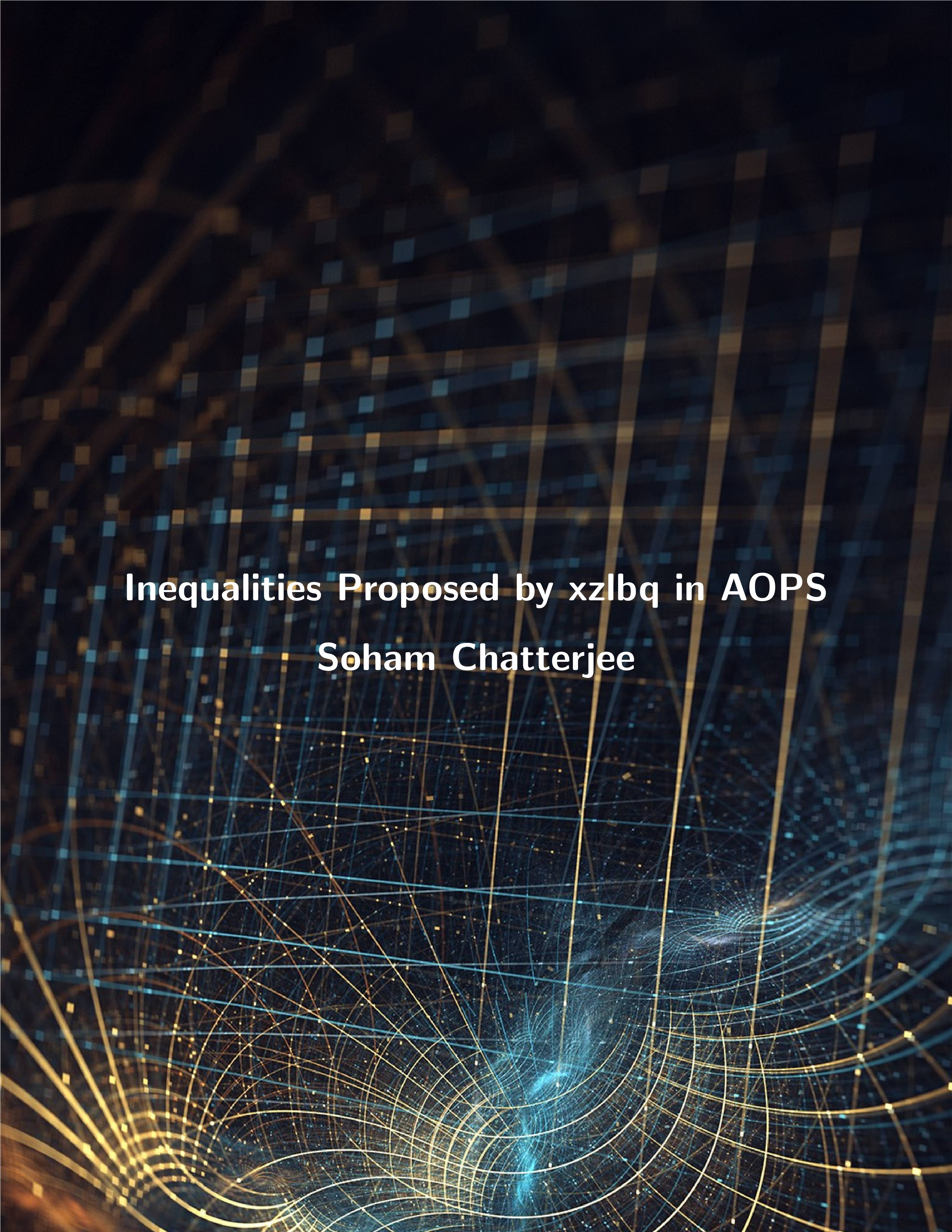

Inequalities Proposed by xzlbq in AOPS

Soham Chatterjee

Inequalities Proposed by xzlbq in AOPS

Soham Chatterjee

sohamchatterjee999@gmail.com

CONTENTS

CHAPTER 1

3–VARIABLE INEQUALITIES

1.1 Solved Problems

1. Let $a, b, c \geq 0$ Prove that

$$\sum \sqrt{\frac{a}{b+c}} \leq \frac{3}{2}\sqrt{\frac{4}{3}+\frac{4}{9}\sum \frac{a}{b+c}}$$

Solution: After squaring,

$$\implies 2\sum \sqrt{\frac{ab}{(b+c)(a+c)}} \leq 3$$

By AM-GM,

$$LHS \leq \sum \left(\frac{a}{a+c}+\frac{b}{b+c}\right) = 3$$

[Solved by spanferkel]

2. Let a, b, c be positive real numbers. and $a+b^2+c^3=1$, Prove that the following inequality holds

$$(1-a)(1-b^2)(1-c^3) \geq 8ab^2c^3$$

Solution: Let $x=a, y=b^2, z=c^3$, ineq becomes

$$x+y+z=1,\ (1-x)(1-y)(1-z) \geq 8xyz$$

i.e.

$$(x+y)(y+z)(z+x) \geq 8xyz$$

which is true

[Solved by kuing]

3. Let a, b, c be real positive numbers and $a+b+c=1$, Prove that

$$2 \geq \frac{(1-a)^3}{3a+b+c}+\frac{(1-b)^3}{3b+c+a}+\frac{(1-c)^3}{3c+a+b} \geq \frac{8}{15}$$

Solution: By Cauchy, we have

$$\sum \frac{(b+c)^4}{(b+c)(b+c+3a)} \geq \frac{\left(\sum(b+c)^2\right)^2}{\sum(b+c)(b+c+3a)} \geq \frac{15}{8}.$$

Denote $\sum a^2 = x$, $\sum ab = y$. Then the inequality is equivalent to

$$9x^2 + 6xy \geq 17y^2,$$

which is obvious, as $\sum a^2 \geq \sum ab$ with equality if and only if $a = b = c$.
We have $3a + b + c \geq a + b + c$ and so on. Hence

$$\sum \frac{(b+c)^3}{b+c+3a} \leq \sum \frac{(b+c)^3}{a+b+c} \leq 2 \Leftrightarrow 3\sum(a^2b + ab^2) + 12abc \geq 0,$$

which is true. Equality holds if and only if $a = b = 0$, $c = 1$ and permutations.

[Solved by hedeng123]

4. Let $a, b, c > 0$,such that $a + b + c = 1$,prove that:

$$3 \geq \sqrt{a + b^2 + c^2} + \sqrt{b + c^2 + a^2} + \sqrt{c + a^2 + b^2} \geq \sqrt{5}$$

Solution: For the left :Use Cauchy-Schwarz ineq, we have:

$$\begin{aligned}\sqrt{a + b^2 + c^2} + \sqrt{b + c^2 + a^2} + \sqrt{c + a^2 + b^2} &\leq \sqrt{3(a + b + c + 2(a^2 + b^2 + c^2))} \\ &\leq \sqrt{3(a + b + c + 2(a + b + c)^2)} = 3\end{aligned}$$

Equality holds when (a,b,c)=(1,0,0)
For the right : Use Mincowski ineq, we have:

$$\begin{aligned}&(\sqrt{a + b^2 + c^2} + \sqrt{b + c^2 + a^2} + \sqrt{c + a^2 + b^2})^2 \\ &= a + b + c + 2(a^2 + b^2 + c^2) + 2\sum \sqrt{(a + b^2 + c^2)(b + c^2 + a^2)} \\ &= 1 + 2(a^2 + b^2 + c^2) + 2\sum \sqrt{(a^2 + b^2 + c^2 + a(b + c))(b^2 + c^2 + a^2 + b(c + a))} \\ &\geq 1 + 2(a^2 + b^2 + c^2) + 2\sum \left(a^2 + b^2 + c^2 + \sqrt{ab(c + a)(c + b)}\right) \\ &= 1 + 8(a^2 + b^2 + c^2) + 2\sum \sqrt{ab(c + a)(c + b)}\end{aligned}$$

We need to prove:

$$\begin{gathered}1 + 8(a^2 + b^2 + c^2) + 2\sum \sqrt{ab(c + a)(c + b)} \geq 5 \\ \Longleftrightarrow 2(a^2 + b^2 + c^2) + \sum \sqrt{ab(c + a)(c + b)} \geq 4(ab + bc + ca)\end{gathered}$$

