Contents

***Abstract*—This manual is an introduction to control systems based on GATE problems.Links to sample Python codes are available in the text.**

1 Stability

2 Routh Hurwitz Criterion

3 Compensators

4 Nyquist Plot

5 State space model

5.0.1. Consider the circuit given below, here R_1= 1KΩ , R_2= 2KΩ , C_1= C_2= $1mF$.Both the capacitors were charged to 1 C. Find Differential equations for q_1 and q_2.

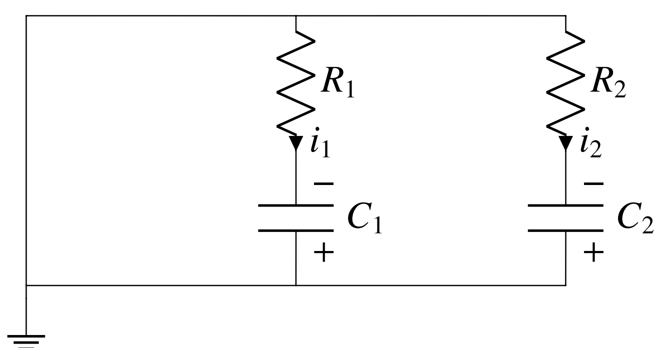

Fig. 5.0.1: Circuit diagram

Solution: We know that, i_1 = -$\dot{q}_1$, i_2 = -$\dot{q}_2$ ('-' as charge is depleting)
Applying KVL on 1st branch we get:

$$-\dot{q}_1 R_1 - \frac{q_1}{c_1} = 0 \tag{5.0.1.1}$$

$$\dot{q}_1 = -q_1 \tag{5.0.1.2}$$

Now, Applying KVL 0n 2nd branch we get:

$$-\dot{q}_2 R_2 - \frac{q_2}{c_2} = 0 \tag{5.0.1.3}$$

$$\dot{q}_2 = -2q_2 \tag{5.0.1.4}$$

If I represent this as state space model
Taking $x = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$
And here $x(0) = \begin{pmatrix} q_1(0) \\ q_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
∴ State space model can be represented as

$$\dot{x} = \begin{pmatrix} -q_1 \\ -2q_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} x \tag{5.0.1.5}$$

5.0.2. Consider the linear system :

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} x \tag{5.0.2.1}$$

with initial condition: x(0) = $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Find x(t)

(A) x(t) = $\begin{pmatrix} e^{-t} & te^{-2t} \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(B) x(t) = $\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(C) x(t) = $\begin{pmatrix} e^{-t} & t^2 e^{-2t} \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(D) x(t) = $\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Solution: Given expression is the state equation, as it can be written in the following form

$$\dot{x} = Ax + BU \tag{5.0.2.2}$$

Here, A = $\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$ and U = 0
Solution to this can be given by

$$x(t) = \phi(t)x(0) \tag{5.0.2.3}$$

Where $\phi(t)$ is called the state-transition matrix and is given by

$$\phi(t) = \mathscr{L}^{-1}\left(\left(sI - A\right)^{-1}\right) \tag{5.0.2.4}$$

$$\left(sI - A\right) = \begin{pmatrix} s+1 & 0 \\ 0 & s+2 \end{pmatrix} \tag{5.0.2.5}$$

$$\left(sI - A\right)^{-1} = \begin{pmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{pmatrix} \tag{5.0.2.6}$$

$$\mathscr{L}^{-1}\left(\left(sI-A\right)^{-1}\right)=\begin{pmatrix}e^{-t} & 0\\ 0 & e^{-2t}\end{pmatrix} \tag{5.0.2.7}$$

Given x(0) = $\begin{pmatrix}1\\1\end{pmatrix}$

$$\therefore\ x(t)=\begin{pmatrix}e^{-t} & 0\\ 0 & e^{-2t}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} \tag{5.0.2.8}$$

$$\therefore\ option(D) \tag{5.0.2.9}$$

**Derivation for state transition matrix:*

$$\dot{x}=Ax \tag{5.0.2.10}$$

Applying laplace transform, we get

$$sX(s)-x(0)=AX(s) \tag{5.0.2.11}$$

$$\left(sI-A\right)X(s)=x(0) \tag{5.0.2.12}$$

$$X(s)=\left(sI-A\right)^{-1}x(0) \tag{5.0.2.13}$$

$$\tag{5.0.2.14}$$

Taking inverse laplace transform

$$x(t)=\mathscr{L}^{-1}\left(\left(sI-A\right)^{-1}\right)x(0) \tag{5.0.2.15}$$

→ Where $\mathscr{L}^{-1}\left(\left(sI-A\right)^{-1}\right)$ is called State transition matrix.

