Shortlisted Problems

(with solutions)

IGMO (Instagram Math Olympiad) 2020

18th December, 2020 – 17th February, 2021

Note from the Leader:

Being the first rendition of what we hope to continue for the coming few years, the team acknowledges that the difficulty of the questions, the wording, the procedure and the marking rubric may not have been perfect. It's important to consider that IGMO (inspired by how the idea came through mathematics accounts on Instagram working together) was carried out by members who haven't had any professional experience in the making of Mathematical Olympiads. While most of our members are familiar with Olympiad problems, we decided to make IGMO a bit **spicier** than the conventional Olympiads there are. This means we decided to include problems that were a bit unconventional for Olympiads. This did cause some participants to be annoyed and some questions (such as Round 1's P2 and Round 2's P2 and P5) were disliked by a few, but also enjoyed by an almost equal number of people.

We acknowledge that there were some issues with wording (especially with Round 1's P2) and that some domains were unspecified in problems (such as Round 1's P3). These issues, along with any other issues the team was made aware of, were noted and will not occur again. If there are some other issues with the problems/solutions in the shortlist, or just any suggestion/comment anyone wants to let the team know, please email igmoteam1@gmail.com. As this is the first rendition, any form of feedback is vital and encouraged; even if you've got only a single sentence to say, we urge you to email us and let us know your thoughts. It'll only help us improve and make the coming renditions better.

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Round 1 Problems

Round 1 P1

Round 1 P2

Round 1 P3

Round 1 P4

Round 1 P5

Round 1 P6

Round 2 Problems

Round 2 P1

Round 2 P2

Round 2 P3

Round 2 P4

Round 2 P5

Round 2 P6

You can click on these boxes to be taken to the questions. The solution will be right under the question so make sure to not get the problems spoiled.

Algebra

Problem 1 :

Prove the inequality which states that if you let $x_1, x_2, \ldots, x_n$ be positive real numbers, with $n \geq 2$, then you have the inequality

$$\frac{x_1}{x_2 + x_3 + \cdots + x_n} + \frac{x_2}{x_1 + x_3 + x_4 + \cdots + x_n} + \cdots + \frac{x_n}{x_1 + x_2 + \cdots + x_{n-1}} \geq \frac{n}{n-1}$$

Problem posed by @creative_math_

Solution :

We'll use Chebyshev's inequality here. WLOG we can assume that $x_1 \leq x_2 \leq \cdots x_{n-1} \leq x_n$ and then set $a_k = x_k$. From this we also know that

$$\frac{1}{x_2 + x_3 + \cdots + x_{n-1} + x_n} \leq \frac{1}{x_1 + x_3 + \cdots + x_{n-1} + x_n} \leq \cdots \leq \frac{1}{x_1 + x_2 + \cdots + x_{n-1}}$$

and we can set our b_k to be these expressions as so

$$b_k \times \sum_{i \in \mathbb{N}/\{k\}} x_i = 1$$

Both a_k and b_k are increasing sequences so we can apply Chebyshev's inequality to give

$$n \sum a_k b_k \geq \left(\sum a_k \right) \left(\sum b_k \right) \Rightarrow nI \geq n + I$$

Where I is the inequality expression (you can obtain $n + I$ after expanding the product)

$$\therefore I \geq \frac{n}{n-1}$$

7 marks

Of course, any other valid method can be used and that secures marks too.

Problem 2 :

Let a_1, a_2, b_2, b_3, c_1, c_2 be positive numbers such that $a_1 b_1 - c_1^2 > 0$ and $a_2 b_2 - c_2^2 > 0$. Prove that

$$\frac{8}{(a_1 + a_2)(b_1 + b_2) - (c_1 + c_2)^2} \leq \frac{1}{a_1 b_1 - {c_1}^2} + \frac{1}{a_2 b_2 - {c_2}^2}$$

Problem posed by @pepemaths

Solution :

Let $D_1 = a_1b_1 - {c_1}^2$ and $D_2 = a_2b_2 - {c_2}^2$.

$$\frac{8}{(a_1+a_2)(b_1+b_2)-{(c_1+c_2)}^2} - \frac{1}{a_1b_1-{c_1}^2} - \frac{1}{a_2b_2-{c_2}^2}$$

$$= \frac{8}{a_1b_1-{c_1}^2+a_2b_2-{c_2}^2+a_1b_2+a_2b_1-2c_1c_2} - \frac{1}{a_1b_1-{c_1}^2} - \frac{1}{a_2b_2-{c_2}^2}$$

$$= \frac{8}{D_1+D_2+a_1b_2+a_2b_1-2c_1c_2} - \frac{1}{D_1} - \frac{1}{D_2}$$

$$\leq \frac{8}{D_1+D_2+a_1b_2+a_2b_1-(\frac{b_2{c_1}^2}{b_1}+\frac{b_1{c_2}^2}{b_2})} - \frac{1}{D_1} - \frac{1}{D_2}$$

$$= \frac{8}{(D_1+D_2)+(\frac{b_2}{b_1}D_1+\frac{b_1}{b_2}D_2)} - \frac{1}{D_1} - \frac{1}{D_2}$$

$$\leq \frac{8}{2\sqrt{D_1D_2}+2\sqrt{D_1D_2}} - \frac{1}{D_1} - \frac{1}{D_2}$$

$$= \frac{2}{\sqrt{D_1D_2}} - \frac{1}{D_1} - \frac{1}{D_2} = -{(\frac{1}{\sqrt{D_1}} - \frac{1}{\sqrt{D_2}})}^2 \leq 0$$

So

$$\frac{8}{(a_1+a_2)(b_1+b_2)-{(c_1+c_2)}^2} \leq \frac{1}{a_1b_1-{c_1}^2} + \frac{1}{a_2b_2-{c_2}^2}$$

7 marks

Problem 3 :

Find all $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfying:

$$\text{(a) } f(x) + f\left(\frac{1}{x}\right) = 1, \text{ for all x} \in \mathbb{Q}^+$$

$$\text{(b) } f(1+2x) = \frac{1}{2}f(x), \text{ for all x} \in \mathbb{Q}^+$$

proving that you have found all such $f : \mathbb{Q}^+ \to \mathbb{Q}^+$

Problem posed by @creative.math_solving

Solution :

Taking $x = 1$ in (a), we get $f(1) = 1/2$. If we set $x = 1/2$ in (a) and (b), we see that

$$f\left(\frac{1}{2}\right) + f(2) = 1; \quad f(2) = \frac{1}{2} f\left(\frac{1}{2}\right)$$

Solving for $f(2)$ and $f\left(\frac{1}{2}\right)$, we obtain

$$f(2) = \frac{1}{3}, \quad f\left(\frac{1}{2}\right) = \frac{2}{3}$$

Taking $x = 1$ in (b), we see that

$$f(3) = \frac{1}{2} f(1) = \frac{1}{4}$$

Now if we use (a), we compute

$$f\left(\frac{1}{3}\right) = 1 - f(3) = \frac{3}{4}$$

Similarly substituting $x = 1/4$ in (b) we see that

$$f\left(\frac{3}{2}\right) = \frac{1}{2} f\left(\frac{1}{4}\right)$$

and taking $x = 3/2$ in (a) we obtain

$$f(4) = \frac{1}{2} f\left(\frac{3}{2}\right)$$

If we eliminate $f\left(\frac{3}{2}\right)$ from these two relations, we get

$$f(4) = \frac{1}{4} f\left(\frac{1}{4}\right)$$

But we know from (a) that

$$f(4) + f\left(\frac{1}{4}\right) = 1$$

It follows that

$$f(4) = \frac{1}{5}, \quad \text{and} \quad f\left(\frac{1}{4}\right) = \frac{4}{5}$$

These in turn leads to

$$f\left(\frac{3}{2}\right) = \frac{2}{5}, \text{ and } f\left(\frac{2}{3}\right) = \frac{3}{5}$$

An inspection of the values so far obtained reveals that $f(x) = \dfrac{1}{1+x}$ is possibly the required function. We show that this indeed is the solution of our functional equation. We adopt the following procedure to prove our claim by induction. For each rational $r = p/q \in \mathbb{Q}^+$ with $gcd(p,q) = 1$, we define $d(r) = p+q$ which is a natural number. We show that $f(r) = \frac{1}{1+r}$ by using induction on $d(r)$. We have so far verified this claim for all $r \in \mathbb{Q}^+$ for which $d(r) \leq 5$ holds.