By AM-GM ineq , we have:

$$\sum \sqrt{ab(c + a)(c + b)} \geq 2\sum \sqrt{abc\sqrt{ab}} \geq 6\sqrt[3]{a^2b^2c^2} \geq \frac{18abc}{a + b + c}$$

So this ineq can be rewriten:

$$a^2 + b^2 + c^2 + \frac{9abc}{a + b + c} \geq 2(ab + bc + ca)$$

It's Schur inequality
Equality holds when $a = b = c = \frac{1}{3}$

[Solved by minhhoang]

5. Let a,b,c be real positive numbers and $a+b+c=1$, Prove that

$$\frac{9}{2}(2-a)(2-b)(2-c) \geq (3-a)(3-b)(3-c)+(1-a)(1-b)(1-c) \geq \frac{104}{25}(2-a)(2-b)(2-c)$$

Solution:

$$\begin{aligned}
&9\prod(2-a)-2\prod(3-a)-2\prod(1-a)\\
=&9\prod(a+2b+2c)-2\prod(2a+3b+3c)-2\prod(b+c)\\
=&25abc+10\sum ab(a+b)
\end{aligned}$$

$$\begin{aligned}
&25\prod(3-a)+25\prod(1-a)-104\prod(2-a)\\
=&25\prod(2a+3b+3c)+25\prod(b+c)-104\prod(a+2b+2c)\\
=&22(a^3+b^3+c^3-3abc)+6(2a^3+2b^3+2c^3-\sum ab(a+b))
\end{aligned}$$

[Solved by kuing]

6. $x,y,z>0$, prove that

$$\sqrt{1+4\,\frac{xyz}{(y+z)\,(z+x)\,(x+y)}} \geq \prod_{cyc} \frac{y+z-x}{\sqrt{(y+z)\,(x+y+z)}}$$

Solution: We need to prove that

$$\sqrt{(x+y+z)\sum_{cyc}(x^2y+x^2z+2xyz)} \geq \sum_{cyc}(x+y-z)\sqrt{(x+z)(y+z)}$$

By C-S

$$\sqrt{(x+y+z)\sum_{cyc}(x^2y+x^2z+2xyz)} = \sqrt{(x+y+z)\sum_{cyc}x(y+z)^2} \geq \sum_{cyc}x(y+z) = 2(xy+xz+yz)$$

Thus, it remains to prove that

$$2(xy+xz+yz) \geq \sum_{cyc}(x+y-z)\sqrt{(x+z)(y+z)}$$

or

$$\sum_{cyc}(x+y-z)\left(\sqrt{x+z}-\sqrt{y+z}\right)^2 \geq \sum_{cyc}(x+y-z)(x+z+y+z)-4(xy+xz+yz)$$

or

$$\sum_{cyc}\frac{(x-y)^2(x+y-z)}{\left(\sqrt{x+z}+\sqrt{y+z}\right)^2} \geq 0$$

Let $x \geq y \geq z$.

Hence,

$$\begin{aligned}
&\sum_{cyc} \frac{(x-y)^2(x+y-z)}{\left(\sqrt{x+z}+\sqrt{y+z}\right)^2} \\
\geq\ & \frac{(x-z)^2(x+z-y)}{\left(\sqrt{x+y}+\sqrt{y+z}\right)^2} + \frac{(y-z)^2(y+z-x)}{\left(\sqrt{x+y}+\sqrt{x+z}\right)^2} \\
\geq\ & \frac{(y-z)^2(x-y)}{\left(\sqrt{x+y}+\sqrt{y+z}\right)^2} + \frac{(y-z)^2(y-x)}{\left(\sqrt{x+y}+\sqrt{x+z}\right)^2} \\
=\ & (y-z)^2(x-y)\left(\frac{1}{\left(\sqrt{x+y}+\sqrt{y+z}\right)^2} - \frac{1}{\left(\sqrt{x+y}+\sqrt{x+z}\right)^2}\right) \geq 0
\end{aligned}$$

[Solved by arqady]

7. Let x, y, z are positive real numbers, prove that:

$$\frac{z}{x}+\frac{z}{y}+\frac{x+y}{z}+\frac{64}{3}\frac{xyz}{(y+z)(z+x)(x+y)} \geq \frac{20}{3}$$

Solution: Let $x = tz,\ y = uz$ and $p = t+u, q = tu$, that we have $0 < q \leq \dfrac{p^2}{4}$ and inequality becomes

$$f(q) = (3p^2-20p+64)\left(q+\frac{p(3p^2-14p-20)}{2(3p^2-20p+64)}\right)^2 + \frac{p^2(-9p^2+84p+92)(p-2)^2}{4(3p^2-20p+64)} \geq 0$$

Easy to prove that

$$3p^2-20p+64 > 0$$

so

(a) If

$$-9p^2+84p+92 \geq 0$$

then ineq clear holds;

(b) If

$$-9p^2+84p+92 < 0$$

we have

$$3p^2-14p-20 > 28p+\frac{92}{3}-14p-20 = 14p+\frac{32}{3} > 0$$

so we get

$$f(q) > f(0)$$

, but

$$f(0) = 3p^2(p+1) > 0$$

So ineq holds.

[Solved by kuing]

8. Let x, y, z are positive real numbers, prove that:

$$\frac{y+z}{x}+\frac{z+x}{y}+\frac{x+y}{z} \geq 3+2\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)$$

Solution 1:

$$\begin{aligned}
&\sum \frac{y+z}{x} \ge 3+2\sum \frac{x}{y+z}\\
\iff &\sum \left(\frac{y+z}{x}-1-\frac{2x}{y+z}\right) \ge 0\\
\iff &\sum \frac{(x+y+z)(y+z-2x)}{x(y+z)} \ge 0\\
\iff &\sum \frac{y-x+z-x}{x(y+z)} \ge 0\\
\iff &\sum \left(\frac{y-x}{x(y+z)}+\frac{x-y}{y(z+x)}\right) \ge 0\\
\iff &\sum \frac{(x-y)^2 z}{xy(y+z)(z+x)} \ge 0
\end{aligned}$$

[Solved by kuing]

Solution 2:

$$\begin{aligned}
&\frac{y+z}{x}+\frac{z+x}{y}+\frac{x+y}{z} \ge 3+2\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)\\
\iff &\begin{array}{l}(x-y)^2(y-z)^2(z-x)^2+4\sum y^2z^2(x-y)(x-z)\\ \quad +2xyz\sum x(x-y)(x-z)+3xyz\sum (y+z)(x-y)(x-z) \ge 0\end{array}\\
\iff &\sum_{sym} x^4y^2+\sum_{sym} x^3y^3 \ge \sum_{sym} x^3y^2z+\sum_{sym} x^2y^2z^2
\end{aligned}$$

obvious true.

[Solved by hedeng123]

9. Let x,y,z are positive real numbers,prove that:

$$\frac{1}{3} \ge \frac{1}{(y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)}+\frac{1}{\left(\frac{1}{y}+\frac{1}{z}\right)(x+y+z)}$$

Solution:

$$\left(y^2-yz+z^2\right)x^2+(y+z)\left(z^2-3yz+y^2\right)x+yz\left(y^2-yz+z^2\right) \ge 0$$

[Solved by xzlbq]

10. Let x, y, z are positive real numbers,and $x+y+z=1, t \ge 1$,prove that:

$$\frac{x^2}{x^2-2x+t}+\frac{y^2}{y^2-2y+t}+\frac{z^2}{z^2-2z+t} \ge \frac{3}{9t-5}$$

Solution: When $0<x<1, t \ge 1$, we have

$$\frac{x^2}{x^2-2x+t} \ge \frac{(54t-18)x-9t+1}{(9t-5)^2} \iff \frac{(3x-1)^2((2-6t)x+9t^2-t)}{(9t-5)^2(x^2-2x+t)} \ge 0$$

so we have

$$\frac{x^2}{x^2-2x+t}+\frac{y^2}{y^2-2y+t}+\frac{z^2}{z^2-2z+t} \ge \frac{(54t-18)(x+y+z)-27t+3}{(9t-5)^2}=\frac{3}{9t-5}$$

[Solved by kuing]

