CHAPTER 2

Abstract Rewriting

Contents

The principle of rewriting comes from combinatorial algebra. It was introduced by Thue when he considered systems of transformation rules on combinatorial objects such as strings, trees or graphs in order to solve the word problem, [Thu14]. Given a collection of objects and a system of transformation rules on these objects, the word problem is

> Instance: *given two objects,*
>
> Question: *can one of these objects be transformed to the other by means of a finite number of applications of the transformation rules?*

Dehn described the word problem for finitely presented groups, [Deh10] and Thue studied this problems for strings, which correspond to the word problem for finitely presented monoids, [Thu14]. Note that it was only much later, that the problem was shown to be undecidable, independently by Post [Pos47] and Markov [Mar47a, Mar47b]. Afterwards, the word problem have been considered in many contexts in algebra and in computer science.

Far beyond the precursor works on this decidabilty problem on strings, rewriting theory has been mainly developed in theoretical computer science, producing numerous variants corresponding to different syntaxes of the formulas being transformed: strings in a monoid, [BO93, GM18], paths in a graph, terms in an algebraic theory, [BN98, Klo92, Ter03], terms modulo, λ-terms, trees, Boolean circuits, [Laf03], graph grammars, etc. Rewriting appears also on various forms in algebra, for commutative algebras, [Buc65, Buc87], Lie algebras, [Shi62], with the notion of Gröbner-Shirshov bases, or associative algebras, [Bok76, Ber78, Mor94, Ufn95, GHM19] and operads, [DK10], as well as on topological objects, such as Reidemeister moves, knots or braids, [Bur01], or in higher-dimensional categories, [GM09, GM12, Mim10, Mim14].

Many of the basic definitions and fundamental properties of all these forms of rewriting can be stated on the most abstract version of rewriting defined by a family of binary relations on a set, called an *abstract rewriting system*. Newman introduced the first formalisation of these computational systems in a seminal article [New42].

This chapter introduces this abstract version of rewriting and the main abstract rewriting properties used in these lectures, namely *confluence*, Section 2.2, and *termination*, Section 2.3. We explain the principle of diagrammatic reasoning to prove confluence. All these notions are presented in a rather introductory way, we refer the reader to [BN98, Klo92, Ter03] for a complete account on the abstract rewriting theory. We present Newman's Lemma, [New42], proving confluence from local confluence for terminating rewriting systems in Section 2.4.

We then present two formalisations of abstract rewriting. In Section 2.5, we explain a formalisation of abstract rewriting theory in the theory of Kleene algebras introduce in [Str02]. Kleene algebras provide a formal setting to capture abstract rewriting properties in which deduction in diagrammatic reasoning is replaced by algebraic calculation. Finally, in Section 2.6 we describe the polygraphic interpretation of abstract rewriting systems and we show how to extend the abstract rewriting diagrammatic reasoning into a calculus of coherences in two-dimensional groupoids. We refer to [ABG+24] for a complete account on the polygraphic interpretation of rewriting systems.

2.1. Abstract rewriting systems

Abstract rewriting systems are the simplest computational model. They formalise transformations on terms by abstracting their syntactic nature. The inspiration for this computational model has many origins, including combinatorial algebra [Thu14, Deh10], combinatorial topology [Ale30, New31] or lambda calculus [CR36]. Newman introduced the first formalisation of these computational systems in a seminal article [New42].

> « *The name "combinatorial theory" is often given to branches of mathematics in*

> *which the central concept is an equivalence relation defined by means of certain "allowed transformations" or "moves". A class of objects is given, and it is declared of certain pairs of them that one is obtained from the other by a "move"; and two objects are regarded as "equivalent" if, and only if, one is obtainable from the other by a series of moves. For example, in the theory of free groups the objects are words made from an alphabet a, b, $\cdots$, a^{-1}, b^{-1}, $\cdots$, and a move is the insertion or removal of a consecutive pair of letters xx^{-1} or $x^{-1}x$. In combinatorial topology the objects are complexes, and the allowed moves are "breaking an edge" by the insertion of a new vertex, or the reverse of this process. In Church's "conversion calculus" the rules II and III are "moves" of this kind.*
>
> *In many such theories the moves fall naturally into two classes, which may be called "positive" and "negative". Thus in the free group the cancelling of a pair of letters may be called a positive move, the insertion negative; in topology the breaking of an edge, in the conversion calculus the application of Rule II (elimination of a λ), may be taken as the positive moves. In theories that have this dichotomy it is always important to discover whether there is what may be called a "theorem of confluence", namely, whether if A and B are "equivalent" it follows that there exists a third object, C, derivable both from A and from B by positive moves only. A closely connected problem is the search for "end-forms", or "normal forms", i.e. objects which admit no positive move. It is obvious that in a theory in which the confluence theorem holds no equivalence class can contain more than one end-form, but there remains the question whether in such a class any random series of positive moves must terminate at the end-form, or whether infinite series of moves may also exist.* »
>
> Maxwell Newman, in *On theories with a combinatorial definition of "equivalence"*, Annals of Mathematics, Vol. 43, No. 2, April, 1942, [New42].

In this article, Newman introduced a "general theory of sets of moves", or transformations, formulated in the relational systems setting. Newman's motivation was to provide a unified formal framework for formalizing the transformation operations used by Alexander in combinatorial topology [Ale30], see also [New31], and by Church and Rosser for the β-reduction relation in λ-calculus in [CR36].

2.1.1. Abstract Rewriting Systems. An *abstract rewriting system*, or ARS for short, is a data $(A, \rightarrow_I)$ made of a set A and a sequence $\rightarrow_I$ of binary relations on A indexed by a set I, that is,

$$\rightarrow_I := (\rightarrow_\alpha)_{\alpha\in I}, \quad \text{and} \quad \rightarrow_\alpha \subseteq A \times A. \tag{2.1.2}$$

The relation is called a *reduction*, or a *rewrite* relation on A. An element (a, b) in the relation $\rightarrow$ will be denoted by

$$a \rightarrow b,$$

and we said that *a reduces* to b, that b is a *one-step reduct* of a, or that a is a *one-step expansion* of b. An element of $\longrightarrow$ is called a *reduction step*. In most cases the elements of A have a syntactic or graphical nature (string, term, tree, graph, polynomial...). We will denoted by $\equiv$ the syntactical or graphical identity.

2.1.3. Reduction sequence. A *reduction sequence*, or *rewriting sequence*, with respect to a reduction relation $\longrightarrow$ is a finite or infinite sequence of reduction steps

$$a_0 \longrightarrow a_1 \longrightarrow a_2 \longrightarrow \ldots$$

If we have a reduction sequence

$$a \equiv a_0 \longrightarrow a_1 \longrightarrow a_2 \longrightarrow \ldots \longrightarrow a_{n-1} \longrightarrow a_n \equiv b$$

we say that *a reduces to b*. The *length* of a finite reduction sequence is the number of its reduction steps.

