frac{z}{z-e^{i\theta}} - \frac{z}{z-e^{-i\theta}}\right]$$

$$= \frac{1}{2i}\left[\frac{z(z-e^{i\theta}) - z(z-e^{-i\theta})}{(z-e^{i\theta})(z-e^{-i\theta})}\right]$$

$$= \frac{z}{2i}\left[\frac{z - z - e^{i\theta} + e^{-i\theta}}{z^2 - ze^{-i\theta} - ze^{i\theta} + e^{i\theta}e^{-i\theta}}\right]$$

$$= \frac{z}{2i}\left[\frac{-(e^{i\theta} - e^{-i\theta})}{z^2 - z(e^{+i\theta}) - ze^{i\theta} + 1}\right]$$

$$= \frac{z}{2i}\left[\frac{2i \sin\theta}{z^2 - z(e^{i\theta} + e^{-i\theta}) + 1}\right]$$

$$= \boxed{\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}}$$

Note:-

1 Learn this

1) $Z(na^n) = \frac{az}{(z-a)^2}$

2) $Z(n^2a^n) = \frac{az^2 + a^2z}{(z-a)^3}$

3) $Z(a^n(\cos(n\theta))) = \frac{z(z-a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$

4) $Z(a^n \sin(n\theta)) = \frac{za\sin\theta}{z^2 - 2az\cos\theta + a^2}$

5) $Z(u_n) = U(z)$ then

$$Z(u_{n+k}) = z^k\left[U(z) - u_0 - u_1 z^{-1} \dots - u_{(k-1)} z^{-(k-1)}\right]$$

Special case:-

$K=1$ $\quad Z(u_{n+1}) = z\,[U(z) - u_0]$

$K=2$ $\quad Z(u_{n+2}) = z^2\,[U(z) - u_0 - u_1 z^{-1}]$

1A) find z transform of the following :-

$$u_n = \left(\frac{1}{2}\right)^n + (1/3)^n$$

$$z\left(\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n\right) = \left[z\left(1/2\right)^n + z\left(1/3\right)^n\right]$$

$\boxed{z(a^n) = \frac{z}{z-a}}$

$$= \left[\frac{z}{z-1/2} + \frac{z}{z-1/3}\right]$$

$$= \boxed{\frac{2z}{2z-1} + \frac{3z}{3z-1}}$$

2a) $2n + 5\sin\left(\frac{n\pi}{4}\right) - 3a^4$

$$z\left(2n + 5\sin\left(\frac{n\pi}{4}\right) - 3a^4\right)$$

$$= z(2n) + z\left(5\sin\left(\frac{n\pi}{4}\right)\right) - 3\left(z(a^4 \cdot 1)\right)$$

$$= 2\left(\frac{z}{(z-1)^2}\right) + 5\left(\frac{z\sin(\pi/4)}{z^2 - 2z\cos\frac{\pi}{4} + 1}\right) - 3a^4 \cdot \frac{z}{(z-1)}$$

$$= \frac{2z}{(z-1)^2} + \frac{5z \cdot 1/\sqrt{2}}{z^2 - 2z \cdot \frac{1}{\sqrt{2}} + 1} - \frac{3a^4 z}{(z-1)}.$$

$$= \frac{2z}{(z-1)^2} + \frac{5z}{\sqrt{2}(z^2 - \sqrt{2}z + 1)} - \frac{3a^4 z}{(z-1)}$$

3) $u_n = (\cos\theta + i\sin\theta)^n$

we have Euler's formula,

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n$$

$$Z(\cos\theta + i\sin\theta)^n = Z(e^{i\theta})^n$$

$$Z(e^{i\theta})^n = \frac{z}{z - e^{i\theta}} = \frac{z}{z - (\cos\theta + i\sin\theta)}$$

