A stochastic optimization algorithm for revenue maximization in a service system with balking customers

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July 3, 2025

Motivation and Setup

- Amusement park / Ice-cream truck / Ride-sharing service
- Joining decisions typically based on: $\begin{cases} \text{Waiting time} \\ \text{Price} \end{cases}$
- Balking customers may remain unobserved by service operator
- Operator desires to maximize earnings

Basic model

- Customers arrive according to a Possion process with rate Λ
- Each customer is equipped with service time $S \sim G(\cdot)$
- A single server

Impatience modelling framework

- Each customer is also equipped with disutility threshold Y sampled from disutility threshold CDF $H(\cdot)$
- Incoming customer observes
  (a) admission price p
  (b) prospective waiting time V
- Disutility evaluation function $\phi : \mathcal{P} \times \mathbb{R}_+ \mapsto \mathbb{R}_+$
- Joins iff $Y \geq \phi(p, V)$, otherwise balks

Examples

Example 1:

- $\phi(p, V) = \theta_1 p + \theta_2 V$ for $\theta_1, \theta_2 > 0$
- $H(x) = 1 - e^{-x}$.

Example 2:

- $\phi(p, V) = \theta_1 p^2 + \theta_2 pV + \theta_3 V^2$ for $\theta_1, \theta_2, \theta_3 > 0$
- $H(x) = \frac{x}{(x+1)^2}$

Equivalent formulation

- Incoming customer observes p, V
- Joins with probability $H\big(\phi(p, V)\big)$
- With abuse of notation, we denote joining probability by $H(p, V)$

Revisiting Example 1 (Running example for the talk):

- $\phi(p, V) = \theta_1 p + \theta_2 V$ for $\theta_1, \theta_2 > 0$
- $H(x) = 1 - e^{-x}$.
- Then, $H(p, V) = e^{-\theta_1 p - \theta_2 V}$

Notations

- $\widetilde{A}_0 := 0$, effective arrival times $\left\{\widetilde{A}_k\right\}_{k\geq 1}$, interarrival times $\{A_k\}_{k\geq 1}$
- Service times $\{S_k\}_{k\geq 1}$
- $\overline{W}_0 := 0$, workloads just after effective arrivals $\left\{\overline{W}_k\right\}_{k\geq 1}$, workloads just before effective arrivals $\{\underline{W}_k\}_{k\geq 1}$
- $F_p(\cdot; w) \equiv$ CDF of A_k, when admission price $= p$ and $\overline{W}_{k-1} = w$
- $F_p^{-1}(\zeta; w) \equiv$ a realization of A_k, when admission price $= p$, $\overline{W}_{k-1} = w$ for $\zeta \sim$ Uniform$[0, 1]$

Interarrival time CDF $F_p(\cdot; w)$

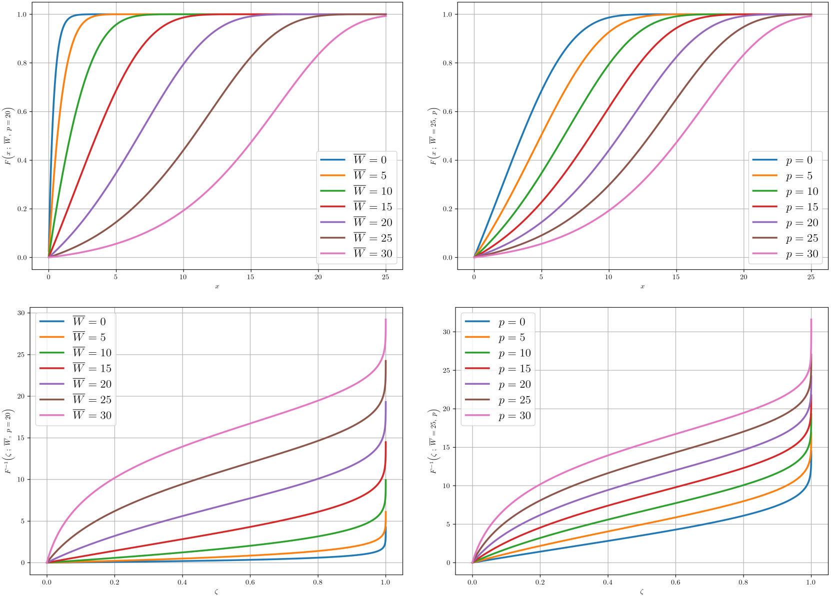
$F_p(\ell; w)$ is the probability that given a current workload level w, the subsequent interarrival time is atmost ℓ , i.e.,