11. Let $x,y,z\geq 0$,$t>0$,and $x+y+z=txyz$,prove that:

$$xy+yz+zx+x+y+z\geq \frac{3(3+\sqrt{3t})}{t}$$

Solution: The inequality is equivalent to:

$$(x+y+z)^2+(xy+yz+zx)(x+y+z)\geq 9xyz+3\sqrt{3xyz(x+y+z)}$$

we have :

$$xy+yz+zx\geq \sqrt{3xyz(x+y+z)}$$

then let's prove that

$$(x+y+z)^2+(xy+yz+zx)(x+y+z)\geq 9xyz+3(xy+yz+zx)$$

but:

$$(xy+yz+zx)(x+y+z)\geq 9xyz$$

and

$$(x+y+z)^2\geq 3(xy+yz+zx)$$

[Solved by anonyme93]

1.2 Unsolved Problems

1. Let a,b,c be a positive real numbers such that $abc=1$ Prove that

$$(a^3+1)(b^3+1)(c^3+1)\geq \frac{(a+b)^2(b+c)^2(c+a)^2}{(a+b+c)(bc+ca+ab)-1}$$

2. Let $a,b,c\geq 0$ Prove this

$$\sum\sqrt{\frac{a}{b+c}}\geq\sqrt{9-3\sum\frac{b+c}{2a+b+c}}$$

3. Let a,b,c be positive real numbers,and a+b+c=3. Prove that the following inequality holds

$$\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}\geq\left(\frac{2}{3}+\frac{\sqrt{2}}{3}\right)\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)-\frac{1}{4}-\frac{\sqrt{2}}{2}$$

4. Let a,b,c be real positive numbers and $a+b+c=1$, find $f(t)$ and $g(t)$

$$f(t)\geq\frac{(1-a)^t}{ta+b+c}+\frac{(1-b)^t}{tb+c+a}+\frac{(1-c)^t}{tc+a+b}\geq g(t)$$

5. Let a,b,c be real positive numbers and $a+b+c=1$, Prove that

$$\frac{11}{18}\geq\frac{(1-a)(2-b)}{(3-c)^2}+\frac{(1-b)(2-c)}{(3-a)^2}+\frac{(1-c)(2-a)}{(3-b)^2}\geq\frac{15}{32}$$

6. Let a,b,c be real positive numbers and $a+b+c=1$, Prove that

$$\left(\frac{5\sqrt{6}}{6}+\frac{\sqrt{2}}{6}\right)\sqrt{(2-a)(2-b)(2-c)} \geq \sqrt{(1-a)(1-b)(1-c)}$$
$$+\sqrt{(3-a)(3-b)(3-c)} \geq \frac{3\sqrt{2}}{2}\sqrt{(2-a)(2-b)(2-c)}$$

7. $x,y,z>0$, $x+y>\frac{z}{5}$, $y+z>\frac{x}{5}$, $z+x>\frac{y}{5}$, prove that

$$\frac{y+z-x}{\sqrt{y+z}}+\frac{z+x-y}{\sqrt{z+x}}+\frac{x+y-z}{\sqrt{x+y}}>0$$

8. Let $f(x,y,z)$ fits as $f(x,y,z)=f(x,z,y)$ or $f(x,y,z)=f(z,y,x)$ or $f(x,y,z)=f(y,x,z)$ then

$$\begin{vmatrix} f(x,y,z) & f(y,z,x) & f(z,x,y) \\ f(z,y,x) & f(x,z,y) & f(y,x,z) \\ f(x,y,z)f(z,y,x) & f(y,z,x)f(x,z,y) & f(z,x,y)f(y,x,z) \end{vmatrix} \geq 0$$

9. Let x,y,z are positive real numbers, $a,b,c\geq 0$, and $a\leq b+c$, prove that:

$$\frac{1}{(by+cz)(\frac{a}{x}+\frac{b}{y}+\frac{c}{z})}+\frac{1}{(\frac{b}{y}+\frac{c}{z})(ax+by+cz)}\leq\frac{2}{(b+c)(a+b+c)}$$

10. Let $a,b,c>0$,

$$\sqrt{\frac{2a}{b+c}}+\sqrt{\frac{2b}{c+a}}+\sqrt{\frac{2c}{a+b}}\leq\frac{(a+b+c)^2}{bc+ca+ab}$$

11. Let $a,b,c>0$,

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}\leq\frac{\sqrt{6}}{2}\frac{a+b+c}{\sqrt{ab+bc+ca}}$$

12. Let a,b,c be positive real numbers. Prove that

$$\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{a}}+\sqrt{\frac{a+b}{b+c}}+\sqrt{\frac{b+c}{c+a}}+\sqrt{\frac{c+a}{a+b}}\leq\sqrt{\frac{2\,a+b+c}{a}}+\sqrt{\frac{2\,b+c+a}{b}}+\sqrt{\frac{2\,c+a+b}{c}}$$

13. Prove that:

$$0\leq(2304\,y+2304\,z)\,x^3+\left(-1154\,y^2-1154\,z^2-764\,yz\right)x^2$$
$$-2\,(y+z)\,(577\,z^2-770\,yz+577\,y^2)\,x+2\,(2\,y^2-959\,yz+2\,z^2)\,(y-z)^2$$

14. $x,y,z>0$, prove that

$$\sqrt{x\,(y+z)}+\sqrt{y\,(z+x)}+\sqrt{z\,(x+y)}\leq\frac{3}{2}\sqrt{3}\sqrt{\frac{(y+z)\,(z+x)\,(x+y)}{x+y+z}}$$

15. $x,y,z>0$, prove

$$\frac{y\sqrt{(z+x)\,(x+y)}z}{x^2\,(y+z)}+\frac{z\sqrt{(x+y)\,(y+z)}x}{y^2\,(z+x)}+\frac{x\sqrt{(y+z)\,(z+x)}y}{z^2\,(x+y)}\geq 3$$

16. Let x,y,z are positive real numbers, prove that:

$$\frac{z}{x}+\frac{z}{y}+\frac{x+y}{z}\geq 1+\frac{y+z}{z+x}+\frac{z+x}{x+y}+\frac{x+y}{y+z}$$

17. Let x,y,z are positive real numbers, prove that:

$$\frac{1}{3}\geq\frac{1}{(2y+3z)(\frac{1}{x}+\frac{2}{y}+\frac{3}{z})}+\frac{1}{(\frac{2}{y}+\frac{3}{z})(x+2y+3z)}$$

CHAPTER 2

4–VARIABLE INEQUALITIES

2.1 Solved Problems

1. Let a,b,c be positive numbers, $t \geq 0$prove that:

$$\frac{a+t}{b+t}+\frac{b+t}{c+t}+\frac{c+t}{a+t} \leq \frac{a}{b}+\frac{b}{c}+\frac{c}{a}$$

Solution 1: It's equivalent to

$$(t^3+(a+b+c)t^2)\sum_{cyc}(a^2c-abc)+t\sum_{cyc}(a^3c^2-a^2b^2c) \geq 0$$

[solved by arqady]

Solution 2: Without loss of generality, we can assume that $c=\min\{a,b,c\}$, and we have

$$\begin{aligned}&\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-\left(\frac{a+t}{b+t}+\frac{b+t}{c+t}+\frac{c+t}{a+t}\right)\\ &=\left(\frac{1}{ab}-\frac{1}{(a+t)\,(b+t)}\right)(a-b)^2+\left(\frac{1}{ac}-\frac{1}{(a+t)\,(c+t)}\right)(a-c)\,(b-c)\end{aligned}$$

so, ineq clear true.

[solved by kuing]

Solution 3: Let

$$f(t)=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-\left(\frac{a+t}{b+t}+\frac{b+t}{c+t}+\frac{c+t}{a+t}\right)$$

where $(t \geq 0)$. Then we have

$$f'(t)=\frac{a+t}{(b+t)^2}+\frac{b+t}{(c+t)^2}+\frac{c+t}{(a+t)^2}-\frac{1}{a+t}-\frac{1}{b+t}-\frac{1}{c+t}$$

easy to prove that

$$\frac{a+t}{(b+t)^2}+\frac{b+t}{(c+t)^2}+\frac{c+t}{(a+t)^2} \geq \frac{1}{a+t}+\frac{1}{b+t}+\frac{1}{c+t}$$

Thus, we get $f'(t) \geq 0$, so we have

$$f(t) \geq f(0) = 0$$

and get the result.