4Q)

$$Z(e^{-an}\sin(n\theta)) = Z((e^a)^{-n}\sin(n\theta))$$

$$= Z(a^{-n}u_n) = U(az) = U(z)\Big|_{z = e^a z}$$

$a = e^a \quad u_n = \sin(n\theta)$

$$= Z(u_n)\Big|_{z = e^a z}$$

$$Z(\sin(n\theta))\Big|_{z = e^a z}$$

$$= \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}\Big|_{z = e^a z}$$

$$= \frac{e^a z\sin\theta}{e^{2a}z^2 - 2(e^a z)\cos\theta + 1}.$$

$$\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$$

$$= \cos\frac{n\pi}{2}\cdot\cos\frac{\pi}{4} - \sin\frac{n\pi}{2}\cdot\sin\frac{\pi}{4}$$

$$= \cos\frac{n\pi}{2}\cdot\frac{1}{\sqrt{2}} - \sin\frac{n\pi}{2}\cdot\frac{1}{\sqrt{2}}.$$

$$= \frac{1}{\sqrt{2}}\left[\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right]$$

$$Z\left(\left(\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right)\right) = Z\left[\frac{1}{\sqrt{2}}\left(\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left[Z\cos\frac{n\pi}{2} - Z\sin\frac{n\pi}{2}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{z(z-\cos\pi/2)}{z^2-2z\cos\pi/2+1} - \frac{z\sin\pi/2}{z^2-2z\cos\cancel{\frac{\pi}{2}}^{0}+1}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{z^2-z}{z^2+1}\right]$$

$$= \boxed{\frac{z^2-z}{\sqrt{2}(z^2+1)}}$$

6Q) $\cos h(n\theta)$

$$\boxed{\cosh x = \frac{e^{ix} - e^{-x}}{2}}$$

$$= Z\left[\frac{e^{n\theta} - e^{-n\theta}}{2}\right]$$

$$= \frac{1}{2}\left[Ze^{n\theta} + Z(e^{-n\theta})\right]$$

$$= \frac{1}{2}\left[Z(e^{\theta})^n + (e^{-\theta})^n\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{\theta}} + \frac{z}{z-e^{-\theta}}\right]$$

$$= \frac{1}{2}\left[\frac{z(z-e^{-\theta}) + z(z-e^{\theta})}{(z-e^{\theta})(z-e^{-\theta})}\right]$$

$$= \frac{z}{2}\left[\frac{2z - e^{-\theta} - e^{\theta}}{z^2 - ze^{-\theta} - ze^{\theta} + e^{\theta}e^{-\theta}}\right]$$

$$= \frac{z}{2}\left[\frac{2z - (e^{\theta} + e^{-\theta})}{z^2 - z(e^{\theta} + e^{-\theta}) + 1}\right]$$

$$= \frac{z}{2}\left[\frac{2z - 2\cos h\theta}{z^2 - 2z\cosh\theta + 1}\right]$$

$$= \frac{z[z - \cosh\theta]}{z^2 - 2z\cosh\theta + 1}$$

78) $Z(n^2 e^{n\theta})$

$= Z(n^2 (e^\theta)^n)$

$Z(n^2 a) = \dfrac{az^2 + az^2}{(z-a)^3}$

$= \dfrac{e^\theta z^2 + (e^\theta)^2}{(z - e^\theta)^3} = \dfrac{z e^\theta (z + e^\theta)}{(z - e^\theta)^3}$

89) $Z(nU_n) = -z \dfrac{d}{dz} U(z)$ Prove it.

$z^{-n} \times z/z = z^{(-n-1)} z$

$Z(nU_n) = \sum_{n=0}^{\infty} n U_n z^{-n} = -z \sum_{n=0}^{\infty} U_n (-n) z^{(-n-1)}$

$= -z \sum_{n=0}^{\infty} U_n \dfrac{d}{dz} (z^{-n})$

$d/dz (z^{-n}) = z^{-n-1} (-n)$

$= -z \dfrac{d}{dz} \sum_{n=0}^{\infty} U_n (z^{-n})$

$Z(nU_n) = \boxed{-z \dfrac{d}{dz} U(z)}$

By induction $\boxed{Z(n^P U_n) = -z^P \dfrac{d^P}{dz^P} U(z)}$

V.V
V.IMP **Initial value Theorem:-**

If $Z(u_n) = U(z)$ then

$$u_0 = \lim_{z \to \infty} U(z)$$

we know that $Z(u_n) = U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$

i.e $= U(z) = u_0 + u_1 z^{-1} + u_2 z^{-2} + u_3 z^{-3} + \ldots$

Taking lt $z \to \infty$.

