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UNIT – I

VECTOR CALCULUS

Vector algebra (review):

Scalar: is a quantity that is fully described by a magnitude only. It is described by just a single number.

Examples: Speed, distance, volume, mass, temperature, power, energy, and time.

Vector: is a quantity that has both a magnitude and a direction. Vector quantities are important in the study of motion.

Examples: Force, velocity, acceleration, displacement, and momentum.

What is the difference between a scalar and vector? A vector quantity has a direction and a magnitude, while a scalar has only a magnitude.

You can tell if a quantity is a vector by whether or not it has a direction associated with it.

Example questions: Is it a scalar or a vector?

1) The football player was running 10 miles an hour towards the end zone.

This is a vector because it represents a magnitude (10 mph) and a direction (towards the end zone). This vector represents the velocity of the football player.

2) The volume of that box at the west side of the building is 14 cubic feet.

This is a scalar. It might be a bit tricky as it gives the location of the box at the west side of the building, but this has nothing to do with the direction of the volume which has a magnitude of 14 cubic feet.

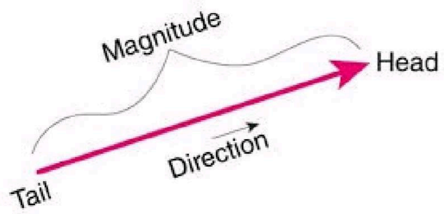
3) The temperature of the room was 15 degrees Celsius.

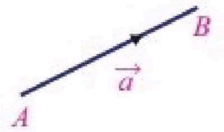
This is a scalar, there is no direction.

4) The car accelerated north at a rate of 4 meters per second squared.

This is a vector as it has both direction and magnitude. We also know that acceleration is a vector quantity.

How to Draw a Vector: A vector is drawn as an arrow with a head and a tail (also called initial and terminal points of a vec. The magnitude of the vector is often described by the length of the arrow. The arrow points in the direction of the vector.





The length or magnitude of the vector $\vec{a}$ is the length of the line segment AB and is denoted by $|\vec{a}| = a$.

How to Write a Vector: Vectors are generally written as boldface letters. They can also be written with an arrow over the top of the letter.

<u>**Types of vectors:**</u>

1. Null vectors or zero vectors: A vector whose magnitude is zero is called a null vector and is denoted by $\vec{0}$.

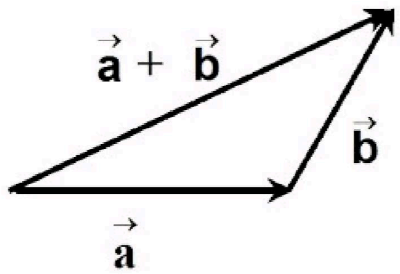
2. Unit vector: A vector whose magnitude is unit is called a unit vector.
Unit vector corresponding to the vector $\vec{a}$ is $\hat{a}$. If $\vec{a}$ is any vector and $\hat{a}$ is a unit vector in the direction of $\vec{a}$ then $\vec{a} = |\vec{a}|\,\hat{a} \Rightarrow \hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$.

3. Position vector: the position vector of a point *P* with respect to an arbitrary chosen origin *O* is the vector $\overrightarrow{OP}$.

4. Negative of a vector: a vector having the same magnitude as that of a given vector $\vec{a}$ but having opposite direction is called the negative of the vector $\vec{a}$ and is denoted by $-\vec{a}$
Clearly if $\vec{a} = \overrightarrow{PQ}$, then $-\vec{a} = \overrightarrow{QP}$.

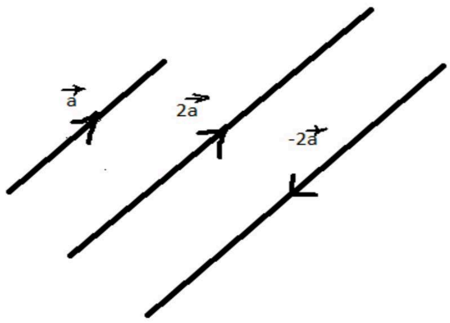
Vector algebra:

Addition of vectors: Let $\vec{a}$ and $\vec{b}$ be two given vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{AB}$. Then the vector $\overrightarrow{OB}$ represents the sum (or resultant) of $\vec{a}$ and $\vec{b}$. i.e., $\vec{a} + \vec{b}$.



Multiplication of a vector by a scalar:

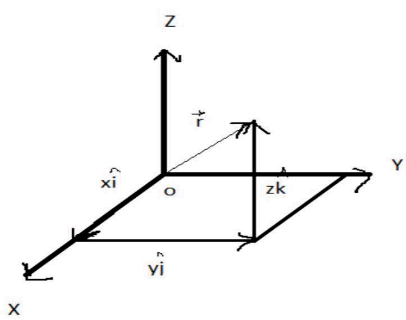
Let $\vec{a}$ be a vector and *m* be a scalar. Then their product $m\,\vec{a}$ is a vector whose magnitude is $|m|$ times that of $\vec{a}$ and direction is the same or opposite to $\vec{a}$ according to *m* is positive or negative.



Rectangular unit vectors:

The rectangular unit vectors are the unit vectors having the direction of the positive X, Y and Z axes of a three dimensional rectangular coordinate system and are denoted by $\hat{i}, \hat{j}$ & $\hat{k}$ respectively.

If $\vec{r}$ is a position vector from *O* to the point (x, y, z), then $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ and $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$.



The dot or scalar product of two vectors:

$\vec{a}\bullet\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$, $0 \le \theta \le \pi$. Where θ is the angle between $\vec{a}$ and $\vec{b}$. Since $\vec{a}$, $\vec{b}$ and $\cos\theta$ are scalars $\vec{a}\bullet\vec{b}$ is a scalar quantity.

Properties:

1. $\vec{a}\bullet\vec{b} = \vec{b}\bullet\vec{a}$

2. If $\vec{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$ & $\vec{b} = b_1\,\hat{i} + b_2\,\hat{j} + b_3\,\hat{k}$, then

$$\vec{a}\bullet\vec{b} = a_1\,b_1 + a_2\,b_2 + a_3\,b_3$$

$$\vec{a}\bullet\vec{a} = a_1^2 + a_2^2 + a_3^2 = a^2$$

$$\vec{b}\bullet\vec{b} = b_1^2 + b_2^2 + b_3^2 = b^2$$

3. If $\vec{a}\bullet\vec{b} = 0$ and $\vec{a}$, $\vec{b}$ are not null vectors, then $\vec{a}$ and $\vec{b}$ are perpendicular.

4. $\hat{i}\bullet\hat{i} = \hat{j}\bullet\hat{j} = \hat{k}\bullet\hat{k} = 1$

$$\hat{i}\bullet\hat{j} = \hat{j}\bullet\hat{k} = \hat{k}\bullet\hat{i} = 0$$

5. $\cos\theta = \dfrac{\vec{a}\bullet\vec{b}}{|\vec{a}|\,|\vec{b}|}$ gives the angle θ between two vectors $\vec{a}$ and $\vec{b}$.

The cross or vector product of two vectors:

$\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\bullet\hat{n}$, $0 \le \theta \le \pi$. Where $\hat{n}$ is a unit vector normal to the plane of $\vec{a}$ and $\vec{b}$ such that the vectors $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed system.

Properties:

1. $\vec{a}\times\vec{b} \ne \vec{b}\times\vec{a}$ but $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$

2. If $\vec{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$ & $\vec{b} = b_1\,\hat{i} + b_2\,\hat{j} + b_3\,\hat{k}$, then

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

3. If $\vec{a}\times\vec{b} = \vec{0}$ and $\vec{a}$, $\vec{b}$ are not null vectors, then $\vec{a}$ and $\vec{b}$ are parallel.

4. $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$

$$\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \hat{j} \qquad \Rightarrow\ \hat{j}\times\hat{i} = -\hat{k},\ \hat{k}\times\hat{j} = -\hat{i},\ \hat{i}\times\hat{k} = -\hat{j}.$$

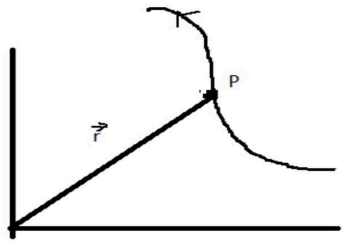
Differentiation of vectors:

If a vector $\vec{r}$ varies continuously as a scalar variable t changes, then $\vec{r}$ is said to be a function of t and is written as $\vec{r} = \vec{f}(t)$. Just as in scalar calculus, we define derivative of a vector function $\vec{r} = \vec{f}(t)$ as $\frac{d\vec{r}}{dt}$ which is a vector quantity and its derivative is denoted by $\frac{d^2\vec{r}}{dt^2}$.

Note: If $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$, where x, y, z are scalar functions of t, then

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\,\hat{i} + \frac{dy}{dt}\,\hat{j} + \frac{dz}{dt}\,\hat{k}.$$

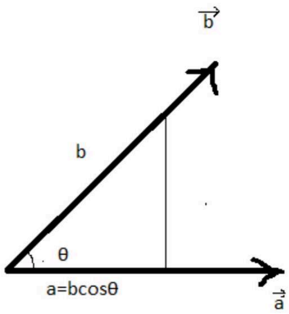
Velocity and acceleration:



If $'t'$ is a scalar variable which denotes time and $\vec{r}$ is the position vector of a moving particle P, then the velocity of the particle at P is $\vec{v} = \frac{d\vec{r}}{dt}$ and its direction is along the tangent at P. The acceleration of the particle at P is $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$.

Component of a vector on the other (or projection):

Let $\vec{a}$ and $\vec{b}$ be any two vectors then $\vec{a} \bullet \vec{b} = a\, b\, \cos\theta$, where $|\vec{a}| = a$, $|\vec{b}| = b$ and $b\, \cos\theta$ is the projection of the vector $\vec{b}$ in the direction of $\vec{a}$.



i.e., the component of $\vec{b}$ in the direction of $\vec{a} = b\, \cos\theta$

$$= \frac{\vec{a} \bullet \vec{b}}{a}$$

$$= \frac{\vec{a}}{a} \bullet \vec{b}$$

$$= \hat{a} \bullet \vec{b}$$

Problems:

1. Determine the magnitude of velocity and acceleration at $t = 0$ of a particle moving along a curve whose parametric equations are $x = e^{-t}, y = 2\cos 3t, z = 2\sin 3t,$ where t is the time.

Ans: Let $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ be the position vector of a moving particle along a curve.

Given $x = e^{-t}, y = 2\cos 3t, z = 2\sin 3t,$ where t is the time.

$$\Rightarrow \vec{r} = e^{-t}\,\hat{i} + 2\cos 3t\,\hat{j} + 2\sin 3t\,\hat{k}$$

$$\Rightarrow \vec{v} = \frac{d\vec{r}}{dt} = -e^{-t}\,\hat{i} - 6\sin 3t\,\hat{j} + 6\cos 3t\,\hat{k}$$

$$\Rightarrow \vec{a} = \frac{d\vec{v}}{dt} = e^{-t}\,\hat{i} - 18\cos 3t\,\hat{j} - 18\sin 3t\,\hat{k}$$

At $t = 0,\quad \vec{v} = -\hat{i} + 0\,\hat{j} + 6\,\hat{k}$

$$\vec{a} = \hat{i} - 18\,\hat{j} + 0\,\hat{k}$$

$$\Rightarrow v = |\vec{v}| = \sqrt{(-1)^2 + 0^2 + 6^2} = \sqrt{37}$$

$$\Rightarrow a = |\vec{a}| = \sqrt{1^2 + (-18)^2 + 0^2} = \sqrt{325}$$

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2. A particle moves along the curve $x = 2t^2, y = t^2 - 4t, z = 3t - 5$ where t is the time. Find the components of its velocity and acceleration at time $t = 1$ in the direction $\hat{i} - 3\,\hat{j} + 2\,\hat{k}$.

