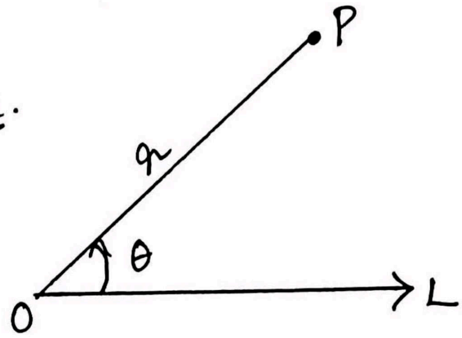
DIFFERENTIAL CALCULUS

Polar coordinates:

The initial reference point 'O' in the plane is called the pole. The line OL drawn through O is called the initial line.



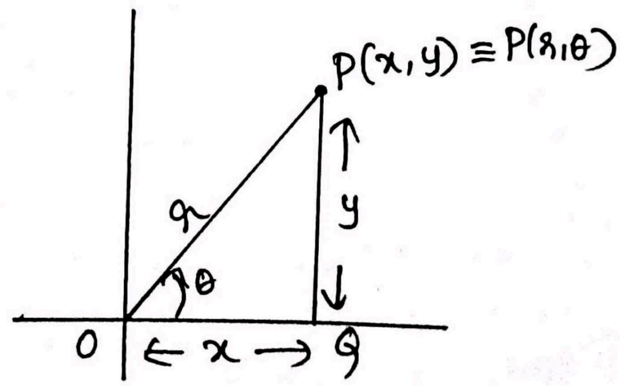
Let 'P' be any given point. Then the length OP denoted by 'r' is called the radius vector and the angle $\theta = \angle LOP$ (measured in anticlockwise direction) is called the vectorial angle.

The pair r & θ represented by $P(r,\theta)$ is called the polar coordinates of the point P.

Note: 'r' is positive ($\because$ it is the length) & θ lies between 0 and 2π.

Relation between the cartesian coordinates (x,y) & the polar coordinates (r,θ):

From the figure, we have



$\cos\theta = \dfrac{x}{r} \Rightarrow \boxed{x = r\cos\theta}$ — (1)

$\sin\theta = \dfrac{y}{r} \Rightarrow \boxed{y = r\sin\theta}$ — (2)

Further, $x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2$

$$\Rightarrow \boxed{r = \sqrt{x^2+y^2}}$$

& $\frac{(2)}{(1)}$ is $\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta}$ $\Rightarrow \tan\theta = \left(y/x\right)$

$$\Rightarrow \boxed{\theta = \tan^{-1}\left(y/x\right)}$$

From the above relations, it is evident that 'r' is a function of θ. The equation $r=f(\theta)$ or $f(r,\theta)$ = a constant, is called the equation in the polar form or polar curve of the curve

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Angle between radius vector & tangent :

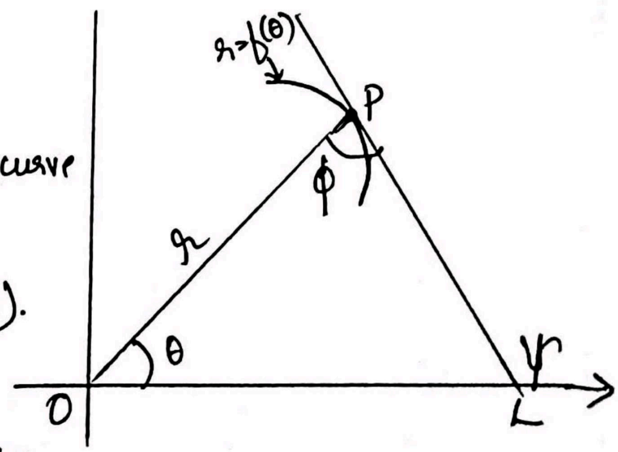
Q: With usual notation prove that $\boxed{\tan\phi = r\frac{d\theta}{dr}}$

Let $P(r,\theta)$ be any point on a polar curve $r=f(\theta)$.

Then $\angle XOP = \theta$ and $OP = r$.

Let PL be the tangent to the curve at P subtending an angle ψ with the initial line (i.e x-axis).

And ϕ be the angle between the radius vector OP and the tangent PL.



i.e $\angle OPL = \phi$

From the figure $\psi = \phi + \theta$

$$\Rightarrow \tan\psi = \tan(\phi+\theta)$$

$$\tan\psi = \frac{\tan\phi + \tan\theta}{1 - \tan\phi\tan\theta} \quad \text{---} (1)$$

Let (x, y) be the cartesian coordinates of P, so that $x = r\cos\theta$, $y = r\sin\theta$.

By the geometrical meaning of derivative

$$\frac{dy}{dx} = \text{slope of PL} = \tan\psi$$

$$\text{i.e } \tan\psi = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r\cos\theta + \sin\theta\frac{dr}{d\theta}}{-r\sin\theta + \cos\theta\frac{dr}{d\theta}}$$

Let $r' = \dfrac{dr}{d\theta}$

$$\Rightarrow \tan\psi = \frac{r\cos\theta + \sin\theta\, r'}{-r\sin\theta + \cos\theta\, r'} \quad \text{---} (2)$$

Dividing Nr & Dr of RHS of (2) by $r'\cos\theta$, we get

$$\tan\psi = \frac{\frac{r}{r'} + \tan\theta}{-\frac{r}{r'}\tan\theta + 1} \quad \text{---} (3)$$

Comparing (1) & (3), we get $\tan\phi = \dfrac{r}{r'}$

$$\Rightarrow \boxed{\tan\phi = \frac{r}{\left(\frac{dr}{d\theta}\right)} = r\frac{d\theta}{dr}}$$

Find the angle between the radius vector and the tangent for the following curves.

<1> $r = a(1-\cos\theta)$; a is a constant.

Ans:

Method 1:

We have $\tan\phi = r \cdot \frac{d\theta}{dr}$

$$= \frac{r}{\left(\frac{dr}{d\theta}\right)}$$

$$= \frac{\cancel{a}(1-\cos\theta)}{\cancel{a}\sin\theta} = \frac{\cancel{2}\sin^{\cancel{2}}\theta/2}{\cancel{2}\cancel{\sin\theta/2}\cos\theta/2}$$

$$\tan\phi = \tan\theta/2$$

$$\Rightarrow \boxed{\phi = \theta/2}$$

'OR' Method 2: Given $r = a(1-\cos\theta)$

$\Rightarrow \log r = \log(a(1-\cos\theta))$

$\log r = \log a + \log(1-\cos\theta)$

diff w.r.t 'θ', we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1-\cos\theta} \cdot \sin\theta = \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}$$

$$= \cot\theta/2$$

Now $\tan\phi = r \cdot \frac{d\theta}{dr} = \frac{1}{\cot\theta/2} = \tan\theta/2 \Rightarrow \boxed{\phi = \theta/2}$

2. $r = a(1+\cos\theta)$

Ans: Given $r = a(1+\cos\theta)$

$$\Rightarrow \frac{dr}{d\theta} = -a\sin\theta$$

$$\therefore \tan\phi = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{\cancel{a}(1+\cos\theta)}{-\cancel{a}\sin\theta} = \frac{\cancel{2}\cos^2\theta/2}{-\cancel{2}\sin\theta/2\,\cancel{\cos\theta/2}}$$

$$\tan\phi = -\cot\theta/2 = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$$\Rightarrow \phi = \frac{\pi}{2}+\frac{\theta}{2}$$

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3. $r = ae^{\theta\cot\alpha}$, α & a are constants.

Ans: Given $r = ae^{\theta\cot\alpha}$

$$\Rightarrow \frac{dr}{d\theta} = a\,e^{\theta\cot\alpha}\cdot\cot\alpha$$

$$\therefore \tan\phi = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{\cancel{ae^{\theta\cot\alpha}}}{\cancel{ae^{\theta\cot\alpha}}\cdot\cot\alpha} = \frac{1}{\cot\alpha} = \tan\alpha$$

$$\Rightarrow \phi = \alpha$$

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4. $\frac{2a}{r} = 1-\cos\theta$

Ans: Given $r = \frac{2a}{1-\cos\theta} \Rightarrow \frac{dr}{d\theta} = -\frac{2a}{(1-\cos\theta)^2}\cdot\sin\theta$

$$\therefore \tan\phi = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{\frac{2a}{(1-\cos\theta)}}{\frac{-2a}{(1-\cos\theta)^2}\sin\theta} = -\frac{(1-\cos\theta)}{\sin\theta}$$

$$\tan\phi = \frac{-(1-\cos\theta)}{\sin\theta} = -\frac{\cancel{2}\sin^{\cancel{2}}\theta/2}{\cancel{2\sin\theta/2}\cdot\cos\theta/2} = \frac{-\sin\theta/2}{\cos\theta/2}$$

$$\tan\phi = -\tan\theta/2 = \tan(\pi - \theta/2)$$

$$\therefore \phi = \pi - \theta/2$$

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5. $a^n = r^n \cos n\theta$, a is constant

<u>Ans:</u> Given $a^n = r^n \cos n\theta$

$$\Rightarrow \log a^n = \log(r^n \cdot \cos n\theta)$$

$$n \log a = n \log r + \log \cos n\theta$$

<u>diff</u> w.r.t 'θ', we get

$$0 = n \frac{1}{r}\frac{dr}{d\theta} + \frac{1}{\cos n\theta} \cdot -\sin n\theta \cdot n$$

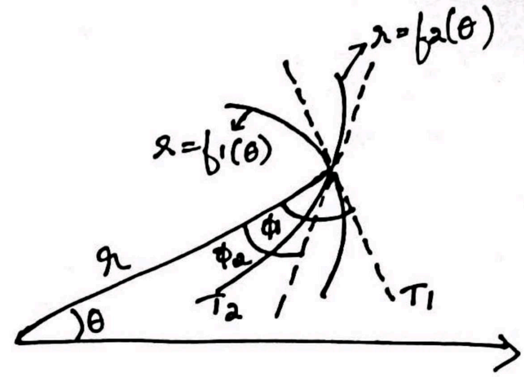
$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = +\tan n\theta$$

$$\Rightarrow \tan\phi = r \cdot \frac{d\theta}{dr} = \frac{1}{\tan n\theta} = \cot n\theta$$

$$= \tan\left(\frac{\pi}{2} - n\theta\right)$$

$$\therefore \phi = \frac{\pi}{2} - n\theta$$

Angle of intersection of two polar curves:



Let $r=f_1(\theta)$ and $r=f_2(\theta)$ be two curves intersecting at a point P.

Let T_1 & T_2 be the tangents drawn to the curve at the point P.

Let ϕ_1 be the angle between the radius vector OP and the tangent T_1 and ϕ_2 be the angle made by the radius vector OP with T_2.

Then the angle between the two tangents T_1 & T_2 is $|\phi_1-\phi_2|$. This angle is determined by using the formula

$$\tan|\phi_1-\phi_2| = |\tan(\phi_1-\phi_2)| = \left|\frac{\tan\phi_1-\tan\phi_2}{1+\tan\phi_1\cdot\tan\phi_2}\right|$$

Note: The angle between $r=f_1(\theta)$ & $r=f_2(\theta)$ at P is $\frac{\pi}{2}$ iff $|\phi_1-\phi_2| = \pi/2$. or equivalently $\tan\phi_1\cdot\tan\phi_2=-1$.

[$\because$ $\tan|\phi_1-\phi_2| = \tan\frac{\pi}{2} = \infty$ & from R.H.S of eqn

$\tan\frac{\pi}{2}=\infty$, when $\tan\phi_1\cdot\tan\phi_2=-1$]

Find the angle of intersection between the following curves

1) $r = a(1+\sin\theta)$ & $r = b(1-\sin\theta)$

Ans: Consider $r = a(1+\sin\theta)$

then $\tan\phi_1 = \dfrac{r}{\left(\frac{dr}{d\theta}\right)} = \dfrac{a(1+\sin\theta)}{a\cos\theta}$

$$= \frac{1+\cos\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)}$$

$$= \frac{2\cos^2\left(\frac{\pi}{4}-\theta/2\right)}{2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cdot\cos\left(\frac{\pi}{4}-\theta/2\right)}$$

$$= \cot\left(\pi/4-\theta/2\right)$$

$$= \tan\left(\frac{\pi}{2}-\left(\frac{\pi}{4}-\theta/2\right)\right)$$

$$\tan\phi_1 = \tan\left(\frac{\pi}{4}+\theta/2\right)$$

$$\Rightarrow \phi_1 = \frac{\pi}{4}+\frac{\theta}{2} \quad \text{--- (1)}$$

Now consider, $r = b(1-\sin\theta)$

$\Rightarrow \tan\phi_2 = \dfrac{r}{\left(\frac{dr}{d\theta}\right)} = \dfrac{b(1-\sin\theta)}{-b\cos\theta} = -\dfrac{(1-\sin\theta)}{\cos\theta}$

$$= \frac{-\left[1-\cos\left(\frac{\pi}{2}-\theta\right)\right]}{\sin\left(\frac{\pi}{2}-\theta\right)}$$

$$\tan\phi_2 = \frac{-2\sin^2\left(\frac{\pi}{4}-\theta/2\right)}{2\sin\left(\frac{\pi}{4}-\theta/2\right)\cos\left(\frac{\pi}{4}-\theta/2\right)}$$

$$= -\tan\left(\frac{\pi}{4}-\theta/2\right)$$

$$\tan\phi_2 = \tan\left(\theta/2-\frac{\pi}{4}\right) \quad \left[\because \tan(-\theta) = -\tan\theta\right.$$

$$\Rightarrow \phi_2 = \frac{\theta}{2}-\frac{\pi}{4} \quad -(2)$$

From (1) & (2),

the angle of intersection of the given two curves is $|\phi_1-\phi_2| = \left|\frac{\pi}{4}+\frac{\theta}{2}-\left(\frac{\theta}{2}-\frac{\pi}{4}\right)\right| = \left|\frac{2\pi}{4}\right| = \frac{\pi}{2}$

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2. $r = 2\sin\theta$ & $r = \sin\theta+\cos\theta$

<u>Ans:</u> Consider $r = 2\sin\theta$

$$\Rightarrow \tan\phi_1 = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{2\sin\theta}{2\cos\theta} = \tan\theta$$

$$\Rightarrow \phi_1 = \theta$$

Now consider $r = \sin\theta+\cos\theta$

$$\Rightarrow \tan\phi_2 = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{\sin\theta+\cos\theta}{\cos\theta-\sin\theta} = \frac{\frac{\sin\theta+\cos\theta}{\cos\theta}}{\frac{\cos\theta-\sin\theta}{\cos\theta}}$$

$$= \frac{\tan\theta+1}{1-\tan\theta} = \tan\left(\frac{\pi}{4}+\theta\right)$$

$$\Rightarrow \phi_2 = \frac{\pi}{4}+\theta$$

$\therefore$ the angle of intersection between two curves is

$$|\phi_1 - \phi_2| = \left|\theta - \left(\frac{\pi}{4} + \theta\right)\right| = \left|-\frac{\pi}{4}\right| = \frac{\pi}{4}.$$