$$\lim_{z \to \infty} U(z) = u_0 + 0 + 0 + \ldots = u_0.$$

Note:-

1) $\lim_{z \to \infty} z\,(U(z) - u_0) =$

we know that $U(z) = u_0 + \frac{u_1}{z} + \frac{u_2}{z} + \ldots\ldots$

$$U(z) - u_0 = \frac{u_1}{z} + \frac{u_2}{z^2} + \ldots\ldots$$

Multiply both sides by 'z'.

$$z\,(U(z) - u_0) = u_1 + \frac{u_2}{z} + \frac{u_3}{z^2} + \ldots\ldots$$

let $\lim_{z \to \infty}$ $\lim_{z \to \infty} z\,(U(z) - u_0) = u_1 + 0 + 0 + \ldots\ldots$

$$\boxed{\lim_{z \to \infty} z\,(U(z) - u_0) = u_1}$$

Final Value Theorem :-

$Z(u_n) = U(z)$ then $\lim_{n\to\infty} u_n = \lim_{z\to 1} (z-1)U(z)$.

Proof: consider $Z(u_{n+1} - u_n)$

$$= Z(u_{n+1}) - Z(u_n)$$

$$= z(U(z) - u_0) - U(z).$$

$$Z(u_{n+1} - u_n) = (z-1)U(z) - u_0 z \quad \text{—— (1)}.$$

By Definition:

$$Z(u_{n+1} - u_n) = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n} \quad \text{—— (2)}$$

$\therefore$ (1) = (2)

$$(z-1)U(z) - u_0 z = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}.$$

Take $\lim_{z\to 1}$

$$\lim_{z\to 1} (z-1)U(z) - u_0 = \sum_{n=0}^{\infty} (u_{n+1} - u_n)$$

$$= u_1 - u_0 + u_2 - u_1 + u_3 - u_2 + \dots$$

$$\lim_{z\to 1} (z-1)U(z) - u_0 = \lim_{n\to\infty} u_{n+1} - u_0$$

$$\lim_{z\to 1} (z-1)U(z) = \lim_{n\to\infty} u_n$$

$$\therefore \lim_{n\to\infty} u_n = \lim_{z\to 1} (z-1)U(z).$$

Convolution Theorem :-

If $Z^{-1}(U(z)) = u_n$ and

$Z^{-1}(V(z)) = v_n$ then

$$Z^{-1}[U(z) \cdot V(z)] = u_n * v_n = \sum_{m=0}^{\infty} u_m \cdot v_{n-m}$$

consider :- $U(z) \cdot V(z) = \sum_{n=0}^{\infty} u_n z^{-n} \cdot \sum_{n=0}^{\infty} v_n z^{-n}$

$$= \left(u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \cdots \frac{u_n}{z^n} + \cdots\right) \times \left(v_0 + \frac{v_1}{z} + \frac{v_2}{z^2} + \cdots \frac{v_n}{z^n} + \cdots\right)$$

$$= \sum_{n=0}^{\infty} (u_0 v_n + u_1 v_{n-1} + u_2 v_{n-2} + \cdots (u_n v_0)) z^{-n}$$

$$= Z(u_0 v_n + u_1 v_{n-1} + \cdots + u_n v_0)$$

$$= Z(u_0 v_{n-0} + u_1 v_{n-1} + \cdots + u_n v_{n-n})$$

$$= \boxed{Z\left(\sum_{m=0}^{n} u_m v_{n-m}\right)}$$

$$U(z) \cdot V(z) = Z(u_n * v_n).$$

apply Z^{-1}

$$Z^{-1}(U(z) \cdot V(z)) = u_n * v_n.$$

13) If $u(z) = \dfrac{2z^2 + 3z + 12}{(z-1)^4}$

find the value of u_2 and u_3

Let, $u(z) = \dfrac{z^2 (2 + 3/z + 12/z^2)}{z^{4\,2} (1 - 1/z)^4}$

$(z-1)^4 = (z - \frac{z}{z})^4$

$= (z(1 - \frac{1}{z}))^4$

$= z^4 (1 - \frac{1}{z})^0$

By initial value theorem, we have.

