Multivariable Calculus – Integration

Double Integral:

is denoted by $\iint_D f(x,y)\, dA$ —— (1)

gives the volume bounded by the surface $f(x,y)$ over the region (domain) D.

Where D is a closed and bounded domain in the XY-plane bounded by a simple closed curve c. $f(x,y)$ be a given continuous function in D.

For purpose of evaluation (1) is expressed as

$$I = \iint_D f(x,y)\, dA = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\, dx\, dy$$

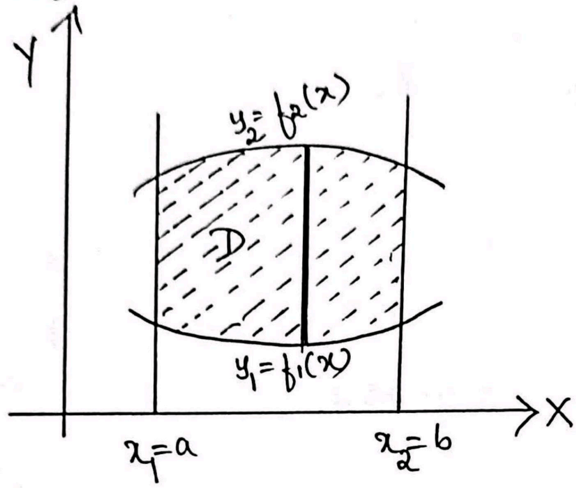
Its value is found as follows:

1) When $y_1 = f_1(x)$, $y_2 = f_2(x)$, $x_1 = a$ & $x_2 = b$:

Then $$I = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_1} f(x,y)\, dy}\, dx}$$

In this case the integration is first performed with respect to 'y' and then with respect to 'x'.

Geometrically:

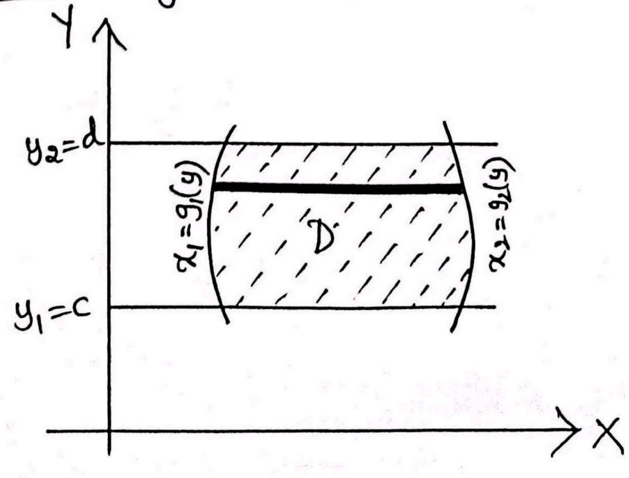


2) When $x_1 = g_1(y)$, $x_2 = g_2(y)$, $y_1 = c$ & $y_2 = d$:

Then $$I = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x,y)\,dx}\, dy}$$

In this case the integration is first performed w.r.t 'x' and then w.r.t 'y'.

Geometrically:

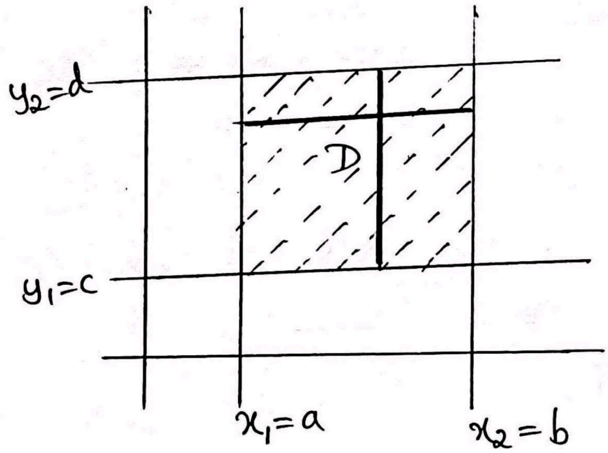


3) When $x_1 = a$, $x_2 = b$, $y_1 = c$ & $y_2 = d$:
(i.e, all the four limits are constants)

$$I = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x,y)\,dx}\,dy} = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x,y)\,dy}\,dx}$$

In this case the order of integration can be done in either way.

<u>Geometrically:</u>



Evaluate the following double integrals:

1. $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx$

<u>Ans:</u> Let $I = \int_{x=0}^{1} \int_{y=x}^{\sqrt{x}} (x^2+y^2)\,dy\,dx$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_{y=x}^{\sqrt{x}} dx$$

$$I = \int_0^1 \left[x^2(\sqrt{x} - x) + \frac{1}{3}\left((\sqrt{x})^3 - x^3\right)\right] dx$$

$$= \int_0^1 \left[\left(x^{5/2} - x^3\right) + \frac{1}{3}\left(x^{3/2} - x^3\right)\right] dx$$

$$= \left[\frac{2}{7} x^{7/2} - \frac{x^4}{4} + \frac{1}{3}\left(\frac{2}{5} x^{5/2} - \frac{x^4}{4}\right)\right]_{x=0}^{1}$$

$$= \frac{2}{7} - \frac{1}{4} + \frac{1}{3}\left(\frac{2}{5} - \frac{1}{4}\right)$$

$$I = \frac{3}{35} \simeq \underline{0.08571}$$

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2. $\displaystyle\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx$

<u>Ans:</u> Let $\displaystyle I = \int_{x=0}^{1} \int_{y=0}^{x^2} e^{y/x}\, dy\, dx$

$$= \int_0^1 \left(\frac{e^{y/x}}{\frac{1}{x}}\right)_{y=0}^{x^2} dx$$

$$= \int_0^1 x\left[e^x - 1\right] dx$$

$$= \left[\left(x e^x - e^x\right) - \left(\frac{x^2}{2}\right)\right]_{x=0}^{1}$$

$$I = (e - 0) - (e - 1) - \frac{1}{2}(1 - 0)$$

$$= e - e + 1 - \tfrac{1}{2}$$

$$I = \tfrac{1}{2}$$

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3. $\displaystyle\int_1^2 \int_0^{2-y} xy \; \underline{dy \; dx}$
   Note

Ans: Let $\displaystyle I = \int_{y=1}^{2} \int_{x=0}^{2-y} xy \; \underline{dx \; dy}$
Note

$$= \int_{y=1}^{2} y \left(\frac{x^2}{2}\right)_{x=0}^{2-y} dy$$

$$= \tfrac{1}{2} \int_1^2 y(2-y)^2 \, dy$$

$$= \tfrac{1}{2} \int_1^2 \left(4y - 4y^2 + y^3\right) dy$$

$$= \tfrac{1}{2} \left[\frac{4y^2}{2} - \frac{4y^3}{3} + \frac{y^4}{4}\right]_{y=1}^{2}$$

$$= \tfrac{1}{2}\left[2(4-1) - \frac{4}{3}(8-1) + \tfrac{1}{4}(16-1)\right]$$

$$= \tfrac{1}{2}\left[6 - \frac{28}{3} + \frac{15}{4}\right]$$

$$= \frac{5}{24} \simeq 0.2083$$

4. $\int_0^2 \int_0^{x^2} x(x^2+y^2)\, dx\, dy$

Note

Ans: Let $I = \int_{x=0}^{2} \int_{y=0}^{x^2} x(x^2+y^2)\, dy\, dx$

$$= \int_0^2 \left(x^3 y + \frac{x y^3}{3}\right)_{y=0}^{x^2} dx$$

$$= \int_0^2 \left(x^5 + \frac{x^7}{3}\right) dx$$

$$= \left[\frac{x^6}{6} + \frac{x^8}{24}\right]_{x=0}^{2}$$

$$= \frac{64}{6} + \frac{256}{24}$$

$$= \frac{64}{3} \simeq 21.33$$

5. $\int_1^4 \int_3^5 x^2 y\, dy\, dx$

Ans: Let $I = \int_{x=1}^{4} \int_{y=3}^{5} x^2 y\, dy\, dx \left(= \int_3^5 \int_1^4 x^2 y\, dx\, dy\right)$