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3. $r = a\log\theta$, $r = \dfrac{a}{\log\theta}$

<u>Ans:</u> Consider $r = a\log\theta$

$$\Rightarrow \tan\phi_1 = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{\cancel{a}\log\theta}{\cancel{a}\left(\frac{1}{\theta}\right)} = \theta\log\theta$$

Now by $r = \dfrac{a}{\log\theta}$

$$\tan\phi_2 = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{\frac{\cancel{a}}{\cancel{\log\theta}}}{-\frac{\cancel{a}}{(\log\theta)^{\cancel{2}}}\cdot\frac{1}{\theta}} = -\theta\log\theta$$

We have,

$$\tan|\phi_1 - \phi_2| = |\tan(\phi_1 - \phi_2)| = \left|\frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1\cdot\tan\phi_2}\right|$$

$$= \left|\frac{\theta\log\theta + \theta\log\theta}{1 + (\theta\log\theta)(-\theta\log\theta)}\right|$$

$$= \left|\frac{2\theta\log\theta}{1 - \theta^2(\log\theta)^2}\right| \quad \text{---} (*)$$

At the point of intersection,

$$a\log\theta = \frac{a}{\log\theta} \Rightarrow (\log\theta)^2 = 1 \Rightarrow \log\theta = 1$$
$$\Rightarrow \theta = e$$

$\therefore$ from (*)

$$\tan|\phi_1-\phi_2| = \left|\frac{2e}{1-e^2}\right|$$

$$\Rightarrow |\phi_1-\phi_2| = \tan^{-1}\left(\left|\frac{2e}{1-e^2}\right|\right)$$ is the angle b/n given 2 curves

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4 $r=\dfrac{a\theta}{1+\theta}$ & $r=\dfrac{a}{1+\theta^2}$

Ans: Consider $r=\dfrac{a\theta}{1+\theta}$

$$\Rightarrow \tan\phi_1 = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{\frac{a\theta}{1+\theta}}{\frac{(1+\theta)a-(a\theta)\cdot 1}{(1+\theta)^2}} = \frac{\cancel{\frac{a\theta}{1+\theta}}}{\frac{\cancel{a}}{(1+\theta)^{\cancel{2}}}}$$

$$= \theta(1+\theta)$$

Now by $r=\dfrac{a}{1+\theta^2}$

$$\tan\phi_2 = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{\frac{\cancel{a}}{\cancel{1+\theta^2}}}{-\frac{\cancel{a}}{(1+\theta^2)^{\cancel{2}}}\cdot 2\theta} = -\frac{(1+\theta^2)}{2\theta}$$

We have,

$$\tan|\phi_1-\phi_2| = \left|\frac{\tan\phi_1-\tan\phi_2}{1+\tan\phi_1\cdot\tan\phi_2}\right| = \left|\frac{\theta(1+\theta)+\frac{(1+\theta^2)}{2\theta}}{1-\theta(1+\theta)\frac{(1+\theta^2)}{2\theta}}\right|$$

$$= \left|\frac{2\theta^2(1+\theta)+(1+\theta^2)}{2\theta-\theta(1+\theta)(1+\theta^2)}\right| \quad \text{—(*)}$$

At the point of intersection,

$$\frac{\cancel{\alpha}\theta}{1+\theta} = \frac{\alpha}{1+\theta^2}$$

$$\Rightarrow (1+\theta^2)\theta = 1+\theta$$

$$\Rightarrow \theta+\theta^3 = 1+\theta \quad \Rightarrow \theta^3 = 1 \quad \Rightarrow \theta = 1$$

$$\therefore (*) \Rightarrow \tan|\phi_1-\phi_2| = \left|\frac{4+2}{2-4}\right| = \left|\frac{6}{-2}\right| = |-3| = 3$$

$$\therefore |\phi_1-\phi_2| = \tan^{-1} 3$$

Prove that the following curves intersect each other orthogonally:

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1) $r = \dfrac{a}{1+\cos\theta}$, $r = \dfrac{b}{1-\cos\theta}$

<u>Ans:</u> Consider $r = \dfrac{a}{1+\cos\theta}$

$$\Rightarrow \tan\phi_1 = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{\cancel{\frac{a}{1+\cos\theta}}}{\frac{-\cancel{a}}{(1+\cos\theta)^{\cancel{2}}}\cdot -\sin\theta} = \frac{1+\cos\theta}{\sin\theta}$$

Now consider $r = \dfrac{b}{1-\cos\theta}$

$$\Rightarrow \tan\phi_2 = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{\frac{\cancel{b}}{\cancel{1-\cos\theta}}}{-\frac{\cancel{b}}{(1-\cos\theta)^{\cancel{2}}}\sin\theta} = -\frac{(1-\cos\theta)}{\sin\theta}$$

$$\Rightarrow \tan\phi_1 \cdot \tan\phi_2 = \frac{1+\cos\theta}{\sin\theta} \cdot \frac{-(1-\cos\theta)}{\sin\theta} = -\frac{1-\cos^2\theta}{\sin^2\theta}$$

$\Rightarrow \tan\phi_1 \cdot \tan\phi_2 = -\frac{\sin^2\theta}{\sin^2\theta} = -1$

$\Rightarrow$ Given two curves intersect each other orthogonally.

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2) $r = a(1+\cos\theta)$, $r = b(1-\cos\theta)$

Ans: Consider $r = a(1+\cos\theta)$

$$\Rightarrow \tan\phi_1 = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{\cancel{a}(1+\cos\theta)}{-\cancel{a}\sin\theta} = \frac{-(1+\cos\theta)}{\sin\theta}$$

Now consider $r = b(1-\cos\theta)$

$$\Rightarrow \tan\phi_2 = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{\cancel{b}(1-\cos\theta)}{\cancel{b}\sin\theta} = \frac{1-\cos\theta}{\sin\theta}$$

$$\Rightarrow \tan\phi_1 \cdot \tan\phi_2 = -\frac{(1+\cos\theta)}{\sin\theta} \cdot \frac{(1-\cos\theta)}{\sin\theta} = -\frac{(1-\cos^2\theta)}{\sin^2\theta} = -1$$

$\Rightarrow$ Given curves intersect at right angles

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3) $r = \frac{a}{\theta}$, $r = a\theta$

Ans: Consider $r = \frac{a}{\theta} \Rightarrow \tan\phi_1 = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{\frac{\cancel{a}}{\cancel{\theta}}}{-\frac{\cancel{a}}{\theta^{\cancel{2}}}} = -\theta$

& $r = a\theta \Rightarrow \tan\phi_2 = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{\cancel{a}\theta}{\cancel{a}} = \theta$

$\Rightarrow \tan\phi_1 \cdot \tan\phi_2 = -\theta^2$

At the point of intersection,

$$\frac{a}{\theta} = a\theta \Rightarrow \theta^2 = 1$$

$\Rightarrow \tan\phi_1 \cdot \tan\phi_2 = -1$

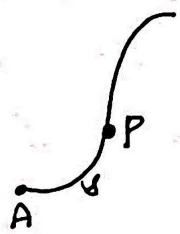
$\therefore$ given curves intersect at right angles

H.W

4. $r^n = a^n \cos n\theta$, $r^n = b^n \sin n\theta$

Arc Length:

Consider a curve C in xy-plane. Let A be a fixed point & P be a variable point on C. Then the distance of P from A along the curve C, denoted by 's' is called the arc length at P w.r.t A.



Derivative of an arc : No derivation will come under derivative of arc

Cartesian curve : $y = f(x)$

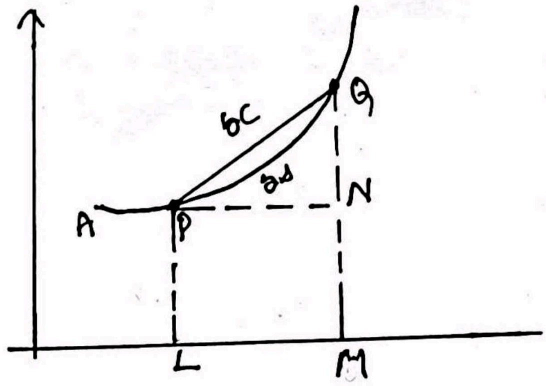
For the curve $y = f(x)$, $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

Proof: Let $P(x, y)$, $Q(x + \delta x, y + \delta y)$ be two neighbouring points on the curve AB.

Let the arc $AP = s$, arc $PQ = \delta s$ & chord $PQ = \delta c$

Draw PL, QM $\perp$rs to the x-axis & $PN \perp QM$.

Then from the right angled ΔPNQ,



$$(PQ)^2 = (PN)^2 + (NQ)^2$$

i.e, $(\delta c)^2 = (\delta x)^2 + (\delta y)^2$

or $\left(\frac{\delta c}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$

$$\therefore \left(\frac{\delta s}{\delta x}\right)^2 = \left(\frac{\delta s}{\delta c} \cdot \frac{\delta c}{\delta x}\right)^2 = \left(\frac{\delta s}{\delta c}\right)^2 \left[1 + \left(\frac{\delta y}{\delta x}\right)^2\right]$$

As $Q \to P$, $\delta c \to 0$, $\delta s \to 0$, $\delta x \to 0$

$$\left(\frac{ds}{dx}\right)^2 = 1\left[1 + \left(\frac{dy}{dx}\right)^2\right]$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$\left[\because y = f(x),\ y + \delta y = f(x + \delta x)\right.$

$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \frac{dy}{dx}$

& $\lim_{\delta c \to 0} \frac{\delta s}{\delta c} = 1$

~~imp~~ **Note:**

I) If the equation of the curve is,

1. $x = f(y)$; then $\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$

$$\left[\because \frac{ds}{dy} = \frac{ds}{dx} \cdot \frac{dx}{dy} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot \frac{dx}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\right]$$

2. in parametric form $x = f(t)$, $y = g(t)$; then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$\left[\because \frac{ds}{dt} = \frac{ds}{dx}\cdot\frac{dx}{dt} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot\frac{dx}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\right.$$

3. <u>in polar form $r = f(\theta)$:</u>

then $$\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$$

4. <u>for $\theta = f(r)$:</u> $$\frac{ds}{dr} = \sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}$$

$$\left[\because \frac{ds}{dr} = \frac{ds}{d\theta}\cdot\frac{d\theta}{dr} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\cdot\frac{d\theta}{dr} = \sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\right.$$

Find $\frac{ds}{d\theta}$ for the following curves:

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1. $r = a(1-\cos\theta)$

<u>Ans:</u> We have for polar curve $r = f(\theta)$

$$\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$$

$$= \sqrt{(a(1-\cos\theta))^2+(a\sin\theta)^2}$$

$$= \sqrt{a^2+a^2\cos\theta + 2a^2\cos\theta + a^2\sin^2\theta}$$

$$\frac{ds}{d\theta} = \sqrt{a^2 + a^2(\cos^2\theta + \sin^2\theta) - 2a^2\cos\theta}$$

$$= \sqrt{2a^2 - 2a^2\cos\theta}$$

$$= \sqrt{2a^2(1-\cos\theta)}$$

$$= \sqrt{2}\, a \sqrt{1-\cos\theta}$$

$$= \sqrt{2}\, a \sqrt{2\sin^2\theta/2} = \sqrt{2}\, a \sqrt{2}\, \sin\theta/2$$

$$\frac{ds}{d\theta} = 2a\sin\theta/2$$

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2. $r^2 = a^2\cos 2\theta$

<u>Ans:</u> We have $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$

$$\frac{ds}{d\theta} = \sqrt{a^2\cos 2\theta + \left(\frac{-a^2\sin 2\theta}{r}\right)^2}$$

$r^2 = a^2\cos 2\theta$

$2r\frac{dr}{d\theta} = -2a^2\sin 2\theta$

$\Rightarrow \frac{dr}{d\theta} = \frac{-a^2\sin 2\theta}{r}$

$$= \sqrt{a^2\cos 2\theta + \frac{a^4\sin^2 2\theta}{a^2\cos 2\theta}}$$

$$= \sqrt{\frac{a^4\cos^2 2\theta + a^4\sin^2 2\theta}{a^2\cos 2\theta}}$$

$$= \sqrt{\frac{a^4}{a^2\cos 2\theta}} = a\sqrt{\frac{1}{\cos 2\theta}} = \underline{\underline{a\sqrt{\sec 2\theta}}}$$

3. Find $\frac{ds}{d\theta}$, $\frac{ds}{dx}$, $\frac{ds}{dy}$ for the cycloid

$x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$.

Ans: The given curve is in parametric form.

i.e $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$

$$\therefore \frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

$$= \sqrt{(a(1-\cos\theta))^2 + (a\sin\theta)^2}$$

$$= \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}$$

$$= \sqrt{a^2 + a^2\cos^2\theta - 2a^2\cos\theta + a^2\sin^2\theta}$$

$$= \sqrt{2a^2 - 2a^2\cos\theta} = \sqrt{2a^2(1-\cos\theta)}$$

$$= \sqrt{2a^2(2\sin^2\theta/2)} = \sqrt{4a^2\sin^2\theta/2}$$

$$\frac{ds}{d\theta} = 2a\sin\theta/2$$

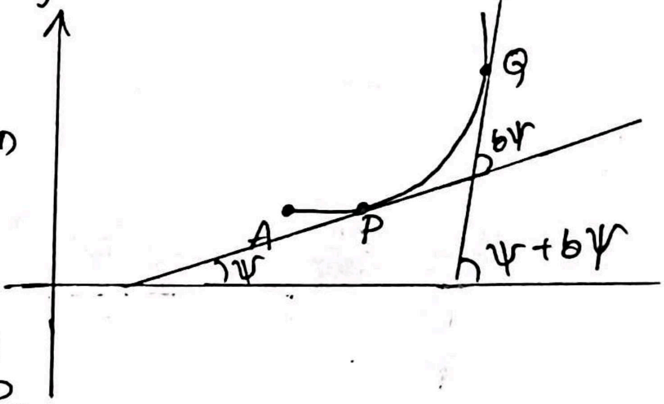
$$\frac{ds}{dx} = \frac{\left(\frac{ds}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{2a\sin\theta/2}{a(1-\cos\theta)} = \frac{2a\sin\theta/2}{2a\sin^2\theta/2}$$

$$= \frac{1}{\sin\theta/2} = \operatorname{cosec}\theta/2$$

$$\frac{ds}{dy} = \frac{\left(\frac{ds}{d\theta}\right)}{\left(\frac{dy}{d\theta}\right)} = \frac{2a \sin\theta/2}{a \sin\theta} = \frac{\cancel{2a} \cancel{\sin\theta/2}}{\cancel{2a} \cancel{\sin\theta/2} \cos\theta/2}$$

$$\frac{ds}{dy} = \frac{1}{\cos\theta/2} = \sec\theta/2$$

<u>Curvature of a curve:</u> is a measure of rate of bending of a curve at a point.