2.1.4. Composition. Given two reduction relations $\longrightarrow_1$ and $\longrightarrow_2$ on A, their *composition* is denoted by $\longrightarrow_1 \cdot \longrightarrow_2$ and defined by

$$a \longrightarrow_1 \cdot \longrightarrow_2 b \quad \text{if} \quad a \longrightarrow_1 c \longrightarrow_2 b, \quad \text{for some } c \text{ in } A.$$

2.1.5. Operations on relations. The *identity relation* is denoted by

$$\overset{0}{\longrightarrow} := \{(a,a) \mid a \in A\}.$$

The *inverse relation* of $\longrightarrow$ is denoted by $\longleftarrow$, or by $\longrightarrow^{-}$, and defined by:

$$\longleftarrow := \{(b,a) \mid a \longrightarrow b\}.$$

A relation is *reflexive* if $\overset{0}{\longrightarrow} \subseteq \longrightarrow$ and *transitive* if $\longrightarrow \cdot \longrightarrow \subseteq \longrightarrow$. The *reflexive closure* of $\longrightarrow$ is denoted by $\overset{\equiv}{\longrightarrow}$ and defined by

$$\overset{\equiv}{\longrightarrow} := \longrightarrow \sqcup \overset{0}{\longrightarrow}.$$

The *symmetric closure* of $\longrightarrow$ is denoted by $\longleftrightarrow$ and defined by

$$\longleftrightarrow := \longrightarrow \sqcup \longleftarrow.$$

The *transitive closure* of $\longrightarrow$ is denoted by $\overset{+}{\longrightarrow}$ and defined by

$$\overset{+}{\longrightarrow} := \bigsqcup_{i>0} \overset{i}{\longrightarrow},$$

where $\overset{i+1}{\rightarrow} := \overset{i}{\rightarrow} \cdot \rightarrow$, for all $i > 0$. The reflexive and transitive closure of $\rightarrow$ is denoted by $\twoheadrightarrow$, or by $\overset{*}{\rightarrow}$, and defined by

$$\twoheadrightarrow := \overset{+}{\rightarrow} \sqcup \overset{0}{\rightarrow} .$$

The reflexive, transitive and symmetric closure of $\rightarrow$ is denoted by $\overset{*}{\leftrightarrow}$, and defined by

$$\overset{*}{\leftrightarrow} := (\leftrightarrow)^{*}.$$

In particular, we have $a \twoheadrightarrow b$ if there is a rewriting sequence from a to b and we have $a \overset{*}{\leftrightarrow} b$ if and only if there is a zig-zag of rewriting sequence from a to b:

$$a \equiv a_0 \leftrightarrow a_1 \leftrightarrow a_2 \leftrightarrow \ldots \leftrightarrow a_{n-1} \leftrightarrow a_n \equiv b.$$

Recall that a binary relation on A is an *equivalence relation* if it is reflexive, symmetric and transitive, and the *equivalence relation generated* by a binary relation $X \subset A \times A$ on A is the intersection of the equivalence relations on A that contain X. By definition, the relation $\overset{*}{\leftrightarrow}$ is equal to the equivalence relation generated by $\rightarrow$.

2.1.6. Category of relations. Given two sets A and B, a *relation from A to B* is a subset of the product $A \times B$. The category of relations, denoted by Rel, is defined as follows

i) objects of Rel are sets,

ii) for sets A and B, the hom-set $\mathsf{Rel}(A, B)$ is the set of relations from A to B,

iii) the composite of two relations $R : A \rightarrow B$ and $S : B \rightarrow C$ is the relation

$$R \cdot S \;:=\; \{\, (a, c) \in A \times C \mid \exists b \in B, ((a, b) \in R) \wedge ((b, c) \in S) \,\},$$

iv) the identity relation on a set A is de diagonal relation

$$\Delta(A) \;:=\; \{\, (a, a) \mid \; a \in A \,\}.$$

In fact, sets and relations form a 2-category, denoted by REL, where 0-cells and 1-cells are defined as above, and a 2-cells are inclusions of relations, that is there is a 2-cell $S \Rightarrow T$ the inclusion $S \subseteq T$ holds. All the constructions in the above subsections on binary relations on a set A belong into the category $\mathsf{REL}(A)$ defined as the category REL, but with a single object A.

The 2-category is thus a category enriched in the category of posets. Such a category is called a 2-*poset* and forms the prototypical example of allegory, a locally posetal 2-category with an involution satisfying some conditions [FS90].

2.2. Confluence

2.2.1. Branchings and confluence pairs. A *branching* (resp. *local branching*) of the relation $\rightarrow$ is an element of the composition $\twoheadleftarrow \cdot \twoheadrightarrow$ (resp. $\leftarrow \cdot \rightarrow$). It is defined by a triple $a \twoheadleftarrow c \twoheadrightarrow b$ (resp. $a \leftarrow c \rightarrow b$) as pictured by the following diagram:

$$\begin{tikzcd}[column sep=small] & c \arrow[dl, twoheadrightarrow] \arrow[dr, twoheadrightarrow] & \\ a & & b \end{tikzcd} \qquad \text{(resp.} \quad \begin{tikzcd}[column sep=small] & c \arrow[dl] \arrow[dr] & \\ a & & b \end{tikzcd} \quad).$$

A *confluence pair* (resp. *local confluence pair*) of the relation $\rightarrow$ is an element of the composition $\twoheadrightarrow \cdot \twoheadleftarrow$ (resp. $\rightarrow \cdot \leftarrow$). It is defined by a triple $a \twoheadrightarrow d \twoheadleftarrow b$ as pictured by the following diagram:

$$\begin{tikzcd}[column sep=small] a \arrow[dr, twoheadrightarrow] & & b \arrow[dl, twoheadrightarrow] \\ & d & \end{tikzcd} \qquad \text{(resp.} \quad \begin{tikzcd}[column sep=small] a \arrow[dr] & & b \arrow[dl] \\ & d & \end{tikzcd} \quad).$$

Note that the relations $\leftarrow \cdot \rightarrow$ and $\twoheadleftarrow \cdot \twoheadrightarrow$ are symmetric.

2.2.2. Commutation and diamond property. Let $\rightarrow_1$ and $\rightarrow_2$ be two relations. We say that

i) $\rightarrow_1$ *commute weakly* with $\rightarrow_2$ if ${}_1\!\leftarrow \cdot \rightarrow_2 \;\subseteq\; \twoheadrightarrow_2 \cdot {}_1\!\twoheadleftarrow$.

ii) $\rightarrow_1$ *commute* with $\rightarrow_2$ if ${}_1\!\twoheadleftarrow \cdot \twoheadrightarrow_2 \;\subseteq\; \twoheadrightarrow_2 \cdot {}_1\!\twoheadleftarrow$.

A relation $\rightarrow$ has the *diamond property* if it commutes with itself. This means that for any local branching $a \leftarrow c \rightarrow b$ there exists a local confluence:

$$\begin{tikzcd}[column sep=small] & c \arrow[dl] \arrow[dr] & \\ a \arrow[dr] & & b \arrow[dl] \\ & d & \end{tikzcd}$$

This property is hard to obtain in general. Let us give the main confluence patterns used in rewriting.

2.2.3. Confluence patterns. A *confluence pattern* is a 2-cell in the category REL. Here are the main confluence patterns used in the sequel expressed algebraically in terms of composition and inclusion of relations, see Section 2.5. A reduction relation $\rightarrow$ is called

i) *Church-Rosser* if $\overset{*}{\leftrightarrow} \;\subseteq\; \twoheadrightarrow \cdot \twoheadleftarrow$,

ii) *confluent* if $\twoheadrightarrow$ satisfies diamond property, that is $\twoheadleftarrow \cdot \twoheadrightarrow \;\subseteq\; \twoheadrightarrow \cdot \twoheadleftarrow$,

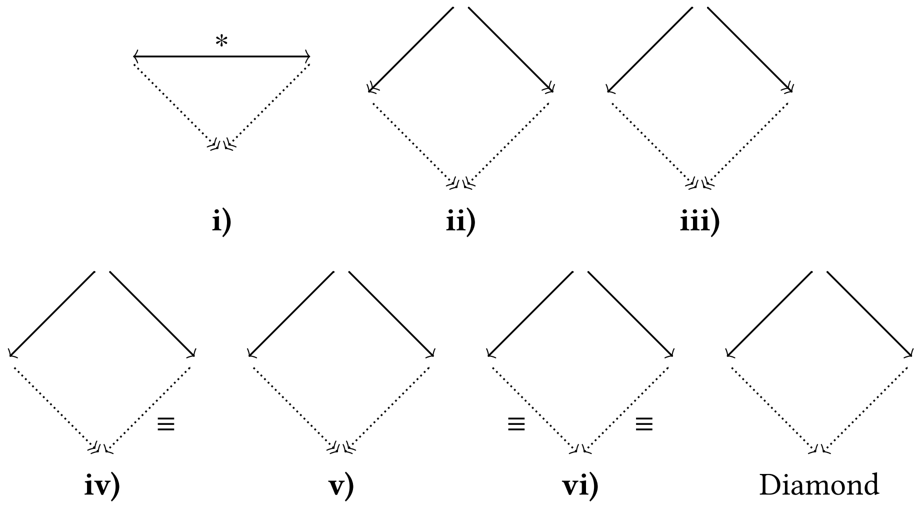
iii) *semi-confluent* if $\leftarrow \cdot \twoheadrightarrow \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$,

iv) *strongly-confluent* if $\leftarrow \cdot \rightarrow \subseteq \twoheadrightarrow \cdot \overset{\equiv}{\leftarrow}$,

