MAT123

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Calculus I
Fall 2024

Contents

1 Course Contents

1. Limits
2. Continuity
3. Differentiability
4. Properties of differentiability
5. Integrability
6. Fundamental theorem of calculus
7. Properties of integrability

2 Introduction

$$\mathbb{R} = \text{Rational Numbers} \cup \text{Irrational Numbers}$$

Real Number Set is an uncountably infinite set.
Functions:

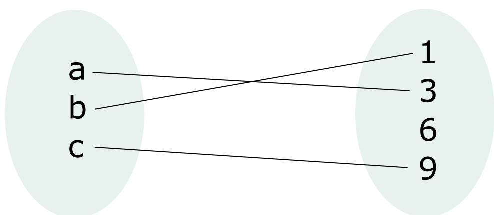
1. Function is a relation.
2. All the elements of the domain must be related to the elements of the co-domain.
3. For all $a \in A$ there exists an element $b \in B$ such that $(a, b) \in f$.

$$\forall\, a \in A\, \exists\, b \in B : (a, b) \in f$$

For $a \in A$ if there exists $(b, c) \in B$ such that $(a, b) \in f$ and $(a, c) \in f$ then $b = c$.

1. $f : \mathbb{R} \to \mathbb{R}\ f(x) = x$
2. $f : \mathbb{R} \to \mathbb{R}\ f(x) = x^2$
3. $f : \mathbb{R} \to \mathbb{R}\ f(x) = |x|$

3 Injective fuctions

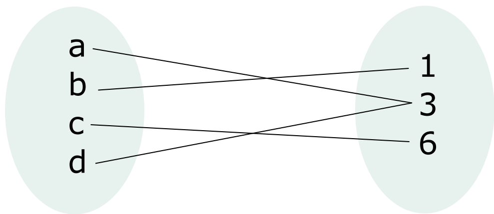


If $f(x) = f(y)$ then $x = y$.

$$f(x) = f(y) \tag{1}$$
$$x^2 = y^2 \tag{2}$$
$$x^2 - y^2 = 0 \tag{3}$$
$$(x + y)(x - y) = 0 \tag{4}$$
$$x = y \text{ or } x = -y \tag{5}$$

Therefore $f(x) = x^2$ is not injective.

4 Surjective function



$$f : A \to B$$
$$f(A) = B$$

5 Bijective Function

Both injective and surjective.

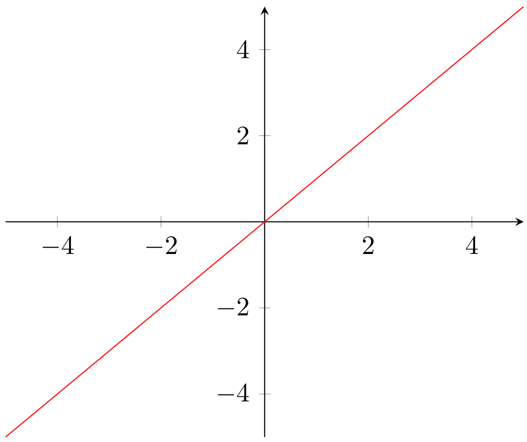
6 Inverse function

If a function is bijective then there exists an inverse of that function.

7 Graphs of functions

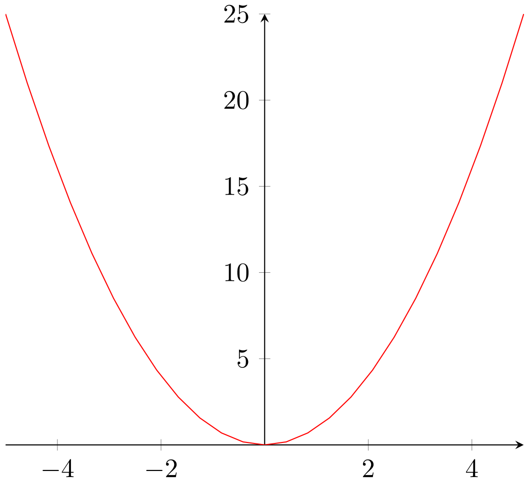
Example 1:

$$f(x) = x$$



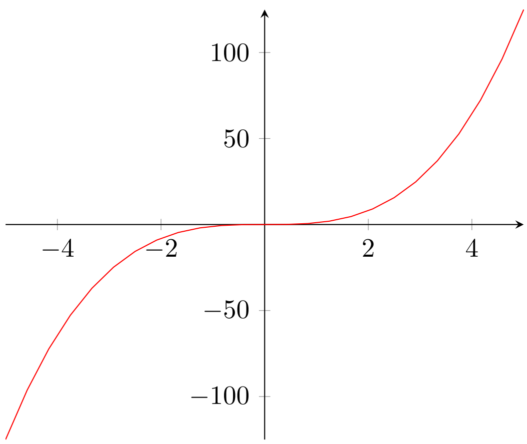
Example 2:

$$f(x) = x^2$$

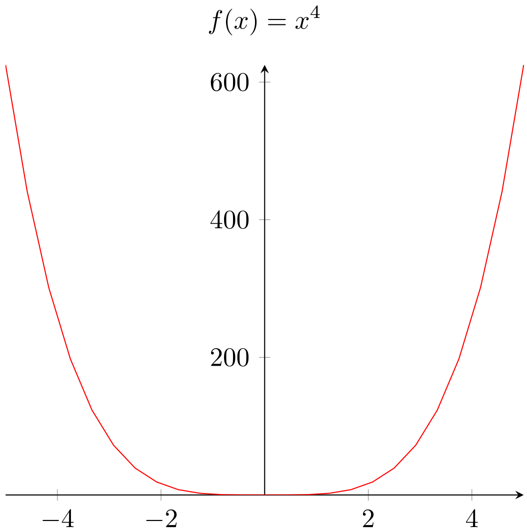


Example 3:

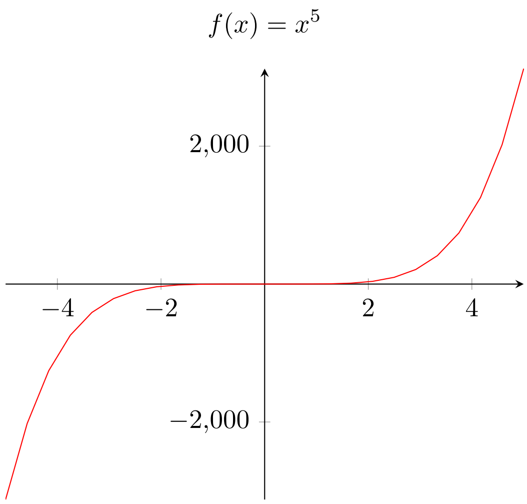
$$f(x) = x^3$$



Example 4:

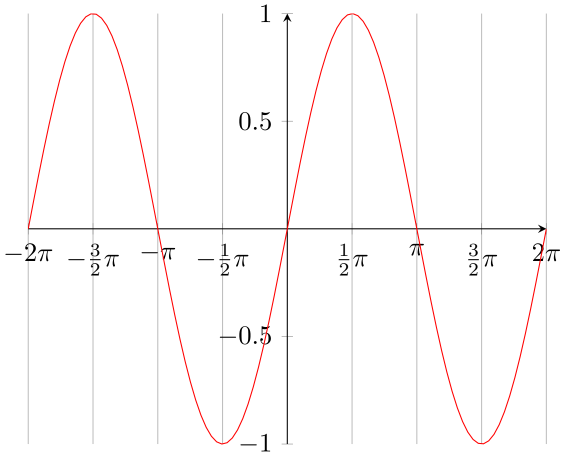


Example 5:



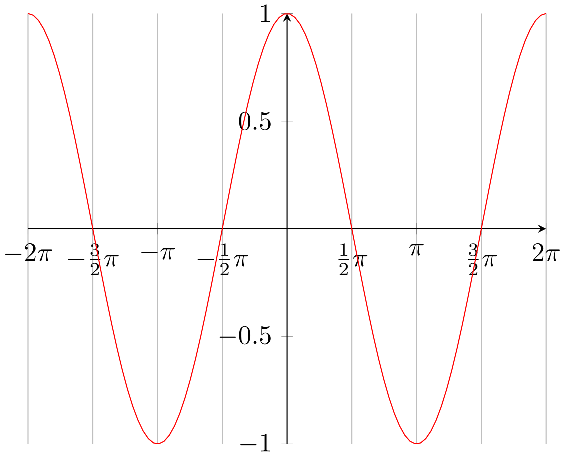
Example 6:

$$f(x) = \sin x$$



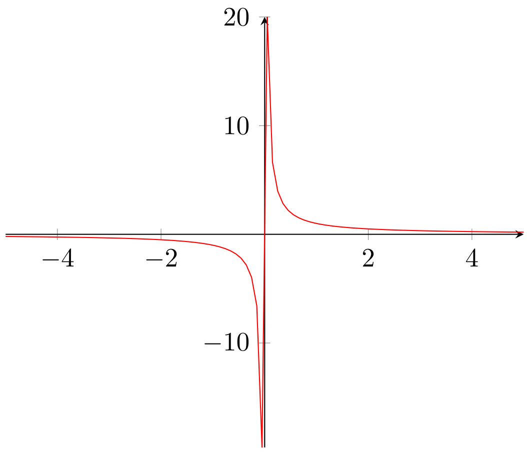
Example 7:

$$f(x) = \cos x = \sin\left(\frac{\pi}{2} - x\right)$$



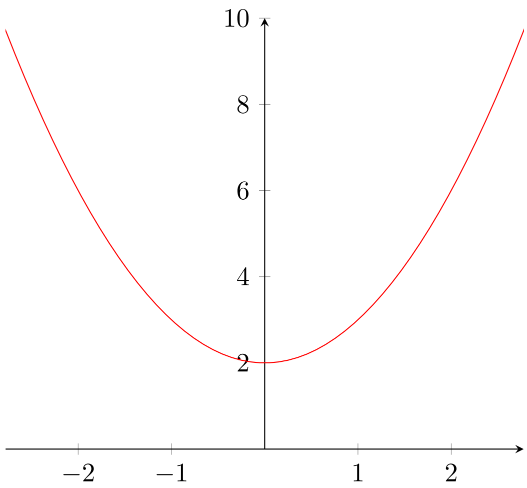
Example 8:

$$f(x) = \frac{1}{x}$$



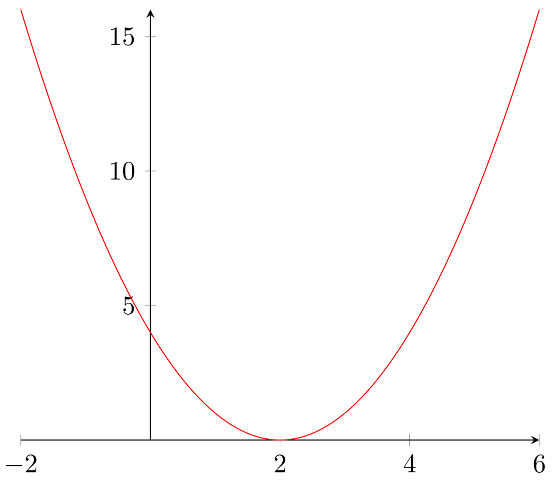
Example 9:

$$f(x) = x^2 + 2$$



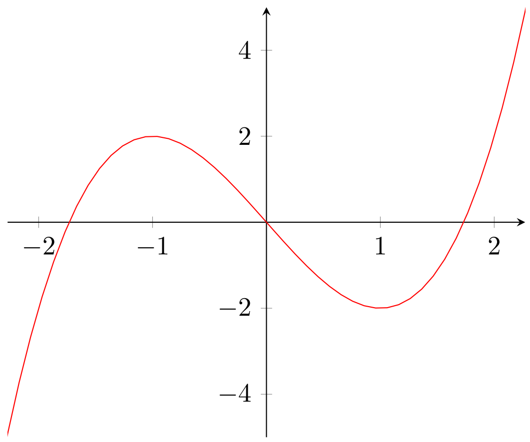
Example 10:

$$f(x) = (x - 2)^2$$



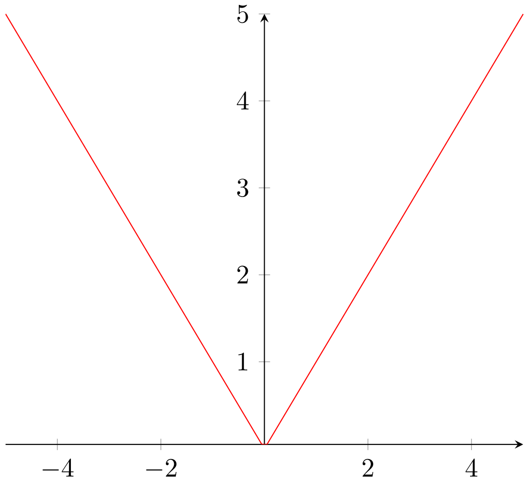
Example 11:

$$f(x) = x^3 - 3x$$



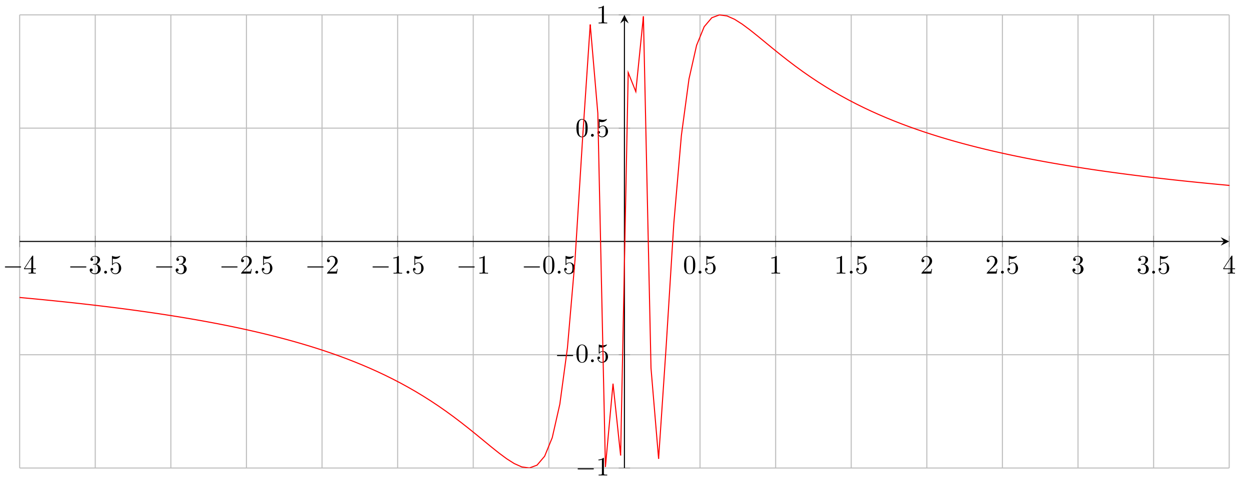
Example 12:

$$f(x) = |x|$$



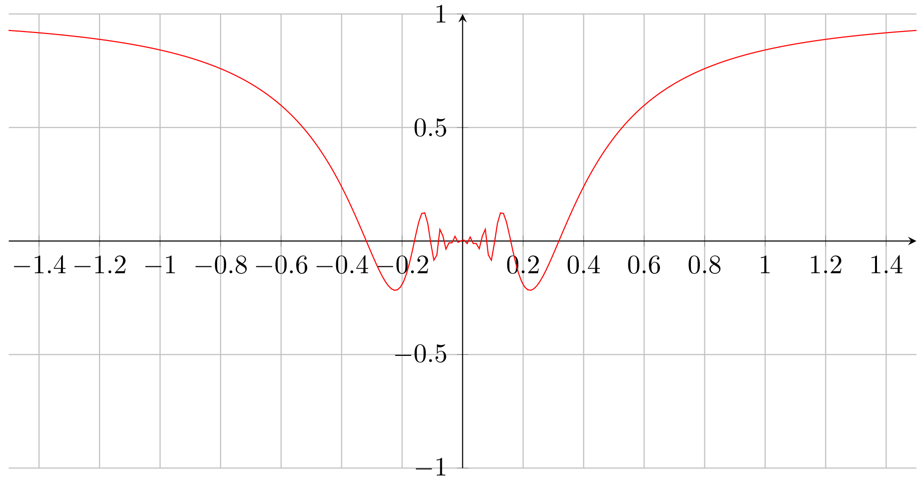
Example 13:

$$f(x) = \sin\left(\frac{1}{x}\right)$$



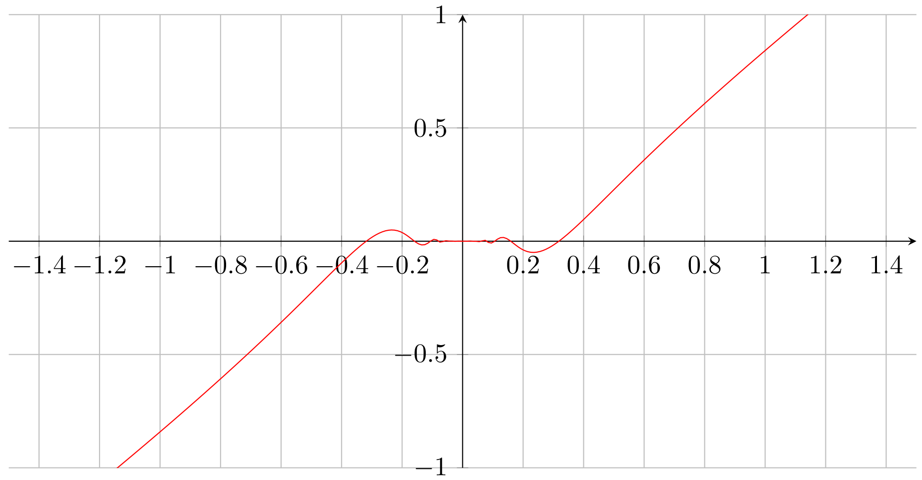
Example 14:

$$f(x) = x\sin\left(\frac{1}{x}\right)$$



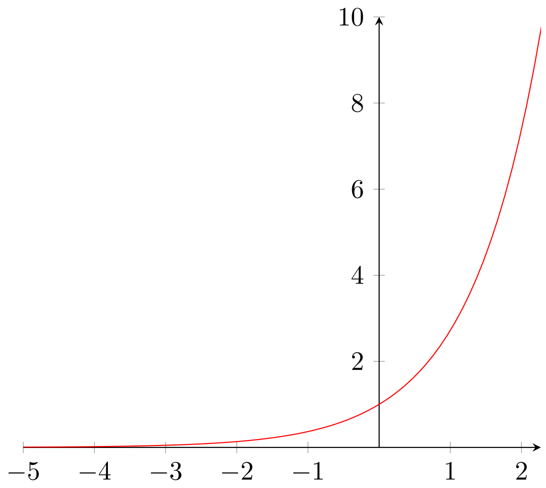
Example 15:

$$f(x) = x^2 \sin\left(\frac{1}{x}\right)$$



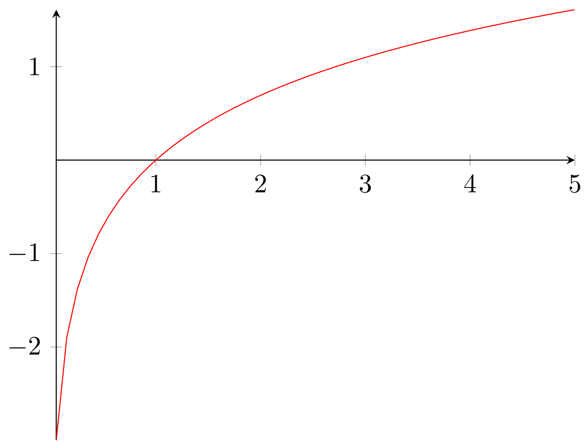
Example 16:

$$f(x) = e^x$$



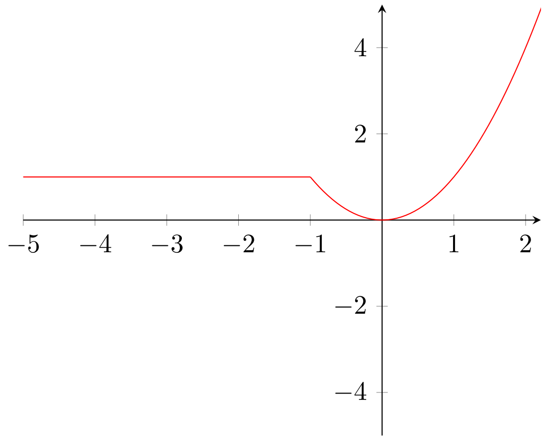
Example 17:

$$f(x) = \ln(x)$$



Example 18:

$$f(x) = \begin{cases} x^2 & \text{if } x \geq -1 \\ 1 & \text{if } x < -1 \end{cases} \tag{6}$$



$$f(x) = \sin\frac{1}{x} \mid f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$$

$$\begin{align}
|x| &< a \tag{1}\\
-a < x &< a \tag{2}\\
-\frac{1}{10} < x\sin\frac{1}{x} &< \frac{1}{10} \tag{3}\\
|\sin\frac{1}{x}| &\leq 1 \tag{4}\\
|x||\sin\frac{1}{x}| \leq |x| = |x| &< \frac{1}{10} \tag{5}\\
|x\sin\frac{1}{x}| &< \frac{1}{10} \tag{6}\\
\therefore -\frac{1}{10} < x &< \frac{1}{10} \tag{7}
\end{align}$$

$$\begin{align}
-\frac{1}{100} < x^2\sin\frac{1}{x} &< \frac{1}{100} \tag{1}\\
|\sin\frac{1}{x}| &\leq 1 \tag{2}\\
|x^2||\sin\frac{1}{x}| \leq |x^2| = |x^2| < \frac{1}{100} = |x|^2 &< \frac{1}{100} \tag{3}\\
\therefore -\frac{1}{10} < x &< \frac{1}{10} \tag{4}
\end{align}$$

8 Limits

Informal: The function f approaches the limit l near a, if we can make $f(x)$ as close as we like to l by requiring that x be sufficiently close to but unequal to a.

Formal: The function f approaches l near a denoted:

$$\lim_{x\to a} f(x) = l$$

For all $\epsilon > 0$ there exists a $\delta > 0$ such that:

$$|f(x) - l| < \epsilon \text{ whenever } |x - a| < \delta$$

$$|x - a| < \delta \implies |f(x) - l| < \epsilon$$

Theorem: If f is a function that approaches to l near a then the limit l is unique.
Proof: Let us consider:

$$\lim_{x \to a} f(x) = l \mid \lim_{x \to a} f(x) = m$$

Let us fix:

$$\epsilon > 0 \mid \epsilon = |\frac{l - m}{2}|$$

Then there exists:

$$\delta_1 > 0 \text{ and } \delta_2 > 0 \text{ such that :}$$

$$|x - a| < \delta_1 \implies |f(x) - l| < |\frac{l - m}{2}|$$

$$|x - a| < \delta_2 \implies |f(x) - m| < |\frac{l - m}{2}|$$

Let:

$$\delta = \min\{\delta_1, \delta_2\}$$

$$\delta \leq \delta_1$$

$$\delta \leq \delta_2$$

$$|x - a| < \delta \implies |f(x) - l| < |\frac{l - m}{2}|$$

$$|x - a| < \delta \implies |f(x) - m| < |\frac{l - m}{2}|$$

$$\begin{aligned} |l - m| &= |l - f(x) + f(x) - m| && (1) \\ &= |l - f(x)| + |f(x) - m| < |\frac{l - m}{2}| + |\frac{l - m}{2}| = |l - m| && (2) \\ &= |l - m| < |l - m| && (3) \end{aligned}$$

This is a contradiction!

$$\therefore l = m$$

Example-1:

$$\lim_{x \to 1} x = 1$$

Rough:

$$|f(x) - l| < \epsilon \tag{1}$$
$$|x - 1| < \epsilon \tag{2}$$

Proof: Let $\epsilon > 0$ is given and $\delta > 0$ such that $\epsilon = \delta$. Now consider:

$$|x - 1| < \delta = \epsilon$$

Example-2:

$$f(x) = \frac{3}{5}x - 2$$
$$\lim_{x \to 1} f(x) = -\frac{7}{5}$$

Rough:

$$|f(x) - l| < \epsilon \tag{1}$$
$$|\frac{3x}{5} - 2 + \frac{7}{5}| < \epsilon \tag{2}$$
$$|\frac{3x}{5} - \frac{3}{5}| < \epsilon \tag{3}$$
$$\frac{3}{5}|x - 1| < \epsilon \tag{4}$$
$$|x - 1| < \frac{5\epsilon}{3} \tag{5}$$

Proof:

Let $\epsilon > 0$ and $\delta = \frac{5\epsilon}{3}$ such that $|x - 1| < \frac{5\epsilon}{3}$.

$$|x - 1| < \frac{5\epsilon}{3} \tag{1}$$
$$|3x - 3| < 5\epsilon \tag{2}$$
$$|\frac{3x}{5} - \frac{3}{5}| < \epsilon \tag{3}$$
$$|\frac{3x}{5} - 2 + \frac{7}{5}| < \epsilon \tag{4}$$
$$|f(x) - (-\frac{7}{5})| < \epsilon \tag{5}$$

Example-3:

$$\lim_{x \to 1} x^2 = 1$$

Rough:

$$
\begin{aligned}
|x^2 - 1| &< \epsilon && (1)\\
|(x+1)(x-1)| &< \epsilon && (2)\\
|x+1||x-1| &< \epsilon && (3)\\
|x-1| &< \frac{\epsilon}{|x+1|} && (4)\\
|x-1| &< \frac{1}{2} && (5)\\
-\frac{1}{2} < x - 1 &< \frac{1}{2} && (6)\\
\frac{1}{2} < x &< \frac{3}{2} && (7)\\
\delta &= min\{\frac{1}{2}, \delta' = \frac{2\epsilon}{3}\} && (8)\\
\text{Where } |x-1| &< \delta' && (9)\\
\delta &\leq \frac{1}{2} && (10)\\
\delta &\leq \delta' && (11)
\end{aligned}
$$

Proof:

Let $\epsilon > 0$ such that $\delta = min\{\frac{1}{2}, \frac{2\epsilon}{3}\}$

$$
\begin{aligned}
|x-1| &< \delta && (1)\\
|x-1||x+1| &< \delta \cdot |x+1| && (2)\\
|x^2-1| &< \delta \cdot |x+1| = \delta \cdot \frac{3}{2} && (3)\\
|x^2-1| &< \frac{2\epsilon}{3} \times \frac{3}{2} && (4)\\
|x^2-1| &< \epsilon && (5)
\end{aligned}
$$

Theorem:

Let:

$$\lim_{x \to a} f(x) = l$$

$$\lim_{x \to a} g(x) = m$$

Then the following are true:

1. $$\lim_{x \to a} (f+g)(x) = l + m$$

2. $$\lim_{x \to a} (f \cdot g)(x) = l \cdot m$$

3.
$$\lim_{x \to a} (\frac{f}{g})(x) = \frac{l}{m}$$

$$m \neq 0$$

Lemma:

If 1)

$$|x - x_0| < \frac{\epsilon}{2} \text{ and } |y - y_0| < \frac{\epsilon}{2}$$

Then 1)

$$|(x + y) - (x_0 + y_0)| < \epsilon$$

Proof:

$$\begin{align} |(x + y) - (x_0 + y_0)| = |(x - x_0) + (y - y_0)| \tag{1} \\ |x - x_0| + |y - y_0| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{2} \end{align}$$

If 2)

$$|x - x_0| < min(1, \frac{\epsilon}{2(|y_0| + 1)}) \text{ and } |y - y_0| < \frac{\epsilon}{2(|x_0| + 1)}$$

Then 2)
$$|xy - x_0y_0| < \epsilon$$

Proof:

$$\begin{align} |xy - x_0y_0| &= |xy - xy_0 + xy_0 - x_0y_0| \tag{1} \\ &= |x(y - y_0) + y_0(x - x_0)| \tag{2} \\ |x(y - y_0) + y_0(x - x_0)| &\leq |x||y - y_0| + |y_0||x - x_0| \tag{3} \\ |x - x_0| &< 1 \tag{4} \\ ||x| - |x_0|| \leq |x - x_0| &< 1 \tag{5} \\ |x| &\leq 1 + |x_0| \tag{6} \\ (1 + |x_0|) \cdot \frac{\epsilon}{2(|x_0| + 1)} + |y| \cdot \frac{\epsilon}{2(|x_0| + 1)} &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{7} \end{align}$$

If 3)

$$|y - y_0| < min(\frac{y_0}{2}, \frac{\epsilon \cdot |y_0|^2}{2})$$

Then 3)

$$|\frac{1}{y} - \frac{1}{y_0}| < \epsilon \mid y \neq 0$$

$$\lim_{x \to a} f(x) = l \mid \lim_{x \to a} g(x) = m$$

1)

$$\lim_{x \to a} (f + g)(x) = l + m$$

Proof:

We know there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that:

$$0 < |x - a| < \delta_1 \implies |f(x) - l| < \frac{\epsilon}{2}$$

$$0 < |x - a| < \delta_2 \implies |g(x) - m| < \frac{\epsilon}{2}$$

$$\text{Let } \delta = \min(\delta_1, \delta_2) \tag{1}$$
$$0 < |x - a| < \delta \implies |f(x) - l| < \frac{\epsilon}{2} \tag{2}$$
$$0 < |x - a| < \delta \implies |g(x) - m| < \frac{\epsilon}{2} \tag{3}$$
$$|f(x) + g(x) - (l + m)| = |f(x) - l + g(x) - m| \tag{4}$$
$$|f(x) - l + g(x) - m| \leq |f(x) - l| + |g(x) - m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{5}$$

2)

$$\lim_{x \to a} (f \cdot g)(x) = l \cdot m$$

Proof:

Given $\epsilon > 0$, we know that there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that:

$$0 < |x - a| < \delta_1 \implies |f(x) - l| < \min\left(1, \frac{\epsilon}{2(|m| + 1)}\right)$$

$$0 < |x - a| < \delta_2 \implies |g(x) - m| < \frac{\epsilon}{2(|l| + 2)}$$

$$\text{Define } \delta = \min(\delta_1, \delta_2) \tag{1}$$
$$\text{Then:} \tag{2}$$
$$0 < |x - a| < \delta \implies |f(x) - l| < \min\left(1, \frac{\epsilon}{2(|m| + 1)}\right) \tag{3}$$
$$0 < |x - a| < \delta \implies |g(x) - m| < \frac{\epsilon}{2(|l| + 1)} \tag{4}$$
$$\therefore |f(x) \cdot g(x) - lm| < \epsilon \text{ [\textbf{Lemma 2}]} \tag{5}$$

3)

$$\lim_{x\to a}(\frac{f}{g})(x) = \frac{l}{m} \implies \lim_{x\to a}(f\cdot\frac{1}{g})(x) = \frac{l}{m}$$

Proof:

$$\lim_{x\to a}(\frac{1}{g})(x) = \frac{1}{m}$$

Given $\epsilon > 0$ there exist $\delta > 0$ such that :

$$0 < |x-a| < \delta \implies |g(x)-m| < \text{min}\left(\frac{|m|}{2}, \frac{\epsilon|m|^2}{2}\right)$$

$$\implies |\left(\frac{1}{g}\right)(x) - \frac{1}{m}| < \epsilon \ [\textbf{Lemma 3}]$$

$$f(x) = x + \frac{1}{x^3} \mid \lim_{x\to 1} f(x) = 2$$

$$g(x) = x \tag{1}$$
$$\lim_{x\to 1} g(x) = 1 \tag{2}$$
$$\lim_{x\to 1} x^3 = 1 \tag{3}$$
$$\lim_{x\to 1} x\cdot x\cdot x = 1\cdot 1\cdot 1 = 1 \tag{4}$$
$$\lim_{x\to 1}\left(\frac{1}{g}\right)^3 = \frac{1}{1} = 1 \tag{5}$$
$$\lim_{x\to 1}(f+g)(x) = l+m \tag{6}$$
$$\therefore \lim_{x\to 1} x + \frac{1}{x^3} = 1+1 = 2 \tag{7}$$

9 Continuity

Definition: A function f is continuous at a point if:

$$\lim_{x\to a} f(x) = f(a)$$

- $\lim_{x\to a} f(x)$ has to exist.
- The function must be defined at a.
- The limit and function value must agree at a.

Formal Definition: A function f is continuous at a point a if:

$$\forall\, \epsilon > 0\, \exists\, \delta > 0 : |x-a| < \delta \implies |f(x)-f(a)| < \epsilon$$

textbfTheorem 1: Let f and g are continuous functions at a. Then the following functions are also continuous.

1.
$$f + g$$

2.
$$f \cdot g$$

3.
$$\frac{1}{g} \quad g(a) \neq 0$$

4. Corollary:
$$\frac{f}{g}$$

Theorem 2: If f is a continuous function at a and g is a continuous function at $f(a)$ then $g \cdot f$ is continuous at a. **Proof:**

$$\lim_{x \to a} (g \cdot f)(x) = (g \cdot f)(a)$$
$$\forall \, \epsilon > 0 \, \exists \, \delta > 0 : |x - a| < \delta \implies |(g \cdot f)(x) - (g \cdot f)(a)| < \epsilon$$

Given g is continuous at $f(a)$:

$$\forall \, \epsilon > 0 \, \exists \, \delta' > 0 : |y - f(a)| < \delta' \implies |g(y) - g(f(a))| < \epsilon$$

Equivalently:

$$|f(x) - f(a)| < \delta' \implies |g(f(x)) - g(f(a))| < \epsilon$$

We know f is continuous at a:

$$\forall \, \epsilon > 0 \, \exists \, \delta > 0 : |x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

Since $\delta' > 0$ we can choose $\epsilon = \delta'$ for the case of the continuity of the function f at a.