Let $P(x, y)$, $Q(x+\delta x, y+\delta y)$ be two neighbouring points on a curve C

Then the rate of bending of the curve at P is called the curvature of the curve at P, denoted by k and is given by

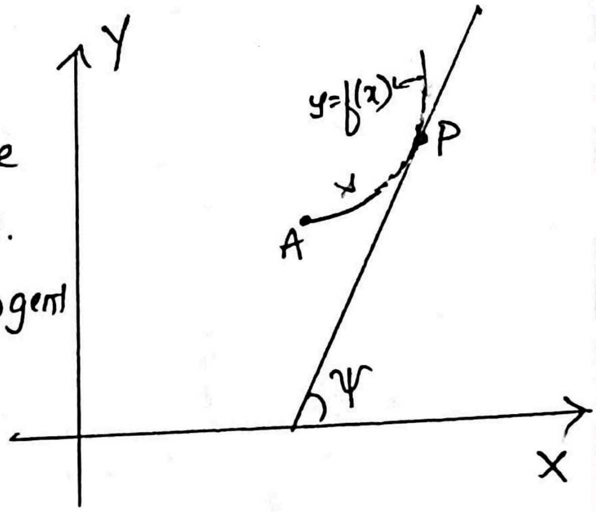
$$k = \frac{d\psi}{ds} = \lim_{Q \to P} \frac{\delta\psi}{\delta s}$$

<u>Radius of curvature:</u> 'ρ'

The reciprocal of the curvature of a curve at any point P is called the radius of curvature at P and is denoted by ρ.

i.e $\rho = \frac{1}{k} = \frac{ds}{d\psi}$

In case of a cartesian curve $y=f(x)$:



Let $P(x,y)$ be a point on the curve such that the arc length of $AP = s$.

Let ψ be an angle made by the tangent at P with the x-axis.

We have, $\tan\psi = \frac{dy}{dx} = y_1$

$$\Rightarrow \tan\psi = y_1 \quad \Rightarrow \psi = \tan^{-1} y_1$$

diff w.r.t x, $\frac{d\psi}{dx} = \frac{1}{1+y_1^2}\frac{dy_1}{dx} = \frac{1}{1+y_1^2} y_2$

$$\therefore \rho = \frac{ds}{d\psi} = \frac{ds}{dx}\cdot\frac{dx}{d\psi} = \sqrt{1+y_1^2}\cdot\frac{1+y_1^2}{y_2}$$

$$\boxed{\rho = \frac{\left[1+y_1^2\right]^{3/2}}{y_2}}, \quad \text{where } y_1 = \frac{dy}{dx} \;\&\; y_2 = \frac{dy_1}{dx}$$

2) In case of parametric form $x=f(t)$, $y=g(t)$:

We have for a cartesian curve $y=f(x)$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} \quad \text{—} (1)$$

Where $y_1 = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'}{x'}$

$$y_2 = \frac{dy_1}{dx} = \frac{\frac{dy_1}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{y'}{x'}\right)}{x'}$$

$$= \frac{\left(\frac{x'y''-y'x''}{(x')^2}\right)}{x'} = \frac{x'y''-y'x''}{(x')^3}$$

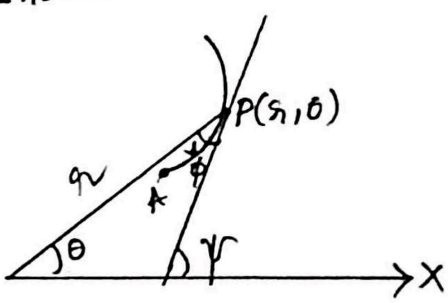
$\therefore$ from ①, $\rho = \dfrac{\left[1 + \left(\frac{y'}{x'}\right)^2\right]^{3/2}}{\dfrac{x'y'' - y'x''}{(x')^3}}$

$$= \frac{\left[\dfrac{(x')^2 + (y')^2}{(x')^2}\right]^{3/2}}{\dfrac{x'y'' - y'x''}{(x')^3}} = \frac{\dfrac{\left[(x')^2 + (y')^2\right]^{3/2}}{(x')^3}}{\dfrac{x'y'' - y'x''}{(x')^3}}$$

$$\boxed{\rho = \frac{\left[(x')^2 + (y')^2\right]^{3/2}}{x'y'' - y'x''}}$$

3. In case of Polar form: $r = f(\theta)$

Consider a polar curve $r = f(\theta)$ as shown in the figure.



We have from the figure,

$\psi = \theta + \phi$

$$\Rightarrow \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds}$$

$$\Rightarrow \frac{1}{\rho} = \frac{d\theta}{ds} + \frac{d\phi}{d\theta} \cdot \frac{d\theta}{ds} = \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right) \quad \text{①}$$

Also, we know that

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{r_1}, \quad \text{where } r_1 = \frac{dr}{d\theta}$$

$$\Rightarrow \phi = \tan^{-1}\left(\frac{r}{r_1}\right)$$

diff w.r.t θ, we get

$$\frac{d\phi}{d\theta} = \frac{1}{1+\left(\frac{r}{r_1}\right)^2} \cdot \left(\frac{r_1 r_1 - r r_2}{r_1^2}\right)$$

$$= \frac{\cancel{r_1^2}}{r_1^2 + r^2}\left(\frac{r_1^2 - r r_2}{\cancel{r_1^2}}\right)$$

$$= \frac{r_1^2 - r r_2}{r_1^2 + r^2} \quad \text{— (2)}$$

Also $\frac{ds}{d\theta} = \sqrt{r^2 + r_1^2}$ — (3)

Substituting (2), (3) in (1), we get

$$\frac{1}{\rho} = \frac{1}{\sqrt{r^2+r_1^2}}\left(1 + \frac{r_1^2 - r r_2}{r_1^2 + r^2}\right)$$

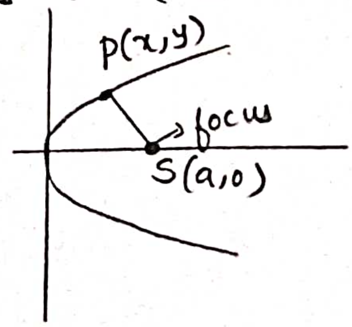
$$= \frac{1}{\sqrt{r^2+r_1^2}}\left(\frac{r_1^2 + r^2 + r_1^2 - r r_2}{r_1^2 + r^2}\right)$$

$$= \frac{r^2 + 2r_1^2 - r r_2}{(r_1^2 + r^2)^{3/2}}$$

$$\therefore \boxed{\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}}$$

---

1) If ρ is the radius of curvature at any point P on the parabola $y^2 = 4ax$ and S be its focus, then show that ρ^2 varies as $(SP)^3$.



Ans: Consider the parabola $y^2 = 4ax$

Diff w.r.t x, we get

$$2yy_1 = 4a$$

$$\Rightarrow y_1 = \frac{4a}{2y} = \frac{2a}{y}$$

$$\Rightarrow y_2 = \frac{dy_1}{dx} = 2a\left(-\frac{1}{y^2}\cdot y_1\right) = -\frac{2a}{y^2}\cdot\frac{2a}{y}$$

$$= -\frac{4a^2}{y^3}$$

We have radius of curvature in cartesian form,

$$\rho = \frac{\left[1+y_1^2\right]^{3/2}}{y_2} = \frac{\left[1+\left(\frac{2a}{y}\right)^2\right]^{3/2}}{-\frac{4a^2}{y^3}} = \frac{\left[\frac{y^2+4a^2}{y^2}\right]^{3/2}}{-\frac{4a^2}{y^3}}$$

$$= \frac{\frac{\left[y^2+4a^2\right]^{3/2}}{\cancel{y^3}}}{-\frac{4a^2}{\cancel{y^3}}} = \frac{\left[4ax+4a^2\right]^{3/2}}{-4a^2}$$

$$= \frac{\left[4a(x+a)\right]^{3/2}}{-4a^2}$$

$$\Rightarrow \rho^2 = \frac{(4a)^3(x+a)^3}{16a^4} = \frac{4}{a}(x+a)^3 \quad \text{—— (1)}$$

By the distance formula,

$$SP = \sqrt{(x-a)^2+(y-0)^2} = \sqrt{x^2-2ax+a^2+y^2}$$

$$= \sqrt{x^2-2ax+a^2+4ax} = \sqrt{x^2+2ax+a^2}$$

$$= \sqrt{(x+a)^2} = x+a$$

$\therefore$ from (1), $\rho^2 = \frac{4}{a}(SP)^3 \Rightarrow \rho^2 \propto (SP)^3$ $\left(\because \frac{4}{a} \text{ is constant}\right)$

$\Rightarrow$ ρ^2 varies as $\underline{(SP)^3}$

Show that the radius of curvature at any point of the cardioid $r = a(1-\cos\theta)$ varies as $\sqrt{r}$.

Ans: Given $r = a(1-\cos\theta)$

R.O.C in polar form is,

$$\rho = \frac{[r^2 + r_1^2]^{3/2}}{r^2 + 2r_1^2 - r r_2}$$

Where $r_1 = \frac{dr}{d\theta} = a\sin\theta$

$\Rightarrow r_2 = \frac{dr_1}{d\theta} = a\cos\theta$

$$\therefore \rho = \frac{[(a(1-\cos\theta))^2 + (a\sin\theta)^2]^{3/2}}{(a(1-\cos\theta))^2 + 2a^2\sin^2\theta - a^2\cos\theta + a^2\cos^2\theta}$$

$$= \frac{[a^2 + a^2\cos^2\theta - 2a^2\cos\theta + a^2\sin^2\theta]^{3/2}}{a^2 + a^2\cos^2\theta - 2a^2\cos\theta + 2a^2\sin^2\theta - a^2\cos\theta + a^2\cos^2\theta}$$

$$= \frac{[2a^2(1-\cos\theta)]^{3/2}}{3a^2(1-\cos\theta)} = \frac{2\sqrt{2}\, a^3}{3a^2}(1-\cos\theta)^{1/2}$$

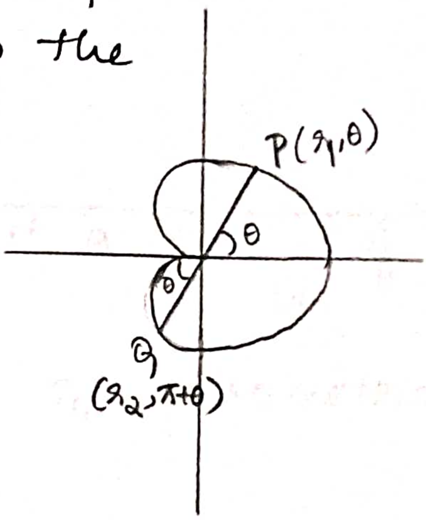
$$= \frac{2\sqrt{2}}{3}\left(\frac{a^{3/2}}{a}\right)(a(1-\cos\theta))^{1/2} = \frac{2\sqrt{2}}{3}\sqrt{a}\sqrt{r}$$

$\rho = \text{constant}\sqrt{r} \Rightarrow \rho \propto \sqrt{r}$

$\Rightarrow \rho$ varies as $\sqrt{r}$

If ρ_1 & ρ_2 are the radii of curvature at the extremities of any chord of the cardioid $r = a(1+\cos\theta)$ which passes through the pole, show that $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$.

Ans: Consider a chord PQ through the pole.



Let ρ_1 and ρ_2 be the radii of curvature at the points P & Q respectively.

R.O.C in polar form is,

$$\rho_1 = \frac{\left[r^2 + r_1^2\right]^{3/2}}{r^2 + 2r_1^2 - r r_2}$$

Where $r_1 = \frac{dr}{d\theta} = -a\sin\theta$

$\Rightarrow r_2 = \frac{dr_1}{d\theta} = -a\cos\theta$

$$\therefore \rho_1 = \frac{\left[(a(1+\cos\theta))^2 + a^2\sin^2\theta\right]^{3/2}}{a^2(1+\cos\theta)^2 + 2a^2\sin^2\theta + (a(1+\cos\theta)\,a\cos\theta)}$$

$$= \frac{\left[a^2 + a^2\cos^2\theta + 2a^2\cos\theta + a^2\sin^2\theta\right]^{3/2}}{a^2 + a^2\cos^2\theta + 2a^2\cos\theta + 2a^2\sin^2\theta + a^2\cos\theta + a^2\cos^2\theta}$$

$$= \frac{\left[2a^2(1+\cos\theta)\right]^{3/2}}{3a^2(1+\cos\theta)} = \frac{2\sqrt{2}\,a^3}{3a^2}(1+\cos\theta)^{1/2}$$

$$\Rightarrow \rho_1^2 = \frac{8a^2}{9}(1+\cos\theta) \quad \text{——} ①$$

At Q, $\rho_2^2 = \frac{8a^2}{9}(1+\cos(\pi+\theta))$ ($\because$ by replace θ by $\pi+\theta$ in ①)

$$\Rightarrow \rho_2^2 = \frac{8a^2}{9}\left(1-\cos\theta\right)$$

$$\Rightarrow \rho_1^2+\rho_2^2 = \left[\frac{8a^2}{9}\left(1+\cos\theta\right)\right]+\left[\frac{8a^2}{9}\left(1-\cos\theta\right)\right]$$

$$\therefore \rho_1^2+\rho_2^2 = \frac{16a^2}{9}.$$

---

4) Find the radius of curvature at a point of the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$

Ans: Given $x = a\cos^3\theta$, $y = a\sin^3\theta$ is in parametric form

$$\therefore \rho = \frac{\left[(x')^2+(y')^2\right]^{3/2}}{x'y''-x''y'}$$

Where

$$x' = \frac{dx}{d\theta} = -3a\cos^2\theta\ \sin\theta$$

$$x'' = \frac{dx'}{d\theta} = -3a\left[\cos^2\theta\cdot\cos\theta + \sin\theta\ 2\cos\theta\,(-\sin\theta)\right]$$

$$= -3a\cos\theta\left[\cos^2\theta - 2\sin^2\theta\right]$$

$$y' = \frac{dy}{d\theta} = 3a\sin^2\theta\ \cos\theta$$

$$y'' = \frac{dy'}{d\theta} = 3a\left(-\sin^2\theta\ \sin\theta + 2\cos\theta\ \sin\theta\ \cos\theta\right)$$

$$= 3a\sin\theta\left[2\cos^2\theta - \sin^2\theta\right]$$

$$\therefore \rho = \frac{\left[\left(-3a\cos^2\theta\ \sin\theta\right)^2+\left(3a\sin^2\theta\ \cos\theta\right)^2\right]^{3/2}}{-18a^2\sin^2\theta\cos^4\theta + 9a^2\cos^2\theta\sin^4\theta - \left(-9a^2\sin^2\theta\cos^4\theta + 18a^2\cos^2\theta\sin^4\theta\right)}$$

$$=$$

$$\Rightarrow \rho = \frac{\left[9a^2 \cos^4\theta \sin^2\theta + 9a^2 \sin^4\theta \cos^2\theta\right]^{3/2}}{-9a^2 \sin^2\theta \cos^4\theta - 9a^2 \cos^2\theta \sin^4\theta}$$

$$= \frac{\left[9a^2 \sin^2\theta \cos^2\theta \left(\cos^2\theta + \sin^2\theta\right)\right]^{3/2}}{-9a^2 \sin^2\theta \cos^2\theta \left(\cos^2\theta + \sin^2\theta\right)}$$

$$= \frac{\left(9a^2 \sin^2\theta \cos^2\theta\right)^{3/2}}{-9a^2 \sin^2\theta \cos^2\theta}$$

$$= -\left(9a^2 \sin^2\theta \cos^2\theta\right)^{1/2} = -3a \sin\theta \cos\theta$$

$\therefore$ $\rho = 3a \sin\theta \cos\theta$ (taking the absolute value)

5) Prove that radius of curvature of a circle is a constant.