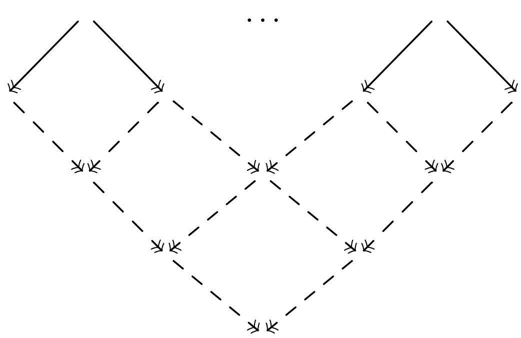
v) *locally confluent*, also called *weakly confluent*, if $\leftarrow \cdot \rightarrow \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$,

vi) *subcommutative* if $\leftarrow \cdot \rightarrow \subseteq \overset{\equiv}{\rightarrow} \cdot \overset{\equiv}{\leftarrow}$.

These patterns are respectively pictured as follows



2.2.4. Remark. The diamond property implies the Church-Rosser property, [New42, Theorem 1]. Note that in [New42] Newman called confluence the Church-Rosser property defined above. He showed that these properties coincide. Obviously, any Church-Rosser property is confluent, and the reverse implication is shown by the following diagram:



2.2.5. Theorem (Church-Rosser Theorem). *For an ARS $(A, \rightarrow)$ the following conditions are equivalent*

i) $\longrightarrow$ *is confluent,*

ii) $\longrightarrow$ *is semi-confluent,*

iii) $\longrightarrow$ *has the Church-Rosser property.*

Proof. Obviously **i)** implies **ii)**. Prove that **iii)** implies **i)**. Suppose that $\longrightarrow$ is Church-Rosser. Given a branching $a \twoheadleftarrow c \twoheadrightarrow b$, we have $a \overset{*}{\leftrightarrow} b$. Hence by the Church-Rosser property, there is a confluence pair $a \twoheadrightarrow d \twoheadleftarrow b$, hence $\longrightarrow$ is confluent. Prove that **ii)** implies **iii)**. Suppose that $\longrightarrow$ is semi-confluent and let $a \overset{*}{\leftrightarrow} b$. Prove by induction on the length of the sequence of reductions between a and b, that there is a confluence pair $a \twoheadrightarrow d \twoheadleftarrow b$. This is obvious when the sequence is of length 0, that is $a \equiv b$, or when the sequence is of length 1, that is $a \longrightarrow b$ or $a \longleftarrow b$. Let consider a sequence of reductions $a \overset{n-1}{\leftrightarrow} b' \overset{1}{\leftrightarrow} b$. By induction hypothesis, there is a confluence pair $a \twoheadrightarrow d \twoheadleftarrow b'$. If $b \longrightarrow b'$, that is

$$
\begin{tikzcd}
a \arrow[rr, leftrightarrow, "n-1"] \arrow[dr, twoheadrightarrow, bend right] \arrow[dr, phantom, "\text{Induction}"] & & b' \arrow[dl, twoheadrightarrow, bend left] & b \arrow[l, "1"'] \\
& d & &
\end{tikzcd}
$$

by induction, this gives a confluence pair $a \twoheadrightarrow d \twoheadleftarrow b$. In the other case, if $b' \longrightarrow b$, by semi-confluence, there is a confluence pair $d \twoheadrightarrow d' \twoheadleftarrow b$:

$$
\begin{tikzcd}
a \arrow[rr, leftrightarrow, "n-1"] \arrow[dr, twoheadrightarrow, bend right] \arrow[dr, phantom, "\text{Induction}"] & & b' \arrow[r, "1"] \arrow[dl, twoheadrightarrow, bend left] \arrow[d, phantom, "\text{Semi-Confluence}"] & b \arrow[ddl, twoheadrightarrow, bend left] \\
& d \arrow[dr, twoheadrightarrow, bend right] & & \\
& & d' &
\end{tikzcd}
$$

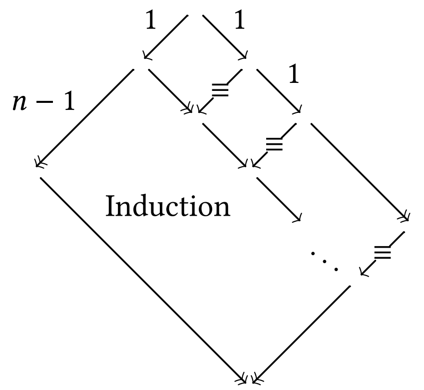
hence, by induction we have a confluence pair $a \twoheadrightarrow d' \twoheadleftarrow b$. It follows that the relation $\longrightarrow$ is Church-Rosser. □

2.2.6. Example. The following ARS is locally confluent and not having the Church-Rosser property.

$$
\begin{tikzcd}
b & a \arrow[l] \arrow[r, bend left] & a' \arrow[l, bend left] \arrow[r] & b'
\end{tikzcd}
$$

2.2.7. Theorem ([Hue80, Lemma 2.5]). *An ARS is confluent if and only if it is strongly confluent.*

Proof. The proof is given by the following deduction confluence diagram



□

2.2.8. Exercise. Let A be a set and let $\to_1$ and $\to_2$ be two reduction relations on A.

1. Prove that the confluence of $\to_1$ and $\to_2$ does not imply the confluence of $\to_1 \cup \to_2$.

2. Prove that

$$\to_1 \subseteq \to_2 \subseteq \twoheadrightarrow_1 \quad \text{implies} \quad \twoheadrightarrow_1 = \twoheadrightarrow_2 .$$

3. Prove that if $\to_1 \subseteq \to_2 \subseteq \twoheadrightarrow_1$ and $\to_2$ is strongly confluent, then $\to_1$ is confluent.

4. Prove that if $\to_1$ and $\to_2$ are confluent and commute, then the relation $\to_1 \cup \to_2$ is also confluent.

2.3. Normalisation and termination

Let $(A, \to)$ be an ARS.

2.3.1. Normal form. An element a in A is in *normal form*, or *irreductible*, with respect to $\to$ if there is no b in A such that $a \to b$. It is *reductible* if it is no irreductible. We denote by $\mathrm{NF}(\to)$ the set of normal forms in A with respect to $\to$.

2.3.2. Normalizing. An element a in A is *(weakly) normalizing* if $a \twoheadrightarrow b$ for some b in $\mathrm{NF}(\to)$. Then we say that *a has a normal form b* and b is called a *normal form of a*. The relation $\to$ is *(weakly) normalizing* if every element a in A is normalizing.

2.3.3. Termination. An element a in A is *strongly normalizing* if every reduction sequence starting from a is finite. The relation $\longrightarrow$ is *strongly normalizing*, or *terminating*, or *noetherian* if every a in A is strongly normalizing. Any terminating relation is normalizing. Note that the converse is false as shown by the following ARS

$$\begin{tikzcd}
a_0 \arrow[r] \arrow[d] & a_1 \arrow[r] \arrow[dl] & a_2 \arrow[r] \arrow[dll] & a_3 \arrow[r] \arrow[dlll] & a_4 \arrow[r] \arrow[dllll] & a_5 \arrow[r] \arrow[dlllll] & \cdots \arrow[dllllll] \\
b
\end{tikzcd}$$

2.3.4. Convergence. We say that $\longrightarrow$ is *convergent*, or *complete*, *canonical*, *uniquely terminating*, if $\longrightarrow$ is confluent and terminating.