Suppose this result is true for all $r \in \mathbb{Q}^+$ such that $d(r) \leq N$. Take any $r = p/q \in \mathbb{Q}^+$ such that $gcd(p,q) = 1$ and $d(r) = p+q = N+1$. We have

$$f\left(\frac{q}{p}\right) = f\left(1 + 2\left(\frac{q-p}{2p}\right)\right) = \frac{1}{2} f\left(\frac{q-p}{2p}\right)$$

If q and p are both odd, then 2 divides $q-p$. Thus

$$d\left(\frac{q-p}{2p}\right) \leq \frac{q-p}{2} + p = \frac{q+p}{2} \leq N$$

By induction hypothesis, we obtain

$$f\left(\frac{q-p}{2p}\right) = \frac{1}{1+\frac{q-p}{2p}} = \frac{2p}{q+p}$$

Thus we get

$$f\left(\frac{q}{p}\right) = \frac{p}{p+q}$$

and

$$f\left(\frac{p}{q}\right) = 1 - f\left(\frac{q}{p}\right) = \frac{q}{p+q} = \frac{1}{1+\frac{p}{q}}$$

Suppose p and q have different parity. If $q - p > 2p$, then

$$f\left(\frac{q-p}{2p}\right) = f\left(1 + 2\left(\frac{q-3p}{4p}\right)\right) = \frac{1}{2} f\left(\frac{q-3p}{4p}\right)$$

and hence

$$f\left(\frac{q}{p}\right) = \frac{1}{2} f\left(\frac{q-p}{2p}\right) = \frac{1}{2^2} f\left(\frac{q-(2^2-1)\,p}{2^2 p}\right)$$

Let s_1 be the least positive integer such that $2^{s_1}p > q - (2^{s_1}-1)\,p$. Using the above procedure we arrive at

$$f\left(\frac{q}{p}\right) = \frac{1}{2^{s_1}} f\left(\frac{q-(2^{s_1}-1)\,p}{2^{s_1} p}\right)$$

Put $q_1 = 2^{s_1}p$ and $p_1 = q - (2^{s_1}-1)\,p$. Then we can express $f\left(q_1/p_1\right)$ by

$$f\left(\frac{q_1}{p_1}\right) = 1 - f\left(\frac{p_1}{q_1}\right) = 1 - 2^{s_1} f\left(\frac{q}{p}\right)$$

We observe that

$$d\left(\frac{q_1}{p_1}\right) = q_1 + p_1 = q + p = N + 1$$

and $gcd\left(p_1, q_1\right) = gcd(p, q) = 1$. Let s_2 be the least positive integer such that

$$2^{s_2} p_1 > q_1 - \left(2^{s_2} - 1\right) p_1$$

Then we obtain

$$f\left(\frac{q_1}{p_1}\right) = \frac{1}{2^{s_2}} f\left(\frac{p_2}{q_2}\right)$$

where $q_2 = 2^{s_2} p_1$ and $p_2 = q_1 - \left(2^{s_2} - 1\right) p_1$. Thus we obtain

$$\begin{aligned} f\left(\frac{q_2}{p_2}\right) &= 1 - f\left(\frac{p_2}{q_2}\right) \\ &= 1 - 2^{s_2} f\left(\frac{q_1}{p_1}\right) \\ &= 1 - 2^{s_2}\left(1 - f\left(\frac{p_1}{q_1}\right)\right) \\ &= 1 - 2^{s_2} + 2^{s_2 + s_1} f\left(\frac{q}{p}\right) \end{aligned}$$

Continuing this process, we get a sequence $\langle (p_k, q_k) \rangle$ such that

1. $gcd\left(p_k, q_k\right) = 1$ for all k
2. $p_k + q_k = p + q = N + 1$, for all k
3. $p_k = q_{k-1} - \left(2^{s_k} - 1\right) p_{k-1}$ and $q_k = 2^{s_k} p_{k-1}$ (here $p_0 = p$ and $q_0 = q$) where s_k is the least positive integer such that $2^{s_k} p_{k-1} > q_{k-1} - \left(2^{s_k} - 1\right) p_{k-1}$
4. $2^{s_k} f\left(\frac{q_{k-1}}{p_{k-1}}\right) = f\left(\frac{p_k}{q_k}\right)$

Now there are only finitely many solutions to the equation $a + b = N + 1$ with $gcd(a, b) = 1$. Hence there must be repetitions in the in the sequence $\langle (p_k, q_k) \rangle$. Let us suppose

$$(p_m, q_m) = (p_{m+t}, q_{m+t})$$

For convenience, let us also introduce

$$2^{s_{m+t} + s_{m+t-1} + \cdots + s_{m+t-r}} = u_r$$

We then obtain

$$\begin{aligned}
f\left(\frac{p_{m+t}}{q_{m+t}}\right) &= 2^{s_{m+t}} f\left(\frac{q_{m+t-1}}{p_{m+t-1}}\right) \\
&= 2^{s_{m+t}} - 2^{s_{m+t}} f\left(\frac{p_{m+t-1}}{q_{m+t-1}}\right) \\
&= u_0 - u_1 + u_1 f\left(\frac{p_{m+t-2}}{q_{m+t-2}}\right) \\
&\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \\
&= u_0 - u_1 + \cdots + (-1)^{t-1} u_{t-1} + (-1)^t u_{t-1} f\left(\frac{p_m}{q_m}\right)
\end{aligned}$$

Now using $p_{m+t} = p_m$ and $q_{m+t} = q_m$ we solve for $f\left(\frac{p_m}{q_m}\right)$:

$$f\left(\frac{p_m}{q_m}\right) = \frac{u_0 - u_1 + \cdots + (-1)^{t-1} u_{t-1}}{1 - (-1)^t u_t}$$

However, we also have

$$\begin{aligned}
p_{m+t} &= q_{m+t-1} - \left(2^{s_m+t} - 1\right) p_{m+t-1} \\
&= q_{m+t-1} + p_{m+t-1} - 2^{s_{m+t}} p_{m+t-1} \\
&= (p_m + q_m) - 2^{s_{m+t}} p_{m+t-1}
\end{aligned}$$

An easy induction gives

$$p_{m+t} = (p_m + q_m)\left\{1 - u_0 + u_1 - u_2 + \cdots + (-1)^{t-1} u_{t-2}\right\} + (-1)^t u_{t-1} p_m$$

Using $p_{m+t} = p_m$, we obtain

$$\frac{p_m}{p_m + q_m} = \frac{1 - u_0 + u_1 - u_2 + \cdots + (-1)^{t-1} u_{t-2}}{1 - (-1)^t u_{t-1}}$$

But we also note that

$$\begin{aligned}
f\left(\frac{q_m}{p_m}\right) &= 1 - f\left(\frac{p_m}{q_m}\right) \\
&= 1 - \left\{\frac{u_0 - u_1 + u_2 - \cdots + (-1)^{t-1} u_{t-1}}{1 - (-1)^t u_{t-1}}\right\} \\
&= \frac{1 - u_0 + u_1 - u_2 + \cdots + (-1)^{t-1} u_{t-2}}{1 - (-1)^t u_{t-1}} \\
&= \frac{p_m}{p_m + q_m}
\end{aligned}$$

Thus it follows that

$$f\left(\frac{p_m}{q_m}\right) = 1 - f\left(\frac{q_m}{p_m}\right) = \frac{q_m}{p_m + q_m}$$

On the other hand we also observe that

$$f\left(\frac{p_m}{q_m}\right) = 2^{s_m} f\left(\frac{q_{m-1}}{p_{m-1}}\right)$$

so that

$$\begin{aligned} f\left(\frac{q_{m-1}}{p_{m-1}}\right) &= \frac{1}{2^{s_m}} f\left(\frac{p_m}{q_m}\right) \\ &= \left(\frac{1}{2^{s_m}}\right)\left(\frac{q_m}{p_m + q_m}\right) \\ &= \frac{p_{m-1}}{p_m + q_m} \\ &= \frac{p_{m-1}}{p_{m-1} + q_{m-1}} \end{aligned}$$