$$\begin{aligned} F_p(\ell; w) &= \mathbb{P}\left(A_k \leq \ell \;\middle|\; p, \overline{W}_{k-1} = w\right) \\ &= 1 - \mathbb{P}\left(A_k > \ell \;\middle|\; p, \overline{W}_{k-1} = w\right) \\ &= 1 - \mathbb{P}\left(N\left(\widetilde{A}_{k-1}, \widetilde{A}_{k-1} + \ell\right] = 0 \;\middle|\; p, \overline{W}_{k-1} = w\right) \end{aligned}$$

Observe that $N(\cdot)$, conditioned on p and $\overline{W}_{k-1}$, is a thinned Poisson process, entailing for our running example that

$$F_p(\ell; w) = 1 - \exp\left(-\int_0^\ell \Lambda\, e^{-\theta_1 p - \theta_2 (w-t)^+} \mathrm{d}t\right).$$

Visualization of the CDF $F_p(\ell; w)$ and $F^{-1}(\zeta; w)$



Some Assumptions

Assumption 1

The joining probability $H : \mathcal{P} \times \mathbb{R}_+ \mapsto [0, 1]$ *satisfies*

1. $\lim_{p \to \infty} H(p, V) = \lim_{V \to \infty} H(p, V) = 0$
2. $H(\cdot, V)$ *is continuously differentiable*
3. $H(p, \cdot)$ *is differentiable*

Assumption 2

There exists a random variable $\Xi[\zeta]$ *such that for any* $m \geq 1$, $\gamma_m := \mathbb{E}\left[(\Xi[\zeta])^m\right] < 1$ *and satisfying*

$$0 <_{a.s.} 1 - \nabla_w F_p^{-1}(\zeta; W) <_{a.s.} \Xi[\zeta]$$

Interesting model results

Suppose that two M/G/1+H(p, V) queues are coupled with random initial workloads $\overline{W}_0^1 < \infty$, and $\overline{W}_0^2 < \infty$. Then,

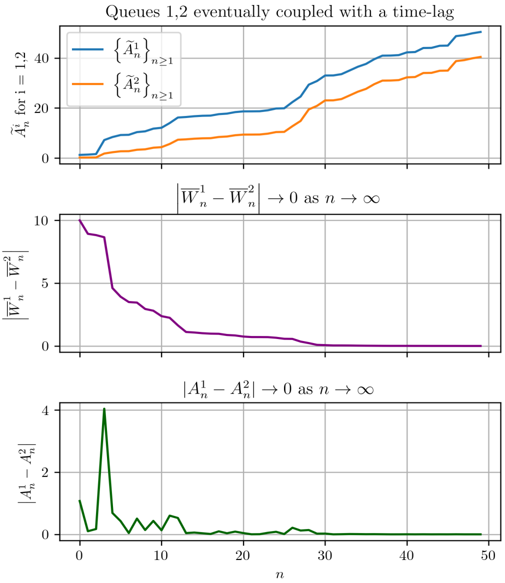
(a) For all $n \geq 1$, and $m \geq 0$, $\mathbb{E}\left|\overline{W}_n^1 - \overline{W}_n^2\right|^m \leq \gamma_m^n \, \mathbb{E}\left|\overline{W}_0^1 - \overline{W}_0^2\right|^m$