2.3.5. Normal form property. The relation $\longrightarrow$ has the *normal form property* if for any a in A and any normal form b in A

$$a \overset{*}{\leftrightarrow} b \quad \text{implies} \quad a \twoheadrightarrow b.$$

The relation $\longrightarrow$ has the *unique normal form property* if for all normal forms a and b in A

$$a \overset{*}{\leftrightarrow} b \quad \text{implies} \quad a \equiv b.$$

2.3.6. Semi-convergence. We say that $\longrightarrow$ is *semi-convergent*, or *semi-complete*, if $\longrightarrow$ has the unique normal form property and is normalizing. If $(A, \longrightarrow)$ is semi-convergent, then every element a in A reduces to a unique normal form denoted by $\widehat{a}$.

2.3.7. Confluence and unicity of the normal form. If $\longrightarrow$ is confluent, every element has at most one normal form. As an immediate consequence of the equivalence of the Church-Rosser property and the confluence property, Proposition 2.2.5, we have

2.3.8. Theorem. *For an ARS $(A, \longrightarrow)$ the following implications hold:*

i) *The normal form property implies the unique normal form property.*

ii) *If $\longrightarrow$ is confluent then $\longrightarrow$ has the normal form property.*

iii) *If $\longrightarrow$ is semi-convergent then it is confluent.*

For a confluent ARS $(A, \longrightarrow)$, two elements a and b in A are equivalent, that is $a \overset{*}{\leftrightarrow} b$, if and only if there are *joignable*, that is $a \twoheadrightarrow \cdot \twoheadleftarrow b$. The test of joignability may be not possible when the relation is not terminating. For example, how to test the joignability of $-n$ and n in the following example:

$$\cdots \longleftarrow -2 \longleftarrow -1 \longleftarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots$$

Let us show that normalisation suffices to determine joignability.

If $\longrightarrow$ is normalizing and confluent, every element a in A has a unique normal form denoted by $\widehat{a}$.

2.3.9. Theorem. *If $\longrightarrow$ is normalizing and confluent then we have*

$$a \overset{*}{\leftrightarrow} b \qquad \textit{if and only if} \qquad \widehat{a} \equiv \widehat{b}.$$

As a consequence of the previous result, for a normalizing and confluent ARS $(A, \longrightarrow)$ an equivalence test of two elements a and b in A is to check the syntactical equality of their normal forms $\widehat{a}$ and $\widehat{b}$. If the normal forms are computable and the syntactic identity is decidable then the equivalence is decidable.

2.3.10. Exercise. Prove Theorem 2.3.8 and Theorem 2.3.9.

2.3.11. Examples. The following ARS

$$\begin{tikzcd}
a \arrow[rr, bend left] \arrow[dr] & & a' \arrow[ll, bend left] \arrow[dl] \\
& b &
\end{tikzcd}$$

is confluent, not terminating and admits a unique normal form. The ARS

$$\begin{tikzcd}
c & b \arrow[l] \arrow[r] & a \arrow[r, bend left] & a' \arrow[l, bend left]
\end{tikzcd}$$

is not confluent, not terminating and admits a unique normal form.

2.3.12. Example. Let $A = \mathbb{N} - \{0, 1\}$. Consider the relation on A defined by the

$$\{(m, n) \mid m > n \ \text{ and } \ n \ \text{ divides } \ m\}.$$

Then m is in $\mathrm{NF}(\longrightarrow)$ if and only if m is prime. An element p is a normal form of m if and only if p is a prime factor of m. We have $m \twoheadrightarrow \cdot \twoheadleftarrow n$ if and only if m and n are note relatively prime. The transitive closure of $\longrightarrow$ coincide woth $\longrightarrow$ because $>$ and divide relations are already transitive. We have $\overset{*}{\leftrightarrow} = A \times A$ and $\longrightarrow$ terminates and it is not confluent.

2.3.13. Example. Let $A = \{a, b\}^*$ be the free monoid on $\{a, b\}$. We consider the relation $\to$ defined by the set

$$\{(ubav, uabv) \mid u, v \in A\}.$$

Then an element of A is in normal form if and only if it is of the form $a^n b^m$ for $n, m \in \mathbb{N}$. Not that the relation $\to$ terminates and its confluent. Thus every element of A has a unique normal form and we have $w \twoheadrightarrow \cdot \twoheadleftarrow w'$ if and only if $w \overset{*}{\leftrightarrow} w'$ if and only if w and w' contain the same number of as and bs. We will see in the next chapter on string rewriting that such an ARS can be specified by only one rewriting rule $ba \;\to\; ab$.

2.3.14. Exercise, [Jan88]. Consider the set $\mathbb{N} \times \mathbb{N}$ with the reduction relation $\to_1$ defined by $(x, y) \to_1 (x', y')$ if

$$(\,(x' = x - 2) \;\text{ and }\; (y' = y \geqslant 1)\,) \quad \text{or} \quad (\,(x' = x + 2) \;\text{ and }\; (y' = y - 1)\,)\,.$$

1. Show that $\to_1$ is terminating.

2. Show that $\to_1$ is not confluent.

3. Define a reduction relation $\to_2$ on $\mathbb{N} \times \mathbb{N}$ that is terminating, confluent and equivalent to $\to_1$, that is the relations $\overset{*}{\leftrightarrow}_1$ and $\overset{*}{\leftrightarrow}_2$ are equal.

2.3.15. Noetherian induction. The principle of induction for natural numbers ensures that a property $\mathcal{P}(n)$ holds for all natural numbers n if we can show that $\mathcal{P}(n)$ holds under the hypothesis that $\mathcal{P}(m)$ holds for all $m < n$. The principle is a consequence of the fact that there is no infinitely descending chain of natural numbers.

Recall from [Hue80] that the *Noetherian induction* principle for an ARS $(A, \to)$ can be stated as follows. Given a property $\mathcal{P}$ on elements of A, then

$$\forall\, a \in A,\; \big(\,\forall\, b \in A,\; a \overset{+}{\to} b \;\text{ implies }\; \mathcal{P}(b)\,\big) \quad \text{implies} \;\; \mathcal{P}(a)$$

implies

$$\forall a \in A, \;\; \mathcal{P}(a).$$

With this principle, the property $\mathcal{P}(a)$ is proved for all elements a in A by proving that the property $\mathcal{P}(b)$ holds for any element b in A such that there is a rewriting sequence $a \twoheadrightarrow b$.

2.3.16. Theorem. *If $\to$ terminates then the principle of Noetherian induction holds*

Proof. Suppose that the principle of induction does not hold, that is

$$\forall\, a \in A,\; \big(\,\forall\, b \in A,\; a \overset{+}{\to} b \;\text{ implies }\; \mathcal{P}(b)\,\big) \quad \text{implies} \;\; \mathcal{P}(a)$$

holds and that there exist an element c in A such that $\mathcal{P}(c)$ does not hold. Then there exists c' such that $c \overset{+}{\to} c'$ and $\mathcal{P}(c')$ does not hold. In this way, we construct an infinite reduction sequence starting on c. Hence, the reduction relation $\to$ does not terminate. $\square$

Conversely, if the noetherian induction principe holds for an ARS $(A, \to)$, then it terminates. It suffices to apply the induction principle to the property:

$$\mathcal{P}(a) \ \equiv \ (\text{there is no infinite reduction sequence starting on } a)\,.$$

2.3.17. Exercise. Let $(A, \to)$ be an ARS. The relation $\to$ is called *finitely branching* if each element a of A has only finitely many direct successors, that is elements b such that $a \to b$. The relation is called *globally finite* if the relation $\overset{+}{\to}$ is finitely branching, that is each element a in A has only finitely many successors.

1. Suppose that the relation $\to$ is terminating and finitely branching. Prove that it is globally finite.

2. Show that it is not true that a finitely branching relation is terminating if it is globally finite.

A relation is *acyclic* if there is no element a in A such that $a \overset{+}{\to} a$.

3. Show that any acyclic relation is terminating if it is globally finite.

4. Show that a finitely branching and acyclic relation is terminating if and only if it is globally finite.

2.3.18. Exercise. Let $(A, \to)$ be an ARS such that every element a in A has a unique irreducible descendant. Prove that the relation $\to$ is confluent.