Ans: Let the position vector of the particle be $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$

$$\Rightarrow \vec{r} = (2t^2)\,\hat{i} + (t^2 - 4t)\,\hat{j} + (3t - 5)\,\hat{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = (4t)\,\hat{i} + (2t - 4)\,\hat{j} + 3\,\hat{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = 4\,\hat{i} + 2\,\hat{j} + 0\,\hat{k}$$

At $t = 1,\quad \vec{v} = 4\,\hat{i} - 2\,\hat{j} + 3\,\hat{k}$

$$\vec{a} = 4\,\hat{i} + 2\,\hat{j} + 0\,\hat{k}$$

$\therefore$ the components of velocity in the direction $\hat{i} - 3\hat{j} + 2\hat{k}$ at time $t = 1$

$$= \frac{(\hat{i} - 3\hat{j} + 2\hat{k}) \bullet \vec{v}}{\sqrt{14}}$$

$$= \frac{(\hat{i} - 3\hat{j} + 2\hat{k}) \bullet (4\hat{i} - 2\hat{j} + 3\hat{k})}{\sqrt{14}}$$

$$= \frac{16}{\sqrt{14}}$$

The components of acceleration in the direction $\hat{i} - 3\hat{j} + 2\hat{k}$ at time $t = 1$

$$= \frac{(\hat{i} - 3\hat{j} + 2\hat{k}) \bullet \vec{a}}{\sqrt{14}}$$

$$= \frac{(\hat{i} - 3\hat{j} + 2\hat{k}) \bullet (4\hat{i} + 2\hat{j} + 0\hat{k})}{\sqrt{14}}$$

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3. A particle moves along the curve $x = t^2, y = -t^3, z = t^4$ where t is the time variable. Find it velocity and acceleration at $t = 1$.

Ans: At $t = 1$, $\vec{v} = 2\hat{i} - 3\hat{j} + 4\hat{k}$

$\vec{a} = 2\hat{i} - 6\hat{j} + 12\hat{k}$

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4. A particle moves along the curve $x = 1 - t^3, y = 1 + t^2, z = 2t - 5$ where t is the time variable. Find the components of velocity and acceleration at time $t = 2$ in the direction given by $2\hat{i} + \hat{j} + 2\hat{k}$.

Ans: $\vec{v} = -\frac{16}{3}, \vec{a} = -\frac{22}{3}$ $at\ t = 2.$

Unit tangent vector: $\hat{T}$

Let $\vec{r}(t)$ be a differentiable vector valued function and $\vec{v}(t) = \frac{d\vec{r}}{dt}$ be the velocity vector. Then, the unit tangent vector is the unit vector in the direction of the velocity vector and is given by

$$\hat{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}.$$

Note: $\hat{T}$ is the unit tangent vector to the curve pointing in the direction of motion.

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1. Given the curve $x = t^2 + 2,\ y = 4t - 5,\ z = 2t^2 - 6t$ find the unit tangent vector at the point $t = 2$.

Ans: Let the position vector of the particle be $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$

$$\vec{r}(t) = (t^2 + 2)\,\hat{i} + (4t - 5)\,\hat{j} + (2t^2 - 6t)\,\hat{k}$$

$$\vec{v}(t) = 2t\,\hat{i} + 4\,\hat{j} + (4t - 6)\,\hat{k}$$

At $t = 2,\ \vec{v}(t) = 4\,\hat{i} + 4\,\hat{j} + 2\,\hat{k}$

$$\therefore\ \hat{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{4\,\hat{i} + 4\,\hat{j} + 2\,\hat{k}}{\sqrt{16 + 16 + 4}} = \frac{4\,\hat{i} + 4\,\hat{j} + 2\,\hat{k}}{6} = \frac{2\,\hat{i} + 2\,\hat{j} + \hat{k}}{3}.$$

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2. Find unit tangent vector at any point on the curve $x = 3\,cost,\ y = 3\,sint,\ z = 4t$ where t is a parameter. Also find unit tangent vector at $t = \frac{\pi}{2}$.

Ans: $\hat{T}(t) = \dfrac{-3\,sint\,\hat{i} + 3\,cost\,\hat{j} + 4\,\hat{k}}{5}$

$$\hat{T}\left(\frac{\pi}{2}\right) = \frac{-3\,\hat{i} + 4\,\hat{k}}{5}.$$

The vector operator del $'\nabla'$ **:**

$$\nabla = \hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\,\hat{i} + \frac{\partial}{\partial y}\,\hat{j} + \frac{\partial}{\partial z}\,\hat{k}$$

Gradient of a scalar field $'\nabla\phi'$ **:**

Let ϕ be a scalar point function, then the gradient of ϕ, denoted as $\nabla\phi$ or $grad\,\phi$, is defined as

$$\nabla\phi = \left(\frac{\partial}{\partial x}\,\hat{i} + \frac{\partial}{\partial y}\,\hat{j} + \frac{\partial}{\partial z}\,\hat{k}\right)\phi$$

$$= \left(\frac{\partial\phi}{\partial x}\,\hat{i} + \frac{\partial\phi}{\partial y}\,\hat{j} + \frac{\partial\phi}{\partial z}\,\hat{k}\right)$$

Here $\nabla\phi$ defines a vector field.

Level surface: If a surface $\phi(x, y, z) = c$ is drawn through any point *P* such that at each point on the surface, the function has the same value as at *P* then such a surface is called a level surface through P.

Note:

1. If $\phi(x, y, z)$ is a scalar function then $\nabla \phi$ is a vector normal to the surface $\phi(x, y, z) = c$.

2. A unit vector normal to the surface $\phi(x, y, z) = c$ is given by $\hat{n} = \dfrac{\nabla \phi}{|\nabla \phi|}$.

Directional derivatives of a scalar point function:

The rate of change of a scalar point function ϕ in the direction $\hat{n}$ is called the directinal derivative in the direction $\hat{n}$. It is denoted by $\dfrac{d\phi}{dr}$ and is defined as $\dfrac{d\phi}{dr} = \nabla \phi \bullet \hat{n}$. Where $\hat{n}$ is a unit vector in any given direction.

Note:

1. The maximum value of the directional derivative of ϕ is $|\nabla \phi|$ and is in the direction $\nabla \phi$.

2. Since the angle between 2 surfaces at a point is the angle between their normal at that point, we have if $\phi_1(x, y, z) = c_1$ and $\phi_2(x, y, z) = c_2$ are two surfaces then the angle between their normals is given by $\cos\theta = \dfrac{\nabla \phi_1 \bullet \nabla \phi_2}{|\nabla \phi_1|\ |\nabla \phi_2|}$.

Problems:

1. Find $\nabla \phi$ and $|\nabla \phi|$ at $(1, 2, -1)$ if $\phi = x^3 + y^3 + z^3 + 3xyz$.

Ans: Given $\phi = x^3 + y^3 + z^3 + 3xyz$

$$\nabla \phi = \left(\frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \right)$$

$$= (3x^2 + 3yz)\hat{i} + (3y^2 + 3xz)\hat{j} + (3z^2 + 3xy)\hat{k}$$

At $(1, 2, -1)$

$$|\nabla \phi| = \sqrt{9 + 81 + 81} = \sqrt{171}.$$

2.Determine the directional derivative of $f = xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$.

Ans: Directional derivative $= \dfrac{df}{dr} = \nabla f \bullet \hat{n}$

Where $\hat{n}$ is the unit vector along $\hat{i} + 2\hat{j} + 2\hat{k}$.

$i.e., \hat{n} = \dfrac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1+4+4}} = \dfrac{\hat{i} + 2\hat{j} + 2\hat{k}}{3}$

$$\nabla f = \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) = y^2\hat{i} + (2xy + z^3)\hat{j} + (3yz^2)\hat{k}$$

At $(2,-1,1)$, $\nabla f = \hat{i} - 3\hat{j} - 3\hat{k}$

$\therefore$ the directional derivative of f at the point $(2,-1,1)$ in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$ is

$$\frac{df}{dr} = \nabla f \bullet \hat{n} = (\hat{i} - 3\hat{j} - 3\hat{k}) \bullet \frac{(\hat{i} + 2\hat{j} + 2\hat{k})}{3}$$

$$= \frac{1-6-6}{3} = -\frac{11}{3}.$$

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3. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direction of the vector $2\hat{i} - \hat{j} - 2\hat{k}$.

Ans: $\dfrac{37}{3}$

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4. In what direction from $(3,-1,2)$ is the directional derivative of $\phi = \log(x^2 + y^2 + z^2)$ maximum? Find also the magnitude of this maximum?

Ans: We have the directional derivative is maximum along the direction $\nabla\phi$.

Given $\phi = \log(x^2 + y^2 + z^2)$

$$\nabla\phi = \frac{1}{x^2 + y^2 + z^2}$$

At $(3,-1,2)$, $\nabla\phi = \dfrac{1}{14}(6\hat{i} - 2\hat{j} + 4\hat{k})$

Hence $|\nabla\phi| = \dfrac{1}{14}\sqrt{36+4+16} = \dfrac{\sqrt{56}}{14}$.

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5. What is the greatest rate of increase of $u = x^2 + yz^2$ at the point $(1,-1,3)$?

Ans: $|\nabla u| = \sqrt{121}$.

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6. Find the angle between surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at $(1,2,2)$.

Ans: Let $\phi_1 = x^2 + y^2 + z^2 - 9$

$\phi_2 = x^2 + y^2 - z - 3$

$\Rightarrow \nabla\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\nabla\phi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$

At $(1,2,2)$, $\nabla\phi_1 = 2\hat{i}+4\hat{j}+4\hat{k}$

$\nabla\phi_2 = 2\hat{i}+4\hat{j}-\hat{k}$

Now the angle between the given two surfaces is given by

$$\cos\theta = \frac{\nabla\phi_1 \bullet \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|} = \frac{(2\hat{i}+4\hat{j}+4\hat{k})(2\hat{i}+4\hat{j}-\hat{k})}{\sqrt{4+16+16}\,\sqrt{4+16+1}}$$

$$= \frac{4+16-4}{\sqrt{36}\,\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

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7. Calculate the angle between the normals to the surface $xy = z^2$ at $(4,1,2)$ and $(3,3,-3)$.

Ans: At $(4,1,2)$, $\nabla\phi_1 = \hat{i}+4\hat{j}-4\hat{k}$ and at $(3,3,-3)$, $\nabla\phi_2 = 3\hat{i}+3\hat{j}+6\hat{k}$.

$$\therefore \cos\theta = \frac{-9}{\sqrt{33}\,\sqrt{54}} \Rightarrow \theta = \cos^{-1}\left(\frac{-9}{\sqrt{33}\,\sqrt{54}}\right).$$

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8. what is the angle between the tangents to the curve $x = t,\ y = t^2,\ z = t^4$ at the points where $t = 1$ and

Ans: Let $\vec{r}(t) = t\,\hat{i} + t^2\,\hat{j} + t^4\,\hat{k} \Rightarrow \vec{v}(t) = \hat{i} + 2t\,\hat{j} + 4t^3\,\hat{k}$

$$\hat{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{\hat{i}+2t\,\hat{j}+4t^3\,\hat{k}}{\sqrt{1+4t^2+16t^6}}$$

At $t = 1$, $\hat{T}_1(t) = \dfrac{\hat{i}+2\hat{j}+4\hat{k}}{\sqrt{21}}$ and at $t = 2$, $\hat{T}_2(t) = \dfrac{\hat{i}+4\hat{j}+32\hat{k}}{\sqrt{1041}}$

$$\cos\theta = \frac{\hat{T}_1 \bullet \hat{T}_2}{|\hat{T}_1|\,|\hat{T}_2|} = \frac{137}{\sqrt{21}\sqrt{1041}} \Rightarrow \theta = \cos^{-1}\left(\frac{137}{\sqrt{21}\sqrt{1041}}\right).$$

Gradient and Hessian

A **vector** is a matrix having <u>either one row or one column</u>.
The number of elements in a row vector or a column vector is its dimension, and the elements are called components.

