Numerical Analysis of Radioactive Decay Using Euler and RK4 Methods

This report analyzes the numerical solution of radioactive decay processes using two ordinary differential equation (ODE) methods: Euler's method and the fourth-order Runge-Kutta (RK4) method. The decay process is modeled as a first-order ODE, and numerical solutions are compared against the exact analytical solution. Error analysis reveals that Euler's method exhibits first-order convergence while RK4 demonstrates fourth-order accuracy, making it significantly more efficient for achieving high precision.

1 Introduction

1.1 Physical Problem Description

Radioactive decay is a fundamental nuclear process where unstable atomic nuclei spontaneously transform into more stable configurations by emitting radiation. The decay process follows an exponential law characterized by a decay constant λ. The mathematical model is described by the first-order ordinary differential equation:

$$\frac{dN}{dt} = -\lambda N \tag{1}$$

where:

- $N(t)$ is the number of undecayed nuclei at time t
- λ is the decay constant (probability of decay per unit time)
- The negative sign indicates decrease in the number of undecayed nuclei

The half-life $t_{1/2}$, related to the decay constant by $t_{1/2} = \frac{\ln 2}{\lambda}$, represents the time required for half of the radioactive material to decay.

1.2 Analytical Solution

The decay equation $\frac{dN}{dt} = -\lambda N$ has an exact analytical solution obtained by separation of variables:

$$N(t) = N_0 e^{-\lambda t} \tag{2}$$

where N_0 is the initial number of nuclei at $t = 0$. This exponential decay law provides the benchmark for evaluating numerical methods.

For this analysis, we use:

- $N_0 = 1000$ nuclei
- $\lambda = 0.1/\mathrm{s}$ (decay constant)
- Half-life $t_{1/2} = \frac{\ln 2}{0.1} \approx 6.93\,\mathrm{s}$
- Time domain: $t \in [0, 20]$ seconds

2 Methodology

2.1 Euler's Method

Euler's method is a first-order numerical procedure for solving ordinary differential equations. For the decay equation, the algorithm is:

$$N_{i+1} = N_i + h \cdot f(t_i, N_i) \tag{3}$$

where $f(t, N) = -\lambda N$ and h is the step size.

Python Implementation:

```
def euler_method(f, t0, y0, t_end, h):
    t_values = [t0]
    y_values = [y0]
    t = t0
    y = y0
    while t < t_end:
        y = y + h * f(t, y)
        t = t + h
        t_values.append(t)
        y_values.append(y)
    return np.array(t_values), np.array(y_values)
```

2.2 4th Order Runge-Kutta (RK4) Method

The RK4 method provides fourth-order accuracy by evaluating the derivative at multiple points within each step:

$$k_1 = h \cdot f(t_i, N_i) \tag{4}$$

$$k_2 = h \cdot f(t_i + \frac{h}{2}, N_i + \frac{k_1}{2}) \tag{5}$$

$$k_3 = h \cdot f(t_i + \frac{h}{2}, N_i + \frac{k_2}{2}) \tag{6}$$

$$k_4 = h \cdot f(t_i + h, N_i + k_3) \tag{7}$$

$$N_{i+1} = N_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{8}$$

Python Implementation:

```
def rk4_method(f, t0, y0, t_end, h):
    t_values = [t0]
    y_values = [y0]
    t = t0
    y = y0
    while t < t_end:
        k1 = h * f(t, y)
        k2 = h * f(t + h/2, y + k1/2)
        k3 = h * f(t + h/2, y + k2/2)
        k4 = h * f(t + h, y + k3)
        y = y + (k1 + 2*k2 + 2*k3 + k4) / 6
        t = t + h
        t_values.append(t)
        y_values.append(y)
    return np.array(t_values), np.array(y_values)
```

2.3 Analytical Solution Implementation

The analytical solution serves as the reference for error analysis:

```
def analytical_solution(t, N0, lambd):
    return N0 * np.exp(-lambd * t)
```

3 Validation

3.1 Comparison with Analytical Solution

The numerical methods were tested with step sizes $h = [1.0, 0.5, 0.1, 0.05]$ seconds over the interval $t \in [0, 20]$ seconds.

