IYMC Pre-Final Round 2020

Shreenabh Agrawal

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Problem A.1

Find all points (x, y) where the functions $f(x), g(x), h(x)$ have the same value:

$$f(x) = 2^{x-5} + 3, \quad g(x) = 2x - 5, \quad h(x) = \frac{8}{x} + 10$$

Solution A.1

Solving for $g(x)$=$h(x)$,

$$2x - 5 = \frac{8}{x} + 10$$

Multiply by x, $(x \neq 0)$

$$\begin{aligned}
2x^2 - 15x - 8 &= 0 \\
2x^2 - 16x + x - 8 &= 0 \\
2x(x-8) + 1(x-8) &= 0 \\
(2x+1)(x-8) &= 0
\end{aligned}$$

$$\boxed{x = 8, x = -\frac{1}{2}}$$

$f\left(-\frac{1}{2}\right)$ is irrational while $g\left(-\frac{1}{2}\right)$ and $h\left(-\frac{1}{2}\right)$ are rational. Hence, they cannot be equal.

$$\begin{aligned}
f(8) &= 2^{8-5} + 3 = 11, \\
g(8) &= 2 \times 8 - 5 = 11, \\
h(8) &= \frac{8}{8} + 10 = 11
\end{aligned}$$

$\therefore$ (8,11) is the only point where $f(x), g(x), h(x)$ have the same value.

Problem A.2

Determine the roots of the function $f(x) = \left(5^{2x} - 6\right)^2 - \left(5^{2x} - 6\right) - 12$.

Solution A.2

Let $5^{2x} - 6$ be t. We have:

$$\begin{aligned}
t^2 - t - 12 &= 0 \\
t^2 - 4t + 3t - 12 &= 0 \\
t(t-4) + 3(t-4) &= 0 \\
(t+3)(t-4) &= 0
\end{aligned}$$

$$\boxed{t = -3, t = 4}$$

Case I

Let $t = -3$,

$$\begin{aligned}
5^{2x} - 6 &= -3 \\
5^{2x} &= 3 \\
2x &= \log_5 3 \\
x &= \frac{1}{2} \log_5 3
\end{aligned}$$

Case II

Let $t = 4$,

$$\begin{aligned}
5^{2x} - 6 &= 4 \\
5^{2x} &= 10 \\
2x &= \log_5 10 \\
x &= \frac{1}{2} \log_5 10
\end{aligned}$$

$$\boxed{x = \frac{1}{2} \log_5 3, \frac{1}{2} \log_5 10}$$

Problem A.3

Find the derivative $f'_m(x)$ of the following function with respect to x:

$$f_m(x) = \left(\sum_{n=1}^{m} n^x \cdot x^n \right)^2$$

Solution A.3

By chain rule of derivatives we have,

$$f(g(x)) = f'(g(x))g'(x)$$

$$\begin{aligned} f_m(x) &= \left(\sum_{n=1}^{m} n^x \cdot x^n \right)^2 \\ f'_m(x) &= 2 \left(\sum_{n=1}^{m} n^x \cdot x^n \right) \left(\sum_{n=1}^{m} n^x \cdot x^n \right)' \end{aligned}$$

Now by product rule of derivatives we have,

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

$$f'_m(x) = 2 \left(\sum_{n=1}^{m} n^x \cdot x^n \right) \left(\sum_{n=1}^{m} \left(n^x \cdot x^n \cdot \ln(n) + n^{x+1} x^{n-1} \right) \right)$$

$$\boxed{f'_m(x) = 2 \left(\sum_{n=1}^{m} n^x \cdot x^n \right) \left(\sum_{n=1}^{m} n^x \cdot x^{n-1} (x \ln(n) + n) \right)}$$

Problem A.4

Find at least one solution to the following equation:

$$\frac{\sin\left(x^2-1\right)}{1-\sin\left(x^2-1\right)} = \sin(x) + \sin^2(x) + \sin^3(x) + \sin^4(x) + \cdots$$