(b) Let $\left\{A_k^i\right\}_{k\geq 1}$, $\left\{\widetilde{A}_k^i\right\}_{k\geq 1}$ for i = 1,2 be the effective interarrival, arrival times in the two queues. Then,

$$\mathbb{E}\left|A_n^1 - A_n^2\right| \leq \gamma_1^n \, \mathbb{E}\left|\overline{W}_0^1 - \overline{W}_0^2\right|,$$

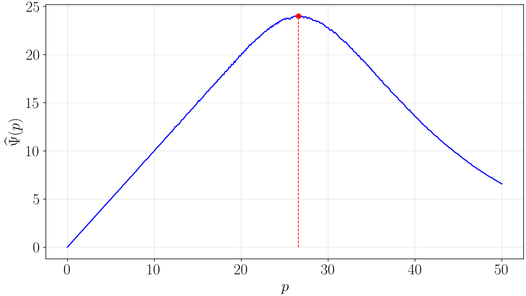
$$\mathbb{E}\left|\widetilde{A}_n^1 - \widetilde{A}_n^2\right| \leq \left(\frac{1-\gamma_1^n}{1-\gamma_1}\right) \mathbb{E}\left|\overline{W}_0^1 - \overline{W}_0^2\right|.$$

Visualization of model results



Problem Description

- Let $\Psi(p) \equiv$ expected revenue generated per unit time in stationarity when admission price is p
- Let p^* be the optimal price, i.e. $p^* = \arg\max_{p \in \mathcal{P}} \Psi(p)$



- **Goal:** Learn p^* and minimize the regret while doing so

Stochastic gradient descent algorithm

Theorem 1

Let $A_\infty(p)$ be the steady-state interarrival-time in our system with admission price p. Then, $\Psi(p) = \frac{p}{\mathbb{E}[A_\infty(p)]}$.

- Closed form expression for $\mathbb{E}[A_\infty(p)]$ is unavailable
- Let $\nabla\Psi(p)$ be the gradient of $\Psi(p)$ with respect to price. Then,

$$\nabla\Psi(p) = \frac{1}{\mathbb{E}[A_\infty(p)]} - p\frac{\nabla\,\mathbb{E}[A_\infty(p)]}{\mathbb{E}[A_\infty(p)]^2}$$

- We need estimators for $\mathbb{E}[A_\infty(p)]$ and $\nabla\,\mathbb{E}[A_\infty(p)]$
- We show that $\nabla\,\mathbb{E}[A_\infty(p)] = \mathbb{E}[\nabla A_\infty(p)]$.

Stochastic gradient descent algorithm

- Start at time 0 with an empty system
- Let $\{T_k^*\}_{k\geq 1}$ be an increasing sequence of window sizes
- We set admission price p_{k-1} for a duration of length $T_k \geq_{\text{a.s.}} T_k^*$
- Let $\overline{T}_m = T_1 + \cdots + T_m$. Then, set price p_{k-1} during $\left[\overline{T}_{k-1}, \overline{T}_k\right)$
- Suppose we observe interarrival times $A_1 = F_p^{-1}(\zeta_1; \overline{W}_0)$, $\cdots$, $A_N = F^{-1}(\zeta_N; \overline{W}_{N-1})$
- Estimator for $\mathbb{E}[A_\infty(p_{k-1})]$ is $\widehat{A_\infty}(p_{k-1}) = \frac{1}{N}\sum_{i=1}^N F_p^{-1}\left(\zeta_i; \overline{W}_{i-1}\right)$
- Estimator for $\nabla\,\mathbb{E}[A_\infty(p_{k-1})]$ is $\widehat{\nabla A_\infty}(p_{k-1}) = \frac{1}{N}\sum_{i=1}^N \nabla_p F_p^{-1}\left(\zeta_i; \overline{W}_{i-1}\right)$