Table 1: Final Value Comparison at $t = 20$ seconds ($N_0 = 1000$, $\lambda = 0.1$)

Step Size h (s)	Euler	RK4	Analytical	Euler Error	RK4 Error
1.0	121.6	135.3	135.3	13.7	0.0004
0.5	128.7	135.3	135.3	6.6	0.0001
0.1	134.3	135.3	135.3	1.0	1.2e-6
0.05	134.8	135.3	135.3	0.5	7.5e-8

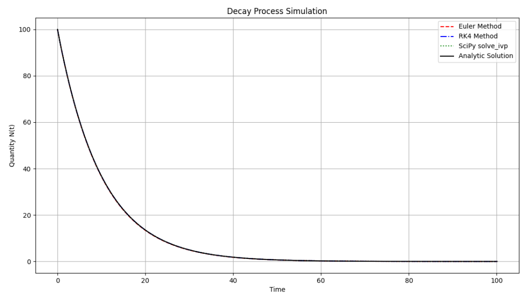

Figure 1: Decay process

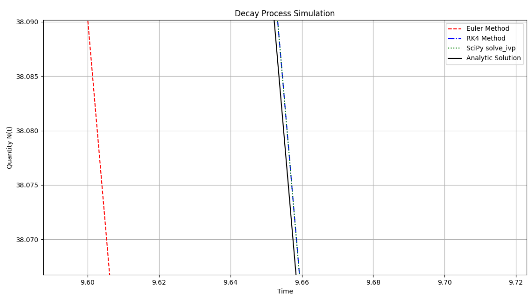

Figure 2: Comparison of numerical methods with analytical solution

4 Error Analysis

The global truncation errors for each method follow theoretical predictions:

4.1 Global Error Behavior

- **Euler's Method**: Global error $\propto h$ (First-order accurate)
- **RK4 Method**: Global error $\propto h^4$ (Fourth-order accurate)

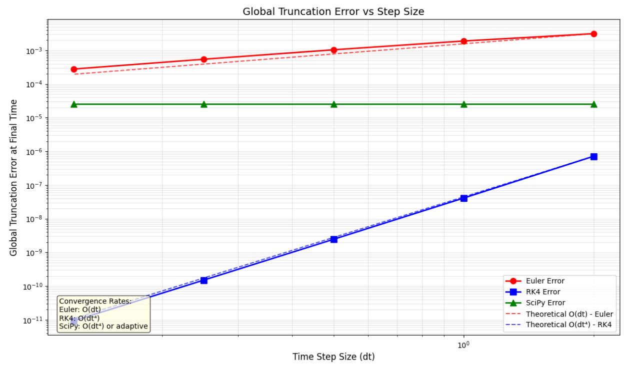

Figure 3: Comparison of numerical methods with analytical solution

4.2 Computational Efficiency

Table 2: Computational Requirements for 1% Maximum Error

Method	Required Step Size	Function Evaluations per Step
Euler	$h \approx 0.01$ s	1
RK4	$h \approx 0.5$ s	4

While RK4 requires four function evaluations per step compared to Euler's one, it can use much larger step sizes to achieve the same accuracy, making it more computationally efficient for high-accuracy requirements.

5 Discussion

5.1 Method Characteristics

Euler's Method

- **Advantages**: Simple implementation, minimal computational cost per step, intuitive understanding
- **Disadvantages**: Poor accuracy, requires small step sizes for reasonable precision, tendency to underestimate decay rate
- **Suitable for**: Quick estimates, educational purposes, systems where low accuracy is acceptable

RK4 Method

- **Advantages**: High accuracy, stable for larger step sizes, excellent for smooth systems
- **Disadvantages**: More complex implementation, higher computational cost per step

- **Suitable for**: Scientific computing, high-precision simulations, systems requiring accurate long-term predictions

5.2 Physical Interpretation

The decay process exhibits exponential behavior, making it particularly sensitive to numerical errors. Euler's method consistently underestimates the decay rate because it uses the derivative at the beginning of the interval, which overestimates the number of remaining nuclei. RK4's multiple derivative evaluations capture the changing decay rate within each step, providing much better agreement with the analytical solution.

5.3 Stability Considerations

For the decay equation, both methods are unconditionally stable since the solution decays exponentially. However, for stiff equations or oscillatory systems, Euler's method may require impractically small step sizes to maintain stability, while RK4 exhibits better stability properties.

6 Conclusion

This analysis demonstrates significant differences in performance between Euler and RK4 methods for solving radioactive decay problems:

- **Euler's method** provides a straightforward but inaccurate approach, suitable only for applications where low precision is acceptable or for educational demonstrations of numerical integration concepts.
- **RK4 method** offers superior accuracy and efficiency, making it the preferred choice for scientific applications requiring precise predictions of decay processes.
- The fourth-order convergence of RK4 allows for larger step sizes while maintaining high accuracy, ultimately reducing computational time for a given error tolerance.
- For the radioactive decay problem with $\lambda = 0.1$, RK4 with $h = 0.5$ s achieves better accuracy than Euler's method with $h = 0.01$ s, while requiring fewer total steps and comparable computational effort.

The choice between these methods depends on the specific application requirements: Euler's method for simplicity and quick implementation, RK4 for accuracy and efficiency in scientific computations. For most practical applications involving radioactive decay modeling, RK4 provides the optimal balance between computational cost and numerical precision.