This gives

$$f\left(\frac{p_{m-1}}{q_{m-1}}\right) = 1 - f\left(\frac{q_{m-1}}{p_{m-1}}\right) = \frac{q_{m-1}}{p_{m-1} + q_{m-1}}$$

Continuing this process by induction, we arrive at

$$f\left(\frac{p}{q}\right) = f\left(\frac{p_0}{q_0}\right) = \frac{q_0}{p_0 + q_0} = \frac{q}{p + q}$$

Thus we finally obtain

$$f\left(\frac{p}{q}\right) = \frac{q}{p + q} = \frac{1}{1 + \frac{p}{q}}$$

showing that $f(r) = \frac{1}{1+r}$ for all positive rationals r

Problem 4 :

Given that $x_1, x_2 \cdots, x_k$ are positive reals such that $\sum_{i=1}^{k} x_i^{n-1} = k - 1$, prove that

$$\frac{x_1^n}{x_2 + x_3 + \cdots + x_k} + \frac{x_2^n}{x_1 + x_3 + \cdots + x_k} + \cdots + \frac{x_k^n}{x_1 + x_2 + \cdots + x_{k-1}} \geq 1$$

Problem posed by @creative_math_

Solution :

Firstly, notice that the inequality is cyclic. This means that we can, WLOG, assume that we have a decreasing sequence of x_i, or in other words $x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_k$. This implies

$$\begin{array}{ccccc} x_1^n & \geq & x_2^n & \geq \cdots \geq & x_k^n \\ \dfrac{1}{x_2+x_3+\cdots+x_k} & \geq & \dfrac{1}{x_1+x_3+\cdots+x_k} & \geq \cdots \geq & \dfrac{1}{x_1+x_2+\cdots+x_{k-1}} \end{array}$$

Applying Chebyshev's Inequality gives us

$$kI \geq (x_1^n+x_2^n+\cdots+x_k^n)\left(\frac{1}{x_2+x_3+\cdots+x_k}+\frac{1}{x_1+x_3+\cdots+x_k}+\frac{1}{x_1+x_2+\cdots+x_{k-1}}\right)$$

Where I is the expression in the original inequality. Multiplying it out, we get one I and the expression on the R.H.S

$$(k-1)S \geq \frac{x_1^n+x_2^n+\cdots+x_{k-1}^n}{x_1+x_2+\cdots+x_{k-1}}+\frac{x_2^n+x_3^n+\cdots+x_k^n}{x_2+x_3+\cdots+x_k}+\cdots$$

We see there's k terms in the R.H.S, and we can obtain an inequality for each using Chebyshev's inequality yet again if we consider 2 sequences

$$x_1^{n-1} \geq x_2^{n-1} \geq \cdots \geq x_k^{n-1}$$

$$x_1 \geq x_2 \geq \cdots \geq x_k$$

One thing to be cautious about is that each of the terms in the R.H.S contains only $(k-1)$ number of x_i's, so when we set up a chebyshev inequality for one of them(let's say the first term on the L.H.S), we get

$$(k-1)(x_1^n+x_2^n+\cdots+x_{k-1}^n) \geq (x_1+x_2+\cdots+x_{k-1})(x_1^{n-1}+x_2^{n-1}+\cdots+x_{k-1}^{n-1})$$

Rearranging, we obtain

$$\frac{x_1^n+x_2^n+\cdots+x_{k-1}^n}{x_1+x_2+\cdots+x_{k-1}} \geq \frac{x_1^{n-1}+x_2^{n-1}+\cdots+x_{k-1}^{n-1}}{k-1}$$

Note that we get k such inequalities, and in all the inequalities except 1, an x_i will be present. So if we're adding them up, we'll $(k-1)$ number of x_i's and this will cancel with the $(k-1)$ in the denominator and we obtain

$$(k-1)S \geq x_1^{n-1}+x_2^{n-1}+\cdots+x_k^{n-1}$$

But, as we know that $\displaystyle\sum_{i=1}^{k} x_i^{n-1} = k-1$, we have

$$S \geq 1$$

Discrete Mathematics

Problem 1 :

A computer calculates the nth Fibonacci Number (F_n, where $F_n = F_{n-1} + F_{n-2}$ and $F_0 = 0$, $F_1 = 1$) using "steps". A step is defined to be a single calculation (this means carrying out $a + b$ when we already know a and b counts as one step, though if we don't know what a or b are, carrying out $a + b$ takes *the number of steps to find* a + *the number of steps to find* b + 1, with the 1 coming from calculating $a + b$). The computer can only have F_0 and F_1 permanently stored, and it has to calculate everything else all over again when a request to calculate a new Fibonacci number is made. The advantage of this process is that the **final addition step**(to add one due to calculating $a + b$) gets omitted(by some algorithmic magic). In this algorithm, the steps needed to "call" F_0 and F_1 (these are the only values for which a call counts as a step) are both 1. Given that the computer can only carry out simple addition of 2 numbers at most :

- Find an expression for $S(F_n)$, the number of steps taken to find F_n using this approach if we can only have F_0 and F_1 stored for all $n \geq 2$ where $S(F_0) = 1$ and $S(F_1) = 1$ due to the "calling" feature
- Find an expression for $\gcd(S(F_n), S(F_m)))$ involving only n, m and the gcd function for $n, m \geq 2$

Now, the computer has the ability to "cache" (store) all previously calculated Fibonacci Numbers, but the final step is not omitted anymore. Assuming that we've already calculated F_k for some $2 \leq k \leq n - 1$, and the probability of picking an F_k is equally likely for all k :

- Show that the expected number of steps taken to calculate F_n, $\mathbb{E}(S(F_n))$, using the "cache" feature (if we have F_0 and F_1 stored and there is no calling step) is $\frac{n-1}{2}$

Note : $\mathbb{E}(X) = \sum P(X = x_i)x_i$ here where x_i is any possible value X can take

Problem posed by @creative_math_

Solution :

Firstly, let's calculate some values. $F_2 = F_0 + F_1$ so $S(F_2) = S(F_1) + S(F_0) = 2$. Also, $F_3 = F_2 + F_1 \Rightarrow S(F_3) = S(F_2) + S(F_1)$ and as we know that $S(F_2) = 2$ we get that $S(F_3) = 3$ as well. Similarly, $S(F_4) = 5$ and using induction(not required for marks) one can show that $S(F_n) = F_{n+1}$

2 marks

As $\gcd(F_{n+1}, F_{m+1}) = F_{\gcd(n+1,m+1)}$, we have $\gcd(S(F_m), S(F_n)) = F_{\gcd(n+1,m+1)}$. Candidates are not expected to prove this and simply stating it secures the mark.

1 mark

Assume we already calculated F_2, then to get to F_3 it'd take 1 step $(F_2 + F_1)$ and to get to F_4 it'd take 1 step $(F_3 + F_2)$ as F_3 is now cached so in total it'd take 2 steps from F_2. Continuing it'd take $n-2$ steps to calculate F_n if we'd picked F_2 to calculate before. If we had picked F_3 it would take $n-3$ steps and for some $2 \leq k \leq n-1$ it'd take $n-k$ steps to get to F_n if we'd picked F_k. The probability of picking either $F_2, F_3, \cdots, F_k \cdots, F_{n-1}$ is $\frac{1}{n-2}$. Plugging this into the expression for the expected value we get

$$E(S(F_n)) = \frac{1}{n-2}\Bigg((n-2) + (n-3) + \cdots + 3 + 2 + 1\Bigg) = \frac{(n-2)(n-1)}{2(n-2)} = \frac{n-1}{2}$$

4 marks

Problem 2 :

Given that $f(x) = x + 1$ and $g(x) = 2x$, how many different ways are there of combining $f(x)$ and $g(x)$ (this means doing any number of compositions like $fg(x)$ or $g^3f^2g(x)$ etc) such that the resulting composition is $8x + 8m$ where $m \geq 0$ is an integer?