Stochastic gradient descent algorithm

- Evaluating $\nabla_p F_p^{-1}(\zeta_i; \overline{W}_{i-1})$ requires knowledge of $\nabla_p \overline{W}_{i-1}$
- Set $\nabla_p \overline{W}_0 := 0$
- Then for any $i \in \{\widetilde{N}_{k-1}+1, \cdots, \widetilde{N}_k\}$, $\nabla_p \overline{W}_i$ can be computed through the following recursion

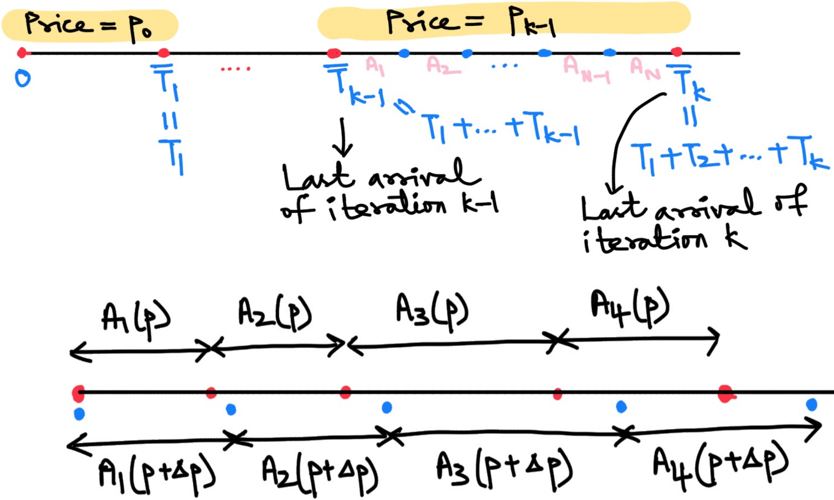
$$\nabla_p \overline{W}_i = \begin{cases} \nabla_p \overline{W}_{i-1} - \nabla_p F_p^{-1}\left(\zeta_i; \overline{W}_{i-1}\right) & \text{if } \underline{W}_i > 0, \\ 0 & \text{if } \underline{W}_i = 0. \end{cases}$$

- Estimator for $\nabla \Psi(p_{k-1})$ is given by

$$\widehat{\nabla \Psi}(p_{k-1}) = \frac{1}{\widehat{A_\infty}(p_{k-1})} - p_{k-1} \frac{\widehat{\nabla A_\infty}(p_{k-1})}{\widehat{A_\infty}(p_{k-1})^2}$$

- Finally, $p_k = p_{k-1} + \eta_k \widehat{\nabla \Psi}(p_{k-1})$

Stochastic gradient descent algorithm



Stochastic Approximation analysis

Theorem 2

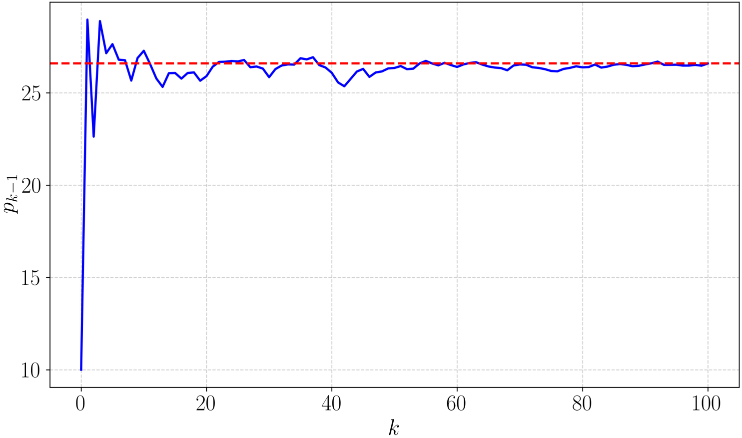
Let $\{T_k^*\}_{k\geq 1}$ *be an increasing sequence of window sizes. Then,*