5.0.3. Draw block diagram for the the above mentioned state model.
Solution: Here, clearly Transfer function is given by

$$X(s)=\left(sI-A\right)^{-1}x(0) \tag{5.0.3.1}$$

Input being constant = x(0), and output is X(s)

$$X_1(s)=\frac{1}{s+1}x_1(0) \tag{5.0.3.2}$$

$$X_2(s)=\frac{1}{s+2}x_2(0) \tag{5.0.3.3}$$

Lets consider a closed loop negative unity feed back system, We know its open loop gain

$$=\frac{G(s)}{1+G(s)} \tag{5.0.3.4}$$

We can re-write the above to transfer equations as

$$X_1(s)=\frac{\frac{1}{s}}{\frac{1}{s}+1}x_1(0) \tag{5.0.3.5}$$

$$X_2(s)=\frac{\frac{1}{s+1}}{\frac{1}{s+1}+1}x_2(0) \tag{5.0.3.6}$$

Below are the block the block diagrams for X_1(s) and X_2(s)

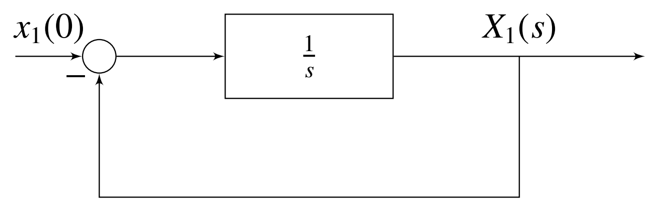

Fig. 5.0.3: block diagram for X_1

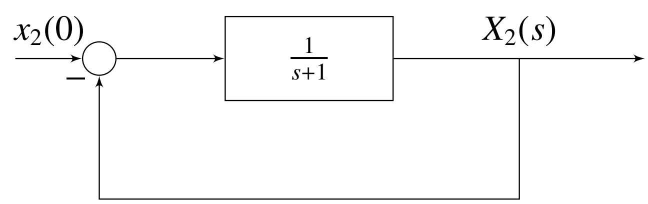

Fig. 5.0.3: block diagram for X_2

5.0.4. Describe how an oscillator works
Solution: Oscillators convert a DC input (the supply voltage) into an AC output (the waveform)
Given Below is basic block diagram

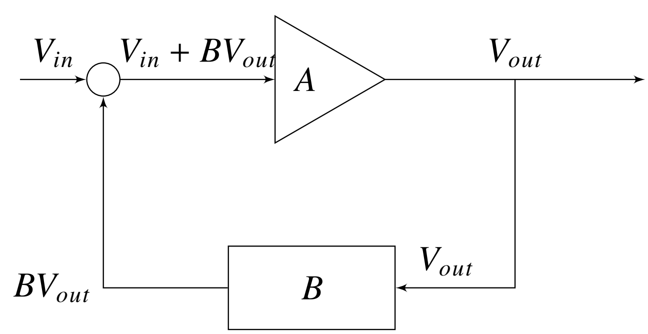

Fig. 5.0.4: block diagram for oscillator

Resonant frequency, is the frequency at which oscillator oscillates, it depends on R/L/C components of the circuit it's been fed back through.