For $m = 1$, the answer is 10. Find a general formula for the number of possibilities in terms of m

Problem posed by @creative_math_

Solution :

There must be 3 applications of $g(x)$ as $8 = 2^3$ and each application can have an arbitrary number of fs applied before and after it. The general form must be :

$$f^l g f^k f^j g f^i(x) = 8x + 8m$$

Expanding gives us

$$8x + 8i + 4j + 2k + l = 8x + 8m \Rightarrow 8i + 4j + 2k + l = 8m$$

As $m \geq 0$, we have $0 \leq i \leq m$ and if we fix an i, we get

$$4j + 2k + l = 8(m - i)$$

From which we obtain that $0 \leq j \leq 2m - 2i$ so for each $i \in \{0, 1, 2, 3 \cdots, m\}$ we have $2m - 2i + 1$ values for j and if we fix a j we get

$$2k + l = 4(2m - 2i - j)$$

This gives us that $0 \leq k \leq 4m - 4i - 2j$ and hence for each j, there are $4m - 4i - 2j + 1$ values for k. Now, if we fix k we get

$$l = 8m - 8i - 4k - 2k$$

so for each k there is only one possible l. This gives us the total number of possibilites to be

$$\sum_{i=0}^{m} \sum_{j=0}^{2m-2i} 4m - 4i - 2k + 1 = \frac{4m^3}{3} + 4m^2 + \frac{11m}{3} + 1 = \frac{(m+1)(2m+1)(2m+3)}{3}$$

Note : The answer is coincidentally $\sum_{k=0}^{m}(2k+1)^2$. The team hasn't been able to figure why yet but perhaps a recurrence can be derived. If we let $N(m)$ be the number of possibilities, all that is left to show is $N(m) = N(m-1) + (2m+1)^2$ and of course, $N(0) = 1$.

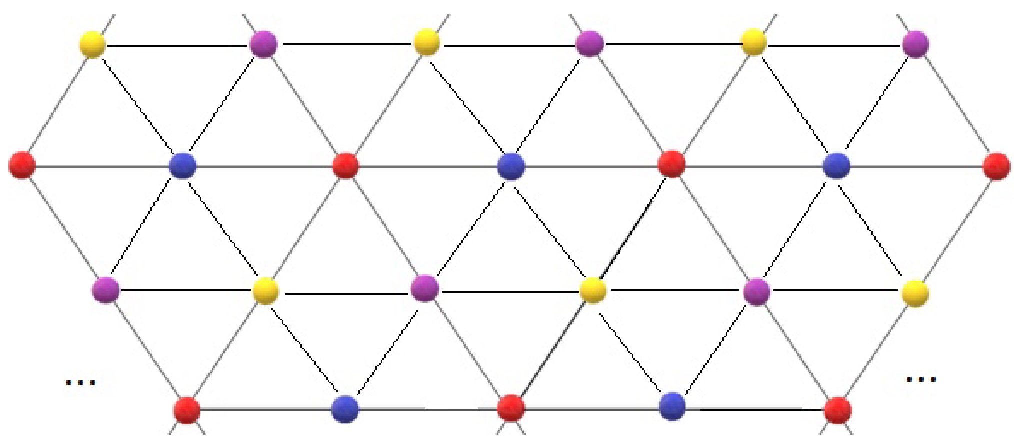
7 marks

Problem 3 :

Three frogs are initially on the vertices of an equilateral triangle with sides length of 1. The frogs can jump over each other in the following way: if frog A at point M jumps over frog B at point N, then frog A will land on point O such that $MN = ON$ and M, N, O are co-linear. By repeated jumping, is it possible that the three frogs eventually move to the vertices of an equilateral triangle with sides length of 10?

Problem posed by @pepemaths

Solution :



Consider the following triangular lattice points where each adjacent points are 1 unit from each other. Suppose the 3 frogs are on 3 adjacent lattice points which form an equilateral triangle with sides length 1 initially. The frogs are on lattice points with three different colours. Note that after each jump, a frog will always land on a lattice point which has the same colour as its original point. But for an equilateral triangle with sides length of 10 units, the three vertices must be lattice points having the same colour. Therefore it is not possible.

7 marks

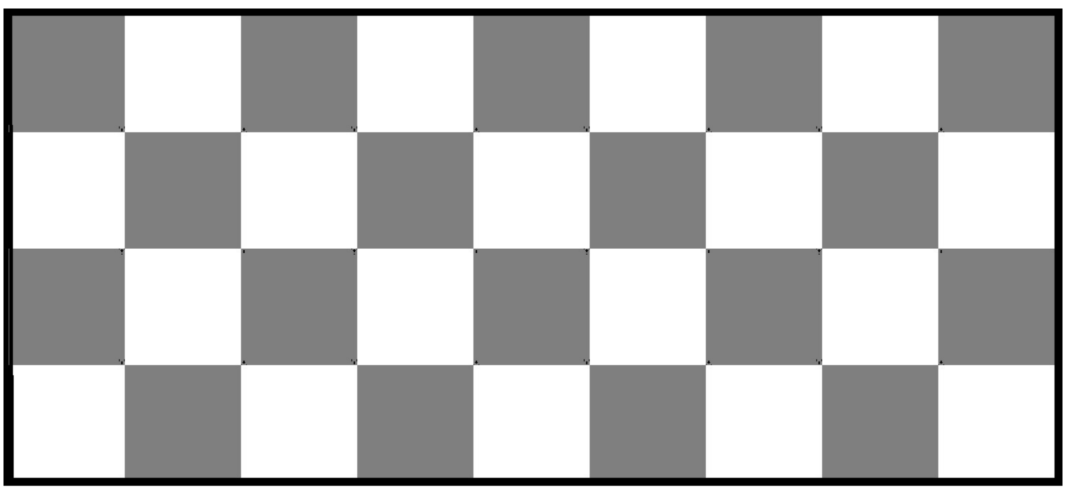
Problem 4 :

In chess, a knight moves either two squares vertically and one square horizontally, or two squares horizontally and one square vertically in each move. Suppose a knight can visit every squares on a $4 \times n$ chessboard exactly once without repeating. Find all the possible values of n.

Problem posed by @pepemaths

Solution :

We claim that there are no possible values of n.



4x9 Example Grid

Colour the chessboard with black and white on alternate squares. Also, we call the squares on the first and forth row as peripheral squares (denoted by P), and those on the second and third row as centre squares (denoted as C). It is obvious that a knight must move from a black square to a white square, and a white square to a black square alternately in each move. The knight always moves from a P square to a C square. It could move from a C square to either a C square or a P square, but since there are equal numbers of C squares and P squares, if in one move, the knight moves from a C square to another C square, the there won't be enough C square for the knight to move to after every P squares. Therefore the knight must move from a P square to a C square, and a C square to a P square alternately in each move. This is only possible if all the P squares are black and all the C squares are white, or all the P squares are white and all the C squares are black, which are both contradictory. So the knight can never visit every squares on a $4 \times n$ chessboard exactly once without repeating.

Problem 5 :

People from 80 countries participated in the 1^{st} round of IGMO. In order to ensure that participants from any of the 80 countries can travel to any one of the remaining 79 countries by at most 2 flights, the countries have an agreement such that among the 80 countries, each country's airport connects to at least n other countries' airport.

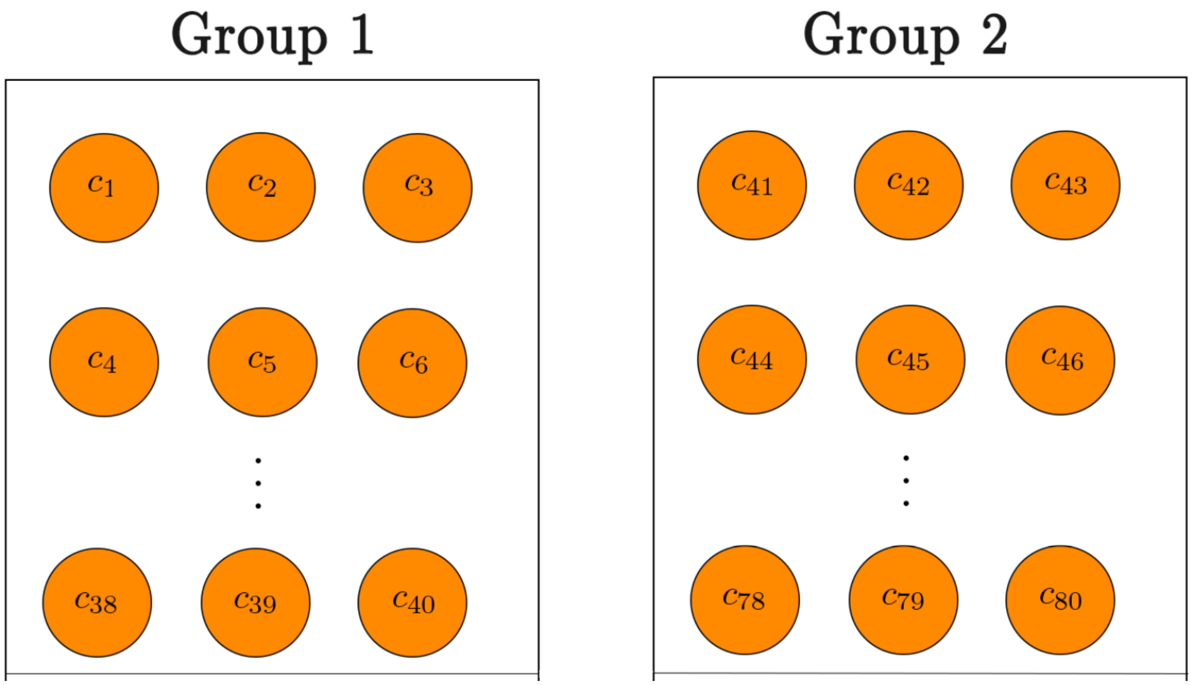
- Find the minimum value of n, proving that it is the minimum
- Prove that for this minimum n, if there isn't a direct flight between 2 countries, then there are at least 2 paths that require 2 flights between the countries