Ans: Consider a circle $x^2+y^2=r^2$

Let $r=a \Rightarrow x^2+y^2=a^2$ — ①

① in parametric form is $x = a\cos\theta$, $y = a\sin\theta$

$$\therefore \text{R.O.C} \quad \rho = \frac{\left[(x')^2+(y')^2\right]^{3/2}}{x'y''-x''y'}$$

where $x' = \frac{dx}{d\theta} = -a\sin\theta \Rightarrow x'' = -a\cos\theta$

$y' = \frac{dy}{d\theta} = a\cos\theta \Rightarrow y'' = -a\sin\theta$

$$\therefore \rho = \frac{\left[a^2\sin^2\theta + a^2\cos^2\theta\right]^{3/2}}{a^2\sin^2\theta - (-a^2\cos^2\theta)} = \frac{\left[a^2\right]^{3/2}}{a^2} = (a^2)^{1/2} = a$$

$\therefore$ radius of curvature of a circle is constant

Find the radius of curvature of

$x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$

Ans: Given $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$

$\Rightarrow x' = a(1-\cos\theta)$ $\Rightarrow x'' = a\sin\theta$

$y' = a\sin\theta$ $\Rightarrow y'' = a\cos\theta$

$\therefore$ R.O.C $\rho = \dfrac{\left[(x')^2 + (y')^2\right]^{3/2}}{x'y'' - y'x''}$

$$= \frac{\left[a^2(1-\cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}}{a(1-\cos\theta)\cdot a\cos\theta - a^2\sin^2\theta}$$

$$= \frac{\left[a^2 + a^2\cos^2\theta - 2a^2\cos\theta + a^2\sin^2\theta\right]^{3/2}}{a^2\cos\theta - a^2\cos^2\theta - a^2\sin^2\theta}$$

$$= \frac{\left[a^2 + a^2(\cos^2\theta + \sin^2\theta) - 2a^2\cos\theta\right]^{3/2}}{a^2\cos\theta - a^2(\cos^2\theta + \sin^2\theta)}$$

$$= \frac{\left[2a^2 - 2a^2\cos\theta\right]^{3/2}}{a^2\cos\theta - a^2} = \frac{\left(2a^2(1-\cos\theta)\right)^{3/2}}{-a^2(1-\cos\theta)}$$

$$= \frac{(2a^2)^{3/2}}{-a^2}(1-\cos\theta)^{1/2} = \frac{2\sqrt{2}\,a^3}{-a^2}\left(2\sin^2\theta/2\right)^{1/2}$$

$$= -2\sqrt{2}\,a\left(\sqrt{2}\,\sin\theta/2\right) = \underline{-4a\sin\theta/2}$$

$\therefore$ radius of curvature $= \underline{4a\sin\theta/2}$

Fundamental Theorems:

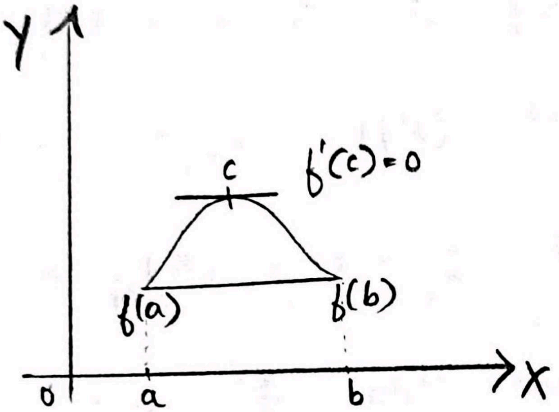
1. **Rolle's theorem:**

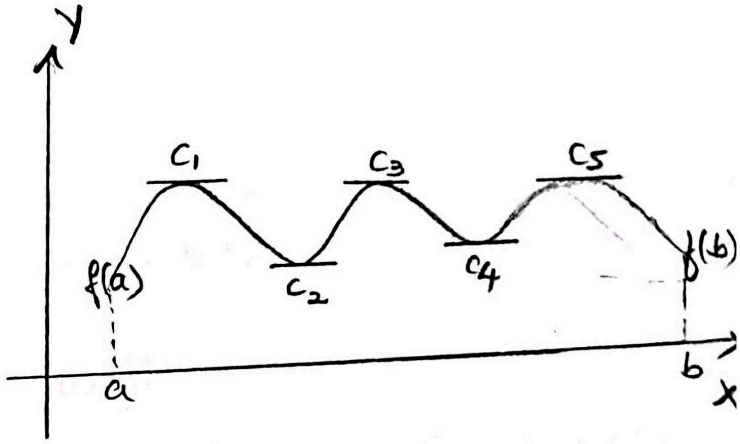
If (i) $f(x)$ is continuous in closed interval $[a,b]$.

(ii) $f'(x)$ exists $\forall x \in (a,b)$ (open interval)

(iii) $f(a) = f(b)$,

then there exists at least one value of $c \in (a,b)$ such that $f'(c) = 0$.





2. **Lagrange's Mean Value theorem: (LMVT)**

If (i) $f(x)$ is continuous in $[a,b]$ and

(ii) $f'(x)$ exists in (a,b), then $\exists$ at least one value c of x in (a,b) such that

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

Note: If a function is differentiable at 'a', then fn is continuous at $x=a$.

Using Lagrange's mean value theorem determine 'c' in the following problems:

1) $f(x) = e^x$ in $[0,1]$

Ans: Given $f(x) = e^x \Rightarrow f'(x) = e^x$, exists in $(0,1)$

$\Rightarrow f(x)$ is continuous in $[0,1]$.

Then by LMVT, $\exists$ atleast one value $c \in (0,1)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(1)-f(0)}{1-0}$$

$$e^c = e^1 - e^0$$

$\Rightarrow e^c = e-1 \Rightarrow c = \log_e(e-1) = 0.5413 \in (0,1)$

Hence satisfies LMVT.

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2) $f(x) = \sin^{-1}x$ in $[0,1]$

<u>Ans:</u> Given $f(x) = \sin^{-1}x$

$\Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}$ exists in $(0,1)$

$\Rightarrow f(x)$ is continuous in $[0,1]$

Then by LMVT, $\exists$ atleast one value $c \in (0,1)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(1)-f(0)}{1-0}$$

$$\frac{1}{\sqrt{1-c^2}} = \sin^{-1}1 - \sin^{-1}0$$

$$\frac{1}{\sqrt{1-c^2}} = \frac{\pi}{2} - 0 \Rightarrow \sqrt{1-c^2} = \frac{2}{\pi}$$

$\Rightarrow 1-c^2 = \frac{4}{\pi^2} \Rightarrow c^2 = 1-\frac{4}{\pi^2}$

$\Rightarrow c = \pm\sqrt{1-\frac{4}{\pi^2}} = \pm 0.771$

$\therefore$ $c = \underline{0.771} \in (0,1)$

3) $f(x) = \log x$ in $[e, e^2]$

Ans: Given $f(x) = \log x$

$\Rightarrow f'(x) = \frac{1}{x}$ exists in (e, e^2)

$\Rightarrow f(x)$ is continuous in $[e, e^2]$,

then by LMVT $\exists$ atleast one value $c \in (e, e^2)$ such that $f'(c) = \frac{f(b) - f(a)}{b-a}$

$$\Rightarrow \frac{1}{c} = \frac{\log e^2 - \log e}{e^2 - e}$$

$$\Rightarrow \frac{1}{c} = \frac{2-1}{e^2 - e} \Rightarrow \frac{1}{c} = \frac{1}{e^2 - e}$$

$$\Rightarrow c = e^2 - e \cong 4.67 \in (e, e^2)$$

H.W 4) $f(x) = x^3 - 2x^2$ in $[2, 5]$

Ans: $c = 3.629$

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5) $f(x) = \tan^{-1} x$ in $[0, 1]$

Ans: $c = 0.5227$

---

3. Cauchy's Mean Value Theorem (CMVT):

If (i) $f(x)$ & $g(x)$ are continuous in $[a, b]$

(ii) $f'(x)$ & $g'(x)$ exist in (a, b)

(iii) $g'(x) \neq 0, \forall x \in (a, b)$,

then there exist at least one value c of x in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Proof: Let us consider a function

$\phi(x) = f(x) - k\,g(x)$, where k is a constant to be determined.

Since $f(x)$ & $g(x)$ are continuous in $[a,b]$, $\phi(x)$ is also continuous in $[a,b]$.

Again since $f'(x)$ & $g'(x)$ exist in (a,b), $\phi'(x) = f'(x) - k\,g'(x)$ also exists in (a,b).

Now $\phi(a) = f(a) - k\,g(a)$ & $\phi(b) = f(b) - k\,g(b)$

$\Rightarrow \phi(a) = \phi(b)$ if

$f(a) - k\,g(a) = f(b) - k\,g(b)$ if

$k\{g(b) - g(a)\} = f(b) - f(a)$ if

$$k = \frac{f(b)-f(a)}{g(b)-g(a)} \quad \text{—①}$$

Thus if $k = \dfrac{f(b)-f(a)}{g(b)-g(a)}$, then $\phi(a) = \phi(b)$

& hence $\phi(x)$ satisfies all the conditions of Rolle's theorem.

$\Rightarrow \exists$ atleast one value $c \in (a,b)$ such that $\phi'(c) = 0$

$\Rightarrow f'(c) - k\,g'(c) = 0 \Rightarrow k = \dfrac{f'(c)}{g'(c)}$ —②

From ① & ②,
$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Verify CMVT for the functions:

1) $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{x}$ in $[a,b]$, $b>a>0$

Ans: Given $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{x}$

$\Rightarrow f'(x) = -\frac{2}{x^3}$ & $g'(x) = \frac{-1}{x^2}$, which exist in (a,b)

$\Rightarrow f(x)$ & $g(x)$ are continuous in $[a,b]$.

Also $g'(x) = \frac{-1}{x^2} \neq 0 \quad \forall x \in (a,b)$

$\therefore$ by CMVT, $\exists\, c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\Rightarrow \frac{-\frac{2}{c^3}}{-\frac{1}{c^2}} = \frac{\frac{1}{b^2}-\frac{1}{a^2}}{\frac{1}{b}-\frac{1}{a}}$$

$$\Rightarrow \frac{2}{c} = \frac{\frac{a^2-b^2}{a^2b^2}}{\frac{a-b}{ab}} \Rightarrow \frac{2}{c} = \frac{\frac{\cancel{(a-b)}(a+b)}{a^2b^2}}{\frac{\cancel{(a-b)}}{ab}}$$

$$\Rightarrow \frac{2}{c} = \frac{a+b}{ab}$$

$$\Rightarrow c = \frac{2ab}{a+b} \in (a,b)$$

($\because c = \frac{2ab}{a+b}$ is the harmonic mean of a & b.)

---

2) $f(x) = e^x$, $g(x) = e^{-x}$ in $[a,b]$

Ans: Given $f(x) = e^x$ & $g(x) = e^{-x}$ in $[a,b]$

$\Rightarrow f'(x) = e^x$ & $g'(x) = -e^{-x}$, exist in (a,b)

$\Rightarrow f(x)$ & $g(x)$ are continuous in $[a,b]$

Also $g'(x) = -e^{-x} \neq 0 \quad \forall x \in (a,b)$

$\therefore$ by CMVT, $\exists$ atleast one $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

$$\Rightarrow \frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}} \Rightarrow \frac{e^c}{-\frac{1}{e^c}} = \frac{e^b - e^a}{\left(\frac{1}{e^b} - \frac{1}{e^a}\right)}$$

$$\Rightarrow -e^{2c} = \frac{e^b - e^a}{\left(\frac{e^a - e^b}{e^a e^b}\right)} = \frac{-\cancel{(e^a - e^b)}}{\left(\frac{\cancel{e^a - e^b}}{e^{a+b}}\right)} = -e^{a+b}$$

$$\Rightarrow 2c = a+b \Rightarrow c = \frac{a+b}{2} \in (a,b)$$

$(\because c = \frac{a+b}{2}$ is the arithmetic mean of a & b).

---

3) $f(x) = x^3$, $g(x) = x^2$ in $[1,2]$

Ans: Here $f'(x) = 3x^2$ & $g'(x) = 2x$ exist in $(1,2)$

$\Rightarrow f(x)$ & $g(x)$ are continuous in $[1,2]$.

Also $g'(x) = 2x \neq 0$ in $(1,2)$.

$\therefore$ By CMVT, $\exists$ atleast one $c \in (1,2)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)} \Rightarrow \frac{3c^{\cancel{2}}}{2\cancel{c}} = \frac{2^3 - 1^3}{4-1} = \frac{7}{3} \Rightarrow c = \frac{14}{9} \in (1,2)$$

H.W 4) $f(x) = x^4$, $g(x) = x^2$ in $[a,b]$ (Ans: $c = \sqrt{\frac{a^2+b^2}{2}} \in (a,b)$

——— x ——— x ——— x ———

4. Find $\frac{ds}{dx}$ for $y^2 = 4ax$

<u>Ans:</u> Given $y^2 = 4ax$ — ①

We have $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$ — ⊛

<u>diff</u> ① w.r.t x, we get

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$$

∴ from ⊛

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{2a}{y}\right)^2}$$

$$= \sqrt{1+\frac{4a^2}{y^2}}$$

$$= \sqrt{1+\frac{4a^2}{4ax}} \quad (\because \text{from ①}$$

$$= \sqrt{1+\frac{a}{x}}$$

Taylor's series expansion for a function of single variable.

