Unit - III : Numerical methods

Roots of algebraic and transcedental equations:

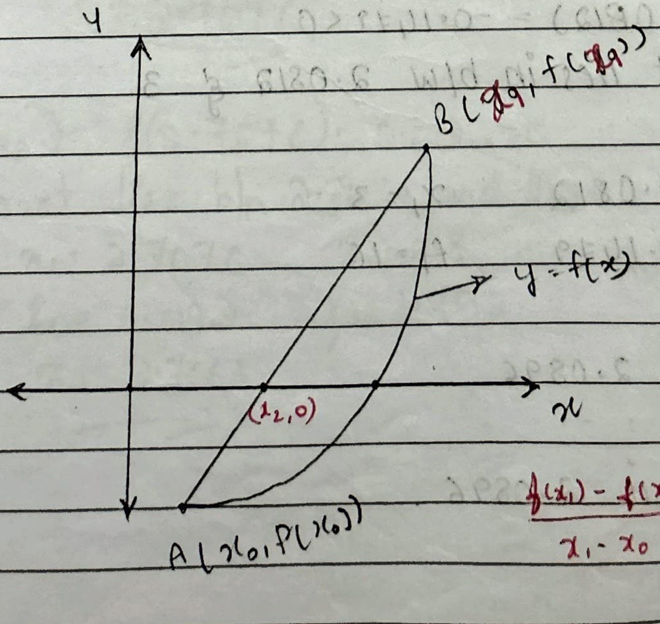
An equation involving algebraic terms is called an algebraic
If an equation contains, trigonometric, exponential, or logarithmic terms then it is called transedental equation.

Intermediate value property:

If $f(x)$ is continuous in the interval $[a,b]$ and $f(a)$ & $f(b)$ are of opposite signs then the equation $f(x)=0$ has atleast one route between $x=a$ & $x=b$.

Regular Falsi methode (or) method of false position:

Geometrically this method is equivalent to replacing the curve $y=f(x)$ by a chord that passes through a point $A(x_0, f(x_0))$, $B(y_0, f(y_0))$.



$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0}$$

where, $f_0 = f(y_0) = y_0$

$f_1 = f(x_1) = y_1$

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f_1 - 0}{x_1 - x_2}$$

$$x_1 - x_2 = \frac{f_1 (x_1 - x_0)}{f_1 - f_0}$$

1. Use the method of false position to find the roots of $x^3 - 2x - 5 = 0$ in $(2, 3)$. Carry out 3 equations.

Let $f(x) = x^3 - 2x - 5$

$f(2) = 2^3 - 2(2) - 5 = -1 < 0$

$f(3) = 3^3 - 2(3) - 5 = 16 > 0$

$\therefore$ A root lies b/w 2 & 3

let $x_0 = 2$ $\quad x_1 = 3$

$f_0 = -1$ $\quad f_1 = 16$

using regular - Falsi method

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \frac{(2 \times 16) - (-1 \times 3)}{16 - (-1)} = 2.0588$$

$f(x_2) = f(2.0588) = -0.3910 < 0$

A root lies in b/w 2.0588 & 3.

Let $x_0 = 2.0588$ $\quad x_1 = 3$

$f_0 = -0.3910$ $\quad f_1 = 16$

$x_3 = 2.08122$

$f(x_3) = f(2.0812) = -0.1479 < 0$

$\therefore$ A root lies in b/w 2.0812 & 3.

Let $x_0 = 2.0812$ $\quad x_1 = 3$

$f_0 = -0.1479$ $\quad f_1 = 16$

$x_4 = 2.0896$

$\therefore$ The root is 2.0896.

2.7065

2. $f(x) = x^3 - 4x - 9 = 0$ in $(2.7, 2.8)$

= let $f(x) = x^3 - 4x - 9$

$f(2.7) = -0.117 < 0$

$f(2.8) = 1.752 > 0$

$\therefore$ A root lies b/n 2.7 and 2.8

Let $x_0 = 2.7$, $x_1 = 2.8$

$f_0 = -0.117$, $f_1 = 1.752$

using Regula Falsi method

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0}$$

$$= \frac{(2.7 \times 1.752) - (2.8 \times -0.117)}{1.752 + 0.117}$$

$$x_2 = 2.7062$$

$f(x_2) = f(2.7062) = -0.0291 < 0$

A root lies b/n 2.7062 and 2.8

Let $x_0 = 2.7062$ $x_1 = 2.8$

$f_0 = -0.0291$ $f_1 = 1.752$

$$\therefore x_3 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0}$$

$$= 2.7076$$

$f(x_3) = f(2.7076) = 0.0192 > 0$

A root lies b/n 2.7076 and 2.8

$x_0 = 2.7076$ $x_1 = 2.8$

$f_0 = 0.0192$ $f_1 = 1.752$

$x_4 = 2.7066$

3. Using the method of false position find the root of $x \log_{10} x - 1.2 = 0$.

Let $f(x) = x \log_{10} x - 1.2 = 0$.