Solution A.4

As $|\sin(x)| \leq 1$, R.H.S. can be written as sum of infinite geometric progression as:

$$\frac{\sin(x^2-1)}{1-\sin(x^2-1)} = \frac{\sin(x)}{1-\sin(x)}$$

One solution can be (in principal interval):

$$\sin(x^2-1) = \sin(x)$$
$$x^2 - 1 = x$$

Where, $-\pi \leq x^2 - 1 \leq \pi \implies x \in [-\sqrt{\pi+1}, \sqrt{\pi+1}]$,

$$x^2 - x - 1 = 0$$

By quadratic formula, $x = \frac{1\pm\sqrt{1+4}}{2} \implies \frac{1-\sqrt{5}}{2}$ This satisfies the initial conditions also. Thus we have,

$$\boxed{x = \frac{1-\sqrt{5}}{2}}$$

Problem B.1

Consider the following sequence of successive numbers of the 2^k -th power:

$$1, 2^{2^k}, 3^{2^k}, 4^{2^k}, 5^{2^k}, \ldots$$

Show that the difference between the numbers in this sequence is odd for all $k \in \mathbb{N}$.

Solution B.1

Natural powers of odd numbers are odd and natural powers of even numbers are even. This is because raising a number to its power does not introduce/remove new factors of 2. Hence parity is retained. This applies to 2^k-th powers too. Thus, the sequence will be like:

$$1, even, odd, even, odd, even, \ldots$$

Here, as we can observe, difference of two consecutive terms is always odd. This is because:

$$even - odd = odd$$

$$odd - even = odd$$

Hence proved, the difference between the numbers in this sequence is odd for all $k \in \mathbb{N}$.

Problem B.2

Prove this identity between two infinite sums (with $x \in \mathbb{R}$ and $n!$ stands for factorial):

$$\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)^2 = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

Solution B.2

We know the Taylor series expansion:

$$e^x = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)$$

Squaring both sides we have,

$$e^{2x} = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)^2$$

But by replacing x with $2x$ in the original Taylor series expansion, we can write e^{2x} as:

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

Hence proved:

$$\boxed{\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)^2 = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}}$$

Problem B.3

You have given a function $\lambda : \mathbb{R} \to \mathbb{R}$ with the following properties ($x \in \mathbb{R}, n \in \mathbb{N}$)

$$\lambda(n) = 0, \quad \lambda(x+1) = \lambda(x), \quad \lambda\left(n + \frac{1}{2}\right) = 1$$

Find two functions $p, q : \mathbb{R} \to \mathbb{R}$ with $q(x) \neq 0$ for all x such that $\lambda(x) = q(x)(p(x)+1)$

Solution B.3

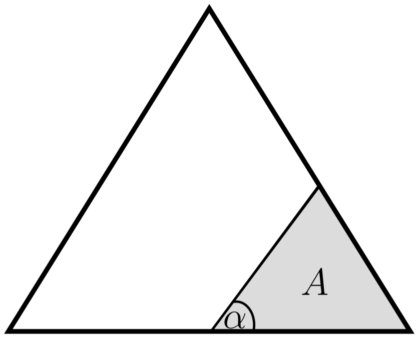
In this problem, we make use of the 'Fractional Part Function' — {x}.
We have $\lambda(n) = 0$ and {n} = 0 (by definition of F.P.F). So let us designate $q(x) = \{x\}$.
Next, we have $\lambda(x+1) = \lambda(x)$, this implies both $q(x)$ and $p(x)$ are purely composed of $\{x\} + c$ where c is a numerical constant. We have already proved that in $q(x)$, $c = 0$.
We are given: $\lambda\left(n + \frac{1}{2}\right) = 1$. We also know that, $\left\{n + \frac{1}{2}\right\} = \left\{\frac{1}{2}\right\}$. Hence:

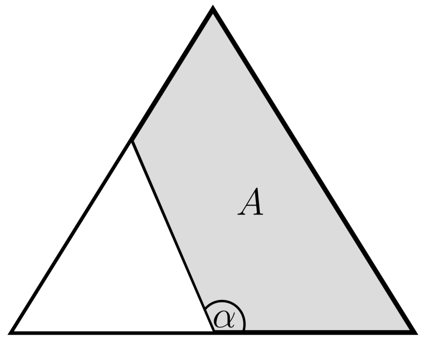
$$\begin{aligned} \lambda\left(n + \frac{1}{2}\right) &= \left\{n + \frac{1}{2}\right\}\left(\left\{n + \frac{1}{2}\right\} + c + 1\right) = 1 \\ \frac{1}{2}\left(\frac{1}{2} + c + 1\right) &= 1 \\ \frac{3}{4} + \frac{c}{2} &= 1 \\ c &= \frac{1}{2} \end{aligned}$$