[solved by kuing]

2. Let $x, y, z > 0$,

$$f(t) = \frac{y+z}{(y+z)^2 + tx(x+y+z)} + \frac{z+x}{(z+x)^2 + ty(x+y+z)} + \frac{x+y}{(x+y)^2 + tz(x+y+z)} - \frac{18}{(4+3t)(x+y+z)}$$

if $t \geq 2$, then: $f(t) \geq 0$ else if $\frac{5}{3} \geq t \geq \frac{52}{57} - \frac{16}{57}\sqrt{7}$ then: $f(t) \leq 0$

Solution:

$$f(t) = \frac{y+z}{(y+z)^2 + tx(x+y+z)} + \frac{z+x}{(z+x)^2 + ty(x+y+z)} + \frac{x+y}{(x+y)^2 + tz(x+y+z)} - \frac{18}{(4+3t)(x+y+z)}$$

if $t \geq 2$, then: $f(t) \geq 0$

$$\begin{aligned} f(t) \geq 0 \iff & (6t^2 - 10t)\sum(x^4(x-y)(x-z)) + (-33t + 13t^2 + 3t^3 + 4)\sum(x^3(y+z)(x-y)(x-z)) \\ & + (6 - 32t - 13t^2 + 15t^3)(x-y)^2(y-z)^2(z-x)^2 \\ & + (8 - 36t - 80t^2 + 48t^3)\sum(y^2z^2(x-y)(x-z)) \\ & + (-146t + 12t^2 + 36t^3 + 24)xyz\sum(x(x-y)(x-z)) \\ & + (12 - 63t - 36t^2 + 36t^3)xyz\sum((y+z)(x-y)(x-z)) \geq 0 \end{aligned}$$

easy prove $t \geq 2$

$$\begin{aligned} & 6t^2 - 10t \geq 0 \\ & -33t + 13t^2 + 3t^3 + 4 \geq 0 \\ & 6 - 32t - 13t^2 + 15t^3 \geq 0 \\ & 8 - 36t - 80t^2 + 48t^3 \geq 0 \\ & -146t + 12t^2 + 36t^3 + 24 \geq 0 \\ & 12 - 63t - 36t^2 + 36t^3 \geq 0 \end{aligned}$$

so $t \geq 2$ then $f(t) \geq 0$.
we can prove $\frac{52}{57} - \frac{16}{57}\sqrt{7} \leq t \leq \frac{5}{3}$

$$\begin{aligned} & 6t^2 - 10t \leq 0 \\ & -33t + 13t^2 + 3t^3 + 4 \leq 0 \\ & 6 - 32t - 13t^2 + 15t^3 \leq 0 \\ & 8 - 36t - 80t^2 + 48t^3 \leq 0 \\ & -146t + 12t^2 + 36t^3 + 24 \leq 0 \\ & 12 - 63t - 36t^2 + 36t^3 \leq 0 \end{aligned}$$

so $\frac{52}{57} - \frac{16}{57} \leq t \leq \frac{5}{3}$ then $f(t) \leq 0$

[solved by hedeng123]

2.2 Unsolved Problems

1. Letx_1, x_2, x_3, x_4 be real positive numbers and$x_1 + x_2 + x_3 + x_4 = 1$, Prove that

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \frac{x_3}{x_4} + \frac{x_4}{x_1} \geq \frac{1 - x_1}{1 - x_2} + \frac{1 - x_2}{1 - x_3} + \frac{1 - x_3}{1 - x_4} + \frac{1 - x_4}{1 - x_1}$$

2. Let $a, b, c, d > 0$, prove that

$$\frac{4}{\sqrt{13}} = \sqrt{\frac{a}{a + 4\,(b + c + d)}} + \sqrt{\frac{b}{b + 4\,(a + c + d)}} + \sqrt{\frac{c}{c + 4\,(d + a + b)}} + \sqrt{\frac{d}{d + 4\,(a + b + c)}}$$

   equality hold when $a = b = c = \frac{d}{27}$

3. Let x_1, x_2, x_3, x_4 are positive real numbers, prove that:

$$\frac{1}{6} \geq \frac{1}{(x_2 + x_3 + x_4)\,(x_1{}^{-1} + x_2{}^{-1} + x_3{}^{-1} + x_4{}^{-1})} + \frac{1}{(x_2{}^{-1} + x_3{}^{-1} + x_4{}^{-1})\,(x_1 + x_2 + x_3 + x_4)}$$

4. Let $x_1, x_2, ..., x_n$ are positive real numbers,$n \geq 3$, prove that:

$$\frac{2}{n(n-1)} \geq \frac{1}{\sum_{i=2}^{n} x_i \sum_{i=1}^{n} x_i^{-1}} + \frac{1}{\sum_{i=2}^{n} x_i^{-1} \sum_{i=1}^{n} x_i}$$

5. Let x_1, x_2, x_3, x_4 are positive real numbers, prove that:

$$\frac{1}{4} \geq \frac{1}{(x_2 + x_1)\,(x_1{}^{-1} + x_2{}^{-1} + x_3{}^{-1} + x_4{}^{-1})} + \frac{1}{(x_1{}^{-1} + x_2{}^{-1})\,(x_1 + x_2 + x_3 + x_4)}$$

CHAPTER 3

MORE THAN 4 VARIABLE INEQUALITIES

3.1 Solved Problems

1. Let $x_1, x_2, x_3, x_4 \geq 0$,prove that:

$$2(x_1+x_2+x_3+x_4)^2(x_1^2x_2^2+x_2^2x_3^2+x_3^2x_4^2+x_4^2x_1^2) \\ \geq (x_1x_2+x_2x_3+x_3x_4+x_1x_4)^2(x_1^2+x_2^2+x_3^2+x_4^2+x_1x_2+x_2x_3+x_3x_4+x_1x_4)$$

Solution:

$$0 \leq (x_1+x_2+x_3+x_4+x_5)^2\left({x_1}^2{x_2}^2+{x_2}^2{x_3}^2+{x_3}^2{x_4}^2+{x_4}^2{x_5}^2+{x_5}^2{x_1}^2\right) \\ -(x_1\,x_2+x_2\,x_3+x_3\,x_4+x_4\,x_5+x_5\,x_1)^3$$

3.2 Unsolved Problems

1. Let a, b, c, k_1, k_2, k_3be real positive numbers and $a+b+c=1$ give the minimum and the maximum of

$$\frac{a+k_1}{b+k_2}+\frac{b+k_2}{c+k_3}+\frac{c+k_3}{a+k_1}$$

in terms of k_1, k_2, k_3

2. a, b, c, x, y, z be positive numbers

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{a}+\frac{a}{b}+\frac{b}{c}+\frac{c}{x} \geq 3+\frac{x+a}{y+b}+\frac{y+b}{z+c}+\frac{z+c}{x+a}$$

3. Let x_1, x_2, x_3, x_4, x_5 be positive real numbers, prove that

$$\frac{1}{(x_1+x_2+x_3)(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}+\frac{1}{x_5})}+\frac{1}{(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3})(x_1+x_2+x_3+x_4+x_5)} \leq \frac{2}{15}$$

4. Let $x_1, x_2, x_3, x_4, x_5, x_6$ be positive real numbers, prove that

$$\frac{1}{9} \leq \frac{1}{(x_1+x_2+x_3)\left({x_1}^{-1}+{x_2}^{-1}+{x_3}^{-1}+{x_4}^{-1}+{x_5}^{-1}+{x_6}^{-1}\right)} \\ +\frac{1}{\left({x_1}^{-1}+{x_2}^{-1}+{x_3}^{-1}\right)(x_1+x_2+x_3+x_4+x_5+x_6)}$$

CHAPTER 4

N–VARIABLE INEQUALITIES

4.1 Solved Problems

4.2 Unsolved Problems

1. Let $x_i > 0, i = 1, ..., n, n \geq 3, S = \sum_{i=1}^{n} x_i$,prove that
$$n\sqrt{\frac{1}{1+(n-1)\,n}} \geq \sum_{i=1}^{n} \sqrt{\frac{x_i}{x_i + n\,(S - x_i)}}$$
equality hold $x_1 = x_2 ... = \dfrac{x_n}{(n-1)^2}$

2. Prove or disprove that if $x_i > 0$ for $i = 1, 2, \cdots, n$ where $n \geq 3$ and $x_{n+1} = x - 1, x_{n+2} = x_2$ then
$$\sum_{i=1}^{n} x_i \sum_{i=1}^{n} \frac{1}{x_i + x_{i+1}} - \frac{n^2}{2} \geq \sum_{i=1}^{n} \frac{x_i}{x_{i+1} + x_{i+1}} - \frac{n}{2}$$