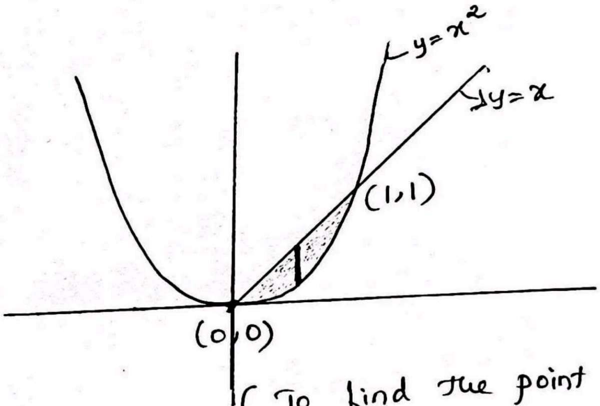
$$= \int_1^4 x^2 \left(\frac{y^2}{2}\right)_{y=3}^{5} dx = \int_1^4 x^2 \left(\frac{25-9}{2}\right) dx$$

$$= 8\left[\frac{x^3}{3}\right]_{x=1}^{4} = \frac{8}{3}\left[64-1\right] = 168$$

6. Evaluate $\iint_D xy(x+y)\,dx\,dy$, if D is the region bounded between $y=x^2$ and $y=x$.

<u>Ans:</u> Given D is the region bounded between $y=x^2$ and $y=x$.

(By the rough sketch of D, we have)



$$I = \iint_D xy(x+y)\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} xy(x+y)\,dy\,dx$$

$$= \int_0^1 \left(x^2\frac{y^2}{2} + \frac{xy^3}{3}\right)_{y=x^2}^{x} dx$$

$$= \int_0^1 \left[\frac{x^2}{2}(x^2-x^4) + \frac{x}{3}(x^3-x^6)\right] dx$$

$$= \int_0^1 \left(\frac{x^4}{2} - \frac{x^6}{2} + \frac{x^4}{3} - \frac{x^7}{3}\right) dx$$

$$= \left(\frac{x^5}{10} - \frac{x^7}{14} + \frac{x^5}{15} - \frac{x^8}{24}\right)_{x=0}^{1}$$

$$= \frac{1}{10} - \frac{1}{14} + \frac{1}{15} - \frac{1}{24} = \frac{3}{56} \simeq 0.05357$$

To find the point of intersection

$x^2 = x \Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

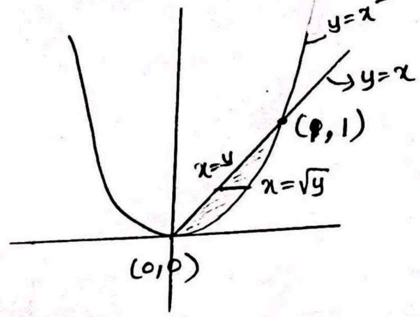
$\Rightarrow x=0,\ x=1$

$\Rightarrow y=0,\ y=1$

Note: Here 'I' can also obtained as

$$I = \int_{y=0}^{1}\int_{x=y}^{\sqrt{y}} xy(x+y)\,dx\,dy$$

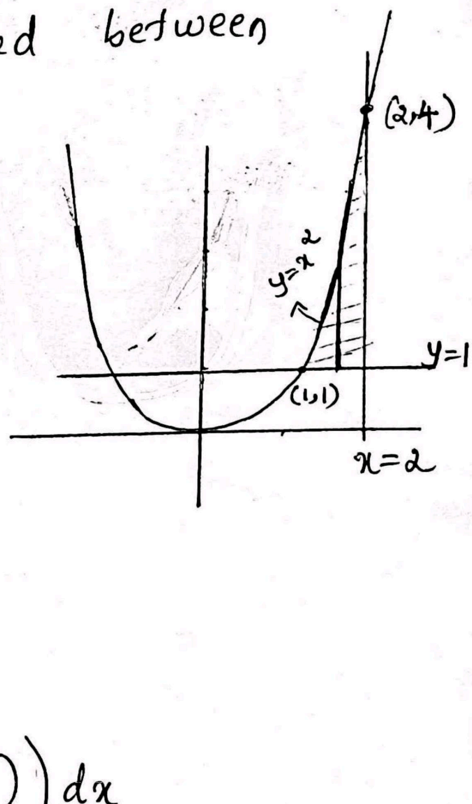
$$= \frac{3}{56}$$



7. Evaluate $\iint_D (x^2+y^2)\,dx\,dy$, if D is the region bounded between $x=2$, $y=1$ and $y=x^2$.

Ans: Let $I = \iint_D (x^2+y^2)\,dx\,dy$

Given D is the region bounded between $x=2$, $y=1$ and $y=x^2$.



From the rough sketch of D

$$I = \int_{x=1}^{2}\int_{y=1}^{x^2} (x^2+y^2)\,dy\,dx$$

$$= \int_{x=1}^{2}\left[x^2y + \frac{y^3}{3}\right]_{y=1}^{x^2} dx$$

$$= \int_{x=1}^{2}\left(x^2(x^2-1) + \frac{1}{3}(x^6-1)\right) dx$$

$$I = \int_{x=1}^{2} \left(x^4 - x^2 + \frac{x^6}{3} - \frac{1}{3} \right) dx$$

$$= \left[\frac{x^5}{5} - \frac{x^3}{3} + \frac{x^7}{21} - \frac{x}{3} \right]_{x=1}^{2}$$

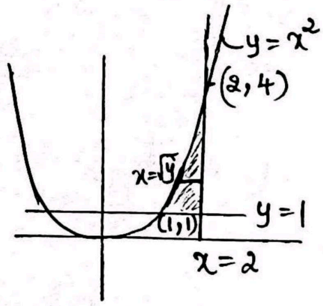
$$= \frac{1}{5}\left[32-1\right] - \frac{1}{3}\left[8-1\right] + \frac{1}{21}(128-1) - \frac{1}{3}(2-1)$$

$$= \frac{31}{5} - \frac{7}{3} + \frac{127}{21} - \frac{1}{3}$$

$$= \frac{1006}{105} \simeq 9.5809$$

Note: Here 'I' can also obtained as

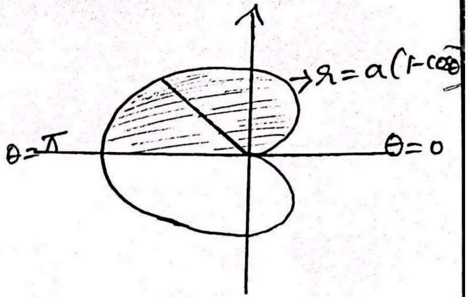
$$I = \int_{y=1}^{4} \int_{x=\sqrt{y}}^{2} (x^2+y^2)\, dx\, dy$$



$$= \frac{1006}{105}$$

8. Evaluate $\iint_D r \sin\theta \, dr\, d\theta$, if D is the region bounded by the cardiod $r = a(1-\cos\theta)$ above the initial line.