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

about the point 'a'.

When $a=0$, we have

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

this is called Maclaurin's series expansion of $f(x)$.

---

1. Obtain Taylor's series of $\sin x$ in powers of $(x - \pi/2)$ up to terms containing $(x-\pi/2)^4$.

Ans: Given $f(x) = \sin x$ & $a = \frac{\pi}{2} \Rightarrow f(\frac{\pi}{2}) = 1$

$f'(x) = \cos x \Rightarrow f'(\frac{\pi}{2}) = \cos \pi/2 = 0$

$f''(x) = -\sin x \Rightarrow f''(\frac{\pi}{2}) = -1$

$f'''(x) = -\cos x \Rightarrow f'''(\frac{\pi}{2}) = 0$

$f^{iv}(x) = \sin x \Rightarrow f^{iv}(\frac{\pi}{2}) = 1$

The Taylor's series expansion of $f(x)$ upto 4^{th} power is

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^4}{4!} f^{iv}(a) + \cdots$$

$$\sin x = 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} - \cdots$$

2. Obtain Taylor's expansion of $\log x$ about $x=1$ upto three non-vanishing terms.

Ans: The Taylor's series expansion of $f(x)$ is given by,

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^4}{4!} f^{iv}(a) + \cdots$$

Where $a = 1$, $f(x) = \log x \Rightarrow f(1) = \log 1 = 0$

$f'(x) = \frac{1}{x} \left[= x^{-1}\right] \Rightarrow f'(1) = 1$

$f''(x) = \frac{-1}{x^2} \left[= -x^{-2}\right] \Rightarrow f''(1) = -1$

$f'''(x) = \frac{2}{x^3} \left[= 2x^{-3}\right] \Rightarrow f'''(1) = 2$

$f^{iv}(x) = \frac{-6}{x^4} \Rightarrow f^{iv}(1) = -6$

$$\therefore \log x = \frac{(x-1)}{1!} - \frac{(x-a)^2}{2!} + 2\frac{(x-a)^3}{3!} - 6\frac{(x-a)^4}{4!} + \cdots$$

$$\Rightarrow \log x = \frac{(x-1)}{1} - \frac{(x-a)^2}{2} + \frac{(x-a)^3}{3} - \frac{(x-a)^4}{4} + \cdots$$

II. Obtain the Maclaurin's expansion of the following

1. e^x

Ans: Maclaurin's series expansion of $f(x)$ is given by

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \cdots$$

Where $f(x) = e^x \Rightarrow f(0) = e^0 = 1$

$f'(x) = e^x \Rightarrow f'(0) = 1$

$f''(x) = e^x \Rightarrow f''(0) = 1$

$f'''(x) = e^x \Rightarrow f'''(0) = 1$

$f^{iv}(x) = e^x \Rightarrow f^{iv}(0) = 1$

$$\therefore e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

---

2. $\sin x$

Ans: The Maclaurin's expansion of $f(x)$ is given by

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

Here $f(x) = \sin x \Rightarrow f(0) = \sin 0 = 0$

$f'(x) = \cos x \Rightarrow f'(0) = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$

$f^{iv}(x) = \sin x \Rightarrow f^{iv}(0) = 0$

$$\therefore \sin x = \frac{x}{1!} - \frac{x^3}{3!} + \cdots$$

H.W 3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

---

4. $\tan x$

Ans: The Maclaurin's expansion of $f(x)$ is given by

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

Here $f(x) = \tan x \Rightarrow f(0) = \tan 0 = 0$

$\Rightarrow f'(x) = \sec^2 x \Rightarrow f'(0) = \sec^2 0 = 1$

$\Rightarrow f''(x) = 2 \sec x (\sec x \tan x) \Rightarrow f''(0) = 0$

$\Rightarrow f'''(x) = 2\left[\sec^2 x \cdot \sec^2 x + \tan x (2 \sec x \cdot \sec x \cdot \tan x)\right]$

$\Rightarrow f'''(0) = 2[1+0] = 2$

$$\therefore \tan x = \frac{x}{1!} + 2\frac{x^3}{3!} + \cdots$$

$$= \frac{x}{1} + \frac{x^3}{3} + \cdots$$

---

5. $\tan^{-1} x$

Ans: Maclaurins series expansion of $f(x)$ is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

Here $f(x) = \tan^{-1} x \Rightarrow f(0) = \tan^{-1} 0$

$f'(x) = \frac{1}{1+x^2} \Rightarrow f'(0) = 1$

$f''(x) = -\frac{1}{(1+x^2)^2} \cdot 2x \Rightarrow f''(0) = 0$

$f'''(x) = -2\left[\frac{1}{(1+x^2)^2} \cdot 1 + x \frac{(-2)}{(1+x^2)^3} \cdot 2x\right] \Rightarrow f'''(0) = -2$

$$\therefore \tan^{-1} x = \frac{x}{1!} - 2\left(\frac{x^3}{3!}\right) + \cdots = \frac{x}{1} - \frac{x^3}{3} + \cdots$$

---

6. $\log(1+x)$

Ans: Maclaurins series expansion of $f(x)$ is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

Here $f(x) = \log(1+x) \Rightarrow f(0) = \log 1 = 0$

$$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} \Rightarrow f''(0) = -1$$

$$\Rightarrow f'''(0) = 2$$

$$f'''(x) = \frac{2}{(1+x)^3}$$

$$\Rightarrow f^{iv}(0) = -6$$

$$f^{iv}(x) = \frac{-6}{(1+x)^4}$$

$$\therefore \log(1+x) = \frac{x}{1!} - \frac{x^2}{2!} + 2\frac{x^3}{3!} - 6\frac{x^4}{4!} + \cdots$$

$$\log(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

H.W
7. $e^x \cos x$

H.W
8. $e^{\sin x}$

3. Obtain Taylor's series of $\cos 2x$ in powers of $\left(x-\frac{\pi}{2}\right)$ up to terms containing $\left(x-\frac{\pi}{2}\right)^4$.

Ans: The Taylor's series expansion of

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Where $f(x) = \cos 2x \Rightarrow f\left(\frac{\pi}{2}\right) = -1$

$f'(x) = -\sin 2x\,(2) \Rightarrow f'\left(\frac{\pi}{2}\right) = 0$

$f''(x) = -\cos 2x\,(4) \Rightarrow f''\left(\frac{\pi}{2}\right) = 4$

$f'''(x) = \sin 2x\,(8) \Rightarrow f'''\left(\frac{\pi}{2}\right) = 0$

$f^{iv}(x) = \cos 2x\,(16) \Rightarrow f^{iv}\left(\frac{\pi}{2}\right) = -16$

$f^{v}(x) = -\sin 2x\,(32) \Rightarrow f^{v}\left(\frac{\pi}{2}\right) = 0$

$$\therefore \cos 2x = -1 + \frac{\left(x-\frac{\pi}{2}\right)^2}{2!}4 + \frac{\left(x-\frac{\pi}{2}\right)^4}{4!}(-16) + \cdots$$

$$= -1 + 2\left(x-\frac{\pi}{2}\right)^2 - 4\left(x-\frac{\pi}{2}\right)^4 + \cdots$$

9. $\log(1+2x)$

<u>Ans:</u> The Maclaurin's series expansion of

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

Given $f(x) = \log(1+2x) \Rightarrow f(0) = \log 1 = 0$

$$\Rightarrow f'(x) = \frac{1}{1+2x}(2) \qquad \Rightarrow f'(0) = 2$$

$$\Rightarrow f''(x) = \frac{-1}{(1+2x)^2}\cdot(4) \qquad \Rightarrow f''(0) = -4$$

$$\Rightarrow f'''(x) = \frac{2}{(1+2x)^3}(8) \qquad \Rightarrow f'''(0) = 16$$

$$\therefore \log(1+2x) = \frac{x}{1!}(2) + \frac{x^2}{2!}(-4) + \frac{x^3}{3!}(16) + \cdots$$

$$= 2x - 2x^2 + \frac{8}{3}x^3 + \cdots$$

Partial Differentiation

Function of two variables:

If for every x & y, a unique value $f(x,y)$ is associated then f is said to be a function of the two independent variables x & y and is denoted by $z = f(x,y)$.

III^{ly} we can write a function of 3 or more variables $x, y, t, \dots$ as $z = f(x, y, t, \dots)$

Partial differentiation:

A partial derivative of a function of several variables is its derivative with respect to one of these variables, with the others held constant.

Partial differentiation is the process of finding partial derivatives.

Let $z = f(x,y)$ be a function of two independent variables x & y. Then $\frac{\partial z}{\partial x}$ gives the partial derivative of z w.r.t x treating y as constant.

III^{ly} the partial derivative of z w.r.t y is $\frac{\partial z}{\partial y}$.

Notation: The partial derivative $\frac{\partial z}{\partial x}$ is also denoted by $\frac{\partial f}{\partial x}$ or $f_x(x,y)$ or f_x so on.

In the same way we can define the partial derivatives of 2nd (or higher) order

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \quad , \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\, \partial x} \quad , \quad \frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

Note: If $z = f(x,y)$, then $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y\, \partial x}$.

I Find the first and second order partial derivatives of

1. $z = x^3 + y^3 - 3axy$

Ans: $\frac{\partial z}{\partial x} = 3x^2 - 3ay$

$\frac{\partial z}{\partial y} = 3y^2 - 3ax$

$\frac{\partial^2 z}{\partial x^2} = 6x$

$\frac{\partial^2 z}{\partial y \partial x} = -3a$

$\frac{\partial^2 z}{\partial y^2} = 6y$

---

2. $z = e^x \cos y$

Ans: $\frac{\partial z}{\partial x} = e^x \cos y$

$\frac{\partial z}{\partial y} = -e^x \sin y$

$\frac{\partial^2 z}{\partial x^2} = e^x \cos y$

$\frac{\partial^2 z}{\partial y^2} = -e^x \cos y$

$$\frac{\partial^2 z}{\partial y \partial x} = -e^x \sin y$$

3. $z = \tan^{-1}\left(\frac{y}{x}\right)$

Ans: $$\frac{\partial z}{\partial x} = \frac{1}{1+(y/x)^2} \cdot y\left(-\frac{1}{x^2}\right) = \frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right) = \frac{x^2}{x^2+y^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = -y\left[-\frac{1}{(x^2+y^2)^2} \cdot 2x\right] = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = x\left[-\frac{1}{(x^2+y^2)^2} \cdot 2y\right] = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 z}{\partial y \partial x} = -\left[\frac{(x^2+y^2) \cdot 1 - y(2y)}{(x^2+y^2)^2}\right] = \frac{-(x^2-y^2)}{(x^2+y^2)^2}$$

---

4. If $u = \log(x^2+y^2)$. Verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Ans: Given $u = \log(x^2+y^2)$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} \cdot 2x \quad \& \quad \frac{\partial u}{\partial y} = \frac{1}{x^2+y^2} \cdot 2y$$

$$\Rightarrow \frac{\partial^2 u}{\partial y \partial x} = 2x\left(-\frac{1}{(x^2+y^2)^2} \cdot 2y\right) = \frac{-4xy}{(x^2+y^2)^2} \quad \text{—} (1)$$

$$\& \quad \frac{\partial^2 u}{\partial x \partial y} = 2y\left(-\frac{1}{(x^2+y^2)^2} \cdot 2x\right) = \frac{-4xy}{(x^2+y^2)^2} \quad \text{—} (2)$$

From (1) & (2), we have $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

If $u = \dfrac{e^{x+y}}{e^x + e^y}$ then show that $u_x + u_y = u$.

Ans: Given $u = \dfrac{e^{x+y}}{e^x+e^y}$

$$\Rightarrow \quad u_x = \frac{\partial u}{\partial x} = \frac{(e^x+e^y)\, e^{x+y} - e^{x+y}(e^x)}{(e^x+e^y)^2}$$

$$= \frac{e^{x+y}\left[e^x + e^y - e^x\right]}{(e^x+e^y)^2} = \frac{e^y \cdot e^{x+y}}{(e^x+e^y)^2}$$

$$\& \quad u_y = \frac{\partial u}{\partial y} = \frac{(e^x+e^y)\, e^{x+y} - e^{x+y}(e^y)}{(e^x+e^y)^2}$$

$$= \frac{e^{x+y}\left[e^x + e^y - e^y\right]}{(e^x+e^y)^2} = \frac{e^x \, e^{x+y}}{(e^x+e^y)^2}$$

Now consider,

$$LHS = u_x + u_y = \frac{e^y\, e^{x+y}}{(e^x+e^y)^2} + \frac{e^x\, e^{x+y}}{(e^x+e^y)^2}$$

$$= \frac{e^{x+y}}{(e^x+e^y)^{\cancel{2}}}\left[\cancel{e^y+e^x}\right] = \frac{e^{x+y}}{e^x+e^y}$$

$$= u = R.H.S.$$

---

6. If $z = f(x+ct) + g(x-ct)$, prove that

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}.$$

Ans: Given $z = f(x+ct) + g(x-ct)$

$$\Rightarrow \frac{\partial z}{\partial x} = f'(x+ct) + g'(x-ct)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + g''(x-ct) \quad \text{— (1)}$$

$$\& \quad \frac{\partial z}{\partial t} = cf'(x+ct) - c\, g'(x-ct)$$

$$\Rightarrow \frac{\partial^2 z}{\partial t^2} = c^2 f''(x+ct) + c^2 g''(x-ct) \quad \text{— (2)}$$

From (1) & (2), $\dfrac{\partial^2 z}{\partial t^2} = c^2 \dfrac{\partial^2 z}{\partial x^2}$

---

7. If $z = e^{ax+by} f(ax-by)$, prove that $b\dfrac{\partial z}{\partial x} + a\dfrac{\partial z}{\partial y} = 2abz$.

Ans: Given $z = e^{ax+by} f(ax-by)$

$$\Rightarrow \frac{\partial z}{\partial x} = a e^{ax+by} f'(ax-by) + af(ax+by)\, e^{ax+by}$$

$$\Rightarrow b\frac{\partial z}{\partial x} = ab\, e^{ax+by} \left[f'(ax-by) + f(ax+by) \right] \quad \text{— (1)}$$

$$\& \quad \frac{\partial z}{\partial y} = -be^{ax+by} f'(ax-by) + bf(ax-by)\, e^{ax+by}$$

$$\Rightarrow a\frac{\partial z}{\partial y} = ab\, e^{ax+by} \left[-f'(ax-by) + f(ax-by) \right] \quad \text{— (2)}$$

Adding (1) & (2), we get

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2ab\, e^{ax+by} f(ax-by)$$

$$= 2ab\, z$$

If $f = \log\left(\frac{x+y}{xy}\right)$. Verify that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

Ans: Given $f = \log\left(\frac{x+y}{xy}\right) = \log\left(\frac{1}{y} + \frac{1}{x}\right)$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{1}{\left(\frac{1}{y}+\frac{1}{x}\right)}\left[-\frac{1}{x^2}\right]$$

$$= \frac{xy}{x+y}\left(-\frac{1}{x^2}\right) = \frac{-y}{x(x+y)}$$

$$\frac{\partial f}{\partial y} = \frac{1}{\left(\frac{1}{y}+\frac{1}{x}\right)}\left(-\frac{1}{y^2}\right) = \frac{-xy}{x+y}\left(\frac{1}{y^2}\right) = \frac{-x}{y(x+y)}$$

$$\frac{\partial^2 f}{\partial y \partial x} = -\frac{1}{x}\left[\frac{(x+y) - y(1)}{(x+y)^2}\right] = -\frac{1}{(x+y)^2} \quad \text{—(1)}$$

$$\& \quad \frac{\partial^2 f}{\partial x \partial y} = -\frac{1}{y}\left[\frac{(x+y) - x}{(x+y)^2}\right] = -\frac{1}{(x+y)^2} \quad \text{—(2)}$$

From (1) & (2), $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

---

9) If $w = \log\sqrt{x^2+y^2}$, find first order partial derivatives of w.

