BT6270 Computational Neuroscience

Computational Neuroscience Assignment 3

Hopf Oscillators

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Hopf oscillator :

A Hopf oscillator without any external inputs can be expressed as follows in complex state variable representation.

In cartesian coordinates:

$$\frac{dx}{dt} = x\left(\alpha + \beta_1 r^2 + \frac{\epsilon\,\beta_2 r^5}{1-\epsilon\, r^2}\right) - \omega y$$

$$\frac{dy}{dt} = y\left(\alpha + \beta_1 r^2 + \frac{\epsilon\,\beta_2 r^5}{1-\epsilon\, r^2}\right) + \omega x$$

In polar coordinates :

$$\frac{dr}{dt} = \alpha + \beta_1 r^2 + \frac{\epsilon\,\beta_2 r^5}{1-\epsilon\, r^2}$$

$$\frac{d\phi}{dt} = \omega$$

Depending on the values of α, β_1, β_2 , the oscillator is characterized into 4 difference hopf variants

$\alpha = 0 \quad \beta_1 < 0 \quad \beta_2 = 0$	Critical Hopf
$\alpha > 0 \quad \beta_1 < 0 \quad \beta_2 = 0$	Supercritical Hopf
$\alpha < 0 \quad \beta_1 > 0 \quad \beta_2 < 0$ Local maxima > 0	Supercritical double limit cycle
$\alpha < 0 \quad \beta_1 > 0 \quad \beta_2 < 0$ Local maxima < 0	Subcritical double limit cycle

Throughout the rest of this article we will be referring to the **Supercritical** Hopf regime with $\alpha = \mu \quad \beta_1 = 1 \quad \beta_2 = 0$.

The equations for which come out to

$$\frac{dx}{dt} = \mu(1 - r^2)x - \omega y$$

$$\frac{dy}{dt} = \mu(1 - r^2)y + \omega x$$

In polar coordinates can be represented as

$$\frac{dr}{dt} = \mu(1 - r^2)r$$

$$\frac{d\phi}{dt} = \omega$$

Complex Coupled :

If two oscillators are coupled together with equal natural frequencies with real coupling coefficients, they are going to phase lock in phase(0) or out of phase by $(2n + 1)\pi$. This is determined by the polarity of the coupling and the initialization.

When two Hopf oscillators with identical frequencies are coupled bilaterally with complex coefficients they exhibit phase locking. The equation for both the oscillators can be represented as :

$$\frac{dr_1}{dt} = \mu(1 - r_1^2) \cdot r_1 + A \cdot r_2 \cdot cos(\phi_2 - \phi_1 + \theta)$$

$$\frac{dr_2}{dt} = \mu(1 - r_2^2) \cdot r_2 + A \cdot r_1 \cdot cos(\phi_1 - \phi_2 - \theta)$$

$$\frac{d\phi_1}{dt} = \omega + A \cdot \frac{r_2}{r_1} sin(\phi_2 - \phi_1 + \theta)$$

$$\frac{d\phi_2}{dt} = \omega + A \cdot \frac{r_1}{r_2} sin(\phi_1 - \phi_2 - \theta)$$

Here the coupling constant is given by $W = Ae^{i\theta}$ and $W^* = Ae^{-i\theta}$. Where A and θ are the magnitude and angle of coupling coefficient respectively. After a long time, or at steady state, $\phi_1 - \phi_2$ approaches any of the $2n\pi + \theta$ solution.

We model these equations numerically using Euler's Integration with Δt = 0.1 msec on python. The values of r_1, r_2, ϕ_1, ϕ_2 are randomized to observe the plot between $(\phi_1 - \phi_2 - \theta)$ Vs time. The set values are $\omega = 5$ and $\theta = \{-47°, 98°\}$. And for various initial values and A values , we plot the required graphs and check how a particular graph varies with A[1]

[1] The values of the initial angles and radii were picked at random with A = 0.5 and random.seed(1000)

For theta = – 47:

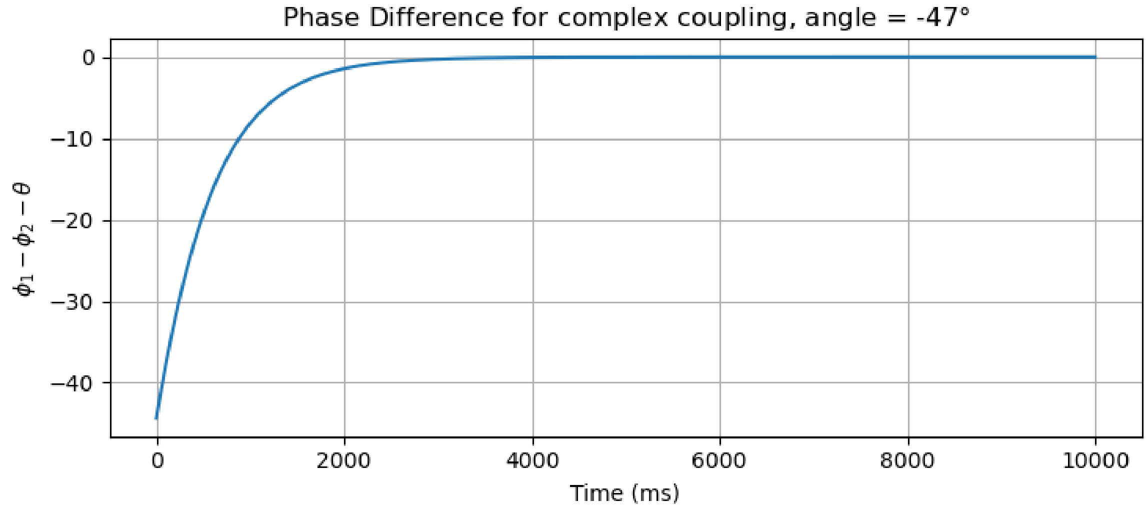

Fig 1a : Phase difference with θ = – 47°

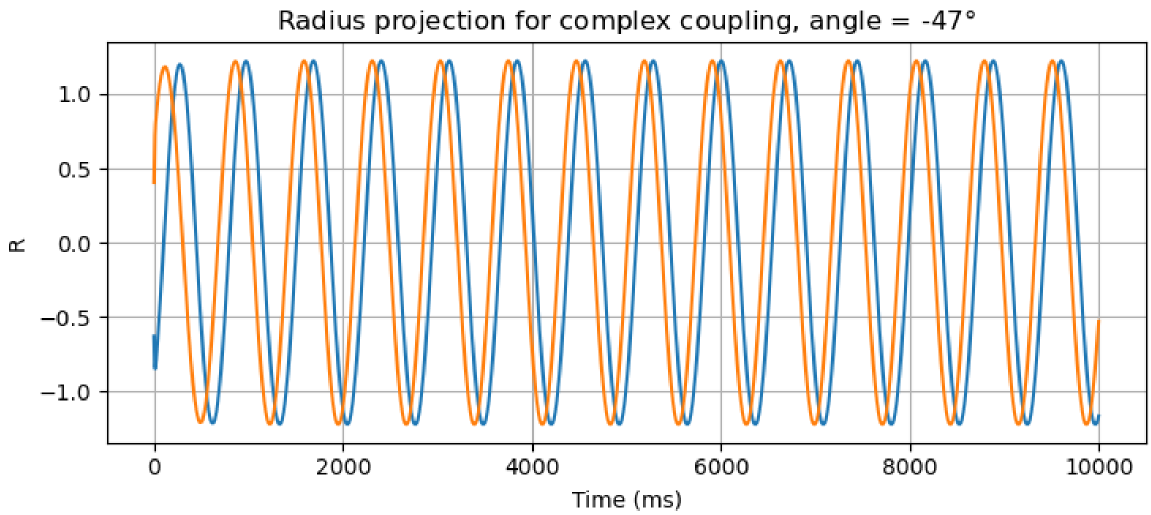

Fig 1b : Radius projections of both oscillators with θ = – 47°

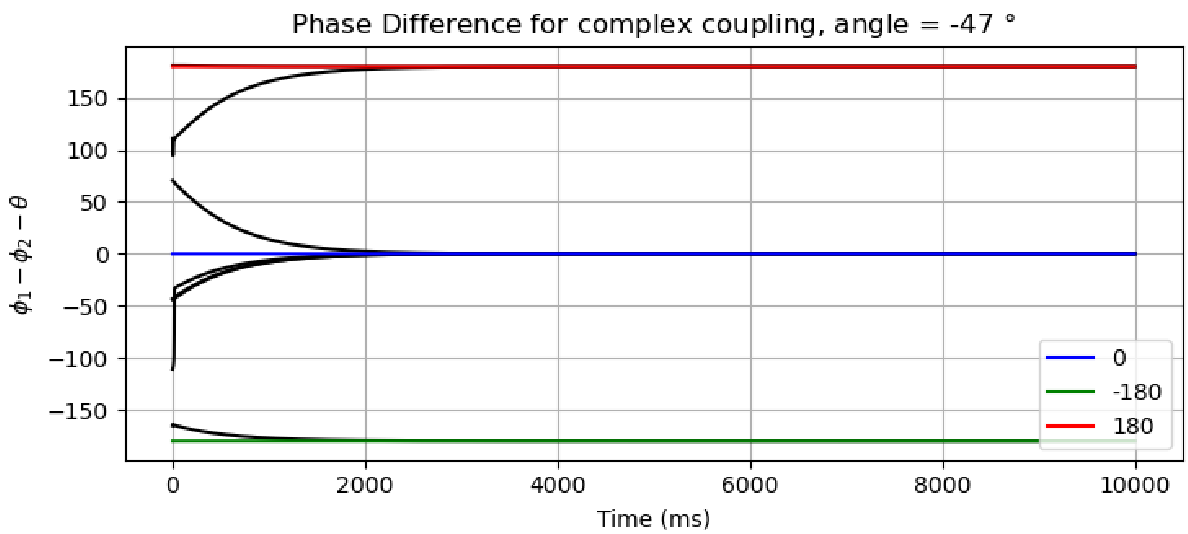

Fig 1c : Phase difference for multiple ϕ angles with θ = – 47°

For theta = 98 :