Ans: By the rough sketch of D



$$I = \iint_D r \sin\theta \, dr \, d\theta$$

$$= \int_{\theta=0}^{\pi} \int_{r=0}^{a(1-\cos\theta)} r \sin\theta \, dr \, d\theta$$

$$= \int_{\theta=0}^{\pi} \sin\theta \left(\frac{r^2}{2}\right)_0^{a(1-\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi} a^2 (1-\cos\theta)^2 \sin\theta \, d\theta$$

$$= \frac{a^2}{2}\int_{t=0}^{2} t^2 \, dt$$

put $1-\cos\theta = t$
$\Rightarrow \sin\theta \, d\theta = dt$
When $\theta=0$, $t=0$
$\theta=\pi$, $t=2$

$$= \frac{a^2}{2}\left[\frac{t^3}{3}\right]_0^2$$

$$= \frac{a^2}{6}\left[8-0\right] = \frac{4a^2}{3}$$

H.W

9. $I = \iint_D (x^2+y^2)\, dx\, dy$ where D is bounded by $y=x$ and $y^2=4x$ [Ans: $\frac{768}{35}$

Change of order of integration:

$$I = \iint_D f(x,y)\, dA \quad \text{---(1)}$$

- The limits of integration can be fixed from a rough sketch of the domain of integration.

Then (1) can be evaluated as

$$I = \int_{x=a}^{b} \int_{y=y_1(x)}^{y_2(x)} f(x,y)\, dy\, dx \quad \text{---(2)}$$

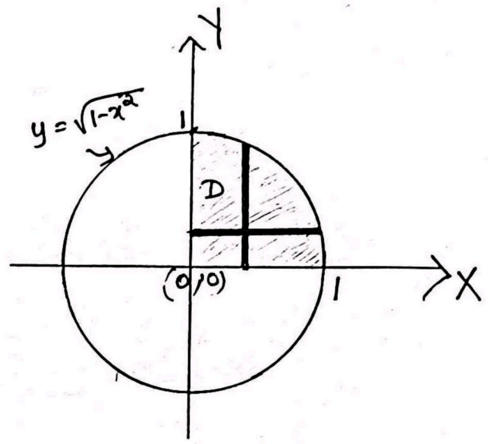
'OR'

$$I = \int_{y=c}^{d} \int_{x=x_1(y)}^{x_2(y)} f(x,y)\, dx\, dy \quad \text{---(3)}$$

In each specific problem, depending upon the type of the domain D and/or the nature of integrand, choose either of the form (2) or (3) whichever is easier to evaluate. Thus in several problems, the evaluation of double integrals becomes easier with the change of order of integration, which of course, changes the limits of integration also.

Change the order of integration and hence evaluate the following double integrals.

(1) $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dy \, dx$



Ans: By the rough sketch of the region D, we have the required region is the portion of the circle in the 1st quadrante

$\because y = \sqrt{1-x^2}$
$\Rightarrow y^2 = 1 - x^2$
$\Rightarrow x^2 + y^2 = 1$
$\Rightarrow x = \sqrt{1-y^2}$

$$\therefore I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} y^2 \, dy \, dx$$

$$= \int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} y^2 \, dx \, dy$$

$$= \int_0^1 y^2 \left[x\right]_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 y^2 \sqrt{1-y^2} \, dy$$

$$= \int_0^{\pi/2} \sin^2\theta \sqrt{1-\sin^2\theta} \, \cos\theta \, d\theta$$

put $y = \sin\theta$
$\Rightarrow dy = \cos\theta \, d\theta$
when $y=0$, $\theta=0$
$y=1$, $\theta=\pi/2$

$$I = \int_0^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta$$

$$= \frac{1}{4}\left(\frac{1}{2} \cdot \pi/2\right) \qquad \left(\because \text{by reduction formula } \int_0^{\pi/2} \sin^m x \cos^n x \, dx\right.$$

$$\underline{I = \frac{\pi}{16}}$$

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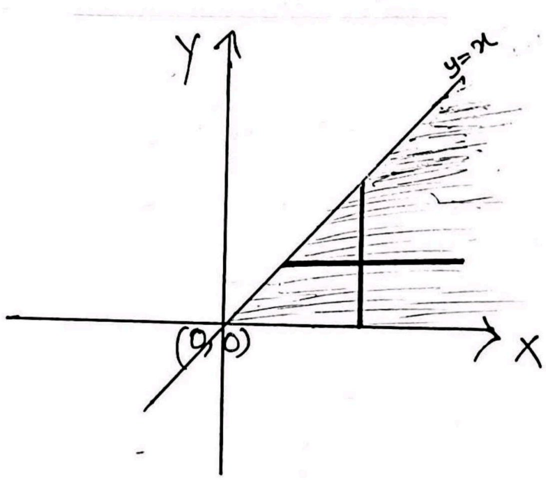
2) $\displaystyle\int_0^{\infty}\int_0^{x} x e^{-\frac{x^2}{y}} \, dy \, dx$



<u>Ans:</u> By the rough sketch of the region D, the required region is the portion under the line $y = x$ and above +ve x-axis, in the I quadrant.

$$\therefore \; I = \int_{x=0}^{\infty}\int_{y=0}^{x} x e^{-\frac{x^2}{y}} \, dy \, dx$$

$$= \int_{y=0}^{\infty}\int_{x=y}^{\infty} x e^{-x^2/y} \, dx \, dy$$

$$= \int_{y=0}^{\infty}\int_{t=y}^{\infty} e^{-t} \, y/2 \, dt \, dy$$

put $x^2/y = t$

$\Rightarrow \frac{2x}{y} dx = dt$

$\Rightarrow x \, dx = \frac{y}{2} dt$

When $x = y$, $t = y$

$x = \infty$, $t = \infty$

$$= \frac{1}{2}\int_{y=0}^{\infty} y \left[\frac{e^{-t}}{-1}\right]_y^{\infty} dy$$

$$I = -\tfrac{1}{2} \int_{y=0}^{\infty} y\left(0 - e^{-y}\right) dy$$

$$= \tfrac{1}{2} \int_{y=0}^{\infty} y e^{-y} \, dy$$

$$= \tfrac{1}{2} \left[\cancel{-ye^{-y}} - e^{-y} \right]_0^{\infty} \qquad \left\{ \begin{array}{l} \because \text{ by integration by parts} \\ e^{-\infty} = 0 \end{array} \right.$$

$$= \tfrac{1}{2} \left[-(0 - 1) \right]$$

$$I = \tfrac{1}{2}$$

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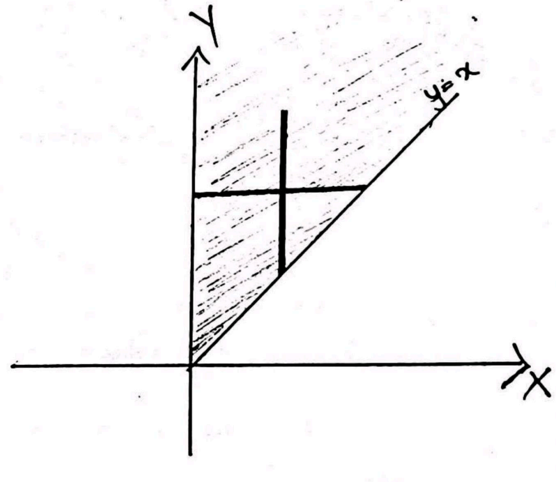
3) $\int_0^{\infty} \int_x^{\infty} \frac{e^{-y}}{y} \, dy \, dx$



Ans: By the rough sketch of the given region D, the required region is the portion above the line $y = x$ and +ve. y-axis in the I quadrant.

$$\therefore I = \int_{x=0}^{\infty} \int_{y=x}^{\infty} \frac{e^{-y}}{y} \, dy \, dx$$