$u_0 = \lim_{z \to \infty} u(z)$

taking $\lim_{z \to \infty} u(z)$

Then $\lim_{z \to \infty} u(z) = u_0 = \dfrac{\lim_{z \to \infty}}{\lim_{z \to \infty}} \left(\dfrac{2 + 3/z + 12/z^2)}{z^2 (1 - 1/z)^4} \right.$

$= \dfrac{2 + 0 + 0}{\infty (1 - 0)} = 0.$

$u_1 = \lim_{z \to \infty} z(u(z) - u_0)) = \lim_{z \to \infty} \dfrac{z(2 + 3/z + 12/z^2}{z^2 (1 - 1/z)^4}$

$= \dfrac{2}{\infty(1)} = 0.$

$\frac{2/16}{2(1)} = \cdot$

$$u_2 = \lim_{z\to\infty} z^2\left(u(z) - u_0 - \frac{u_1}{z}\right)$$

$$= \lim_{z\to\infty} z^2\left(\left(\frac{2+3/z+12/z^2}{z^2(1-1/z)^4}\right) - 0 - 0\right)$$

$$u_2 = \frac{2}{1} = 2$$

$$\boxed{u_2 = 2}$$

$$u_3 = \lim_{z\to\infty} z^3\left(u(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2}\right)$$

$$= \lim_{z\to\infty} z^3\left[\frac{2z^2+3z+12}{(z-1)^4} - \frac{2}{z^2}\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{2z^4+3z^3+12z^2-2(z-1)^4}{(z-1)^4}\right)$$

$$= \lim_{z\to\infty} \frac{z(2z^4+3z^3+12z^2 - 2(a^4-4a^3b+6a^2b^2-4ab^3+b^4)}{z^4(1-1/z)^4}$$

$$= \lim_{z\to\infty} z\,\frac{(2z^4+3z^3+12z^2-2(z^4-4z^3-4z+1)}{z^4(1-1/z)^4}$$

$$= \lim_{z\to\infty} \frac{z(2z^4+3z^3+12z^2-2z^4+8z^3+8z-2)}{(z^4(1-1/z)^4)}$$

$$= \lim_{z\to\infty} \frac{z^4(11+12/z+8/z^3-2/z^4)}{z^4(1-1/z)^4}$$

$$= \frac{11 + 0 + 0 + 0}{1 - 0} = 11$$

$$\boxed{u_3 = 11}$$

Solving Z transforms using Partial fraction method.

This method consists of $\frac{u(z)}{z}$ into partial fractions, then multiplying the result by expression by z and then inverting same.

$$n a^n = z^{-1}\left(\frac{az}{(z-a)^2}\right)$$

$$Z(a^n) = \frac{z}{z-a}$$

i) $\frac{8z^2}{(2z-1)(4z-1)}$

Let

$$u(z) = \frac{8z^2}{(2z-1)(4z-1)}$$

Divide by z

$$\frac{u(z)}{z} = \frac{8z^2/z}{(2z-1)(4z-1)} = \frac{8z}{(2z-1)(4z-1)}.$$

$$= \frac{A}{2z-1} + \frac{B}{4z-1}$$

By Partial fractions :-

$$8z = A(4z-1) + B(2z-1)$$

sub: $z = 1/4$

$$\cancel{8}^{2} \times \frac{1}{\cancel{4}} = A\left(4 \times \frac{1}{4} - 1\right) + B\left(2 \times 1/4 - 1\right)$$

$$2 = 0 + \left(\frac{-B}{2}\right)$$

$$\boxed{B = -4}$$

sub $z = 1/2$

$$8 \times \frac{1}{2} = A\left(4 \times \frac{1}{2} - 1\right) + 0$$

$$\boxed{4 = A}$$

Now sub $A = 4$ and $B = -4$

$$\frac{u(z)}{z} = \frac{4}{2z-1} + \frac{(-4)}{4z-1}$$

$$\frac{u(z)}{z} = \frac{4}{2(z-1/2)} - \frac{4}{4(z-1/4)}$$

$$u(z) = \frac{2z}{z-1/2} - \frac{z}{(z-1/4)}$$

apply the inverse z^{-1}:

$$u_n = 2z^{-1}\left(\frac{z}{z-1/2}\right) - z^{-1}\left(\frac{z}{z-1/4}\right)$$

$$= \boxed{2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n}$$

ii) $\frac{5z}{(2-z)(3z-1)}$

$$\frac{u(z)}{(z)} = \frac{5z}{(2-z)(3z-1)} = \frac{A}{(2-z)} + \frac{B}{(3z-1)}$$

$$\frac{5\cancel{z}}{\cancel{z}} = A(3z-1) + B(2-z)$$

$$\boxed{\text{Put } z = 1/3}$$

$$\frac{5 \times 1/3}{1/3} = 0 + B(2-1/3)$$

$$5 = B(2-1/3)$$

$$\boxed{B=3}$$

Put $z = 2$

$$5 = A(6-1) + 0$$

$$5 = 5A$$

$$A = 1$$

Now

$$\frac{u(z)}{z} = \frac{1}{2-z} + \frac{3}{3z-1}$$

$$\frac{u(z)}{z} = \frac{1}{(2-z)} + \frac{3}{3(z-1/3)}$$

$$\frac{u(z)}{z} = \frac{1}{-(z-2)} + \frac{1}{z-1/3}$$

$$u(z) = \frac{-z}{z-2} + \frac{z}{z-1/3}$$

$$U(z) = \frac{-z}{z-2} + \frac{z}{z-1/3}$$

apply Z^{-1}

$$u_n = -Z^{-1}\left(\frac{z}{z-2}\right) + Z^{-1}\left(\frac{z}{z-1/3}\right)$$

$$= \boxed{-(2)^n + \left(\frac{1}{3}\right)^n}$$

(iii) $\dfrac{z}{(z+3)^2\,(z-2)}$

$$\frac{U(z)}{z} = \frac{1}{(z+3)^2(z-2)} = \frac{A}{z+3} + \frac{B}{(z+3)^2} + \frac{C}{z-2}$$

$$1 = A(z+3)(z-2) + B(z-2) + C(z+3)^2$$

Put $z = -3$

$1 = 0 + B(-5)$

$B = -1/5$

Put $z = 2$

$1 = 0 + 0 + C(2+3)^2$

$1 = 25C$

$C = \dfrac{1}{25}$

Put $z = 0$

$$1 = A(3)(-2) + \left(-\frac{1}{5}\right)(-2) + \frac{1}{25}(3)^2$$

$$1 = -6A + \frac{2}{5} + \frac{9}{25}$$

$$1 = -6A + \frac{10+9}{25}$$

$$1 = -6A + \frac{19}{25}$$

$$\boxed{A = \frac{-1}{25}}$$

$$\frac{u(z)}{z} = \frac{-1}{25(z+3)} - \frac{1}{5(z+3)^2} + \frac{1}{25(z-2)}$$

$$u(z) = \frac{-z}{25(z+3)} - \frac{z}{25(z+3)^2} + \frac{z}{25(z-2)}$$

apply z^{-1}

$$u_n = \frac{-1}{25} z^{-1}\left(\frac{z}{(z+3)}\right) - \frac{z^{-1}}{5}\left(\frac{z \times -3}{(z+3)^2 \times (-3)}\right) + \frac{z^{-1}}{25}\,(-$$

$$u_n = \frac{-1}{25}(-3)^n + \frac{1}{15} n(-3)^n + \frac{1}{25}(2^n)$$

iv) $\frac{z}{z^2+2z+1} = \frac{z}{(z+1)^2} = \frac{z}{(z-(-1))^2}$.

apply z^{-1}

$$u_n = \frac{z^{-1}}{(-1)}\left(\frac{(-1)\,z}{(z-(-1))^2}\right)$$

$$= \boxed{-n(-1)^n}$$

$$= \boxed{n(-1)^{n+1}}$$

Difference Equations

Let $y = f(x)$, then for a change h in x we have $\Delta f(x) = f(x+h) - f(x)$ is called finite forward difference of $f(x)$.