Functions
Notation to indicate that a function f maps elements of a set A to elements of a set B is:
$f: A \longrightarrow B$

A is the function's domain; B contains its range.

Gradient and Hessian of a function in $\mathbb{R}^n$

Consider a differentiable function $f(X): \mathbb{R}^n \longrightarrow \mathbb{R}$ where $X \in \mathbb{R}^n$.

Eg: Let $X = (x, y\ z) \in \mathbb{R}^3$. Define $f(X) = f(x, y, z) = x^2y + xz^3 - 2yz$

Here $f(X): \mathbb{R}^3 \to \mathbb{R}$

The gradient of f w.r.t its vector arguments, denoted by $\nabla f(X)$ and is given by:

$$\nabla f(X) = \begin{bmatrix} \frac{\partial}{\partial x_1} f(X) \\ \frac{\partial}{\partial x_2} f(X) \\ \vdots \\ \frac{\partial}{\partial x_n} f(X) \end{bmatrix} \in \mathbb{R}^n \qquad \{\text{i.e}\}\quad X \in \mathbb{R}^n \Rightarrow X = (x_1, x_2, \dots \dots x_n)$$

Eg: Let $f(x, y, z) = x^2y + xz^3 - 2yz$, then $\nabla f(X) = \begin{bmatrix} \frac{\partial}{\partial x} f(X) \\ \frac{\partial}{\partial y} f(X) \\ \frac{\partial}{\partial z} f(X) \end{bmatrix} = \begin{bmatrix} 2xy + z^3 \\ x^2 - 2z \\ 3xz^2 - 2y \end{bmatrix}$

Note: If $\nabla f(P) = 0$ at any point P, then P is a maximum or minimum or a saddle point of f.

The Hessian of $f(X)$ is a $n \times n$ matrix, denoted by H_f is given by :

$$H_f = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{1 \le i,j \le n} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots\cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots\cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots\cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

If P is any point in $\mathbb{R}^n$, then Hessian of f at that point P is: $H_{f(P)} = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(P)\right)_{1 \le i,j \le n}$

Note that $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$. Hence H_f is a symmetric matrix.

Eg: Let $f(x,y,z) = 2xy^2 - 3x + 4yz^3$, then $H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix}$

Now, $\frac{\partial f}{\partial x} = 2y^2 - 3$, $\frac{\partial f}{\partial y} = 4xy - 4z^3$, $\frac{\partial f}{\partial z} = 12yz^2 - 3$

i.e $H_f = \begin{bmatrix} 0 & 4y & 0 \\ 4y & 4x & 12z^2 \\ 0 & 12z^2 & 24yz \end{bmatrix}$

At the point $P = (1,-1,1)$, we get $H_{f(P)} = \begin{bmatrix} 0 & -4 & 0 \\ -4 & 4 & 12 \\ 0 & 12 & -24 \end{bmatrix}$

1. Let $f(x,y,z) = x^2y^2 + z^2 + 2x - 4y + z$. Find the gradient and Hessian of f. Evaluate these at the point $(1,-1,1)$.

Solution: Given $f(x,y,z) = x^2y^2 + z^2 + 2x - 4y + z$

Then gradient, $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 2xy^2 + 2 \\ 2x^2y - 4 \\ 2z + 1 \end{bmatrix}$

And Hessian $H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix} = \begin{bmatrix} 2y^2 & 4xy & 0 \\ 4xy & 2x^2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

At the point (1,0,1), $\nabla f = \begin{bmatrix} 6 \\ -6 \\ 3 \end{bmatrix}$ and $H_f = \begin{bmatrix} 2 & -4 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

---

2. Evaluate the gradient and Hessian of the function $f(x,y) = x^2 + y^2 + x^3y$ at the point$(1,-1)$.

Solution: Given $f(x,y,z) = x^2 + y^2 + x^3y$

Then gradient, $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x + 3x^2y \\ 2y + x^3 \end{bmatrix}$

And Hessian $H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 + 6xy & 3x^2 \\ 3x^2 & 2 \end{bmatrix}$

3. $f(x,y,z) = x^3 + xy + 4z$ at $(1,2,-1)$

4. $f(x,y) = \sin(x+y)$ at $\left(0,\frac{\pi}{2}\right)$.

5. $f(x,y,z) = x^2e^y$ at $(1,0)$.

6. Let $f(x,y,z) = x^2 - xy + yz^3 - 6z$. Find all the points (x,y,z) such that $\nabla f(x,y,z) = (0,0,0)$.

Solution: $\nabla f(x,y,z) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x - y \\ -x + z^3 \\ 3yz^2 - 6 \end{bmatrix}$

$\nabla f = (0,0,0) \Rightarrow \begin{bmatrix} 2x - y \\ -x + z^3 \\ 3yz^2 - 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 2x = y \qquad z^3 = x \qquad y = \frac{2}{z^2}$

From these 3 equations, we get

$y = 2x = 2z^3 \qquad \therefore \frac{2}{z^2} = 2z^3 \qquad \Longrightarrow z = 1$

$\therefore y = \frac{2}{z^2} = 2$ and $x = \frac{y}{2} = 1$

$\therefore$ the point where $\nabla f = 0$ is $(1,2,1)$.

<u>Directional Derivates:</u>

Let $F: \mathbb{R}^n \to \mathbb{R}^m$ then $F(X)$ has m components $f_1(X), f_2(X), \dots \dots . . f_m(X)$. We write this as $F(X) = (f_1, f_2, \dots \dots f_m)$ where each f_i is a function of n variables.

i.e $F(X) = f_1e_1 + f_2e_2 + \cdots \dots . . + f_me_m$ where $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $e_n = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots \dots . e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

Then $F'(X) = \begin{bmatrix} \nabla f_1(X) \\ \nabla f_2(X) \\ \vdots \\ \nabla f_m(X) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_2} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$ is known as the derivative of F

1. Let $f(x,y,z,w) = (x^2y, xyz, x^2 + y^2 + zw^2)$. Find $F'(X)$ at the point $X = (1,2,-1,2)$.

<u>Solution:</u> Given $f(x,y,z,w) = (x^2y, xyz, x^2 + y^2 + zw^2)$

Let $f_1 = x^2y\,,\ f_2 = xyz$ and , $f_3 = x^2 + y^2 + zw^2$

$\therefore F(X) = (f_1, f_2, f_3)$ where $X = (x, y, z, w)$

Now $F'(X) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} & \frac{\partial f_3}{\partial w} \end{bmatrix} = \begin{bmatrix} 2xy & x^2 & 0 & 0 \\ yz & xz & xy & 0 \\ 2x & 2y & w^2 & 2zw \end{bmatrix}$

At the point $X = (1, 2, -1\,, 2)$, we have

$$F'(X) = F'(1, 2, -1\,, 2) = \begin{bmatrix} 4 & 1 & 0 & 0 \\ -2 & -1 & 2 & 0 \\ 2 & 4 & 4 & 4 \end{bmatrix}$$

2. If $f(x, y\,, z) = (x^2 - y^2\,, 2xy, xz)$, find $F'(1,2,-1)$

<u>Solution:</u> Given $f(x, y\,, z) = (x^2 - y^2\,, 2xy,\ xz)$

Let $f_1 = x^2 - y^2\,,\ f_2 = 2xy\ \ \&\ f_3 = xz$

$\therefore F(X) = (f_1, f_2, f_3)$ where $X = (x, y, z)$

$$F'(X) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x & -2y & 0 \\ 2y & 2x & 0 \\ z & 0 & x \end{bmatrix}$$

At the point $X = (1, 2, -1\,)$, get

$F'(X) = F'(1, 2, -1) = \begin{bmatrix} 2 & -4 & 0 \\ 4 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Del applied to vector point function:

1. Divergence of a vector point function:

The divergence of a differentiable vector point function $\vec{v}$, denoted by div $\vec{v}$ or $\nabla \cdot \vec{v}$ is defined as

$$\text{div } \vec{v} = \nabla \cdot \vec{v} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot \vec{v}$$

$$= \hat{i} \cdot \frac{\partial \vec{v}}{\partial x} + \hat{j} \cdot \frac{\partial \vec{v}}{\partial y} + \hat{k} \cdot \frac{\partial \vec{v}}{\partial z}$$

The divergence of a vector point function is a scalar point function.

Note: If $\vec{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$, then

$$\text{div } \vec{v} = \nabla \cdot \vec{v} = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \left(v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k} \right)$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

2. Curl of a vector point function:

The curl (or rotation) of a differentiable vector point function $\vec{v}$ is denoted by curl $\vec{v}$ or $\nabla \times \vec{v}$ and is defined by

$$\text{curl } \vec{v} = \nabla \times \vec{v} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \times \vec{v}$$

$$= \hat{i} \times \frac{\partial \vec{v}}{\partial x} + \hat{j} \times \frac{\partial \vec{v}}{\partial y} + \hat{k} \times \frac{\partial \vec{v}}{\partial z}$$

Which is a vector function.

Note: If $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$, then curl v

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

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1. If $\phi = 2x^3y^2z^4$ and $\vec{F} = \nabla\phi$, find $\nabla\cdot\vec{F}$ & $\nabla\times\vec{F}$. Evaluate these at $(1,-1,1)$.

Ans: Given $\phi = 2x^3y^2z^4$

$$\vec{F} = \nabla\phi = (6x^2y^2z^4)\hat{i} + (4x^3yz^4)\hat{j} + (8x^3y^2z^3)\hat{k}$$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(6x^2y^2z^4) + \frac{\partial}{\partial y}(4x^3yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3)$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

At $(1,-1,1)$, $\nabla\cdot\vec{F} = 12+4+24 = 40$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6x^2y^2z^4 & 4x^3yz^4 & 8x^3y^2z^3 \end{vmatrix}$$

$$= \hat{i}(16x^3yz^3 - 16x^3yz^3) - \hat{j}(24x^2y^2z^3 - 24x^2y^2z^3)$$
$$+ \hat{k}(12x^2yz^4 - 12x^2yz^4)$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$= \vec{0}$$

At $(1,-1,1)$, $\nabla\times\vec{F} = \vec{0}$

Solenoidal and Irrotational vector field:

Solenoidal vector:

A vector $\vec{F}$ is said to be solenoidal if $\text{div}\,\vec{F} = 0$ or $\nabla \cdot \vec{F} = 0$.

Irrotational vector:

A vector $\vec{F}$ is said to be irrotational if $\nabla \times \vec{F} = \vec{0}$ or $\text{curl}\,\vec{F} = \vec{0}$.

Problems:

1) Show that the vector $2xy\hat{i} + (x^2 + 2yz)\hat{j} + (y^2+1)\hat{k}$ is irrotational.

Ans:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2+2yz & y^2+1 \end{vmatrix}$$

$$= \hat{i}(2y - 2y) - \hat{j}(0-0) + \hat{k}(2x - 2x)$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$= \vec{0}$$

$\therefore \vec{F}$ is irrotational

2) Find 'a' such that

$(3x-2y+z)\hat{i} + (4x+ay-z)\hat{j} + (x-y+2z)\hat{k}$

solenoidal.

Ans:

Let $\vec{F} = (3x-2y+z)\hat{i} + (4x+ay-z)\hat{j} + (x-y+2z)\hat{k}$

Given $\nabla \cdot \vec{F} = 0$

i.e $\frac{\partial}{\partial x}(3x-2y+z) + \frac{\partial}{\partial y}(4x+ay-z) + \frac{\partial}{\partial z}(x-y+2z) = 0$

$\Rightarrow 3+a+2=0$

$\Rightarrow a+5=0 \quad \Rightarrow \underline{a=-5}$

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3) Determine the constants a, b, c so that the function

$\vec{F} = (axy+z^3)\hat{i} + (bx^2+z)\hat{j} + (bxz^2+cy)\hat{k}$

is irrotational.