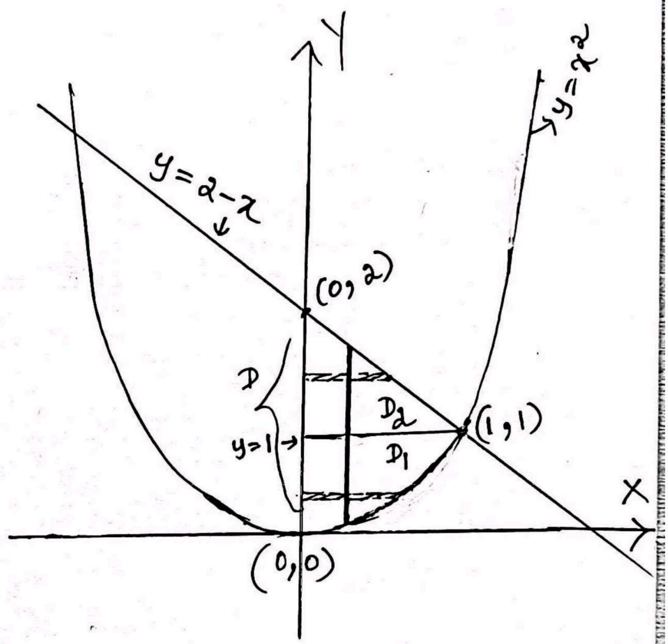
$$= \int_{y=0}^{\infty} \int_{x=0}^{y} \frac{e^{-y}}{y} \, dx \, dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y} \left[x \right]_0^{y} dy$$

$$I = \int_{y=0}^{\infty} \frac{e^{-y}}{y} (y)\, dy$$

$$= \left[\frac{e^{-y}}{-1}\right]_0^{\infty} = -(0-1) = 1$$

4) $\int_0^1 \int_{x^2}^{2-x} xy \, dx\, dy$



Ans: By the rough sketch of the given region D, required region is the portion bounded by the curve $y=x^2$ & the line $y=2-x$ in the I quadrant.

$x^2 = 2-x$
$x^2+x-2=0$
$(x+2)(x-1)=0$
$x=1, -2$

$$I = \int_0^1 \int_{x^2}^{2-x} xy \, dx\, dy$$

$$= \iint_{D=D_1+D_2} xy \, dx\, dy$$

$$I = \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\, dx\, dy + \int_{y=1}^{2}\int_{x=0}^{2-y} xy\, dx\, dy$$

($\because$ Limits : D_1 : x : 0 to $\sqrt{y}$, y : 0 to 1 ;
for D_2 : x : 0 to $2-y$, y : 1 to 2)

$$\therefore I = \int_{y=0}^{1} y \left(\frac{x^2}{2}\right)_0^{\sqrt{y}} dy + \int_{y=1}^{2} y \left(\frac{x^2}{2}\right)_0^{2-y} dy$$

$$= \frac{1}{2}\int_0^1 y\,(y-0)\,dy + \frac{1}{2}\int_{y=1}^{2} y\left((2-y)^2-0\right) dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\int_{y=1}^{2} \left(4y - 4y^2 + y^3\right) dy$$

$$= \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{2}\left(\frac{4y^2}{2} - \frac{4y^3}{3} + \frac{y^4}{4}\right)_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left(2(4-1) - \frac{4}{3}(8-1) + \frac{1}{4}(16-1)\right)$$

$$+ \frac{1}{2}\left(6 - \frac{28}{3} + \frac{15}{4}\right)$$

$$I = \frac{3}{8} \simeq 0.375$$

H.W problems

5) $\int_0^2 \int_1^{e^x} dy\, dx$ (Ans : $I = e^2 - 3$)

6) $\int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx$ (Ans: $I = \frac{1}{24}$)

Change of Variables (Cartesian to Polar)

We have the relation between the cartesian to polar coordinates as $x = r\cos\theta$, $y = r\sin\theta$

In several cases, given double integral can be evaluated easily by changing the variables from cartesian to polar form as follows

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dx\,dy = \int_{\theta_1}^{\theta_2}\int_{r_1=f_1(\theta)}^{r_2=f_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

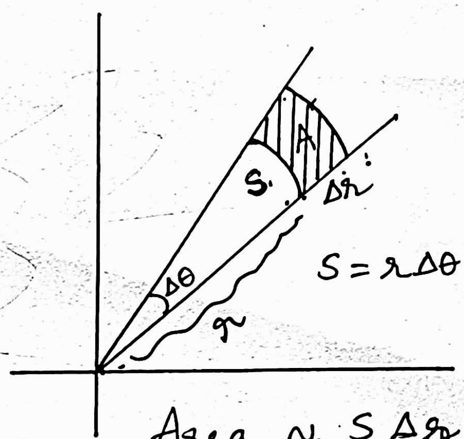
$$= \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\,J\left(\frac{x,y}{r,\theta}\right)dr\,d\theta$$

Reason: for where did the r come from?

Area of the small region given by = arch length $\times \Delta r$

$= r\,\Delta\theta\,\Delta r$

$= r\,\Delta r\,\Delta\theta$



Area $\approx S\,\Delta r$

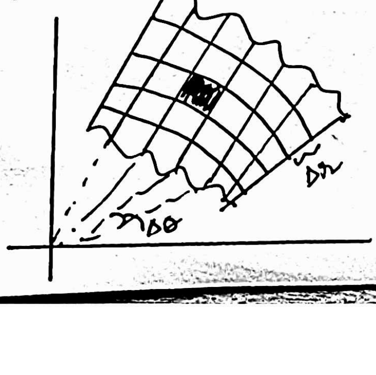
$= (r\,\Delta\theta)\Delta r$

S is the arch length

$S = r\theta$

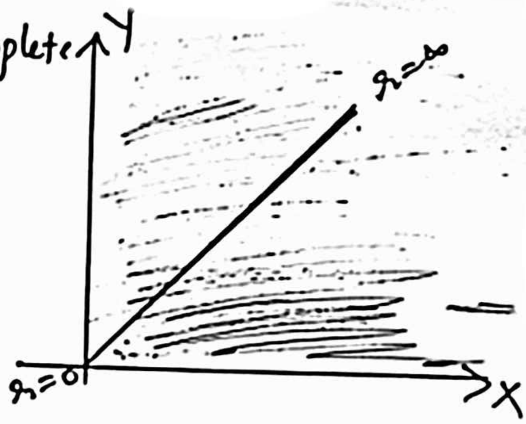
$\therefore$ For the entire region R after taking limit as $\Delta r, \Delta\theta \to 0$

Area $= \iint_R r\,dr\,d\theta$



1) Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$ by changing to polar co-ordinates.

Ans: Given region is the complete I quadrant.



By changing to polar form

$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-(r^2\cos^2\theta + r^2\sin^2\theta)}\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_0^\infty e^{-r^2}\, r\, dr\, d\theta$$

put $r^2 = t$

$\Rightarrow 2r\, dr = dt$

$\Rightarrow r\, dr = \frac{dt}{2}$

$$= \int_{\theta=0}^{\pi/2} \int_0^\infty e^{-t}\, \frac{dt}{2}\, d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi/2} \left(\frac{e^{-t}}{-1}\right)_0^\infty d\theta$$

$$= -\frac{1}{2} \int_{\theta=0}^{\pi/2} (0-1)\, d\theta$$

$$= \frac{1}{2} (\theta)_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

Applications of double integrals:

1. To find area of a plane region D.

$$A = \iint_D dx\,dy \quad ; \quad A = \iint_D r\,dr\,d\theta$$

(Cartesian form) (Polar form)

2. Volume under a surface of a solid as a double integral

Let $z = f(x,y) > 0$ be the equation of a surface, defined on the region D in the XY-plane. Then the double integral of $f(x,y)$ taken over D gives the volume V under the surface $z = f(x,y)$

$$V = \iint_D f(x,y)\,dx\,dy$$

3. To find centre of Gravity (Centroid) of a plane region D.

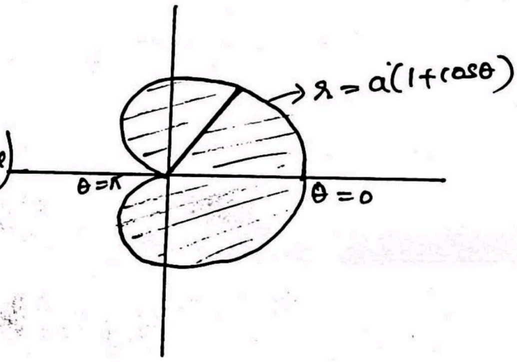
4. Moment of Inertia of a plane region.