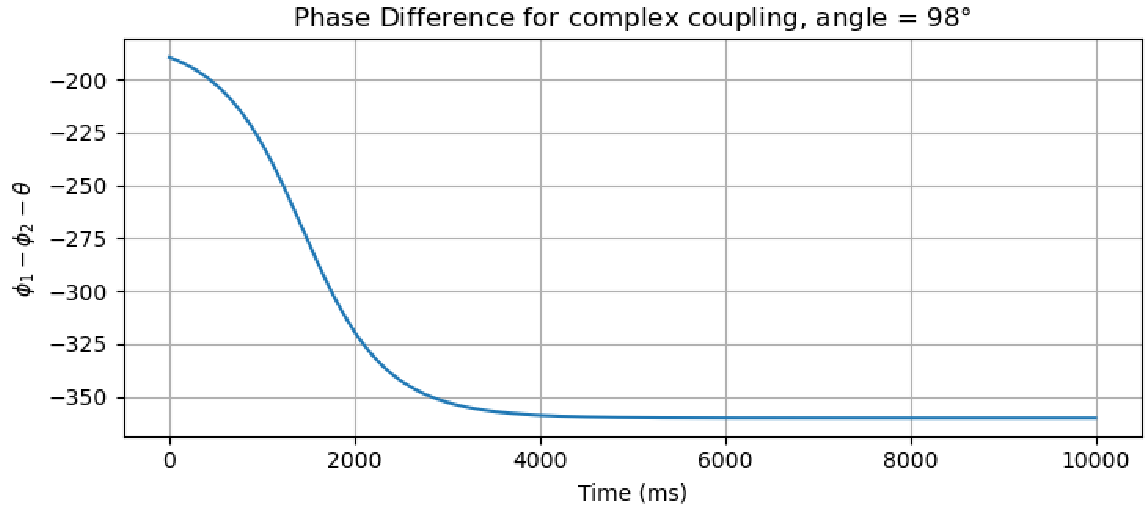

Fig 2a : Phase difference with θ = 98°

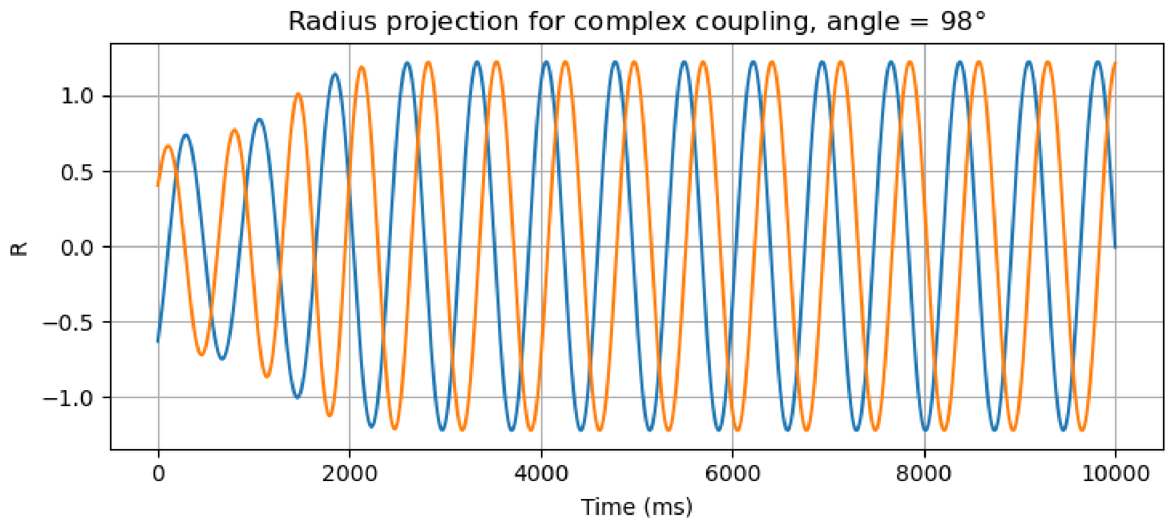

Fig 2b : Radius projections of both oscillators with θ = 98°

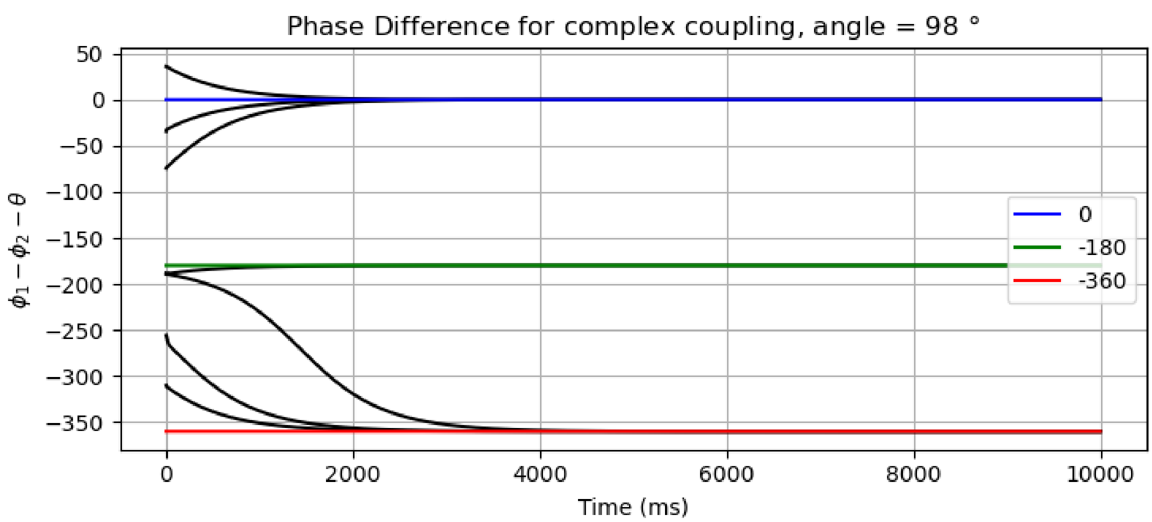

Fig 2c : Phase difference for multiple ϕ angles with θ = 98°

Effect of change in A on Phase Difference:

The magnitude of the coupling constant varies how fast or slow the phase difference approaches any of the $2n\pi + \theta$ solution.

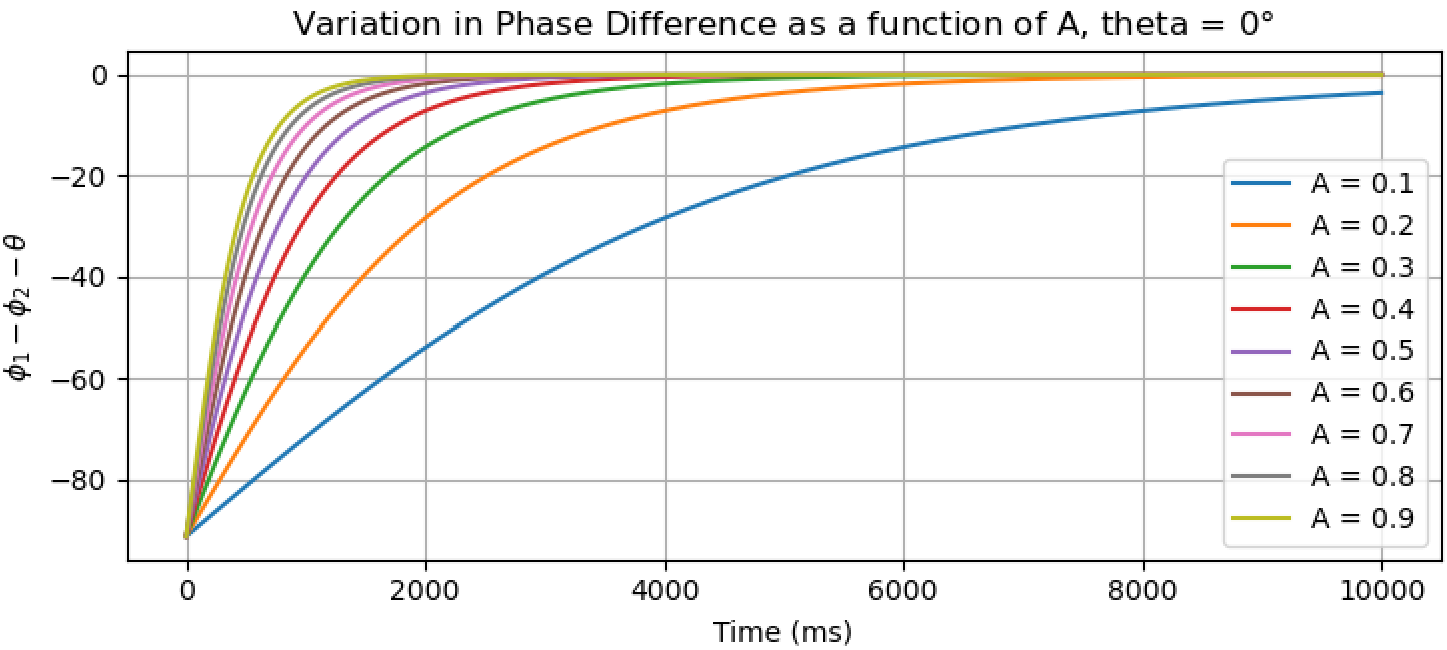

Fig 3 : Variation of A changing the phase difference curve

We see that the larger the value of A the faster the Phase Difference approaches the solution.