$$\boxed{\therefore q(x) = \{x\}, p(x) = \{x\} + 0.5}$$

Problem B.4

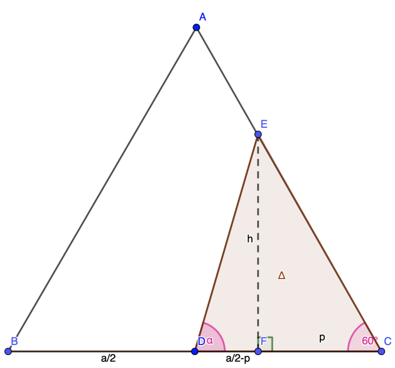
You have given an equal sided triangle with side length a. A straight line connects the center of the bottom side to the border of the triangle with an angle of α. Derive an expression for the enclosed area $A(\alpha)$ with respect to the angle (see drawing).

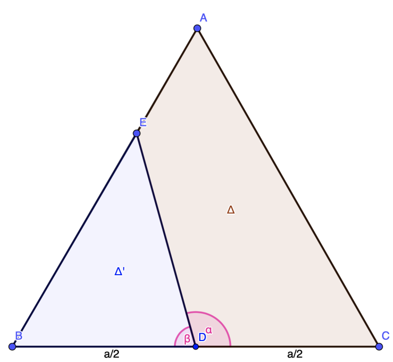




Solution B.4

We do the constructions and labelling as shown (two cases dealt separately).





Case I

In ΔDEC,

$$\begin{aligned}
\tan(60^\circ) &= \frac{EF}{FC} = \frac{h}{p} \\
h &= \sqrt{3}p \\
\tan(\alpha) &= \frac{EF}{DF} \\
&= \frac{h}{\frac{a}{2} - p} \\
&= \frac{\sqrt{3}p}{\frac{a}{2} - p} \\
\frac{a}{2}\tan(\alpha) - p\tan(\alpha) &= \sqrt{3}p \\
\frac{a}{2}\tan(\alpha) &= (\sqrt{3} + \tan(\alpha))p \\
p &= \frac{a\tan(\alpha)}{2(\sqrt{3} + \tan(\alpha))} \\
\therefore h &= \sqrt{3}p \\
&= \frac{\sqrt{3}a\tan\alpha}{2(\sqrt{3} + \tan(\alpha))} \\
\therefore Ar(\Delta DEC)(= \Delta = A(\alpha)) &= \frac{1}{2}\cdot\frac{a}{2}\cdot h
\end{aligned}$$

$$\boxed{A(\alpha) = \frac{\sqrt{3}a^2\tan(\alpha)}{8(\sqrt{3} + \tan(\alpha))}}$$

Note: This result is valid only for $0^\circ \leq \alpha < 90^\circ$.

Case II

$$\begin{aligned}
\alpha + \beta &= 180^\circ \\
\Delta + \Delta' &= \frac{\sqrt{3}a^2}{4}
\end{aligned}$$

From our result in Case-I,

$$\Delta' = \frac{\sqrt{3}a^2\tan(\beta)}{8(\sqrt{3} + \tan(\beta))}$$

Substituting $\beta = 180° - \alpha$,

$$\begin{aligned}\Delta' &= \frac{\sqrt{3}a^2\tan(180° - \alpha)}{8(\sqrt{3} + \tan(180° - \alpha))} \\ &= \frac{\sqrt{3}a^2\tan(\alpha)}{8(\tan(\alpha) - \sqrt{3})}\end{aligned}$$

Using the sum of Δ and Δ',

$$\begin{aligned}\Delta + \frac{\sqrt{3}a^2\tan(\alpha)}{8(\tan(\alpha) - \sqrt{3})} &= \frac{\sqrt{3}a^2}{4} \\ \Delta &= \frac{\sqrt{3}a^2}{4}\left\{1 - \frac{\tan(\alpha)}{2(\tan(\alpha) - \sqrt{3})}\right\}\end{aligned}$$

$$\boxed{A(\alpha) = \frac{\sqrt{3}a^2}{8} \cdot \frac{2\sqrt{3} - \tan(\alpha)}{\sqrt{3} - \tan(\alpha)}}$$

This result is valid only for $90° < \alpha \leq 180°$.