3. Let x_i are positive real numbers for all $i = 1, 2, \cdots, n$, prove that:
$$\sum_{i=1}^{n} \frac{x_i + x_{i+1}}{x_{i+2}} \geq n + 2\sum_{i=1}^{n} \frac{x_i}{x_{i+1} + x_{i+2}}$$

4. Let $x_1, x_2, ..., x_n > 0,\ n \geq 2, k \geq n - 1,$, prove that:
$$\frac{2}{k+n-1} \geq \frac{x_1}{kx_1 + \sum\limits_{i=2}^{n} x_i} + \frac{x_1^{-1}}{kx_1^{-1} + \sum\limits_{i=2}^{n} x_i^{-1}}$$

5. Let $x_1, x_2, ..., x_n > 0,\ n \geq 2, k \geq n - 1,$, prove that:
$$\frac{\sum\limits_{i=2}^{n} x_i}{kx_1 + \sum\limits_{i=2}^{n} x_i} + \frac{\sum\limits_{i=2}^{n} x_i^{-1}}{kx_1^{-1} + \sum\limits_{i=2}^{n} x_i^{-1}} \geq \frac{2(n-1)}{k+n-1}$$

6. Let $x_1, x_2, ..., x_n$ are positive real numbers,$n \geq 3$, prove that:

$$\frac{2}{n(n-1)} \geq \frac{1}{\sum_{i=2}^{n} x_i \sum_{i=1}^{n} x_i^{-1}} + \frac{1}{\sum_{i=2}^{n} x_i^{-1} \sum_{i=1}^{n} x_i}$$

7. Let $x_i > 0,\ i = 1, 2, \cdots, n$

$$\frac{2}{kn} \geq \frac{1}{\sum_{i=1}^{k} x_i \sum_{i=1}^{n} x_i^{-1}} + \frac{1}{\sum_{i=1}^{k} x_i^{-1} \sum_{i=1}^{n} x_i}$$

or

$$\frac{2}{kn} \leq \frac{1}{\sum_{i=1}^{k} x_i \sum_{i=1}^{n} x_i^{-1}} + \frac{1}{\sum_{i=1}^{k} x_i^{-1} \sum_{i=1}^{n} x_i}$$

8. Let $x_i \geq 0, i = 1, 2, ..., n,\ n \geq 3, x_{n+1} = x_i$, prove that:

$$\frac{n^2}{4} \geq \sum_{i=1}^{n} \frac{x_i}{x_i + x_{i+1}} \cdot \sum_{i=1}^{n} \frac{x_i^{-1}}{x_i^{-1} + x_{i+1}^{-1}} \geq n - 1 + 2^n \left(\frac{n^2}{4} - n + 1 \right) \prod_{i=1}^{n} \frac{x_i}{x_i + x_{i+1}}$$

9. Let $x_1, x_2, ..., x_n \geq 0, n \geq 3, \sum_{i=1}^{n} x_i = S, x_0 = x_n$,prove that:

$$\sum \frac{1}{x_i^t} \left(\frac{1}{S - x_{i-1}} - \frac{1}{S - x_i} \right) \geq 0 (t \geq 0)$$

and

$$\sum \frac{1}{x_i^t} \left(\frac{1}{S - x_{i-1}} - \frac{1}{S - x_i} \right) \leq 0 (t \leq 0)$$

CHAPTER 5

GEOMETRIC INEQUALITIES

5.1 List of Symbols

a, b, c	$\Longrightarrow$	The lengths of sides of a triangle
R	$\Longrightarrow$	The circumradius
r	$\Longrightarrow$	The radius of the incircle
r_a, r_b, r_c	$\Longrightarrow$	The radii of excircles
h_a, h_b, h_c	$\Longrightarrow$	The altitudes
m_a, m_b, m_c	$\Longrightarrow$	The lengths of medians
w_a, w_b, w_c	$\Longrightarrow$	The bisectors of angles
S	$\Longrightarrow$	The area of a triangle

5.2 Solved Problems

1. Prove that in $\triangle ABC$, with the usual notations, the following inequality holds:

$$\frac{1}{8}\,a\left((s-b)^{-1}+(s-c)^{-1}\right)h_a+\frac{1}{2}\,h_a\geq w_a$$

Solution:

$$\left(\frac{1}{8}\,a\left((s-b)^{-1}+(s-c)^{-1}\right)h_a+\frac{1}{2}\,h_a\right)^2-{w_a}^2$$
$$=\frac{1}{16}\,\frac{(b-c)^2\,(a+b+c)\,(b+c-a)\left((b-c)^2\,(a+b+c)\,(b+c-a)+a^4+12\,(s-b)\,a^2\,(s-c)\right)}{(a+c-b)\,a^2\,(a+b-c)\,(b+c)^2}$$

2. In triangle, prove

$$\frac{\cos(A)\cos(B)\cos(A-B)}{1+\cos(C)}+\frac{\cos(A)\cos(C)\cos(A-C)}{1+\cos(B)}+\frac{\cos(B)\cos(C)\cos(B-C)}{1+\cos(A)}\geq 4\,\cos(A)\cos(B)\cos(C)$$

Solution:

$$u3^2\left(9\,r^2+s^2\right)+2\,u1\,r^2\left(8\,r^3+11\,Rr^2+12\,rR^2+4\,R^3\right)+2\,u10\,\left(24\,r^3+38\,Rr^2+40\,rR^2+7\,s^2R\right)+2\,u3\,R\left(16\,R^3+7\,rs^2\right)$$

[Proposed by xzlbq]

3. In acute triangle, prove

$$\sum\frac{\cos B\cos C}{1+\cos 2A}\geq\frac{3}{2}$$

[Proposed by xzlbq]

Solution 1:

$$u3^2\left(4\,R^2+r^2+s^2+4\,Rr\right)+20\,u4\,u3\,R^2\,(2\,R+r+s)+u10\,\left(8\,r^3+32\,Rr^2+8\,rs^2\right)+16\,u3\,R^3r$$

[Proposed by xzlbq]

Solution: By AM-GM,

$$\sum\frac{\cos B\cos C}{1+\cos 2A}=\frac{1}{2}\sum\frac{\cos B\cos C}{\cos^2 A}\geq\frac{3}{2}$$

[Proposed by sqing]

4. Let ABC be a triangle,prove that:

$$r_a+r_b+r_c\geq w_aw_bw_c\left(\frac{1}{w_a^2}+\frac{1}{w_b^2}+\frac{1}{w_c^2}\right)\geq\sqrt{3}s$$

Solution:

$$r_a+r_b+r_c\geq w_aw_bw_c\left(\frac{1}{w_a^2}+\frac{1}{w_b^2}+\frac{1}{w_c^2}\right)$$

$$\iff (r_a+r_b+r_c)^2 \geq \left(w_a w_b w_c\left(\frac{1}{w_a^2}+\frac{1}{w_b^2}+\frac{1}{w_c^2}\right)\right)^2 \cdots (*)$$

substitution $a=x+y, b=y+z, c=z+x$

$$(*) \iff \sum_{sym}\left(x^3y^2-x^2y^2z\right) \geq 0$$

from Muirhead theorem the inequality holds

[Proposed by hedeng123]

5.3 Unsolved Problems

1. In any triangle
$$\frac{a}{b}+\frac{b}{c}+\frac{m_a}{m_b}+\frac{m_b}{m_c} \geq 4$$

2. In acute triangle
$$\frac{a}{b}+\frac{b}{c}+\frac{w_a}{w_b}+\frac{w_b}{w_c} \geq 4$$

3. In any triangle
$$\left(\frac{2\,r_a+w_a}{r_a\,(b+c)}+\frac{2\,r_b+w_b}{r_b\,(c+a)}+\frac{2\,r_c+w_c}{r_c\,(a+b)}\right)a \geq \frac{9}{2}\,\frac{2\,w_b+2\,w_c-w_a}{w_a+w_b+w_c}$$

4. In any triangle
$$\left(\frac{2\,r_a+w_a}{r_a\,(b+c)}+\frac{2\,r_b+w_b}{r_b\,(c+a)}+\frac{2\,r_c+w_c}{r_c\,(a+b)}\right)a \geq \frac{9}{2}\,\frac{2\,h_b+2\,h_c-h_a}{h_a+h_b+h_c}$$