For, theta = – 47, the coupling constant $W = 0.5 \times e^{-i\,0.8203}$ and $W^* = 0.5 \times e^{i\,0.8203}$

For, theta = 98, the coupling constant $W = 0.5 \times e^{1.71042\,i}$ and $W^* = 0.5 \times e^{-1.71042\,i}$

Power Coupled :

We know that when the natural frequencies of the oscillators are equal they show complex coupling factor. However when the frequencies are not the same, the phase difference cannot be considered as $\phi_1 - \phi_2$.

When the coupled oscillators are in a ratio $m : n$, the phase difference becomes $m\phi_1 - n\phi_2$.

We can generalize this equation to account for other phases. The normalized phase difference is defined as follows

$$\Psi_{12} = \Psi_{21} = \frac{\phi_1}{\omega_1} - \frac{\phi_2}{\omega_2}$$

Coupling a pair of Hopf oscillators through power coupling gives us the following equations in polar coordinates.

In polar coordinates :

$$\frac{dr_1}{dt} = \mu(1 - r_1^2)r_1 + A_{12} \cdot r_2^{\frac{\omega_1}{\omega_2}} \cdot cos\omega_1(\frac{\phi_2}{\omega_2} - \frac{\phi_1}{\omega_1} + \frac{\theta_{12}}{\omega_1\omega_2})$$

$$\frac{dr_2}{dt} = \mu(1 - r_2^2)r_2 + A_{21} \cdot r_1^{\frac{\omega_2}{\omega_1}} \cdot cos\omega_2(\frac{\phi_1}{\omega_1} - \frac{\phi_2}{\omega_2} + \frac{\theta_{21}}{\omega_1\omega_2})$$

$$\frac{d\phi_1}{dt} = \omega_1 + A_{12} \cdot \frac{r_2^{\frac{\omega_1}{\omega_2}}}{r_1} sin\omega_1(\frac{\phi_2}{\omega_2} - \frac{\phi_1}{\omega_1} + \frac{\theta_{12}}{\omega_1\omega_2})$$

$$\frac{d\phi_2}{dt} = \omega_2 + A_{21} \cdot \frac{r_1^{\frac{\omega_2}{\omega_1}}}{r_2} sin\omega_2(\frac{\phi_1}{\omega_1} - \frac{\phi_2}{\omega_2} + \frac{\theta_{21}}{\omega_1\omega_2})$$

Here the $W_{12} = A_{12} \cdot e^{i\frac{\theta_{12}}{\omega_2}}$ is the weight of the power coupling from 2nd to the 1st oscillator, and $W_{21} = A_{21} \cdot e^{i\frac{\theta_{21}}{\omega_1}}$ is the weight of the power coupling from the 1st to the second oscillator. We now observe the variation in phase difference and the radial projection of the oscillators with each timestep. We model these equations numerically using Euler's Integration with Δt = 0.1 msec on python. The values of r_1, r_2, ϕ_1, ϕ_2 are randomized to observe the plot between $(\frac{\phi_1}{\omega_1} - \frac{\phi_2}{\omega_2} + \frac{\theta_{21}}{\omega_1\omega_2})$ Vs time. The set values are $\omega_1 = 5$ & $\omega_2 = 15$ and $\theta = \{-47°, 98°\}$. We should observe a steady state as the time step increases. (A_{12} & $A_{21} = 0.4$)

For theta = – 47 :

For theta = –47 , $W_{12} = 0.4 \times e^{-0.0546\,i}$ & $W_{21} = 0.4 \times e^{0.164\,i}$