$f(1) = 1(0) - 1.2$
$= -1.2$

$f(2) = 2\log(2) - 1.2$
$= -0.5979$

$f(3) = 3\log(3) - 1.2$.
$= 0.2313$

$\therefore$ The root lies between 2 & 3.

Let $x_0 = 2$, $x_1 = 3$
$f_0 = -0.5979$ $f_1 = 0.2313$

Using method of false position,

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0}$$

$$x_2 = \frac{2(0.2313) - 3(-0.5979)}{0.2313 + 0.5979}$$

$x_2 = 2.7210$

$f(x_2) = f(2.7210) = -0.0170$

$x_0 = 2.7210$ $x_1 = 3$
$f_0 = -0.0170$ $f_1 = 0.2313$
$x_3 = 2.74021$

$f(x_3) = f(2.7402) =$
$x_0 = 2.7402$ $x_1 = 3$
$f_0 = -389045 \times 10^{-4}$ $f_1 = 0.2313$

$x_4 = 2.74063$

Hence the root is 2.7406.

4. Using regular falsi method find the root of $x.e^x - 3 = 0$. Carry out 3 equations.

Let $f(x) = xe^x - 3 = 0$

$f(0) = 0(e^0) - 3 = 0$ $\quad f(1) = 1(e^1) - 3$

$f(0) = -3$ $\quad = -0.28171$

$f(2) = 2e^2 - 3$

$= 11.77811$

$\therefore$ The root lies btw 1 & 2.

Let $x_0 = 1$ $\quad x_1 = 2$

$f_0 = -0.28171$ $\quad f_1 = 11.77811$

Using regular falsi method, $x_2 = \dfrac{x_0 f_1 - x_1 f_0}{f_1 - f_2}$

$= \dfrac{1(11.77811) + 2(0.28171)}{11.77811 + 0.28171}$

$= 1.02335$

$f(x_2) = f(1.02335) = -0.15252$

$\therefore$ root lies btw 1.02335 & 2

$x_0 = 1.02335$ $\quad x_1 = 2$

$f_0 = -0.15252$ $\quad f_1 = 11.77811$

$x_3 = 1.03583$

$f(x_3) = f(1.03583) = -0.08160$

$\therefore$ root lies btw 1.03583 & 2.

$x_0 = 1.03583$ $\quad x_1 = 2$

$f_0 = -0.08160$ $\quad f_1 = 11.77811$

$x_4 = 1.04246$.

Hence the root is 1.04246.

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5. Using regular falsi method find the root of $xe^x = \cos x$ in $(0,1)$.

Let $f(x) = \cos x - xe^x$

$f(0) = \cos(0) - 0e^0$ $\quad f(1) = \cos(1) - e$

$f(0) = 1$ $\quad = -2.17797$

$\therefore$ The root lies in between 0 & 1.

$x_0 = 0$ $\quad x_1 = 1$

$f_0 = 1$ $\quad f_1 = -2.17797$

$$x_2 = \frac{x_0 f_1 - f_0 x_1}{f_1 - f_0}$$

$$x_2 = \frac{0 - 1(1)}{-2.17797 - 1} = \frac{1}{3.17791}$$

$x_2 = 0.31466$

$f(x_2) = f(0.31466) = 0.51988$

The root lies btw 0.31466 & 1.

$x_0 = 0.31466$ $\quad x_1 = 1$

$f_0 = 0.51988$ $\quad f_1 = -2.17797$.

$x_3 = 0.44672$

$f(x_3) = f(0.44672) = 0.20356$

$\therefore$ The root lies btw 0.44672 & 1.

$x_0 = 0.44672$ $\quad x_1 = 1$

$f_0 = 0.20356$ $\quad f_1 = -2.17797$

$x_4 = 0.49401$

$\therefore$ The root is 0.49401.

6. Using regular falsi method find 4th rooth of 32 in the interval (2,3).

$f(x) = x^4 - 32$

$f(2) = 2^4 - 32 = -16$ $\quad$ $f(3) = 3^4 - 32 = 49$

root lies b/w 2 & 3

$x_0 = 2$ $\quad$ $x_1 = 3$

$f_0 = -16$ $\quad$ $f_1 = 49$

By regular falsi method, $x_2 = \dfrac{x_0 f_1 - x_1 f_0}{f_1 - f_0}$

$x_2 = \dfrac{2(49) - 3(-16)}{49 - (-16)} = 2.24615$

$f(x_2) = f(2.24615) = -6.54605$

$\therefore$ root lies b/w 2.24615 & 3.