2.4. Proving confluence

The local confluence does not generally imply confluence, however these properties are equivalent for terminating rewriting systems. This result is also due to Newman.

2.4.1. Theorem (Newman's lemma, [New42, Theorem 3]). *A terminating ARS is confluent if and only if it is locally confluent.*

The original proof of this result by Newman used a complicated combinatorial topological arguments. A short proof by Noetherian induction is given by Huet in [Hue80]. Due to this proof, Newman's Lemma is also called the *diamond lemma*.

Proof. Suppose that $\to$ is locally confluent and terminating. We prove its confluence by Noetherian induction. Given a_0 in A, we suppose that for all a with $a_0 \twoheadrightarrow a$ and for all branching

$$\begin{tikzcd}
 & a_1 \\
a \arrow[ur, twoheadrightarrow, bend left] \arrow[dr, twoheadrightarrow, bend right] & \\
 & a_2
\end{tikzcd}$$

there exists a confluence

$$\begin{tikzcd}[row sep=small] a_1 \arrow[dr, bend left, two heads] & \\ & t \\ a_2 \arrow[ur, bend right, two heads] & \end{tikzcd}$$

Let us consider a branching

$$\begin{tikzcd}[row sep=small] & a' \\ a_0 \arrow[ur, bend left, two heads] \arrow[dr, bend right, two heads] & \\ & a'' \end{tikzcd}$$

The cases $a' \equiv a_0$ or $a'' \equiv a_0$ are obvious. In the other case, the length of the reductions $a_0 \twoheadrightarrow a'$ and $a_0 \twoheadrightarrow a''$ are greater than 1:

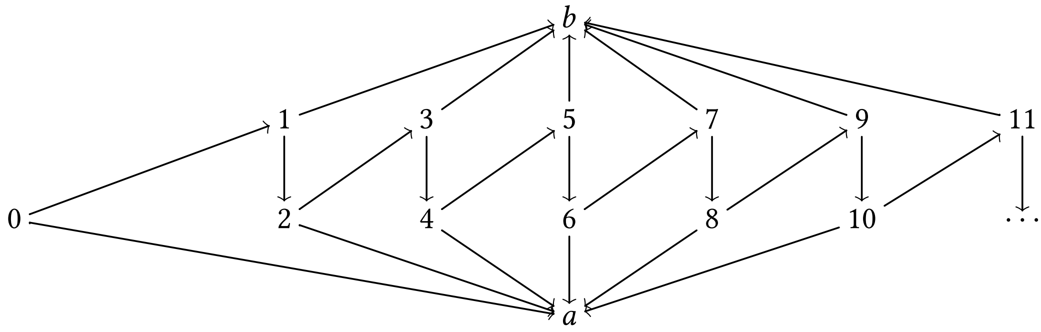
$$\begin{tikzcd} & & a_0 \arrow[dl, bend right, "1"'] \arrow[dr, bend left, "1"] \arrow[d, phantom, "\text{loc. confl.}"] & & \\ & a'_1 \arrow[dl, bend right, two heads] \arrow[r, two heads] \arrow[d, phantom, "\text{ind. hyp.}"] & a_2 \arrow[dl, bend left, two heads] \arrow[d, phantom, "\text{ind. hyp.}"] & a''_1 \arrow[l, bend left, two heads] \arrow[dr, bend left, two heads] & \\ a' \arrow[r, bend right, two heads] & d' \arrow[r, bend right, two heads] & d & & a'' \arrow[ll, bend left, two heads] \end{tikzcd}$$

We conclude using the induction hypothesis and local confluence. $\square$

2.4.2. Example, [Hue80]. The following examples illustrate that the requirement of noetherianity is necessary to prove confluence from local confluence. The following ARS is locally confluent but not confluent

$$\begin{tikzcd} b & a \arrow[l] \arrow[r, bend left] & a' \arrow[l, bend left] \arrow[r] & b' \end{tikzcd}$$

The following ARS with $2n \longrightarrow a$, $2n+1 \longrightarrow b$ and $n \longrightarrow n+1$ for all n in $\mathbb{N}$ without cycle is local confluent but not confluent:



It is locally confluent but not confluent.

2.5. Abstract abstract confluence

In previous section we proved confluence results such a the Church-Rosser Theorem 2.2.5 and Newman Theorem 2.4.1diagrammatically by transforming a branching scheme into a confluence scheme. In this section, we present the algebraic framework of Kleene algebras allowing to formulate the diagrammatical proofs of confluence into calculational proofs. In particular, we present the algebraic formulation of Church-Rosser and Newman theorem given by Struth in [Str02, Str06] and Desharnais-Moller-Struth in [DMS11]. The content of this section is largely inspired by these articles.

2.5.1. Kleene algebras. A Kleene algebra provides an algebraic setting to capture abstract rewriting properties. Its elements represent relations and their compositions and unions are respectively described by a product and a sum satisfying compatibilities conditions. The deductions in diagrammatic reasoning are then replaced by algebraic calculations in this structure. We begin by recalling the structure of Kleene algebras.

A *semiring* is a structure $(S, +, 0, \cdot, 1)$ made of

i) a commutative monoid $(S, +, 0)$,

ii) a monoid $(S, \cdot, 1)$,

satisfying the following compatibility conditions

iii) (*distributivity*) $x(y + y') = xy + xy'$ and $(x + x')y = xy + x'y$, for all $x, x', y, y' \in S$,

iv) (*neutrality*) $0x = 0 = x0$, for every $x \in S$, that is x is the *zero* element of S.

A *dioid* is a semiring S in which addition is idempotent:

v) (*idempotence*): $x + x = x$, for every $x \in S$.

In a dioid, the relation defined by

$$x \leqslant y \quad \Longleftrightarrow \quad x + y = y, \quad \text{for all } x, y \in S,$$

is a partial order on S, with respect to which the operations $+$ and $\cdot$ are monotone, and 0 is minimal. In particular, we have, for all $x, y, z \in S$,

$$x + y \leqslant z \quad \Longleftrightarrow \quad (x \leqslant z) \wedge (y \leqslant z). \tag{2.5.2}$$

A *Kleene algebra* is a dioid K equipped with an operation $(-)^* : K \to K$ satisfying the following two families of axioms

vi) *(Unfold axioms)* $1 + xx^* \leqslant x^*$ and $1 + x^*x \leqslant x^*$, for every $x \in K$,

vii) *(Induction axioms)* $z + xy \leqslant y \Rightarrow x^*z \leqslant y$ and $z + yx \leqslant y \Rightarrow zx^* \leqslant y$, for all $x, y, z \in K$.

The map $(-)^*$ is called the *Kleene star operation* of K.

2.5.3. Example. The *relation Kleene algebra* on a set A is the following structure

$$K(A) := (\mathcal{P}(A \times A), \sqcup, \cdot, \emptyset_A, Id_A, (-)^*),$$

where

i) the operation $\sqcup$ is the set-theoretical union whose unity $\emptyset_A$ is the empty set,

ii) the composition $\cdot$ is the relational composition defined by

$$(a, b) \in R \cdot S \quad \text{iff} \quad (a, c) \in R \text{ and } (c, b) \in S, \text{ for some } c \in A,$$

see (2.1.4), whose unity $Id_A = \{(a, a) \mid a \in A\}$ is the identity relation on A, see (2.1.5),

iii) the operation $(-)^*$ is the reflexive transitive closure operation defined by:

$$R^* := \bigsqcup_{i \in \mathbb{N}} R^i, \quad \text{with} \quad R^0 = Id_A \quad \text{and} \quad R^{i+1} = R \cdot R^i.$$

iv) $\leqslant$ is the set-theoretical inclusion.

2.5.4. Lemma ([Str02, Lemma 3]). *Let K be a Kleene algebra. For all $x, y, z \in K$, the following conditions hold*

i) $1 = 1^*$,

ii) $1 \leqslant x^*$,

iii) $x^*x^* = x^*$,

iv) $x \leqslant x^*$,

v) $x^{**} = x^*$,

vi) $xz \leqslant zy$ *implies* $x^*z \leqslant zy^*$,

vii) $zy \leqslant xz$ *implies* $zy^* \leqslant x^*z$,

viii) $(x + y)^* = x^*(yx^*)^*$.