Total Derivative:

Total differential of a function f of three variables x, y, z denoted by df & is defined as

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

Composite function:

If $z = f(x,y)$ where $x = \phi(t)$, $y = \psi(t)$ then z is called a composite function of the single variable t.

i.e, $z \to (x,y) \to t$

If $z = f(x,y)$ where $x = \phi(u,v)$, $y = \psi(u,v)$ then z is called a composite function of the two variables u & v.

Total derivative rule:

If $z = f(x,y)$ where $x = \phi(t)$ & $y = \psi(t)$ then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

, this is called the total derivative of z.

z
x, y
t

Chain rule for partial derivatives

If $z = f(x,y)$ where $x = \phi(u,v)$ & $y = \psi(u,v)$ then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}$$

z
x, y
u, v

Problems:

1) If $u = x^2 - y^2$, where $x = 2r - 3s + 4$ & $y = -r + 8s - 5$.

Find $\frac{\partial u}{\partial r}$ & $\frac{\partial u}{\partial s}$

<u>Ans:</u> Given $u = x^2 - y^2$, $x = 2r - 3s + 4$ & $y = -r + 8s - 5$

$$\left[\begin{array}{c} u \\ \downarrow\ \downarrow \\ x,\ y \\ \downarrow r, s \end{array}\right.$$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= 2x\,(2) + (-2y)\,(-1) = 4x + 2y$$

$$\& \quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= 2x\,(-3) + (-2y)\,(8) = -6x - 16y$$

---

2) If $u = e^x \sin yz$, where $x = t^2$, $y = t - 1$, $z = \frac{1}{t}$

find $\frac{du}{dt}$ at $t = 1$

<u>Ans:</u> Given $u = e^x \sin yz$, $x = t^2$, $y = t - 1$, $z = \frac{1}{t}$

$$\Rightarrow \frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt}$$

$$\left[\begin{array}{c} \because\ u \\ \downarrow\ \downarrow\ \downarrow \\ x\ \ y\ \ z \\ \searrow t \swarrow \end{array}\right.$$

$$= e^x \sin yz\,(2t) + e^x \cos yz\,(z)(1) + e^x \cos yz\,(y)\left(-\frac{1}{t^2}\right)$$

$$= e^{t^2} \sin\left(\frac{t-1}{t}\right)(2t) + e^{t^2} \cos\left(\frac{t-1}{t}\right)\left(\frac{1}{t}\right) + e^{t^2} \cos\left(\frac{t-1}{t}\right)\left(\frac{t-1}{t^2}\right)$$

At $t = 1$, $\frac{du}{dt} = 0 + e\,(1) + 0 = e$

---

3) If $V(r, s, t)$ where $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$,

show that $x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} \; \frac{\partial V}{\partial z} = 0$

<u>Ans:</u> Given $V(r, s, t)$ & $r = \frac{x}{y}$, $s = \frac{y}{z}$,

$t = \frac{z}{x}$.

$$\left[\begin{array}{c} \because\ V \\ \downarrow\ \downarrow\ \downarrow \\ r, s, t \\ \downarrow\ \downarrow\ \downarrow \\ x, y, z \end{array}\right.$$

$$\Rightarrow \frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial V}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial V}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= \frac{\partial V}{\partial r}\left(\frac{1}{y}\right) + \frac{\partial V}{\partial s}(0) + \frac{\partial V}{\partial t}\left(\frac{-z}{x^2}\right)$$

$$\Rightarrow x\frac{\partial V}{\partial x} = \frac{x}{y}\frac{\partial V}{\partial r} - \frac{z}{x}\frac{\partial V}{\partial t} \quad \text{—(1)}$$

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial V}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial V}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial V}{\partial r}\left(\frac{-x}{y^2}\right) + \frac{\partial V}{\partial s}\left(\frac{1}{z}\right) + \frac{\partial V}{\partial t}(0)$$

$$\Rightarrow y\frac{\partial V}{\partial y} = \frac{-x}{y}\frac{\partial V}{\partial r} + \frac{y}{z}\frac{\partial V}{\partial s} \quad \text{—(2)}$$

$$\frac{\partial V}{\partial z} = \frac{\partial V}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial V}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial V}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= \frac{\partial V}{\partial r}(0) + \frac{\partial V}{\partial s}\left(\frac{-y}{z^2}\right) + \frac{\partial V}{\partial t}\left(\frac{1}{x}\right)$$

$$\Rightarrow z\frac{\partial V}{\partial z} = -\frac{y}{z}\frac{\partial V}{\partial s} + \frac{z}{x}\frac{\partial V}{\partial t} \quad \text{—(3)}$$

From (1) + (2) + (3),

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = \frac{x}{y}\frac{\partial V}{\partial r} - \frac{z}{x}\frac{\partial V}{\partial t} - \frac{x}{y}\frac{\partial V}{\partial r} + \frac{y}{z}\frac{\partial V}{\partial s} - \frac{y}{z}\frac{\partial V}{\partial s} + \frac{z}{x}\frac{\partial V}{\partial t}$$

$$= 0$$

4) If $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$, show that

$$\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$$

Let $r = 2x - 3y$, $s = 3y - 4z$, $t = 4z - 2x$

$\Rightarrow u = f(r, s, t)$

$\because$ $u \to r, s, t \to x, y, z$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}(2) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}(-2)$$

$$\Rightarrow \frac{1}{2}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \quad \text{—— (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}\cdot(-3) + \frac{\partial u}{\partial s}(3) + \frac{\partial u}{\partial t}(0)$$

$$\Rightarrow \frac{1}{3}\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \quad \text{—— (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(-4) + \frac{\partial u}{\partial t}(4)$$

$$\Rightarrow \frac{1}{4}\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \quad \text{—— (3)}$$

Consider (1) + (2) + (3), gives

$$\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = +\frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} - \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}$$

$$= 0$$

5) If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

Ans: Let $r = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$

$s = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$

$\left[\because u \to r, s \to x, y, z \right.$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x}$

$= \frac{\partial u}{\partial r}\left(-\frac{1}{x^2}\right) + \frac{\partial u}{\partial s}\left(-\frac{1}{x^2}\right)$

$\Rightarrow x^2 \frac{\partial u}{\partial x} = -\frac{\partial u}{\partial r} - \frac{\partial u}{\partial s}$ —— ①

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y}$

$= \frac{\partial u}{\partial r}\left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial s}(0)$

$\Rightarrow y^2 \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}$ —— ②

$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial z}$

$= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left(\frac{1}{z^2}\right)$

$\Rightarrow z^2 \frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}$ —— ③

Adding ① + ② + ③, we get

$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial r} - \frac{\partial u}{\partial s} + \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} = 0$

H.W.

6) If $u = f\left(xz, \frac{y}{z}\right)$, S.T. $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} - z\frac{\partial u}{\partial z} = 0$

7) If $u = f(x, y)$ & $x = r\cos\theta$, $y = r\sin\theta$.

S.T. $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$

Ans: Given $u = f(x, y)$ & $x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\left[\because \begin{array}{c} u \\ \swarrow \searrow \\ x, y \\ \downarrow \\ r, \theta \end{array}\right.$$

$$= \frac{\partial u}{\partial x} \cos\theta + \frac{\partial u}{\partial y} \sin\theta$$

$$\Rightarrow \left(\frac{\partial u}{\partial r}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 \cos^2\theta + \left(\frac{\partial u}{\partial y}\right)^2 \sin^2\theta + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\sin\theta\cos\theta \quad \text{--- (1)}$$

Also $$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

$$= \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

$$\Rightarrow \left(\frac{\partial u}{\partial \theta}\right)^2 = r^2\sin^2\theta\left(\frac{\partial u}{\partial x}\right)^2 + r^2\cos^2\theta\left(\frac{\partial u}{\partial y}\right)^2 - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} \cdot r^2\cos\theta\sin\theta$$

$$\Rightarrow \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 \sin^2\theta + \left(\frac{\partial u}{\partial y}\right)^2 \cos^2\theta - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\sin\theta\cos\theta \quad \text{--- (2)}$$

Adding (1) & (2), we get

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2\left[\cos^2\theta + \sin^2\theta\right] + \left(\frac{\partial u}{\partial y}\right)^2\left[\sin^2\theta + \cos^2\theta\right]$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

Explicit function: Is a function in which the dependent variable has been given explicitly in terms of the independent variable.
It is denoted by $y = f(x)$

Eg: $y = 5x^3 - 3$

Implicit function: is a function in which the dependent variable has not been given explicitly in terms of the independent variable. It is denoted by $f(x, y) = 0$.

Eg: $x^2 + y^2 = 1$

Differentiation of Implicit function:

If $f(x, y) = C$ is an implicit relation between x & y which defines a differentiable function of x, then

$$df = \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-f_x}{f_y}$$

This is a formula for the 1st differential coefficient of an implicit function.

1) If $xy^3 - yx^3 = 6$ is the equation of a curve, find the slope & the equation of the tangent line at the point $(1, 2)$.

Ans: Let $f(x,y) = xy^3 - yx^3 - 6$, then we have ~~Differentiating~~ $\cancel{xy^3 - yx^3 = 6}$ ~~implicitly w.r.t 'x', we get.~~

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{-(y^3 - 3x^2y)}{(3xy^2 - x^3)}$$

Slope at $(1, 2)$, $\frac{dy}{dx} = \frac{-(8-6)}{3(4)-1} = \frac{-2}{11} = m$

Equation of the tangent line at $(1, 2)$ is

$$y - y_0 = m(x - x_0)$$

$$y - 2 = \frac{-2}{11}(x - 1)$$

$$\Rightarrow 11y - 22 = -2x + 2 \Rightarrow \underline{2x + 11y - 24 = 0}$$

Jacobian:

Jacobian matrix is a matrix of partial derivatives. Jacobian is the determinent of the jacobian matrix. The main use of Jacobian is found in the transfor-mation of coordinates.

Definition: If u & v are functions of two independent variables x & y, then the Jacobian of u, v w.r.t x, y denoted by $J\left(\frac{u,v}{x,y}\right)$ or $\frac{\partial(u,v)}{\partial(x,y)}$ & is defined as

$$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Similarly the Jacobians of 3 function u, v, w of 3 independent variables x, y, z is defined as

$$J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

In a similar way, Jacobian of n functions in n variables can be defined.

Properties of Jacobians

1) If $J = \frac{\partial(u,v)}{\partial(x,y)}$ & $J' = \frac{\partial(x,y)}{\partial(u,v)}$, then $JJ' = 1$

2) Chain rule of Jacobians:

If u, v are function of r, s and r, s are functions of x, y, then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$$

Problems:

1) If $x = r\cos\theta$, $y = r\sin\theta$, find $J = \frac{\partial(x,y)}{\partial(r,\theta)}$

& $J' = \frac{\partial(r,\theta)}{\partial(x,y)}$ & show that $JJ' = 1$.

Ans: Given $x = r\cos\theta$, $y = r\sin\theta$.

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta.$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

We have $r = \sqrt{x^2+y^2}$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$J' = \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{2\sqrt{x^2+y^2}}\, 2x & \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y \\ \frac{1}{1+(y/x)^2}\left(\frac{-y}{x^2}\right) & \frac{1}{1+(y/x)^2}\, \frac{1}{x} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{(x^2+y^2)\sqrt{x^2+y^2}} + \frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$= \frac{1}{\cancel{(x^2+y^2)}\sqrt{x^2+y^2}} \cancel{(x^2+y^2)}$$

$$= \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$$

Consider $J \cdot J' = r\left(\frac{1}{r}\right) = 1$

---

2) If $u = x + \frac{y^2}{x}$, $v = \frac{y^2}{x}$ prove that $JJ' = 1$

Ans: Given $u = x + \frac{y^2}{x}$, $v = \frac{y^2}{x}$ — (1)

$$\therefore J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 - \frac{y^2}{x^2} & \frac{2y}{x} \\ \frac{-y^2}{x^2} & \frac{2y}{x} \end{vmatrix}$$

$$\therefore J = \frac{2y}{x} - \frac{2y^3}{x^3} + \frac{2y^3}{x^3} = \frac{2y}{x}$$

From (1), $u - v = x \Rightarrow \boxed{x = u - v}$

& $v = \frac{y^2}{x} \Rightarrow y^2 = vx$

$= v(u-v)$

$\Rightarrow \boxed{y = \sqrt{v(u-v)}}$

$$\therefore J' = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$J' = \begin{vmatrix} 1 & -1 \\ \frac{1}{2\sqrt{v(u-v)}} \cdot v & \frac{1}{2\sqrt{v(u-v)}} (u-2v) \end{vmatrix}$$

$$= \frac{u-2v}{2\sqrt{v(u-v)}} + \frac{v}{2\sqrt{v(u-v)}}$$

$$= \frac{1}{2\sqrt{v(u-v)}} \left[u-2v+v \right]$$

$$= \frac{1}{2y} (u-v) = \frac{x}{2y}$$

Consider $J \cdot J' = \left(\frac{2y}{x}\right) \cdot \left(\frac{x}{2y}\right) = 1$

---

3. If $x = u(1-v)$, $y = uv$ then S.T. $JJ' = 1$

Ans: Given $x = u(1-v)$, $y = uv$ } — (1)

$\Rightarrow$ $x+y = u \Rightarrow \boxed{u = x+y}$

$y = uv \Rightarrow \boxed{v = \frac{y}{u} = \frac{y}{x+y}}$ } (2)

From (1),

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$J = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = (1-v)u + vu = u$$

From ②,

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ y \cdot \frac{-1}{(x+y)^2} & \frac{(x+y) - y(1)}{(x+y)^2} \end{vmatrix}$$

$$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2}$$

$$= \frac{x+y}{(x+y)^2} = \frac{1}{x+y} = \frac{1}{u}$$

$$\therefore JJ' = u\left(\frac{1}{u}\right) = 1$$

---

4) If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$. S.T.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4.$$

Ans: Given $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

$$= \frac{-yz}{x^2}\left[\frac{x^2yz}{y^2z^2} - \frac{x^2}{yz}\right] - \frac{z}{x}\left[\frac{-xyz}{yz^2} - \frac{xy}{yz}\right] + \frac{y}{x}\left[\frac{xz}{yz} + \frac{xyz}{y^2z}\right]$$

$$= \frac{-yz}{x^2}\left[\frac{x^2}{yz} - \frac{x^2}{yz}\right] - \frac{z}{x}\left[-\frac{x}{z} - \frac{x}{z}\right] + \frac{y}{x}\left[\frac{x}{y} + \frac{x}{y}\right]$$

$$= 0 - \frac{z}{x}\left[\frac{-2x}{z}\right] + \frac{y}{z}\left[\frac{2x}{y}\right] = 2 + 2 = 4$$

(8

5) If $u = x^2 - 2y^2$, $v = 2x^2 - y^2$, where $x = r\cos\theta$, $y = r\sin\theta$ show that $\dfrac{\partial(u,v)}{\partial(r,\theta)} = 6r^3 \sin 2\theta$.