$x_0 = 2.24615$ $\quad$ $x_1 = 3$

$f_0 = -6.54605$ $\quad$ $f_1 = 49.$

$x_3 = 2.33499$

$f(x_3) = f(2.33499) = -2.27375$

$\therefore$ root lies b/w 2.33499 & 3

$x_0 = 2.33499$ $\quad$ $x_1 = 3$

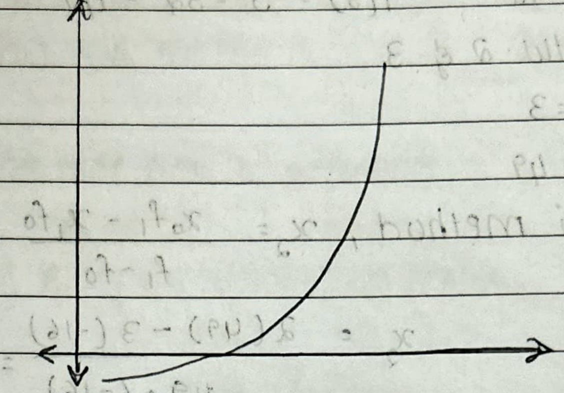
$f_0 = -2.27375$ $\quad$ $f_1 = 49$

$x_4 = 2.36448$

$\therefore$ The root is 2.36448.

Newton - Raphson method:

Newton - Raphson method



Geometrically this method consists in replacing part of the curve between the point a and x-axis by means of tangent to the curve at a.

Formula for newtons - Raphson method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2 \ldots$$

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1. Using newton raphson method obtain the root of $x^3 - 3x - 5 = 0$, near $x_0 = 3$. Carry out 3 iterations.

Let $f(x) = x^3 - 3x - 5$ $\quad f(x_0) = f(3) = 13$

$f'(x) = 3x^2 - 3$ $\quad f'(x_0) = f'(3) = 24$

By Newton raphson method,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 3 - \frac{13}{24}$$

$$x_1 = 2.45833.$$

$f(x_1) = 2.48164$

$f'(x_1) = 15.13015$

$$x_2 = 2.29431$$

$f(x_2) = 0.19399$

$f'(x_2) = 12.79157.$

$$x_3 = 2.2791.$$

Hence the root is 2.2791.

2. $x^3 - 2x - 11 = 0$, $x_0 = 2$, 3 eqn's

$f(x) = x^3 - 2x - 11$ $\qquad f(x_0) = 2^3 - 2(2) - 11 = -7$

$f'(x) = 3x^2 - 2$ $\qquad f'(x_0) = 3(2)^2 - 2 = 10$

Using NR method:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 2 - \left(\frac{-7}{10}\right)$$

$$x_1 = 2.7$$

$f(x_1) = 3.283$

$f'(x_1) = 19.87$

$$x_2 = 2.5347$$

$f(x_2) = 0.21529$

$f'(x_2) = 17.2741$

$$x_3 = 2.5274$$

Hence the root is 2.5274

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3. $f(x) = \tan x - 1.5x$, $x_0 = 1$

$f(x) = \tan x - 1.5x$

$f'(x) = \sec^2 x - 1.5$

$= (1 - \tan^2 x) - 1.5$

$= \tan^2 x - 0.5$

$f(x_0) = 0.0574$

$f'(x_0) = 1.92551$

Using NR method,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 1 - \frac{0.0574}{1.92551}$$

$$x_1 = 0.9701$$

$f(x_1) = 4.3641 \times 10^{-3}$

$f'(x_1) = 1.6301$

$$x_2 = 0.9674$$

Hence the root is 0.9674.

4. Evaluate $\sqrt{12}$ to 4 decimal places by Newtons iterative method near $x_0 = 3$.

$x = \sqrt{12}$

$x^2 = 12$

$x^2 - 12 = 0$

$\therefore f(x) = x^2 - 12$

$f'(x) = 2x$

$f(x_0) = -3$

$f'(x_0) = 6$

using NR method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = 3 - \left(\frac{-3}{6}\right)$$

$$x_1 = 3.5$$

$f(x_1) = 0.25.$

$f'(x_1) = 7$

$$x_2 = 3.5 - \frac{0.25}{7} = 3.4642.$$

$f(x_2) = 6.8164 \times 10^{-4}$

$f'(x_2) = 6.9284$

$$x_3 = 3.4641$$

Hence the root is $3.4641.$ //

5. $2x = \cos x + 3, \quad x_0 = \frac{\pi}{2}$

$f(x) = \cos x - 2x + 3 \qquad f(x_0) = -0.1415.$

$f'(x) = -\sin x - 2 \qquad f'(x_0) = -3$

$$x_1 = 1.5236$$

$f(x_1) = -2.1192 \times 10^{-5}$

$f'(x_1) = -2.9988$

$$x_2 = 1.5235$$

$f(x_2) = -1.5193 \times 10^{-3}$

$f'(x_2) = -2.9988$

$$x_3 = 1.5229$$

Hence the root is $1.5229.$

6. $\cos x = xe^x$, $x_0 = 0.5$

$f(x) = \cos x - xe^x$

$f'(x) = -\sin x - (xe^x + e^x)$ $\qquad f(x_0) = 0.0532.$

$= -\sin x - e^x(x+1)$ $\qquad f'(x_0) = -2.9525$

Using NR method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = 0.5 - \left(\frac{0.0532}{-2.9525}\right)$$

$$x_1 = 0.5180$$

$f(x_1) = -7.3827 \times 10^{-4}$

$f'(x_1) =$

Numerical solution of Ordinary differential Equations

(i) Taylor's series method of order four.