Using double integrals:

1) find the area bounded by the cardiod

$r = a(1+\cos\theta)$

Ans: $A = 2\int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\,dr\,d\theta$



($\because$ curve is symmetric w.r.t initial line)

$$A = 2\int_{\theta=0}^{\pi}\left(\frac{r^2}{2}\right)_0^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi} a^2(1+\cos\theta)^2\, d\theta$$

$$= a^2\int_0^{\pi}\left(2\cos^2\theta/2\right)^2 d\theta$$

$$= 4a^2\int_0^{\pi}\cos^4\theta/2\; d\theta$$

put $\theta/2 = t$

$d\theta = 2\,dt$

when $\theta=0,\ t=0$

$\theta=\pi,\ t=\pi/2$

$$= a^2\int_0^{\pi/2}\cos^4 t\; 2\,dt$$

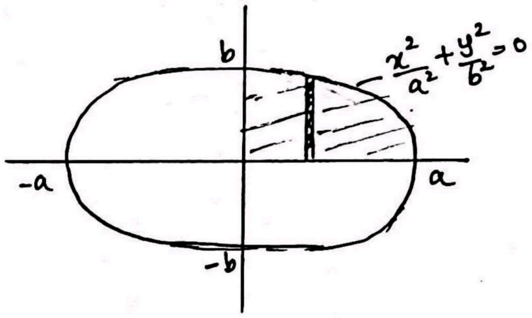
$$= 8a^2\left[\frac{3}{4}\,\frac{1}{2}\,\pi/2\right]$$

$$= \frac{3\pi a^2}{2} \text{ sq. units}$$

2) Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

<u>Ans:</u> Since the curve is symmetric w.r.t x & y axis

Required area = 4 x area in the I quadrant



$$A = 4 \times \left[\int_{x=0}^{a}\int_{y=0}^{b/a\sqrt{a^2-x^2}} dy\, dx\right]$$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\Rightarrow \quad y^2 = b^2\left(1-\frac{x^2}{a^2}\right)$

$\Rightarrow \quad y = \frac{b}{a}\sqrt{a^2-x^2}$

$$= 4\int_{x=0}^{a} (y)_0^{b/a\sqrt{a^2-x^2}}\, dx$$

$$= \frac{4b}{a}\int_{x=0}^{a}\sqrt{a^2-x^2}\, dx$$

put $x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

when $x=0,\ \theta=0$

$x=a,\ \theta=\pi/2$

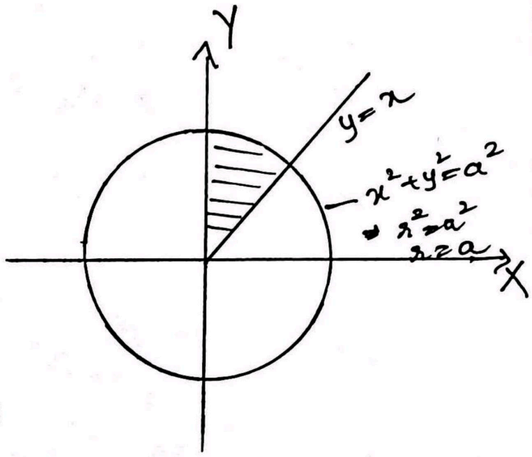
$$= \frac{4b}{a}\int_0^{\pi/2}\sqrt{a^2-a^2\sin^2\theta}\; a\cos\theta\, d\theta$$

$$= 4b\int_0^{\pi/2}(a\cos\theta)\cos\theta\, d\theta$$

$$= 4ab\int_0^{\pi/2}\cos^2\theta\, d\theta$$

$$= 4ab\left[\frac{1}{2}\ \frac{\pi}{2}\right] = \underline{\underline{\pi ab}} \text{ sq units}$$

3) Find the area of the region bounded by the positive y-axis, the arc of the circle $x^2+y^2=a^2$ and above the line $y=x$.



Ans: Because of symmetry of the circle in the I quadrant about the line $y=x$

$$\left.\begin{array}{l}\text{Required area}\\ A\end{array}\right\} = \frac{1}{2}\times\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r\,dr\,d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2}\left(\frac{r^2}{2}\right)_0^a d\theta$$

$$= \frac{a^2}{4}\int_{\theta=0}^{\pi/2} d\theta$$

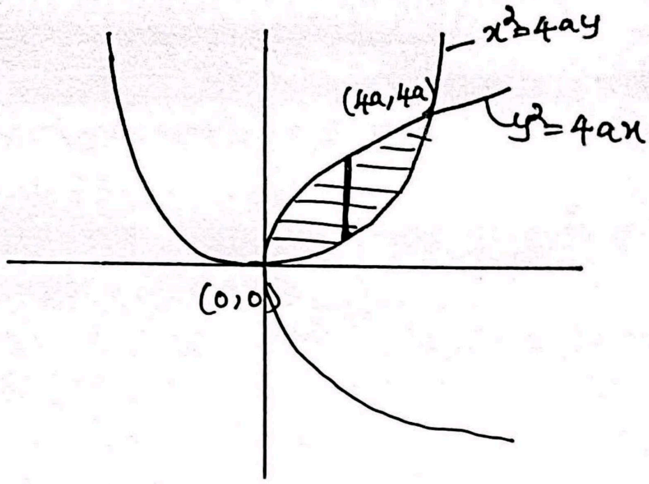
$$= \frac{a^2}{4}(\theta)_0^{\pi/2}$$

$$= \frac{a^2}{4}\left(\frac{\pi}{2}\right) = \frac{\pi a^2}{8} \text{ sq. units}$$

---

4. Show that the area between the parabolas $y^2=4ax$ and $x^2=4ay$ is $\frac{16}{3}a^2$.

<u>Ans:</u>



$$A = \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} dy\, dx$$

$$= \int_{x=0}^{4a} \left[y \right]_{x^2/4a}^{2\sqrt{ax}} dx$$

$$= \int_0^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a} \right) dx$$

$$= \left[2\sqrt{a} \frac{x^{3/2}}{3/2} - \frac{1}{4a} \frac{x^3}{3} \right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3} \quad 8a\sqrt{a} - \frac{1}{12a} 64a^3$$

$$= \frac{32a^2}{3} - \frac{32}{6}a^2$$

$$= \frac{32a^2}{6}$$

$$= \frac{16}{3}a^2$$

$y = \frac{x^2}{4a}$; $y^2 = 4ax$

$\Rightarrow y^2 = \frac{x^4}{16a^2}$

Now $\frac{x^4}{16a^2} = 4ax$

$x^4 = 64a^3x$

$x^4 - 64xa^3 = 0$

$x(x^3 - 64a^3) = 0$

$x = 0 \qquad x^3 = 64a^3$

$x = 4a$

Triple Integrals:

Triple integral is a generalization of a double integral. Let V be a given 3D domain in space, bounded by a closed surface S.

Let $f(x,y,z)$ be a continuous function in V of the rectangular co-ordinates x, y, z.

Then triple integral of 'f' over the domain V, denoted by

$$\iiint_V f(x,y,z)\, dx\, dy\, dz$$

Evaluation of a triple integral:

Let the equations of the surfaces bounding a regular domain V below and above be $z = z_1(x,y)$ and $z = z_2(x,y)$ & let the projection D of V onto xy-plane be bounded by the curves $y = y_1(x)$, $y = y_2(x)$, $x = a$ and $x = b$.