For $\alpha = 90°$, the shaded area is half of the total area and equal to $\frac{\sqrt{3}a^2}{8}$. This can also be shown by evaluating the limit as $\alpha \to 90°$ in any of the formulae derived in the cases above.

To sum up,

$$\boxed{A(\alpha) = \begin{cases} \dfrac{\sqrt{3}a^2\tan(\alpha)}{8(\sqrt{3} + \tan(\alpha))}, & 0° \leq \alpha < 90° \\ \dfrac{\sqrt{3}a^2}{8}, & \alpha = 90° \\ \dfrac{\sqrt{3}a^2(2\sqrt{3} - \tan(\alpha))}{8(\sqrt{3} - \tan(\alpha))}, & 90° < \alpha \leq 180° \end{cases}}$$

Problem C.1

Let $\pi(N)$ be the number of primes less than or equal to N (example: $\pi(100) = 25$). The famous prime number theorem then states (with $\sim$ meaning asymptotically equal):

$$\pi(N) \sim \frac{N}{\log(N)}$$

Proving this theorem is very hard. However, we can derive a statistical form of the prime number theorem. For this, we consider random primes which are generated as follows:

(i) Create a list of consecutive integers from 2 to N.
(ii) Start with 2 and mark every number > 2 with a probability of $\frac{1}{2}$.
(iii) Let n be the next non-marked number. Mark every number $> n$ with a probability of $\frac{1}{n}$.
(iv) Repeat (iii) until you have reached N.

All the non-marked numbers in the list are called random primes.

(a) Let q_n be the probability of n being selected as a random prime during this algorithm. Find an expression for q_n in terms of q_{n-1}.
(b) Prove the following inequality of q_n and q_{n+1}:

$$\frac{1}{q_n} + \frac{1}{n} < \frac{1}{q_{n+1}} < \frac{1}{q_n} + \frac{1}{n-1}$$

(c) Use the result from (b) to show this inequality:

$$\sum_{k=1}^{N} \frac{1}{k} < \frac{1}{q_N} < \sum_{k=1}^{N} \frac{1}{k} + 1$$

(d) With this result, derive an asymptotic expression for q_n in terms of n.
(e) Let $\tilde{\pi}(N)$ be the number of random primes less than or equal to N. Use the result from (d) to derive an asymptotic expression for $\tilde{\pi}(N)$ i.e. the prime number theorem for random primes.

Solution C.1

(a)

Let,
U_k be the event of 'k' being unmarked (or being selected as a random prime).
M_k be the event of 'k' being marked.

Clearly, either a number is marked or unmarked. So, $P(U_k) + P(M_k) = 1$.
Also, as per definition, $P(U_k) = q_k$.

Now, two cases are possible for 'n'. Either 'n-1' is marked, or it is unmarked. By the total probability theorem, we have:-

$$P(U_n) = q_n = P(U_{n-1}) \cdot P\left(\frac{U_n}{U_{n-1}}\right) + P\left(M_{n-1}\right) \cdot P\left(\frac{U_n}{M_{n-1}}\right)$$

Let's calculate each one of these one by one:-

$$\begin{aligned}
P(U_{n-1}) &= q_{n-1} \\
P\left(\frac{U_n}{U_{n-1}}\right) &= \text{Probability of 'n' being unmarked given 'n-1' is unmarked} \\
&= 1 - (\text{Probability of 'n' being marked given 'n-1' is unmarked}) \\
&= \left(1 - \frac{1}{n-1}\right) q_{n-1} \\
&= \left(\frac{n-2}{n-1}\right) q_{n-1} \\
P(M_{n-1}) &= 1 - P(U_{n-1}) \\
&= 1 - q_{n-1} \\
P\left(\frac{U_n}{M_{n-1}}\right) &= \text{Probability of 'n' being marked given 'n-1' is marked} \\
&= q_{n-1}
\end{aligned}$$