Consider the initial value problem (I.V.P)

$\frac{dy}{dx} = f(x,y)$; $y(x_0) = y_0$ then,

$$y_1 = y(x_1) = y(x_0+h) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \ldots$$

1. Using Taylor's series method of order four, compute y at $x = 0.1$

$y' = x+y$, $y(0) = 1$, $h = 0.1$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

we have to find $y(x_1) = y(0.1) = y_1$

Given,

$y' = x+y$ $\quad$ $y_0' = 1$

$y'' = 1+y'$ $\quad$ $y'' = 2$

$y''' = y''$ $\quad$ $y''' = 2$

$y^{iv} = y'''$ $\quad$ $y^{iv} = 2$

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \ldots$$

$$y_1 = 1 + (0.1)(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(2) + \ldots$$

$$y_1 = 1 + 0.1 + 0.01 + 3.3333 \times 10^{-4} + 8.3333 \times 10^{-6}$$

$$y_1 = 1.1103$$

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2. Using Taylor series method of order four, compute y at $x=0.1$, given $y' = 2y + 3e^x$, $y(0)=0$, $h=0.1$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

we have to find $y(x_1) = y(0.1) = y_1$

Given,

$y' = 2y + 3e^x$ $\quad y_0' = 3$

$y'' = 2y' + 3e^x$ $\quad y_0'' = 9$

$y''' = 2y'' + 3e^x$ $\quad y_0''' = 21$

$y^{iv} = 2y''' + 3e^x$ $\quad y_0^{iv} = 45$

$$y_1 = y(0.1) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} \dots$$

$$y_1 = 0 + 0.1(3) + \frac{(0.1)^2}{2!}\cdot 9 + \frac{(0.1)^3}{3!}(21) + \frac{(0.1)^4}{4!}(45) + \dots$$

$$y_1 = 0.3 + 0.045 + 3.5\times10^{-3} + 1.875\times10^{-4}$$

$$y_1 = 0.3486$$

3. Using TSM of order four, compute y at 0.1, given $y' = xy + 1$, $y(0) = 1$, $h = 0.1$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

we have to find $y(x_1) = y(0.1) = y_1$

Given, $y' = xy + 1$ $\quad y_0' = 1$

$y'' = xy' + y$ $\quad y_0'' = 1$

$y''' = xy'' + y' + y' = xy'' + 2y'$ $\quad y_0''' = 2$

$y^{iv} = xy''' + y'' + 2y''$ $\quad y_0^{iv} = 3$

$$y_1 = y(0.1) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \dots$$

$$y_1 = 1 + 0.1(1) + \frac{(0.1)^2}{2!}(1) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(3) + \dots$$

$y_1 = 1 + 0.1 + 5 \times 10^{-3} + 3.3333 \times 10^{-4} + 1.25 \times 10^{-5}$

$y_1 = 1.1053.$

Modified Euler's method:

Consider the IVP $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_1$.
we need to find y at $x_1 = x_0 + h$. i.e, $y(x_1) = y_1$

To find initial approximation for $y_1 = y_1^{(0)}$

$$y_1^{(0)} = y_0 + h(f(x_0, y_0))$$

This value of y_0 is successfully improved to the desired degree of accuracy by applying ME formula where the successive approximations are denoted by $y_1^{(0)}, y_1^{(1)}, y_1^{(2)} \ldots$

$$y_1^{(1)} = y_0 + \frac{h}{2}(f(x_0, y_0) + f(x_1, y_1^{(0)}))$$

$$y_1^{(2)} = y_0 + \frac{h}{2}(\underset{\wedge}{} f(x_0, y_0) + f(x_1, y_1^{(1)}))$$ & so on.

Thus we get y_1

1. Using modified Euler's method find $y(0.1)$ by solving the equation $\frac{dy}{dx} = x - y^2$, $y(0) = 1$, taking $h = 0.1$. carry out 2 modification.

Given $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

we have to find, $y(0.1) = y(x_1) = y_1$

Initial approximation to y_1:

$y_1(0) = y_0 + h f(x_0, y_0)$

$= 1 + 0.1(0 - 1)$

$= 0.9$

Using modified Euler's method.