Then $$I = \int_{x=a}^{x=b} \left[\int_{y=y_1(x)}^{y=y_2(x)} \left\{ \int_{z=z_1(x,y)}^{z=z_2(x,y)} f(x,y,z)\, dz \right\} dy \right] dx$$

Evaluate the following triple integrals:

(1) $\int_0^1 \int_0^2 \int_1^2 x^2 yz \, dx \, dy \, dz$

Ans: $I = \int_0^1 \int_0^2 yz \left(\frac{x^3}{3}\right)_1^2 dy \, dz$

$= \int_0^1 \int_0^2 yz \left(\frac{8-1}{3}\right) dy \, dz$

$= \frac{7}{3} \int_0^1 z \left[\frac{y^2}{2}\right]_0^2 dz$

$= \frac{7}{3} \int_0^1 z \left[\frac{4-0}{2}\right] dz$

$= \frac{14}{3} \left[\frac{z^2}{2}\right]_0^1$

$= \frac{14}{3} \left[\frac{1}{2}\right] = \frac{7}{3}$

2> $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x+y+z) \, dy \, dx \, dz$

Ans: $$I = \int_{z=-1}^{1} \int_{x=0}^{z} \int_{y=x-z}^{x+z} (x+y+z)\, dy\, dx\, dz$$

$$= \int_{z=-1}^{1} \int_{x=0}^{z} \left(xy + \frac{y^2}{2} + zy \right)_{x-z}^{x+z} dx\, dz$$

$$= \int_{z=-1}^{1} \int_{x=0}^{z} \left(x\left[x+z-(x-z)\right] + \frac{1}{2}\left[(x+z)^2-(x-z)^2\right] + z\left[x+z-(x-z)\right] \right) dx\, dz$$

$$= \int_{z=-1}^{1} \int_{x=0}^{z} \left(2xz + \frac{1}{2}(4xz) + z(2z) \right) dx\, dz$$

$$= \int_{z=-1}^{1} \left(2z\left(\frac{x^2}{2}\right) + 2z\left(\frac{x^2}{2}\right) + 2z^2 x \right)_{x=0}^{z} dz$$

$$= \int_{z=-1}^{1} \left(2z\left(z^2\right) + 2z^2(z) \right) dz$$

$$= \int_{z=-1}^{1} 4z^3\, dz$$

$$= 4\left[\frac{z^4}{4}\right]_{-1}^{1} = (1-1) = 0$$

3) $$\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$$

Ans: $$I = \int_{x=0}^{a} \int_{y=0}^{x} \int_{z=0}^{x+y} e^{x+y+z}\, dz\, dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} e^{x+y} \left(e^{z}\right)_0^{x+y} dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} e^{x+y} \left(e^{x+y} - 1\right) dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} \left(e^{2x+2y} - e^{x+y}\right) dy\, dx$$

$$= \int_{x=0}^{a} \left(\frac{e^{2x+2y}}{2} - \frac{e^{x+y}}{1}\right)_{y=0}^{x} dx$$

$$= \int_{x=0}^{a} \left[\frac{1}{2}\left(e^{4x} - e^{2x}\right) - \left(e^{2x} - e^{x}\right)\right] dx$$

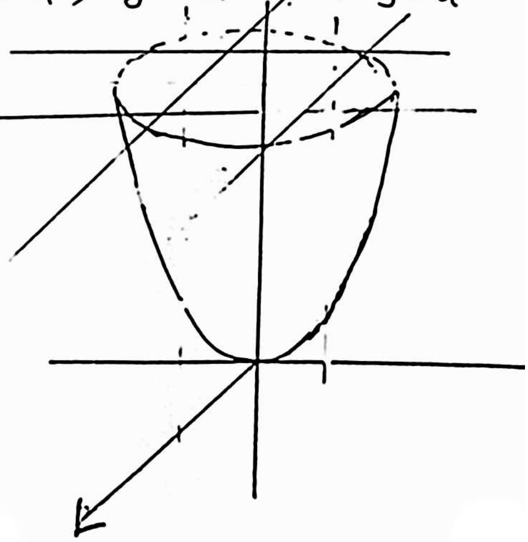
$$= \left[\frac{1}{2}\left(\frac{e^{4x}}{4}\right) - \frac{1}{2}\left(\frac{e^{2x}}{2}\right) - \frac{e^{2x}}{2} + e^{x}\right]_{x=0}^{a}$$

$$= \frac{1}{8}\left(e^{4a} - 1\right) - \frac{1}{4}\left(e^{2a} - 1\right) - \frac{1}{2}\left(e^{2a} - 1\right) + \left(e^{a} - 1\right)$$

$$= \frac{e^{4a}}{8} - \frac{3}{4}e^{2a} + e^{a} - \frac{3}{8}$$

7. Find the volume of the region bounded by

$z = x^2+y^2$, $z=0$, $x=-a$, $x=a$, $y=-a$ and $y=a$.



Ans: $V = \int_{x=-a}^{a}\int_{y=-a}^{a}\int_{z=0}^{x^2+y^2} dz\, dy\, dx$

$$V = \int_{x=-a}^{a}\int_{y=-a}^{a} [z]_0^{x^2+y^2}\, dy\, dx$$

$$V = \int_{x=-a}^{a}\int_{y=-a}^{a} (x^2+y^2)\, dy\, dx$$

$$= \int_{x=-a}^{a} \left[x^2 y + \frac{y^3}{3}\right]_{-a}^{a} dx$$

$$= \int_{x=-a}^{a} \left(x^2 (a-(-a)) + \frac{1}{3}(a^3-(-a^3))\right) dx$$

$$= \int_{x=-a}^{a} \left(2ax^2 + \frac{2a^3}{3}\right) dx$$

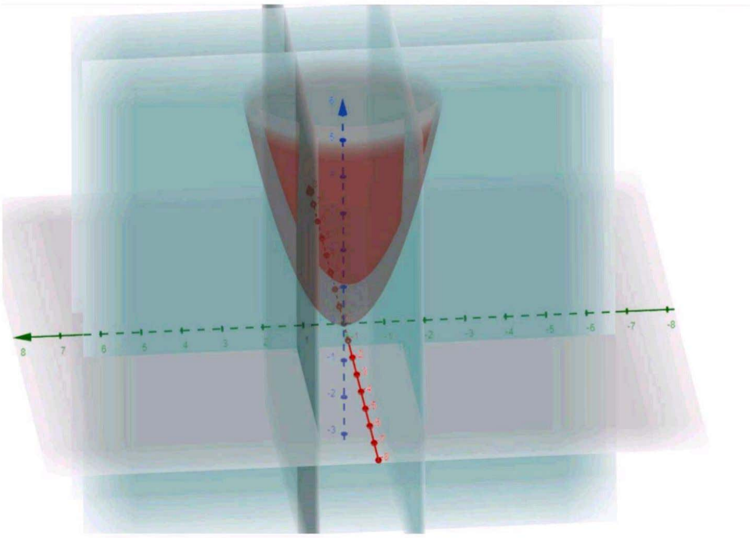
$$= \left[\frac{2a\,x^3}{3} + \frac{2a^3}{3}x\right]_{-a}^{a}$$

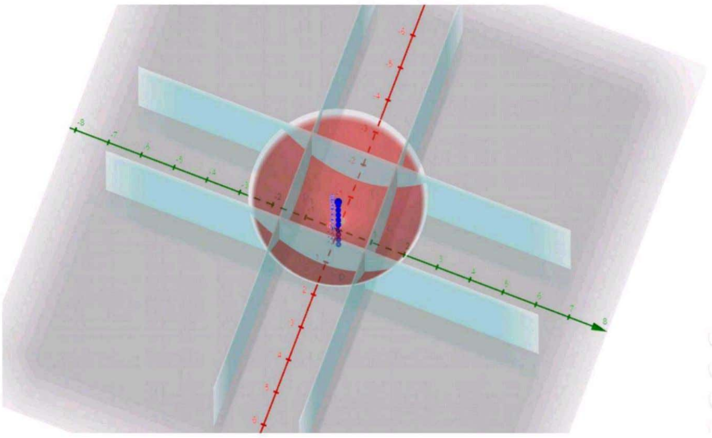
$$= \frac{2a}{3}\left(a^3-(-a^3)\right) + \frac{2a^3}{3}\left(a-(-a)\right)$$

$$= \frac{4a^4}{3} + \frac{4a^4}{3}$$

$$V = \frac{8a^4}{3} \text{ cubic units}$$

4. Volume bounded by the paraboloid $z = x^2 + y^2$ and 4 given lines





Beta - Gamma Function

Gamma Function : is defined as

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx \quad (n>0) \qquad \text{—— (1)}$$