5. In any triangle
$$\frac{81}{2}\,(s-a)\,ar \leq w_a\,(w_a+w_b+w_c)^2$$

6. In any triangle
$$\sqrt{\cot\frac{B}{2}\cot\frac{C}{2}} \geq \frac{27}{2}\,\frac{\cos\frac{B}{2}+\cos\frac{C}{2}-\cos\frac{A}{2}}{(\cos\frac{B}{2}+\cos\frac{C}{2}+\cos\frac{A}{2})^2}$$

7. In any triangle
$$\frac{r_a}{r_b+r_c}+\frac{\cos A}{\cos B+\cos C}+\sin\frac{A}{2} \geq \frac{3}{2}$$

8. In any triangle
$$sw_a+w_bw_c\tan\frac{A}{2} \geq bw_b+cw_c;$$

9. In any triangle
$$\frac{w_a}{r_a}+\frac{w_b+w_c}{w_a} \geq 3(\frac{a}{b+c}+\frac{w_a}{w_b+w_c}) \geq 3$$

10. In any triangle
$$m_bm_c \geq \frac{\sqrt{3}}{12}a(9r+\sum h_a)$$

11. In any triangle

$$\frac{1}{2} \geq e_b(1-e_b) + e_c(1-e_c) \geq \frac{r}{R}$$

where

$$(e_b, e_c) = \left(\sin\frac{B}{2}, \sin\frac{C}{2}\right), \left(\frac{r_b}{r_b+w_b}, \frac{r_c}{r_c+w_c}\right), \left(\frac{r}{w_b-r}, \frac{r}{w_c-r}\right)$$

12. In any triangle

$$\cos\frac{B-C}{2}\cos^{-1}\frac{A}{2} \geq (\sin\frac{A}{2}+\sin\frac{B}{2})\cos^{-1}\frac{C}{2}+(\sin\frac{A}{2}+\sin\frac{C}{2})\cos^{-1}\frac{B}{2}-(\sin\frac{B}{2}+\sin\frac{C}{2})\cos^{-1}\frac{A}{2} \geq \frac{2r}{R}\cos^{-1}\frac{A}{2}$$

13. In any triangle

$$a(b+c) \geq \frac{4}{9}(w_b+w_c)\sum w_a$$

14. In any triangle

$$\frac{w_a-r}{w_b+w_c} - \frac{s-a}{s} \geq \frac{1}{2}\left(\frac{w_a}{w_b+w_c} + \frac{a}{b+c} - 1\right) \geq 0$$

15. In any triangle

$$2\sin\frac{B}{2}\sin\frac{C}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \geq \cos\frac{A}{2}(\sin B + \sin C)$$

16. In any triangle

$$\frac{3}{2}\sum\frac{w_a}{a} \geq \sum\cos^{-1}\frac{A}{2} - \sum\tan\frac{A}{2} \geq \sqrt{3}$$

17. In any triangle

$$\frac{9}{16}\frac{(b+c)^2}{s(s-a)} \geq 1 + \frac{l_b+l_c}{l_a}$$

where

$$(l_a, l_b, l_c) = (w_a, w_b, w_c), (m_a, m_b, m_c)$$

18. In any triangle

$$\frac{w_b}{s-b} + \frac{w_c}{s-c} \geq 3\sqrt{3}\left(\frac{l_b^{-1}+l_c^{-1}}{l_a^{-1}+l_b^{-1}+l_c^{-1}}\right)$$

where

$$(l_a, l_b, l_c) = (h_a, h_b, h_c), (w_a, w_b, w_c)$$

19. In any triangle

$$\frac{s}{w_b+w_c} \geq \cos\frac{A}{2}$$

20. In any triangle

$$\frac{b+c}{w_b+w_c} \geq \frac{3\sqrt{3}}{2}\frac{bc}{s^2}$$

21. In any triangle

$$\sum(w_a-w_b)(w_a-w_c)(w_a-h_a) \geq 0$$

22. In any triangle

$$2\sum\frac{r_b+r_c}{w_a} \leq \sum\frac{(r_a+r_b)(r_a+r_c)}{w_bw_c}$$

23. In any triangle

$$2\sum \frac{r_b + r_c}{w_a} + \frac{1}{8}\frac{1}{r^2}(s^2 - 31r^2 + 2Rr) \geq \sum \frac{(r_a + r_b)(r_a + r_c)}{w_b w_c}$$

24. In any triangle

$$\sum \frac{w_a - kr}{w_b + w_c} \geq \frac{3}{2} - \frac{1}{2}k\,(k \geq 0)$$

25. In any triangle

$$bw_b + cw_c \geq \frac{1}{2}aw_a + 2S + \frac{27}{2}\frac{R}{s}r^2$$

26. In any triangle

$$(a-b)[w_b + tw_c - (t+1)w_a] + (b-c)[w_c + tw_a - (t+1)w_b] \geq 0$$

where $(11 \geq t \geq -\frac{5}{6})$

27. In any triangle

$$\frac{(w_b + r_b)(w_c + r_c)}{(w_a + w_b)(w_a + w_c)} \geq \frac{r_a bc}{(r_a + w_a)^2 r} \geq 1$$

28. In any triangle

$$\sum \frac{\cos\frac{B}{2} + \cos\frac{C}{2}}{\sin\frac{B}{2} + \sin\frac{C}{2}} = \prod \frac{\cos\frac{B}{2} + \cos\frac{C}{2}}{\sin\frac{B}{2} + \sin\frac{C}{2}}$$

29. In any triangle

$$w_a\left(\frac{w_b}{r_b} + \frac{w_c}{r_c}\right) \geq w_b + w_c$$

30. In any triangle

$$\frac{3\sqrt{3}}{8}bc(b+c) \geq w_a^2(w_b + w_c)$$

31. In any triangle

$$\frac{\cos A}{\cos B + \cos C} + \frac{\sin A}{\sin B + \sin C} \geq \frac{1}{2} + \cos A$$

32. In any triangle

$$\sum \frac{w_a}{b+c} \geq \frac{1}{2}\sum \frac{-w_a + w_b + w_c}{a}$$

33. In any triangle

$$\frac{1}{2}\sqrt{\frac{(a+b)(b+c)(c+a)}{s}} + 2r \geq \sum \frac{a(s-a)}{w_a}$$

34. In any triangle

$$\frac{1}{2}R\left(\sum \frac{w_b w_c}{w_a}\right)^2 \geq \sum w_a^3$$

35. In any triangle

$$\prod \cos\frac{B-C}{2} \geq \prod\left(\cos A + \sin\frac{A}{2}\right)$$

36. In any triangle

$$\prod\left(\frac{b}{w_b} + \frac{c}{w_c}\right) \geq \frac{32}{9}\sum \frac{a}{h_a}$$

37. In any triangle

$$\sum \cos\frac{A}{2} \geq \frac{1}{2}\sum \frac{w_b+w_c}{a} \geq \frac{27}{16}\prod \frac{s-b}{w_b}+\frac{s-c}{w_c} \geq \sum \sin A$$

38. In any triangle

$$\frac{5}{2}\sum \frac{r_a}{r_b+r_c} \geq \sum (\sin\frac{B}{2}+\sin\frac{C}{2}-\sin\frac{A}{2})^2+3 \geq \frac{5}{2}\sum \frac{a}{b+c}$$

39. In any triangle

$$\sum \frac{w_a}{w_b} \geq 1+\frac{2}{3}\sum \frac{w_a}{h_a}$$

40. In any triangle

$$\sum w_a \geq \sum \frac{r(w_b+w_c)(r_a+w_a)}{bc} \geq \sum h_a$$

41. In any triangle

$$\prod \cos\frac{A}{2} \geq \prod \left(\cos\frac{B}{2}+\cos\frac{C}{2}-\cos\frac{A}{2}\right) \geq \prod \sin A$$

42. In any triangle

$$\sum w_a^5 \geq rs^5$$

43. In any triangle

$$(a+b+c)bc \geq (b+c)(w_b+w_c)w_a$$

44. In any triangle

$$\frac{2}{3}\sum h_a \geq \sum \frac{h_b+h_c}{w_a} \geq 6r$$

45. In any triangle

$$\sum w_a^3 w_b \geq 3(\frac{w_a}{w_b}+\frac{w_b}{w_c}+\frac{w_c}{w_a})S^2$$

46. In any triangle

$$\left(\sum \frac{w_a}{a}\right)^2 \geq \frac{55}{12}+\frac{13}{35}\sum \frac{b+c}{a}$$