Oscillators work because they overcome the losses of their positive feedback circuit either in the form of a capacitor, inductor or both. In other words, an oscillator is a an amplifier which uses positive feedback that generates an output frequency without the use of an input signal.
Oscillators gain can be given as follows:
From the above circuit we have :
A = Amplifiers gain, B = positive feedback gain, G = total gain of the circuit

$$A(V_{in} + BV_{out}) = V_{out} \tag{5.0.4.1}$$
$$G = \frac{V_{out}}{V_{in}} = \frac{A}{1 - AB} \tag{5.0.4.2}$$

We get,
sustained oscillations when $AB = 1$
Exponentially decaying oscillation for $AB < 1$
Exponentially increasing oscillation for $AB > 1$ (unstable)

5.0.5. Hartley oscillator:
The Hartley oscillator is one of the classical LC feedback circuits,i.e feedback is made of LC components.Below here we can see a general form of any LC-type oscillator:

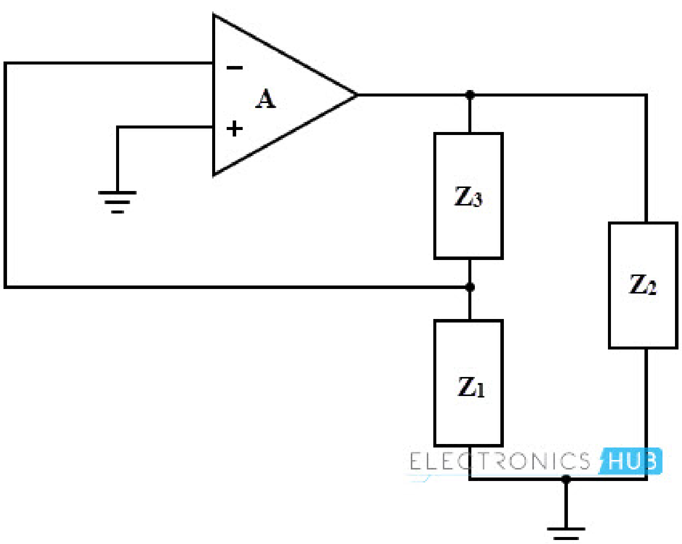

Fig. 5.0.5: block diagram for oscillator

For any LC oscillator,

$$Z_1 = jX_1 \tag{5.0.5.1}$$
$$Z_2 = jX_2 \tag{5.0.5.2}$$
$$Z_3 = jX_3 \tag{5.0.5.3}$$
$$\tag{5.0.5.4}$$

We know that feedback gain is B, i.e, $\frac{V_0}{V_f} = B$
Applyting voltage divider rule we get

$$B = \frac{Z_1}{Z_1 + Z_3} \tag{5.0.5.5}$$

Consider the below circuit

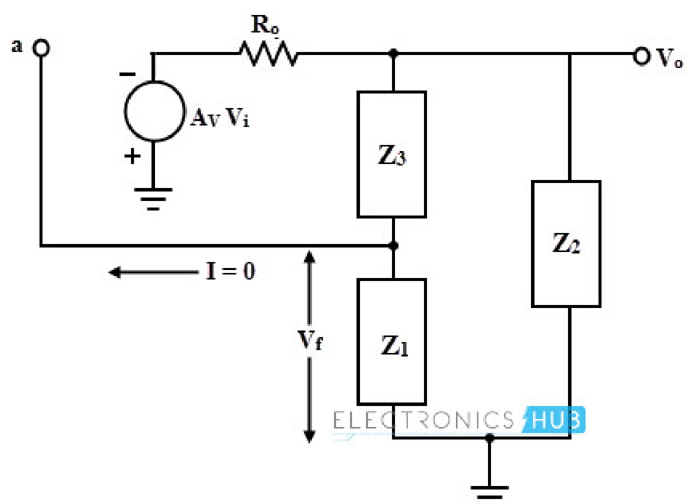

Fig. 5.0.5: block diagram for oscillator

$$A = \frac{V_o}{V_{in}} = \frac{AZ_L}{R_o + Z_L} \tag{5.0.5.6}$$
$$where \tag{5.0.5.7}$$
$$Z_L = \frac{(Z_2 + Z_3)Z_1}{Z_1 + Z_2 + Z_3} \tag{5.0.5.8}$$
$$\tag{5.0.5.9}$$