1. $\nu_k := \mathbb{E}\left[\widehat{\nabla\Psi}(p_{k-1})^2\right] = \mathcal{O}(1)$
2. $\left|\beta_k\right| := \left|\mathbb{E}\left[\widehat{\nabla\Psi}(p_{k-1})\middle|\, \mathcal{F}_{k-1}\right] - \nabla\Psi(p_{k-1})\right| = \mathcal{O}\left(\eta_{k-1}\frac{1}{{T_{k-1}^*}^2}\right)$

Ongoing work:

- $-\Psi(\cdot)$ may not be strongly convex. At best K-convex, and unimodal
- Are there results/techniques to analyze this SA?
- If not, what's the best we can do?

Numerical results



Regret Analysis

- Let $N(\cdot)$ be the counting process corresponding to effective arrivals
- Let $W(\cdot)$ be the workload process in a M/G/1 + $\mathrm{H}_\theta(p, V)$ queue
- Let $\mathscr{F} = (\mathscr{F}_k)_{k\geq 0}$ be a filtration such that $\mathscr{F}_0 = \Phi$, and $\mathscr{F}_k$ consists of all queueing data from the first k iterations of the algorithm, i.e. from time 0 till time $\overline{T}_k$.
- Let $R(L)$ be the regret accumulated after performing L iterations of our algorithm. Then,

$$\begin{aligned} R(L) &= \sum_{k=1}^{L} \Psi(p^*)\,\mathbb{E}\left[T_k\right] - \mathbb{E}\left[p_{k-1}\,\mathbb{E}\left[N\left(\overline{T}_{k-1}, \overline{T}_k\right]\middle|\,\mathscr{F}_{k-1}\right]\right] \\ &= R_1(L) + R_2(L) \end{aligned}$$

Regret Analysis

$$R_1(L) = \sum_{k=1}^{L} \Psi(p^*)\, T_k^* - \mathbb{E}\left[\Psi(p_{k-1})\right] T_k^*$$

$$R_2(L) = \sum_{k=1}^{L} \mathbb{E}\left[\Psi(p_{k-1})\right] T_k^* - \mathbb{E}\left[p_{k-1}\, \mathbb{E}\left[N\left(\overline{T}_{k-1}, \overline{T}_{k-1} + T_k^*\right] \middle| \mathscr{F}_{k-1}\right]\right]$$

Theorem 3

The regret of sub-optimal price $R_1(L)$ and the regret of transience $R_2(L)$ decay to 0 at the following rates

$$R_1(L) = \mathcal{O}\left(\sum_{k=1}^{L} T_k^*\, \mathbb{E}\left|p^* - p_{k-1}\right|\right), \quad R_2(L) = \mathcal{O}\left(\sum_{k=1}^{L} \eta_k \frac{1}{T_k^{*2}}\right)$$

Ongoing/Future Work

- Is it possible to say anything theoretically for M/G/s queues?
- Learning the optimal price in a model with unknown parameters?
- Stochastic approximation?

References

- Suri R., and Zazanis M.A. “Perturbation analysis gives strongly consistent sensitivity estimates for the M/G/1 queue.” Management Science 34.1 (1988): 39-64.
- Glasserman, P. “Gradient estimation via perturbation analysis”. Vol. 116. Springer Science & Business Media, 1990.
- Naor, Pinhas. "The regulation of queue size by levying tolls.” Econometrica: journal of the Econometric Society (1969): 15-24.
- Chen X., Liu Y., and Hong G. “An online learning approach to dynamic pricing and capacity sizing in service systems.” Operations Research (2023).
- Inoue Y., Ravner L., and Mandjes M. “Estimating customer impatience in a service system with unobserved balking.” Stochastic Systems 13.2 (2023): 181-210.
- Bodas S.A., Mandjes M., and Ravner L. “Statistical inference for a service system with non-stationary arrivals and unobserved balking.” arXiv preprint arXiv:2311.16884 (2023).

THANK YOU!