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(1)} = 1 + \frac{0.1}{2}\left[(0-1)^2 + (0.1 - 0.9^2)\right]$$

$$y_1^{(1)} = 0.91$$

$$y_1^{(2)} = 0.9132$$

$$y_1 = y(0.1) = 0.9132. \;//$$

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2. Using modified Eulers method find when $x = 0.1$ given that $y' = x + y$, $y(0) = 1$, $h = 0.1$ carry out a modifications

Given, $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

we have to find $y(0.1) = y(x_1) = y_1$

Initial approximation to y_1

$$y_1(0) = y_0 + h(f(x_0, y_0))$$

$$y_1^0 = 1 + 0.1(1)$$

$$y_1^0 = 2.1$$

Using modified Euler's method,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(1)} = 1 + \frac{0.1}{2}\left[1 + 1.2\right]$$

$$y_1^{(1)} = 1.11$$

$$y_1^{(2)} = 1 + \frac{0.1}{2}\left[1 + 1.21\right]$$

$$y_1^{(2)} = 1.1105$$

//

3. Solve the differential equation $\frac{dy}{dx} = -xy^2$ under the initial condition $y(0) = 2$ using ME method at point $x = 0.1$. take step size $h = 0.1$. Carry out 2 modifications.

Given,

$x_0 = 0,\ y_0 = 2,\ h = 0.1$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

To find $y(0.1) = y(x_1) = y_1$

initial approximation to y_1.

$y_1^0 = y_0 + h(f(x_0, y_0))$

$y_1^0 = 2$

Using modified euler's method,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$y_1^{(1)} = 2 + \frac{0.1}{2}[0 + (-0.1 \times 2^2)]$$

$$y_1^{(1)} = 1.98$$

using MEM,

$$y_1^{(2)} = 2 + \frac{0.1}{2}[0 + (-0.1 \times (1.98)^2)]$$

$$y_1^{(2)} = 1.9804$$

$$y_1 = y(0.1) = 1.9804$$

$1.0512.$

4. Using ME method compute $y(0.05)$, $y' = x^2 + y$, $y(0) = 1$, $h = 0.05$ take single stage.

Given, $x_0 = 0$, $y_0 = 1$, $h = 0.05$

$$x_1 = x_0 + h = 0 + 0.05 = 0.05$$

we have to find $y(0.05) = y(x_1) = y_1$

Intial approximation,

$$y_1^{(0)} = y_0 + h\,(f(x_0, y_0))$$

$$y_1^{(0)} = 1 + 0.05\,(1)$$

$$y_1^{(0)} = 1.05.$$

Using MEM,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(1)} = 1 + \frac{0.05}{2}\left[1 + 1.0525\right]$$

$$y_1^{(1)} = 1.0513.$$

using MEM,

$$y_1^{(2)} = 1 + \frac{0.05}{2}\left[1 + 1.0513\right]$$

$$y_1^{(2)} = 1.0512 \quad //$$

Fourth order Runge - Kutta method :

For finding the increment $k(y)$ corresponding to an increment $h(x)$ of x by RK method from $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ ------(i)

Suppose we wish to find an approximate solution of this problem at $x_1 = x_0 + h$, where h is step length.

Then the solution of (i) is

$$y_1 = y_0 + k$$

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

//

1. Employe RK method to find soln of IIVP,

$\frac{dy}{dx} = x+y$, $y(0)=1$, $h=0.2$, $x=0.2$, take $k=0.2$.

$x_0=0$, $y_0=1$, $h=0.2$

$x_1 = x_0+h = 0+0.2 = 0.2$

To find $y(0.2) = y(x_1) = y_1$

$k_1 = hf(x_0, y_0)$

$k_1 = 0.2(0+1)$

$k_1 = 0.2$

$k_2 = hf(x_0+\frac{h}{2}, y_0+\frac{k_1}{2})$

$k_2 = 0.2(0.1+1.1)$

$k_2 = 0.24$

$k_3 = hf(x_0+\frac{h}{2}, y_0+\frac{k_2}{2})$

$k_3 = 0.2(0.1+1.12)$

$k_3 = 0.244$

$k_4 = hf(x_0+h, y_0+k_3)$

$k_4 = 0.2(0.2+0.244)$

$k_4 = 0.2888$

$y_1 = y_0+k$

$y_1 = y_0+\frac{1}{6}[k_1+2k_2+2k_3+k_4]$

$y_1 =$

$y_1 = 1.2428$

3. By using 4th order RK method find the solution of $y' = 1 + y^2$, $y(0) = 0$, $h = 0.2$. Find $y(0.2)$.

$x_0 = 0,\ y_0 = 0,\ h = 0.2$

$x_1 = x_0 + h = 0 + 0.2 = 0.2$

We need to find $y(0.2) = y(x_1) = y_1$

$k_1 = hf(x_0, y_0)$

$= 0.2\ (1)$

$= 0.2.$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$k_2 = 0.2 f(0.1, 0.1)$

$k_2 = 0.2(1 + 0.1^2)$

$k_2 = 0.202.$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$

$k_3 = 0.2\ f(1 + 0.101)$

$k_3 = 0.20204$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$k_4 = 0.2 f(0.2, 0.20204)$

$k_4 = 0.2($

$k_4 = 0.208164$

$y_1 = y_0 + k$

$= y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$y_1 = 0 + \frac{1}{6}[0.2 + 2(0.202) + 2(0.20204) + 0.208164]$

$y_1 = 0.202707$ //

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Note :- $U_{ij} = U(x,y) = U(ih, jk)$.
For a square mesh $h=k$.

