Differential Equation Modeling and Simulation of a First-Order RC Discharge Circuit

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1 Introduction

When analyzing electrical circuits, transient behavior plays a crucial role, especially when energy storage elements like capacitors are involved. In this project, we model the transient discharge behavior of a simple RC (resistor-capacitor) circuit, where a capacitor initially charged to a known voltage discharges through a resistor. We derive the governing differential equation, solve it analytically, simulate the behavior using Python and LTspice, and discuss the expected results.

2 Problem Description and Original Circuit

The original circuit setup consists of a DC voltage source $V_1 = 50$V, a switch S_1, a $10k\Omega$ resistor, and a $400\mu F$ capacitor. The capacitor is initially charged through the voltage source.

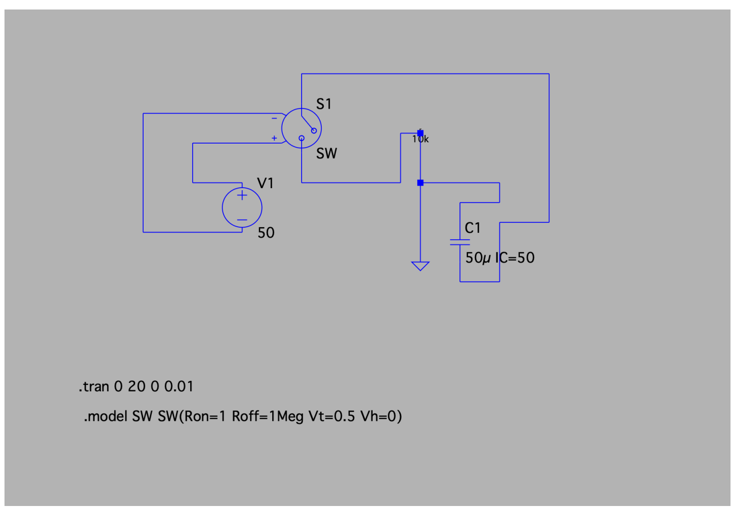

Figure 1: Original Circuit: Capacitor charging setup with switch

2.1 Circuit Behavior

- **For** $t < 0$: The switch S_1 is open. The capacitor is connected to the DC source and charged to an initial voltage $V_0 = 50$V.

- **At** $t = 0$: The switch closes. The DC source is effectively disconnected, and the capacitor starts discharging through the resistor R.

- **For** $t > 0$: The governing differential equation $\frac{dV_C(t)}{dt} + \frac{1}{RC}V_C(t) = 0$ describes the voltage decay across the capacitor.

3 Problem Setup

The circuit consists of a capacitor $C = 400\mu F$ initially charged to a voltage $V_0 = 50V$ and a resistor $R = 10k\Omega$ connected in series. No external voltage source or switch is present after $t = 0$.

- **For** $t < 0$: The capacitor is pre-charged to $V_0 = 50V$ and isolated (no current flows).

- **At** $t = 0$: The capacitor starts discharging through the resistor.

- **For** $t > 0$: The capacitor voltage $V_C(t)$ decreases exponentially according to the derived differential equation.

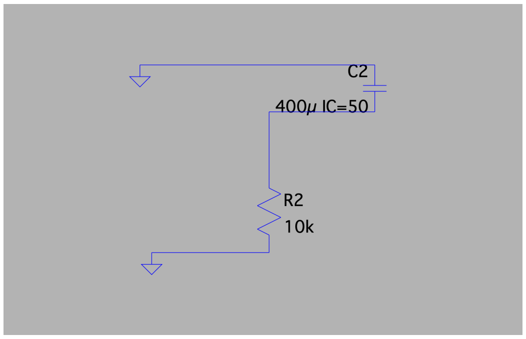

Figure 2: RC Discharge Circuit

4 Theoretical Analysis

4.1 Establishing the Differential Equation

Applying Kirchhoff's Voltage Law (KVL) around the loop gives:

$$V_C(t) + i(t)R = 0 \tag{1}$$

where:

- $V_C(t)$ is the voltage across the capacitor,
- $i(t)$ is the current through the resistor.

The current through the capacitor is related to the voltage by:

$$i(t) = C\frac{dV_C(t)}{dt} \tag{2}$$

Substituting (2) into (1):

$$V_C(t) + RC\frac{dV_C(t)}{dt} = 0 \tag{3}$$

Rearranging:

$$\frac{dV_C(t)}{dt} + \frac{1}{RC}V_C(t) = 0 \tag{4}$$

This is a first-order linear homogeneous differential equation.