Now,we know that $AB = 1$ for sustained oscillations, putting the the above terms in the equation
on solving,

$$AB = \frac{Z_1 Z_2 A}{(Z_1 + Z_2 + Z_3)A + Z_1(Z_2 + Z_3)} \tag{5.0.5.10}$$
$$Z_1 = jX_1, Z_2 = jX_2, Z_3 = jX_3 \tag{5.0.5.11}$$
$$\tag{5.0.5.12}$$

putting that in we get

$$AB = \frac{AX_1X_2}{X_1(X_2 + X_3) - jR_o(X_1 + X_2 + X_3)} \tag{5.0.5.13}$$

For hartley oscillator,

$$Z_1 = j\omega L_1(inductor) \tag{5.0.5.14}$$
$$Z_2 = j\omega L_2(inductor) \tag{5.0.5.15}$$
$$Z_3 = \frac{1}{j\omega C}(capacitor) \tag{5.0.5.16}$$
$$\tag{5.0.5.17}$$

Since, AB is real

$$X_1 + X_2 + X_3 = 0 \tag{5.0.5.18}$$
$$\tag{5.0.5.19}$$

For, Hartley oscillator, substituting terms in above equation

$$\omega L_1 + \omega L_2 = \frac{1}{\omega C} \tag{5.0.5.20}$$
$$\omega = \frac{1}{\sqrt{(L_1 + L_2)(C)}} \tag{5.0.5.21}$$
$$f = \frac{1}{2\pi\sqrt{(L_1 + L_2)(C)}} \tag{5.0.5.22}$$

$$B = \frac{Z_1}{Z_1 + Z_3} = \frac{Z_1}{Z_2} \tag{5.0.5.23}$$
$$= \frac{L_1}{L_2} \tag{5.0.5.24}$$
$$A = \frac{L_2}{L_1} \tag{5.0.5.25}$$

Given below is circuit for Hartley oscillator built using opamp

5.0.6. For Hartley oscillator frequency generated can be given as

$$f = \frac{1}{2\pi\sqrt{(L_1 + L_2)C}} \tag{5.0.6.1}$$

Taking,

$$L_1 = 1\mu H \tag{5.0.6.2}$$
$$L_2 = 1\mu H \tag{5.0.6.3}$$
$$C = 1.2pF \tag{5.0.6.4}$$

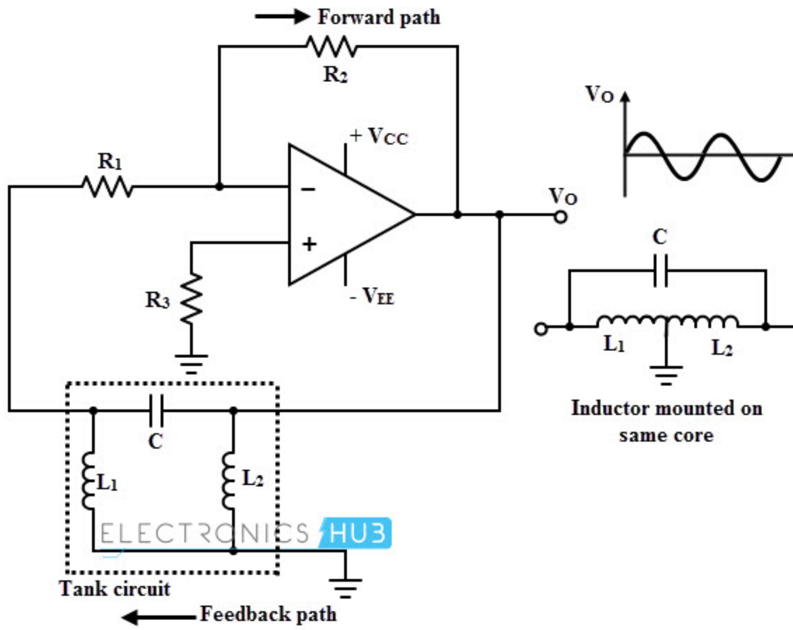

Fig. 5.0.5: block diagram for oscillator

We get f = 103 MHz
Feedback factor for Hartley given by:

$$Feedbackfactor = \frac{L_1}{L_2} = 1 \tag{5.0.6.5}$$

W.K.T, $AB = 1$
$\therefore$ Minimum amplification Gain = 1