Suppose $y_n = f(x_n)$; $n = 0, 1, 2, \ldots$

Define $\boxed{\Delta y_n = y_{n+1} - y_n}$

called the first forward difference.

further,

$$\Delta^2 y_n = \Delta(\Delta y_n)$$
$$= \Delta(y_{n+1} - y_n)$$
$$= \Delta y_{n+1} - \Delta y_n$$
$$= y_{n+2} - y_{n+1} - (y_{n+1} - y_n)$$
$$= y_{n+2} - 2y_{n+1} + y_n$$

2nd (FD)

Similarly :

$$\Delta^3 y_n = y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n$$

IMP **Definition:-**

A difference equation is a relation b/w the difference of an unknown function $y = f(x)$ at several values of the independent variable x.

General form :

It is given by:

$$a_r y_{n+r} + a_{r-1} y_{n+r-1} + \dots + a_1 y_{n+1} + a_0 y_n = \phi(n).$$

where $n = 0, 1, 2 \dots$

and $a_0, a_1, \dots a_r$ are all constants (independent of n) is called linear difference equation of order 'r'.

Solution by Z transform :

Step 1 :

Take z transform on both sides of difference eqn.

Step 2 :-

Use difference conditions and solution $u(z)$

Step 3 :- apply Partial fraction method.

Step 4 :- Take z^{-1} transform on both sides to obtain u_n.

19) Solve by using z transform

i) $u_{n+2} - 4u_{n+1} + 3u_n = 3^n$

given $u_0 = 0$ $u_1 = 1$

apply z transform on both sides

$$Z(u_{n+2} - 4u_{n+1} + 3u_n) = Z(3^n)$$

$$Z\,u_{n+2} - 4Z\,u_{n+1} + 3Z\,u_n = Z\,3^n$$

$$z^2\left(u(z) - u_0 - \frac{u_1}{z}\right) - 4z(u(z) - u_0) + 3u(z) = \frac{z}{z-3}$$

$$z^2 u(z) - z^2(0) - z^2\frac{(1)}{z} - 4z\,u(z) + 4z(0) + 3u(z) = \frac{z}{z-3}$$

$$z^2 u(z) - z - 4z\,u(z) + 3u(z) = \frac{z}{z-3}$$

$$u(z)(z^2 - 4z + 3) = \frac{z}{z-3} + z.$$

$$(z^2 - 4z + 3)\,u(z) = \frac{z + z^2 - 3z}{z-3} = \frac{z(z-2)}{(z-3)}.$$

$$u(z) = \frac{z(z-2)}{(z-3)(z^2 - 4z + 3)}.$$

$$u(z) = \frac{z(z-3)}{(z-3)(z-3)(z-1)} = \frac{z(z-2)}{(z-3)^2(z-1)}$$

$$\frac{u(z)}{z} = \frac{(z-2)}{(z-3)^2(z-1)} = \frac{A}{(z-3)} + \frac{B}{(z-3)^2} + \frac{C}{(z-1)}$$

Put $\boxed{z=3}$

$$z-2 = A(z-3)(z-1) + B(z-1) + C(z-3)^2$$

$$3-2 = 0 + B(3-1) + 0$$

$$1 = 2B$$

$$\boxed{B = 1/2}$$

Put $\boxed{z=1}$

$$1-2 = 0+0+C(-1-3)^2$$

$$-1 = C(-2)^2$$

$$-1 = 4C$$

$$C = -1/4$$

Put $z=0$

$$0-2 = A(0-3)(0-1) + B(0-1) + C(0-3)^2$$

$$-2 = A(3) + \frac{1}{2}(-1) \frac{-1}{4}(-3)^2$$

$$-2 = 3A - \frac{1}{2} - \frac{9}{4}.$$

$$-2 = 3A - \left(\frac{2+9}{4}\right)$$

$$-2 = 3A - \frac{11}{4}$$

$$3A = -2 + \frac{1}{2} - 9\left(\frac{-1}{4}\right)$$

$$= \frac{-8+2+9}{4}$$

$$3A = 3/4$$

$$\boxed{A = 1/4}$$

$$\frac{1/4\, z}{(z-3)} + \frac{-1/2\, z}{(z-3)^2} + \frac{-1/4\, z}{(z-1)}$$

apply z^{-1}

$$z^{-1}(u(z)) = u_n = \frac{1}{4} z^{-1}\left(\frac{z}{z-3}\right) + \frac{1}{2\times 3} z^{-1}\left(\frac{3z}{(z-3)^2}\right)$$