47. In any triangle

$$\left(\sum \frac{w_a}{a}\right)^2 \geq \frac{55}{12}+\frac{13}{35}\left(1-\frac{2r}{R}\right)$$

48. In any triangle

$$\sum r_a \sum r_a^{-1} \geq \sum \frac{w_b+w_c}{w_a-r} \geq \sum a \sum a^{-1}$$

49. In any triangle

$$(2r_a+w_a)(w_a-r)(w_b+w_c) \geq 4w_a w_b w_c$$

50. In any triangle

$$w_c r_c + w_b r_b \geq w_a(w_b+w_c)$$

51. In any triangle

$$0 \geq (r_a w_a - w_b w_c)(r_b w_b - w_c w_a)(r_c w_c - w_a w_b)$$

52. In any triangle

$$\sum w_b w_c \geq S\sqrt{9+6\sum \frac{w_a}{h_a}}$$

53. In any triangle

$$\sqrt{3} \geq \sum \frac{s-a}{w_a} \geq \frac{2}{3} \sum \cos \frac{A}{2}$$

54. In any triangle

$$\frac{4}{5} w_a + \frac{9}{20} \frac{a(b+c)}{w_a} \geq w_b + w_c$$

55. In any triangle

$$\sum \frac{w_a}{r_a} \sum \frac{r_a}{w_a} \geq \sum \frac{a}{b} \sum \frac{r_a}{r_b}$$

56. In any triangle

$$(w_b + r_b)(w_c + r_c) - (w_a + w_b)(w_a + w_c) \geq 5[r_c w_c + r_b w_b - w_a(w_b + w_c)] \geq 0$$

57. In any triangle

$$\sum \frac{w_a}{a} \sum \frac{a}{w_a} \geq \sum w_a \sum \frac{1}{w_a} \geq \frac{3\sqrt{3}}{2} \sum \frac{a}{w_a}$$

58. In any triangle

$$12Rr \geq \frac{\prod w_b + w_c}{\sum w_a} \geq \frac{8\sqrt{3}}{3} S$$

59. In any triangle

$$\sum \frac{a}{w_a} \geq \sum \frac{w_b^{-1} + w_c^{-1}}{b^{-1} + c^{-1}} \geq \sum \cos^{-1} \frac{A}{2} \geq 2\sqrt{3}$$

60. In any triangle

$$(a+b)(w_a + w_b) + (a+c)(w_a + w_c) \geq (b+c)(w_b + w_c) + 4ah_a$$

61. In any triangle

$$\frac{s-a}{l_a^2} + \frac{s-b}{l_b^2} \geq \frac{s-c}{l_c^2} + \frac{l_c^2}{(s-c)s^2}$$

where

$$\begin{aligned}(l_a, l_b, l_c) &= (h_a, h_b, h_c), \text{ in Acute triangle} \\ &= (w_a, w_b, w_c) \\ &= \left(\frac{s(s-a)}{w_a}, \frac{s(s-b)}{w_b}, \frac{s(s-c)}{w_c} \right) \\ &= (m_a, m_b, m_c)\end{aligned}$$

62. In any triangle

$$w_b w_c bc + w_c w_a ca \geq w_a w_b ab + \frac{4}{3}(w_a + w_b + w_c)^2 r^2$$

63. In any triangle

$$k_2 \left(\sum \frac{a}{b+c} - \frac{3}{2} \right) \geq \sum \frac{w_a - 2r}{w_a - r} - \frac{3}{2} \geq k_1 \left(\sum \frac{a}{b+c} - \frac{3}{2} \right)$$

$max(k_1) =?$ when $min(k_2) = 3$.

64. In any triangle

$$(w_a + w_b + w_c)(2r_a + w_a) \geq 9 w_b w_c$$

65. In any triangle

$$\sum\left(\frac{r_a h_a}{w_b w_c}-\frac{h_a}{w_a}\right)(a-b)(a-c)\geq 0$$

66. In any triangle

$$\frac{1}{2}\sum((r_b-r_c)^2)\geq\sum(w_b-r_b)(r_c-w_c)\geq\frac{1}{2}\sum((b-c)^2)$$

67. In any triangle

(a)

$$\sum\frac{r_a}{r_a+2w_a}\geq 1$$

(b)

$$\sum\frac{r_a}{r_a+2^{-1}w_a}\leq 2$$

(c)

$$\sum\frac{r_a}{r_a+2m_a}\geq 1$$

68. In any triangle

$$\frac{9}{8}\frac{b+c}{w_a+w_b+w_c}\geq\cos\frac{A}{2}$$

69. In any triangle

$$w_a-r\geq\left(\frac{l_b l_c-r_a r}{r_a+l_a}\right)\left(\frac{b+c}{a}\right)$$

where

$$(l_a,l_b,l_c)=(h_a,h_b,h_c),(w_a,w_b,w_c)$$

70. In any triangle

$$\frac{a}{b+c}\frac{l_a}{l_a-r}\geq\frac{(w_b+w_c)^2}{(b+c)^2}$$

where

$$(l_a,l_b,l_c)=(h_a,h_b,h_c),(w_a,w_b,w_c)$$

71. In any triangle

$$\frac{w_a-r}{b+c-a}+\frac{w_b-r}{c+a-b}+\frac{w_c-r}{a+b-c}\geq\sqrt{3}$$

72. In any triangle

$$2\sum\cos\frac{A}{2}\geq\frac{b+c-a}{w_a-r}+\frac{c+a-b}{w_b-r}+\frac{a+b-c}{w_c-r}\geq 2\sum\sin A$$

73. In any triangle

$$\frac{1}{r}+\frac{1}{h_a}\geq\frac{1}{w_b}+\frac{1}{w_c}$$

74. In any triangle

$$1+\frac{2(h_a+h_b+h_c)}{w_a+w_b+w_c}\geq\sum\frac{h_a}{w_a}$$

75. In any triangle

$$1+\frac{1}{3}\sum\sin^{-1}\frac{A}{2}\geq\sum\cos^{-1}\frac{B-C}{2}$$

76. Prove nice triangle Inequality:

$$\sqrt{w_b w_c}(\frac{1}{r_a} + \frac{1}{w_a}) \geq 2$$

77. Prove nice triangle Inequality:

$$\sqrt{r_a w_a}(\frac{1}{w_b} + \frac{1}{w_c}) \geq 2$$

78. Let ABC be a triangle. Prove that

$$\frac{3\sqrt{3}}{2} \geq \frac{3}{8}\sum \frac{\sin B + \sin B}{\cos\frac{B}{2}\cos\frac{C}{2}} \geq \sum \cos\frac{A}{2}$$

79. Let ABC be a triangle,prove that:

$$\sum \frac{a}{w_a} \geq \sum \frac{(b+c)r_a}{(w_a+r_a)w_a} \geq \frac{8}{3}\sum \frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\sin B + \sin C} \geq \sum \frac{1}{\cos\frac{A}{2}}$$

80. Let ABC be a triangle,prove that:

$$2\sum \frac{\sin\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}} \geq \sum \frac{\cos B + \cos C}{\sin\frac{B}{2}\sin\frac{C}{2}} \geq \sum \frac{1}{\sin\frac{B}{2}\sin\frac{C}{2}}$$

81. Let ABC be a triangle,prove that:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 4\sum(\frac{h_a}{h_a+h_b})^2 \geq \frac{a+b}{b+c} + \frac{b+c}{c+a} + \frac{c+a}{a+b}$$

82. Let ABC be a triangle,prove that:

$$\frac{w_a^2}{(w_a+r_b)^2} + \frac{w_b^2}{(w_b+r_c)^2} + \frac{w_c^2}{(w_c+r_a)^2} \geq \frac{3}{4}$$

83. Let ABC be a triangle:

$$\frac{h_a^t}{(h_a+r_b)^t} + \frac{h_b^t}{(h_b+r_c)^t} + \frac{h_c^2}{(h_c+r_a)^t} \geq \frac{3}{2^t}$$

prove that : if $t = 4$,then this inequality holds and $t(min) =?$

84. Let ABC be a triangle:

$$\sum a^2 \geq 4\sqrt{3}S + \frac{1}{2}\sum(\frac{a}{b+c} + \frac{b+c}{a})(b-c)^2$$

85. Let ABC be a triangle:

$$\sum a^2 \geq 4\sqrt{3}S + \frac{1}{2}\sum(\frac{a}{b+c} + \frac{b+c}{a})(w_b-w_c)^2$$