From the definition of continuity we know that there exist $\delta > 0$ such that:

$$|x - a| < \delta \implies |f(x) - f(a)| < \delta'$$

Hence $f \cdot g$ is continuous at a.

Theorem 3: Suppose f is continuous at a and $f(a) > 0$. Then there exist a $\delta > 0$ such that $f(x) > 0$ for all x satisfying $|x - a| < \delta$.

Proof:

If f is continuous at a:

$$\forall \, \epsilon > 0 \, \exists \, \delta > 0 : |x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

Let $\epsilon = f(a) > 0$. Then there exists $\delta > 0$ such that:

$$|x - a| < \delta \implies |f(x) - f(a)| < f(a)$$
$$-f(a) < f(x) - f(a) < f(a)$$
$$0 < f(x) < 2f(a)$$
$$\therefore |x - a| < \delta \implies f(x) > 0$$

10 Three Hard Theorems

Theorem 1: If f is continuous at $[a, b]$ and $f(a) < 0 < f(b)$ then there is some x in $[a, b]$ such that $f(x) = 0$.

Theorem 2: If f is continuous at $[a, b]$ then f is bounded above on $[a, b]$, that there is some number N such that $f(x) \leq N$ for all $x \in [a, b]$.

Theorem 3: If f if continuous at $[a, b]$, then there is some number y in $[a, b]$ such that $f(y) \geq f(x)$ for all $x \in [a, b]$.

Theorem 4: If f is continuous at $[a, b]$ and $f(a) < c < f(b)$ then there is some $x \in [a, b]$ such that $f(x) = c$.

Proof:

Let us define $g = f - c$. g is continuous at $[a, b]$.

$$[g(a) = f(a) - c] < 0 < [f(b) - c = g(b)]$$

$$g(a) < 0 < g(b)$$

Then according to **Theorem 1** there exist an $x \in [a, b]$ such that $g(x) = 0$.

$$g(x) = 0$$

$$f(x) - c = 0$$

$$f(x) = c$$

Proven.

Theorem 5: If f is continuous at $[a, b]$ and $f(a) > c > f(b)$ then there is some $x \in [a, b]$ such that $f(x) = c$. **Proof:**

If f is continuous then $-f$ is continuous.

$$-f(a) < -c < -f(b)$$

By **Theorem 4** we know $\exists\, x \in [a, b]$ such that:

$$-f(x) = -c$$

$$f(x) = c$$

Proven. Theorem 6: If f is continuous on $[a, b]$, then f is bounded below on $[a, b]$. That is, there is some number L such that $f(x) \geq L$ for all $x \in [a, b]$. **Proof:**

If f is continuous then $-f$ is continuous.

According to **Theorem 2** we know there exists an N such that $-f(x) \leq N \forall x \in [a, b]$.

$$-f(x) \leq N$$

$$f(x) \geq -N$$

Let us define $L = -N$. Then:

$$f(x) \geq L \;\forall\, x \in [a, b]$$

Proven.

Theorem 7: If f is continuous at $[a, b]$ then there is some $y \in [a, b]$ such that $f(y) \leq f(x)\, \forall\, x \in [a, b]$. **Proof:**

If f is continuous then $-f$ is continuous.

According to **Theorem 3**:

$$\exists\, y \in [a, b] : -f(y) \geq -f(x)\, \forall\, x \in [a, b]$$
$$f(y) \leq f(x)\, \forall\, x \in [a, b]$$

Proven.

Theorem 8: There exist a square root for all +ve real numbers. **Proof:**

Consider the function $f(x) = x^2$. Let us choose $\alpha > 0$

$f(b) > \alpha$ for a positive real number.

For $\alpha > 1$ we can take $a = b$.

$$f(0) < \alpha < f(b)$$

$$f : [0, b] \to \mathbb{R}$$

We know from **Theorem 4**:

$$\exists\, x \in [0, b] : f(x) = \mathbb{R}$$

Proven.

11 Least Upper Bound

$$f(x) > 0\, \exists\, \delta > 0 : f(x) > 0\, \forall\, x - \delta < x < x + \delta$$

Upper Bound of a Set: A number m is an upper bound of a set A if $x \leq m\, \forall\, x \in A$.

Least Upper Bound (Supremum): Supremum of a set A is a number s if:

1. s is an upper bound.
2. if b is any other upper bound then $s \leq b$.

Remark: If s is a least upper bound for A then:

$$\forall \epsilon > 0\, \exists\, x \in A : s - \epsilon < x$$

Every non-empty bounded above set of $\mathbb{R}$ has a least upper bound.

Theorem: If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$ then there is some number $x \in [a, b] : f(x) = 0$.

Proof:

$$A = \{x : a \leq x \leq b \text{ and } f < 0 \text{ on } [a, x]\}$$

A is bounded above by b and has atleast one member a. Hence it has a least upper bound (α).

1) We know $f(\alpha) < 0 \implies \exists\, \delta > 0 : f(x) < 0 \,\forall\, a - \delta < x < a + \delta$

2) Now α is the least upper bound of A.

$$\forall\, \epsilon > 0 \,\exists\, x_0 \in A : \alpha - \epsilon < x_0$$

From 1) we assume $-\delta < \alpha < x < +\delta$.

But from 2) we know $x \leq \alpha$. $\therefore$ **a contradiction arises.** $f(\alpha)$ **cannot be less than** 0**.**

Axiom of Completeness: every non-empty subset of real numbers that is bounded above has a least upper bound.

Theorem 1: If f is continuous at a, then $\exists\, \delta > 0 : f$ is bounded above on the interval $[a - \delta, a + \delta]$.

Proof:

From the definition of continuity we know:

$$\forall\, \epsilon > 0 \,\exists\, \delta > 0 : |x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

Let $\epsilon = 1$.

$$\begin{aligned}
|x - a| < \delta &\implies |f(x) - f(a)| < 1 \\
&\implies -1 < f(x) - f(a) < 1 \\
&\implies f(x) < 1 + f(a) = M \\
-\delta < x - a < \delta &\implies f(x) < M \\
a - \delta < x < a + \delta &\implies f(x) < M
\end{aligned}$$

Theorem 2: If f is continuous on $[a, b]$ then f is bounded.

Proof:

*If f is continuous at a then $\exists\, \delta > 0 : f$ is bounded on $[a - \delta, a + \delta]$.

$$A = \{x \in [a, b] : f \text{ is bounded on } [a, x]\}$$

A is non-empty so it has a least upper bound.

Let $\alpha = \sup A$.

Task-1: Show that $\alpha = b$.

For the sake of contradiction let $\alpha < b$.

$$\exists\, \delta > 0 : \alpha - \delta < x < \alpha + \delta$$

$$\alpha < y < \alpha + \delta$$

.

$$\therefore \alpha = b$$

.

We know f is continuous on $[a, b]$. There exists a $\delta > 0$ such that f is bounded for all x' such that $-b - \delta < x' \leq b$.

f is bounded on $[a, x]$ and f is bounded on $[x, b]$.