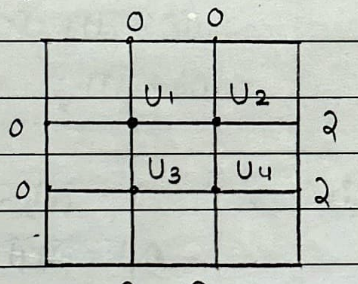
Numerical solution of PDE

(i) Numerical solution of Laplace equation :

Laplace equation is $U_{xx} + U_{yy} = 0$.
Standard five point formula, is .,

$$U_{ij} = \frac{1}{4}\left[U_{(i-1,j)} + U_{(i+1,j)} + U_{(i,j-1)} + U_{(i,j+1)}\right]$$

1. Solve the laplace equation $U_{xx} + U_{yy} = 0$ given the values of $U_{(x,y)}$ on the boundary of the square mesh in the figure given below.



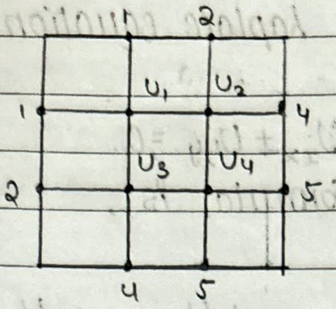
$$U_{i,j} = \frac{1}{4}\left[U_{(i-1,j)} + U_{(i+1,j)} + U_{(i,j-1)} + U_{(i,j+1)}\right]$$

$U_1 = [0 + 0 + U_2 + U_3]\frac{1}{4}$
$U_2 = [0 + U_1 + U_4 + 2]\frac{1}{4}$
$U_3 = [0 + U_1 + U_4 + 2]\frac{1}{4}$
$U_4 = [U_2 + U_3 + 4]\frac{1}{4}$

$4U_1 = U_2 + U_3 \Rightarrow 4U_1 - 2U_2 + 0U_4 = 0$
$4U_2 = U_1 + U_4 + 2 \Rightarrow 4U_1 + 4U_2 - U_4 = 2$
$4U_3 = U_1 + U_4 + 2 \Rightarrow 0U_1 - 2U_2 + 4U_4 = 4$
$4U_4 = U_2 + U_3 + 4$

$U_1 = 0.5, \quad U_2 = U_3 = 1, \quad U_4 = 1.5$

2. Solve $U_{xx} + U_{yy} = 0$ for the square mesh with the boundary values as shown.



$$U_{ij} = \frac{1}{4}\left[U_{(i-1,j)} + U_{(i+1,j)} + U_{(i,j-1)} + U_{(i,j+1)} \right]$$

$4U_1 = U_2 + U_3 + 2$ ---- (i)
$4U_2 = U_1 + U_4 + 6$ ---- (ii)
$4U_3 = U_1 + U_4 + 6$ ---- (iii)
$4U_4 = U_2 + U_3 + 10$ ---- (iv)

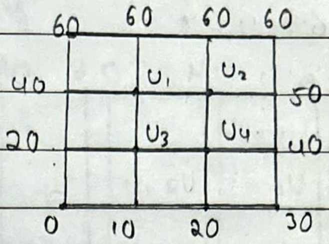
From (ii) & (iii) $U_2 = U_3$.

$\therefore$ eq (i) = $4U_1 - U_2 - U_2 = 2 \Rightarrow 4U_1 - 2U_2 - 0U_4 = 2$ ----- (v)
eq (ii) = $4U_2 - U_1 - U_4 = 6 \Rightarrow -U_1 + 4U_2 - U_4 = 6$ ---- (vi)
eq (iii) = $4U_4 - U_2 - U_2 = 10 \Rightarrow 0U_1 - 2U_2 + 4U_4 = 10$ ---- (vii)

Solving (v), (vi) & (vii)

$U_1 = 2, \; U_2 = U_3 = 3, \; U_4 = 4$ //

3. Solve $U_{xx} + U_{yy} = 0$ for the square mesh with boundary values as shown.



$$U_{(i,j)} = \frac{1}{4}\left[U_{(i-1,j)} + U_{(i+1,j)} + U_{(i,j-1)} + U_{(i,j+1)} \right]$$

$4U_1 = 40 + U_2 + 60 + U_3 = U_2 + U_3 + 100 = 4U_1 - U_2 - U_3 - 100 = 0$

$4U_2 = U_1 + U_4 + 110 \qquad = -U_1 = -4U_2 + U_4 + 110$

$4U_3 = U_1 + U_4 + 30 \qquad = -U_1 = -4U_3 + U_4 + 30$

$4U_4 = U_2 + U_3 + 60$

eq (i) - (iv) $\qquad$ (i) $\Rightarrow 4(U_4 + 10) = U_2 + U_3 + 100$

$4U_1 - 4U_4 = 40 \qquad -U_2 - U_3 + 4U_4 = 60$ --- (v)