Plugging in these values,

$$\begin{aligned}
q_n &= (q_{n-1})^2 \left(\frac{n-2}{n-1}\right) + (1 - q_{n-1})\,(q_{n-1}) \\
&= q_{n-1} \left\{1 - q_{n-1}\left(1 - \frac{n-2}{n-1}\right)\right\}
\end{aligned}$$

$$\boxed{q_n = q_{n-1}\left(1 - \frac{q_{n-1}}{n-1}\right)}$$

(b)

From (a),

$$\begin{aligned}
q_{n+1} &= q_n \left(1 - \frac{q_n}{n}\right) \\
\frac{1}{q_{n+1}} &= \frac{1}{q_n} \cdot \left(\frac{n - q_n + q_n}{n - q_n}\right) \\
&= \frac{1}{q_n}\left(1 + \frac{q_n}{n - q_n}\right) \\
&= \frac{1}{q_n} + \frac{1}{n - q_n}
\end{aligned}$$

To prove:-

$$\frac{1}{q_n} + \frac{1}{n} < \frac{1}{q_n} + \frac{1}{n - q_n} < \frac{1}{q_n} + \frac{1}{n-1}$$

$$\frac{1}{n} < \frac{1}{n - q_n} < \frac{1}{n-1}$$

$$n > n - q_n > n - 1$$

$$0 < q_n < 1$$

Which is true (by definition) because q_n is a probability.
Hence proved,

$$\boxed{\frac{1}{q_n} + \frac{1}{n} < \frac{1}{q_{n+1}} < \frac{1}{q_n} + \frac{1}{n-1}}$$

(c)

From (b),

$$\frac{1}{q_n} + \frac{1}{n} < \frac{1}{q_{n+1}} < \frac{1}{q_n} + \frac{1}{n-1}$$

Putting n=N-1,

$$\frac{1}{q_{N-1}} + \frac{1}{N-1} < \frac{1}{q_N} < \frac{1}{q_{N-1}} + \frac{1}{N-2}$$

Similarly

$$\frac{1}{N-2} + \frac{1}{q_{N-2}} < \frac{1}{q_{N-1}} < \frac{1}{q_{N-2}} + \frac{1}{N-3}$$

$\therefore$ We can write,

$$\cdots < \frac{1}{N-2} + \frac{1}{N-1} + \frac{1}{q_{N-2}} < \frac{1}{q_N} < \frac{1}{q_{N-2}} + \frac{1}{N-2} + \frac{1}{N-3} < \cdots$$

We can keep substituting this till we arrive at q_2. We know, $q_2 = 1$.
Upon substituting it simplifies to:-

$$\boxed{\sum_{k=1}^{N} \frac{1}{k} < \frac{1}{q_N} < \sum_{k=1}^{N} \frac{1}{k} + 1}$$

(d)

We can write $\sum_{k=1}^{n} \frac{1}{k} = H_n$ where H_k is the 'k'-th harmoinc number.

$$\therefore H_n < \frac{1}{q_n} < H_{n+1}$$

$$\frac{1}{H_n} > q_n > \frac{1}{H_{n+1}}$$

We know $H_n \sim \log(N) + \gamma$ where γ is the Euler-Mascheroni constant.

$$\therefore \frac{1}{\log(n) + \gamma} > q_n > \frac{1}{\log(N) + \gamma + 1}$$

As $n \to \infty$, both L.H.S. and R.H.S. approach $\frac{1}{\log(N)}$ as constant terms can be ignored.

$$\therefore \boxed{q_n \sim \frac{1}{\log(N)}}$$

(e)

As q_n is the probability of n being a random prime, the number of random primes less than or equal to N will just be the sum of probabilities:-

$$\tilde{\pi}(N) = \sum_{k=2}^{N} q_k$$

From (d), this is equivalent to:

$$\boxed{\tilde{\pi}(N) = \sum_{k=2}^{N} \frac{1}{\log(k)}}$$

Problem C.2

(a)

What are the values of H_n, F_n and $\mathbb{F}_n$ for $n = 1, 2, 3$?