Note: For two airports A and B to be connected, it must be possible to have a flight from A to B and a flight from B to A. This adds 1 to both the tallies of connections from the airports. Also note that for this problem, each country has only 1 airport.

Problem posed by @saitejasomu , @pepemaths and @creative_math_

Solution :

Inspired by Ore's Theorem, we claim $n = 40$ is the minimum value of n. Let's then prove that $n = 39$ is impossible. Consider two groups of countries,



The groups are arranged such that for any country c_i in Group 1, each country it has a flight to is also in Group 1. This means that there exists a flight from c_i to $c_1, c_2, c_3, \ldots, c_{i-1}, c_{i+1}, \ldots, c_{39}, c_{40}$. As it can have only 39 flights emerging from it, a country c_j in Group 1 can only travel to a country c_k in Group 1 and hence stays in Group 1. Hence $n = 39$ cannot be our solution.

Let us now examine $n = 40$. Suppose that we want to travel from country c_A to c_B in at most 2 flights, if there exists a connection from c_A to c_B, we are done. If there doesn't, then we have 40 flights from c_A and 40 flights from c_B, with no flight from c_A to c_B and vice versa. In total, there are 80 possible flights with 78 other countries to connect to, which means there exists a c_n that has a flight from c_A to it and a flight from c_B to it(Pigeon-Hole Principle). As there exists a flight from c_B to it, thee also exists a flight from c_n to c_B and we can move travel between c_A and c_B as so : $c_a \to c_n \to c_B$. Hence we can move from any country to the other in at most 2 flights and therefore $n = 40$.

5 marks

As c_A and c_B are not connected, there are 80 flights and 78 other countries. A country cannot have more than 1 flight between each other (there cannot be 2 flights from c_A to c_n), there must exist 2 countries connected to both c_A and c_B (Pigeon-Hole Principle). Hence there are two paths from c_A to c_B and both require 2 flights.

2 marks

Geometry

Problem 1 :

A non-rectangular trapezoid is called a ”Pepe trapezoid” if

1. It has integral side lengths AND
2. An ellipse with integral lengths of semi-major axis and semi-minor axis can be inscribed in the trapezoid such that the major axis or minor axis of the ellipse is perpendicular to the bases of the trapezoid.

Prove or disprove that there exists infinitely many non-similar Pepe trapezoids.

Problem posed by @pepemaths

Solution :

Suppose an ellipse is inscribed in an isosceles trapezoid $WXYZ$ with bases WX and YZ ($YZ > WX$). The lateral sides XY and ZW are equal. We denote the length of axis perpendicular to the bases as $2a$, the length of axis parallel to the bases as $2b$, respectively. Let $WX = s$, $YZ = t$, $XY = ZW = u$.
Height of the trapezoid is $\sqrt{u^2 - (\frac{t-s}{2})^2} = \frac{\sqrt{4u^2-(t-s)^2}}{2}$. So

$$a = \frac{\sqrt{4u^2 - (t-s)^2}}{4} \tag{1}$$

Then apply an affine transformation which dilates/contracts the figure along the major axis of the ellipse with a scale of $\frac{b}{a}$. The ellipse is transformed to a circle with radius b. Suppose points W, X, Y, Z are transformed to W', X', Y', Z', respectively. Note that after the transformation, $W'X' = s$, $Y'Z' = t$, $X'Y' = Z'W'$. Moreover, $W'X'Y'Z'$ is a tangential trapezoid, by Pitot's theorem, $W'X' + Y'Z' = X'Y' + Z'W'$. Hence, $X'Y' = Z'W' = \frac{s+t}{2}$.
Height of this triangle is $\sqrt{(\frac{s+t}{2})^2 - (\frac{t-s}{2})^2} = \sqrt{st}$. So

$$b = \frac{\sqrt{st}}{2} \tag{2}$$

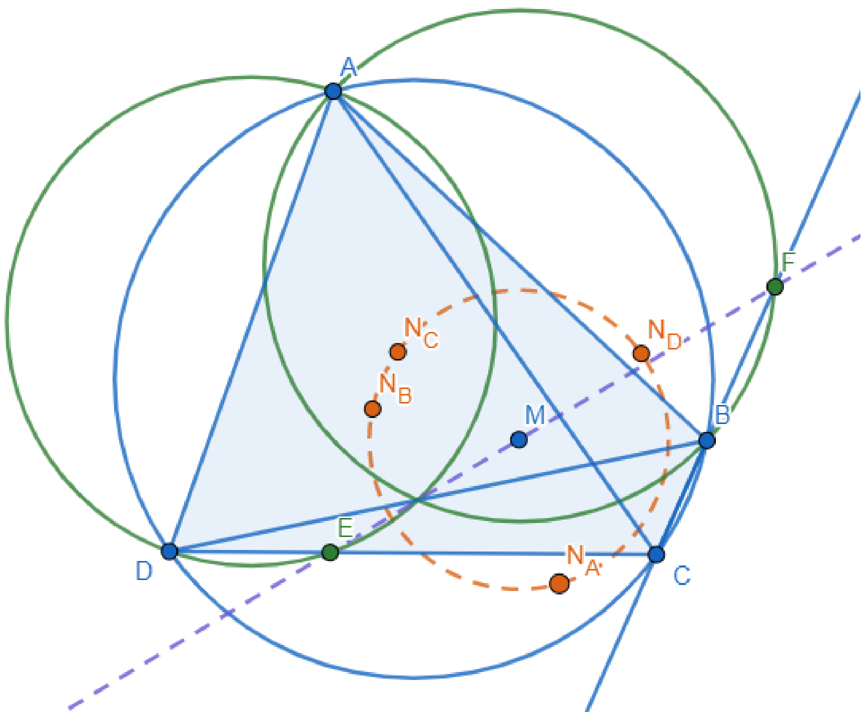
So a Pepe trapezoid is a trapezoid which u, s, t, $\frac{\sqrt{4u^2-(t-s)^2}}{2}$ and $\frac{\sqrt{st}}{2}$ are all integers. It is easy to verify that if $u = 4m^2+4n^2$, $s = 2mn$, $t = 18mn$, where m and n are positive integers such that $m > n$, then $\frac{\sqrt{4u^2-(t-s)^2}}{4} = 2(m^2 - n^2)$, $\frac{\sqrt{st}}{2} = 3mn$, which are also integers. By substituting different m and n, we can generate infinitely many non-similar Pepe trapezoids.

Problem 2 :

If $ABCD$ is a cyclic quadrilateral; N_1, N_2, N_3 and N_4 are the nine-point centres of $\triangle ABC$, $\triangle BCD$, $\triangle CDA$ and $\triangle DAB$, respectively. A circle with AD as diameter meets CD again at point E. Another circle with AB as diameter meets BC again at point F. Prove that N_1, N_2, N_3 and N_4 are concyclic and their circumcircle is bisected by line EF.