12 Derivative

$$f'(x_1) = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$$
$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$\begin{aligned}
f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
(x-a)f'(a) &= (x-a) \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
\lim_{x \to a}(x-a)f'(a) &= \lim_{x \to a}(x-a) \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
&= \lim_{x \to a}(x-a)\frac{f(x) - f(a)}{x - a} \\
0 \times f'(a) &= \lim_{x \to a} f(x) - f(a) \\
0 &= \lim_{x \to a} f(x) - \lim_{x \to a} f(a) \\
\therefore \lim_{x \to a} f(x) &= f(a)
\end{aligned}$$

Theorem: Let g is differentiable at a and f is differentiable at $g(a)$ then $(f \circ g)$ is differentiable at a and is given by:

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$$

Proof:

$$\Phi(h) = \begin{cases} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} & \text{if } g(a+h) - g(a) \neq 0 \\ f'(g(a)) & \text{if } g(a+h) - g(a) = -1 \end{cases} \tag{8}$$

We know f is differentiable at $g(a)$. This means that:

$$\lim_{k \to 0} \frac{f(g(a) + k) - f(g(a))}{k} = f'(g(a))$$

Thus, if $\epsilon > 0$ there is some number $\delta' > 0$ such that, for all k:

$$0 < |k| < \delta' \implies \left| \frac{f(g(a) + k) - f(g(a))}{k} - f'(g(a)) \right| < \epsilon$$

Now g is differentiable at a, hence continuous at a, so there is a $\delta > 0$ such that for all h:

$$|h| < \delta \implies |g(a+h) - g(a)| < \delta'$$

Consider now any h with $|h| < \delta$. If $k = g(a+h) - g(a) \neq 0$ then:

$$\Phi(h) = \frac{f(g(a+h)) - f(g(a)))}{g(a+h) - g(a)} = \frac{f(g(a) + k) - f(g(a))}{k}$$

It follows that $|k| < \delta'$. Hence:

$$|\Phi(h) - f'(g(a))| < \epsilon$$

On the other hand. if $g(a+h) - g(a) = 0$, then $\Phi(h) = f'(g(a))$, so it is surely true that:

$$|\Phi(h) - f'(g(a))| < \epsilon$$

We have therefore proved that:

$$\lim_{h\to 0} \Phi(h) = f'(g(a))$$

If $h \neq 0$, then we have:

$$\frac{f(g(a+h)) - f(g(a))}{h} = \Phi(h) \cdot \frac{g(a+h) - g(a)}{h}$$

$$\begin{aligned}\therefore (f \circ g)'(a) = \lim_{h\to 0} \frac{f(g(a+h)) - f(g(a))}{h} &= \lim_{h\to 0} \Phi(h) \cdot \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} \\ &= f'(g(a)) \cdot g'(a)\end{aligned}$$

Theorem: Let f be any function defined on $[a, b]$. If x is a maximum point for f on (a, b) and f is differentiable at x, then $f'(x) = 0$.

Definition: A function f has a maximum at $x \in A$ on A if $f(x) \geq f(y) \,\forall\, y \in A$. Similarly a function f has a minimum at $x \in A$ if $f(x) \leq f(h) \,\forall\, y \in A$.

Proof:

Choose h such that $x + h \in (a, b)$.

$$f(x) \geq f(x+h)$$

$$\Rightarrow f(x+h) - f(x) \leq 0$$

If $h > 0$ then:

$$\frac{f(x+h) - f(x)}{h} \leq 0$$

$$\Rightarrow \lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h} \leq 0$$

$$\Rightarrow f'(x) \leq 0$$

Again, if $h < 0$

$$\frac{f(x+h) - f(x)}{h} \geq 0$$

$$\Rightarrow \lim_{h\to 0^-} \frac{f(x+h) - f(x)}{h} \geq 0$$

$$\Rightarrow f'(x) \geq 0$$

Hence, $f'(x) = 0$.

Local maximum of a function f at x on the set A if $\exists \delta > 0 : f(x) \geq f(y) \forall y \in [x - \delta, x + \delta]$.

Theorem: If f is defined on (a, b) and has a local maximum or minimum at x and f is differentiable at x then $f'(x) = 0$.

Definition: A critical point of a function f is a number x such that $f'(x) = 0$. The number $f(x)$ itself is called the critical value of f.

To find the maximum or minimum of a function f we need to consider 3 different points:

1. Critical points of $f \in [a, b]$.
2. The endpoints of $[a, b]$.
3. Points $x \in [a, b]$ such that f is not differentiable at x.

Rolle's Theorem: If f is continuous on $[a, b]$ and differentiable on (a, b) then there is a number $x \in [a, b]$ such that $f'(x) = 0$ given $f(a) = f(b) \mid a \neq b$.

Proof: It follows from the continuity of f on $[a, b]$ that f has a maximum and minimum value on $[a, b]$.

Mean Value Theorem: If f is continuous on $[a, b]$ and differentiable on (a, b). Then there is an $x \in (a, b)$ such that:

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

Proof:

Let:

$$h(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a)$$

Clearly h is continuous on $[a, b]$ and differentiable on (a, b), and:

$$h(a) = f(a)$$

$$h(b) = f(b) - \left[\frac{f(b) - f(a)}{b - a}\right](b - a) = f(a)$$

Consequently, we may apply Rolle's Theorem to h and conclude that there is some $x \in (a, b)$ such that:

$$0 = h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow f'(x) = \frac{f(b) - f(a)}{b - a}$$

Corollary 1: If f is differentiable on (a, b) and f is defined on $[a, b]$. If $f'(x) = 0$ for all $x \in [a, b]$ then f is a constant function.

Proof:

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

$$\begin{aligned}
&\Rightarrow 0 = \frac{f(b) - f(a)}{b - a} \\
&\Rightarrow f(b) - f(a) = 0 \\
&\Rightarrow f(a) = f(b)
\end{aligned}$$

Corollary 2: If $f'(x) = g'(x)\ \forall x \in [a, b]$ then $f(x) = g(x) + c$.

Proof:

$$\begin{aligned}
h(x) &= f(x) - g(x) \\
\Rightarrow h'(x) &= f'(x) - g'(x) = 0 \\
&\Rightarrow h(x) = c \\
&\Rightarrow f(x) - g(x) = c \\
&\Rightarrow f(x) = g(x) + c
\end{aligned}$$

A function is increasing if $y \geq x \implies f(y) \geq f(x)$. A function is decreasing if $y \geq x \implies f(y) \leq f(x)$.

Theorem: A function is increasing if $f'(x) \geq 0$.