Ans: Given $u = x^2 - 2y^2$, $v = 2x^2 - y^2$

$x = r\cos\theta$, $y = r\sin\theta$

u, v

$\downarrow$

x, y

$\downarrow$

r, θ

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & -4y \\ 4x & -2y \end{vmatrix} \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= (-4xy + 16xy)(r\cos^2\theta + r\sin^2\theta)$$

$$= 12xyr$$

$$= 12(r\cos\theta)(r\sin\theta)r = 12r^3 \sin\theta\cos\theta$$

$$= 6r^3 (2\sin\theta\cos\theta)$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = 6r^3 \sin 2\theta$$

6) If $u = x+y+z$, $uv = y+z$ & $uvw = z$,
find the value of $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.

Ans: $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \quad \text{--- (1)}$$

Given $\boxed{z = uvw}$

$uv = y+z \Rightarrow y = uv - z \Rightarrow \boxed{y = uv - uvw}$

$u = x+y+z \Rightarrow x = u - y - z = u - (uv - uvw) - uvw$

$$\boxed{x = u - uv}$$

$\therefore$ from (1)

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1-v & -u & 0 \\ v - vw & u - uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$= (1-v)\left[(u-uw)\,uv - ((-uv)\,uw)\right] - (-u)\left[(v-vw)\,uv - ((-uv)(vw))\right]$

$= (1-v)\left[u^2v - \cancel{u^2vw} + \cancel{u^2vw}\right] + u\left[uv^2 - \cancel{uv^2w} + \cancel{uv^2w}\right]$

$= (1-v)\,u^2v + u^2v^2 = u^2v - \cancel{u^2v^2} + \cancel{u^2v^2} = \underline{u^2v}$

Taylor's theorem for functions of two variables:

We have Taylor's series expansion for a function $f(x)$ of single variable 'x' about the point 'a' is

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

$|||^{ly}$ Taylor's series expansion for a function $f(x,y)$ of two independent variables x & y about the point (a,b) is given by

$$\begin{aligned} f(x,y) = {} & f(a,b) + \frac{1}{1!}\left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right] \\ & + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) \right. \\ & \left. + (y-b)^2 f_{yy}(a,b)\right] + \\ & + \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b) \right. \\ & \left. + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right] \\ & + \cdots \qquad \text{———} (*) \end{aligned}$$

Maclaurin's series expansion of $f(x,y)$:

put $a=0$, $b=0$ in $(*)$, we get

$$\begin{aligned} f(x,y) = {} & f(0,0) + \frac{1}{1!}\left[x f_x(0,0) + y f_y(0,0)\right] + \\ & + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + \\ & + \frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + \right. \\ & \left. y^3 f_{yyy}(0,0)\right] \\ & + \cdots \end{aligned}$$

Expand the following functions upto 2nd degree terms:

1) $f(x,y) = \sin(xy)$ about the point $(1, \pi/2)$.

Ans: Given $(a,b) = (1, \pi/2)$

$f(x,y) = \sin xy \Rightarrow f(1,\pi/2) = \sin \pi/2 = 1$

$f_x(x,y) = y\cos xy \Rightarrow f_x(1,\pi/2) = \pi/2 \cdot \cos \pi/2 = 0$

$f_y(x,y) = x\cos xy \Rightarrow f_y(1,\pi/2) = 1 \cdot \cos\pi/2 = 0$

$f_{xx}(x,y) = -y^2 \sin xy \Rightarrow f_{xx}(1,\pi/2) = -(\pi/2)^2 \sin \pi/2 = -\frac{\pi^2}{4}$

$f_{xy}(x,y) = -xy\sin xy + \cos xy \Rightarrow f_{xy}(1,\pi/2) = -\pi/2(1) + 0 = -\pi/2$

$f_{yy}(x,y) = -x^2 \sin xy \Rightarrow f_{yy}(1,\pi/2) = -(1)^2 \sin\pi/2 = -1$

$\therefore$ by Taylor's series expansion,

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

$$\sin xy = 1 + \frac{1}{1!}\left[(x-1)(0) + (y-\pi/2)(0)\right] + \frac{1}{2!}\left[(x-1)^2\left(\frac{-\pi^2}{4}\right) + 2(x-1)(y-\pi/2)(-\pi/2) + (y-\pi/2)^2(-1)\right] + \cdots$$

$$\sin xy = 1 + \frac{1}{2!}\left[(x-1)^2\left(\frac{-\pi^2}{4}\right) - \pi(x-1)(y-\pi/2) - (y-\pi/2)^2\right]$$

2. $f(x,y) = x^2 + xy + y^2$ in power of $(x-1)$ & $(y-2)$.

Ans: Given $(a,b) = (1,2)$

$f(x,y) = x^2 + xy + y^2 \Rightarrow f(1,2) = 7$

$f_x(x,y) = 2x + y \Rightarrow f_x(1,2) = 4$

$f_y(x,y) = x + 2y \Rightarrow f_y(1,2) = 5$

$f_{xx}(x,y) = 2 \Rightarrow f_{xx}(1,2) = 2$

$f_{xy}(x,y) = 1 \Rightarrow f_{xy}(1,2) = 1$

$f_{yy}(x,y) = 2 \Rightarrow f_{yy}(1,2) = 2$

$\therefore$ by Taylor's series expansion

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b) + \cdots\right]$$

$$= 7 + \frac{1}{1!}\left[(x-1)\,4 + (y-2)\,5\right] + \frac{1}{2!}\left[(x-1)^2\,2 + 2(x-1)(y-2) + (y-2)^2 \cdot 2\right] + \cdots$$

H.W

3. $f(x,y) = e^{xy}$ about $(1,1)$

Expand the following functions upto 3rd degree terms.

1. $f(x,y) = e^y \log(1+x)$

Ans: Here $(a,b) = (0,0)$

$f(x,y) = e^y \log(1+x) \Rightarrow f(0,0) = e^0 \log(1) = 0$

$f_x(x,y) = e^y \left(\frac{1}{1+x}\right) \Rightarrow f_x(0,0) = e^0 \left(\frac{1}{1}\right) = 1$

$f_y(x,y) = e^y \log(1+x) \Rightarrow f_y(0,0) = e^0 . \log 1 = 0$

$f_{xx}(x,y) = e^y \left(-\frac{1}{(1+x)^2}\right) \Rightarrow f_{xx}(0,0) = e^0 . \left(-\frac{1}{1}\right) = -1$

$f_{xy}(x,y) = e^y \left(\frac{1}{1+x}\right) \Rightarrow f_{xy}(0,0) = 1$

$f_{yy}(x,y) = e^y \log(1+x) \Rightarrow f_{yy}(0,0) = 0$

$f_{xxx}(x,y) = e^y \left(\frac{2}{(1+x)^3}\right) \Rightarrow f_{xxx}(0,0) = 2$

$f_{xxy}(x,y) = e^y \left(-\frac{1}{(1+x)^2}\right) \Rightarrow f_{xxy}(0,0) = -1$

$f_{xyy}(x,y) = e^y \left(\frac{1}{1+x}\right) \Rightarrow f_{xyy}(0,0) = 1$

$f_{yyy}(x,y) = e^y \log(1+x) \Rightarrow f_{yyy}(0,0) = 0$

$\therefore$ by Maclaurins series expansion

$$f(x,y) = f(0,0) + \frac{1}{1!}\left[x\, f_x(0,0) + y\, f_y(0,0)\right]$$

$$+\frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy\, f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$

$$+\frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2y\, f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$$

$$+ \cdots$$

$$e^y \log(1+x) = x + \frac{1}{2!}\left[-x^2 + 2xy\right] + \frac{1}{3!}\left[2x^3 - 3x^2y + 3xy^2\right] + \cdots$$

---

2. $f(x,y) = \cos x \cos y$

Ans: Here $(a,b) = (0,0)$

$f(x,y) = \cos x \cos y \Rightarrow f(0,0) = 1$

$f_x(x,y) = -\sin x \cos y \Rightarrow f_x(0,0) = 0$

$f_y(x,y) = -\cos x \sin y \Rightarrow f_y(0,0) = 0$

$f_{xx}(x,y) = -\cos x \cos y \Rightarrow f_{xx}(0,0) = -1$

$f_{xy}(x,y) = +\sin x \sin y \Rightarrow f_{xy}(0,0) = 0$

$f_{yy}(x,y) = -\cos x \cos y \Rightarrow f_{yy}(0,0) = -1$

$f_{xxx}(x,y) = +\sin x \cos y \Rightarrow f_{xxx}(0,0) = 0$

$f_{xxy}(x,y) = +\cos x \sin y \Rightarrow f_{xxy}(0,0) = 0$

$f_{xyy}(x,y) = \sin x \cos y \Rightarrow f_{xyy}(0,0) = 0$

$f_{yyy}(x,y) = \cos x \sin y \Rightarrow f_{yyy}(0,0) = 0$

$\therefore$ by Maclaurin's series expansion

$$f(x,y) = f(0,0) + \left[x\, f_x(0,0) + y\, f_y(0,0)\right] +$$

$$+ \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy\, f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] +$$

$$+ \frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2y\, f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$$

$$+ \cdots$$

$$\cos x \cos y = 1 + \frac{1}{2!}\left[-x^2 - y^2\right] + \cdots$$

H.W

3. $f(x,y) = e^{ax+by}$.

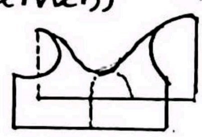
Maxima & Minima of functions of two variables:

1) **Maximum value:**
For a function $f(x,y)$ if $f(a,b) > f(a+h, b+k)$ for small values of h & k (+ve or −ve), then $f(x,y)$ has maximum value at (a,b)

2) **Minimum value:**
For a function $f(x,y)$ if $f(a,b) < f(a+h, b+k)$ for small values of h and k, then $f(x,y)$ has minimum value at (a,b).

3) **Extreme value:**
A maximum or a minimum value of a function is called an extreme value or extremum.

4) **Saddle point:**
A point of a surface which falls for displacement in certain direction & raises for displacement in other direction is called saddle point.



Necessary conditions: The necessary conditions for $f(x,y)$ to have maximum or minimum value at (a,b) are
$f_x(a,b) = 0$ and $f_y(a,b) = 0$

Sufficient conditions:
If $f_x(a,b) = 0$, $f_y(a,b) = 0$, $f_{xx}(a,b) = r$, $f_{xy}(a,b) = s$, $f_{yy}(a,b) = t$, then

1. $f(a,b)$ is maximum value if $rt - s^2 > 0$ & $r < 0$.
2. $f(a,b)$ is minimum value if $rt - s^2 > 0$ & $r > 0$.
3. $f(a,b)$ is not extremum if $rt - s^2 < 0$
[i.e (a,b) is a saddle point]

1. Find the extreme values of $x^2+y^2+6x-12$.

Ans: Given $f(x,y)=x^2+y^2+6x-12$.

$\Rightarrow f_x(x,y) = 2x+6 \Rightarrow f_{xx} = 2$ & $f_{xy}=0$

$f_y(x,y) = 2y \Rightarrow f_{yy} = 2$

$\Rightarrow f_x = 0 \Rightarrow 2x+6=0 \Rightarrow x=-3$

$f_y = 0 \Rightarrow 2y=0 \Rightarrow y=0$

$\therefore (a,b) = (-3,0)$

Let $r = f_{xx} = 2$, $s = f_{xy} = 0$, $t = f_{yy} = 2$

At $(-3,0) \Rightarrow r=2,\ s=0,\ t=2$.

$\Rightarrow rt-s^2 = 4-0 = 4>0$ & $r=2>0$

$\Rightarrow f(x,y)$ has minima at $(-3,0)$ & minimum value

$f(-3,0) = (-3)^2+0+6(-3)-12 = 9-18-12 = -21$

---

2. Find the maxima, minima & saddle point of $x^4+y^4-x^2-y^2+1$.

Ans: Given $f(x,y) = x^4+y^4-x^2-y^2+1$

$\Rightarrow f_x(x,y) = 4x^3-2x$

$f_y(x,y) = 4y^3-2y$

$r = f_{xx}(x,y) = 12x^2-2$

$s = f_{xy}(x,y) = 0$

$t = f_{yy}(x,y) = 12y^2-2$

Consider $f_x(x,y)=0$ & $f_y(x,y)=0$

i.e $4x^3 - 2x = 0 \Rightarrow 2x(2x^2-1) = 0 \Rightarrow x = 0,\ x = \pm\frac{1}{\sqrt{2}}$

$4y^3 - 2y = 0 \Rightarrow 2y(2y^2-1) = 0 \Rightarrow y = 0,\ y = \pm\frac{1}{\sqrt{2}}$

$\therefore$ required (a,b) such that $f_x = 0$ & $f_y = 0$ are,

$(0,0),\ \left(0, \pm\frac{1}{\sqrt{2}}\right),\ \left(\pm\frac{1}{\sqrt{2}}, 0\right),\ \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$

Consider $rt - s^2 = (12x^2-2)(12y^2-2)$ —— ①

At $(0,0)$: from ① $rt - s^2 = (-2)(-2) = 4 > 0$ & $r = -2 < 0$

$\Rightarrow f(x,y)$ has maximum at $(0,0)$ & maximum value is $f(0,0) = 1$

At $\left(0, \pm\frac{1}{\sqrt{2}}\right)$: from ① $rt - s^2 = (-2)(4) = -8 < 0$

$\Rightarrow f(x,y)$ has saddle points at $\left(0, \pm\frac{1}{\sqrt{2}}\right)$.