Ans: Given $\nabla \times \vec{F} = \vec{0}$

$$\Rightarrow \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy+z^3 & bx^2+z & bxz^2+cy \end{vmatrix} = \vec{0}$$

$= \hat{i}(c-1) - \hat{j}(bz^2-3z^2) + \hat{k}(2bx-ax) = \vec{0}$

$\Rightarrow \quad c-1=0 \quad ; \quad -bz^2+3z^2=0 \quad ; \quad 2bx-ax=0$

$\Rightarrow \quad c=1 \quad ; \quad b=3 \quad ; \quad 2b-a=0$

$\Rightarrow a=2b$

$\Rightarrow a=2(3)$

$a=6$

$\therefore \; a=6, \quad b=3, \quad c=1$

H.W

4) Find the value of 'a' if the vector field $(ax^2y+yz)\hat{i}+(xy^2-xz^2)\hat{j}+(2xy-2x^2y^2)\hat{k}$ has zero divergence. Find the curl of the vector field when it has zero divergence.

<u>Ans:</u> Hint $\nabla\cdot\vec{F}=0$ gives $a=-1$.

and when $a=-1$,

$\nabla\times\vec{F}=(2x-4x^2y+2xz)\hat{i}-(y-4xy^2)\hat{j}+(y^2+x^2-z^2-z)\hat{k}$

UNIT - I

Differential Equations: D.E.

Defn: A differential equation is an equation which involves differential coefficients or differentials of an unknown function.

Eg: 1) $\left(\frac{dy}{dx}\right)^2 + 3\frac{dy}{dx} + 2 = 0$

2) $\frac{d^2y}{dx^2} + k^2 y = 0$

where $y = f(x)$ is an unknown function. Here 'y' is reffered as dependent variable & x as independent variable.

Defn: The order of a D.E. is the order of the highest derivative present in the equation.

The degree of a D.E. is the power of the highest order derivative occuring in the equation, after clearing the fractional powers.

Egs: 1) $\frac{dy}{dx} = \cos x$, is of order 1 & degree one.

2) $\frac{d^2y}{dx^2} + k^2 y = 0$, is of order 2 & degree 1.

3) $\left[\frac{d^2w}{dx^2}\right]^3 - xy\frac{dw}{dx} + w = 0$ is of order 2 & degree 3.

4) $L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{c} i = E \cos \omega t$

is of order 2 & degree 1.

5) $$\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = c$$

raising the power by 2 both on NR. & DR.

$$\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3}{\left(\frac{d^2y}{dx^2}\right)^2} = c^2$$

$$\Rightarrow \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = c^2 \left(\frac{d^2y}{dx^2}\right)^2$$

is of order 2 & degree 2.

6) $$5 \left(\frac{dy}{dx}\right)^{2/3} = 7 \left(\frac{d^2y}{dx^2}\right)^{3/4}$$

raising power by 12 we get

$$5^{12} \left(\frac{dy}{dx}\right)^8 = 7^{12} \left(\frac{d^2y}{dx^2}\right)^9$$

is of order 2 & degree 9.

Solution of a differential equations

1) A solution of a D.E. is a relation between the variables which satisfies the given D.E.

2) The general solution of a D.E. is the solution involved with arbitrary constants, where the numbers of arbitrary constants present in the solution is equal to the order of the D.E.

3) A particular solution is that which can be obtained from the general solution by giving particular values to the arbitrary constants.

Solution of D.Es of first order and first degree

In general a 1^{st} order & first degree equation will be of the form

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad M(x, y)\,dx + N(x, y)\,dy = 0.$$

1. Variable Separable Differential Equation:

If the given first order D.E. $\frac{dy}{dx} = f(x,y)$ ——(1) can be put in the form $P(x)\,dx + Q(y)\,dy = 0$. ——(2)

i.e, the coefficients of 'dx' is a function of the variable 'x' only & the coefficients of 'dy' is a function of 'y' only then the given equation is said to be in the variable separable form.

Integrating (2) on both sides, we get

$$\int P(x)\,dx + \int Q(y)\,dy = c,$$

gives the general solution of the D.E. (1).

3. **Linear Equation:**

A D.E is said to be linear if the dependent variable and its differential coefficients occur only in the first degree & not multiplied together.

Linear equation in 'y': A first order linear D.E. is of the form

$$\boxed{\frac{dy}{dx} + Py = Q}$$

Where P and Q are functions of 'x' alone, can be solved by finding the $\boxed{I.F = e^{\int P\,dx}}$ & solution of which is given by,

$$\boxed{y\,(I.F.) = \int Q\,(I.F)\,dx + C}$$

Linear equation in 'x': An equation of the form $\boxed{\frac{dx}{dy} + Px = Q}$, where P and Q are functions of y only is called a linear equation in 'x'. Can be solved by finding $\boxed{I.F. = e^{\int P\,dy}}$ & solution of which is given by

$$\boxed{x\,(I.F.) = \int Q\,(I.F.)\,dy + C}$$

1. Solve $\frac{dy}{dx} + y \tan x = \sec x$

Ans: is a linear equation in 'y'.

Where $P = \tan x$ & $Q = \sec x$

$\therefore$ $I.F. = e^{\int P dx} = e^{\int \tan x \, dx} = e^{\log \sec x} = \sec x$

$\therefore$ the solution is given by

$$y\,(I.F) = \int Q\,(I.F)\,dx + C$$

$$\Rightarrow y \cdot \sec x = \int \sec^2 x \, dx + C$$

$$= \tan x + C$$

---

2. Solve $dx + x\,dy = e^{-y} \sec^2 y \, dy$

Ans: ($\div$ dy b.s) $\frac{dx}{dy} + x = e^{-y} \sec^2 y$

is a linear equation in 'x'.

Where $P = 1$ & $Q = e^{-y} \sec^2 y$

$\therefore$ $I.F. = e^{\int P dy} = e^{\int dy} = e^{y}$.

$\therefore$ the solution is given by

$$x \cdot (I.F.) = \int Q\,(I.F)\,dy + C$$

$$x e^{y} = \int e^{-y} \sec^2 y \,(e^{y})\,dy + C$$

$$= \tan y + C$$

Second And Higher Order Linear Differential Equations with Constant Coefficients

General linear D.E. of the n^{th} order is of the form

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + p_{n-1} \frac{dy}{dx} + p_n y = X \quad \text{--- (1)}$$

where $p_1, p_2, \ldots, p_n$ and X are functions of x only.

Linear D.E. with constant coefficients are of the form

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + k_{n-1} \frac{dy}{dx} + k_n y = X \quad \text{--- (2)}$$

where $k_1, k_2, \ldots, k_n$ are constants.

Differential operator 'D':

Denoting $\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \ldots, \frac{d^n}{dx^n}$ by

$$D, D^2, D^3, \ldots, D^n.$$

so that $\frac{dy}{dx} = Dy, \frac{d^2y}{dx^2} = D^2 y, \ldots, \frac{d^n y}{dx^n} = D^n y$

The equation (2) above can be written in the symbolic form as,

$$D^n y + k_1 D^{n-1} y + k_2 D^{n-2} y + \cdots + k_{n-1} Dy + k_n y = X$$

$$\Rightarrow \left(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_{n-1} D + k_n\right) y = X$$

i.e, $f(D)\, y = X$ —— (3)

The complete solution of (3) will be of the form $y = y_c + y_p$

Where the part 'y_c' is called the complementary function (C.F.) & the part y_p is called the particular integral (P.I.) of (3).

i.e. $y =$ C.F. + P.I.

<u>Working procedure to solve the equation:</u>

<u>I:</u> To find the complementary function of $f(D)y = X$, consider $f(D)\, y = 0$ and write the auxiliary equation $f(m) = 0$

i.e $m^n + k_1 m^{n-1} + k_2 m^{n-2} + \cdots + k_{n-1} m + k_n = 0$

Let $m_1, m_2, \ldots, m_n$ be its roots.

<u>II:</u> Write the C.F. as follows

	Roots of A·E $f(m)=0$	Complementary function
Case I	When $m_1, m_2, \ldots, m_n$ are real and different roots	$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$
Case II	When $m_1, m_1, m_3, \ldots, m_n$ have two real and equal roots i.e, $m_1 = m_2$	$y_c = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$
Case III	When $m_1, m_1, m_1, m_4, \ldots, m_n$ have three real and equal roots i.e, $m_1 = m_2 = m_3$	$y_c = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}$
Case IV	When $\alpha \pm i\beta, m_3, m_4, \ldots, m_n$ have a pair of imaginary roots i.e $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$	$y_c = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$
Case V	When $\alpha \pm i\beta, \alpha \pm i\beta, m_5, \ldots, m_n$ have 2 pairs of equal imaginary roots i.e, $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, $m_3 = \alpha + i\beta$, $m_4 = \alpha - i\beta$	$y_c = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x] + c_5 e^{m_5 x} + \cdots + c_n e^{m_n x}$

Solve the following differential equations:

1) $(D^2+10D+25)y=0$ —(1)

Ans: A.E. of (1) is $f(m)=0$

i.e, $m^2+10m+25=0$

$(m+5)^2=0$

$\Rightarrow m=-5,-5$

$\therefore$ G.S of (1) is,

$y=(c_1+c_2x)e^{-5x}$

[or $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\dfrac{-10 \pm \sqrt{100-100}}{2}$

$-5,-5$

$\Rightarrow -5$ is a repeating root

2) $(D^2+D-2)y=0$ —(1)

Ans: A.E of (1) is $f(m)=0$

$m^2+m-2=0$

$(m+2)(m-1)=0$

$\Rightarrow m=1,-2$

$\therefore$ G.S of (1) is,

$y=c_1e^{x}+c_2e^{-2x}$

[OR $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\dfrac{-1 \pm \sqrt{1+8}}{2}$

$\dfrac{-1 \pm 3}{2}$

$\dfrac{-1+3}{2}, \dfrac{-1-3}{2}$

$1,-2$

3) $(D^2-6D+13)y=0$ —(1)

Ans: A.E of (1) is $f(m)=0$

$m^2-6m+13=0$

$m=\dfrac{-(-6) \pm \sqrt{36-52}}{2}$ $\left(\because \text{by } \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} \text{ formula}\right.$

$$m = \frac{6 \pm \sqrt{-16}}{2} \qquad \sqrt{-1} = i$$

$$= \frac{6 \pm 4i}{2} = 3 \pm 2i \qquad (\text{Here } \alpha = 3,\ \beta = 2$$

$\therefore$ G.S of (1) is,

$$y = e^{3x}\left[c_1 \cos 2x + c_2 \sin 2x\right]$$

---

4) $(D^3+1)y = 0$ —(1)

Ans: A.E of (1) is $m^3+1 = 0$ —(2)

$m = -1$ is a root of (2) by inspection

then by synthetic division

$$\begin{array}{r|rrrr} -1 & 1 & 0 & 0 & 1 \\ & & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & \boxed{0} \end{array}$$

$\therefore$ $m^3+1 = (m+1)(m^2-m+1) = 0$

$m^2 - m + 1 = 0$

$$\frac{+1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}\,i}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$\therefore$ $m = -1,\ \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ are roots of (2)

$\therefore$ G.S of (1) is, $y = c_1 e^{-x} + e^{\frac{1}{2}x}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right]$

5) $(D^3 - 2D^2)y = 0$, —(1) given $y(0) = -3$, $y'(0) = 0$
& $y''(0) = 12$

Ans: A.E of (1) is $f(m) = 0$

i.e, $m^3 - 2m^2 = 0$

$m^2(m-2) = 0$

$\Rightarrow m = 0, 0, 2$ are the roots

$\therefore$ G.S of (1) is

$y = (c_1 + c_2 x)e^{0x} + c_3 e^{2x}$

$= c_1 + c_2 x + c_3 e^{2x}$ —(2)