Reduction formula for $\Gamma(n)$:

$$\Gamma(n+1) = n\,\Gamma(n)$$

Proof: Consider $\Gamma(n+1) = \int_0^\infty e^{-x} x^{n}\, dx$ ($\because$ by (1)

$$= \left[x^n\left(\frac{e^{-x}}{-1}\right)\right]_{x=0}^{\infty} - \int_0^\infty n x^{n-1}\left(\frac{e^{-x}}{-1}\right) dx$$

$$= (0-0) + n\int_0^\infty e^{-x} x^{n-1}\, dx$$

$$\Gamma(n+1) = n\,\Gamma(n)$$

Value of $\Gamma(n)$ in terms of factorial:

$$\Gamma(n+1) = n!$$

Proof: Using $\Gamma(n+1) = n\,\Gamma(n)$ successively, we get

$$\Gamma(n+1) = n\,\Gamma(n)$$

$$= n\,(n-1)\,\Gamma(n-1)$$

$$= n\,(n-1)\,(n-2)\,\Gamma(n-2)\,, \dots$$

$$\Gamma(n+1) = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1\ \Gamma(1) \longrightarrow (*)$$

Consider $\Gamma(1) = \int_0^\infty e^{-x} x^0\, dx = \left(\frac{e^{-x}}{-1}\right)_0^\infty$

$$= -(0-1) = 1$$

$$\therefore\ \Gamma(n+1) = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$

$$= n!$$

<u>Note:</u> $\Gamma(n)\,\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$

---

1. Obtain $\int_0^\infty x^n e^{-a^2x^2}\, dx$ in terms of Gamma function.

<u>Ans:</u> put $a^2x^2 = t \Rightarrow x = \dfrac{\sqrt{t}}{a} = \dfrac{t^{1/2}}{a}$

$$\Rightarrow dx = \frac{1}{a}\,\tfrac{1}{2}\,t^{-1/2}\, dt$$

When $x=0$, $t=0$;

$x=\infty$, $t=\infty$

Ans: $I = \int_0^\infty x^n e^{-a^2x^2}\,dx$

$$= \int_0^\infty \left(\frac{t^{1/2}}{a}\right)^n e^{-t}\, \frac{1}{2a}\, t^{-1/2}\,dt$$

$$= \frac{1}{2}\,\frac{1}{a^{n+1}} \int_0^\infty e^{-t}\, t^{\frac{n-1}{2}}\,dt$$

$$= \frac{1}{2a^{n+1}}\, \Gamma\left(\frac{n-1}{2}+1\right)$$

$$= \frac{1}{2a^{n+1}}\, \Gamma\left(\frac{n+1}{2}\right)$$

---

2. Evaluate $\int_0^\infty x^4 e^{-x^2}\,dx$

Ans: Put $x^2 = t \Rightarrow x = \sqrt{t} \Rightarrow x = t^{1/2}$

$\Rightarrow dx = \frac{1}{2} t^{-1/2}\,dt$

$$I = \int_0^\infty x^4 e^{-x^2}\,dx$$

$$= \int_0^\infty t^2 e^{-t}\, \frac{1}{2}\, t^{-1/2}\,dt$$

$$= \frac{1}{2} \int_0^\infty e^{-t}\, t^{3/2}\,dt = \frac{1}{2}\,\Gamma\left(\frac{3}{2}+1\right)$$

$$= \frac{1}{2}\,\Gamma\left(\frac{5}{2}\right) = \frac{1}{2}\,\frac{3}{2}\,\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\,\frac{3}{2}\,\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{3\sqrt{\pi}}{8}$$

3. Evaluate $\int_0^\infty x^6 e^{-2x}\, dx$

Ans: put $2x = t \Rightarrow x = t/2$

$\Rightarrow dx = 1/2\, dt$

$$\therefore I = \int_0^\infty x^6 e^{-2x}\, dx$$

$$= \int_0^\infty \left(t/2\right)^6 e^{-t}\, 1/2\, dt$$

$$= \frac{1}{2^7} \int_0^\infty e^{-t} t^6\, dt$$

$$= \frac{1}{128} \Gamma(6+1)$$

$$= \frac{1}{128} \Gamma(7) = \underline{\underline{\frac{6!}{128}}}$$

---

4. $\int_0^\infty e^{-x^2}\, dx$

Ans: put $x^2 = t \Rightarrow x = \sqrt{t}$

$\Rightarrow x = t^{1/2}$

$\Rightarrow dx = 1/2\, t^{-1/2}\, dt$

$$\therefore I = \int_0^\infty e^{-x^2}\, dx = \int_0^\infty e^{-t}\, 1/2\, t^{-1/2}\, dt$$

$$= 1/2 \int_0^\infty e^{-t} t^{-1/2}\, dt = 1/2\, \Gamma(-1/2+1) = 1/2\, \Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

Beta Function : is defined as

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad , \text{ where } m,n>0.$$

Properties:

1. $\beta(m,n) = \beta(n,m)$

Ans: $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

put $x = 1-y \Rightarrow dx = -dy$

when $x=0,\ y=1$ & $x=1,\ y=0$

$$\therefore \beta(m,n) = -\int_1^0 (1-y)^{m-1}\, y^{n-1}\,dy$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy$$

$$= \beta(n,m)$$

---

2. $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\,d\theta$

Ans: $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$

when $x=0$, $\theta=0$ & $x=1$, $\theta=\pi/2$

$$\therefore \beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2+1}\theta \, \cos^{2n-2+1}\theta \, d\theta$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \, d\theta$$

3. $$\int_0^{\pi/2} \sin^p\theta \, \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Ans: We have

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \, d\theta \quad \text{——— (1)}$$

put $2m-1=p$, $2n-1=q$

$\Rightarrow m = \frac{p+1}{2}$, $n = \frac{q+1}{2}$

$\therefore$ from (1) $$\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta \, \cos^q\theta \, d\theta$$

$$\Rightarrow \int_0^{\pi/2} \sin^p\theta \, \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

4. Relation between Beta and Gamma function

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Ans: We have $\Gamma(m) = \int_0^\infty e^{-t}\, t^{m-1}\, dt$

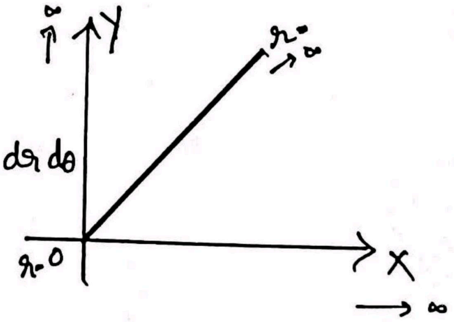
$$\Rightarrow \Gamma(m) = 2\int_0^\infty e^{-x^2}\, x^{2m-1}\, dx \quad \text{——①} \qquad \left(\because \text{Put } t = x^2 \text{ so that } dt = 2x\,dx\right)$$

$$|||^{ly} \quad \Gamma(n) = 2\int_0^\infty e^{-y^2}\, y^{2n-1}\, dy$$

$$\therefore \Gamma(m)\Gamma(n) = 2\int_0^\infty e^{-x^2}\, x^{2m-1}\, dx \cdot 2\int_0^\infty e^{-y^2}\, y^{2n-1}\, dy$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\, x^{2m-1}\cdot y^{2n-1}\, dx\, dy$$