2.5.5. Exercice. Prove Lemma 2.5.4.

2.5.6. Lemma ([Str02, Proposition 1]). *Let K be a Kleene algebra. For all $x, y \in K$, the following conditions hold*

i) $yx^* \leqslant x^*y^*$ *implies* $(x + y)^* \leqslant x^*y^*$,

ii) $y^*x^* \leqslant x^*y^*$ *implies* $(x + y)^* \leqslant x^*y^*$,

iii) $yx \leqslant xy$ *implies* $(x + y)^* \leqslant x^*y^*$.

Proof. The proof of condition **i)** in [Str02] is as follows. By the induction axioms (2.5.2-**vii)**), $xy + z \leqslant y$ implies $x^*z \leqslant y$. It thus suffices to show that

$$(x + y)x^*y^* + 1 \leqslant x^*y^*,$$

for all $x, y \in K$. Following (2.5.2), we thus have to show that

$$xx^*y^* \leqslant x^*y^*, \qquad yx^*y^* \leqslant x^*y^*, \qquad \text{and} \qquad 1 \leqslant x^*y^*. \tag{2.5.7}$$

Let us prove the first equality. Following (2.5.4-**iv)**) and monotonicity of composition, we have

$$xx^*y^* \leqslant x^*x^*y^* = x^*y^*,$$

where the equality is given by (2.5.4-**iii)**). The second equality in (2.5.7) is a consequence of the assumption, $yx^* \leqslant x^*y^*$ and the monotonicity of composition: $yx^*y^* \leqslant x^*y^*y^* = x^*y^*$. The thirst equality in (2.5.7) is a consequence of (2.5.4-**ii)**) and monotonicity: $1 \leqslant x^* = x^*1 \leqslant x^*y^*$. □

2.5.8. Exercice. Prove conditions (2.5.6-**i)**) and (2.5.6-**ii)**).

2.5.9. The following result is an algebraic formulation of the Church-Rosser Theorem in Kleene algebras [Str02, Thm. 4].

2.5.10. Theorem (Church-Rosser Theorem *à la* Struth, 2002). *For all x, y in a Kleene algebra K, we have*

$$y^*x^* \leqslant x^*y^* \;\Leftrightarrow\; (x + y)^* \leqslant x^*y^*.$$

$$\begin{tikzcd} & \cdot \arrow[dl, "y^*"'] \arrow[dr, "x^*"] & \\ \cdot \arrow[dr, dotted, "x^*"'] & & \cdot \arrow[dl, dotted, "y^*"] \\ & \cdot & \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} \cdot \arrow[rr, leftrightarrow, "(x+y)^*"] \arrow[dr, dotted, "x^*"'] & & \cdot \arrow[dl, dotted, "y^*"] \\ & \cdot & \end{tikzcd}$$

Proof. One implication is given by (2.5.6-**ii)**). The reverse implication deduces from the inequality $x^*y^* \leqslant (x+y)^*$ proved by the following implications

$$1 \leqslant x^* \quad \Rightarrow \quad y \leqslant yx^* \quad \Rightarrow \quad x^*y^* \leqslant x^*(yx^*)^* = (x+y)^*,$$

where the last equality is given by (2.5.4-**viii)**).

We have thus proved that $y^*x^* \leqslant (x+y)^* \leqslant x^*y^*$ and the reverse implication as a consequence. □

2.6. Coherent confluence

This section presents a coherent formulation of confluence results of Church-Rosser Theorem 2.2.5 and Newman Theorem 2.4.1. The aim is to reach confluence with cells that fill the confluence diagrams in the diagrammatic reasoning. In order to formulate this construction, the proof of confluence of an ARS is represented by a two-dimensional cell which will be obtained as a composition of cells representing the successive stages of the proof (hypothesis of the statement, induction hypothesis, etc.).

We will see in later lectures that this approach allows us to obtain a setting for proving coherence by rewriting. Indeed, a double application of the Church-Rosser theorem with respect to normalized confluence pairs proves the Squier theorem for ARS, Theorem 2.6.12.

All these constructions can be achieved in the categorical setting of polygraphs. In this section, we introduce low-dimensional polygraphs.

2.6.1. One-dimensional polygraphs. A 1-*polygraph* is a directed graph X, *i.e.*, a diagram of sets and maps

$$X_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} X_1.$$

The elements of X_0 and X_1 are called the 0-*generators* and 1-*generators* of X, respectively. If there is no confusion, we just write $X = (X_0, X_1)$. A 1-generator a of X will be represented by an arrow as follows

$$s_0(a) \xrightarrow{a} t_0(a).$$

A 1-polygraph is *finite* if it has finitely many 0-cells and 1-cells.

2.6.2. Note that the notion of 1-polygraph is equivalent to the notion of ARS given in (2.1.1).

2.6.3. Free categories. The *free category* on a 1-polygraph X is the category denoted by X_1^* and defined as follows:

i) its 0-cells are the 0-generators of X,

ii) the 1-cells of X_1^* with source x and target y are the finite sequences, possibly empty,

$$x \xrightarrow{a_1} x_1 \xrightarrow{a_2} x_2 \xrightarrow{a_3} \quad \cdots \quad \xrightarrow{a_{n-1}} x_{n-1} \xrightarrow{a_n} y \tag{2.6.4}$$

of composable 1-generators of X,

iii) the 0-composition is the concatenation of 1-cells,

iv) the identity on a 0-cell x is the empty sequence of 1-generators from x to x.

The 1-cell in (2.6.4) is called a *rewriting path* with respect to the polygraph X. If the polygraph X has only one 0-cell, then the category X_1^* is the free monoid on X_1, see (1.1.7).

2.6.5. Two-dimensional polygraphs. A 2-*polygraph* is a triple $X = (X_0, X_1, X_2)$ made of a 1-polygraph (X_0, X_1) and a cellular extension X_2 of the free category X_1^*. In other terms, a 2-polygraph X is a 2-graph

$$\begin{tikzcd} X_0 & X_1^* \arrow[l, shift right, "s_0"'] \arrow[l, shift left, "t_0"] & X_2 \arrow[l, shift right, "s_1"'] \arrow[l, shift left, "t_1"] \end{tikzcd}$$

whose 0-cells and 1-cells form a free 1-category. The elements of X_k are called the *k-generators of the 2-polygraph X* and X is *finite* if it has finitely many generators in every dimension.

A 2-generator α can be pictured by a 2-sphere

$$\begin{tikzcd} s_0(\alpha) \arrow[r, bend left, "s_1(\alpha)"] \arrow[r, bend right, "t_1(\alpha)"'] \arrow[r, phantom, "\Downarrow\alpha"] & t_0(\alpha) \end{tikzcd}$$

The *category presented* by a 2-polygraph X is the category denoted by $\overline{X}$ and defined by

$$\overline{X} := X_1^*/X_2.$$

If **C** is a category, a *presentation of* **C** is a 2-polygraph X such that **C** is isomorphic to $\overline{X}$. In that case, the 1-cells of X are *the generators of* **C**, and the 2-cells of X are *the relations of* **C**.