$\Rightarrow y' = c_2 + 2c_3 e^{2x}$ —(3)

$\Rightarrow y'' = 4c_3 e^{2x}$ —(4)

Given $y(0) = -3$, from (2) $y(0) = c_1 + c_3$

$\Rightarrow c_1 + c_3 = -3$ —(5)

$y'(0) = 0$, from (3) $0 = c_2 + 2c_3$ —(6)

$y''(0) = 12$, from (4) $12 = 4c_3 \Rightarrow c_3 = 3$

$\Rightarrow$ from (5) $c_1 = -3 - c_3 = -3 - 3 = -6$

from (6) $c_2 = -2c_3 = -2(3) = -6$

$\therefore$ from (2), $y = -6(1+x) + 3e^{2x}$

6) $(D^4+5D^2+4)y=0$ — (1)

Ans: A.E. of (1) is $f(m)=0$

$$m^4+5m^2+4=0$$

$$m^4+(m^2+4m^2)+4=0$$

$$m^2(m^2+1)+4(m^2+1)=0 \Rightarrow (m^2+1)(m^2+4)=0$$

$$\Rightarrow m^2+1=0 \;,\; m^2+4=0$$

$$\Rightarrow m^2=-1 \;,\; m^2=-4$$

$$\Rightarrow m=\pm i, \pm 2i$$

Here

$\pm i = 0 \pm 1\cdot i \Rightarrow \alpha=0, \beta=1$

$\pm 2i = 0 \pm 2\cdot i \Rightarrow \alpha=0, \beta=2$

$\therefore$ G.S of (1) is

$$y_c = e^{0x}\left[c_1\cos x + c_2\sin x\right] + e^{0x}\left[c_3\cos 2x + c_4 \sin 2x\right]$$

$$\Rightarrow y = c_1\cos x + c_2\sin x + c_3\cos 2x + c_4\sin 2x$$

Home work problems:

7) $(16D^3-11D-5)y=0$

Ans: $y = c_1e^{x} + e^{-\frac{1}{2}x}\left[c_2\cos\frac{1}{4}x + c_3\sin\frac{1}{4}x\right]$

8) $(D^3-2D^2+4D-8)y=0$ [Ans: $y=c_1e^{2x}+c_2\cos 2x + c_3\sin 2x$]

9) $(D^4-4D^2+4)y=0$ [Ans: $y=(c_1+c_2x)e^{\sqrt{2}x}+(c_3+c_4x)e^{-\sqrt{2}x}$

10) $(D^2+1)^3y=0$ [Ans: $(c_1+c_2x+c_3x^2)\cos x + (c_4+c_5x+c_6x^2)\sin x$

Note: 1) $DX = \frac{d}{dx} X$

2) $\frac{1}{D} X = \int X \, dx$

Particular Integral 'y_p':

Given the D.E. $f(D)\, y = X$ —— (1)

Then the particular integral is given by

$$y_p = \frac{1}{f(D)} X$$

where $f(D)$ & $\frac{1}{f(D)}$ are inverse operators of each other.

Rules for finding the particular integral:

Given the D.E. $f(D) y = X$, then we have

$$y_p = \frac{1}{f(D)} X$$

Case I: When $X = e^{ax}$

then $y_p = \frac{1}{f(D)} X = \frac{1}{f(a)} e^{ax}$, for $f(a) \neq 0$

if $f(a) = 0$, then

$$y_p = \frac{1}{f(D)} e^{ax} = x \cdot \frac{1}{f'(a)} e^{ax}, \text{ for } f'(a) \neq 0$$

If $f'(a)=0$, then

$$y_p = \frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} e^{ax}, \text{ for } f''(a) \neq 0$$

& so on.

Solve the following linear differential equations:

1) Solve $(D^2+3D-4)y = 12e^{2x}$ ——(1)

Ans: Complete solution of (1) is $y = y_c + y_p$

To find y_c: A.E. of (1) is $f(m)=0$

$$m^2+3m-4=0$$

$$\Rightarrow m = \frac{-3 \pm \sqrt{9+16}}{2} = \frac{-3 \pm \sqrt{25}}{2} = \frac{-3 \pm 5}{2}$$

$$\Rightarrow m = 1, -4$$

$$\therefore y = y_c = c_1 e^{x} + c_2 e^{-4x} \quad \text{——(2)}$$

or $m^2+3m-4=0$
$\Rightarrow m^2+4m-m-4=0$
$\Rightarrow m(m+4)-1(m+4)=0$
$\Rightarrow (m-1)(m+4)=0 \Rightarrow m=1,-4$

To find y_p: from (1)

$$y_p = \frac{1}{D^2+3D-4} 12e^{2x} = 12 \frac{1}{(2)^2+(3\times 2)-4} e^{2x}$$

$$= 12 \cdot \frac{1}{6} e^{2x} = 2e^{2x}$$

$$\therefore y = y_c + y_p$$

$$= c_1 e^{x} + c_2 e^{-4x} + 2e^{2x} \text{ is the G.S of (1)}$$

2) $(D-3)^2 y = e^{3x}$ —— (1)

Ans: A.E. of (1) is $f(m)=0$

$(m-3)^2=0 \Rightarrow m=3,3$

$\therefore y_c = (c_1 + c_2 x)e^{3x}$ —— (2)

Now from (1), $y_p = \frac{1}{(D-3)^2} e^{3x}$

$\Rightarrow y_p = x^2 \frac{1}{f''(3)} e^{3x}$

$= \frac{x^2}{2} e^{3x}$ —— (3)

$\because$ Here $f(D) = (D-3)^2$

$f'(D) = 2(D-3)$

$f''(D) = 2$

$\Rightarrow f(3) = 0,\ f'(3)=0$

& $f''(3) = 2$

$\therefore$ G.S of (1) is

$y = y_c + y_p = (2)+(3)$

$y = (c_1 + c_2 x)e^{3x} + \frac{x^2}{2} e^{3x}$

---

3) $(D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{-3x}$ —— (1)

Ans: Complete solution of (1) is $y = y_c + y_p$

To find y_c: A.E. of (1) is $f(m)=0$

i.e $m^3 - 6m^2 + 11m - 6 = 0$ —— (2)

$m=1$ is a root of (2) by inspection.

Now by synthetic division.

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$\therefore \quad m^3 - 6m^2 + 11m - 6 = 0$

$\Rightarrow \quad (m-1)(m^2 - 5m + 6) = 0$

$\Rightarrow \quad (m-1)(m-3)(m-2) = 0$

$\Rightarrow \quad m = 1, 2, 3$

$\therefore \quad y_c = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} \quad -(3)$

<u>To find y_p:</u> from (1)

$$y_p = \frac{1}{D^3 - 6D^2 + 11D - 6} \left(e^{-2x} + e^{-3x}\right)$$

$$= \frac{1}{D^3 - 6D^2 + 11D - 6} e^{-2x} + \frac{1}{D^3 - 6D^2 + 11D - 6} e^{-3x}$$

$$= \frac{1}{(-2)^3 - 6(-2)^2 + 11(-2) - 6} e^{-2x} + \frac{1}{(-3)^3 - 6(-3)^2 + 11(-3) - 6} e^{-3x}$$

$$= \frac{1}{-8 - 24 - 22 - 6} e^{-2x} + \frac{1}{-27 - 54 - 33 - 6} e^{-3x}$$

$$= -\frac{1}{60} e^{-2x} - \frac{1}{120} e^{-3x} \quad -(4)$$

$\therefore \quad y = y_c + y_p$

$$= c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} - \frac{1}{60}\left[e^{-2x} + \frac{e^{-3x}}{2}\right]$$

4) $(4D^2-1)y = e^{x/2} + 12e^{x} + 4$ —(1)

<u>Ans:</u> Complete solution of (1) is $y = y_c + y_p$

<u>To find y_c:</u> A.E. of (1) is $f(m)=0$

i.e, $4m^2-1=0 \Rightarrow m^2 = 1/4 \Rightarrow m = \pm 1/2$

$\therefore y_c = c_1 e^{-1/2 x} + c_2 e^{1/2 x}$ —(2)

<u>To find y_p:</u> from (1) $\Rightarrow y_p = \frac{1}{4D^2-1}\left[e^{x/2} + 12e^{x} + 4\right]$

$$y_p = \frac{1}{4D^2-1} e^{x/2} + 12\frac{1}{4D^2-1} e^{x} + 4 \frac{1}{4D^2-1} e^{0x}\Big]$$

$$= x \frac{1}{f'(1/2)} e^{x/2} + 12 \frac{1}{(4-1)} e^{x} + 4 \frac{1}{(0-1)} e^{0x}$$

$$= \frac{x e^{x/2}}{4} + 4e^{x} - 4 \quad \text{—(3)}$$

$\Big[\because f(D) = 4D^2-1$

$f'(D) = 8D$

$\Rightarrow f(1/2) = 4(1/2)^2 - 1 = 1-1 = 0$

$\Rightarrow f'(1/2) = 8(1/2) = 4$

$\therefore$ G.S of (1) is

$y = y_c + y_p$

$$y = c_1 e^{-1/2 x} + c_2 e^{1/2 x} + \frac{x e^{x/2}}{4} + 4e^{x} - 4$$

---

5) $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

<u>Ans:</u> $(c_1 + c_2 x)e^{x} + c_3 e^{-2x} + \frac{x}{9} e^{-2x} + \frac{x^2 e^{x}}{6} - \frac{e^{-x}}{4}$

Hint : $\sinh x = \frac{e^{x} - e^{-x}}{2}$ $\left[\text{III}^{ly} \; \cosh x = \frac{e^{x}+e^{-x}}{2}\right]$

Case II: When $X = \sin(ax+b)$ or $\sin ax$

& when $X = \cos(ax+b)$ or $\cos ax$

$X = \sin(ax+b)$ or $\sin ax$	$X = \cos(ax+b)$ or $\cos ax$
1) $y_p = \frac{1}{f(D^2)} X$ $= \frac{1}{f(-a^2)} \sin(ax+b)$ for $f(-a^2) = 0$ [i.e replace D^2 by $-a^2$]	1) $y_p = \frac{1}{f(D^2)} X = \frac{1}{f(-a^2)} \cos(ax+b)$ for $f(-a^2) \neq 0$
2) If $f(-a^2) = 0$ $y_p = x \frac{1}{f'(-a^2)} \sin(ax+b)$, $f'(-a^2) \neq 0$	2) If $f(-a^2) = 0$, $y_p = x \frac{1}{f'(-a^2)} \cos(ax+b)$, $f'(-a^2) \neq 0$
3) If $f'(-a^2) = 0$, $y_p = x^2 \frac{1}{f''(-a^2)} \sin(ax+b)$, for $f''(-a^2) \neq 0$	3) If $f'(-a^2) = 0$, $y_p = x^2 \frac{1}{f''(-a^2)} \cos(ax+b)$ for $f''(-a^2) \neq 0$
& so on	& so on

1) $(D^2+36)\,y = 4\cos 6x \quad \text{——} (1)$

Ans: Complete solution of (1) is $y = y_c + y_p$

To find y_c: A.E. of (1) is $f(m)=0$

$$m^2+36=0 \Rightarrow m^2 = -36$$
$$\Rightarrow m = \pm 6i$$

$$\therefore y_c = e^{0x}\left[c_1 \cos 6x + c_2 \sin 6x\right]$$
$$= c_1 \cos 6x + c_2 \sin 6x$$