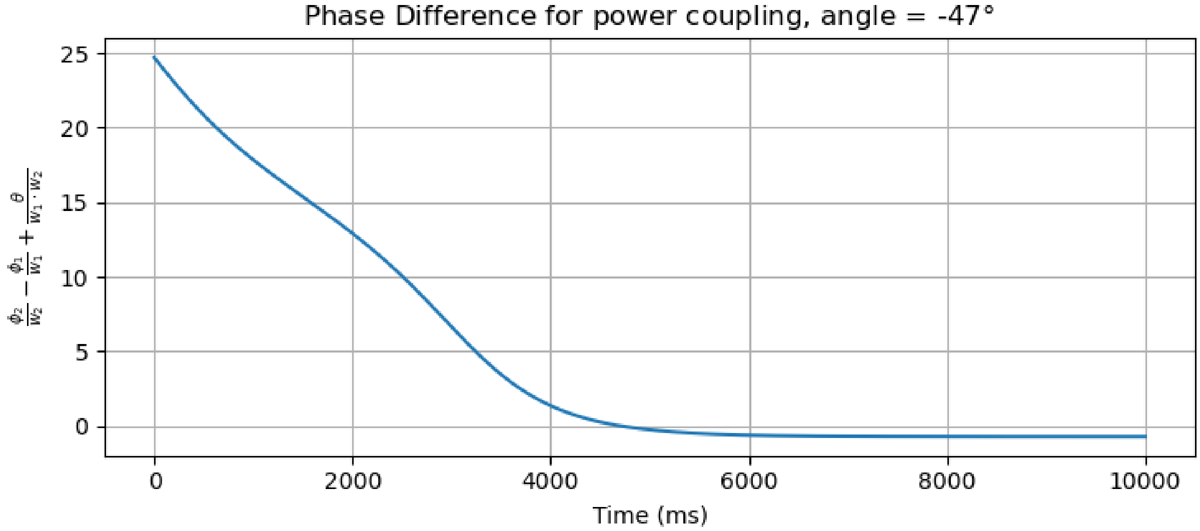

Fig 4a: Phase difference with θ = – 47°

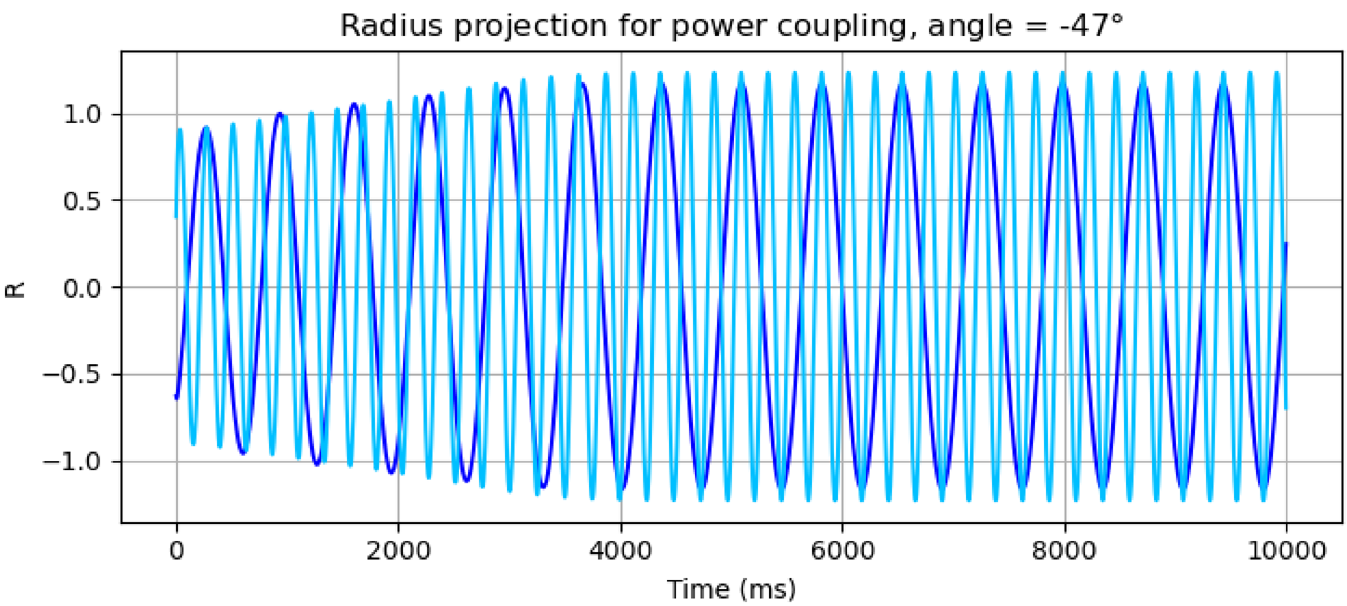

Fig 4b : Radius projections of both oscillators with θ = –47°

For theta = 98 [2]:

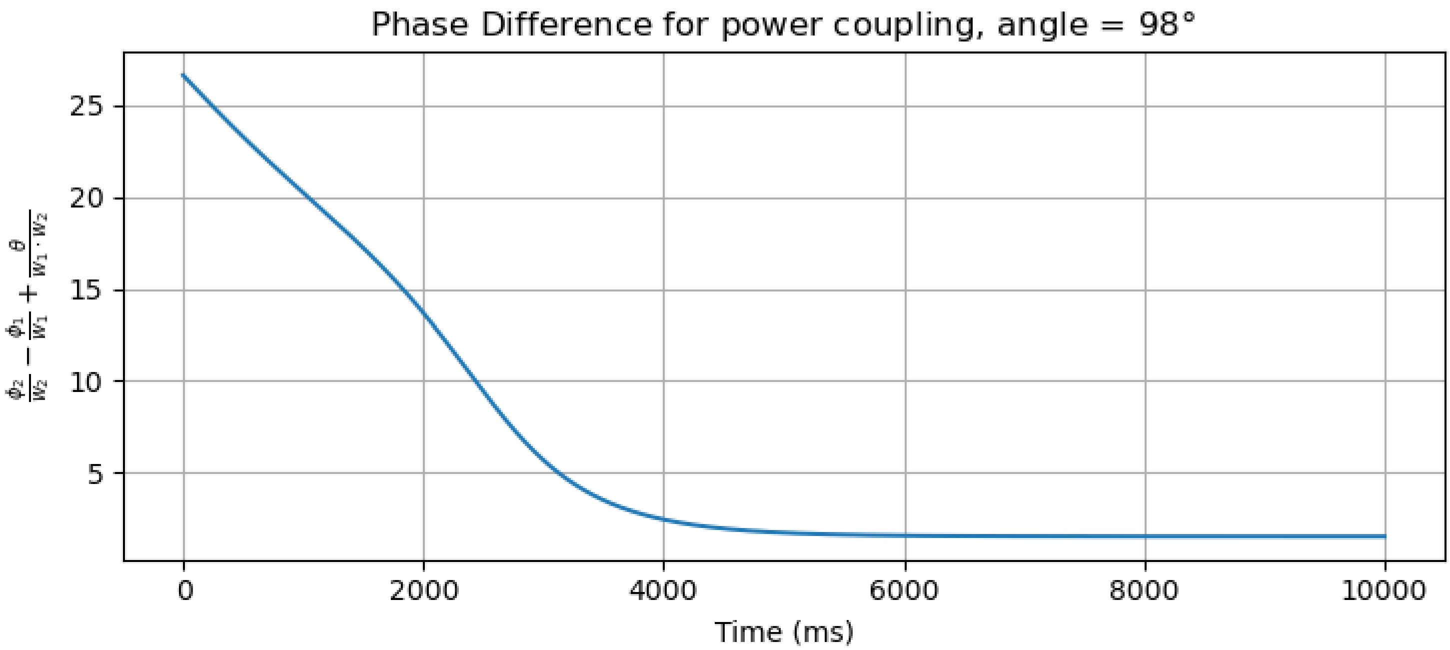

Fig 5a: Phase difference with θ = 98 °

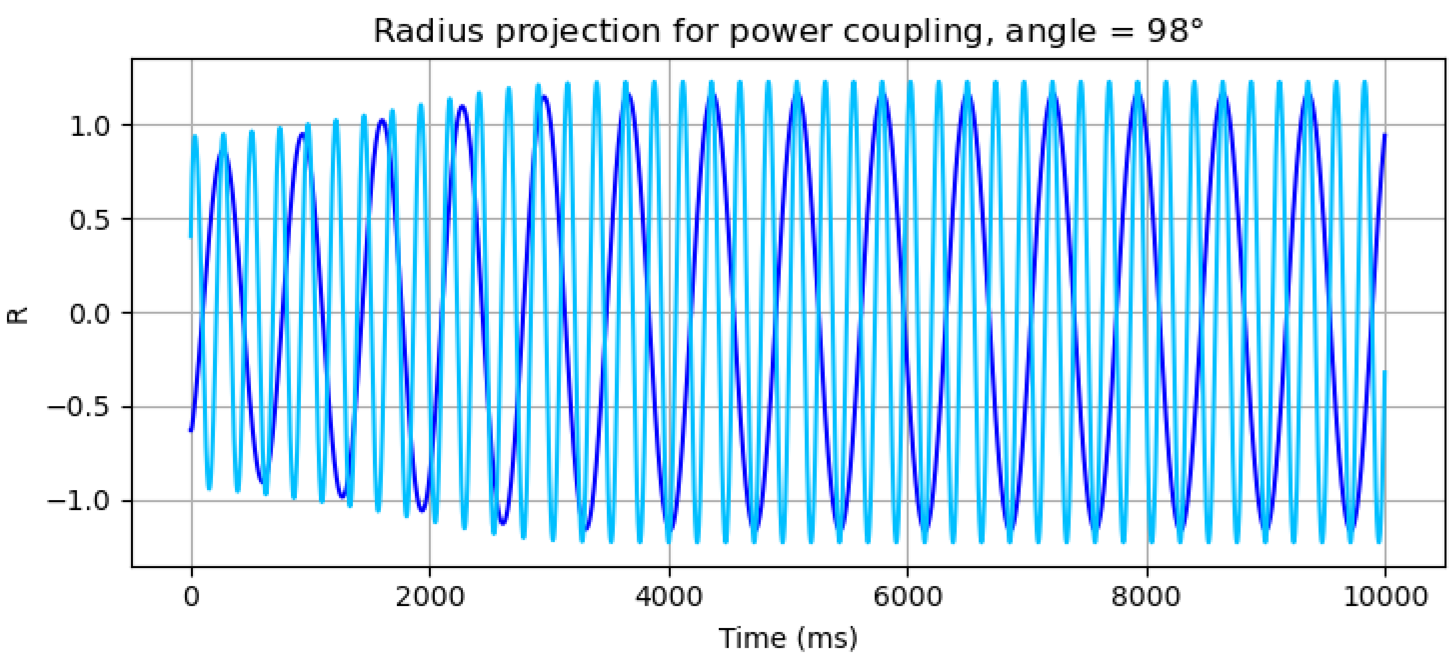

Fig 5b : Radius projections of both oscillators with θ = 98 °

For theta = 98 , $W_{12} = 0.4 \times e^{0.1140\, i}$ & $W_{21} = 0.4 \times e^{-0.3420\, i}$

[2] We initialize A12 and A21 both to be 0.4

For theta = 0 :

We now observe how different initial points will behave in such an oscillators, we assume that $\theta = 0$

