Blur Model and Distribution

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Blurring Model and Blur Kernel

Consider the following image blur model -

$$\boldsymbol{b} = \boldsymbol{k} \otimes \boldsymbol{l} + \boldsymbol{\epsilon}$$

Where, $\boldsymbol{b}$ is observed blurred image, $\boldsymbol{l}$ is the latent sharp image, $\boldsymbol{k}$ is the *Blur Kernel* or *Point Spread Function*, $\boldsymbol{\epsilon}$ is noise and $\otimes$ denotes convolution operator. Our aim is to estimate the latent image (l). In general, the blur kernel k is also unknown. So the problem of estimating l and k from b becomes an *ill-possed* problem.

In our work, we will assume some special parametric forms of blur kernel.

1. **Gaussian Kernel**: We assume a truncated Gaussian kernel defined over finite square grid.In this case we have only one parameter h.

$$k(x, y) = \frac{1}{2\pi h^2} e^{-\frac{x^2+y^2}{2h^2}}$$

2. **Circular Gaussian Kernel**: Instead of considering a truncated Gaussian kernel over a finite square grid, we will consider a truncation over a circular region. Consequently we have two parameters h and r.

$$k(x, y) = \frac{1}{2\pi h^2} e^{-\frac{x^2+y^2}{2h^2}} \times \mathrm{I}_{\{x^2+y^2 \,\leq\, r^2\}}$$

3. **Circular Cauchy Kernel**: Similarly we can define Cauchy kernel over circular region. The kernel should be such that univariate marginals are Cauchy in both directions.

$$k(x, y) = \frac{h}{2\pi} \frac{1}{(x^2 + y^2 + h^2)^{3/2}} \times \mathrm{I}_{\{x^2+y^2 \,\leq\, r^2\}}$$

   In this case also, we have two parameters h and r. But, r is controlled by the parameter θ.

4. **Disc Kernel**: We can also define disc kernel in the similar manner. In this case, we have only one parameter r.

$$k(x, y) = \frac{1}{\pi r^2} \times \mathrm{I}_{\{x^2+y^2 \,\leq\, r^2\}}$$

Remark: The parameter r in both the circular Gaussian and Cauchy characterizes the radius of blur circle (i.e. $c_{diam}/2$) and we have already noted in last pdf that $c_{diam} = a_{diam} f \left| \frac{d - d_{focus}}{d(d_{focus} - f)} \right|$ and scale parameter $h = \kappa \times c_{diam}$. Hence, **we cannot change h and r independently**.

Prior on Natural Image and Maximum Likelihood Estimation

Assuming a specific parametric form of blur kernel reduces the number of parameter in the model. But still the problem remains ill-possed problem. Thus, we need to assume a prior for the latent image. Our prior is not exactly defined on latent image intensities. But defined on image gradients (horizontal or vertical). Thus, we rewrite the model as

$$\delta_h \otimes \boldsymbol{b} = \boldsymbol{k} \otimes (\delta_h \otimes \boldsymbol{l}) \ + \delta_h \otimes \boldsymbol{\epsilon} \quad \text{and} \quad \delta_v \otimes \boldsymbol{b} = \boldsymbol{k} \otimes (\delta_v \otimes \boldsymbol{l}) \ + \delta_v \otimes \boldsymbol{\epsilon}$$

With $\delta_h = [-1, 1]$ and $\delta_v = [-1, 1]^T$. To keep notation simple, we will henceforth take the model

$$\boldsymbol{y} = \boldsymbol{k} \otimes \boldsymbol{x} + \boldsymbol{n}$$

Where, $\boldsymbol{y}$ denotes the gradient (vertical/horizontal) of observed blur image, $\boldsymbol{x}$ denotes the gradient of original latent image and $\boldsymbol{n}$ denotes the gradient of noise. In frequency domain, this reduces to (for all ω) -

$$\boldsymbol{Y_\omega} = \boldsymbol{K_\omega X_\omega} + \boldsymbol{N_\omega}$$

Where, capital letters indicate the corresponding frequency domain representation of small letter symbols.Note that, $\omega = (\omega_1, \omega_2)$. As the former one is 2D image domain model. Our model assumes that $X_\omega \sim \mathcal{CN}(0, \sigma^2 g_\omega)$ and $N_\omega \sim \mathcal{CN}(0, \eta^2 h_\omega)$ independently for all ω. It the follows that $Y_\omega \sim \mathcal{CN}(0, \eta^2 h_\omega + \sigma^2 |K_\omega|^2 g_\omega)$. As we have data on Y_ω. We can use maximum likelihood method to estimate the parameter of blur kernel. For that we need to find likelihood of Y_ω's or equivalently $|Y_\omega|^2$'s.

Result: *If $Z \sim \mathcal{CN}(0, \sigma^2)$. Then, $Re(Z)$ and $Im(Z)$ follows $\mathcal{N}(0, \frac{\sigma^2}{2})$ independently. Hence, $|Z|^2 = Re^2(Z) + Im^2(Z) \sim \frac{\sigma^2}{2}\chi^2_2$. As, $\chi^2_2 \equiv Exp(\lambda = \frac{1}{2})$, it follows that $|Z|^2 \sim Exp(\lambda = \frac{1}{\sigma^2})$.*

Using the above result, $|Y_\omega|^2 \sim \text{Exp}(\lambda_\omega = \frac{1}{\eta^2 h_\omega + \sigma^2 |K_\omega|^2 g_\omega})$ for all ω and they are asymptotically independent. If $f_\theta(Y_\omega)$ denotes the pdf of $\text{Exp}(\lambda_\omega = \frac{1}{\eta^2 h_\omega + \sigma^2 |K_\omega|^2 g_\omega})$ for given parameters θ (say,) of blur kernel. Then likelihood of $|Y_\omega|^2$ is given by -

$$f_\theta(Y) = \prod_\omega f_\theta(Y_\omega)$$

Our aim is to maximize $f_\theta(Y)$ or equivalently $log(f_\theta(Y)) = \sum_\omega log(f_\theta(Y_\omega))$ as a function of θ.