$U_1 - U_4 = 10$

$U_1 = U_4 + 10$

(ii) $\Rightarrow$

$4U_2 = (U_4 + 10) + U_4 + 110$

$4U_2 = 0U_3 + 2U_4 + 120$

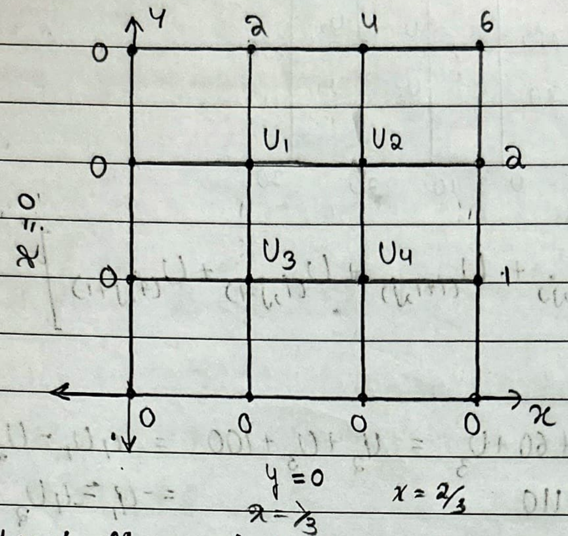
$4U_2 - 2U_4 = 120$ --- (vi)

(iii) $\Rightarrow 0U_2 + 4U_3 - 2U_4 = 40$ --- (vii)

$\therefore U_2 = 46.66$, $U_3 = 26.666$, $U_4 = 33.333$ $\quad U_1 = 43.333$

//

4. Solve the laplace equation $U_{xx}+U_{yy}=0$ for $0<x,y<1$, $u(x,0)=u(0,y)=0$, $u(x,1)=6x$, $0<x\le 1$ & $u(1,y)=3y$, $0<y<1$. Divide the region into 9 square meshes. $6x$ $6x$



$h = 1/3$

$u(x,1)=6x$

$u(ih,1)=6x$

$u(i/_3,1)=\frac{6}{3}i$

$\therefore u(i/_3,1)=2i$

$i=1,2,3$

$u(1,y)=3y$

$u(1,j/3)=3j/_3$

$u(1,j/3)=j$

$j=1,2.$

Standard five point formula:

$$U_{ij}=\frac{1}{4}\left[U_{(i+1,j)}+U_{(i-1,j)}+U_{(i,j+1)}+U_{(i,j-1)}\right]$$

$$4U_1 = 0+U_2+2+U_3 \quad = U_2+U_3+2 \text{ ---- (i)}$$
$$4U_2 = U_1+2+4+U_4 \quad = U_1+U_4+6 \text{ ---- (ii)}$$
$$4U_3 = 0+U_4+2+U_1 \quad = U_1+U_4+2 \text{ ---- (iii)}$$
$$4U_4 = U_3+1+U_2+0 \quad = U_2+U_3+1 \text{ ---- (iv)}$$

(i) − (iv) ⇒ $4U_1-U_4=1$

$U_1-U_4=1/4$

$U_1=U_4+0.25$ -----

(i) ⇒ $4(U_4+0.25)=U_2+U_3+2$

$-U_2-U_3+4U_4=1$ ----→ (v)

(ii) ⇒ $4\cancel{e}U_2 = U_4+0.25+6+U_4$

$4U_2+0U_3-2U_4=6.25$ ----→ (vi)

(iii) ⇒ $4U_3=U_4+2+(U_4+0.25)$

$0U_2+4U_3-2U_4=0.25$ ---→ (vii)

Solving (v), (vi), (vii) ...

$U_2=2$, $U_3=0.5$, $U_4=0.875$, $U_1=1.125$ //

$u_1 = 1/2 \quad u_2 = u_3 = 1 \quad u_4 = 1.5$

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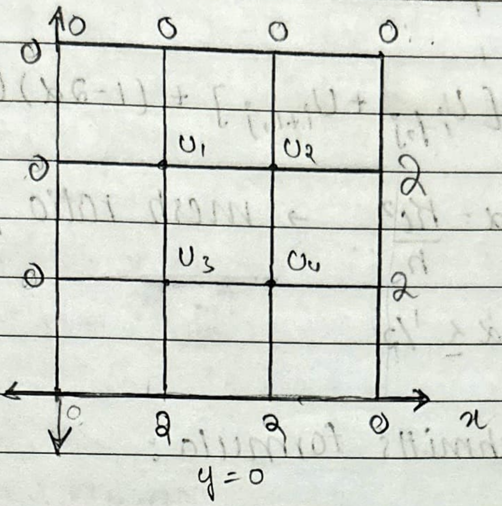
5. Solve the laplace equation $u_{xx} + u_{yy} = 0$, $0 < x < 1$, $0 < y < 1$, given $u(x,1) = u(0,y) = 0$ & $u(x,0) = 9(x - x^2)$, $u(1,y) = 9(y - y^2)$ take $h = 1/3$.



