IGMO 2020 Round 2 Day 1 + Day 2

16th February 2021

Question	**Points**
1	7
2	7
3	7
4	7
5	7
6	7
Total	42

Instructions:

1. This examination contains 3 pages, including this page.
2. You have **five(5) hours** to submit your solutions starting from when you accessed the paper.
3. Submit your answers in the form that came with the E-Mail. If your solutions are written, you are asked to scan your answers using CamScanner or TapScanner. If your solutions are typed using LaTeX(not recommended), then you are asked to send the solutions to **each** question as a separate PDF (or a screenshot of each answer) in the corresponding submission section
4. You are not allowed to disclose any questions on any online forums until 13:00 GMT 17th February. Do not participate or attempt the paper along with someone else, each contestant should be individual.

ROUND 2 QUESTIONS

Problem 1 :

A sphere of radius r can be inscribed in a tetrahedron. The distances between the centroid of the tetrahedron and its four faces are w, x, y and z. Prove that $wxyz \geq r^4$.

Problem 2 :

Given that $f(x) = x + 1$ and $g(x) = 2x$, how many different ways are there of combining $f(x)$ and $g(x)$ (this means doing any number of compositions like $fg(x)$ or $g^3 f^2 g(x)$ etc) such that the resulting composition is $8x + 8m$ where $m \geq 0$ is an integer?

Find a general formula for the number of possibilities in terms of m.

Problem 3 :

Let's define a function $\phi : \mathbb{N} \to \mathbb{N}$, where $0 \notin \mathbb{N}$, as follows

$$\phi(n) = \sum_{k=1}^{n} k!$$

Let $\mathbb{V}$ be defined as the set of all triplets $(x, y, z) \in \mathbb{N}$ such that $\phi(x) = y^{z+1}$. For a triplet x, y, z (denoted by v) in $\mathbb{V}$, we define

$$f_v(n) = 8\left[\frac{xy}{8}\right]\lfloor\sqrt{n}\rfloor + \frac{zn}{z + x - y}.$$

($[x]$ is **fractional part** of x and $\lfloor x \rfloor$ is greatest integer less than x)

Show that for any $v \in \mathbb{V}$ and $m \in \mathbb{N}$ the sequence

$$m,\ f_v(m),\ f_v(f_v(m)),\ f_v(f_v(f_v(m))), ...$$

contains at least one square of a natural number. Please note that $[x]$ here refers to the **fractional part** of x, it can also be denoted as $\{x\}$ but it is denoted as $[x]$ here.

TURN OVER FOR P4-6

Problem 4 :

People from 80 countries participated in the 1^{st} round of IGMO. In order to ensure that participants from any of the 80 countries can travel to any one of the remaining 79 countries by at most 2 flights, the countries have an agreement such that among the 80 countries, each country's airport connects to at least n other countries' airport.

- Find the minimum value of n, proving that it is the minimum
- Prove that for this minimum n, if there isn't a direct flight between 2 countries, then there are at least 2 paths that require 2 flights between the countries

Note: For two airports A and B to be connected, it must be possible to have a flight from A to B and a flight from B to A. This adds 1 to both the number of connections from the airports. Also note that for this problem, each country has only 1 airport.

Problem 5 :

A non-rectangular trapezoid is called a "Pepe trapezoid" if

(a) It has integral side lengths AND

(b) An ellipse (that is not a circle) with integral lengths of semi-major axis and semi-minor axis can be inscribed in the trapezoid such that the major axis or minor axis of the ellipse is perpendicular to the bases of the trapezoid.

Prove or disprove that there exists infinitely many non-similar Pepe trapezoids.

Problem 6 :

Find all $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfying:

$$f(x) + f\left(\frac{1}{x}\right) = 1, \text{ for all } x \in \mathbb{Q}^+$$

$$f(1+2x) = \frac{1}{2}f(x), \text{ for all } x \in \mathbb{Q}^+$$