$$-\frac{1}{4} z^{-1}\left(\frac{z}{z-1}\right)$$

$$u_n = \frac{1}{4}(3)^n + \frac{1}{6} n(3)^n - \frac{1}{4}(1)^n$$

ii) $u_{n+2} + 2u_{n+1} + u_n = n, \quad u_0 = u_1 = 0$

apply z of both sides,

$$z\,u_{n+2} + 2z\,u_{n+1} + z\,u_n = z(n)$$

$$z^2\left(u(z) - \cancel{u_0}^{0} - \cancel{\frac{u_1}{z}}^{0}\right) + 2z\left(u(z) - \cancel{u_0}^{0}\right) + u(z) = \frac{z}{(z-1)^2}$$

$$z^2(u(z)) + 2z\,u(z) + u(z) = \frac{z}{(z-1)^2}$$

$$u(z)(z^2 + 2z + 1) = \frac{z}{(z-1)^2}$$

$$u(z) = \frac{z}{(z-1)^2 (z^2+2z+1)} = \frac{z}{(z-1)^2 (z+1)^2}.$$

Divide by z

$$\frac{u(z)}{z} = \frac{1}{(z-1)^2(z+1)^2} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{(z+1)} + \frac{D}{(z+1)^2}$$

$$1 = A(z-1)(z+1)^2 + B(z+1)^2 + C(z+1)(z-1)^2 + D(z-$$

$z = 1$

$1 = 0 + B(2)^2 + 0 + 0$

$1 = 4B$

$B = 1/4$

$z = -1$

$1 = 0 + 0 + 0 + D(-2)^2$

$1 = 4D$

$\boxed{D = \frac{1}{4}}$

$z = 0$

$1 = A(-1)(1)^2 + B(1) + C(1)(-1)^2 + D(-1)^2$

$1 = -A + B + C + D$

$1 = -A + \frac{1}{4} + C + \frac{1}{4}$

$1 = -A + C + \frac{2}{4}$

$-A + C = \frac{2}{4} - \frac{4}{4}$

$-A + C = \frac{-1}{2}$

$\boxed{C - A = 1/2}$ — ①

$z = 2$

$1 = A(1)(3)^2 + B(3)^2 + C(3)(1)^2 + D(1)^2$

$1 = 9A + 9B + 3C + D$

$1 = 9A + 9B + 3C + \frac{1}{4}$

$1 = (9A) + (3C + \frac{10}{4})$

$1 = 9A + 3C + \frac{5}{2}$

$9A + 3C = 1 - \frac{5}{2}$

$\boxed{9A + 3C = \frac{-3}{2}} \rightarrow$ ②

$3A + C = \frac{-1}{9}$

$C = 1/4$

$A = -1/4$

$$u(z) = \frac{-1/4\, z}{(z-1)} + \frac{1/4\, z}{(z-1)^2} + \frac{1/4\, z}{(z-1)^2} + \frac{1/4\, z}{(z+1)} + \frac{1/4}{(z+1)}$$

$$z^{-1} \cdot (u(z)) = u_n = \frac{1}{4}\left[z^{-1}\frac{-z}{(z-1)} + z^{-1}\frac{z}{(z-1)^2} + z^{-1}\frac{z}{(z+1)} - (z^{-1}\frac{z}{z+1}\right.$$

$$= \frac{1}{4}\left[-(1)^n + n(1)^n + (-1)^n + n(-1)^n\right]$$

$$= \frac{1}{4}\left[-1^n + n1^n + (-1)^n - n(-1)^n\right]$$

$$= \frac{1}{4}\left[-1 + n(-1)^n - n(-1)^n\right]$$

iii) $u_{n+2} - 2u_{n+1} + u_n = 2^n$

Given $u_0 = 2,\ u_1 = 1$

iv) $u_{n+2} - 5u_{n+1} + 6u_n = 1$

Given $u_0 = 0,\ u_1 = 1$

iii) apply Z on B.S:

$$Z(u_{n+2}) - 2Z\, u_{n+1} + Z\, u_n = Z(2^n)$$

$$z^2\left(u(z) - u_0 - \frac{u_1}{z}\right] - 2z(u(z) - u_0) + u(z) = \frac{z}{z-2}$$

$$\Rightarrow z^2 u(z) - 2z^2 - z - 2z\, u(z) + u_z + u_z = \frac{z}{z-2}$$

$$u_z(z^2 - 2z + 1) = \frac{z}{z-2} + 2z^2 - z$$

$$u(z) = \frac{z + 2z^2(z-2) - z(z-2)}{(z-2)(z^2 - 2z + 1)}$$

$$u(z) = \frac{z + 2z^3 - 4 - z^2 + 2z}{(z-2)(z-1)^2}$$

$$u(z) = \frac{2z^3 - z^2 + 3z - 4}{(z-2)(z-1)^2}$$

$$2z^3 - z^2 + 3z - 4 = \frac{A}{z-2} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$z = 2$

$$16 - 4 + 6 - 4 = A(z-1)^2 + B(z-1)(z-2) + C(z-2)$$

$$16 - 2 = A(1)^2$$

$$\boxed{14 = A}$$

$z = 1$

$$2 - 1 + 3 - 4 = C(1-2)$$

$$1 - 1 = -C$$

$$0 = -C$$

$$\boxed{C = 0}$$

$$\boxed{z = 0}$$

$$-4 = A(-1)^2 + B(-1)(-2) + C(-2)$$

$$-4 = A + 2B - 2C$$

$$-4 = 14 + 2B - 0$$

$$-18 = 2B$$

$$\boxed{B = -9}$$

$$\frac{u(z)}{z} = \frac{14}{z-2} + \frac{-9}{(z-1)} + 0$$

$$u(z) = \frac{14z}{z-2} - \frac{9z}{(z-1)}$$

$$z^{-1}u(z) = 14z^{-1}(z/z-2) - 9z^{-1}(z/z-2)$$

$$u_n = 14(2)^n - 9(2)^n$$

iv) $Z u_{n+2} - 5Z u_{n+1} + 6Z u_n = Z(1)$

$$z^2\left(u(z) - \cancel{u_0}^{\,0} - \frac{u_1}{z}\right) = 5z\left(u(z) - \cancel{u_0}^{\,0}\right) + 6u(z)$$

$$z^2 u(z) - z - 5z\,u(z) + 6u(z) = \frac{z}{z-1}$$

$$u(z)(z^2 - 5z + 6) = \frac{z}{z-1} + z$$

$$\frac{u(z)}{z} = \frac{z + z^2 - z}{(z-1)(z^2-5z+6)} = \frac{z}{(z-1)(z-2)(z-3)}$$

$$z = \frac{A}{(z-1)} + \frac{B}{(z-2)} + \frac{C}{(z-3)}$$

$$z = A(z-2)(z-3) + B(z-1)(z-3) + C(z-1)(z-2)$$

$z = 1$

$1 = A(-1)(-2) + 0 + 0$

$1 = 2A$

$\boxed{A = 1/2}$

$z = 2$

$2 = B(1)(-1)$

$2 = -B$

$\boxed{B = -2}$

$z = 3$

$3 = C(2)(1)$

$\boxed{C = 3/2}$

$$\frac{u(z)}{z} = \frac{1/2}{(z-1)} + \frac{(-2)}{(z-2)} + \frac{3/2}{(z-3)}$$

$$u(z) = \frac{\frac{1}{2}z}{z-1} - \frac{2z}{z-2} + \frac{3/2\,z}{z-3}$$

$$z^{-1}(U(z)) = \frac{1}{2} z^{-1}\left(\frac{z}{z-1}\right) - 2z^{-1}\left(\frac{z}{z-2}\right) + \frac{3}{2} z^{-1}\left(\frac{z}{z-3}\right)$$

$$\boxed{u_n = \frac{1}{2}(1)^n - 2(2)^n + \frac{3}{2}(3)^n}$$