Problem posed by @pepemaths

Solution :



We shall prove this using complex numbers. Introduce a complex plane. Let 0 be the centre of the circumcircle of ABCD, and let the circle be a unit circle. $n_1 = \frac{a+b+c}{2}$, $n_2 = \frac{b+c+d}{2}$, $n_3 = \frac{c+d+a}{2}$, $n_4 = \frac{d+a+b}{2}$. Let $m = \frac{a+b+c+d}{2}$, it is easy to verify that $|n_1 - m| = |n_2 - m| = |n_3 - m| = |n_4 - m| = \frac{1}{2}$. So N_1, N_2, N_3 and N_4 are concyclic. The centre of the circumcircle is point M, represented by $m = \frac{a+b+c+d}{2}$, and the radius of the circle is $\frac{1}{2}$.

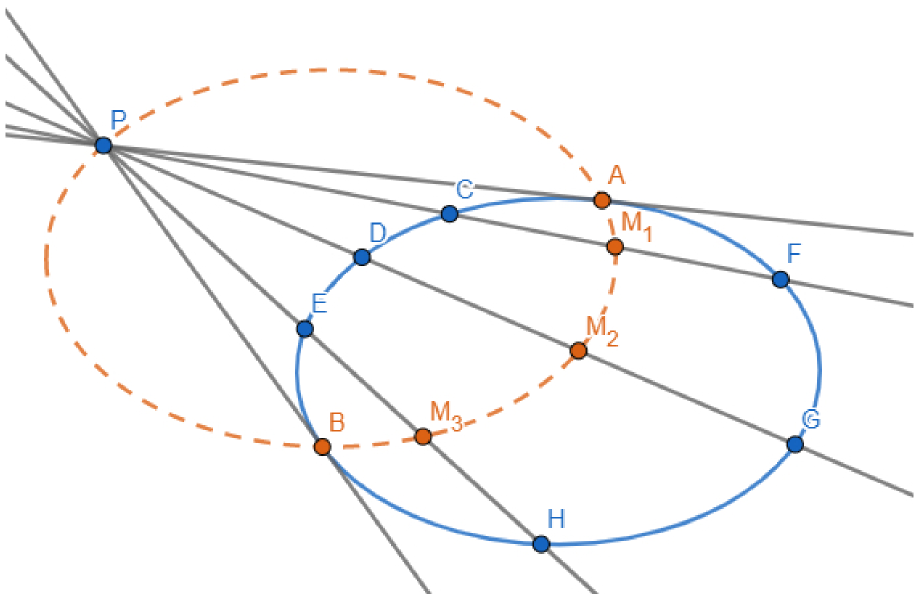
Note that AE is perpendicular to CD, AF is perpendicular to BC, so line EF is the Simson line of $\triangle BCD$ with pole A. By a well known property of Simson line, we know that EF passes through the mid-point of A and the orthocentre of $\triangle BCD$, which is represented by $\frac{a+(b+c+d)}{2} = \frac{a+b+c+d}{2}$. So EF passes through the centre of the circumcircle of $N_1N_2N_3N_4$. It bisects the circumcircle.

Problem 3 :

For any given ellipse ω, let P be a point external to it. A and B are points on ω such that PA and PB are tangent to ω. C, D and E are points on ω such that $AC = CD = DE = EB$. Lines PC, PD, PE meets ω again at points F, G, H, respectively. M_1, M_2, M_3 are mid-points of CF, DG and EH, respectively. Prove that P, A, B, M_1, M_2 and M_3 lie on an ellipse.

Problem posed by @pepemaths

Solution :



Apply an affine transformation which transforms the ellipse into a circle. Suppose P, A, B, C, D, E, F, G, H, M_1, M_2, M_3 are transformed to P', A', B', C', D', E', F', G', H', M_1', M_2', M_3', respectively. Note that $P'A'$ and $P'B'$ are tangent to the circle. C', D' and E' are points on the circle such that $A'C' = C'D' = D'E' = E'B'$. Lines $P'C'$, $P'D'$, $P'E'$ meets the circle again at points F', G', H', respectively. M_1', M_2', M_3' are mid-points of $C'F'$, $D'G'$ and $E'H'$, respectively.

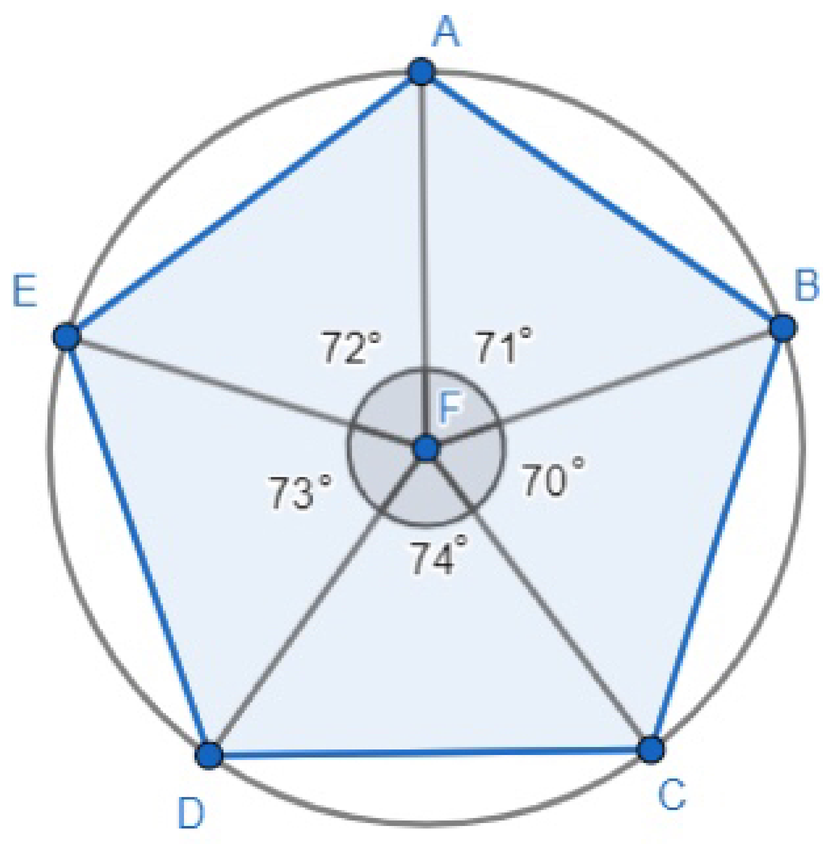
Let O be the centre of the circle. We claim that O, P', A', B', M_1', M_2' and M_3' are concyclic. $\angle PAO = \angle PBO = 90^o$. $\angle PAO + \angle PBO = 180^o$. O, P', A', B' are concyclic. M_1', M_2' and M_3' are mid-points of chords of the circle. $\angle P'M_1'O = \angle P'M_2'O = \angle P'M_3'O$. So P', O, M_1', M_2', M_3' are concyclic. Overall, O, P', A', B', M_1', M_2' and M_3' are concyclic. Before the affine transformation, P, A, B, M_1, M_2 and M_3 must lie on an ellipse.

Problem 4 :

Some points are drawn on a plane such that the points do not have equal distances to each other. For each point, a line is drawn to connect it with its nearest point. Find the maximum possible number of lines that a point is connected with.