Solution

We know $H_n = \sum_{k=1}^{n} \frac{1}{k}$.

$$\begin{aligned}
H_1 &= \sum_{k=1}^{1} \frac{1}{k} = 1 \\
H_2 &= \sum_{k=1}^{2} \frac{1}{k} = 1 + \frac{1}{2} = \frac{3}{2} \\
H_3 &= \sum_{k=1}^{3} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}
\end{aligned}$$

We know $F_{n+2} = F_{n+1} + F_n$.

$$\begin{aligned}
F_1 &= 1(\text{given}) \\
F_2 &= F_1 + F_0 = 1 + 0 = 1 \\
F_3 &= F_2 + F_1 = 1 + 1 = 2
\end{aligned}$$

We know $\mathbb{F}_n = \sum_{k=1}^{n} \frac{1}{F_k}$.

$$\begin{aligned}
\mathbb{F}_1 &= \sum_{k=1}^{1} \frac{1}{F_k} = \frac{1}{F_1} = 1 \\
\mathbb{F}_2 &= \sum_{k=1}^{2} \frac{1}{F_k} = \frac{1}{F_1} + \frac{1}{F_2} = 1 + 1 = 2 \\
\mathbb{F}_3 &= \sum_{k=1}^{3} \frac{1}{F_k} = \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} = 1 + 1 + \frac{1}{2} = \frac{5}{2}
\end{aligned}$$

(b)

Determine the hyperharmonic number $H_3^{(10)}$ (Tip: use Equation 4) and $F_2^{(3)}$.

Solution

We know that $H_n^{(r)} = \sum_{t=1}^{n} \left(\begin{array}{c} n+r-t-1 \\ r-1 \end{array} \right) \frac{1}{t}$
Putting $n = 3, r = 10$ in this,

$$\begin{aligned} H_3^{(10)} &= \sum_{t=1}^{3} \left(\begin{array}{c} 3+10-t-1 \\ 10-1 \end{array} \right) \frac{1}{t} \\ &= \sum_{t=1}^{3} \left(\begin{array}{c} 12-t \\ 9 \end{array} \right) \frac{1}{t} \\ &= \left(\begin{array}{c} 11 \\ 9 \end{array} \right) + \left(\begin{array}{c} 10 \\ 9 \end{array} \right) \frac{1}{2} + \left(\begin{array}{c} 9 \\ 9 \end{array} \right) \frac{1}{3} \\ &= 55 + 10 \cdot \frac{1}{2} + \frac{1}{3} \\ &= \frac{181}{3} \end{aligned}$$

We know that $F_n^{(r)} = \sum_{k=0}^{n} F_k^{(r-1)}$.

$$\begin{aligned} F_2^{(3)} &= \sum_{k=0}^{2} F_k^{(2)} \\ &= F_0^{(2)} + F_1^{(2)} + F_2^{(2)} \\ &= 0 + 1 + F_2^{(2)} \\ F_2^{(2)} &= \sum_{k=0}^{2} F_k^{(1)} \\ &= F_0^{(1)} + F_1^{(1)} + F_2^{(1)} \\ &= 0 + 1 + F_2^{(1)} \\ F_2^{(1)} &= \sum_{k=0}^{2} F_k^{(0)} \\ &= F_0^{(0)} + F_1^{(0)} + F_2^{(0)} \\ &= 0 + 1 + F_2 \\ &= 2 \\ F_2^{(2)} &= 0 + 1 + 2 = 3 \\ F_2^{(3)} &= 0 + 1 + 3 = 4 \end{aligned}$$

$$\therefore \boxed{H_3^{(10)} = \frac{181}{3}, F_2^{(3)} = 4}$$

(c)

Use the definition of $x^{\underline{m}}$ to simplify the following fraction: $\frac{x^{\underline{m+1}}-x^{\underline{m}}}{x^{\underline{m}}+x^{\underline{m+1}}}$.

Solution

It can be easily shown that:

$$x^{\underline{m+1}} = x^{\underline{m}} \cdot (x-m)$$

$$\therefore \frac{x^{\underline{m+1}}-x^{\underline{m}}}{x^{\underline{m}}+x^{\underline{m+1}}} = \frac{\cancel{x^{\underline{m}}}(x-m)-\cancel{x^{\underline{m}}}}{\cancel{x^{\underline{m}}}+\cancel{x^{\underline{m}}}(x-m)} = \frac{x-m-1}{x-m+1}$$

(d)

Present the proof of Theorem 1 step-by-step by applying Equation 6.