Proof:

$$\begin{aligned}
f'(x) &= \frac{f(b) - f(a)}{b - a} \\
&\Rightarrow \frac{f(b) - f(a)}{b - a} \geq 0 \\
&\Rightarrow f(b) \geq f(a)
\end{aligned}$$

Theorem: Suppose $f'(a) = 0$. If $f''(a) > 0$ then f has a local minimum at a.

$$\begin{aligned}
f''(a) = \lim_{h \to 0} \frac{f'(a+h) - f'(a)}{h} &= \lim_{h \to 0} \frac{f'(a+h)}{h} \\
\Rightarrow \lim_{h \to 0} \frac{f'(a+h)}{h} &\geq 0
\end{aligned}$$

If $h > 0$ then $f'(a + h) \geq 0$.

If $h < 0$ then $f'(a + h) \leq 0$.

Cauchy Mean Value Theorem: If f and g are continuous functions on $[a, b]$ and differentiable on (a, b) then there is a number x in $[a, b]$ such that:

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x)}{g'(x)}$$

L'Hospital's Rule: Suppose:

$$\lim_{x \to a} f(x) = 0 \mid \lim_{x \to a} g(x) = 0$$

and suppose that

$$\lim_{x \to a} \frac{f'(x)}{g'(x)}$$

exists. Then,

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

exists and

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Proof: Given

$$\lim_{x \to a} \frac{f'(x)}{g'(x)}$$

exists. Then,

1. there is an interval $(a - \delta, a + \delta)$ such that $f'(x)$ and $g'(x)$ exist for all $x \in (a - \delta, a + \delta)$ except perhaps for $x = a$.
2. in this interval $g'(x) \neq 0$ except for $x = a$.

$$(a, a + \delta) : a < x < a + \delta$$

$$[a, x] \text{ and apply mean value theorem on } g$$

$$g'(x_1) = \frac{g(x) - g(a)}{x - a}$$

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(x)}{g'(x)}$$

$$\Rightarrow \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$$

$$\Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Definition: A fuction f is convex on an interval if for a, x, b in the interval with $a < x < b$ we have:

$$\frac{f(x) - f(a)}{x - a} < \frac{f(b) - f(a)}{b - a} [\text{convex}]$$

$$\frac{f(x) - f(a)}{x - a} > \frac{f(b) - f(a)}{b - a} [\text{concave}]$$

1. The graph of f lies above the line at $(a, f(a))$ except the point $(a, f(a))$. $(a, f(a))$ is called the contact point.
2. If $a < b$, then the slope of the tangent line at $(a, f(a))$ is less than the slope of the tangent line at $(b, f(b))$, that f' is increasing.

13 Inverse Functions

Let $f : A \to B$.

There exist a function f^{-1} which is the inverse of f if the function f is bijective. A function f is bijective if it is both surjective and injective.

Definition: For any function f, the inverse of f, denoted by f^{-1} is the set of all pairs (a, b) for which the pair (b, a) is in f.

Theorem: If f^{-1} is a function if f is one-one.

Proof: Let f is one-one.

$$
\begin{gathered}
(a, b) \mid (c, b) \\
b = f(a) \mid b = f(c) \\
f(a) = f(c) \\
a = c
\end{gathered}
$$

Theorem: If f is continuous and one-one on an interval then f is either decreasing or increasing.

Theorem: If f is continuous and one-one on an interval, then f^{-1} is also continuous.

Proof:

$$
\begin{gathered}
\lim_{x \to b} f^{-1}(x) = f^{-1}(b) \\
\forall \epsilon > 0 \exists \delta > 0 : |x - b| < \delta \implies |f^{-1}(x) - f^{-1}(b)| < \epsilon \\
\Rightarrow b - \delta < x < b + \delta \implies -\epsilon + f^{-1}(b) < f^{-1}(x) < \epsilon + f^{-1}(b) \\
\Rightarrow -\delta + f(a) < x < \delta + f(a) \implies -\epsilon + a < f^{-1}(x) < \epsilon + a \\
\delta = min\{f(a) - f(a - \epsilon), f(a + \epsilon) - f(a)\} \\
a - \epsilon < a < a + \epsilon \\
\Rightarrow f(a - \epsilon) < f(a) < f(a + \epsilon) \\
f(a - \epsilon) \leq f(a) - \delta \mid f(a) + \delta \leq f(a + \epsilon) \\
f(a) - \delta < x < f(a) + \delta \\
\Rightarrow f^{-1}(f(a - \epsilon)) < f^{-1}(x) < f^{-1}(f(a + \epsilon)) \\
\Rightarrow a - \epsilon < f^{-1}(x) < a + \epsilon \\
\Rightarrow -\epsilon < f^{-1}(x) - a < \epsilon \\
\Rightarrow -\epsilon < f^{-1}(x) - f^{-1}(b) < \epsilon \\
\Rightarrow |f^{-1}(x) - f^{-1}(b)| < \epsilon
\end{gathered}
$$

14 Integrals

Partition: Let $a < b$, A partition of the interval $[a, b]$ is a finite collection of points in $[a, b]$ such that one of which is a and one of which is b.

Suppose f is bounded on [a,b] and $P = \{t_0, t_1, \dots, t_n\}$ is a partition of $[a, b]$. Let:

$$m_i = \inf\{f(x) : t_{i-1} \leq x \leq t_i\}$$
$$M_i = \sup\{f(x) : t_{i-1} \leq x \leq t_i\}$$

The lower sum of f for P, denoted by $L(f, P)$ is defined as:

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})$$

The upper sum of f for P, denoted by $U(f, P)$ is defined as:

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})$$

$$\implies L(f, P) \leq U(f, P)$$

Lemma: If Q contains P, then:

$$L(f, P) \leq L(f, Q)$$

$$U(f, P) \geq U(f, Q)$$

Proof: Let P is a partition of $[a, b]$, $P = \{a = t_0, t_1, \dots, t_n = b\}$ and $Q = \{a = t_0, t_1, \dots, t_{k-1}, \mu, t_k, \dots, t_n = b\}$.

$$m_i = \inf\{f(x) : t_{i-1} \leq x \leq t_i\}$$
$$m' = \inf\{f(x) : t_{k-1} \leq x \leq \mu\}$$
$$m'' = \inf\{f(x) : \mu \leq x \leq t_k\}$$

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}) = \sum_{i=1}^{k-1} m_i(t_i - t_{i-1}) + m_k(t_k - t_{k-1}) + \sum_{i=k}^{n} m_i(t_i - t_{i-1})$$

$$L(f, Q) = \sum_{i=1}^{k-1} m_i(t_i - t_{i-1}) + m'(\mu - t_{k-1}) + m''(t_k - \mu) + \sum_{i=k}^{n} m_i(t_i - t_{i-1})$$

Let $P = P_1 \cup P_2$. So:

$$L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2)$$

$$\implies L(f, P_1) \leq U(f, P_2)$$

A fucntion f bounded on a close interval $[a, b]$ is said to be **integrable** if:

$$M = S$$

$$\text{Lower Sum} = \text{Upper Sum}$$

Denoted by:

$$\int_a^b f \, dx$$

$$L(f, P) \leq M = \int_a^b f \, dx = S \leq U(f, P)$$