Problem posed by @pepemaths

Solution :



We claim that the maximum possible number is 5.
Refer to the figure consisting the vertices of a cyclic pentagon (A, B, C, D and E). Let F be a point slightly deviated from the circumcentre of the pentagon such that AF,BF, CF, DF and EF are not equal to each other, it is easy to verify that A, B, C, D and E should be connected with F. So there are five lines connected with F, hence it is possible to have 5 lines connected to a point.
We shall show that 6 is not possible. Consider any 3 points, A, B and C. Suppose there are 2 lines connected to B. WLOG, assume $AB < BC$. B must be the nearest point of A,

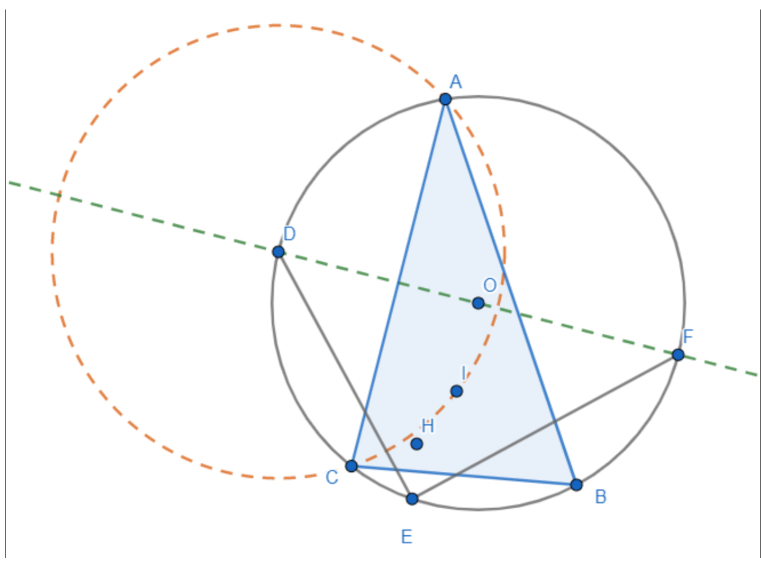
so there is a line connecting A and B, and $AB < AC$. If not, then $AC < AB < BC$, the nearest point of C must be A, then only one line is connected to B, which contradicts with our assumption.
By assumption, there is also a line connecting B and C, since $AB < BC$, the nearest point of B cannot be C, so the nearest point of C must be B, therefore $BC < AC$. Overall we have, $AB < BC < AC$. So $\angle ABC > 60$. If there are 6 lines connecting to a point, all the angles formed by adjacent lines are $> 60°$, the sum of the 6 angles are greater than $360°$, which is impossible.

Problem 5 :

Let I, H, O be the incentre, orthocentre and circumcentre of $\triangle ABC$ respectively. D is the circumcentre of $\triangle AIC$. H is reflected along BC and AB to E and F respectively. Prove that D, O, F are collinear if and only if DE is perpendicular to EF.

Problem posed by @pepemaths

Solution :



By incentre-excentre lemma, D is on the circumcircle of $\triangle ABC$. Also, it is a well known property of orthocentre that E and F are on the circumcircle of $\triangle ABC$. So, D, O, F are collinear if and only if DF is the diameter of circumcircle of $\triangle ABC$ if and only if DE is perpendicular to EF.

Problem 6 :

A sphere of radius r can be inscribed in a tetrahedron. The distances between the centroid of the tetrahedron and its four faces are w, x, y and z. Prove that $wxyz \geq r^4$.

Problem posed by @pepemaths

Solution :

Let the tetrahedron be $ABCD$ and G be the centroid. Let the areas of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$ and $\triangle ABC$ be s_1, s_2, s_3 and s_4 respectively. Let the distances between G and $\triangle BCD$, $\triangle CDA$, $\triangle DAB$ and $\triangle ABC$ be w, x, y and z respectively. Let V be the volume of the tetrahedron.

Note that the volumes of tetrahedrons $GBCD$, $GCDA$, $GDAB$ and $GABC$ are equal. We have $\frac{ws_1}{3} = \frac{xs_2}{3} = \frac{ys_3}{3} = \frac{zs_4}{3}$.

Also, $V = \frac{ws_1}{3} + \frac{xs_2}{3} + \frac{ys_3}{3} + \frac{zs_4}{3} = \frac{rs_1}{3} + \frac{rs_2}{3} + \frac{rs_3}{3} + \frac{rs_4}{3} = \frac{r(s_1+s_2+s_3+s_4)}{3}$.

$$\frac{ws_1}{3} = \frac{xs_2}{3} = \frac{ys_3}{3} = \frac{zs_4}{3} = \frac{1}{4}\frac{r(s_1+s_2+s_3+s_4)}{3} \geq \frac{r\sqrt[4]{s_1s_2s_3s_4}}{3}$$

$$ws_1 = xs_2 = ys_3 = zs_4 \geq r\sqrt[4]{s_1s_2s_3s_4}$$

$$ws_1xs_2ys_3zs_4 \geq (r\sqrt[4]{s_1s_2s_3s_4})^4 = r^4s_1s_2s_3s_4$$

$$wxyz \geq r^4$$

Number Theory

Problem 1 :

For any natural number n and all natural numbers d dividing $2n^2$ show that $n^2 + d$ is not the square of a natural number

Problem posed by @mathinity

Solution :

Suppose that

$$\exists_{x \in \mathrm{N}} : n^2 + d = x^2$$

since d divides $2n^2$, we know that

$$\exists_{k \in \mathbb{N}} : 2n^2 = dk$$

By multiplying both sides of the first equation by k^2 we obtain

$$k^2 x^2 = k^2 n^2 + k^2 d = n^2 \left(k^2 + 2k\right)$$

Above equation implies that $k^2 + 2k$ is a square number. This is a contradiction as

$$k^2 < k^2 + 2k < (k+1)^2$$

and a square cannot lie between 2 consecutive squares.

Alternative Solution :

if $d|2n^2$, then we have $dk = 2n^2$. Suppose now that

$$n^2 + d = m^2$$

For some $m \in \mathbb{N}$. Clearly, $m^2 > n^2$. Multiplying both sides of the equation by k, we get

$$n^2 k + dk = m^2 k$$

$$n^2(k+2) = m^2 k \Rightarrow \frac{n^2}{m^2} = \frac{k}{k+2}$$

If k is even, $k = 2a$ for some $a \in \mathbb{N}$, then we'd have

$$\frac{n^2}{m^2} = \frac{a}{a+1}$$

As a and $a+1$ are co-prime, $n^2 = a$ and $m^2 = a+1$, but this can't happen as squares must be at least 3 apart.

If k is odd, then k and $k+2$ are co-prime and hence $n^2 = k$ and $m^2 = k+2$ which is again a contradiction. To prove that two consecutive odd numbers are co-prime, assume that they share a common prime p_1 in their factorization. Then we'd have

$$k + 2 = xp_1 \quad , \quad k = yp_1$$

Subtracting, we have

$$p_1(x - y) = 2$$

As $p_1 > 2$ (as they're odd), this is a contradiction

Problem 2 :

Let's define a function $\phi : \mathbb{N} \to \mathbb{N}$, where $0 \notin \mathbb{N}$, as follows

$$\phi(n) = \sum_{k=1}^{n} k!$$

Let $\mathbb{V}$ be defined as the set of all triplets $(x, y, z) \in \mathbb{N}$ such that $\phi(x) = y^{z+1}$. For a triplet x, y, z (denoted by v) in $\mathbb{V}$, we define

$$f_v(n) = 8\left[\frac{xy}{8}\right]\lfloor\sqrt{n}\rfloor + \frac{zn}{z + x - y}.$$

($[x]$ is **fractional part** of x and $\lfloor x \rfloor$ is greatest integer less than x)

Show that for any $v \in \mathbb{V}$ and $m \in \mathbb{N}$ the sequence

$$m,\ f_v(m),\ f_v(f_v(m)),\ f_v(f_v(f_v(m))), ...$$

contains at least one square of a natural number. Please note that $[x]$ here refers to the **fractional part** of x, it can also be denoted as $\{x\}$ but it is denoted as $[x]$ here.

Problem posed by @mathinity

Solution 2 :

Firstly let's find all elements of $\mathbb{V}$. Let's do 2 cases

1. $z = 1$. We can notice that

$$\forall_{x \geq 4} : \phi(x) \equiv 3 \pmod{10}$$

and

$$\forall_{y \in \mathbb{N}} : y^2 \not\equiv 3 \pmod{10},$$

Hence all $(x, y, 1)$ for $x \geq 4$ can't be an element of $\mathbb{V}$. By checking $(1, y, 1), (2, y, 1)$ and $(3, y, 1)$ we find out that the only triples that satisfy the equation $\phi(x) = y^{z+1}$ are

$$(1, 1, 1) \quad \text{and } (3, 3, 1)$$

2. $z > 1$. Of course if we set $x = y = 1$, then we have infinitely many new elements of $\mathbb{V}$

$$\forall_{z>1} : (1, 1, z) \in \mathbb{V}$$

Let's set $x, y > 1$ and split this one into 3 cases

- $x \in \{2, ..., 6\}$. We can notice, that $\phi(x)$ is divisible by 3, but not by 27, so for $x \in \{2, ..., 6\}$, $(x, y, z) \notin \mathbb{V}$.
- $x = 7$. We can see that $\phi(7) = 5913$ is divisible by 73, but not by 73^2, so $(7, y, z)$ can't be an element of $\mathbb{V}$.
- $x > 7$. We can observe, that $\forall_{n>8} : n! \equiv 0 \pmod{27}$, so

$$\forall_{x>7} : \phi(x) \equiv \phi(8) \equiv 46233 \equiv 9 \pmod{27}.$$

Hence $\phi(x)$ is divisible by 9, but not by 27, so for $x > 7$, $(x, y, z) \notin \mathbb{V}$.