At $\left(\pm 1/\sqrt{2}, 0\right)$: from ① $rt - s^2 = (4)(-2) = -8 < 0$

$\Rightarrow f(x,y)$ has no extremum (or has saddle points) at $\left(\pm\frac{1}{\sqrt{2}}, 0\right)$.

At $\left(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}\right)$: from ①, $rt - s^2 = (4)(4) = 16 > 0$

& $r = 4 > 0$

$\Rightarrow f(x,y)$ has minimum value at $\left(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}\right)$

& minimum value is $f\left(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}\right) = \frac{1}{4} + \frac{1}{4} - \frac{1}{2} - \frac{1}{2} + 1$

$= \frac{1}{2}$

Find the extreme values of the following functions

1) $f = 1 - x^2 - y^2$

Ans: $f_x = -2x$

$f_y = -2y$

$r = f_{xx} = -2$

$s = f_{xy} = 0$

$t = f_{yy} = -2$

Consider $f_x = 0 \Rightarrow x = 0$

$f_y = 0 \Rightarrow -2y = 0 \Rightarrow y = 0$

$\therefore (a, b) = (0, 0)$

Now consider $rt - s^2 = (-2)(-2) = 4 > 0$ & $r < 0$

$\Rightarrow f(x, y)$ has maxima at $(0, 0)$ & the maximum value is $f(0, 0) = 1$.

---

3) $f = x^2 + y^2$

Ans: $\left.\begin{array}{l} f_x = 2x \\ f_y = 2y \end{array}\right\} \Rightarrow r = f_{xx} = 2$, $s = f_{xy} = 0$, $t = f_{yy} = 2$

Consider $f_x = 0 \Rightarrow x = 0$ & $f_y = 0 \Rightarrow y = 0$

$\therefore (a, b) = (0, 0)$

Now consider $rt - s^2 = 4 > 0$ & $r = 2 > 0$

$\Rightarrow f(x, y)$ has minima at $(0, 0)$ & the minimum value is $f(0, 0) = 0$.

4) $f = xy$

<u>Ans:</u> $\left.\begin{array}{l} f_x = y \\ f_y = x \end{array}\right\} \Rightarrow r = f_{xx} = 0, \quad s = f_{xy} = 1, \quad t = f_{yy} = 0$

Consider $f_x = 0 \Rightarrow y = 0$

$f_y = 0 \Rightarrow x = 0$

$\therefore (a, b) = (0, 0)$

Now consider $rt - s^2 = 0 - 1 = -1 < 0$

$\Rightarrow f(x, y)$ has no extreme values at $(0, 0)$

Stationary points:

A point (a,b) at which $\frac{\partial f}{\partial x}=0$ & $\frac{\partial f}{\partial y}=0$ is called a stationary point. The value $f(a,b)$ is said to be stationary value.

Note: Every extreme value is stationary but the converse need not be true.

Lagrange's method of undetermind multipliers:

In many situations we may be required to find the maximum or minimum value of a function whose variables are connected by some relation or condition known as constraint.

Thus if $f(x,y,z)$ is a function of x,y,z which is to be examined for maximum or minimum value where x,y,z are related by a known constraint

$g(x,y,z)=0$

Working rule:

1. Write $F = f(x,y,z) + \lambda\, g(x,y,z)$ where λ is a parameter known as Lagrange's multiplier.
2. Find $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$
3. Solve the equations $\frac{\partial F}{\partial x}=0$, $\frac{\partial F}{\partial y}=0$, $\frac{\partial F}{\partial z}=0$ together with $g(x,y,z)=0$.

The values of x,y,z so obtained will give the required stationary values.

Disadvantage:
Nature of the stationary point can not be determined. Further investigation needed.

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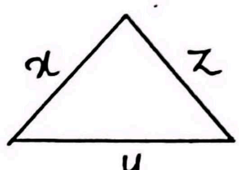
1. If the perimeter of a triangle is a constant, prove that the area of this triangle is maximum when the triangle is equilateral.

Ans: Let x, y, z be the sides of a triangle, then we have



Perimeter $P = x + y + z$

Area $A = \sqrt{s(s-x)(s-y)(s-z)}$

Where $s = \dfrac{x+y+z}{2} = \dfrac{P}{2} \Rightarrow P = 2s$

$\Rightarrow x + y + z = 2s$

Let $f(x,y,z) = s(s-x)(s-y)(s-z)$ [Here f represent area which is to be examined]

& $g(x,y,z) = x + y + z - 2s$

Consider $F = f(x,y,z) + \lambda\, g(x,y,z)$

$F = s(s-x)(s-y)(s-z) + \lambda(x+y+z-2s)$

$\Rightarrow \dfrac{\partial F}{\partial x} = -s(s-y)(s-z) + \lambda = 0 \Rightarrow \lambda = s(s-y)(s-z)$ —— (1)

$\dfrac{\partial F}{\partial y} = -s(s-x)(s-z) + \lambda = 0 \Rightarrow \lambda = s(s-x)(s-z)$ —— (2)

$\dfrac{\partial F}{\partial y} = -s(s-x)(s-y) + \lambda = 0 \Rightarrow \lambda = s(s-x)(s-y)$ —— (3)

From ① & ②,

$$s(s-y)(s-z) = s(s-x)(s-z)$$

$$\Rightarrow (s-y) = (s-x) \Rightarrow \boxed{x = y} \quad \text{— ④}$$

From ② & ③,

$$s(s-x)(s-z) = s(s-x)(s-y)$$

$$\Rightarrow (s-z) = (s-y) \Rightarrow \boxed{y = z} \quad \text{— ⑤}$$

From ④ & ⑤

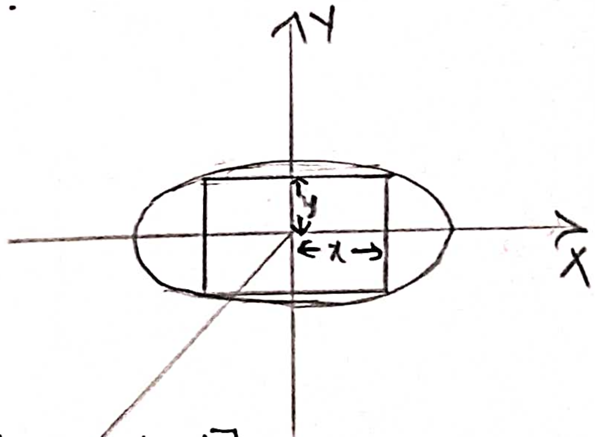
$$x = y = z$$

$\Rightarrow$ the triangle is equilateral

$\therefore$ the area of the triangle is maximum when the triangle is equilateral.

---

2. Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.



Ans: Let $2x, 2y, 2z$ be the length, breadth and height of the parallelopiped.

So that the volume

$V = 8xyz = f(x,y,z)$ [here volume to be examined]

Subject to the condition $g(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Consider $F = f(x,y,z) + \lambda\, g(x,y,z)$

$$F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

$$\Rightarrow \frac{\partial F}{\partial x} = 8yz + \lambda\left(\frac{2x}{a^2}\right) = 0 \Rightarrow \lambda = \frac{-8yz}{\left(\frac{2x}{a^2}\right)} = \frac{-4yza^2}{x} \quad \text{——— (1)}$$

$$\frac{\partial F}{\partial y} = 8xz + \lambda\left(\frac{2y}{b^2}\right) = 0 \Rightarrow \lambda = \frac{-8xz}{\left(\frac{2y}{b^2}\right)} = \frac{-4xzb^2}{y} \quad \text{——— (2)}$$

$$\frac{\partial F}{\partial z} = 8xy + \lambda\left(\frac{2z}{c^2}\right) = 0 \Rightarrow \lambda = \frac{-8xy}{\frac{2z}{c^2}} = \frac{-4xyc^2}{z} \quad \text{——— (3)}$$

From (1) & (2),

$$-\frac{4yza^2}{x} = -\frac{4xzb^2}{y} \Rightarrow \boxed{\frac{x^2}{a^2} = \frac{y^2}{b^2}} \quad \text{— (4)}$$

From (2) & (3),

$$-\frac{4xzb^2}{y} = \frac{-4xyc^2}{z} \Rightarrow \boxed{\frac{y^2}{b^2} = \frac{z^2}{c^2}} \quad \text{— (5)}$$

Also $g(x,y,z) = 0 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\Rightarrow \frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1 \Rightarrow \frac{3x^2}{a^2} = 1 \Rightarrow x^2 = \frac{a^2}{3}$$

$\Rightarrow \quad x = \frac{a}{\sqrt{3}}$

from ④ , $y^2 = \frac{b^2}{3} \Rightarrow y = \frac{b}{\sqrt{3}}$

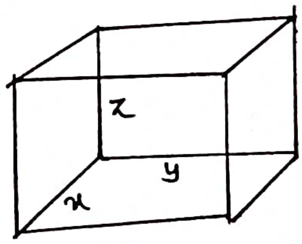
from ⑤ , $z^2 = \frac{c^2}{3} \Rightarrow z = \frac{c}{\sqrt{3}}$

$\therefore \; V = 8xyz = 8 \frac{a}{\sqrt{3}} \frac{b}{\sqrt{3}} \frac{c}{\sqrt{3}} = \frac{8abc}{3\sqrt{3}}$

---

3. A rectangular box open at the top is to have volume of 32 cubic feet. Find the dimensions of the box requiring least material for its construction.

Ans: Let x, y, z be the length, breadth and height of the rectangular box respectively. Then the total surface area of the rectangular open box.



$f(x,y,z) = xy + 2yz + 2zx$ (related to dimension)

$g(x,y,z) = xyz - 32 \quad (\because V = xyz = 32)$

By Lagrange's multiplier method

$F = f(x,y,z) + \lambda\, g(x,y,z)$

$= (xy + 2yz + 2zx) + \lambda\,(xyz - 32)$

$$\Rightarrow \frac{\partial F}{\partial x} = (y+2z) + \lambda(yz) = 0 \Rightarrow \lambda = \frac{-(y+2z)}{yz} \quad \text{—(1)}$$

$$\frac{\partial F}{\partial y} = (2z+x) + \lambda(xz) = 0 \Rightarrow \lambda = \frac{-(x+2z)}{xz} \quad \text{—(2)}$$

$$\frac{\partial F}{\partial z} = (2y+2x) + \lambda(xy) = 0 \Rightarrow \lambda = \frac{-2(x+y)}{xy} \quad \text{—(3)}$$

From (1) & (2),

$$\frac{y+2z}{y\cancel{z}} = \frac{x+2z}{x\cancel{z}}$$

$$xy + 2xz = xy + 2yz \Rightarrow 2xz = 2yz$$

$$\Rightarrow \boxed{x = y} \quad \text{—(4)}$$

From (2) & (3),

$$\frac{x+2z}{\cancel{x}z} = \frac{2(x+y)}{\cancel{x}y}$$

$$\Rightarrow xy + 2zy = 2xz + 2yz$$

$$\Rightarrow xy = 2xz \Rightarrow \boxed{y = 2z} \quad \text{—(5)}$$

$$\Rightarrow \text{from (4) \& (5)}, \quad x = y = 2z \quad \text{—(6)}$$

Consider $g(x,y,z) = 0 \Rightarrow xyz = 32$

$$\Rightarrow x(x)\left(\frac{x}{2}\right) = 32 \Rightarrow x^3 = 64$$

$$\Rightarrow x = 4$$

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$\Rightarrow$ from (4), $x=4$, $y=4$ & $z=2$

Hence the dimensions of the box are

$x=4$, $y=4$ & $z=2$

---

4. Suppose a closed rectangular box has length twice its breadth & has constant volume V. Determine the dimensions of the box requiring least surface area.

Ans: Let x, y, z be the length, breadth & height of a rectangular box.

Given $x = 2y \Rightarrow V = xyz = (2y)yz = 2y^2z$

$\Rightarrow y^2z = \frac{V}{2}$

The surface area S is given by

$S = 2xy + 2yz + 2zx$

$= 4y^2 + 2yz + 4yz = 4y^2 + 6yz$

Let $f(x,y) = 4y^2 + 6yz$ (related to dimension)

subject to $g(x,y) = y^2z - \frac{V}{2}$

Consider $F = f(x,y) + \lambda g(x,y)$

$= (4y^2 + 6yz) + \lambda\left(y^2z - \frac{V}{2}\right)$

$\Rightarrow \frac{\partial F}{\partial y} = (8y + 6z) + \lambda(2yz) = 0 \Rightarrow \lambda = \frac{-(8y+6z)}{2yz}$ — (1)

$$\frac{\partial F}{\partial z} = 6y + \lambda(y^2) = 0 \Rightarrow \lambda = \frac{-6y}{y^2} \quad \text{——} (2)$$

From (1) & (2),

$$\frac{8y+6z}{2yz} = \frac{6}{y}$$

$$\Rightarrow \frac{4}{z} + \frac{3}{y} = \frac{6}{y} \Rightarrow \frac{4}{z} = \frac{6}{y} - \frac{3}{y} = \frac{3}{y}$$

$$\Rightarrow \frac{4}{z} = \frac{3}{y} \Rightarrow \boxed{z = \frac{4y}{3}}$$

Now $g(x,y) = 0 \Rightarrow y^2 z = \frac{V}{2}$

$$\Rightarrow y^2\left(\frac{4y}{3}\right) = \frac{V}{2}$$

$$\Rightarrow \frac{8y^3}{3} = V \Rightarrow y = \left(\frac{3V}{8}\right)^{1/3}$$

$$\Rightarrow z = \frac{4}{3}\left(\frac{3V}{8}\right)^{1/3}$$

$\therefore$ dimensions are $x = 2\left(\frac{3V}{8}\right)^{1/3}$, $y = \left(\frac{3V}{8}\right)^{1/3}$, $z = \frac{4}{3}\left(\frac{3V}{8}\right)^{1/3}$

$\therefore$ the least surface area is

$$S = 4y^2 + 6yz = 4\left(\frac{3V}{8}\right)^{2/3} + 6\left(\frac{3V}{8}\right)^{1/3}\frac{4}{3}\left(\frac{3V}{8}\right)^{1/3}$$

$$= 4\left(\frac{3V}{8}\right)^{2/3} + 8\left(\frac{3V}{8}\right)^{2/3} = 12\left(\frac{3V}{8}\right)^{2/3}$$