To find y_p:

$$y_p = \frac{1}{D^2+36}(4\cos 6x)$$

$$= 4\left[x \frac{1}{f'(D)} \cos 6x\right]$$

$f(D) = D^2+36$

Replacing $D^2 = -6^2$
$= -36$

$f(D)=0$

Now $f'(D) = 2D$

& $\frac{1}{D}\,x = \int x\,dx$

$$= 4\left[x \frac{1}{2D} \cos 6x\right]$$

$$= 2x \int \cos 6x\,dx$$

$$= 2x\left[\frac{\sin 6x}{6}\right]$$

$$= \frac{x \sin 6x}{3}$$

$$\therefore y = y_c + y_p = c_1 \cos 6x + c_2 \sin 6x + \frac{x \sin 6x}{3}$$

is the required G.S. of (1)

2) $(D^3 + D^2 + D + 1)y = \cos 2x$ —(1)

Ans: Complete solution of (1) is $y = y_c + y_p$

To find y_c: Consider A.E. of $f(m) = 0$

$$m^3 + m^2 + m + 1 = 0$$

$$m^2(m+1) + 1(m+1) = 0$$

$$(m^2+1)(m+1) = 0$$

$$\Rightarrow m = \pm i, -1$$

$$\therefore y_c = [C_1 \cos x + C_2 \sin x] + C_3 e^{-x}$$

To find y_p:

$$y_p = \frac{1}{D^3 + D^2 + D + 1} \cos 2x$$

$$= \frac{1}{}$$

$$= \frac{1}{-3(D+1)} \cos 2x$$

$$= -\frac{1}{3} \frac{D-1}{D^2-1} \cos 2x$$

$$= -\frac{1}{3} \frac{D-1}{-4-1} \cos 2x$$

$$= \frac{1}{15}[-\sin 2x(2) - \cos 2x]$$

$$= \frac{1}{15}[-2 \sin 2x - \cos 2x]$$

$f(D) = D^3 + D^2 + D + 1$

Replacing D^2 by $-2^2 = -4$

$f(D) = -4D - 4 + D + 1$

$= -3D - 3$

$= -3[D+1]$

$$\therefore y = C_1 \cos x + C_2 \sin x + C_3 e^{-x} + \frac{1}{15}[-2 \sin 2x - \cos 2x]$$

3) $(D^2-4D+3)y = \sin 3x \cdot \cos 2x$ ——— ①

Ans: Complete solution of ① is $y = y_c + y_p$

To find y_c: A.E. of ① is $f(m) = 0$

$$m^2 - 4m + 3 = 0$$

$$(m-3)(m-1) = 0$$

$$m = 1, 3$$

$$\therefore y_c = c_1 e^{x} + c_2 e^{3x}$$

To find y_p: $y_p = \dfrac{1}{D^2-4D+3} \sin 3x \cdot \cos 2x$

$$\Rightarrow y_p = \frac{1}{D^2-4D+3}\left[\tfrac{1}{2}(\sin 5x + \sin x)\right]$$

$$\Rightarrow y_p = \tfrac{1}{2}\left[\frac{1}{D^2-4D+3}\sin 5x + \frac{1}{D^2-4D+3}\sin x\right]$$

$$\Rightarrow y_p = \tfrac{1}{2}\left[\frac{1}{-25-4D+3}\sin 5x + \frac{1}{-1-4D+3}\sin x\right]$$

$$\Rightarrow y_p = \tfrac{1}{2}\left[\frac{1}{-[4D+22]}\sin 5x + \frac{1}{2-4D}\sin x\right]$$

$$= \tfrac{1}{2}\left[\frac{(4D-22)}{-(16D^2-484)}\sin 5x + \frac{(2+4D)}{4-16D^2}\sin x\right]$$

$$= \tfrac{1}{2}\left[\frac{(4D-22)}{-(-400-484)}\sin 5x + \frac{(2+4D)}{20}\sin x\right]$$

$$= \tfrac{1}{2}\left[\frac{20\cos 5x - 22\sin 5x}{884} + \frac{2\sin x + 4\cos x}{20}\right]$$

$$y_p = \frac{10\cos 5x - 11\sin 5x}{884} + \frac{\sin x + 2\cos x}{20}$$

$\therefore$ G.S of (1)

$$y = y_c + y_p$$

$$= c_1 e^{x} + c_2 e^{3x} + \frac{10\cos 5x - 11\sin 5x}{884} + \frac{\sin x + 2\cos x}{20}$$

Case III: When $X = x^m$, where m is a positive integer.

i.e, to get $y_p = \frac{1}{f(D)} x^m$, we write this in the form $[f(D)]^{-1} x^m$ and expand $[f(D)]^{-1}$ as far as D^m & neglect the remaining higher powers of D.

Note: to expand $[f(D)]^{-1}$ we generally use the following two formulae

① $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots$

② $(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \cdots$

Problems

1) Solve $(D^2+1)y = x^3$ —— ①

Ans: The complete solution of ① $y = y_c + y_p$

To find y_c : A·E of ① $m^2+1=0$

$m^2 = -1 \Rightarrow m = \pm i$ $(0 \pm i \Rightarrow \alpha = 0, \beta = 1)$

$y_c = C_1 \cos x + C_2 \sin x$

To find y_p: $y_p = \frac{1}{D^2+1} x^3$

$= [1+D^2]^{-1} x^3$

$= [1 - D^2 + D^4 - \cdots] x^3$

$= x^3 - 6x$

∴

$f(x) = x^3$

$Dx^3 = 3x^2$

$D^2x^3 = 6x$

$D^3x^3 = 6$

$D^4x^3 = 0$

$$\therefore \quad y = y_c + y_p$$
$$= c_1 \cos x + c_2 \sin x + x^3 - 6x$$

---

2) $$\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$$

$$\Rightarrow (D^2 + D)y = x^2 + 2x + 4 \quad \text{---(1)}$$

Ans: Complete solution of (1) is $y = y_c + y_p$

To find y_c: A.E of (1) is

$$m^2 + m = 0$$
$$\Rightarrow m(m+1) = 0 \quad \Rightarrow m = 0, -1$$

$$\therefore y_c = c_1 e^{0x} + c_2 e^{-x}$$
$$= c_1 + c_2 e^{-x}$$

To find y_p: $y_p = \frac{1}{D^2 + D}(x^2 + 2x + 4)$

$$= \frac{1}{D}\left[\frac{1}{(1+D)}(x^2 + 2x + 4)\right]$$

$$= \frac{1}{D}\left[(1+D)^{-1}(x^2 + 2x + 4)\right]$$

$$y_p = \frac{1}{D}\left[(1 - D + D^2)(x^2 + 2x + 4)\right]$$

$$\Rightarrow y_p = \frac{1}{D}\left[(x^2 + 2x + 4) - (2x + 2) + (2)\right]$$

$$= \frac{1}{D}\left[x^2 + 4\right] = \int (x^2 + 4)\,dx = \frac{x^3}{3} + 4x$$

$$\therefore y = y_c + y_p = c_1 + c_2 e^{-x} + \frac{x^3}{3} + 4x$$

3) $(D^2+2D+2)y = 1+3x+x^2 \quad \text{—— (1)}$

Ans: Complete solution of (1) is $y = y_c + y_p$.

To find y_c: A.E of (1) is,

$$m^2+2m+2=0$$

$$m = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2}{2} \pm \frac{\sqrt{-4}}{2}$$

$$= -1 \pm i$$

$$\therefore \; y_c = e^{-x}\left[c_1 \cos x + c_2 \sin x\right]$$

To find y_p:
$$y_p = \frac{1}{D^2+2D+2}(1+3x+x^2)$$

$$= \frac{1}{2\left[1+\left(D+\frac{D^2}{2}\right)\right]}(1+3x+x^2)$$

$$= \frac{1}{2}\left[1-\left(D+\frac{D^2}{2}\right)+\left(D+\frac{D^2}{2}\right)^2\right](1+3x+x^2)$$

$$= \frac{1}{2}\left[1-D-\frac{D^2}{2}+D^2\right](1+3x+x^2)$$

$$= \frac{1}{2}\left[\left(1-D+\frac{D^2}{2}\right)(1+3x+x^2)\right]$$

$$= \frac{1}{2}\left[(1+3x+x^2)-(3+2x)+\frac{1}{2}(2)\right]$$

$$= \frac{x^2+x-1}{2}$$

$$\therefore \; y = y_c + y_p$$
$$= e^{-x}\left[c_1 \cos x + c_2 \sin x\right] + \left(\frac{x^2+x-1}{2}\right)$$

4) $(2D^2+2D+3)y = x^2+2x-1$

Ans: Complete solution of ① is $y = y_c + y_p$

To find y_c: A.E. of ①, $2m^2+2m+3=0$

$$m = \frac{-2 \pm \sqrt{4-24}}{4} = \frac{-2 \pm \sqrt{-20}}{4}$$

$$= -\tfrac{1}{2} \pm \frac{\sqrt{5}}{2} i \qquad (\because \sqrt{-20} = \sqrt{-(4\times5)} = 2\sqrt{5}\, i$$

$$\therefore y_c = e^{-\frac{1}{2}x}\left[c_1 \cos\frac{\sqrt{5}}{2}x + c_2 \sin\frac{\sqrt{5}}{2}x\right] \quad —②$$

To find y_p: $y_p = \frac{1}{2D^2+2D+3}(x^2+2x-1)$

$$y_p = \frac{1}{3}\left[\frac{1}{\left[1+\left(\frac{2D}{3}+\frac{2D^2}{3}\right)\right]}(x^2+2x-1)\right]$$

$$= \frac{1}{3}\left[\left(1+\left(\frac{2D}{3}+\frac{2D^2}{3}\right)\right)^{-1}(x^2+2x-1)\right]$$

$$= \frac{1}{3}\left[1-\left(\frac{2D}{3}+\frac{2D^2}{3}\right)+\left(\frac{2D}{3}+\frac{2D^2}{3}\right)^2\right](x^2+2x-1)$$

$$= \tfrac{1}{3}\left[\left(1-\frac{2D}{3}-\frac{2D^2}{3}+\frac{4D^2}{9}\right)(x^2+2x-1)\right]$$

$$= \tfrac{1}{3}\left[\left(1-\frac{2D}{3}-\frac{2D^2}{9}\right)(x^2+2x-1)\right]$$

$$= \tfrac{1}{3}\left[(x^2+2x-1)-\tfrac{2}{3}(2x+2)-\frac{2}{9}(2)\right]$$

$$= \tfrac{1}{3}\left[x^2+\frac{2x}{3}-\frac{25}{9}\right] \quad —③$$

$\therefore y = y_c + y_p$
$= ② + ③$

2) $(D^2+3D+2)y = 3e^{-2x} + \cos 3x$ —(1)

Ans: Complete soln of (1) is $y = y_c + y_p$

To find y_c : A.E. of (1) is $f(m)=0$

$m^2+3m+2=0$

$(m+2)(m+1)=0 \Rightarrow m = -1, -2$

$\therefore\ y_c = c_1 e^{-x} + c_2 e^{-2x}$

To find y_p: from (1)

$$y_p = \frac{1}{D^2+3D+2}\left(3e^{-2x} + \cos 3x\right)$$

$$= 3\,\frac{1}{D^2+3D+2}\,e^{-2x} + \frac{1}{D^2+3D+2}\cos 3x$$

$$= 3\,x\,\frac{1}{f'(-2)}\,e^{-2x} + \frac{1}{-9+3D+2}\cos 3x$$

$\because f(D) = D^2+3D+2$

$f(-2) = 4-6+2=0$

$\therefore f'(D) = 2D+3$

$f'(-2) = -4+3=-1$

$$= \frac{3x}{-1}\,e^{-2x} + \frac{1}{3D-7}\cos 3x$$

$$= -3x\,e^{-2x} + \left[\frac{3D+7}{9D^2-49}\cos 3x\right]$$

Replacing $D^2=-3^2=-9$ in the 2nd term

$$= -3xe^{-2x} + \left[\frac{(3D+7)\cos 3x}{-81-49}\right]$$

$$= -3xe^{-2x} - \frac{1}{130}\left[-9\sin 3x + 7\cos 3x\right]$$

$$= -3xe^{-2x} + \frac{1}{130}\left[9\sin 3x - 7\cos 3x\right]$$

$$\therefore y = c_1e^{-x} + c_2e^{-2x} - 3xe^{-2x} + \frac{1}{130}\left[9\sin 3x - 7\cos 3x\right]$$