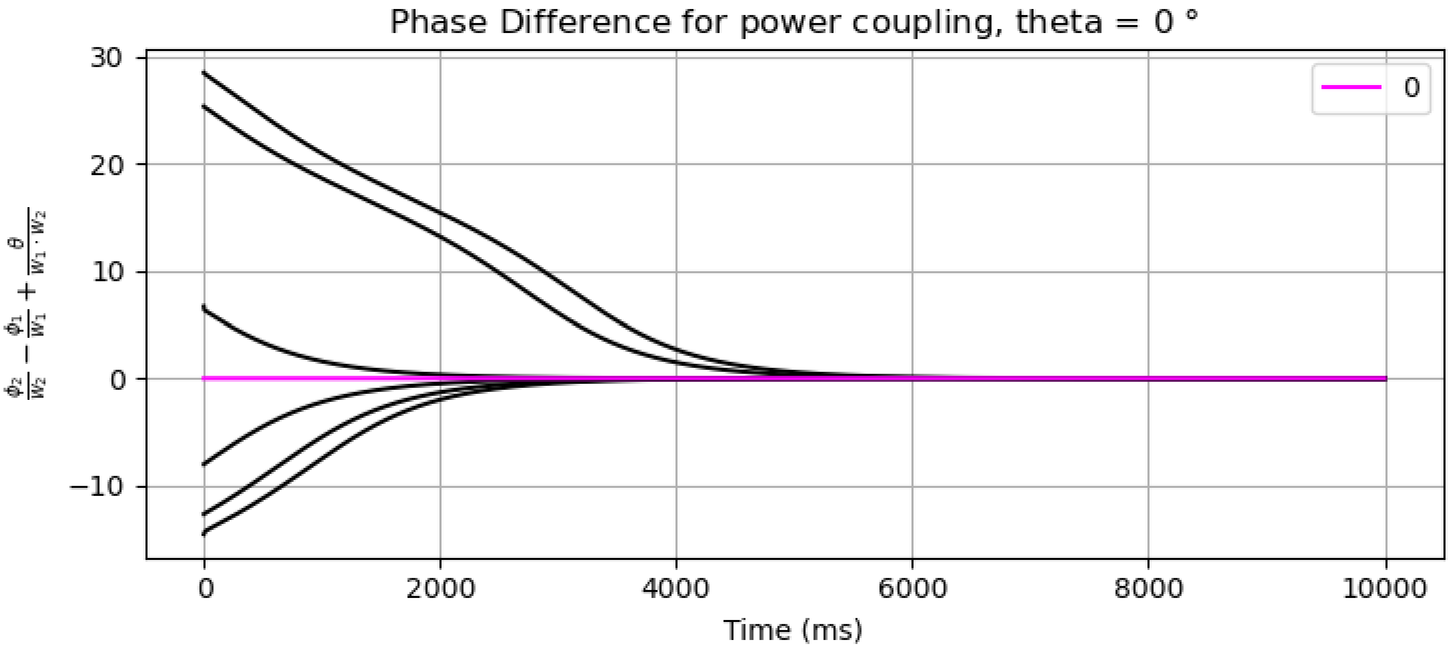

Fig 6 : All initial points approach zero as time increases

References :

1. Biswas, D., Pallikkulath, S., & Chakravarthy, V. S. (2021). A Complex-Valued Oscillatory Neural Network for Storage and Retrieval of Multidimensional Aperiodic Signals. *Frontiers in Computational Neuroscience*, *15*. https://doi.org/10.3389/fncom.2021.551111

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