Solution

Theorem 1:-

$$\sum_{k=0}^{n-1} \mathbb{F}_k = n\mathbb{F}_n - \sum_{k=0}^{n-1} \frac{k+1}{F_{k+1}}$$

Equation 6:-

$$\sum_a^b u(x)\Delta v(x)\delta_x = u(x)v(x)\big|_a^{b+1} - \sum_a^b Ev(x)\Delta u(x)\delta_x$$

Taking, $u(k) = \mathbb{F}_k$ and $\delta v(k) = 1$,

$$\begin{aligned}
\delta u(k) &= \mathbb{F}_{k+1} - \mathbb{F}_k \\
&= \sum_{r=1}^{k+1} \frac{1}{F_r} - \sum_{r=1}^{k} \frac{1}{F_r} \\
&= \frac{1}{F_{k+1}} + \cancel{\sum_{r=1}^{k} \frac{1}{F_r}} - \cancel{\sum_{r=1}^{k} \frac{1}{F_r}} \\
&= \frac{1}{F_{k+1}} \\
\delta v(k) &= 1 \\
\implies v(k) &= k \\
Ev(k) &= v(k+1) = k+1
\end{aligned}$$

Applying Equation (6) and substituting our values,

$$\begin{aligned}\sum_{k=0}^{n-1} \mathbb{F}_k &= k\mathbb{F}_k\big|_0^n - \sum_{k=0}^{n-1} v(k+1)\Delta\mathbb{F}_k \\ &= n\mathbb{F}_n - \sum_{k=0}^{n-1} \frac{k+1}{F_{k+1}}\end{aligned}$$

Hence proved.

(e)

Show that $\mathbb{F}_n^{(r)} - \mathbb{F}_{n-2}^{(r)} = \mathbb{F}_n^{(r-1)} + \mathbb{F}_{n-1}^{(r-1)}$.

Solution

We know that $\mathbb{F}_n^{(r)} = \mathbb{F}_n^{(r-1)} + \mathbb{F}_{n-1}^{(r)}$.
Putting $n-1 \to n$,

$$\mathbb{F}_{n-1}^{(r)} = \mathbb{F}_{n-1}^{(r-1)} + \mathbb{F}_{n-2}^{(r)}$$

$$\begin{aligned}L.H.S. &= \mathbb{F}_n^{(r)} - \mathbb{F}_{n-2}^{(r)} \\ &= \mathbb{F}_n^{(r-1)} + \mathbb{F}_{n-1}^{(r-1)} + \cancel{\mathbb{F}_{n-2}^{(r)}} - \cancel{\mathbb{F}_{n-2}^{(r)}} \\ &= R.H.S.\end{aligned}$$

Hence proved.

(f)

Determine the Euclidean norm of the circulant matrix Circ (1, 1, 0, 0).

Solution

From the definition of circulant matrix,

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

From the definition of Euclidean norm,

$$\begin{aligned}\|C\|_E &= \left(\sum_{i=1}^{4} \sum_{j=1}^{4} |c_{ij}|^2 \right)^{\frac{1}{2}} \\ &= \sqrt{8}\end{aligned}$$

$$\boxed{\|C\|_E = 2\sqrt{2}}$$

(g)

Show that for $u(k) = \mathbb{F}_k^2$ we get $\Delta u(k) = \frac{1}{F_{k+1}} \left(2\mathbb{F}_k + \frac{1}{F_{k+1}} \right)$.

Solution

$$\begin{aligned}\Delta u(k) &= u(k+1) - u(k) \\ &= \mathbb{F}_{k+1}^2 - \mathbb{F}_k^2\end{aligned}$$

We know,

$$\begin{aligned}\mathbb{F}_{k+1} &= \mathbb{F}_k + \frac{1}{F_{k+1}} \\ \therefore \mathbb{F}_{k+1}^2 &= \mathbb{F}_k^2 + 2\frac{\mathbb{F}_k}{F_{k+1}} + \frac{1}{F_{k+1}^2}\end{aligned}$$

$$\boxed{\mathbb{F}_{k+1}^2 - \mathbb{F}_k^2 = \Delta u(k) = \frac{1}{F_{k+1}} \left(2\mathbb{F}_k + \frac{1}{F_{k+1}} \right)}$$

(h)