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}(r\cos\theta)^{2m-1}(r\sin\theta)^{2n-1}\, r\, dr\, d\theta \quad \left[\because \text{by changing to polar coordinates}\right]$$



$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\, r^{2m+2n-2+1}\cos^{2m-1}\theta\,\sin^{2n-1}\theta\, dr\, d\theta$$

$$= \left(2\int_{\theta=0}^{\pi/2}\cos^{2m-1}\theta\,\sin^{2n-1}\theta\, d\theta\right)\left(2\int_0^\infty e^{-r^2}\, r^{2(m+n)-1}\, dr\right)$$

$$= \beta(n,m)\ \Gamma(m+n) \quad (\because \text{from ①}$$

$$\Gamma(m)\cdot\Gamma(n) = \beta(m,n)\ \Gamma(m+n)$$

$$\therefore\ \beta(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

Value of $\Gamma(1/2) = \sqrt{\pi}$:

Proof: We have $\beta(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$ & — ①

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\ d\theta \quad \text{— ②}$$

From ①, $\beta(1/2, 1/2) = \dfrac{\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(1/2+1/2)} = (\Gamma(1/2))^2$ — ③

$(\because \Gamma(1) = 1)$

From ②, $\beta(1/2, 1/2) = 2\int_0^{\pi/2} \sin^0\theta \cdot \cos^0\theta\ d\theta$

$$= 2\int_0^{\pi/2} d\theta = 2\,(\theta)_0^{\pi/2}$$

$$= 2\left(\pi/2\right) = \pi \quad \text{— ④}$$

Comparing ③ & ④, $(\Gamma(1/2))^2 = \pi$

$$\Rightarrow \Gamma(1/2) = \sqrt{\pi}$$

1) Prove that $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \cdot \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi$

Ans:

$$I = \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \cdot \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}}$$

$$= \left(\int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta\, d\theta\right) \cdot \left(\int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta\, d\theta\right)$$

$$= \left(\frac{1}{2}\beta\left(\frac{\frac{1}{2}+1}{2}, \frac{0+1}{2}\right)\right) \cdot \left(\frac{1}{2}\beta\left(\frac{-\frac{1}{2}+1}{2}, \frac{0+1}{2}\right)\right)$$

$$= \left(\frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{2}\right)\right) \cdot \left(\frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right)\right)$$

$$= \left[\frac{1}{2}\frac{\Gamma(3/4)\,\Gamma(1/2)}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)}\right] \cdot \left[\frac{1}{2}\frac{\Gamma(1/4)\cdot\Gamma(1/2)}{\Gamma(1/4+1/2)}\right]$$

$$= \left[\frac{1}{2}\frac{\cancel{\Gamma(3/4)}\,\Gamma(1/2)}{\Gamma\left(\frac{5}{4}\right)}\right] \cdot \left[\frac{1}{2}\frac{\Gamma(1/4)\,\Gamma(1/2)}{\cancel{\Gamma(3/4)}}\right]$$

$$= \cancel{\left(\frac{1}{4}\right)}\left(\Gamma(1/2)\right)^2 \frac{\cancel{\Gamma(1/4)}}{\cancel{(1/4)}\cancel{\Gamma(1/4)}}$$

$\left(\because \Gamma(n+1) = n\Gamma(n)\right.$
$\left.\Gamma(5/4) = \Gamma(1/4+1) = 1/4\,\Gamma(1/4)\right)$

$$I = (\sqrt{\pi})^2$$

$$= \pi$$

Evaluate interms of beta function

2) $\int_0^{\pi/2} \sin^3\theta \cos^{1/2}\theta \, d\theta$

Ans: $I = \int_0^{\pi/2} \sin^3\theta \cos^{1/2}\theta \, d\theta$

$$= \frac{1}{2} \beta\left(\frac{3+1}{2}, \frac{\frac{1}{2}+1}{2}\right)$$

$$= \frac{1}{2} \beta\left(2, \frac{3}{4}\right)$$

$$= \frac{1}{2} \frac{\Gamma(2)\,\Gamma(3/4)}{\Gamma\left(2+3/4\right)}$$

$$= \frac{1}{2} \; 1! \; \frac{\Gamma(3/4)}{\Gamma(11/4)}$$

$\left(\because \Gamma(n+1) = n! \Rightarrow \Gamma(2) = 1!\right)$

$$= \frac{1}{2} \frac{\Gamma(3/4)}{\frac{7}{4}\,\frac{3}{4}\,\Gamma(3/4)}$$

$$I = \frac{1}{2}\left(\frac{4}{3}\right)\left(\frac{4}{7}\right) = \frac{8}{21}$$

$\because \Gamma(n+1) = n\Gamma(n)$

$\Gamma(7/4) = \Gamma(3/4+1)$

$= 3/4\,\Gamma(3/4)$

$\Gamma(11/4) = \Gamma(7/4+1)$

$= 7/4\,\Gamma(7/4)$

3) $\int_0^{\pi/2} \sin^7\theta \cos^6\theta \, d\theta$

Ans: $I = \int_0^{\pi/2} \sin^7\theta \cos^6\theta \, d\theta$

$$= \frac{1}{2} \beta\left(\frac{7+1}{2}, \frac{6+1}{2}\right)$$

$$= \frac{1}{2} \beta(4, 7/2)$$

$$= \frac{1}{2} \frac{\Gamma(4)\,\Gamma(7/2)}{\Gamma(4+7/2)} = \frac{1}{2} \frac{\Gamma(1/4)\,\Gamma(7/2)}{\Gamma(15/4)}$$

$$= \frac{1}{2} \frac{3! \;\; 5/2 \;\; 3/2 \;\; 1/2 \; \Gamma(1/2)}{\frac{13}{2} \cdot \frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot 5/2 \cdot 3/2 \cdot 1/2 \, \Gamma(1/2)}$$

$$= \frac{3}{\frac{13}{2} \cdot \frac{11}{2} \cdot 9/2 \cdot 7/2}$$

$$= \frac{48}{9009}$$

$$= \frac{16}{3003}$$

$\Gamma(n+1) = n!$

$\because \Gamma(n+1) = n\Gamma(n)$

$\Gamma(7/2) = 5/2\,\Gamma(5/2)$

$= 5/2 \;\, 3/2 \; \Gamma(3/2)$

$= 5/2 \cdot 3/2 \;\, 1/2 \; \Gamma(1/2)$

$|||^{ly}\; \Gamma\left(\frac{15}{2}\right) = \frac{13}{2} \cdot \frac{11}{2} \cdot \frac{9}{2} \cdot 7/2 \cdot$

$5/2 \cdot 3/2 \cdot 1/2 \; \Gamma(1/2)$

4) $\int_0^1 x^7 (1-x^4)^3 \, dx$

Ans: put $x^4 = \sin^2\theta$

$\Rightarrow x = \sin^{1/2}\theta$

$\Rightarrow dx = \frac{1}{2} \sin^{-1/2}\theta \cos\theta \, d\theta$

when $x=0$, $\theta=0$ & $x=1$, $\theta=\pi/2$

$$\therefore I = \int_0^1 x^7 (1-x^4)^3 \, dx$$

$$= \int_0^{\pi/2} (\sin^{1/2}\theta)^7 (1-\sin^2\theta)^3 \, \tfrac{1}{2} \sin^{-1/2}\theta \cos\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{7/2-1/2}\theta \cos^{6+1}\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^3\theta \cos^7\theta \, d\theta$$

$$= \frac{1}{2}\left[\frac{1}{2}\beta\left(\frac{3+1}{2}, \frac{7+1}{2}\right)\right]$$

$$= \frac{1}{4}\beta(2, 4)$$

$$= \frac{1}{4}\frac{\Gamma(2)\Gamma(4)}{\Gamma(6)}$$

$$= \frac{1}{4}\frac{1!\,3!}{5!} = \frac{1}{80} \qquad (\because \Gamma(n+1) = n!$$