It follows that

$$\mathbb{V} = \{(1, 1, z) \in \mathbb{N}^3 : z \in \mathbb{N}\} \cup \{(3, 3, 1)\}$$

Let's take $v \in \mathbb{V}$ and put it into the definition of $f_v(n)$. We can see that

$$\forall_{v\in\mathbb{V}} : f_v(n) = 8 \left\lceil \frac{xy}{8} \right\rceil \lfloor \sqrt{n} \rfloor + \frac{zn}{z + x - y} = \lfloor \sqrt{n} \rfloor + n =: f(n)$$

Let's set $m \in \mathbb{N}$. We can define $k := \lfloor \sqrt{m} \rfloor$. It means that $m = k^2 + j$ for some $j \in \{0, 1, ..., 2k\}$. Let's break this up into 2 cases

1. $j \in \{0, 1, ..., k\}$. To prove the thesis we'll need the following lemma

Lemma 0.1. *If $f^r(m) = f(f^{r-1}(m))$, then the equation*

$$f^{2r}(m) = (k + r)^2 + (j - r)$$

holds true for all $r \in \{0, ..., j\}$

Proof. We will perform a proof by induction with respect to r. If we set $r = 1$, then we obtain

$$f^2(m) = f(f(m)) = f(m + k) = f(k^2 + k + j) = k^2 + k + j + \lfloor \sqrt{k^2 + k + j} \rfloor.$$

Since $j \in \{0, 1, ..., k\}$, we have $k^2 + k + j < k^2 + 2k + 1 = (k + 1)^2$, so

$$f^2(m) = k^2 + k + j + \lfloor \sqrt{k^2 + k + j} \rfloor = k^2 + 2k + j = (k + 1)^2 + (j - 1)$$

Now that we have our base case, we can do the induction step. Let's assume that the lemma is true for r. We need to prove the lemma for $r + 1$. We have

$$\begin{aligned} f^{2(r+1)}(m) &= f(f(f^{2r}(m))) = f(f((k + r)^2 + j - r)) \\ &= f((k + r)^2 + j - r + \lfloor \sqrt{(k + r)^2 + j - r} \rfloor) =: F \end{aligned}$$

Like in the base case we can see that $(k+r)^2 + j - r < (k+r+1)^2$, so

$$F = f((k+r)^2 + j - r + k + r) = f((k+r)^2 + k + j)$$
$$= (k+r)^2 + k + j + \lfloor \sqrt{(k+r)^2 + k + j)} \rfloor$$

Once again we can notice that $(k+r)^2 + k + j < (k+r+1)^2$, so

$$F = (k+r)^2 + k + j + k + r = (k + (r+1))^2 + (j - (r+1))$$

□

By the **Lemma 1.1** we know that $f^{2j}(m)$ is a square of a natural number.

2. $j \in \{k+1, k+2, ..., 2k\}$. By considering $f(m)$ instead of m we reduce this case to the 1^{st} case. Hence proved.

Problem 3 :

Find $a, b, c, d \in \mathbb{N}$ such that: $a + 2^b + 3^c = 3d! + 1$ and exist p, q prime such that : $a = (p+1)(2p+1) = (q+1)(q-1)^2$

Problem posed by @creative.math_solving

Solution :

We have

$$(p+1)(2p+1) = (q+1)(q-1)^2$$
$$2p^2 + 3p = q^3 - q^2 - q$$
$$p(2p+3) = q(q^2 - q - 1)$$

Thus $p \mid q^2 - q - 1$ and $q \mid 2p+3$ indeed $2p+3 = qk \iff p = \frac{qk-3}{2}$. Notice $q \equiv \frac{3}{k} \mod \frac{qk-3}{2}$

$$q^2 - q - 1 \equiv 0 \mod \frac{qk-3}{2}$$
$$\frac{9}{k^2} - \frac{3}{k} \equiv 1 \mod \frac{qk-3}{2}$$
$$9 - 3k \equiv k^2 \mod \frac{qk-3}{2}$$

Thus $k^2 + 3k - 9 \geq \frac{qk-3}{2}$ indeed $2k + 6 - \frac{15}{k} \geq q$ indeed $2k + 6 \geq q$ thus $k \geq \frac{q-6}{2}$

Now

$$\begin{aligned}
2p^2+3p &= q^3-q^2-q \\
\frac{(qk-3)^2}{2}+\frac{3}{2}(qk-3) &= q^3-q^2-q \\
(q\left(\frac{q-6}{2}\right)-3)^2+3(q\left(\frac{q-6}{2}\right)-3) &\leq 2q^3-2q^2-2q \\
\frac{q^4}{4}-3q^3+\frac{15q^2}{2}+9q &\leq 2q^3-2q^2-2q \\
\frac{q^4}{4}-5q^3+\frac{19q^2}{2}+11q \leq 0 &
\end{aligned}$$

In fact the trivial without using computing bound is $25 > q$. But let us use computing and so $17 \geq q \geq 5$.
Nice, now we have $q \in \{5, 7, 11, 13, 17\}$ checking when $2p^2+3p = q^3-q^2-q$ we get $(p, q) = (31, 13)$ i.e $a = 2016$.
Thus we have

$$2015 + 2^b + 3^c = 3d!$$

With $2015 < 3d!$ we get $d \geq 7$. Also $2 + (-1)^b \equiv 0 \mod 3$ yielding b is even. So $b \geq 2$ and thus $-1 + (-1)^c \equiv 0 \mod 4$ yielding c is even i.e $(b, c) = (2x, 2y)$. Now write as $2015 + 4^x + 9^y = 3d!$ so

$$4^x \equiv 1 \mod 9$$

Thus $x = 3u$. Yielding $2015 + 64^u + 9^y = 3d!$. So

$$2^y \equiv 0 \mod 7$$

Which is impossible, so there are no solutions.

Problem 4 :

Let k be a positive real number such that $\lfloor kn^2 \rfloor$ is perfect square for all $n \in \mathbb{N}$ Show that k must be a perfect square.

Problem posed by @creative.math_solving

Solution :

If k is irrational, let $p > 1$ be any positive integer. If k is rational, let p be a prime not dividing its denominator. Let m be the smallest positive integer such that $\lfloor kp^{2m} \rfloor > p$. Let $\lfloor kp^{2m+2n} \rfloor = a_n^2$ for all integers $n \geq 0$.

$$a_{n+1}^2 = \lfloor kp^{2m+2n+2} \rfloor \geq p^2 \lfloor kp^{2m+2n} \rfloor = p^2 a_n^2$$

As $a_n > p$ for all n

$$\begin{aligned} a_{n+1}^2 &= \lfloor kp^{2m+2n+2} \rfloor \\ &\leq kp^{2m+2n+2} \\ &< p^2(kp^{2m+2n} - 1) + 2p^2 + 1 \\ &< p^2 a_n^2 + 2pa_n + 1 \end{aligned}$$

From the two inequalities above $p^2 a_n^2 \leq a_{n+1}^2 < p^2 a_n^2 + 2pa_n + 1 \implies pa_n \leq a_{n+1} < pa_n + 1 \implies pa_n = a_{n+1}$ which by induction means $a_n = p^n q$ for all n, where q is some positive integer.

$$\begin{aligned} \lfloor kp^{2m+2n} \rfloor &= p^{2n} q^2 \\ p^{2n} q^2 - 1 < kp^{2m+2n} &\leq p^{2n} q^2 - 1 \\ q^2 - \frac{1}{p^{2n}} < kp^{2m} &\leq q^2 \end{aligned}$$

By taking n to infinity, $kp^{2m} = q^2$ so kp^{2m} must be a perfect square, so k must be too.