Use the theorems from the article to prove the following identity:

$$\sum_{k=1}^{n-1} k^{\underline{m}} (\mathbb{F}_k)^2 = \frac{n^{\underline{m+1}}}{m+1} \mathbb{F}_n^2 - \sum_{k=0}^{n-1} \frac{(k+1)^{\underline{m+1}}}{(m+1)F_{k+1}} \left(2\mathbb{F}_k + \frac{1}{F_{k+1}} \right)$$

Solution

Comparing this to Equation 6 from the article,

$$u(k) = \mathbb{F}_k^2, \Delta v(k) = k^{\underline{m}}$$

$$\therefore v(k) = \sum k^{\underline{m}} \delta_k = \frac{k^{\underline{m+1}}}{m+1}$$

From part (g) of this question, we also know that:

$$\Delta u(k) = \frac{1}{F_{k+1}} \left(2\mathbb{F}_k + \frac{1}{F_{k+1}} \right)$$

Using Equation (6), we have,

$$\begin{aligned}\sum_{k=1}^{n-1} k^{\underline{m}} (\mathbb{F}_k)^2 &= \frac{k^{\underline{m+1}}}{m+1} \left. \mathbb{F}_k^2 \right|_1^n - \sum_{k=1}^{n-1} \frac{(k+1)^{\underline{m+1}}}{(m+1)F_{k+1}} \left(2\mathbb{F}_k + \frac{1}{F_{k+1}} \right) \\ &= \frac{n^{\underline{m+1}}}{m+1} \mathbb{F}_n^2 - \frac{1^{\underline{m+1}}}{m+1} \mathbb{F}_1^2 - \sum_{k=1}^{n-1} \frac{(k+1)^{\underline{m+1}}}{(m+1)F_{k+1}} \left(2\mathbb{F}_k + \frac{1}{F_{k+1}} \right)\end{aligned}$$

We can show $\Delta\mathbb{F}_0^2 = 1 = \mathbb{F}_1^2$.

$$\therefore \frac{1^{\underline{m+1}}}{m+1}\mathbb{F}_1^2 = \frac{(0+1)^{\underline{m+1}}}{m+1}\Delta\mathbb{F}_0^2$$

Substituting this in the above relation, we get,

$$\boxed{\sum_{k=1}^{n-1} k^{\underline{m}}(\mathbb{F}_k)^2 = \frac{n^{\underline{m+1}}}{m+1}\mathbb{F}_n^2 - \sum_{k=0}^{n-1} \frac{(k+1)^{\underline{m+1}}}{(m+1)F_{k+1}}\left(2\mathbb{F}_k + \frac{1}{F_{k+1}}\right)}$$

Hence proved.

(i)

Use Equation 1 and Theorem 5 to show the following:

$$\sum_{k=0}^{n-1} \frac{\mathbb{F}_k}{k+1} = \mathbb{F}_n + \sum_{k=0}^{n-1}\left(\frac{\mathbb{F}_n H_k}{n} - \frac{H_{k+1}}{F_{k+1}}\right)$$

Solution

From Theorem (5) we have,

$$\sum_{k=0}^{n-1} \frac{\mathbb{F}_k}{k+1} = H_n\mathbb{F}_n - \sum_{k=0}^{n-1} \frac{H_{k+1}}{F_{k+1}}$$

From Equation (1) we have,

$$nH_n - n = \sum_{k=1}^{n-1} H_k = \sum_{k=0}^{n-1} \frac{H_k}{n} + 1$$

$$\begin{aligned} H_n\mathbb{F}_n - \sum_{k=0}^{n-1} \frac{H_{k+1}}{F_{k+1}} &= \left(\sum_{k=0}^{n-1} \frac{H_k}{n} + 1\right)\mathbb{F}_n - \sum_{k=0}^{n-1} \frac{H_{k+1}}{F_{k+1}} \\ &= \mathbb{F}_n + \sum_{k=0}^{n-1} \frac{\mathbb{F}_n H_k}{n} - \sum_{k=0}^{n-1} \frac{H_{k+1}}{F_{k+1}} \\ &= \mathbb{F}_n + \sum_{k=0}^{n-1}\left(\frac{\mathbb{F}_n H_k}{n} - \frac{H_{k+1}}{F_{k+1}}\right) \end{aligned}$$

Hence proved.