2.6.6. Free groupoid. The *free groupoid* on a 1-polygraph X is the category denoted by $X_1^\top$ presented by the 2-polygraph whose

i) set of 0-generators is X_0,

ii) set of 1-generators is $X_1 \sqcup X_1^-$, where

$$X_1^- \;:=\; \{a^- : t_0(a) \to s_0(a) \mid a \in X_1\},$$

iii) set of 2-generators is

$$I_2 \;:=\; \{a \star_0 a^- \Rightarrow 1_{s_0(a)}, a^- \star_0 a \Rightarrow 1_{t_0(a)} \mid a \in X_1\}.$$

2.6.7. $(2,0)$-polygraphs. A $(2,0)$*-polygraph* is a triple $X = (X_0, X_1, X_2)$ made of a 1-polygraph (X_0, X_1) and a cellular extension X_2 of the free groupoid $X_1^\top$, that is a 2-graph

$$\begin{tikzcd} X_0 & X_1^\top \arrow[l, shift right, "s_0"'] \arrow[l, shift left, "t_0"] & X_2. \arrow[l, shift right, "s_1"'] \arrow[l, shift left, "t_1"] \end{tikzcd}$$

2.6.8. Free 2-categories. The *free 2-category* on a 2-polygraph X is the 2-category, denoted by X_2^*, and defined as follows:

i) the 0-cells of X_2^* are the ones of X,

ii) for all 0-cells x, y, the 1-category $X_2^*(x, y)$ is defined as

a) the free 1-category on the 1-polygraph whose

a-1) 0-generators are the 1-cells in $X_1^*(x, y)$,

a-2) 1-generators are the whiskered 2-cells of the form

$$\begin{tikzcd} x \arrow[r, "w"] & x' \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & y' \arrow[r, "w'"] & y \end{tikzcd}$$

with $\alpha : u \Rightarrow v$ in X_2 and w and w' in X_1^*,

b) quotiented by the congruence generated by the cellular extension made of all the possible

$$\alpha w v \star_1 u' w \beta \;\equiv\; u w \beta \star_1 \alpha w v',$$

for $\alpha : u \Rightarrow u'$ and $\beta : v \Rightarrow v'$ in X_2 and w in X_1^*:

$$
\begin{array}{ccc}
\begin{tikzcd} {} \arrow[r, bend left, "u"] \arrow[r, bend right, "u'"'] \arrow[r, phantom, "\Downarrow\alpha"] & {} \arrow[r, "w"] & {} \arrow[r, "v"] & {} \end{tikzcd}
& &
\begin{tikzcd} {} \arrow[r, "u"] & {} \arrow[r, "w"] & {} \arrow[r, bend left, "v"] \arrow[r, bend right, "v'"'] \arrow[r, phantom, "\Downarrow\beta"] & {} \end{tikzcd}
\\
\star_1 & = & \star_1
\\
\begin{tikzcd} {} \arrow[r, "u'"] & {} \arrow[r, "w"] & {} \arrow[r, bend left, "v"] \arrow[r, bend right, "v'"'] \arrow[r, phantom, "\Downarrow\beta"] & {} \end{tikzcd}
& &
\begin{tikzcd} {} \arrow[r, bend left, "u"] \arrow[r, bend right, "u'"'] \arrow[r, phantom, "\Downarrow\alpha"] & {} \arrow[r, "w"] & {} \arrow[r, "v'"] & {} \end{tikzcd}
\end{array}
$$

iii) for all 0-cells x, y, z of X the 0-composition functor $\star_0^{x,y,z}$ is given by the concatenation on 1-cells and, on 2-cells, as follows:

$$
\begin{aligned}
&\left(u_1\alpha_1u_1' \star_1 \cdots \star_1 u_m\alpha_mu_m'\right) \star_0 \left(v_1\beta_1v_1' \star_1 \cdots \star_1 v_n\beta_nv_n'\right) \\
&= \quad u_1\alpha_1u_1'v_1s(\beta_1)v_1' \star_1 \cdots \star_1 u_m\alpha_mu_m'v_1s(\beta_1)v_1' \\
&\quad \star_1 u_mt(\alpha_m)u_m'v_1\beta_1v_1' \star_1 \cdots \star_1 u_mt(\alpha_m)u_m'v_n\beta_nv_n'
\end{aligned}
$$

where

$$
\begin{array}{ccccc}
\begin{array}{c}
\begin{tikzcd} x \arrow[r, "u_1"] & {} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\alpha_1"] & {} \arrow[r, "u_1'"] & y \end{tikzcd} \\
\star_1 \\ \vdots \\ \star_1 \\
\begin{tikzcd} x \arrow[r, "u_m"] & {} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\alpha_m"] & {} \arrow[r, "u_m'"] & y \end{tikzcd}
\end{array}
& \star_0 &
\begin{array}{c}
\begin{tikzcd} y \arrow[r, "v_1"] & {} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\beta_1"] & {} \arrow[r, "v_1'"] & z \end{tikzcd} \\
\star_1 \\ \vdots \\ \star_1 \\
\begin{tikzcd} y \arrow[r, "v_n"] & {} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\beta_n"] & {} \arrow[r, "v_n'"] & z \end{tikzcd}
\end{array}
& \equiv &
\begin{array}{c}
\begin{tikzcd} x \arrow[r, "u_1"] & {} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\alpha_1"] & {} \arrow[r, "u_1'"] & y \arrow[r, "v_1"] & {} \arrow[r, "s(\beta_1)"] & {} \arrow[r, "v_1'"] & z \end{tikzcd} \\
\star_1 \\ \vdots \\ \star_1 \\
\begin{tikzcd} x \arrow[r, "u_m"] & {} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\alpha_m"] & {} \arrow[r, "u_m'"] & y \arrow[r, "v_1"] & {} \arrow[r, "s(\beta_1)"] & {} \arrow[r, "v_1'"] & z \end{tikzcd} \\
\star_1 \\
\begin{tikzcd} x \arrow[r, "u_m"] & {} \arrow[r, "t(\alpha_m)"] & {} \arrow[r, "u_m'"] & y \arrow[r, "v_1"] & {} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\beta_1"] & {} \arrow[r, "v_1'"] & z \end{tikzcd} \\
\star_1 \\ \vdots \\ \star_1 \\
\begin{tikzcd} x \arrow[r, "u_m"] & {} \arrow[r, "t(\alpha_m)"] & {} \arrow[r, "u_m'"] & y \arrow[r, "v_n"] & {} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\beta_n"] & {} \arrow[r, "v_n'"] & z \end{tikzcd}
\end{array}
\end{array}
$$

iv) for every 0-cell x, the identity 1-cell 1_x is the one of X_1^*.

2.6.9. Free 2-groupoid. Given a $(2,0)$-polygraph X, the *free* $(2,0)$*-category*, or *free* 2*-groupoid*, on X, denoted by $X_2^{\top}$, is defined as the free 2-category on X, and whose all 1-cells and 2-cells are invertible. Explicitly,

i) its underlying 1-category is the free 1-groupoid $X_1^{\top}$,

ii) for all 0-cells x, y, the 1-groupoid $X_2^{\top}(x,y)$ is defined by the quotient

$$X_2^{\top}(x,y) \ := \ (X_2 \sqcup X_2^-)^*(x,y)/\mathrm{Inv}(X_2),$$

where

a) $(X_2 \sqcup X_2^-)^*(x,y)$ is the 1-category defined in (2), with

$$X_2^- \ := \ \{\alpha^- : t_1(\alpha) \Rightarrow s_1(\alpha) \mid \alpha \in X_2\},$$

b) $\mathrm{Inv}(X_2)$ is the congruence generated by by the cellular extension made of all the possible

$$u\alpha v \star_1 u\alpha^- v \ \equiv \ 1_{us_1(\alpha)v} \qquad \text{and} \qquad u\alpha^- v \star_1 u\alpha v \ \equiv \ 1_{ut_1(\alpha)v},$$

for all 2-generator α of X and 1-cells u, v of $X_1^{\top}$ such that $s_0(u) = x$ and $t_0(v) = y$.

2.6.10. Acyclicity. A $(2,0)$-polygraph X is acyclic when for every 1-sphere

$$\begin{tikzcd} x \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] & y \end{tikzcd}$$

in $X_1^{\top}$, there is a 2-cell $f : u \Rightarrow v$ in the 2-groupoid $X_2^{\top}$.

2.6.11. Generating confluences. Squier's completion procedure provides a way to extend a convergent 1-polygraph into an acyclic $(2,0)$-polygraph.