86. Let ABC be a triangle, prove that:

$$\sum \frac{r_a}{r_b} \geq \sum \frac{h_b h_c}{w_a^2} \geq \frac{\sum\cot\frac{A}{2}}{\sum\tan\frac{A}{2}} \geq \sum \frac{a}{b} \geq \sqrt{3}\sqrt{\frac{\sum\cot\frac{A}{2}}{\sum\tan\frac{A}{2}}} \geq \sum \frac{a+b}{b+c}$$

87. Let ABC be a triangle, prove that:

$$\sum \frac{a}{b} \geq 2\sum \frac{h_a}{h_a+3r} \geq \sum \frac{a+b}{b+c}$$

88. Let ABC be a triangle, prove that:

$$\frac{\sin A}{\cos\frac{B}{2}}+\frac{\sin B}{\cos\frac{C}{2}}+\frac{\sin C}{\cos\frac{A}{2}}\le 3$$

89. Let ABC be a triangle, prove that:

$$2\sum\sin\frac{A}{2}\ge\frac{\sin A}{\cos\frac{B}{2}}+\frac{\sin B}{\cos\frac{C}{2}}+\frac{\sin C}{\cos\frac{A}{2}}$$

90. In any triangle

$$\frac{\sqrt{3}}{2}\sum\frac{a}{w_a}\ge\sum\frac{a}{b}$$

91. Let ABC is a triangle, w_a, w_b, w_c are bisectors of angles, h_a, h_b, h_c are altitudes respectively, r is radius of the incircle

$$h_a+h_b+h_c\ge 3(\frac{(s-b)(s-c)w_a}{w_bw_c}+\frac{(s-c)(s-a)w_b}{w_cw_a}+\frac{(s-a)(s-b)w_c}{w_aw_b})\ge 9r$$

92. Let ABC is a triangle, w_a, w_b, w_c are bisectors of angles, h_a, h_b, h_c are altitudes respectively, r is radius of the incircle, prove that:

$$\frac{s-a}{w_a}+\frac{s-b}{w_b}+\frac{s-c}{w_c}+\frac{9}{8}\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\ge\frac{9}{8}\frac{w_a+w_b+w_c}{s}$$

93. Let ABC is a triangle, w_a, w_b, w_c are bisectors of angles, prove that:

$$\frac{w_a}{h_a}+\frac{w_b}{h_b}+\frac{w_c}{hc}\ge 1+\frac{4abc}{(a+b)(b+c)(c+a)}(\frac{a}{w_a}\cot\frac{A}{2}+\frac{b}{w_b}\cot\frac{B}{2}+\frac{c}{w_c}\cot\frac{C}{2}-2)$$

94. In any triangle

$$\left(\frac{1}{w_a-r}+\frac{1}{w_b-r}+\frac{1}{w_c-r}\right)(w_a+w_b+w_c)\ge\frac{27}{2}$$

95. In any triangle

$$\left(\frac{1}{h_a-r}+\frac{1}{h_b-r}+\frac{1}{h_c-r}\right)(h_a+h_b+h_c)\ge\frac{27}{2}$$

96. In any triangle

$$\frac{1}{2}\left(\frac{1}{w_a-2r}+\frac{1}{w_b-2r}+\frac{1}{w_c-2r}\right)\ge\frac{1}{w_a-r}+\frac{1}{w_b-r}+\frac{1}{w_c-r}$$

97. Let a, b, c denote the sides of a triangle opposite the angles $\angle A, \angle B$, and $\angle C$, respectively. Let R the circumradius of the triangle. If angle $\angle A > 90^\circ$, h_a is altitudes prove that

$$\frac{a^2}{m_a^2}+\frac{8}{3}\frac{h_a}{m_a}\ge 4$$

98. Let a, b, c denote the sides of a triangle opposite the angles $\angle A, \angle B$, and $\angle C$, respectively. Let R the circumradius of the triangle. If angle $\angle A > 90^\circ$ prove that

$$\frac{b}{m_b}+\frac{c}{m_c}\le\frac{4\sqrt{3}}{3}$$

99. Let a, b, c denote the sides of a triangle opposite the angles $\angle A, \angle B$, and $\angle C$, respectively. Let R the circumradius of the triangle. If angle $\angle A > 90^\circ$, h_a is altitudes prove that

$$m_b + m_c - 2m_a \geq \frac{22}{3}(m_a - h_a)$$

100. In any triangle

$$\frac{w_b + w_c}{w_a} + 9\frac{r_a + w_a}{2r_a + w_a} \geq 8$$

101. In any trianle

$$\frac{w_b^t + w_c^t}{w_a^t} + 9\frac{r_a^t + w_a^t}{2r_a^t + w_a^t} \geq 8$$

where $t \in \mathbb{R}$

102. Let is a triangle, w_a, w_b, w_c be bisectors of angles,$a = max(a, b, c)$, prove that:

$$\frac{w_a^{-1}}{w_b^{-1} + w_c^{-1}} \geq \frac{r_a}{r_a + w_a}$$

whether holds

$$\frac{w_a^{-1}}{w_b^{-1} + w_c^{-1}} \geq \frac{r_a}{r_a + m_a}$$

103. Let ABC is a triangle, $a = max(a, b, c)$, prove that:

$$\frac{a}{b + c} + \frac{h_a}{h_b + h_c} \geq 1$$

104. Let is a triangle, w_a, w_b, w_c be bisectors of angles,prove that:

$$\frac{w_a^{-1}}{w_b^{-1} + w_c^{-1}} \geq \sin\frac{A}{2}$$

105. Let is a triangle, w_a, w_b, w_c be bisectors of angles,prove that:

$$\frac{\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}}{\sin\frac{A}{2}} \geq \frac{4w_a + w_b + w_c}{a}$$

106. Let is a triangle, w_a, w_b, w_c be bisectors of angles,prove that:

$$\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right)\cos\frac{A}{2} \leq \frac{4w_a + w_b + w_c}{8r}$$

107. In any triangle

$$\left(\frac{1}{w_a} + \frac{1}{w_b} + \frac{1}{w_c}\right)\frac{1}{m_a} \geq \frac{16}{(b + c)^2}$$

108. In any triangle

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \leq \frac{3(4R + r)}{s^2}$$

109. Let ABC be a triangle,prove that:

$$\frac{b+c}{a}+\frac{x_b+x_c}{x_a}\geq\frac{x_b+x_c}{x_a}\frac{b+c}{a}$$

where

$$(x_a,x_b,x_c)=(m_a,m_b,m_c)\,,(w_a,w_b,w_c)\,,\left(\frac{s(s-a)}{w_a},\frac{s(s-b)}{w_b},\frac{s(s-c)}{w_c}\right)$$

110. Let ABC be a triangle,prove that:

$$\frac{3}{2}\frac{b+c}{a+b+c}\geq\frac{\frac{w_b}{s-b}+\frac{w_c}{s-c}}{\frac{a}{w_a}+\frac{b}{w_b}+\frac{c}{w_c}}$$

111. In an triangle

$$\frac{R}{2r}\geq\frac{1}{6}\frac{r_a+r_b+r_c}{h_a}+\frac{1}{6}\frac{m_a}{r}\geq\frac{m_a}{h_a}$$

112. In a triangle

$$\frac{R}{2r}\geq\frac{r}{h_a}+\frac{1}{2}\frac{b^2+c^2}{h_a(h_b+h_c)}\geq\frac{m_a}{h_a}$$

113. In acute triangle, prove

$$\sum\frac{\sin B\sin C\cos{(B-C)}}{1+\cos 2A}\geq\frac{9}{2}$$

Equality holds when $a=b=kc\,k=\frac{1}{8}(1+\sqrt{17})\,k=\frac{1}{8}(-1+\sqrt{17})$

114. In acute triangle, prove that

$$\sum\left(\sqrt{\frac{1}{\cos{(A)}\,(1+\cos{(A)})}}\right)\geq 9\,\frac{1}{\sin{(A)}+\sin{(B)}+\sin{(C)}}$$

115. In acute triangle, prove that

$$\sqrt{\frac{\cos{(B-C)}}{\cos{(A)}\,(1+\cos{(A)})}}+\sqrt{\frac{\cos{(A-C)}}{\cos{(B)}\,(1+\cos{(B)})}}+\sqrt{\frac{\cos{(A-B)}}{\cos{(C)}\,(1+\cos{(C)})}}\geq 2\sqrt{3}$$