$h = 1/3$

$x = i \cdot \frac{1}{3}$

$u(x,0) = 9(x - x^2)$

$u(i/3, 0) = 9\left(\frac{i}{3} - \frac{i^2}{9}\right)$

$u(i/3, 0) = 9\left(\frac{3i - i^2}{9}\right)$

$u(i/3 = 0) = i(3 - i)$

$i = 1, 2, 3.$

$1(3-1)$

$= 2$

$u(x,1) = u(0,y)$

$x, y = 1, \quad x, y$

$2(3-2)$

$2(1)$

$3(3-3)$

$9\left(j/3 - \frac{j^2}{9}\right)$

$= j(3-j)$

$=$

$4u_1 = 0 + 0 + u_2 + u_3$

$4u_2 = 0 + u_1 + u_4 + 2$

$4u_3 = 0 + u_1 + 2 + u_4$

$4u_4 = u_3 + u_2 + 2 + 2$

$u_2 = u_3.$

(i) $\Rightarrow 4u_1 - 2u_2 + 0u_4 = 0$

(ii) $\Rightarrow -u_1 + 4u_2 + u_4 = 2$

(iv) $\Rightarrow 0u_1 - 2u_2 + 4u_4 = 4$

$u_1 = 0.5$

$u_2 = u_3 = 1$

$u_4 = 1.5$ //

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Numerical solution of heat equation:

$$U_t = c^2 U_{xx} \qquad \left[\frac{\partial U_t}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}\right]$$

(i) Schmitt's explicit formula:

$$U_{i,j+1} = \alpha\left[U_{i-1,j} + U_{i+1,j}\right] + (1-2\alpha)U_{i,j}$$

where $\alpha = \frac{kc^2}{h}$ → mesh ratio parameter

$$0 < \alpha \leq 1/2$$

(ii) Bendre - Schmitt's formula:

$$U_{i,j+1} = \frac{1}{2}\left[U_{i-1,j} + U_{i+1,j}\right]$$

1. Find the numerical solution of $U_{xx} = 2U_t$ when $U(0,t) = 0$, $U(4,t) = 0$, $u(x,0) = x(4-x)$ taking $h = k = 1$. Find the values upto $t = 3$.

Given $U_{xx} = 2U_t$

$$U_t = \frac{1}{2}U_{xx}$$

$$\therefore c^2 = \frac{1}{2}$$

$$\alpha = \frac{kc^2}{h^2} = \frac{1 \times 1/2}{1} = 1/2$$

$$U_{i,j+1} = \frac{1}{2}\left[U_{i-1,j} + U_{i+1,j}\right] \text{ ---- (i)}$$

when $x = 0$ to $x = 4$

$ih = 0$ $ih = 4$

$i = 0$ to $i = 4$

$t = 3$

$jk = 3$, $j = 3$ or $j = 0, 1, 2, 3$

j \ i	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	3/2	2	3/2	0
3	0	1	3/2	1	0

Given $u(x,0) = x(4-x)$

$u(i,0) = i(4-i)$

$i = 1,2,3$

$i=1$

$u(1,0) = 1(4-1)$

$= 3$

put $i=1, j=0$ in eq (i)

$U_{i,j+1} = \frac{1}{2}[U_{i-1,j} + U_{i+1,j}]$

$= \frac{1}{2}[4] = 2.$ //

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

Q. Solve the BVP $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ & boundary conditions

$u(0,t) = 0 = u(8,t)$

$u(x,0) = 4x - \frac{1}{2}x^2$

$x = i$, $i = 0,1,2 \dots 8$

$t = \frac{1}{8}j$, $j = 0,1,2$

Given, $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$

$\therefore c^2 = 4$

$\alpha = \frac{kc^2}{h^2} = \frac{\frac{1}{8}\times 4}{1} = \frac{1}{2}$

$U_{i,j+1} = \frac{1}{2}\left[U_{i-1,j} + U_{i+1,j}\right]$ ---(i)

j \ i	0	1	2	3	4	5	6	7	8
0	0	7/2	6	15/2	8	15/2	6	7/2	0
1	0	3	11/2	7	15/2	7	11/2	3	0
2	0	11/4	5	13/2	7	13/2	5	11/4	0

$u(x,0) = 4x - \frac{1}{2}x^2$

$u(i,0) = 4i - \frac{1}{2}i^2$, $i = 1,2 \dots 7$

$i = 1$,

$u(1,0) = 4(1) - \frac{1}{2}(1) = \frac{7}{2}$

put $i=1$, $j=0$ in eq (i)

$U_{1,1} = \frac{0+6}{2} = 3$