A *family of generating confluences* of a convergent 1-polygraph X is a cellular extension of the free 1-groupoid $X_1^{\top}$ that contains exactly one 2-generator

$$\begin{tikzcd} & v \arrow[dr, "f'"] \arrow[dd, phantom, "\Downarrow A_{f,g}"] & \\ u \arrow[ur, "f"] \arrow[dr, "g"'] & & u' \\ & w \arrow[ur, "g'"'] & \end{tikzcd}$$

for every critical branching (f, g) of X.

Note that, a confluent 1-polygraph always has a family of generating confluences. However, such a family is not necessarily unique, since the 2-generator $A_{f,g}$ can be directed in the reverse way and, for a given branching (f, g), we can have several possible 2-cells f' and g' with the required shape. Later, we will define the notion of normalisation strategies that provide a deterministic way to construct a family of generating confluences.

A *Squier's completion* of a convergent 1-polygraph X is the $(2, 0)$-polygraph, denoted by $\mathcal{S}(X)$, defined by $\mathcal{S}(X) = (X, \Gamma)$, where Γ is a chosen family of generating confluences of X.

2.6.12. Theorem. *Any Squier's completion $\mathcal{S}(X)$ of a 1-polygraph X is acyclic.*

Proof. We proceed in three steps.

2.6.13. Step 1. We prove that, for every parallel 1-cells u and v of X_1^* whose common target is a normal form, there exists a 2-cell from u to v in $\mathcal{S}(X)_2^\top$. We proceed by noetherian induction on the common source x of u and v, using the termination of X. Let us assume that x is a normal form: then, by definition, both 1-cells u and v must be equal to the identity of x, so that $1_{1_x} : 1_x \Rightarrow 1_x$ is a 2-cell of $\mathcal{S}(X)_2^\top$ from u to v.

Now, let us fix a 0-cell x with the following property: for any 0-cell y such that x rewrites into y and for any parallel 1-cells $u, v : y \to \widehat{y} = \widehat{x}$ of X_1^*, there exists a 2-cell from u to v in $\mathcal{S}(X)_2^\top$. Let us consider parallel 1-cells $u, v : x \to \widehat{x}$ and let us prove the result by progressively constructing the following composite 2-cell from u to v in $\mathcal{S}(X)_2^\top$:

$$\begin{tikzcd}
 & x_1 \arrow[dr, "u'_1"] \arrow[drrr, bend left=10, "u_2"] & & & \\
x \arrow[ur, "u_1"] \arrow[dr, "v_1"'] \arrow[rrrr, bend left=60, "u"] \arrow[rrrr, bend right=60, "v"'] \arrow[rrrr, phantom, bend left=45, "="] \arrow[rrrr, phantom, bend right=45, "="] & \Downarrow A & x' \arrow[rr, "w"] \arrow[ur, phantom, "\Downarrow B"] \arrow[dr, phantom, "\Downarrow C"] & & \widehat{x} \\
 & y_1 \arrow[ur, "v'_1"'] \arrow[urrr, bend right=10, "v_2"'] & & &
\end{tikzcd}$$

Since x is not a normal form, we can decompose $u = u_1 \star_0 u_2$ and $v = v_1 \star_0 v_2$ so that u_1 and v_1 are rewriting steps. They form a local branching (u_1, v_1) and we build the 1-cells u'_1 and v'_1, together with the 2-generator A in $\mathcal{S}(X)$. Then, we consider a 1-cell w from x' to $\widehat{x}$ in X_1^*, that must exist by confluence of X and since $\widehat{x}$ is a normal form. We apply the induction hypothesis to the parallel 1-cells u_2 and $u'_1 \star_1 w$ in order to get B and, symmetrically, to the parallel 1-cells $u'_1 \star_1 w$ and v_2 to get C.

2.6.14. Step 2. We prove that every 1-sphere of $X_1^\top$ is the boundary of a 2-cell of $\mathcal{S}(X)_2^\top$. First, let us consider a 1-cell $u : x \to y$ in X_1^*. Using the confluence of X, we choose 1-cells

$$\sigma_x \;:\; x \;\to\; \widehat{x} \qquad \text{and} \qquad \sigma_y \;:\; y \;\to\; \widehat{y} = \widehat{x}$$

in X_2^*. By construction, the 1-cells $u \star_0 \sigma_y$ and σ_x are parallel and their common target $\widehat{x}$ is a normal form. Thus by Step 1, there exists a 2-cell in $\mathcal{S}(X)_2^\top$ from $u \star_0 \sigma_y$ to σ_x or, equivalently, a 2-cell σ_u from u to $\sigma_x \star_0 \sigma_y^-$ in $\mathcal{S}(X)_2^\top$, as in the following diagram:

$$\begin{tikzcd}
x \arrow[rr, bend left, "u"] \arrow[dr, "\sigma_x"'] & \arrow[d, phantom, "\Downarrow\sigma_u"] & y \\
& \widehat{x} \arrow[ur, "\sigma_y^-"'] &
\end{tikzcd}$$

Moreover, the free $(2,0)$-category $\mathcal{S}(X)_2^\top$ contains a 2-cell σ_{u^-} from u^- to $\sigma_y \star_0 \sigma_x^-$, given as the following composite

$$\sigma_{u^-} = u^- \star_0 \sigma_u^- \star_0 \sigma_y \star_0 \sigma_x^-$$

$$\begin{tikzcd}
& & \widehat{x} \arrow[dr, "\sigma_y^-"] & & & \\
y \arrow[r, "u^-"] & x \arrow[ur, "\sigma_x"] \arrow[rr, bend right, "u"'] & \arrow[u, phantom, "\Downarrow\sigma_u^-"] & y \arrow[r, "\sigma_y"] & \widehat{x} \arrow[r, "\sigma_x^-"] & x
\end{tikzcd}$$

2.6.15. Step 3. Now, let us consider a general 1-cell $u : x \to y$ of $X_1^\top$. By construction of the 1-groupoid $X_1^\top$, the 1-cell u can be decomposed into a "zig-zag", that is non-unique in general,

$$x \xrightarrow{u_1} y_1 \xrightarrow{v_1^-} x_2 \xrightarrow{u_2} (\cdots) \xrightarrow{v_{n-1}^-} x_n \xrightarrow{u_n} y_n \xrightarrow{v_n^-} y$$

where each u_i and v_i is a 1-cell of X_1^*. We define σ_u as the following composite 2-cell of $\mathcal{S}(X)_2^\top$, with source u and target $\sigma_x \star_0 \sigma_y^-$:

$$\begin{tikzcd}
x \arrow[rr, "u_1"] \arrow[dr, "\sigma_x"'] & \arrow[d, phantom, "\Downarrow\sigma_{u_1}"] & y_1 \arrow[rr, "v_1^-"] \arrow[dr, "\sigma_{y_1}"'] \arrow[d, phantom, "="] & \arrow[d, phantom, "\Downarrow\sigma_{v_1^-}"] & (\cdots) \arrow[rr, "u_n"] \arrow[dr, "\sigma_{x_n}"'] & \arrow[d, phantom, "\Downarrow\sigma_{u_n}"] & y_n \arrow[rr, "v_n^-"] \arrow[dr, "\sigma_{y_n}"'] \arrow[d, phantom, "="] & \arrow[d, phantom, "\Downarrow\sigma_{v_n^-}"] & y \\
& \widehat{x} \arrow[ur, "\sigma_{y_1}^-"'] \arrow[rr, equal] & & \widehat{x} \arrow[ur, "\sigma_{x_2}^-"'] \arrow[r, equal] & (\cdots) \arrow[r, equal] & \widehat{x} \arrow[ur, "\sigma_{y_n}^-"'] \arrow[rr, equal] & & \widehat{x} \arrow[ur, "\sigma_y^-"'] &
\end{tikzcd}$$

We proceed similarly for any other 1-cell $v : x \to y$ of $X_1^\top$, to get a 2-cell σ_v from v to $\sigma_x \star_0 \sigma_y^-$ in $\mathcal{S}(X)_2^\top$. Thus, the composite $\sigma_u \star_1 \sigma_v^-$ is a 2-cell of the free $(2,0)$-category $\mathcal{S}(X)_2^\top$ from u to v, concluding the proof. □

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