4.2 Solving the Differential Equation

Separating variables:

$$\frac{dV_C(t)}{V_C(t)} = -\frac{dt}{RC} \tag{5}$$

Integrating both sides:

$$\ln|V_C(t)| = -\frac{t}{RC} + k \tag{6}$$

where k is the constant of integration.

Exponentiating:

$$V_C(t) = Ae^{-t/RC} \tag{7}$$

where $A = e^k$.

Applying the initial condition $V_C(0) = V_0$:

$$A = V_0 \tag{8}$$

Thus, the solution is:

$$\boxed{V_C(t) = V_0e^{-t/RC}} \tag{9}$$

5 Python Simulation

Simulation using Python to visualize the discharge behavior:

```
import numpy as np
import matplotlib.pyplot as plt

# Parameters
V0 = 50    # Initial voltage in Volts
R = 10000  # Resistance in Ohms
C = 400e-6 # Capacitance in Farads
tau = R * C # Time constant

# Time points
t = np.linspace(0, 5*tau, 500)

# Voltage across capacitor
Vc = V0 * np.exp(-t/tau)

# Plot
plt.figure(figsize=(8,5))
plt.plot(t, Vc, label=f'$V_0$={V0}V, R={R}$\Omega$, C={C}F')
plt.title('Capacitor Discharge in RC Circuit')
plt.xlabel('Time (s)')
plt.ylabel('Voltage across Capacitor (V)')
plt.grid(True)
plt.legend()
plt.show()
```

6 LTspice Simulation

The circuit is also simulated using LTspice:

- Only a capacitor $400\mu F$ with initial condition $IC = 50V$ and a $10k\Omega$ resistor were used.
- No external voltage source or switch was included.
- A transient analysis (.tran) was performed.

- The capacitor voltage was plotted over time.

7 Simulation and Results Comparison

Screenshots from Python and LTspice simulations are included below for comparison:

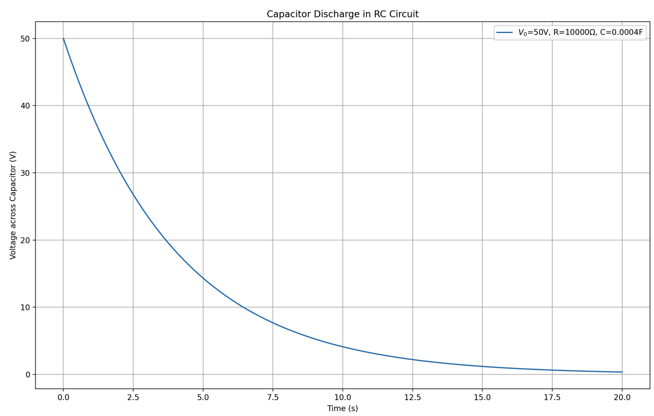

Figure 3: Capacitor Discharge using Python

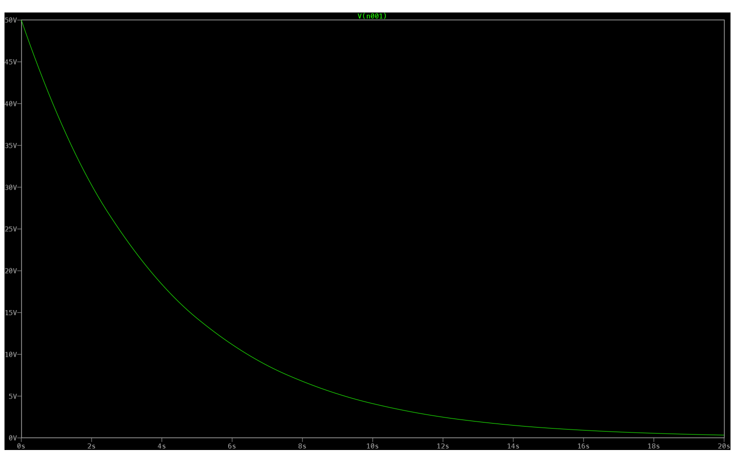

Figure 4: Capacitor Discharge using LTspice

Both simulations demonstrate the expected exponential decay of the capacitor voltage starting from 50V at $t = 0$. Minor differences may arise from numerical integration methods, but overall the two results agree very closely, validating the theoretical solution.

8 Conclusion

This project demonstrated that the transient response of a first-order RC circuit, starting from a pre-charged state without external forcing functions, can be effectively modeled and simulated. The expected exponential decay was confirmed both analytically and through simulations, highlighting the predictive power of differential equations in circuit analysis.