3) $(D^3-D)y = 2x+1+4\cos x+2e^x$ ——(1)

Ans: Complete solution of (1) is $y=y_c+y_p$

To find y_c: A.E. of (1) is $f(m)=0$

$m^3-m=0 \Rightarrow m(m^2-1)=0$

$\Rightarrow m=0, \pm 1$

$\therefore y_c = c_1 e^{0x}+c_2e^{-x}+c_3e^x$

To find y_p: from (1)

$$y_p = \frac{1}{D^3-D}\left[(2x+1)+4\cos x+2e^x\right]$$

$$= \frac{1}{D^3-D}(2x+1)+4\frac{1}{D^3-D}\cos x+2\frac{1}{D^3-D}e^x$$

$$= \frac{1}{-D(1-D^2)}(2x+1)+4\frac{1}{-D-D}\cos x+2x\frac{1}{f'(1)}e^x$$

$$= -\frac{1}{D}\left[1-D^2\right]^{-1}(2x+1)-\frac{4}{2}\left(\frac{1}{D}\cos x\right)+2x\frac{e^x}{2}$$

$\because$ replacing $D^2=-1^2=-1$ in the 2nd term

$$= -\frac{1}{D}(1)(2x+1)-2\int\cos x\,dx+xe^x$$

$f(D)=D^3-D$

$f(1)=1-1=0$

$\therefore f'(D)=3D^2-1$

$f'(1)=3-1=2$

$$= -\int(2x+1)\,dx-2\sin x+xe^x$$

$$y_p = -x^2-x-2\sin x+xe^x$$

$(1-x)^{-1}=1+x+x^2+\cdots$

$(1-D^2)^{-1}=1+D^2+D^4+\cdots$

$\therefore y = c_1+c_2e^{-x}+c_3e^x-x^2-x-2\sin x+xe^x$

$\frac{1}{D}x=\int x\,dx$

Numerical solution of partial differential equations (PDE)

<u>Classification of second order PDE:</u>

The general second order linear PDE in two independent variables *x, y* is of the form

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\,\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+F\,u=0$$

where A, B, C, D, E, F are in general functions of x and y. This equation is said to be

(i) elliptic if $B^2-4\,A\,C<0$,

(ii) parabolic if $B^2-4\,A\,C=0$,

and (iii) hyperbolic if $B^2-4\,A\,C>0$.

Classify the following equations:

1) $\dfrac{\partial^2 u}{\partial x^2}+4\dfrac{\partial^2 u}{\partial x\,\partial y}+4\dfrac{\partial^2 u}{\partial y^2}-\dfrac{\partial u}{\partial x}+2\dfrac{\partial u}{\partial y}=0$

Ans: Here $A=1\,,\,B=4\,,\,C=4$

Now $B^2-4\,A\,C=16-4.1.4=16-16=0$

$\therefore$ the given equation is parabolic.

2) $(1+x^2)\dfrac{\partial^2 u}{\partial x^2}+(5+2\,x^2)\dfrac{\partial^2 u}{\partial x\,\partial t}+(4+x^2)\dfrac{\partial^2 u}{\partial t^2}=0$

Ans: Here $A=1+x^2\,,\,B=5+2\,x^2\,,\,C=4+x^2$

Now $B^2-4\,A\,C=(5+2\,x^2)^2-4\,(1+x^2)(4+x^2)$

$$=25+4\,x^4+20\,x^2-4(4+5\,x^2+x^4)$$

$$=25+4\,x^4+20\,x^2-16-20\,x^2-4\,x^4$$

$$=9>0$$

$\therefore$ the given equation is hyperbolic.

UNIT-III

FIRST AND HIGHER ORDER PARTIAL DIFFERENTIAL EQUATIONS

A partial D.E. is an equation in which there are two or more independent variables and partial differential coefficient w.r.t any of these.

Eg: 1. $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u$

2. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Order: of a PDE is the order of the highest derivative appearing in it.

Standard notation:

We denote: $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$,

$\frac{\partial^2 z}{\partial x^2} = r$, $\frac{\partial^2 z}{\partial x \partial y} = s$, $\frac{\partial^2 z}{\partial y^2} = t$.

Formation of PDE:

The PDE can be formed either by the elimination of arbitrary constants or by the elimination of arbitrary functions from a relation involving three or more variables.

I. Formation of PDE by eliminating arbitrary constants:

Given a relation of the form $f(x,y,z,a,b)=0$, —(1)
where 'z' is a function of x, y and a, b
are arbitrary constants.
Differentiate (1) w.r.t x and y partially
and eliminate the constants a, b to form
the PDE.

In case the number of arbitrary constants are more than the number of independent variables, we need appropriate number of partial derivatives of second & higher order also.

Construct the PDE by eliminating arbitrary constants a, b from the following equations:

---

1) $$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad —(1)$$

Ans: Differentiating (1) partially w.r.t x, we get

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} \Rightarrow p = \frac{x}{a^2} \Rightarrow a^2 = \frac{x}{p}$$

Differentiating (1) partially w.r.t y, we get

$$2\frac{\partial z}{\partial y} = \frac{2y}{b^2} \Rightarrow q = \frac{y}{b^2} \Rightarrow b^2 = \frac{y}{q}$$

Substituting the values of a^2 & b^2 in (1), we get

$$2z = \frac{x^2}{\left(\frac{x}{p}\right)} + \frac{y^2}{\left(\frac{y}{q}\right)} \Rightarrow \underline{2z = px + qy}$$ is the required PDE.

2) $z = ax^2+by^2$ —①

<u>Ans:</u> Differentiating ① partially w.r.t 'x', we get

$\frac{\partial z}{\partial x} = 2ax \Rightarrow p = 2ax \Rightarrow a = \frac{p}{2x}$

Differentiating ① partially w.r.t 'y', we get

$\frac{\partial z}{\partial y} = 2by \Rightarrow q = 2by \Rightarrow b = \frac{q}{2y}$

∴ ① reduces to

$z = \frac{p}{2x}(x^2) + \frac{q}{2y}(y^2) \Rightarrow z = \frac{px}{2} + \frac{qy}{2}$

$\Rightarrow 2z = px+qy$, is the required PDE.

3) $z = (x-a)^2+(y-b)^2+1$ —①

<u>Ans:</u> Differentiating ① partially w.r.t 'x' & 'y' respectively, we get

$p = 2(x-a) \Rightarrow x-a = \frac{p}{2}$

$q = 2(y-b) \Rightarrow y-b = \frac{q}{2}$

Substituting these in ①, we get

$z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2 + 1 \Rightarrow z = \frac{p^2}{4} + \frac{q^2}{4} + 1$

$\Rightarrow 4z = p^2+q^2+4$ is the required PDE

4) $z = ax + by + a^2 + b^2$

Ans: Given $z = ax + by + a^2 + b^2$ — (1)

$\Rightarrow p = a$

& $q = b$

Substituting these in (1), we get

$z = px + qy + p^2 + q^2$ is the required PDE.

5) $z = a \log \left\{ \frac{b(y-1)}{1-x} \right\}$

Ans: Given $z = a \log \left\{ \frac{b(y-1)}{1-x} \right\}$

$\Rightarrow z = a \log b + a \log (y-1) - a \log (1-x)$ — (1)

$\Rightarrow p = \frac{-a}{1-x}(-1) = \frac{a}{1-x} \Rightarrow a = p(1-x)$ — (2)

& $q = \frac{a}{y-1} \Rightarrow a = q(y-1)$ — (3)

Eliminating 'a' from (2) & (3), we get

$p(1-x) = q(y-1)$

$\Rightarrow xp + yq = p + q$ is the required PDE.

II Formation of PDE by eliminating arbitrary Functions:

a) Equations involving one arbitrary function:

Here, PDE, is formed by eliminating arbitrary function by finding 1^{st} order partial derivative.

b) Equations involving two or more arbitrary function:

Here the arbitrary functions can be eliminated by finding partial derivatives of order two or more.

Note: When 'n' is the number of arbitrary functions one may get several PDEs, but generally the one with the least order is chosen.

Form the PDEs by eliminating arbitrary functions:

1) $z = f(x^2 - y^2)$

Ans: Given $z = f(x^2 - y^2)$ — ①

$\Rightarrow p = f'(x^2 - y^2) \cdot 2x$ — ②

& $q = f'(x^2 - y^2) \cdot (-2y)$ — ③

To eliminate $f'(x^2 - y^2)$ from ② & ③,

consider $\frac{②}{③}$, we get

$\frac{p}{q} = \frac{x}{-y} \Rightarrow -py = qx$

$\Rightarrow \underline{xq + yp = 0}$ is the required PDE

2) $x+y+z = f(x^2+y^2+z^2)$

Ans: Given $x+y+z = f(x^2+y^2+z^2)$ — ①

Differentiating ① partial w.r.t x & y respectively we get

$$1+p = f'(x^2+y^2+z^2)\cdot(2x+2zp) \quad \text{— ②}$$

$$\& \quad 1+q = f'(x^2+y^2+z^2)\cdot(2y+2zq) \quad \text{— ③}$$

Now consider $\frac{②}{③}$, gives

$$\frac{1+p}{1+q} = \frac{x+zp}{y+zq}$$

$$\Rightarrow (1+p)(y+zq) = (1+q)(x+zp)$$

$$\Rightarrow y+zq+py+\cancel{pqz} = x+zp+xq+\cancel{zpq}$$

$$\Rightarrow (y-z)p+(z-x)q+(y-x)=0$$

is the required PDE.

H.W 3) $xyz = f(x+y+z)$ (Ans: $px(z-y)+qy(x-z)+z(x-y)=0$)

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4) $z = f(x) + e^y g(x)$

Ans: Given $z = f(x)+e^y g(x)$ — ①

$$\Rightarrow p = f'(x)+e^y g'(x)$$

$$q = e^y g(x) \quad \text{— ②}$$

$r = f''(x) + e^y g''(x)$

$s = e^y g'(x)$

$t = e^y g(x)$ — ③

From ② & ③, we get

$q = t \Rightarrow \frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial y^2}$

$\Rightarrow \frac{\partial z}{\partial y} - \frac{\partial^2 z}{\partial y^2} = 0$, is the required PDE.

---

5) $z = f(x+iy) + g(x-iy)$ — ①

Ans: Given $z = f(x+iy) + g(x-iy)$

$\Rightarrow p = f'(x+iy) + g'(x-iy)$

$q = i f'(x+iy) - i g'(x-iy)$

$r = f''(x+iy) + g''(x-iy)$ — ②

$s = i f''(x+iy) - i g''(x-iy)$

$t = -f''(x+iy) - g''(x-iy)$ — ③

From ② & ③, we get

$r + t = 0 \Rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ is the required PDE

H.W

6) $z = f(y+2x) + g(y-3x)$ [Ans: $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$

c) **Formation of PDE by eliminating arbitrary function F from the equation $F(u, v) = 0$** — (*)

Where u & v are functions of x, y, z.

Here we can eliminate F by differentiating (*) partially w.r.t x & y.

Form the PDE by eliminating arbitrary functions:

---

1. $F(x^2+y^2, z-xy) = 0$.

<u>Ans:</u> Let $u = x^2+y^2$, $v = z-xy$

$\therefore F(x^2+y^2, z-xy) = F(u, v) = 0$ — ①

Differentiating ① w.r.t x & y partially, we get

$$\frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial F}{\partial u} \frac{\partial u}{\partial x} = -\frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial x} \quad \text{— ②}$$

$$\& \quad \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial y} = -\frac{\partial F}{\partial v} \frac{\partial v}{\partial y} \quad \text{— ③}$$

Consider $\frac{②}{③}$, gives

$$\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = \frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} \Rightarrow \frac{2x}{2y} = \frac{p-y}{q-x}$$

$$\Rightarrow x(q-x) = y(p-y)$$

$$\Rightarrow \underline{xq - yp = x^2 - y^2}$$ is the required PDE.

