Control Systems

IN GATE QUESTION , 2018- QUESTION 37

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PROBLEM STATEMENT

Q) Consider the linear system $\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \mathrm{x}$, with initial condition $\mathrm{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The solution x(t) for this system is:

(A) $\mathrm{x(t)} = \begin{bmatrix} e^{-t} & te^{-2t} \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(B) $\mathrm{x(t)} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(C) $\mathrm{x(t)} = \begin{bmatrix} e^{-t} & t^2e^{-2t} \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(D) $\mathrm{x(t)} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Solution

It is of the form $\dot{x} = Ax$. Therefore its solution is

$$\text{x(t)} = \text{e}^{At} x(0)$$

e^{At} is my state transition matrix and is equal to

$$\mathcal{L}^{-1}[\text{sI - A}]^{-1} \qquad \text{(derivation in last slide)}$$

$$\text{A} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \qquad \therefore [\text{sI - A}] = \begin{bmatrix} s+1 & 0 \\ 0 & s+2 \end{bmatrix}$$

$$\rightarrow Adj(sI - A) = \begin{bmatrix} s+2 & 0 \\ 0 & s+1 \end{bmatrix}$$

$$\rightarrow det(sI - A) = (s+1)(s+2)$$

Solution

$$\therefore \text{[sI - A]}^{-1} = \frac{Adj(A)}{det(A)} = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

$$\text{e}^{At} = \begin{bmatrix} \mathcal{L}^{-1}[\frac{1}{s+1}] & 0 \\ 0 & \mathcal{L}^{-1}[\frac{1}{s+2}] \end{bmatrix}$$

$$\text{e}^{At} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

$$\therefore x(t) = e^{At}x(0) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\rightarrow OPTION(D)$$

Derivation of state transition matrix

$$\rightarrow \dot{x} = Ax$$

Taking Laplacian

$$\rightarrow S.X(s) - x(0) = AX(s)$$

$$\therefore X(s) = [SI - A]^{-1}x(0)$$

X(t) $= \mathcal{L}^{-1}[SI - A]^{-1}x(0)$

$\rightarrow$ *Here*, $\mathcal{L}^{-1}$[SI -A]$^{-1}$ is called the state transition matrix