2) $F(x^2+2yz,\ y^2+2xz)=0$

Ans: Let $u=x^2+2yz$, $v=y^2+2xz$

$\therefore F(x^2+2yz,\ y^2+2xz)=F(u,v)=0$ —— ①

Differentiating ① partially w.r.t x & y, we get

$$\frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial x}=0 \Rightarrow \frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial x}=-\frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial x} \quad \text{—— ②}$$

& $$\frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial y}=0 \Rightarrow \frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial y}=-\frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial y} \quad ③$$

Consider $\frac{②}{③}$, we get

$$\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}=\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} \Rightarrow \frac{2x+2yp}{2z+2yq}=\frac{2z+2xp}{2y+2xq}$$

$$\Rightarrow \frac{x+yp}{z+yq}=\frac{z+xp}{y+xq}$$

$$\Rightarrow (x+yp)(y+xq)=(z+xp)(z+yq)$$

$$\Rightarrow xy+x^2q+y^2p+\cancel{xypq}=z^2+zyq+xpz+\cancel{xypq}$$

$\Rightarrow (y^2-xz)p+(x^2-yz)q=z^2-xy$, is the required PDE.

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3) $F(xy+z^2,\ x+y+z)=0$ [Ans: $(y-x)+(2z-x)p+(y-2z)q=0$

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4) $F(x+y+z,\ x^2+y^2+z^2)=0$

Method of seperation of variables:

Working procedure:

This method is illustrated below in respect of a PDE having two independent variables x, y and $u = u(x,y)$.

STEP 1: Assume the solution of the PDE $u = XY$, where $X = X(x)$ & $Y = Y(y)$.

STEP 2: Substitute $u = XY$ in the given PDE, which reduces into ordinary derivatives.

STEP 3: Rearrange the resulting equation such that LHS is a function of x & RHS is a function of y.

STEP 4: Equating LHS & RHS to a common constant k.

STEP 5: Solving the resulting ODEs to obtain $X = X(x)$ & $Y = Y(y)$.

Solve the following PDEs by the method of separation of variables:

1) $$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u, \quad \text{where } u(x,0) = 6e^{-3x} \quad \text{——— (1)}$$

Ans:

Let $u = XT$ be the solution of (1).

Then from (1),

$$\frac{\partial}{\partial x}(XT) = 2\frac{\partial}{\partial t}(XT) + XT$$

$$T\frac{dX}{dx} = 2X\frac{dT}{dt} + XT$$

$\div XT$

$$\frac{1}{X}\frac{dX}{dx} = \frac{2}{T}\frac{dT}{dt} + 1$$

Let $\frac{1}{X}\frac{dX}{dx} = \frac{2}{T}\frac{dT}{dt} + 1 = k$

$\Rightarrow \frac{1}{X}\frac{dX}{dx} = k \quad ; \quad \frac{2}{T}\frac{dT}{dt} + 1 = k$

$\Rightarrow \frac{dX}{X} = k\, dx \quad ; \quad 2\frac{dT}{T} = (k-1)\,dt$ (v.s form)

Integrating we get,

$\log X = kx + c_1' \quad ; \quad 2\log T = (k-1)t + c_2'$

$\Rightarrow X = e^{kx + c_1'}$

$\log T = \frac{(k-1)t + c_2'}{2}$

$T = e^{\frac{1}{2}(k-1)t + \frac{c_2'}{2}}$

$\Rightarrow X = e^{kx + c_1} \quad ; \quad T = e^{\frac{1}{2}(k-1)t + c_2}$

$\therefore u = XT = \left(e^{kx + c_1}\right)\left(e^{\frac{1}{2}(k-1)t + c_2}\right)$

$u = c_3\, e^{kx + \frac{1}{2}(k-1)t}$ —— (2)

Given $u(x,0) = 6e^{-3x}$

from (2), $u(x,0) = c_3 e^{kx}$

$\Rightarrow 6e^{-3x} = c_3 e^{kx} \Rightarrow c_3 = 6, \; k = -3$

$\therefore$ from (2), $u = 6e^{-3x-2t}$ is the required particular solution

2. $y^3 \frac{\partial z}{\partial x} + x^2 \frac{\partial z}{\partial y} = 0$ —— ①

Ans: Let $z = XY$ be the solution of ①.

then from ①,

$$y^3 \frac{\partial}{\partial x}(XY) + x^2 \frac{\partial}{\partial y}(XY) = 0$$

$$y^3 Y \frac{dX}{dx} + x^2 X \frac{dY}{dy} = 0$$

$\div\ XYx^2y^3$, we get

$$\frac{1}{x^2 X}\frac{dX}{dx} + \frac{1}{y^3 Y}\frac{dY}{dy} = 0$$

Let

$$\frac{1}{x^2 X}\frac{dX}{dx} = -\frac{1}{y^3 Y}\frac{dY}{dy} = k$$

$\Rightarrow \frac{1}{x^2 X}\frac{dX}{dx} = k \quad ; \quad -\frac{1}{y^3 Y}\frac{dY}{dy} = k$

$\Rightarrow \frac{dX}{X} = kx^2\,dx \quad ; \quad \frac{dY}{Y} = -ky^3\,dy$ (is in V.S form)

Integrating,

$\Rightarrow \log X = \frac{kx^3}{3} + c_1' \quad ; \quad \log Y = \frac{-ky^4}{4} + c_2'$

$\Rightarrow X = e^{\frac{kx^3}{3} + c_1'} \quad\quad Y = e^{-\frac{ky^4}{4} + c_2'}$

$= e^{\frac{kx^3}{3}} \cdot e^{c_1'} \quad\quad = e^{-\frac{ky^4}{4}} \cdot e^{c_2'}$

$= c_1 e^{\frac{kx^3}{3}} \quad\quad = c_2 e^{-\frac{ky^4}{4}}$

$\therefore z = XY = \left(c_1 e^{\frac{kx^3}{3}}\right)\left(c_2 e^{-\frac{ky^4}{4}}\right) = c_3 e^{k\left(\frac{x^3}{3} - \frac{y^4}{4}\right)}$

is the required solution of ①.

3) $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$ —— ①

Ans: Let $z = XY$ be the solution of ①.

Then from ①,

$$\frac{\partial^2}{\partial x^2}(XY) - 2\frac{\partial}{\partial x}(XY) + \frac{\partial}{\partial y}(XY) = 0$$

$$\Rightarrow Y\frac{d^2X}{dx^2} - 2Y\frac{dX}{dx} + X\frac{dY}{dy} = 0$$

$\div XY$

$$\frac{1}{X}\left[\frac{d^2X}{dx^2} - 2\frac{dX}{dx}\right] = -\frac{1}{Y}\frac{dY}{dy}$$

Let $\frac{1}{X}\left[\frac{d^2X}{dx^2} - 2\frac{dX}{dx}\right] = -\frac{1}{Y}\frac{dY}{dy} = k$

$\Rightarrow \frac{1}{X}\left[\frac{d^2X}{dx^2} - 2\frac{dX}{dx}\right] = k$; $-\frac{1}{Y}\frac{dY}{dy} = k$

$\Rightarrow \frac{d^2X}{dx^2} - 2\frac{dX}{dx} = kX$; $\frac{dY}{Y} = -k\,dy$ (is in V.S form)

$\Rightarrow (D^2 - 2D - k)X = 0$ —— ② ; $\log Y = -ky + c_3'$ (on integrating)

A.E of ② is ; $Y = e^{-ky + c_3'}$

$m^2 - 2m - k = 0$; $Y = c_3 e^{-ky}$

$\Rightarrow m = \frac{2 \pm \sqrt{4+4k}}{2}$

$= 1 \pm \sqrt{1+k}$

$\therefore X = c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}$; $Y = c_3 e^{-ky}$

$\therefore z = XY = \left(c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}\right)\left(c_3 e^{-ky}\right)$

is the required solution of ①.

4) $x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} = 0$ ——— ①

Ans: Let $u = XY$ ——② be the solution of ①.

Substituting ② in ①, we get

$$x^2 \frac{\partial (XY)}{\partial x} + y^2 \frac{\partial (XY)}{\partial y} = 0$$

$$\Rightarrow x^2 Y \frac{dX}{dx} + y^2 X \frac{dY}{dy} = 0$$

$\div XY$, gives

$$\Rightarrow \frac{x^2}{X} \frac{dX}{dx} + \frac{y^2}{Y} \frac{dY}{dy} = 0$$

$$\Rightarrow \frac{x^2}{X} \frac{dX}{dx} = -\frac{y^2}{Y} \frac{dY}{dy} = k \text{ (say)}$$

$$\Rightarrow \frac{x^2}{X} \frac{dX}{dx} = k \quad ; \quad -\frac{y^2}{Y} \frac{dY}{dy} = k$$

$$\Rightarrow \frac{1}{X} dX = \frac{k\,dx}{x^2} \quad ; \quad \frac{1}{Y} dY = -\frac{k}{y^2} dy$$

Integrating we get,

$$\log X = \frac{-k}{x} + c_1 \quad ; \quad \log Y = \frac{k}{y} + c_2$$

$$\Rightarrow X = e^{-\frac{k}{x} + c_1} \quad ; \quad Y = e^{k/y + c_2}$$

$\Rightarrow$ from ②, $u = XY$

$$u = \left(e^{-\frac{k}{x} + c_1}\right)\left(e^{\frac{k}{y} + c_2}\right)$$

$$= e^{k/y - k/x} \cdot e^{c_1 + c_2}$$

$$u = c_3 \, e^{k/y - k/x} = c_3 \, e^{k(1/y - 1/x)}$$

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5) $u_{xx} - u_{yy} = 0$ —— ①

Ans: Let $u = XY$ —— ② be the solution of ①.

Substituting ② in ①, we get

$$\frac{\partial^2 (XY)}{\partial x^2} - \frac{\partial^2 (XY)}{\partial y^2} = 0$$

$$Y \frac{d^2 X}{dx^2} - X \frac{d^2 Y}{dy^2} = 0$$

$\div XY$, we get

$$\frac{1}{X}\frac{d^2 X}{dx^2} - \frac{1}{Y}\frac{d^2 Y}{dy^2} = 0$$

$$\Rightarrow \frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{Y}\frac{d^2 Y}{dy^2} = k \text{ (say)}$$

$$\Rightarrow \frac{1}{X}\frac{d^2 X}{dx^2} = k \quad ; \quad \frac{1}{Y}\frac{d^2 Y}{dy^2} = k$$

$$\Rightarrow \frac{d^2X}{dx^2} = kX \quad ; \quad \frac{d^2Y}{dy^2} = kY$$

$$\Rightarrow (D^2 - k)X = 0 \quad \left(D = \frac{d}{dx}\right) \quad ; \quad (D^2 - k)Y \quad \left(D = \frac{d}{dy}\right)$$

A.E is ; A.E is

$m^2 - k = 0$; $m^2 - k = 0$

$\Rightarrow m^2 = k$; $\Rightarrow m = k$

$\Rightarrow m = \pm\sqrt{k}$; $\Rightarrow m = \pm\sqrt{k}$

$\Rightarrow X = c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}$; $\Rightarrow Y = c_3 e^{\sqrt{k}y} + c_4 e^{-\sqrt{k}y}$

$\therefore$ from (2), $u = XY$

$$u = \left(c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}\right)\left(c_3 e^{\sqrt{k}y} + c_4 e^{-\sqrt{k}y}\right)$$

be the required G.S of (1)